January 2019

Chapter 3.1: Kahler Identities

Huybrechts, *Complex Geometry*

3.1.1 This is a reflection of the fact that Hermitian metrics are not required to be holomorphic. For let M be a complex manifold and take any coordinate chart $U \subset \mathbb{C}^n$. On U, the complex structure of M is as described in Section 1.3. The flat metric g_U is compatible with this complex structure.

Now cover M be charts U_α and let ρ_α be a partition of unity subordinate to this cover. Then $g = \sum_\alpha \rho_\alpha g_{U_\alpha}$ is the desired metric:

$$g(I\cdot, I\cdot) = \sum_\alpha \rho_\alpha g_{U_\alpha}(I\cdot, I\cdot) = \sum_\alpha \rho_\alpha g_{U_\alpha}(\cdot,\cdot) = g(\cdot,\cdot).$$

3.1.2 Because g is Kahler, $dg = 0$. But say

$$dg' = d(e^f)g + e^f dg = d(e^f)g = 0.$$

Because X is connected, this implies that f is constant.

3.1.3 Say a function f is d-harmonic. By degree, $d^* f = 0$, so

$$\langle \Delta_d f, f\rangle = \langle d^* df, f\rangle = \|df\|^2 = 0.$$

Therefore $df = 0$, which implies that $\partial f = \bar\partial f = 0$. The same trick shows that if $\Delta_\partial f = \partial^*\partial f = 0$, then $\partial f = 0$, and if $\Delta_{\bar\partial} f = \bar\partial^*\partial f = 0$, then $\bar\partial f = 0$, so $df = 0$, and therefore $\Delta f = 0$.

Similar arguments work for top forms, since in this case $d^*df = 0$ and similar.

3.1.4 Computation omitted (should be in my papers from class).

3.1.5 This is easily seen using the coordinate representation on page 117 and the observation that, e.g. in U_0, $i^*dz_i = dz_i$ for $0 < i < n$ and $i^*dz_n = 0$.

3.1.6 Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. Then $F_A^*\omega_{FS} = \omega_{FS}$ if and only if $A^*\pi^*\omega_{FS} = \pi^*\omega_{FS}$, where A is the ordinary linear action of A on $\mathbb{C}^{n+1}$. This is because $\pi \circ A = F_A \circ \pi$, so $A^*\pi^*\omega_{FS} = \pi^* F_A^*\omega_{FS}$. Also, π is a subjective submersion, so π^* is injective on forms on $\mathbb{P}^n$: if $\omega \neq 0 \in \Omega_{\mathbb{P}^n}$, then $\omega_{F(p)}(DF_p-, DF_p-)$ must be nonzero for some $p \in \mathbb{C}^{n+1} \setminus \{0\}$ because there is some $F(p)$ where ω does not vanish, and then vectors on which it does not vanish can be "reached" by DF_p.

Now we calculated on pg. 118 that $\pi^*\omega_{FS} = \frac{i}{2\pi}\partial\bar\partial \log\|z\|^2$. Since A is a holomorphic function, it commutes with ∂ and $\bar\partial$. Therefore

$$A^*\pi^*\omega_{FS} = \frac{i}{2\pi}\partial\bar\partial\log\|Az\|^2.$$

When does $\partial\bar{\partial}\log\|Az\|^2 = \partial\bar{\partial}\log\|z\|^2$? We compute that

$$\partial\bar{\partial}\log\|Az\|^2 = \sum_{i,j} \frac{A_{ik}\bar{A}_{jk} dz_i \wedge d\bar{z}_j}{\|Az\|^2} + \mathcal{O}(\|Az\|^{-2}).$$

But scaling A does change $\partial\bar{\partial}\log\|Az\|^2$, so we can ignore the $\mathcal{O}(\|Az\|^{-2})$ term. Then comparing terms with the case $A = I$, we see that we must have $A^H A = \lambda\delta_{ij}$. This A must be a multiple of a unitary matrix.

I believe that it is an error for the book to say that A must be unitary. For A and λA define the same map on $\mathbb{P}^{n+1}$ for $\lambda \neq 0$.

3.1.7 First, I claim that L, d and d^* determine d^c. For

$$[d^*, L] = [\bar{\partial}^* + \partial^*, L] = i(\partial - \bar{\partial}) = -d^c.$$

Now I claim that d^c determines the bidegree decomposition of $\mathcal{A}^*(X)$, and therefore the complex structure, since the eigendecomposition of a diagonalizable operator determines the operator. For let ω be any form of pure type (p,q). Then

$$d^c\omega = I^{-1} \circ d \circ I(\omega) = I^{-1}(-i)^{p-q}(\partial\omega + \bar{\partial}\omega) = (-i)^{p-q-(p+1-q)}\partial\omega + (-i)^{p-q-(p-q-1)}\bar{\partial}\omega.$$

Therefore

$$\partial\omega = (2i)^{-1}(d^c\omega - id\omega).$$

A similar computation allows us to compute $\bar{\partial}\omega$.

Since all degree-1 covectors are realized as linear combinations of ∂ and $\bar{\partial}$ of coordinate functions, this proves that we have determined the way I acts on $\mathcal{A}^1(X)$, which determines how it acts on the tangent bundle.

3.1.8 Also in my class papers.

3.1.9 Let α and β be two forms. We compute

$$\begin{aligned}
\langle \alpha, i\partial\beta \rangle &= \langle -i\partial^*\alpha, \beta \rangle \\
&= \langle [\Lambda, \bar{\partial}]\alpha, \beta \rangle \\
&= \langle \Lambda\bar{\partial}\alpha - \bar{\partial}\Lambda\alpha, \beta \rangle \\
&= \langle \alpha, \bar{\partial}^* L\beta - L\bar{\partial}^*\beta \rangle \\
&= \langle \alpha, [\bar{\partial}^*, L]\beta \rangle.
\end{aligned}$$

The other computation is similar.

3.1.10 The Kahler form on $X \times Y$ is $\pi_X^*\omega_X + \pi_Y^*\omega_Y$. The compatibility, closedness, etc. are easily verified.

3.1.13 The Kahler form is also sympectic. It can be seen that ω^n is nondegenerate for example by Proposition 1.3.12, since there is a coordinate representation around any point at which

$$\omega^n = C \cdot dz_1 \wedge d\bar{z}_1 \wedge ... \wedge dz_n \wedge d\bar{z}_n.$$

However, the space of symplectic forms is not a cone. For if n is odd, both ω and $-\omega$ are symplectic forms, but $\omega^n + -\omega^n = 0$. This cannot happen with Kahler forms because the negative of a Kahler form is not a Kahler form, since the corresponding bilinear form would be negative definite.