General Relativistic Hydrodynamical System in Spherically Symmetric Metric with Horizon Penetrating Coordinate

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We present a horizon penetrating coordinate for spherically symmetric metric. We adopt $G = c = 1$ unit system. Consider usual Schwarzschild line element in BL coordinate for practice

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

where M is a mass. Compare this with usual 3+1 line element form $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^2 + \beta^i dt)(dx^j + \beta^j dt)$ we can identify the lapse $\alpha^2 = \left(1 - \frac{2M}{r}\right)$. As we know, the line element(Eqn. 1) is singular at the horizon ($r = 2M$) and lapse collapse to zero. This can be problematic because equations of motion for the metric can be become exponentially unstable in the presence of a coordinate singularity without some regularization technique.

One way to resolve this problem is to move to a horizon penetrating coordinate system where this singularity is not present. The Kerr-Schild coordinates are one such coordinate system.

For example, Schwarzschild solution in spherical type Kerr-Schild coordinates

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{2}$$

$$\beta^r = \frac{2M}{r + 2M} \tag{3}$$

$$\beta_r = \frac{2M}{r} \tag{4}$$

$$\beta^\theta = \beta^\varphi = 0 \tag{5}$$

$$K_{ij} = \text{diag}\left[-\frac{2M(r + M)}{\sqrt{r^5(r + 2M)}}, 2M\sqrt{\frac{r}{r + 2M}}, K_{\theta\theta}\sin^2\theta\right] \tag{6}$$

Schwarzschild solution in Cartesian type Kerr-Schild coordinate

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{7}$$

$$\beta^i = \frac{2M}{r}\frac{x^i}{r + 2M} \tag{8}$$

$$\beta_i = \frac{2Mx_i}{r^2} \tag{9}$$

$$K_{ij} = \frac{2M}{r^4}\sqrt{\frac{r}{r + 2M}}\left[\left(\frac{M}{r} + 2\right)x_i x_j - r^2\delta_{ij}\right] \tag{10}$$

where $x^i = (x, y, z)$ which is usual spatial Cartesian coordinate. In both cases, we can see lapse is regular at the horizon.

General spherical symmetric line element in polar-areal form

$$ds^2 = -\alpha(r)^2 dt^2 + a(r)^2 dr^2 + r^2 d\Omega^2 \tag{11}$$

where α is referred as lapse function. Compare with above Schwarzschild solution, $\alpha = 1/a$.

Now consider a transformation of the Schwarzschild time t coordinate to a new generic coordinate $\hat{t}$ according to

$$d\hat{t} = dt + a^2\sqrt{1 - \frac{g}{a^2}}dr \tag{12}$$

where $g(r)$ is arbitrary function. Substitute this into $ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$ gives

$$\begin{aligned} ds^2 &= -\alpha^2 \left(d\hat{t} - a^2 \sqrt{1 - \frac{g}{a^2}} dr \right)^2 + a^2 dr^2 + r^2 d\Omega^2 \\ &= -\alpha^2 d\hat{t}^2 + 2\sqrt{1 - \frac{g}{a^2}} d\hat{t} dr + g dr^2 + r^2 d\Omega^2 \end{aligned} \tag{13}$$

Compare this with usual 3+1 framework

$$ds^2 = -\alpha^2 d\hat{t}^2 + \gamma_{ij}(dx^i + \beta^i d\hat{t})(dx^j + \beta^j d\hat{t}) \tag{14}$$

and so into the lapse $\alpha = 1/\sqrt{g}$, the shift $\beta_i = (\sqrt{1 - g/a^2}, 0, 0)$ or $\beta^i = \gamma^{ij}\beta_j$ and the spatial metric of the constant $\hat{t}$ hypersurface $\gamma_{ij} = diag(g, r^2, r^2 \sin^2\theta)$.

If we choose $\alpha = \sqrt{1 - 2M/r} = 1/a$ and $g = 1 + 2M/r$ like in previous (which we will use this), we get

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\hat{t}^2 + \frac{4M}{r} d\hat{t} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \tag{15}$$

which is Schwarzschild in Kerr-Schild coordinate (or Eddington-Finkelstein coordinate). And correspondingly, $\alpha = \sqrt{r/(r+2M)}$, $\beta_i = (2M/r, 0, 0)$, and $\gamma_{ij} = diag(1 + 2M/r, r^2, r^2 \sin^2\theta)$ which are same as above.

As you can see here, the KS (or EF) form of the metric represents an analytic expansion of the Schwarzschild solution from the region $2M < r < \infty$ to $0 < r < \infty$. Thus, we apply this coordinate transformation for our equations.

It is good to rewrite the metric into usual $3+1$ variable form i.e. keep it geometric variables (should be careful of confusion) with considering time dependent case (This is almost same as Marsa and Choptuik's paper). Here, we use t for time coordinate that we used above.

$$ds^2 = (-\alpha^2 + a^2\beta^2) dt^2 + 2a^2\beta dt dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \tag{16}$$

where α, a, b, and β are functions of r and t, and $d\Omega^2$ is the metric of unit sphere. From this, we can calculate non-vanishing components of connection coefficients and Ricci tensors for i, j, and k (spatial indices)

$$\begin{aligned} &\Gamma^r{}_{rr} = \frac{\partial_r a}{a}, \quad \Gamma^r{}_{\theta\theta} = -\frac{rb\partial_r(rb)}{a^2}, \quad \Gamma^\theta{}_{r\theta} = \frac{\partial_r(rb)}{rb} \\ &\Gamma^r{}_{\varphi\varphi} = -\sin^2\theta \frac{rb\partial_r(rb)}{a^2}, \quad \Gamma^\varphi{}_{r\varphi} = \Gamma^\theta{}_{r\theta} \\ &\Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^\varphi{}_{\varphi\theta} = -\cot\theta \end{aligned}$$

$$R^r{}_r = -\frac{2}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right) \tag{17}$$

$$R^\theta{}_\theta = \frac{1}{ar^2b^2}\left[a - \partial_r\left(\frac{rb\partial_r(rb)}{a}\right)\right] \tag{18}$$

$$\tag{19}$$

I. THEORETICAL MODEL

Here, as a beginning set up, we first use the perfect fluid approximation for the matted model. So the stress-energy tensor takes form

$$T_{ab} = (\rho + P)u_a u_b + P g_{ab} \tag{20}$$

where $u^a(r,t)$ is the 4-velocity of a given perfect element, $P(r,t)$ is the isotropic pressure, $\rho(r,t) = \rho_0(r,t)(1 + \epsilon(r,t))$ is the energy density, $\rho_0(r,t)$ is the rest-mass energy density, and $\epsilon(r,t)$ is the specific internal energy.

The equations of motion for this case are derive from the local conservative equations for energy and baryon number such that

$$\nabla_a T^a{}_b = 0 \tag{21}$$
$$\nabla_a(\rho_0 u^a) = 0 \tag{22}$$

First, we define variables.

$$D = \rho_0 W \tag{23}$$
$$E = \rho_0 h W^2 - P \tag{24}$$
$$S = \rho_0 h W^2 v \tag{25}$$
$$\tau = E - D \tag{26}$$

where W is Lorentz factor such that $W = \alpha u^t = 1/\sqrt{1-v^2}$ with fluid velocity $v = (au^r)/(\alpha u^t)$ and $h = 1 + \epsilon + P/\rho_0$ which is specific enthalpy. In our case, $u^a = (u^t, u^r, 0, 0)$

And nonzero components of T^{ab} which we are using

$$T^t{}_t = -E \tag{27}$$
$$T^t{}_r = \frac{a}{\alpha} S \tag{28}$$
$$T^r{}_r = Sv + P \tag{29}$$
$$T^\theta{}_\theta = T^\varphi{}_\varphi = P \tag{30}$$

Using these variables under the metric which we consider, $\nabla_a(\rho_0 u^a) = 0$ (continuity equation) gives

$$\partial_t(\rho_0 u^t) + \Gamma^t{}_{tt}(\rho_0 u^t) + \Gamma^t{}_{tr}(\rho_0 u^r) = 0 \tag{31}$$

In terms of our variables, $\rho_0 u^t = D/\alpha$, $\rho_0 u^r = Dv/a$ so

$$\partial_t(D/\alpha) + \Gamma^t{}_{tt}(D/\alpha) + \Gamma^t{}_{tr}(Dv/a) = 0 \tag{32}$$

$\nabla_a T^a{}_b = 0$ gives

$$\partial_t T^t{}_b + \Gamma^t{}_{tc} T^c{}_b - \Gamma^c{}_{tb} T^t{}_c = 0 \tag{33}$$

The covariant t-component of above equation gives energy equation $T^t{}_t = -E$

$$\begin{aligned}
&\partial_t T^t{}_t + \Gamma^t{}_{tc} T^c{}_t - \Gamma^c{}_{tt} T^t{}_c = 0 \\
&\rightarrow \partial_t T^t{}_t + \Gamma^t{}_{tt} T^t{}_t + \Gamma^t{}_{tr} T^r{}_t - \Gamma^t{}_{tt} T^t{}_t - \Gamma^r{}_{tt} T^t{}_r = 0 \\
&\rightarrow \partial_t E + \Gamma^t{}_{tr} \frac{a^3}{\alpha^3} S + \Gamma^r{}_{tt} \frac{a}{\alpha} S = 0
\end{aligned} \tag{34}$$

Next, consider the covariant r-component

$$\begin{aligned}
&\partial_t T^t{}_r + \Gamma^t{}_{tc} T^c{}_r - \Gamma^c{}_{tr} T^t{}_c = 0 \\
&\rightarrow \partial_t T^t{}_r + \Gamma^t{}_{tt} T^t{}_r + \Gamma^t{}_{tr} T^r{}_r - \Gamma^t{}_{tr} T^t{}_t - \Gamma^r{}_{tr} T^t{}_r = 0 \\
&\rightarrow \partial_t\left(\frac{a}{\alpha} S\right) + \Gamma^t{}_{tt}(\frac{a}{\alpha} S) + \Gamma^t{}_{tr}(Sv + P) + \Gamma^t{}_{tr} E - \Gamma^r{}_{tr} \frac{a}{\alpha} S = 0
\end{aligned}$$

Non-vanishing connection coefficients are evaluated via Mathematica. You can find it here

Now consider the Einstein's equations. Define below quantities that are appearing in the 3+1 equations

$$\rho_{hydro} = n_a n_b T^{ab} = \rho_0 h W^2 - P \tag{35}$$
$$S_i^{hydro} = -\gamma_{ia} n_b T^{ab} = \rho_0 h W u_i \tag{36}$$
$$S_{ij}^{hydro} = \gamma_{ia}\gamma_{ib} T^{ab} = P\gamma_{ij} + \rho_0 h u_i u_j \tag{37}$$
$$S_{hydro} = \gamma^{ij} S_{ij} = 3P + \rho_0 h(W^2 - 1) \tag{38}$$

The Einstein's equations in the ADM form are

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i && (39)\\
\partial_t K^i{}_j &= \alpha(R^i{}_j + KK^i{}_j) - D^i D_j\alpha - 8\pi\alpha\left(S^i{}_j - \frac{1}{2}\delta^i{}_j(S-\rho)\right)\\
&\quad + \beta^k\partial_k K^i{}_j + K^i{}_k\partial_j\beta^k - K^k{}_j\partial_k\beta^i && (40)
\end{aligned}
$$

where D_i is covariant derivative on spatial hypersurface. Momentum and Hamiltonian constraints are

$$
\begin{aligned}
R + K^2 - K_{ij}K^{ij} &= 16\pi\rho && (41)\\
D_i K^i{}_j - D_j K &= 8\pi S_j && (42)
\end{aligned}
$$

Substitute hydro source terms (ρ, S etc) from above then we have

$$
\begin{aligned}
\partial_t K^i{}_j &= \alpha(R^i{}_j + KK^i{}_j) - \gamma^{ik}(\partial_i\partial_k\alpha - \Gamma^l{}_{ik}\partial_l\alpha) - 8\pi\alpha\left(\frac{1}{2}\delta^i{}_j(\rho_0 h - 2P) + \rho_0 h u^i u_j\right)\\
&\quad + \beta^k\partial_k K^i{}_j + K^i{}_k\partial_j\beta^k - K^k{}_j\partial_k\beta^i && (43)\\
\partial_t\gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i && (44)\\
R + K^2 - K_{ij}K^{ij} &= 16\pi(\rho_0 h W^2 - P) && (45)\\
D_i K^i{}_j - D_j K &= 8\pi\rho_0 h W u_j && (46)
\end{aligned}
$$

From our choice of metric/coordinate system, we calculated non-trivial connection coefficients and Ricci tensors. Also, metric form suggests that $\beta^i = (\beta^r, 0, 0)$, $K^i{}_j = diag(K^r{}_r, K^\theta{}_\theta, K^\theta{}_\theta)$. Using these facts, the evolution equations for geometric quantities are

$$
\begin{aligned}
\partial_t a &= -\alpha a K^r{}_r + \partial_r(a\beta^r) && (47)\\
\partial_t b &= -\alpha b K^\theta{}_\theta + \frac{\beta^r}{r}\partial_r(r\beta^r) && (48)\\
\partial_t K^r{}_r &= \beta^r\partial_r K^r{}_r + \alpha K^r{}_r K - \frac{1}{a}\partial_r\left(\frac{\partial_r\alpha}{a}\right) - \frac{2\alpha}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right)\\
&\quad - 4\pi\alpha\left[(1 + 2u^r u_r)\rho_0 h - 2P\right] && (49)\\
\partial_t K^\theta{}_\theta &= \beta^r\partial_r K^\theta{}_\theta + \alpha K^\theta{}_\theta K - \frac{\alpha}{r^2b^2} - \frac{1}{ar^2b^2}\partial_r\left(\frac{\alpha rb\partial_r(rb)}{a}\right)\\
&\quad - 4\pi\alpha(\rho_0 h - 2P) && (50)
\end{aligned}
$$

From constraints

$$
\begin{aligned}
\frac{1}{ar^2b^2}\left[a - \partial_r\left(\frac{rb\partial_r(rb)}{a}\right)\right] - \frac{2}{arb}\partial_r\left(\frac{\partial_r(rb)}{a}\right) + 2K^\theta{}_\theta(K^\theta{}_\theta + 2K^r{}_r) &= 16\pi(\rho_0 h W^2 - P)\\
\frac{\partial_t(rb)}{rb}(K^\theta{}_\theta - K^r{}_r) - \partial_r K^\theta{}_\theta &= 4\pi\rho_0 h W u_r && (51)
\end{aligned}
$$

We can apply different choice of slicing (i.e. gauge choice) to reduce/determine above system. Possible (or simple) choices would be maximal or polar slicing.

A. Initial Data

Our initial NS model is approximated by solution of TOV until Schwarzschild radius.

B. Analytic Case