

Chapter 2.6: Differential Calculus on Complex Manifolds

Huybrechts, *Complex Geometry*

2.6.1 Let (M, I) be an almost-complex manifold and say (U, ϕ) and (V, ψ) are two systems of charts (complex structures) on M inducing the same almost-complex structure. Take charts ϕ and ψ trivializing overlapping neighborhood of M . I claim that $\phi \circ \psi^{-1}$ is holomorphic. This is equivalent to the statement that $D(\phi \circ \psi^{-1})$ is \mathbb{C} -linear. But for any tangent vector v ,

$$D(\phi \circ \psi^{-1})Iv = D\phi \circ D\psi^{-1}Iv = D\phi(ID\psi^{-1}v) = ID\phi(D\psi^{-1}v) = ID(\phi \circ \psi^{-1})v.$$

This is because both coordinate charts are holomorphic with respect to I .

2.6.2 Let S be a surface and $\langle \cdot, \cdot \rangle$ the metric on S . Let (U_i, ϕ_i) be a system of oriented trivializations. As described in Example 1.2.12, there is a unique almost complex structure on each $\phi_i(U_i)$ such that $\langle v, Iv \rangle = 0$, $\|I(v)\| = \|v\|$, and $v, I(v)$ is positively oriented. Since each of these three conditions is preserved by oriented isometries, these local choices of I glue to a unique almost complex structure on S . But I must be integrable, for the complex dimension of the tangent space to S is 1, so $\mathcal{A}^{0,2}(X) = 0$ (Prop.2.6.15). Therefore it follows from Newlander-Neirenberg that the resulting almost complex structure is complex.

2.6.3 The natural complex structure on \mathbb{P}^1 is the one given in each of the standard coordinate charts by multiplying by i . Call this J . It can be checked that for the charts $U_i \subseteq \mathbb{P}^1, i = 0, 1$, the flat metric on $\mathbb{C} \simeq U_i$ is conformal to the Fubini-Study metric. Therefore J satisfies $\langle v, Jv \rangle_{FS} = 0$ and $\|Jv\|_{FS} = \|v\|_{FS}$, and $\{v, Jv\}$ is positively oriented. So this is the same complex structure as given in the previous exercise.

2.6.5 The $\bar{\partial}$ -Poincare lemma on the star-shaped set $M = \mathbb{C}^n$ implies that $H_{\bar{\partial}}^{0,1}(M) = 0$. Now a hypersurface H defines an element of $H^1(M, \Omega^0)$ as follows: locally, H is the vanishing set of f_α defined on an open set U_α . Given the overlap $U_{\alpha\beta}$ of two such charts, the ratio f_α/f_β defines a Čech 1-cocycle $(U_{\alpha\beta}, g_{\alpha\beta})$. The question is then whether $g_{\alpha\beta}$ is the boundary of $(U_\alpha, h|_{U_\alpha})$ for a global holomorphic function h , that is, whether $(U_{\alpha\beta}, g_{\alpha\beta})$ is zero in $H^1(M, \Omega^0)$. But we have seen that $H^1(M, \Omega^0) \simeq H_{\bar{\partial}}^{0,1}(M) = 0$.