# Horizon Penetrating Coordinate for Spherically Symmetric Metric

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We present a horizon penetrating coordinate for spherically symmetric metric. We adopt $G = c = 1$ unit system. Consider usual Schwarzschild line element in BL coordinate for practice

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

where $M$ is a mass. Compare this with usual 3+1 line element form $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^2 + \beta^i dt)(dx^j + \beta^j dt)$ we can identify the lapse $\alpha^2 = \left(1 - \frac{2M}{r}\right)$. As we know, the line element(Eqn. 1) is singular at the horizon ($r = 2M$) and lapse collapse to zero. This can be problematic because equations of motion for the metric can be become exponentially unstable in the presence of a coordinate singularity without some regularization technique.

One way to resolve this problem is to move to a horizon penetrating coordinate system where this singularity is not present. The Kerr-Schild coordinates are one such coordinate system.

For example, Schwarzschild solution in spherical type Kerr-Schild coordinates

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{2}$$

$$\beta^r = \frac{2M}{r + 2M} \tag{3}$$

$$\beta_r = \frac{2M}{r} \tag{4}$$

$$\beta^\theta = \beta^\varphi = 0 \tag{5}$$

$$K_{ij} = \text{diag}\left[-\frac{2M(r + M)}{\sqrt{r^5(r + 2M)}}, 2M\sqrt{\frac{r}{r + 2M}}, K_{\theta\theta}\sin^2\theta\right] \tag{6}$$

Schwarzschild solution in Cartesian type Kerr-Schild coordinate

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{7}$$

$$\beta^i = \frac{2M}{r}\frac{x^i}{r + 2M} \tag{8}$$

$$\beta_i = \frac{2Mx_i}{r^2} \tag{9}$$

$$K_{ij} = \frac{2M}{r^4}\sqrt{\frac{r}{r + 2M}}\left[\left(\frac{M}{r} + 2\right)x_i x_j - r^2\delta_{ij}\right] \tag{10}$$

where $x^i = (x, y, z)$ which is usual spatial Cartesian coordinate. In both cases, we can see lapse is regular at the horizon.

General spherical symmetric line element in polar-areal form

$$ds^2 = -\alpha(r,t)^2 dt^2 + a(r,t)^2 dr^2 + r^2 d\Omega^2 \tag{11}$$

where $\alpha$ is referred as lapse function. Compare with above Schwarzschild solution, $\alpha = 1/a$.

In terms of horizon penetrating coordinate, we can generalize it in 3+1 form

$$ds^2 = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \tag{12}$$

where $\alpha$, $a$, $b$, and $\beta$ are functions of $r$ and $t$, and $d\Omega^2$ is the metric of unit sphere. This is nothing but ingoing Eddington-Finkelstein coordinate system (IEF). Consider Schwarzschild again in IEF

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dV^2 + 2dVdr + r^2 d\Omega^2 \tag{13}$$

Define a timelike coordinate $t = V - r$ then metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{4M}{r} dtdr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \tag{14}$$

Compare this with Eqn. 12, various following metric components can be found

$$\alpha = \sqrt{\frac{r}{r + 2M}} \tag{15}$$

$$\beta = \frac{2M}{r + 2M} \tag{16}$$

$$a = \sqrt{\frac{r + 2M}{r}} \tag{17}$$

and so on. Note that we can also fix the spatial degree of coordinate freedom by introducing a shifting areal coordinate $R \equiv r + f(t)$ where $f(t)$ is some undetermined function.

## I. MODEL

Here, as a beginning set up, we first use the perfect fluid approximation for the matted model. So the stress-energy tensor takes form

$$T_{ab} = (\rho + P)u_a u_b + P g_{ab} \tag{18}$$

where $u^a(r, t)$ is the 4-velocity of a given perfect element, $P(r, t)$ is the isotropic pressure, $\rho(r, t) = \rho_0(r, t)(1 + \epsilon(r, t))$ is the energy density, $\rho_0(r, t)$ is the rest-mass energy density, and $\epsilon(r, t)$ is the specific internal energy.

The equations of motion for this case are derive from the local conservative equations for energy and baryon number such that

$$\nabla_a T^a{}_b = 0 \tag{19}$$

$$\nabla_a(\rho_0 u^a) = 0 \tag{20}$$

We follow usual remaining set-up i.e. employ Euler velocity, using Primitive variables etc for our system