Huybrechts 1.2

Holly Mandel 01/14/2018

1.2.2 Take $\alpha = L^i\tilde{\alpha}$ for $\alpha \in P^{k-2i}$ and $\beta = L^j\tilde{\beta}$ for $\beta \in P^{k-2j}$. Say $i > j$. Now

$$(\alpha, \beta) = L^i\tilde{\alpha} \wedge L^j\tilde{\beta} \wedge \omega^{n-k} = \tilde{\alpha} \wedge \tilde{\beta} \wedge \omega^{n-k+i+j}.$$

Since $i > j$, $i + j \geq 2j + 1$. Therefore $n - k + i + j \geq n - (k - 2j) + 1$. But by Proposition 1.2.30, $\tilde{\beta} \in \text{Ker } L^{n-(k-2j)+1}$. Therefore

$$(\alpha, \beta) = \tilde{\alpha} \wedge L^{n-k+i+j}(\tilde{\beta}) = 0.$$

This proves that the decomposition $\Lambda^k V^* = \oplus L^i P^{k-2i}$ is orthogonal with respect to the Hodge-Riemann pairing.

On the other hand, say $i \neq j$ and $p + q = k - 2i$ and $p' + q' = k - 2j$. Take $\gamma = L^i\tilde{\gamma}$ for $\tilde{\gamma} \in P^{p,q}$ and $\delta = L^i\tilde{\delta}$ for $\tilde{\delta} \in P^{p,q}$. If $i \neq j$, then we cannot have $(p, q) = (q', p')$, for this would imply $p + q = p' + q'$. But

$$(\gamma, \delta) = L^i\tilde{\gamma} \wedge L^j\tilde{\delta} \wedge \omega^{n-k} = \tilde{\gamma} \wedge \tilde{\delta} \wedge \omega^{n-k+i+j}.$$

This last term is zero by bidegree.

1.2.3 Let $z_1, \bar{z}_1, ..., z_n, \bar{z}_n$ be the ordered basis for $V_{\mathbb{C}}$ constructed in the discussion after Lemma 1.2.17. Let $i_1, ...i_p$ and $j_1, ..., j_q$ be ordered collections of indices for $p, q < n$, let $s_1, ..., s_{n-p}, t_1, ..., t_{n-q}$ be the complementary sets of indices, and let σ be the sign of the permutation

$$z_1, \bar{z}_1, ..., z_n, \bar{z}_n \to \bar{z}_{i_1}, ..., \bar{z}_{i_p}, z_{j_1}, ..., z_{j_q}, \bar{z}_{s_1}, ..., \bar{z}_{s_{n-p}}, z_{t_1}, ..., z_{t-n-q}.$$

Now $*$ is characterized by the relation

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle_{\mathbb{C}} \cdot \text{Vol}.$$

Since powers of the $z_i, \bar{z}_i$ form an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, this relation implies that if $\beta = z_{i_1} \wedge ... \wedge z_{i_p} \wedge z_{j_1} \wedge ... \wedge z_{j_q}$, then $*\bar{z}_{i_1} \wedge ... \wedge \bar{z}_{i_p} \wedge z_{j_1} \wedge ... \wedge z_{j_q} = \sigma \bar{z}_{s_1} \wedge ... \wedge \bar{z}_{s_p} \wedge ... \wedge z_{t_1} \wedge ... \wedge z_{t_q}$. By the complex linearity of $*$, this implies that $*(\Lambda^{p,q}V^*) \subseteq \Lambda^{n-q,n-p}V^*$.

FINISH

1.2.4 The product of two primitive forms is not necessarily primitive. For choose a basis $z_1, \bar{z}_1, ..., z_n, \bar{z}_n$ for V as above. Then z_1 and $\bar{z}_1$ are both primitive. For $*z_1$ has degree $n - 1$, so $L(*z_1)$ has degree $n + 1$ and is therefore zero, which implies that $\Lambda(z_1) = *^{-1} \circ L \circ *(z_1) = 0$. A similar argument shows that $\Lambda(\bar{z}_1) = 0$.

On the other hand, $*(z_1 \wedge \bar{z}_1) = z_2 \wedge \bar{z}_2 \wedge ... \wedge z_n \wedge \bar{z}_n$, so

$$*(z_1 \wedge \bar{z}_1) = \frac{i}{2}(\sum_{i=1}^{n} z_1 \wedge \bar{z}_1) \wedge z_2 \wedge \bar{z}_2 \wedge ... \wedge z_n \wedge \bar{z}_n = \frac{i}{2} z_1 \wedge \bar{z}_1 \wedge ... \wedge z_n \wedge \bar{z}_n.$$

Therefore $*^{-1} \circ L \circ *(z_1 \wedge \bar{z}_1) \neq 0$, so $z_1 \wedge \bar{z}_1$ is not primitive.