

General Relativistic Hydrodynamical System in Spherically Symmetric Metric with Horizon Penetrating Coordinate

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We present a horizon penetrating coordinate for spherically symmetric metric. We adopt $G = c = 1$ unit system. Consider usual Schwarzschild line element in BL coordinate for practice

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (1)$$

where M is a mass. Compare this with usual 3+1 line element form $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$ we can identify the lapse $\alpha^2 = \left(1 - \frac{2M}{r}\right)$. As we know, the line element (Eqn. 1) is singular at the horizon ($r = 2M$) and lapse collapse to zero. This can be problematic because equations of motion for the metric can become exponentially unstable in the presence of a coordinate singularity without some regularization technique.

One way to resolve this problem is to move to a horizon penetrating coordinate system where this singularity is not present. The Kerr-Schild coordinates are one such coordinate system.

For example, Schwarzschild solution in spherical type Kerr-Schild coordinates

$$\alpha = \sqrt{\frac{r}{r + 2M}} \quad (2)$$

$$\beta^r = \frac{2M}{r + 2M} \quad (3)$$

$$\beta_r = \frac{2M}{r} \quad (4)$$

$$\beta^\theta = \beta^\varphi = 0 \quad (5)$$

$$K_{ij} = \text{diag} \left[-\frac{2M(r + M)}{\sqrt{r^5(r + 2M)}}, 2M\sqrt{\frac{r}{r + 2M}}, K_{\theta\theta} \sin^2 \theta \right] \quad (6)$$

Schwarzschild solution in Cartesian type Kerr-Schild coordinate

$$\alpha = \sqrt{\frac{r}{r + 2M}} \quad (7)$$

$$\beta^i = \frac{2M}{r} \frac{x^i}{r + 2M} \quad (8)$$

$$\beta_i = \frac{2M x_i}{r^2} \quad (9)$$

$$K_{ij} = \frac{2M}{r^4} \sqrt{\frac{r}{r + 2M}} \left[\left(\frac{M}{r} + 2 \right) x_i x_j - r^2 \delta_{ij} \right] \quad (10)$$

where $x^i = (x, y, z)$ which is usual spatial Cartesian coordinate. In both cases, we can see lapse is regular at the horizon.

General spherical symmetric line element in polar-areal form

$$ds^2 = -\alpha(r)^2 dt^2 + a(r)^2 dr^2 + r^2 d\Omega^2 \quad (11)$$

where α is referred as lapse function. Compare with above Schwarzschild solution, $\alpha = 1/a$.

Now consider a transformation of the Schwarzschild time t coordinate to a new generic coordinate \hat{t} according to

$$d\hat{t} = dt + a^2 \sqrt{1 - \frac{g}{a^2}} dr \quad (12)$$

where $g(r)$ is arbitrary function. Substitute this into $ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$ gives

$$\begin{aligned} ds^2 &= -\alpha^2 \left(d\hat{t} - a^2 \sqrt{1 - \frac{g}{a^2}} dr \right)^2 + a^2 dr^2 + r^2 d\Omega^2 \\ &= -\alpha^2 d\hat{t}^2 + 2\sqrt{1 - \frac{g}{a^2}} d\hat{t} dr + g dr^2 + r^2 d\Omega^2 \end{aligned} \quad (13)$$

Compare this with usual 3+1 framework

$$ds^2 = -\alpha^2 d\hat{t}^2 + \gamma_{ij} (dx^i + \beta^i d\hat{t}) (dx^j + \beta^j d\hat{t}) \quad (14)$$

and so into the lapse $\alpha = 1/\sqrt{g}$, the shift $\beta_i = (\sqrt{1 - g/a^2}, 0, 0)$ or $\beta^i = \gamma^{ij} \beta_j$ and the spatial metric of the constant \hat{t} hypersurface $\gamma_{ij} = \text{diag}(g, r^2, r^2 \sin^2 \theta)$.

If we choose $\alpha = \sqrt{1 - 2M/r} = 1/a$ and $g = 1 + 2M/r$ like in previous (which we will use this), we get

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\hat{t}^2 + \frac{4M}{r} d\hat{t} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \quad (15)$$

which is Schwarzschild in Kerr-Schild coordinate (or Eddington-Finkelstein coordinate). And correspondingly, $\alpha = \sqrt{r/(r + 2M)}$, $\beta_i = (2M/r, 0, 0)$, and $\gamma_{ij} = \text{diag}(1 + 2M/r, r^2, r^2 \sin^2 \theta)$ which are same as above.

As you can see here, the KS (or EF) form of the metric represents an analytic expansion of the Schwarzschild solution from the region $2M < r < \infty$ to $0 < r < \infty$. Thus, we apply this coordinate transformation for our equations.

It is good to rewrite the metric into usual 3 + 1 variable form i.e. keep it geometric variables (should be careful of confusion) with considering time dependent case (This is almost same as Marsa and Choptuik's paper). Here, we use t for time coordinate that we used above.

$$ds^2 = (-\alpha^2 + a^2 \beta^2) dt^2 + 2a^2 \beta dt dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \quad (16)$$

where α , a , b , and β are functions of r and t , and $d\Omega^2$ is the metric of unit sphere. From this, we can calculate non-vanishing components of connection coefficients and Ricci tensors for i, j , and k (spatial indices)

$$\begin{aligned} \Gamma_{rr}^r &= \frac{\partial_r a}{a}, & \Gamma_{\theta\theta}^r &= -\frac{rb \partial_r (rb)}{a^2}, & \Gamma_{r\theta}^\theta &= \frac{\partial_r (rb)}{rb} \\ \Gamma_{\varphi\varphi}^r &= -\sin^2 \theta \frac{rb \partial_r (rb)}{a^2}, & \Gamma_{r\varphi}^\varphi &= \Gamma_{r\theta}^\theta \\ \Gamma_{\varphi\varphi}^\theta &= -\sin \theta \cos \theta, & \Gamma_{\varphi\theta}^\varphi &= -\cot \theta \end{aligned}$$

$$R^r_r = -\frac{2}{arb} \partial_r \left(\frac{\partial_r (rb)}{a} \right) \quad (17)$$

$$R^\theta_\theta = \frac{1}{ar^2 b^2} \left[a - \partial_r \left(\frac{rb \partial_r (rb)}{a} \right) \right] \quad (18)$$

$$(19)$$

I. THEORETICAL MODEL

Here, as a beginning set up, we first use the perfect fluid approximation for the matted model. So the stress-energy tensor takes form

$$T_{ab} = (\rho + P) u_a u_b + P g_{ab} \quad (20)$$

where $u^a(r, t)$ is the 4-velocity of a given perfect element, $P(r, t)$ is the isotropic pressure, $\rho(r, t) = \rho_0(r, t)(1 + \epsilon(r, t))$ is the energy density, $\rho_0(r, t)$ is the rest-mass energy density, and $\epsilon(r, t)$ is the specific internal energy.

The equations of motion for this case are derive from the local conservative equations for energy and baryon number such that

$$\nabla_a T^a_b = 0 \quad (21)$$

$$\nabla_a (\rho_0 u^a) = 0 \quad (22)$$

We are using HRSC so we would like to write this in terms of flux-conservative form such that

$$\partial_t \mathbf{U} + \partial_i \mathbf{F}^i = \Psi \quad (23)$$

where \mathbf{U} ,

$$\mathbf{U} = \begin{pmatrix} \sqrt{\gamma} W \rho_0 \\ \sqrt{\gamma} \alpha T^t_j \\ \alpha^2 \sqrt{\gamma} T^{tt} - \sqrt{\gamma} W \rho_0 \end{pmatrix} \quad (24)$$

and \mathbf{F}^i

$$\mathbf{F}^i = \begin{pmatrix} \sqrt{\gamma} W \rho_0 v^i \\ \sqrt{\gamma} \alpha T^i_j \\ \alpha^2 \sqrt{\gamma} T^{ti} - \sqrt{\gamma} W \rho_0 v^i \end{pmatrix} \quad (25)$$

and ψ

$$\Psi = \begin{pmatrix} 0 \\ \frac{1}{2} \sqrt{\gamma} \alpha T^{ab} g_{ab,j} \\ \alpha^2 \sqrt{\gamma} (T^{at} \partial_a \alpha - \Gamma^0_{ab} T^{ab} \alpha) \end{pmatrix} \quad (26)$$

where W is Lorentz factor such that $W = \alpha u^t$ and $v^i = u^i / u^t$. Under our choice of system, $u^a = (u^t, u^r, 0, 0)$ so we can reduce

$$\partial_t (r^2 ab W \rho_0) + \partial_r (r^2 ab W \rho_0 v^r) = 0 \quad (27)$$

$$\partial_t (r^2 ab \alpha T^t_r) + \partial_r (r^2 ab \alpha T^r_r) = \frac{1}{2} r^2 ab T^{ab} g_{ab,r} \quad (28)$$

$$\partial_t (\alpha^2 r^2 ab T^{tt} - r^2 ab W \rho_0) + \partial_r (\alpha^2 r^2 ab T^{tr} - r^2 ab W \rho_0 v^r) = \alpha r^2 ab (T^{at} \partial_a \alpha - \Gamma^0_{ab} T^{ab} \alpha) \quad (29)$$

Here I omit sin term in the metric determinant because it will be cancelled out anyway

It is useful to define variables like below

$$D = \rho_0 ab W \quad (30)$$

$$E = \rho_0 h W^2 - P \quad (31)$$

$$S = \rho_0 h W^2 v \quad (32)$$

$$\tau = E - D \quad (33)$$

And nonzero components of T^{ab} which we are using

$$T^t_t = -E \quad (34)$$

$$T^t_r = \frac{ab}{\alpha} S \quad (35)$$

$$T^r_r = S v + P \quad (36)$$

$$T^\theta_\theta = T^\varphi_\varphi = P \quad (37)$$

where we define fluid velocity in Eulerian observer

$$v = \frac{ab}{\alpha} v^r = \frac{ab u^r}{\alpha u^t} \quad (38)$$

then also $W = 1/\sqrt{1-v^2}$ and $h = 1 + \epsilon + P/\rho_0$ which is specific enthalpy. Then we can reduce

$$\partial_t(r^2 D) + \partial_r \left(\frac{r^2 \alpha}{ab} Dv \right) = 0 \quad (39)$$

$$\partial_t(r^2 S) + \partial_r \left(\frac{r^2 \alpha}{ab} (Ev + P) \right) = \frac{1}{2} r^2 ab T^{ab} g_{ab,r} \quad (40)$$

$$\partial_t(r^2 \tau) + \partial_r \left(\frac{r^2 \alpha}{ab} (S - Dv) \right) = \alpha r^2 ab (T^{at} \partial_a \alpha - \Gamma^0_{ab} T^{ab} \alpha) \quad (41)$$

or in the form

$$\partial_t \mathbf{u} + \frac{1}{r^2} \partial_r (X r^2 \mathbf{f}) = \psi \quad (42)$$

$$\mathbf{u} = \begin{pmatrix} D \\ S \\ \tau \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} Dv \\ Ev + P \\ S - Dv \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 \\ \frac{ab}{2} ab T^{ab} g_{ab,r} \\ \alpha ab (T^{at} \partial_a \alpha - \Gamma^0_{ab} T^{ab} \alpha) \end{pmatrix} \quad (43)$$

where $X = \alpha/(ab)$ which is purely geometric factor. Some detail evaluation of RHS source terms are in here

Now consider the Einstein's equations. Define below quantities that are appearing in the 3+1 equations

$$\rho_{hydro} = n_a n_b T^{ab} = \rho_0 h W^2 - P \quad (44)$$

$$S_i^{hydro} = -\gamma_{ia} n_b T^{ab} = \rho_0 h W u_i \quad (45)$$

$$S_{ij}^{hydro} = \gamma_{ia} \gamma_{jb} T^{ab} = P \gamma_{ij} + \rho_0 h u_i u_j \quad (46)$$

$$S_{hydro} = \gamma^{ij} S_{ij} = 3P + \rho_0 h (W^2 - 1) \quad (47)$$

The Einstein's equations in the ADM form are

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \quad (48)$$

$$\begin{aligned} \partial_t K^i_j &= \alpha (R^i_j + K K^i_j) - D^i D_j \alpha - 8\pi \alpha \left(S^i_j - \frac{1}{2} \delta^i_j (S - \rho) \right) \\ &\quad + \beta^k \partial_k K^i_j + K^i_k \partial_j \beta^k - K^k_j \partial_k \beta^i \end{aligned} \quad (49)$$

where D_i is covariant derivative on spatial hypersurface. Momentum and Hamiltonian constraints are

$$R + K^2 - K_{ij} K^{ij} = 16\pi \rho \quad (50)$$

$$D_i K^i_j - D_j K = 8\pi S_j \quad (51)$$

Substitute hydro source terms (ρ , S etc) from above then we have

$$\begin{aligned} \partial_t K^i_j &= \alpha (R^i_j + K K^i_j) - \gamma^{ik} (\partial_i \partial_k \alpha - \Gamma^l_{ik} \partial_l \alpha) - 8\pi \alpha \left(\frac{1}{2} \delta^i_j (\rho_0 h - 2P) + \rho_0 h u^i u_j \right) \\ &\quad + \beta^k \partial_k K^i_j + K^i_k \partial_j \beta^k - K^k_j \partial_k \beta^i \end{aligned} \quad (52)$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \quad (53)$$

$$R + K^2 - K_{ij} K^{ij} = 16\pi (\rho_0 h W^2 - P) \quad (54)$$

$$D_i K^i_j - D_j K = 8\pi \rho_0 h W u_j \quad (55)$$

From our choice of metric/coordinate system, we calculated non-trivial connection coefficients and Ricci tensors. Also, metric form suggests that $\beta^i = (\beta^r, 0, 0)$, $K^i_j = \text{diag}(K^r_r, K^\theta_\theta, K^\theta_\theta)$. Using these facts, the evolution equations

for geometric quantities are

$$\partial_t a = -\alpha a K^r_r + \partial_r(a\beta^r) \quad (56)$$

$$\partial_t b = -\alpha b K^\theta_\theta + \frac{\beta^r}{r} \partial_r(r\beta^r) \quad (57)$$

$$\begin{aligned} \partial_t K^r_r &= \beta^r \partial_r K^r_r + \alpha K^r_r K - \frac{1}{a} \partial_r \left(\frac{\partial_r \alpha}{a} \right) - \frac{2\alpha}{arb} \partial_r \left(\frac{\partial_r(rb)}{a} \right) \\ &\quad - 4\pi\alpha [(1 + 2u^r u_r)\rho_0 h - 2P] \end{aligned} \quad (58)$$

$$\begin{aligned} \partial_t K^\theta_\theta &= \beta^r \partial_r K^\theta_\theta + \alpha K^\theta_\theta K - \frac{\alpha}{r^2 b^2} - \frac{1}{ar^2 b^2} \partial_r \left(\frac{\alpha r b \partial_r(rb)}{a} \right) \\ &\quad - 4\pi\alpha(\rho_0 h - 2P) \end{aligned} \quad (59)$$

From constraints

$$\begin{aligned} \frac{1}{ar^2 b^2} \left[a - \partial_r \left(\frac{rb \partial_r(rb)}{a} \right) \right] - \frac{2}{arb} \partial_r \left(\frac{\partial_r(rb)}{a} \right) + 2K^\theta_\theta (K^\theta_\theta + 2K^r_r) &= 16\pi(\rho_0 h W^2 - P) \\ \frac{\partial_t(rb)}{rb} (K^\theta_\theta - K^r_r) - \partial_r K^\theta_\theta &= 4\pi\rho_0 h W u_r \end{aligned} \quad (60)$$

We can apply different choice of slicing (i.e. gauge choice) to reduce/determine above system. Possible (or simple) choices would be maximal or polar slicing.

A. Choice of Gauge

First, we consider maximal slicing i.e. $K = \partial_t K = 0$ then

$$\partial_t a = -\alpha a K^r_r + \partial_r(a\beta^r) \quad (61)$$

$$\partial_t b = \frac{\alpha b}{2} K^r_r + \frac{\beta^r}{r} \partial_r(r\beta^r) \quad (62)$$

$$\begin{aligned} \partial_t K^r_r &= \beta^r \partial_r K^r_r - \frac{1}{a} \partial_r \left(\frac{\partial_r \alpha}{a} \right) - \frac{2\alpha}{arb} \partial_r \left(\frac{\partial_r(rb)}{a} \right) - \frac{2\alpha}{r^2 b^2} - \frac{2}{ar^2 b^2} \partial_r \left(\frac{\alpha r b \partial_r(rb)}{a} \right) \\ &\quad - 8\pi\alpha [(2 + 2u^r u_r)\rho_0 h - 4P] \end{aligned} \quad (63)$$

From constraints

$$\begin{aligned} \frac{1}{ar^2 b^2} \left[a - \partial_r \left(\frac{rb \partial_r(rb)}{a} \right) \right] - \frac{2}{arb} \partial_r \left(\frac{\partial_r(rb)}{a} \right) - \frac{3}{2} (K^r_r)^2 &= 16\pi(\rho_0 h W^2 - P) \\ \partial_r K^r_r - \frac{3\partial_t(rb)}{rb} K^r_r &= 8\pi\rho_0 h W u_r \end{aligned} \quad (64)$$

Fluid EOM parts are same as previous

For lapse, we use

$$\partial_t K = -D^2 \alpha + \alpha(K^{ij} K_{ij} + 4\pi(\rho + S)) + \beta^i D_i K \quad (65)$$

In our choice of gauge, this can be reduced

$$\begin{aligned} D^2 \alpha &= \alpha(K^{ij} K_{ij} + 4\pi(\rho + S)) = \alpha \left(K^{ij} K_{ij} + 8\pi \left[P + \rho_0 h \left(W^2 - \frac{1}{2} \right) \right] \right) \\ &= \alpha \left(2(K^r_r)^2 + 8\pi \left[P + \rho_0 h \left(W^2 - \frac{1}{2} \right) \right] \right) \end{aligned} \quad (66)$$

Also, for shift, we use

$$\partial_t \ln \sqrt{\gamma} = -\alpha K + D_i \beta^i \quad (67)$$

In maximal slicing, this reduces

$$D_i \beta^i = -\partial_t \ln \sqrt{\gamma} \quad (68)$$

or we can write

$$\partial_i \beta^i = -\partial_t \sqrt{\gamma} \quad (69)$$

This shows that the proper volume element $\sqrt{\gamma}$ satisfies a continuity equation in maximal slicing.

In terms of our metric choice and variable

$$\partial_r \beta^r = \frac{b}{2b + \beta^r} \left[\frac{\alpha K^r_r}{2} - \frac{(\beta^r)^2 b}{r} - \frac{\partial_r a}{a} \beta^r \right] \quad (70)$$

Another possible choice to set the lapse is demanding that the ingoing combination of tangent vectors $\vec{\partial}_t - \vec{\partial}_r$ be null. This gives a condition on the metric : $g_{tt} - 2g_{tr} + g_{rr} = 0$. This gives $\alpha = a(1 - \beta)$

B. Initial Data

Our initial NS model is approximated by solution of TOV. After the initial data calculation, an in-going velocity profile is added to drive the star to collapse. We follow the way is described in (<https://arxiv.org/pdf/gr-qc/0107045.pdf>). First, specifying the coordinate velocity

$$U \equiv \frac{dr}{dt} = \frac{u^r}{u^t} \quad (71)$$

of the star. In general, the profile take the algebraic form $U_g(x) = A_0(x^3 - B_0 x)$ where $x \equiv r/R_{star}$ and R_{star} is the radius of the TOV solution.

In this work, we set two profiles

$$U(x) = \begin{cases} U_1(x) = U_{crit}(x^3 - 3x) & x < x_{tlv} \\ U_2(x) = 0 & \text{otherwise} \end{cases} \quad (72)$$

U_{crit} is the amplitude that occurs critical collapse, and x_{tlv} the region that forms black hole.

Our interest is interaction between BH inside of NS we set x_{tlv} is small value such as 1% of size of star i.e. $x_{tlv} = 0.01$ since x is normalized radius by star radius (x_{tlv} must be smaller than 1).

C. Analytic Case

For code test and validation, we use well-known Michel problem