General Relativistic Hydrodynamical System in Spherically Symmetric Metric with Horizon Penetrating Coordinate

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We present a horizon penetrating coordinate for spherically symmetric metric. We adopt $G = c = 1$ unit system. Consider usual Schwarzschild line element in BL coordinate for practice

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

where M is a mass. Compare this with usual 3+1 line element form $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^2 + \beta^i dt)(dx^j + \beta^j dt)$ we can identify the lapse $\alpha^2 = \left(1 - \frac{2M}{r}\right)$. As we know, the line element(Eqn. 1) is singular at the horizon ($r = 2M$) and lapse collapse to zero. This can be problematic because equations of motion for the metric can be become exponentially unstable in the presence of a coordinate singularity without some regularization technique.

One way to resolve this problem is to move to a horizon penetrating coordinate system where this singularity is not present. The Kerr-Schild coordinates are one such coordinate system.

For example, Schwarzschild solution in spherical type Kerr-Schild coordinates

$$\alpha = \sqrt{\frac{r}{r+2M}} \tag{2}$$

$$\beta^r = \frac{2M}{r+2M} \tag{3}$$

$$\beta_r = \frac{2M}{r} \tag{4}$$

$$\beta^\theta = \beta^\varphi = 0 \tag{5}$$

$$K_{ij} = \text{diag}\left[-\frac{2M(r+M)}{\sqrt{r^5(r+2M)}}, 2M\sqrt{\frac{r}{r+2M}}, K_{\theta\theta}\sin^2\theta\right] \tag{6}$$

Schwarzschild solution in Cartesian type Kerr-Schild coordinate

$$\alpha = \sqrt{\frac{r}{r+2M}} \tag{7}$$

$$\beta^i = \frac{2M}{r}\frac{x^i}{r+2M} \tag{8}$$

$$\beta_i = \frac{2Mx_i}{r^2} \tag{9}$$

$$K_{ij} = \frac{2M}{r^4}\sqrt{\frac{r}{r+2M}}\left[\left(\frac{M}{r}+2\right)x_i x_j - r^2\delta_{ij}\right] \tag{10}$$

where $x^i = (x, y, z)$ which is usual spatial Cartesian coordinate. In both cases, we can see lapse is regular at the horizon.

General spherical symmetric line element in polar-areal form

$$ds^2 = -\alpha(r)^2 dt^2 + a(r)^2 dr^2 + r^2 d\Omega^2 \tag{11}$$

where α is referred as lapse function. Compare with above Schwarzschild solution, $\alpha = 1/a$.

Now consider a transformation of the Schwarzschild time t coordinate to a new generic coordinate $\hat{t}$ according to

$$d\hat{t} = dt + a^2\sqrt{1 - \frac{g}{a^2}}dr \tag{12}$$

where $g(r)$ is arbitrary function. Substitute this into $ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$ gives

$$
\begin{aligned}
ds^2 &= -\alpha^2 \left(d\hat{t} - a^2 \sqrt{1 - \frac{g}{a^2}} dr \right)^2 + a^2 dr^2 + r^2 d\Omega^2 \\
&= -\alpha^2 d\hat{t}^2 + 2\sqrt{1 - \frac{g}{a^2}} d\hat{t} dr + g dr^2 + r^2 d\Omega^2 \qquad (13)
\end{aligned}
$$

Compare this with usual 3+1 framework

$$ds^2 = -\alpha^2 d\hat{t}^2 + \gamma_{ij}(dx^i + \beta^i d\hat{t})(dx^j + \beta^j d\hat{t}) \qquad (14)$$

and so into the lapse $\alpha = 1/\sqrt{g}$, the shift $\beta_i = (\sqrt{1 - g/a^2}, 0, 0)$ or $\beta^i = \gamma^{ij}\beta_j$ and the spatial metric of the constant $\hat{t}$ hypersurface $\gamma_{ij} = diag(g, r^2, r^2 \sin^2\theta)$.

If we choose $\alpha = \sqrt{1 - 2M/r} = 1/a$ and $g = 1 + 2M/r$ like in previous (which we will use this), we get

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\hat{t}^2 + \frac{4M}{r} d\hat{t} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \qquad (15)$$

which is Schwarzschild in Kerr-Schild coordinate (or Eddington-Finkelstein coordinate). And correspondingly, $\alpha = \sqrt{r/(r+2M)}$, $\beta_i = (2M/r, 0, 0)$, and $\gamma_{ij} = diag(1 + 2M/r, r^2, r^2 \sin^2\theta)$ which are same as above.

As you can see here, the KS (or EF) form of the metric represents an analytic expansion of the Schwarzschild solution from the region $2M < r < \infty$ to $0 < r < \infty$. Thus, we apply this coordinate transformation for our equations.

It is good to rewrite the metric into usual $3+1$ variable form i.e. keep it geometric variables (should be careful of confusion) with considering time dependent case (This is almost same as Marsa and Choptuik's paper). Here, we use t for time coordinate that we used above.

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \qquad (16)$$

where α, a, b, and β are functions of r and t, and $d\Omega^2$ is the metric of unit sphere. From this, we can calculate non-vanishing components of connection coefficients and Ricci tensors for i, j, and k (spatial indices)

$$
\begin{aligned}
&\Gamma^r{}_{rr} = \frac{\partial_r a}{a}, \quad \Gamma^r{}_{\theta\theta} = -\frac{rb\partial_r(rb)}{a^2}, \quad \Gamma^\theta{}_{r\theta} = \frac{\partial_r(rb)}{rb} \\
&\Gamma^r{}_{\varphi\varphi} = -\sin^2\theta \frac{rb\partial_r(rb)}{a^2}, \quad \Gamma^\varphi{}_{r\varphi} = \Gamma^\theta{}_{r\theta} \\
&\Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^\varphi{}_{\varphi\theta} = -\cot\theta
\end{aligned}
$$

$$
\begin{aligned}
R^r{}_r &= -\frac{2}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right) \qquad (17) \\
R^\theta{}_\theta &= \frac{1}{ar^2b^2}\left[a - \partial_r\left(\frac{rb\partial_r(rb)}{a}\right)\right] \qquad (18) \\
& \qquad (19)
\end{aligned}
$$

I. THEORETICAL MODEL

Here, as a beginning set up, we first use the perfect fluid approximation for the matted model. So the stress-energy tensor takes form

$$T_{ab} = (\rho + P)u_a u_b + P g_{ab} \qquad (20)$$

where $u^a(r,t)$ is the 4-velocity of a given perfect element, $P(r,t)$ is the isotropic pressure, $\rho(r,t) = \rho_0(r,t)(1 + \epsilon(r,t))$ is the energy density, $\rho_0(r,t)$ is the rest-mass energy density, and $\epsilon(r,t)$ is the specific internal energy.

The equations of motion for this case are derive from the local conservative equations for energy and baryon number such that

$$\nabla_a T^a{}_b = 0 \tag{21}$$
$$\nabla_a(\rho_0 u^a) = 0 \tag{22}$$

First, we define variables.

$$D = \rho_0 W \tag{23}$$
$$E = \rho_0 h W^2 - P \tag{24}$$
$$S = \rho_0 h W^2 v \tag{25}$$
$$\tau = E - D \tag{26}$$

where W is Lorentz factor such that $W = \alpha u^t = 1/\sqrt{1-v^2}$ with fluid velocity $v = (au^r)/(\alpha u^t)$ and $h = 1 + \epsilon + P/\rho_0$ which is specific enthalpy. In our case, $u^a = (u^t, u^r, 0, 0)$

And nonzero components of T^{ab} which we are using

$$T^t{}_t = -E \tag{27}$$
$$T^t{}_r = \frac{a}{\alpha} S \tag{28}$$
$$T^r{}_r = Sv + P \tag{29}$$
$$T^\theta{}_\theta = T^\varphi{}_\varphi = P \tag{30}$$

Using these variables under the metric which we consider, $\nabla_a(\rho_0 u^a) = 0$ (continuity equation) gives

$$\partial_t(\rho_0 u^t) + \Gamma^t{}_{tt}(\rho_0 u^t) + \Gamma^t{}_{tr}(\rho_0 u^r) = 0 \tag{31}$$

In terms of our variables, $\rho_0 u^t = D/\alpha$, $\rho_0 u^r = Dv/a$ so

$$\partial_t(D/\alpha) + \Gamma^t{}_{tt}(D/\alpha) + \Gamma^t{}_{tr}(Dv/a) = 0 \tag{32}$$

$\nabla_a T^a{}_b = 0$ gives

$$\partial_t T^t{}_b + \Gamma^t{}_{tc} T^c{}_b - \Gamma^c{}_{tb} T^t{}_c = 0 \tag{33}$$

The covariant t-component of above equation gives energy equation $T^t{}_t = -E$

$$\begin{aligned}
&\partial_t T^t{}_t + \Gamma^t{}_{tc} T^c{}_t - \Gamma^c{}_{tt} T^t{}_c = 0 \\
&\rightarrow \partial_t T^t{}_t + \Gamma^t{}_{tt} T^t{}_t + \Gamma^t{}_{tr} T^r{}_t - \Gamma^t{}_{tt} T^t{}_t - \Gamma^r{}_{tt} T^t{}_r = 0 \\
&\rightarrow \partial_t E + \Gamma^t{}_{tr} \frac{a^3}{\alpha^3} S + \Gamma^r{}_{tt} \frac{a}{\alpha} S = 0
\end{aligned} \tag{34}$$

Next, consider the covariant r-component

$$\begin{aligned}
&\partial_t T^t{}_r + \Gamma^t{}_{tc} T^c{}_r - \Gamma^c{}_{tr} T^t{}_c = 0 \\
&\rightarrow \partial_t T^t{}_r + \Gamma^t{}_{tt} T^t{}_r + \Gamma^t{}_{tr} T^r{}_r - \Gamma^t{}_{tr} T^t{}_t - \Gamma^r{}_{tr} T^t{}_r = 0 \\
&\rightarrow \partial_t \left(\frac{a}{\alpha} S\right) + \Gamma^t{}_{tt}(\frac{a}{\alpha} S) + \Gamma^t{}_{tr}(Sv + P) + \Gamma^t{}_{tr} E - \Gamma^r{}_{tr} \frac{a}{\alpha} S = 0
\end{aligned}$$

Non-vanishing connection coefficients are evaluated via Mathematica. You can find it here

Now consider the Einstein's equations. Define below quantities that are appearing in the 3+1 equations

$$\rho_{hydro} = n_a n_b T^{ab} = \rho_0 h W^2 - P \tag{35}$$
$$S_i^{hydro} = -\gamma_{ia} n_b T^{ab} = \rho_0 h W u_i \tag{36}$$
$$S_{ij}^{hydro} = \gamma_{ia}\gamma_{ib} T^{ab} = P\gamma_{ij} + \rho_0 h u_i u_j \tag{37}$$
$$S_{hydro} = \gamma^{ij} S_{ij} = 3P + \rho_0 h (W^2 - 1) \tag{38}$$

The Einstein's equations in the ADM form are

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \tag{39}$$

$$\begin{aligned}\partial_t K^i{}_j &= \alpha(R^i{}_j + KK^i{}_j) - D^iD_j\alpha - 8\pi\alpha\left(S^i{}_j - \frac{1}{2}\delta^i{}_j(S-\rho)\right)\\ &\quad + \beta^k\partial_k K^i{}_j + K^i{}_k\partial_j\beta^k - K^k{}_j\partial_k\beta^i\end{aligned} \tag{40}$$

where D_i is covariant derivative on spatial hypersurface. Momentum and Hamiltonian constraints are

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \tag{41}$$

$$D_iK^i{}_j - D_jK = 8\pi S_j \tag{42}$$

Substitute hydro source terms (ρ, S etc) from above then we have

$$\begin{aligned}\partial_t K^i{}_j &= \alpha(R^i{}_j + KK^i{}_j) - \gamma^{ik}(\partial_i\partial_k\alpha - \Gamma^l{}_{ik}\partial_l\alpha) - 8\pi\alpha\left(\frac{1}{2}\delta^i{}_j(\rho_0 h - 2P) + \rho_0 h u^i u_j\right)\\ &\quad + \beta^k\partial_k K^i{}_j + K^i{}_k\partial_j\beta^k - K^k{}_j\partial_k\beta^i\end{aligned} \tag{43}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \tag{44}$$

$$R + K^2 - K_{ij}K^{ij} = 16\pi(\rho_0 h W^2 - P) \tag{45}$$

$$D_iK^i{}_j - D_jK = 8\pi\rho_0 h W u_j \tag{46}$$

From our choice of metric/coordinate system, we calculated non-trivial connection coefficients and Ricci tensors. Also, metric form suggests that $\beta^i = (\beta^r, 0, 0)$, $K^i{}_j = diag(K^r{}_r, K^\theta{}_\theta, K^\theta{}_\theta)$. Using these facts, the evolution equations for geometric quantities are

$$\partial_t a = -\alpha a K^r{}_r + \partial_r(a\beta^r) \tag{47}$$

$$\partial_t b = -\alpha b K^\theta{}_\theta + \frac{\beta^r}{r}\partial_r(r\beta^r) \tag{48}$$

$$\begin{aligned}\partial_t K^r{}_r &= \beta^r\partial_r K^r{}_r + \alpha K^r{}_r K - \frac{1}{a}\partial_r\left(\frac{\partial_r\alpha}{a}\right) - \frac{2\alpha}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right)\\ &\quad - 4\pi\alpha\left[(1+2u^r u_r)\rho_0 h - 2P\right]\end{aligned} \tag{49}$$

$$\begin{aligned}\partial_t K^\theta{}_\theta &= \beta^r\partial_r K^\theta{}_\theta + \alpha K^\theta{}_\theta K - \frac{\alpha}{r^2b^2} - \frac{1}{ar^2b^2}\partial_r\left(\frac{\alpha rb\partial_r(rb)}{a}\right)\\ &\quad - 4\pi\alpha(\rho_0 h - 2P)\end{aligned} \tag{50}$$

From constraints

$$\begin{aligned}\frac{1}{ar^2b^2}\left[a - \partial_r\left(\frac{rb\partial_r(rb)}{a}\right)\right] - \frac{2}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right) + 2K^\theta{}_\theta(K^\theta{}_\theta + 2K^r{}_r) &= 16\pi(\rho_0 h W^2 - P)\\ \frac{\partial_t(rb)}{rb}(K^\theta{}_\theta - K^r{}_r) - \partial_r K^\theta{}_\theta &= 4\pi\rho_0 h W u_r\end{aligned} \tag{51}$$

We can apply different choice of slicing (i.e. gauge choice) to reduce/determine above system. Possible (or simple) choices would be maximal or polar slicing.

A. Choice of Gauge

First, we consider maximal slicing i.e. $K = \partial_t K = 0$ then

$$\partial_t a = -\alpha a K^r{}_r + \partial_r(a\beta^r) \tag{52}$$
$$\partial_t b = -\alpha b K^\theta{}_\theta + \frac{\beta^r}{r}\partial_r(r\beta^r) \tag{53}$$
$$\partial_t K^r{}_r = \beta^r \partial_r K^r{}_r - \frac{1}{a}\partial_r\left(\frac{\partial_r \alpha}{a}\right) - \frac{2\alpha}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right) - 4\pi\alpha\left[(1+2u^r u_r)\rho_0 h - 2P\right] \tag{54}$$
$$\partial_t K^\theta{}_\theta = \beta^r \partial_r K^\theta{}_\theta - K - \frac{\alpha}{r^2 b^2} - \frac{1}{ar^2b^2}\partial_r\left(\frac{\alpha r b \partial_r(rb)}{a}\right) - 4\pi\alpha(\rho_0 h - 2P) \tag{55}$$

From constraints

$$\frac{1}{ar^2b^2}\left[a - \partial_r\left(\frac{rb\partial_r(rb)}{a}\right)\right] - \frac{2}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right) - \left[(K^r{}_r)^2 + 2(K^\theta{}_\theta)^2\right] = 16\pi(\rho_0 h W^2 - P)$$
$$\frac{2\partial_t(rb)}{rb}(K^\theta{}_\theta - K^r{}_r) + \partial_r K^r{}_r = 8\pi\rho_0 h W u_r \tag{56}$$

Fluid EOM parts are same as previous

Another possible choice to set the lapse is demanding that the ingoing combination of tangent vectors $\vec{\partial}_t - \vec{\partial}_r$ be null. This gives a condition on the metric : $g_{tt} - 2g_{tr} + g_{rr} = 0$. This gives $\alpha = a(1-\beta)$

B. Initial Data

Our initial NS model is approximated by solution of TOV. After the initial data calculation, an in-going velocity profile is added to drive the star to collapse. We follow the way is described in (https://arxiv.org/pdf/gr-qc/0107045.pdf). First, specifying the coordinate velocity

$$U \equiv \frac{dr}{dt} = \frac{u^r}{u^r} \tag{57}$$

of the star. In general, the profile take the algebraic form $U_g(x) = A_0(x^3 - B_0 x)$ where $x \equiv= r/R_{star}$ and R_{star} is the radius of the TOV solution.

In this work, we set three profiles

$$U(x) = \begin{cases} U_1(x) = U_{crit}(x^3 - 3x) & x < x_{tlv} \\ U_2(x) = \frac{U_{amp}}{2}(x^3 - 3x) & x_{tlv} < x < 1 \\ U_3(x) = \frac{27U_{amp}}{10\sqrt{5}}\left(x^3 - \frac{5x}{3}\right) & \text{otherwise} \end{cases} \tag{58}$$

where U_{amp} is amplitude of ingoing velocity profile, U_{crit} is the amplitude that occurs critical collapse, and x_{tlv} the region that forms black hole.

Our interest is interaction between BH inside of NS so $U_{amp} > U_{crit}$ and we set x_{tlv} is small value such as 1% of size of star

C. Analytic Case

For code test and validation, we use well-known Michel problem