

Chapter 3.1: Kahler Identities

Huybrechts, *Complex Geometry*

3.1.1 This is a reflection of the fact that Hermitian metrics are not required to be holomorphic. For let M be a complex manifold and take any coordinate chart $U \subset \mathbb{C}^n$. On U , the complex structure of M is as described in Section 1.3. The flat metric g_U is compatible with this complex structure.

Now cover M by charts U_α and let ρ_α be a partition of unity subordinate to this cover. Then $g = \sum_\alpha \rho_\alpha g_{U_\alpha}$ is the desired metric:

$$g(I\cdot, I\cdot) = \sum_\alpha \rho_\alpha g_{U_\alpha}(I\cdot, I\cdot) = \sum_\alpha \rho_\alpha g_{U_\alpha}(\cdot, \cdot) = g(\cdot, \cdot).$$

3.1.2 Because g is Kahler, $dg = 0$. But say

$$dg' = d(e^f)g + e^f dg = d(e^f)g = 0.$$

Because X is connected, this implies that f is constant.

3.1.3 Say a function f is d -harmonic. By degree, $d^*f = 0$, so

$$\langle \Delta_d f, f \rangle = \langle d^* df, f \rangle = \|df\|^2 = 0.$$

Therefore $df = 0$, which implies that $\partial f = \bar{\partial} f = 0$. The same trick shows that if $\Delta_\partial f = \partial^* \partial f = 0$, then $\partial f = 0$, and if $\Delta_{\bar{\partial}} f = \bar{\partial}^* \partial f = 0$, then $\bar{\partial} f = 0$, so $df = 0$, and therefore $\Delta f = 0$.

Similar arguments work for top forms, since in this case $d^* df = 0$ and similar.

3.1.4 Computation omitted (should be in my papers from class).

3.1.5 This is easily seen using the coordinate representation on page 117 and the observation that, e.g. in U_0 , $i^* dz_i = dz_i$ for $0 < i < n$ and $i^* dz_n = 0$.

3.1.6 Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$. Then $F_A^* \omega_{FS} = \omega_{FS}$ if and only if $A^* \pi^* \omega_{FS} = \pi^* \omega_{FS}$, where A is the ordinary linear action of A on \mathbb{C}^{n+1} . This is because $\pi \circ A = F_A \circ \pi$, so $A^* \pi^* \omega_{FS} = \pi^* F_A^* \omega_{FS}$. Also, π is a subjective submersion, so π^* is injective on forms on \mathbb{P}^n : if $\omega \neq 0 \in \Omega_{\mathbb{P}^n}$, then $\omega_{F(p)}(DF_p -, DF_p -)$ must be nonzero for some $p \in \mathbb{C}^{n+1} \setminus \{0\}$ because there is some $F(p)$ where ω does not vanish, and then vectors on which it does not vanish can be “reached” by DF_p .

Now we calculated on pg. 118 that $\pi^* \omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log \|z\|^2$. Since A is a holomorphic function, it commutes with ∂ and $\bar{\partial}$. Therefore

$$A^* \pi^* \omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log \|Az\|^2.$$

When does $\partial\bar{\partial}\log\|Az\|^2 = \partial\bar{\partial}\log\|z\|^2$? We compute that

$$\partial\bar{\partial}\log\|Az\|^2 = \sum_{i,j} \frac{A_{ik}\bar{A}_{jk}dz_i \wedge d\bar{z}_j}{\|Az\|^2} + \mathcal{O}(\|Az\|^{-2}).$$

But scaling A does change $\partial\bar{\partial}\log\|Az\|^2$, so we can ignore the $\mathcal{O}(\|Az\|^{-2})$ term. Then comparing terms with the case $A = I$, we see that we must have $A^H A = \lambda\delta_{ij}$. This A must be a multiple of a unitary matrix.

I believe that it is an error for the book to say that A must be unitary. For A and λA define the same map on \mathbb{P}^{n+1} for $\lambda \neq 0$.