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Chapter 4.2: Connections

Huybrechts, *Complex Geometry*

4.2.1

1. We can always define a connection on a complex vector bundle $E \to M$. For if (U_α, ϕ_α) is a locally finite collection of local trivializations of E, there exists a smooth partition of unity subordinate to (U_α). Under this trivialization, a section is just a map $U_\alpha \to \mathbb{C}^n$, so we can apply d to each component. Call this map ∇_α. Then if $s = (s^1, ..., s^n)$ is a section of U_α in these coordinates,

$$\nabla_\alpha(f \cdot s) = (d(fs^1), d(fs^2), ..., d(fs^n)) = (d(f)s^1 + fd(s^1), ..., d(f)s^n + fd(s^n)) = d(f)s + f\nabla_\alpha(s),$$

so ∇_α is a connection on U_α. Then if we define $\nabla = \sum f_\alpha \nabla_\alpha$, ∇ is a connection, for if s is a section, then we can evaluate the locally finite sum:

$$\nabla(fs) = \sum f_\alpha \nabla_\alpha(fs) = \sum f_\alpha \nabla_\alpha(fs) = \sum f_\alpha d(f)s + f\nabla_\alpha(s) = d(f)s + f\nabla(s).$$

Now say $E \to M$ is a Hermitian vector bundle. IF $\sigma^1, ...\sigma^n$ are local sections of E such that $\sigma^1(p), ..., \sigma^n(p)$ form a basis, then locally we can orthogonalize them with respect to the Hermitian metric. Therefore we can cover E with trivializations so that the Hermitian metric is represented by the identity matrix on each trivialization. With respect to such a trivialization, d as defined above is a Hermitian connection:

$$d(H(\sum_{i=1}^n \alpha_i e_i, \sum_{j=1}^n \beta_j e_j)) = d(\sum_{i=1}^n \alpha_i \beta_j) = \sum_{i=1}^n d(\alpha_i)\beta_i + \alpha_i d(\beta_i) = H(d(\sum_{i=1}^n \alpha_i e_i), \sum_{j=1}^n \beta_j e_j) + H(\sum_{i=1}^n \alpha_i e_i, d(\sum_{j=1}^n \beta_j e_j)).$$

The same partition of unity argument as above extends these local Hermitian connections to a global Hermitian connection.

2. Say that $[\nabla - \nabla'](s) = 0$ for all $s \in \Gamma(M, \mathcal{A}^0(E))$. We claim that $[\nabla - \nabla'](t) = 0$ for all $t \in \Gamma(U, \mathcal{A}^0(E))$ for any open $U \subset M$. For say otherwise that $[\nabla - \nabla'](t)$ is nonzero at $p \in U$. Since U is open, we can take a smooth function f that is supported in U and nonvanishing at p. Then $f \cdot t$ gives a global section whose restriction to U is nonzero. But by Proposition 4.2.3,

$$0 = [\nabla - \nabla'](f \cdot t) = f \cdot [\nabla - \nabla'](t).$$

This contradiction shows that there is no such p.

4.2.2

1. Take $s = (s_1, s_2) \in E_1 \oplus E_2$. Then

$$((\nabla_1 + a_1)s_1, (\nabla_1 + a_1)s_2) = \nabla s + (a_1 s_1, a_2 s_2) = \nabla s + a(s)$$

where $a = a_1 \oplus a_2 \in \mathcal{A}^1(\mathrm{End}(E_1) \oplus \mathrm{End}(E_2)) \simeq \mathcal{A}^1(\mathrm{End}(E_1)) \oplus \mathcal{A}^1(\mathrm{End}(E_1))$.

2. $(\nabla_1 + a_1)s_1 \otimes (\nabla_1 + a_1)s_2 = \nabla s + (a_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes a_2)s$.

3. For any $f \in \mathrm{Hom}(E_1, E_2)$ and s a section of E_1,

$$(\nabla_2 + a_2)f(s) - f((\nabla_1 - a_1)s) = \nabla(f)s + a_2 f(s) - f(a_1(s)).$$

4.2.3 Let s_i and t_i be a local sections of E_i. Then

$$\begin{aligned} d(h_1(s_1,t_1) + h_2(s_2,t_2)) &= d(h_1(s_1,t_1)) + d(h_2(s_2,t_2)) \\ &= h_1(\nabla_1(s_1),t_1) + h_1(s_1,\nabla_1(t_1)) + h_2(\nabla_1(s_2),t_2) + h_2(s_2,\nabla_2(t_2)) \\ &= (h_1+h_2)(\nabla(s_1,s_2),(t_1,t_2)) + (h_1+h_2)((s_1,s_2),\nabla(t_1,t_2)) \end{aligned}$$

while

$$\pi^{0,1}(\nabla_1(s_1) + \nabla_2(s_2)) = \pi^{0,1}(\nabla_1(s_1)) + \pi^{0,1}(\nabla_2(s_2)) = \bar{\partial}(s_1 + s_2).$$

The remaining computations are omitted.

4.2.4 Let $e_1, ..., e_n$ be local nonvanishing holomorphic sections of E such that $h(e_i, e_j) = \delta_{ij}$. Locally $\nabla = d + A$ for a matrix of 1-forms $A = (A_{ij})$. Now if ∇ is Hermitian, we have that

$$0 = d(h(e_i,e_j)) = h(Ae_i,e_j) + h(e_i,Ae_j) = A_{ij} + \overline{A_{ji}}.$$

Therefore $A^h = -A$. Now assume that ∇ is compatible with the holomorphic structure. We compute

$$\begin{aligned} \pi^{0,1}((d+A)(e_i)) &= (\sum_j (\bar{\partial}_j e_i) dz_j) e_i + \pi^{0,1}(A_{ij}) e_j \\ &= \pi^{0,1}(A_i j) e_j \\ &= 0, \end{aligned}$$

where the last equality follows because $\pi^{0,1}(\nabla e_j) = \bar{\partial}(e_j) = 0$, since e_j is a holomorphic trivialization of E. Therefore $A_{ij} \in \mathcal{A}^{1,0}(U)$ for all i,j. But $\overline{\mathcal{A}^{1,0}(U)} = \mathcal{A}^{0,1}(U)$. So if ∇ is the Chern connection, then $A = 0$.

4.2.5 The induced connections are hermitian (computation omitted). The second fundamental form is zero, because it is given as

$$\mathrm{pr}_{E/E_i} \circ \mathrm{p}_{E_i} \circ \nabla,$$

and the composition of the two projections is zero.

4.2.6 The determinant bundle is simply the nth tensor power of E, so the connection is just the connection applied to each factor:

$$\nabla(e_1 \otimes ... \otimes e_n) = \nabla(e_1) \otimes e_2 \otimes ... \otimes e_n + ... + e_1 \otimes ... \otimes e_{n-1} \otimes \nabla(e_n).$$

It can be checked that the same formula as the tensor product defines a connection on $\Lambda^2(E)$. I think the idea here is that if $F \subset E$ is a subbundle such that $\nabla(\mathcal{A}^0(F)) \subseteq \mathcal{A}^1(F)$, the connection descends to the quotient.

4.2.7