Huybrechts 1.3

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1.3.1 It can be checked that for any f,

$$f^*dz_i = \frac{df}{dz_i}dz_i + \frac{df}{d\bar{z}_i}d\bar{z}_i \text{ and } f^*d\bar{z}_i = \frac{d\bar{f}}{dz_i}dz_i + \frac{d\bar{f}}{d\bar{z}_i}d\bar{z}_i.$$

For instance, if $f = (u_1 + iv_1, ..., u_n + iv_n)$,

$$\begin{aligned} f^*d\bar{z}_i &= d(x_i \circ f) - i\, d(y_i \circ f) \\ &= \frac{du_i}{dx_j}dx_j + \frac{du_i}{dy_j}dy_j - i\left(\frac{dv_i}{dx_j}dx_j + \frac{dv_i}{dy_j}dy_j\right) \\ &= dx_j(\frac{du_i}{dx_j} - i\,\frac{dv_i}{dx_j}) + dy_j(\frac{du_i}{dy_j} - i\,\frac{dv_i}{dy_j}) \\ &= \frac{d\bar{f}}{dx_j}dx_j - i\,\frac{d\bar{f}}{dy_j}dy_j, \end{aligned}$$

while

$$\begin{aligned} \frac{d\bar{f}}{dz_i}dz_i + \frac{d\bar{f}}{d\bar{z}_i}d\bar{z}_i &= \frac{1}{2}\left(\frac{d(u_i - iv_i)}{dx_j} - i\frac{d(u_i - iv_i)}{dy_j}\right)(dx_j + idy_j) + \frac{1}{2}\left(\frac{d(u_i - iv_i)}{dx_j} + i\frac{d(u_i - iv_i)}{dy_j}\right)(dx_j - idy_j) \\ &= \frac{d\bar{f}}{dx_j}dx_j - i\,\frac{d\bar{f}}{dy_j}dy_j. \end{aligned}$$

Now if f is holomorphic, $\frac{df}{d\bar{z}_i} = \frac{d\bar{f}}{dz_i} = 0$. Therefore

$$f^*dz_{i_1}\wedge...\wedge dz_{i_p}\wedge d\bar{z}_{j_1}\wedge...\wedge d\bar{z}_{j_q} = \sum_{l_1=1}^{n} ... \sum_{l_p=1}^{n}\sum_{m_1=1}^{n} ... \sum_{m_q=1}^{n} \frac{df^{i_1}}{dz_{l_1}}...\frac{df^{i_p}}{dz_{l_p}}\frac{d\bar{f}^{j_1}}{d\bar{z}_{m_1}}...\frac{d\bar{f}^{j_q}}{d\bar{z}_{m_q}}dz_{l_1}\wedge...\wedge dz_{l_p}\wedge d\bar{z}_{m_1}\wedge...\wedge d\bar{z}_{m_q}.$$

The result follows by linearity.

1.3.2 By Proposition 1.2.8, conjugation exchanges $\Lambda^{p,q}U$ and $\Lambda^{q,p}U$. Therefore $\Pi^{p+1,q}(\overline{\beta}) = \overline{\Pi^{p,q+1}\beta}$. Also, since $\frac{d\bar{f}}{d\bar{z}} = \overline{\frac{df}{dz}}$, it follows that $\overline{d\alpha} = d\overline{\alpha}$. Therefore

$$\overline{\delta\alpha} = \overline{\Pi^{p+1,q} \circ d\alpha} = \Pi^{p,q+1}\overline{d\alpha} = \Pi^{p,q+1}d\overline{\alpha} = \bar{\delta}\overline{\alpha}$$

Now say $\alpha = fdz \in \mathcal{A}^{1,0}(U)$ for U an open neighborhood of a bounded disk $B_\epsilon \subseteq \mathbb{C}$. Then $\bar{\alpha} = \bar{f}d\bar{z} \in \mathcal{A}^{0,1}(U)$, so $\bar{\alpha} = \bar{\delta}g$ for g as in Prop 1.3.7. But then $\overline{\bar{\delta}\overline{g}} = \alpha$. So α is δ-exact.

The same algebra gives equivalent statements for Prop. 1.3.8 and Corollary 1.3.9.

1.3.3 Since α is d-closed, it is δ-closed, so $\alpha = \delta\beta$ for some $\beta \in \mathcal{A}_{\mathbb{C}}^{p+q-1}(B)$. We can write

$$\beta = \sum_{k+l=p+q-1} \beta^{k,l} \quad \beta^{k,l} \in \Lambda^{k,l}(B).$$

But since $\alpha \in \Lambda^{p,q}(B)$, $d\beta = d(\beta^{p-1,q} + \beta^{p,q-1})$.

Now by bidegree, $\bar{\delta}\beta^{p-1,q} = \delta(\beta^{p,q-1}) = 0$. Therefore we can write $\beta^{p-1,q} = \bar{\delta}\eta^{p-1,q}$ and $\beta^{p,q-1} = \delta\eta^{p,q-1}$. But then

$$\delta\bar{\delta}(\eta^{p-1,q} - \eta^{p,q-1}) = \delta\bar{\delta}(\eta^{p-1,q}) + \bar{\delta}\delta(\eta^{p,q-1}) = \delta\beta^{p-1,q} + \bar{\delta}\beta^{p,q-1} = \alpha,$$

as desired.

1.3.4 This follows immediately from 1.3.3.

1.3.5 This is a computation (omitted).

1.3.6 Omitted.

1.3.7 $\phi = \frac{i}{2\pi}|z|^2$.

1.3.8 Such an ω satisfies the hypotheses of Proposition 1.3.2, where for each point x, f is simply a translation of 0 to x. Therefore, $d\omega = 0$. Since $\omega \in \mathcal{A}^{1,1}(U)$. Therefore the result follows from 1.3.3.

1.3.9 The only such function if $f = 0$. For if $e^f g - \mathrm{Id} = O(|z|^2)$ and $g = \mathrm{Id} + \tilde{g}$ for $\tilde{g} = O(|z|)^2$, then

$$e^f(\mathrm{Id} + \tilde{g}) - \mathrm{Id} = O(|z|)^2.$$

Since $\tilde{g}(0) = 0$, this implies that $e^f = 1$, so $f(0) = 0$. But this must hold at every point, so $f \equiv 0$.