Horizon Penetrating Coordinate for Spherically Symmetric Metric

Hyun Lim

We present a horizon penetrating coordinate for spherically symmetric metric. We adopt $G = c = 1$ unit system. Consider usual Schwarzschild line element in BL coordinate for practice

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

where M is a mass. Compare this with usual 3+1 line element form $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^2 + \beta^i dt)(dx^j + \beta^j dt)$ we can identify the lapse $\alpha^2 = \left(1 - \frac{2M}{r}\right)$. As we know, the line element(Eqn. 1) is singular at the horizon ($r = 2M$) and lapse collapse to zero. This can be problematic because equations of motion for the metric can be become exponentially unstable in the presence of a coordinate singularity without some regularization technique.

One way to resolve this problem is to move to a horizon penetrating coordinate system where this singularity is not present. The Kerr-Schild coordinates are one such coordinate system.

For example, Schwarzschild solution in spherical type Kerr-Schild coordinates

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{2}$$

$$\beta^r = \frac{2M}{r + 2M} \tag{3}$$

$$\beta_r = \frac{2M}{r} \tag{4}$$

$$\beta^\theta = \beta^\varphi = 0 \tag{5}$$

$$K_{ij} = \mathrm{diag}\left[-\frac{2M(r+M)}{\sqrt{r^5(r+2M)}}, 2M\sqrt{\frac{r}{r+2M}}, K_{\theta\theta}\sin^2\theta\right] \tag{6}$$

Schwarzschild solution in Cartesian type Kerr-Schild coordinate

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{7}$$

$$\beta^i = \frac{2M}{r}\frac{x^i}{r + 2M} \tag{8}$$

$$\beta_i = \frac{2Mx_i}{r^2} \tag{9}$$

$$K_{ij} = \frac{2M}{r^4}\sqrt{\frac{r}{r+2M}}\left[\left(\frac{M}{r} + 2\right) x_i x_j - r^2\delta_{ij}\right] \tag{10}$$

where $x^i = (x, y, z)$ which is usual spatial Cartesian coordinate. In both cases, we can see lapse is regular at the horizon.

General spherical symmetric line element in polar-areal form

$$ds^2 = -\alpha(r)^2 dt^2 + a(r)^2 dr^2 + r^2 d\Omega^2 \tag{11}$$

where α is referred as lapse function. Compare with above Schwarzschild solution, $\alpha = 1/a$.

Now consider a transformation of the Schwarzschild time t coordinate to a new generic coordinate $\hat{t}$ according to

$$d\hat{t} = dt + a^2\sqrt{1 - \frac{g}{a^2}}dr \tag{12}$$

where $g(r)$ is arbitrary function. Substitute this into $ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$ gives

$$\begin{aligned} ds^2 &= -\alpha^2 \left(d\hat{t} - a^2 \sqrt{1 - \frac{g}{a^2}} dr \right)^2 + a^2 dr^2 + r^2 d\Omega^2 \\ &= -\alpha^2 d\hat{t}^2 + 2\sqrt{1 - \frac{g}{a^2}} d\hat{t} dr + g dr^2 + r^2 d\Omega^2 \end{aligned} \tag{13}$$

Compare this with usual 3+1 framework

$$ds^2 = -\alpha^2 d\hat{t}^2 + \gamma_{ij}(dx^i + \beta^i d\hat{t})(dx^j + \beta^j d\hat{t}) \tag{14}$$

and so into the lapse $\alpha = 1/\sqrt{g}$, the shift $\beta_i = (\sqrt{1 - g/a^2}, 0, 0)$ or $\beta^i = \gamma^{ij}\beta_j$ and the spatial metric of the constant $\hat{t}$ hypersurface $\gamma_{ij} = diag(g, r^2, r^2 \sin^2\theta)$.

If we choose $\alpha = \sqrt{1 - 2M/r} = 1/a$ and $g = 1 + 2M/r$ like in previous (which we will use this), we get

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\hat{t}^2 + \frac{4M}{r} d\hat{t} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \tag{15}$$

which is Schwarzschild in Kerr-Schild coordinate (or Eddington-Finkelstein coordinate). And correspondingly, $\alpha = \sqrt{r/(r+2M)}$, $\beta_i = (2M/r, 0, 0)$, and $\gamma_{ij} = diag(1 + 2M/r, r^2, r^2\sin^2\theta)$ which are same as above.

As you can see here, the KS (or EF) form of the metric represents an analytic expansion of the Schwarzschild solution from the region $2M < r < \infty$ to $0 < r < \infty$. Thus, we apply this coordinate transformation for our equations.

I. THEORETICAL MODEL

Here, as a beginning set up, we first use the perfect fluid approximation for the matted model. So the stress-energy tensor takes form

$$T_{ab} = (\rho + P) u_a u_b + P g_{ab} \tag{16}$$

where $u^a(r,t)$ is the 4-velocity of a given perfect element, $P(r,t)$ is the isotropic pressure, $\rho(r,t) = \rho_0(r,t)(1 + \epsilon(r,t))$ is the energy density, $\rho_0(r,t)$ is the rest-mass energy density, and $\epsilon(r,t)$ is the specific internal energy.

The equations of motion for this case are derive from the local conservative equations for energy and baryon number such that

$$\nabla_a T^a{}_b = 0 \tag{17}$$

$$\nabla_a(\rho_0 u^a) = 0 \tag{18}$$

We follow usual remaining set-up i.e. employ Euler velocity, using Primitive variables etc for our system

A. GR-Hydro Equation in HPC

Here we solve GR hydrodynamic equations on Schwarzschild background in KS coordinate system. We follow standard techniques that describe in many literatures. Fluid EOM conservative form

$$\partial_t \mathbf{q} + \frac{1}{r^2}\partial_r(r^2 X \mathbf{f}) = \psi \tag{19}$$

where

$$\mathbf{q} = \begin{bmatrix} D \\ \Pi \\ \Phi \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} Dv \\ v(\Pi + P) + P \\ v(\Phi + P) + P \end{bmatrix}, \quad \psi = \begin{bmatrix} 0 \\ \Sigma \\ -\Sigma \end{bmatrix} \tag{20}$$

where v is Eulerian velocity of fluid such that $v = au^r/(\alpha u^t)$, $X = \alpha/a$ is a purely geometric quantity and

$$D = a\rho_0 W \tag{21}$$
$$\Pi = E - D + S \tag{22}$$
$$\Phi = E - D - S \tag{23}$$
$$S = \rho_0 h W^2 v \tag{24}$$
$$E = \rho_0 h W^2 - P \tag{25}$$

where W is Lorentz factor such that $W = \alpha u^t = 1/\sqrt{1 - v^2}$ and $h = 1 + \epsilon + P/\rho_0$ which is specific enthalpy.

Further, a sufficient set of Einstein's equations for geometric variable α and a are given by the nontrivial component of momentum constraint

$$\partial_t a = -4\pi r \alpha a^2 S \tag{26}$$

and by the polar slicing condition which follows from the demand that metric have the spherically symmetric form for all time

$$\partial_r(\ln \alpha) = a^2 \left[4\pi r(Sv + P) + \frac{m}{r^2}\right] \tag{27}$$

and from Hamiltonian constraint

$$\partial_r a = a^3 \left(4\pi r E - \frac{m}{r^2}\right) \tag{28}$$

To solve these sets of equations on Schwarzschild background in KS, we consider following coordinate transformation in time

$$\hat{t} = t + 2M \ln |\frac{r}{2M} - 1| + k \tag{29}$$

where k is arbitrary constant. So for arbitrary function F

$$\frac{\partial F}{\partial t} = \frac{\partial \hat{t}}{\partial t}\frac{\partial F}{\partial \hat{t}} = \left(1 + \frac{2MXv}{r - 2M}\right)\frac{\partial F}{\partial \hat{t}} \tag{30}$$

We apply this rule to above equations for HPC

B. Initial Data

Our initial NS model is approximated by solution of TOV until Schwarzschild radius.

C. Analytic Case