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Chapter 2.3: Line Bundles and Divisors

Huybrechts, *Complex Geometry*

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1 We assume X is compact and connected. Consider the exact sequence

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0,$$

where the first map is given by inclusion and the second by projection to the quotient. The exact sequence in cohomology then gives the exact sequence

$$H^0(X, \mathcal{M}^*) \to H^0(X, \mathcal{M}^*/\mathcal{O}^*) \to_\delta H^1(X, \mathcal{O}^*).$$

It is checked by diagram chasing that, under the isomorphisms $H^0(X, \mathcal{M}^*/\mathcal{O}^*) \simeq \mathrm{Div}(X)$ and $H^1(X, \mathcal{O}^*) \simeq \mathrm{Pic}(X)$, that δ is the canonical map of divisors to line bundles discussed in the chapter. Therefore this map has a nontrivial kernel exactly when the image of $H^0(X, \mathcal{M}^*)$ under the map on sections induced by the inclusion is nontrivial. But it can be seen that the image of a meromorphic function under this map is zero if and only if the function is locally and therefore globally constant. But a nonzero meromorphic function is constant if and only if satisfies a polynomial relation over $\mathbb{C}^*$.

2 First we note that s is nonsingular on Y. This is because its vanishing locus is a smooth hypersurface, so by the definition of complex submanifold (GH pg. 20), Y is locally given as the vanishing locus of a nondegenerate function h. But by the Nullstellensatz (pg. 11), this means that in some trivialization, h and the coordinate representation of s differ by a nonzero holomorphic function g. A calculation in coordinates then shows that over Y,

$$Ds = Dgh + gDh = gDh.$$

Now take $p \in Y$. By the implicit function theorem, it is possible to choose a chart $(z_1, ..., z_n)$ in a neighborhood U of p such that $Y \cap U$ corresponds to $\{z_n = 0\}$. Choose a trivialization of L over U in the chosen coordinate system and let s_U be the corresponding trivialization of s. I claim that $(z_1, ..., z_n) \to (z_1, ..., z_{n-1}, s_U(z_1, ..., z_n))$ is a diffeomorphism in a neighborhood of p. This follows because the derivative of s_U is normal to $z_n = 0$ and nonzero, so the Jacobian at p is invertible.

Therefore we have a system of charts (U_α, ρ_α) on X such that if U_α intersects Y, the z_n vanishes on Y and such that there exists a trivialization ϕ_α of L over this chart such that $\sigma_\alpha(z_1, ..., z_n) = z_n$. Then if $z' = (z'_1, ..., z'_n)$ is another coordinate chart, $z'_n(z_1, ..., z_{n-1}, 0)$ is identically zero and

$$z'_n(z_1, ..., z_{n-1}, z_n) = \sigma_\beta(z_1, ..., z_n) = g_{\beta\alpha}\sigma_\alpha(z_1, ..., z_n).$$

Therefore the transition function $F_{\alpha\beta}$ between any two such charts takes the form

$$\begin{pmatrix} * & * \\ 0 & g_{\alpha\beta} \end{pmatrix}.$$

Using the characterization of the normal bundle (HY pg. 68), this completes the proof.

3 Let $f_1, ..., f_k$ be the polynomials defining X. Fix $p \in X$ and choose a trivialization containing p. Near p we can assume that $F = (z_1, ..., z_{n-k}, f_1(z), ..., f_k(z))$ is local diffeomorphism. This gives a systen of charts such that the locus of f_i is the $n - k + i$th hyperplane. Trivializations then take the form

$$\begin{pmatrix} * & * \\ 0 & D \end{pmatrix},$$

where D is a diagonal matrix. Thus the normal bundle to the complete intersection is the direct sum of the normal bundles of each hypersurface.

Now we have seen that a homogeneous polynomial of degree d corresponds to a section of $\mathcal{O}(d)$. By the previous exercise, this yields that

$$\mathcal{N}_X = \oplus_{i=1}^k \mathcal{O}(d_k).$$

4 This is proven in Griffiths Harris. FINISH

5 Fix n and d. By projective equivalence, it is enough to show that ϕ is an embedding at any $p \in \mathbb{P}^n$, that is, that $D\phi$ has maximum rank at any point. Pick $p \in U_0 \subset \mathbb{P}^n$, so $\phi(p) \in U_{x_0^d} \subset \mathbb{P}^N$. In the standard coordinates on these charts,

$$\phi(x_{1/0}, ..., x_{n/0}) = (\prod_{i>0} x_{i/0}^{d_i}).$$

We require that $D\phi$ be rank n. But consider the monomial $x_0^{d-1}x_i$, which is $x_{i/0}$ is coordinates. In coordiantes, the derivative of this component is just $dx_{i/0}$. These clearly give n independent columns, so $D\phi$ has rank n.

6 The fiber of $p_1^*(L_1) \otimes p_2^*(L_2)$ over x is $(L_1)_{p_1(x)} \otimes (L_2)_{p_2(x)}$, and the section $s_1^i \otimes s_2^j$ takes the value $s_1^i(p_1(x)) \otimes s_2^j(p_2(x))$. Therefore this statement follows from the fact that if A and B are vector spaces with bases $a_1, ..., a_n$ and $b_1, ..., b_m$, then $a_i \otimes b_j$ form a basis for $A \otimes B$. Recall that this is proven by defining a linear map $f : A \times B \to \mathbb{C}$ sending

$$(\sum s_i a_i, \sum t_j b_j) \mapsto s_i t_j.$$

This map factors through the tensor product to $\tilde{f}$. Therefore if $z = \sum c_{ij} a_i \otimes b_j = 0$, $0 = \tilde{f}(z) = c_{ij}$.

7 By projective equivalence, we can take $x = [1 : 0 : ... : 0]$. Then the linear system is the span of $s_1, ..., s_n$, so ϕ sends $[x_0 : ... : x_n]$ to $[x_1 : ... : x_n]$. This map is a projection of $\mathbb{P}^n$ onto a (projective) hyperplane.

9 The map $\phi_{\mathcal{O}(d)}$ is given in projective coordinates by

$$[x : y] \mapsto [x^3 : x^2y : xy^2 : y^2] = [a : b : c : d].$$

It is easily checked that the image Y of this map is the vanishing locus of the three quadratic polynomials

$$p = ad - bc, q = b^2 - ac, r = c^2 - bd.$$

Since these polynomials are independent over $\mathbb{C}$, no two define Y.

If Y were a complete intersection (f, g), then f and g would have to generate the same ideal in $\mathbb{C}[a, b, c, d]$. Now f and g could not both be quadratic polynomials, for the only homogeneous quadratic polynomials generated by two homogeneous quadratic polynomials are linear combinations of the generators, but p, q and r are linearly independent over $\mathbb{C}$. Therefore one of f and g must be linear. But Y is contained in no hyperplane, for if

$$\sum_{i=0}^{3} a_i x^i y^{3-i} = 0,$$

for all x and y then

$$\sum_{i=0}^{3} a_i x^i = 0$$

for all x, so each $a_i = 0$.

10 Say first that $x \in C$. We can take $x = [1 : 0 : 0] = \phi([1 : 0])$. Then the linear system pulls back to $\mathbb{P}^1$ as the sections xy and y^2, so the map is given by $[x : y] \mapsto [xy : y^2] = [x : y]$. Therefore the map is the identity map on $\mathbb{P}^n \setminus x$, which clearly extends to the identity map on $\mathbb{P}^1$.

Say otherwise that $x \notin C$ and take $x = [0 : 1 : 0]$. Then the linear system pulls back to x^2, y^2, so we have the map $\mathbb{P}^1 \to \mathbb{P}^1$ sending $[x : y] \mapsto [x^2 : y^2]$. This gives a branched double cover of $\mathbb{P}^1$ over itself with branch points 0 and ∞.

11 This follows from Proposition 2.3.34. For if X is not isomorphic to $\mathbb{P}^1$, then the Abel-Jacobi map is injective, so for any two points $\mathcal{O}(x_1) \not\cong \mathcal{O}(x_2)$. But the divisor $x_1 - x_2$ has degree 0, and it is principal only if the corresponding line bundle is $\mathcal{O}$; however, $\mathcal{O}(x_1) \otimes \mathcal{O}(x_2)^{-1} \not\cong \mathcal{O}$.