Huybrechts 2.1

Holly Mandel 06/25/2018

2.1.3

- The algebraic dimension of $\mathbb{P}^n$ is n. For consider the function $[x_0 : .... : x_n] \mapsto x_i/x_0$ for $i = 1, ..., n$. I claim that this is a rational function on $\mathbb{P}^n$. Consider any point $p = [p_0 : ... : p_n]$. If $p_0 \neq 0$ then $p \in U_0$ and on U_0, this function is represented by the obviously holomorphic coordinate function x_i. If $p_0 = 0$, then $p_j \neq 0$ for some $j > 0$, so $p \in U_j$, and z_i/z_j and z_0/z_j are well-defined on U_j and z_0/z_j is invertible. (If $i = j$, then $z_i/z_j = 1$, but the argument still goes through.)

  These n functions are algebraically independent. For say

  $$F(x_1/x_0, ..., x_n/x_0) = 0.$$

  Then $x_0^m F(x_1/x_0, ..., x_n/x_0) = x_0^m G(x_0, x_1, ..., x_n) = 0$ for some polynomial G and some m. This implies that $G = 0$, so $F = 0$.

  The result then follows from the upper bound of Proposition 2.1.9.

- The Weierstrass p function is a nontrivial rational function on the torus. Therefore by Proposition 2.1.9, the algebraic dimension of $\mathbb{T}$ is 1.

- Proposition 2.1.9 does not apply to (non-compact) $\mathbb{C}$. In fact, $K(\mathbb{C})$ has infinite transcendence degree. For $K(\mathbb{C})$ contains the holomorphic functions on $\mathbb{C}$, and e^{z^k} for $k \in \mathbb{N}$ is an algebraically independent set of holomorphic functions on $\mathbb{C}$. This can be seen by considering the asymptotics along the positive real axis.

2.1.4 Let $f : \mathbb{P}^1 \to \mathbb{T}$ be holomorphic. Then, if ϕ_0 is the coordinate on $U_0 \subseteq \mathbb{P}^1$, $f \circ \phi_0^{-1}$ is a holomorphic function $\mathbb{C} \to \mathbb{T}$. But a function on the torus can be interpreted as a periodic function on $\mathbb{C}$, so under this identification, $f \circ \phi_0^{-1}$ is a bounded holomorphic function on $\mathbb{C}$ and therefore constant.

The same argument holds for $\mathbb{P}^n$.

2.1.5 Let

$$\Gamma = \log(\lambda)\mathbb{Z} + i2\pi\mathbb{Z}.$$

Now

$$\text{Exp} : \{z : \text{Im}z \in [0, 2\pi),\ \text{Re}(z) \in [0, \log(\lambda))\} \to \{z : 1 \leq |z| < \lambda\}$$

is a bijective, holomorphic map between a fundamental domain for $\mathbb{C}/\Gamma$ and a fundamental domain for X.