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Chapter 2.5: Blow-ups

Huybrechts, *Complex Geometry*

2.5.1 Here is how to define the map. Take $\alpha \in H^0(X, K_x)$. Let σ be the blow-up map, so $\sigma|_{\hat{X}\setminus E}$ is an isomorphism $\hat{X} \setminus E \simeq X \setminus \{x\}$. Now $a|_{\hat{X}\setminus E} \circ \sigma|_{\hat{X}\setminus E}$ defines a section $\hat{X} \setminus E \to K_{\hat{X}\setminus E}$, because $K_{\hat{X}\setminus E} \simeq K_X$. To show that this section extends across the exceptional divisor E, it is sufficient to show that it is bounded in some local trivialization at each point in E (Riemann Extension Theorem). But this follows because α is "bounded" on X and σ gives a surjection onto $\hat{X} \setminus E$. α is bounded in the sense that there are finitely many trivializations of X and α is bounded in each of these trivializations. This follows because X is compact, so we can cover it in compact sets contained in trivializations. Clearly the extension over E must be unique.

This map is injective because if $\alpha \circ \sigma$ extends to the zero section, it must be zero on $X \setminus \{x\}$, in which case it is zero. It is surjective because a section $\beta : \hat{X} \to K_{\hat{X}}$ induces a section of $X \setminus \{x\}$ by restriction and the isomorphism $\hat{X} \setminus E \simeq X \setminus \{x\}$. This then extends to $\beta' : \hat{X} \setminus E \to K_{\hat{X}\setminus E}$ by the process described above, which agrees with β away from E. The uniqueness of the extension then proves $\beta = \beta'$.

2.5.2 First let $Y = \{x\} \subset X$. Let α be a section $\hat{X} \to \mathcal{O}(E)$. Then $\alpha|_E$ gives a section $\mathbb{P}^n \to \mathcal{O}(E)$. Now by Corollary 2.5.6, $\mathcal{O}(E)|_E \simeq \mathcal{O}(-1)$. Since $\mathcal{O}(-1)$ has no global sections, this must be the zero section. But a section of $\mathcal{O}(E)$ corresponds to a meromorphic function on $\hat{X}$ with at most first order poles along E. Therefore α must be a holomorphic function on $\hat{X}$. Since X is compact, this function must be constant, and since it vanishes at a point, it is the zero section. REVIEW THE CORRESPONDENCE O(E) WITH POLES.

Now let Y Be a general hypersurface.

COME BACK TO THIS.

2.5.3 The action is trivial on the exceptional divisor. This can be seen by COORDINATES.

2.5.4 Coordinates