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Chapter 3.2: Hodge Theory on Kahler Manifolds

Huybrechts, *Complex Geometry*

3.2.1 The Kahler form ω can be written as $L(1)$, where 1 is the function identically equal to 1 on X. Now by Proposition 3.1.12,

$$\Delta L(1) = L\Delta(1) = 0.$$

3.2.2 From the remark we have the map $F : \mathcal{H}^{p,q}(X) \to \mathcal{H}^{n-q,n-p}(X)^*$, given as follows. Let α be an element of $H^{p,q}(X)$. Then $F(\alpha)$ is the linear map $\beta \mapsto \int_X \alpha \wedge \beta$. As discussed in the remark this map is injective (here we use the Hermitian structure). It is then surjective by dimension, since the Hodge star gives $\mathcal{H}^{p,q}(X, g) \simeq \mathcal{H}^{n-q,n-p}(X, g)$.

Now for a Kahler manifold we have the isomorphism $\mathcal{H}^{p,q}_{\bar{\partial}}(X, g) \simeq \mathcal{H}^{p,q}_{d}(X, g) \simeq H^{p,q}(X)$, which completes the proof.

3.2.3 Can't think of much to write for this one!

3.2.6 By Corollary 3.2.12, on a Kahler manifold we have

$$h^k(X) = \sum_{r=0}^{k} h^{k,k-r}(X).$$

But by complex conjugation, $h^{k,k-r}(X) = h^{k-r,k}(X)$. Thus is k is odd,

$$h^k(X) = 2\sum_{r=0}^{\frac{k-1}{2}} h^{k,k-r}(X),$$

so $h^k(X)$ is even.

3.2.7 Let X be a Hopf surface. Then X is diffeomorphic to $S^1 \times S^3$, so $H^1(X, \mathbb{C})$ has dimension 1. However, by the previous exercise, $H^1(K, \mathbb{C})$ must have even dimension.

3.2.8 Let X be a compact Kahler manifold and fix a Kahler metric. Now if α is a $\bar{\delta}$-closed form on X, then it is $\Delta_{\bar{\delta}}$-harmonic. For $H^0(X, \Omega^p_X) = H^{p,0}(X)$ is a subgroup of $\mathcal{A}^{p,0}$. But since X is compact, by Theorem 3.2.8 we have that

$$\mathcal{A}^{p,0} = \mathcal{H}^{p,0}_{\bar{\delta}}(X, g) \oplus \bar{\delta}^*\mathcal{A}^{p,1}(X),$$

so we can write $\alpha = \alpha_1 + \alpha_2$ for α_1 $\bar{\delta}$-harmonic and $\alpha_2 = \bar{\delta}^*\beta$ for some $\beta \in \mathcal{A}^{p,1}(X)$. Now $\bar{\delta}\alpha_1 = 0$ by Lemma 3.2.5. Therefore $\bar{\delta}\alpha = 0$ implies that $\alpha_2 = 0$, for as in the proof of Corollary 3.2.9, $\bar{\delta}\alpha_2 = 0$ implies that

$$0 = \langle \bar{\delta}\bar{\delta}^*\beta, \beta\rangle = \langle \bar{\delta}^*\beta, \bar{\delta}^*\beta\rangle.$$

Therefore $\alpha = \alpha_1 \in \mathcal{H}^{p,0}_{\bar{\delta}}(X, g)$. But since X is Kahler, by Proposition 3.1.12 this implies that α is harmonic. This computation did not depend on the choice of Kahler metric.

3.2.9 This is correct. First we establish that the "$\partial^*\bar{\partial}^*$-lemma" follows from the $\partial\bar{\partial}$ lemma. For say β is a pure form of type (p,q) and $d^*\beta = 0$. Then $d * \beta = 0$ since $*$ is injective. Therefore by the usual lemma, $*\beta$ is d exact if and only if it is $\partial\bar{\partial}$ exact. But $*\beta = d\gamma$ if and only if $\beta = d^*(*\gamma)$, and similarly $*\beta = \partial\bar{\partial}\eta$ if and only if $\beta = \pm\partial^*\bar{\partial}^*(*\eta)$. Thus β is d^* exact if and only if it is $\partial^*\bar{\partial}^*$ exact.

Now by the Riemannian Hodge decomposition, $\alpha \in \mathcal{A}^{p,q}(X) \subseteq \mathcal{A}^{p+q}(X)$ can be written

$$\alpha = d\alpha_1 + \alpha_2 + d^*\alpha_3.$$

Since the decomposition is direct, each component must be of pure type (p,q). Therefore by the $\partial\bar{\partial}$ lemma and its corrolary of the previous paragraph, we can write

$$\alpha = \partial\bar{\partial}\tilde{\alpha}_1 + \alpha_2 + \partial^*\bar{\partial}^*\tilde{\alpha}_3.$$

This proves the first part of Theorem 3.2.8; the second part is similar.

3.2.10 We have seen that

$$\langle\Delta_d\alpha, \alpha\rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2.$$

But by the orthogonality of the decompositions in Theorem 3.2.8,

$$\|d\alpha\|^2 = \|\partial\alpha\|^2 + \|\bar{\partial}\alpha\|^2$$

and

$$\|d^*\alpha\|^2 = \|\partial^*\alpha\|^2 + \|\bar{\partial}^*\alpha\|^2.$$

Thus $\Delta_d\alpha = 0$ implies that $\partial\alpha = \bar{\partial}\alpha = 0$, which implies that $\Delta_\partial\alpha = 0$. The argument is similar for $\bar{\partial}$.

3.2.13 If $\alpha = d^c\beta$ for $\alpha \in \mathcal{A}^{p,q}$, then α is orthogonal to the space of harmonic forms. By the proof of Corollary 3.2.10, this implies that $\alpha = \partial\bar{\partial}\gamma$ for some $\gamma \in \mathcal{A}^{p-1,q-1}$. But $dd^c = 2i\partial\bar{\partial}$.

3.2.16 This is simply the $\partial\bar{\partial}$ lemma applied to the exact form $\omega - \omega'$. We can choose f real because $\omega - \omega'$ is purely imaginary. ELABORATE.