Some Picard Groups

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1. $\mathbb{P}^n$: Recall that $\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}^*)$. We know from algebraic topology that $H^k(\mathbb{P}^n, M) = M$ for k even and 0 otherwise, $M \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{C}\}$. Therefore the exponential sheaf sequence

$$H^1(\mathbb{P}^n, \mathcal{O}) \to H^1(\mathbb{P}^n, \mathcal{O}^*) \to H^2(\mathbb{P}^n, \mathbb{Z})$$

shows that $\mathrm{Pic}(X) = \mathbb{Z}$ and that projective line bundles are determined by their image in $H^2(\mathbb{P}^n, \mathbb{Z})$, which is the first Chern class.

2. **Hypersurfaces in $\mathbb{P}^n$, $n \geq 4$:** . By Chow's theorem, a projective hypersurface is cut out by a section of $\mathcal{O}(d)$ for $d > 0$. Therefore we can apply the Lefschetz hyperplane theorem (GH pg 156) to find that

$$H^1(X, \mathcal{O}) \simeq H^1(X, \mathbb{C}) \simeq H^1(\mathbb{P}^n, \mathcal{O}) = 0,$$

and (because $\dim(X) \geq 3$),

$$H^2(X, \mathcal{O}) \simeq H^2(X, \mathbb{C}) \simeq H^2(\mathbb{P}^n, \mathcal{O}) = 0.$$

Therefore using the exponential sequence over X we have that $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$. But the Lefschetz hyperplane theorem, which is true for integer cohomology, gives that $\mathbb{Z} \simeq H^2(\mathbb{P}^n, \mathbb{Z}) \simeq H^2(X, \mathbb{Z})$. Therefore $\mathrm{Pic}(X) \simeq \mathbb{Z}$.

3. **Complete intersection of k hypersurfaces in $\mathbb{P}^n$, $n \geq 3+k$:** We claim that $\mathrm{Pic}(X) \simeq \mathbb{Z}$. The proof proceeds by induction with the first case being the previous bullet. Let F_k be the polynomial of degree d_k defining S_k, the kth hypersurface. Let $X^j = S_1 \cap ... \cap S_j$. The requirement that the intersection be complete guarantees that X^k is a smooth hypersurface of X^{k-1}. Therefore by Lefschetz and an inductive hypothesis $H^i(X^k, \mathcal{O}) \simeq H^i(X^{k-1}, \mathcal{O}) \simeq 0$ for $i = 1, 2$ and $H^2(X^{k-1}, \mathbb{Z}) \simeq \mathbb{Z}$, so be the exponential sequence and Lefschetz again, $H^2(X^k, \mathbb{Z}) \simeq \mathbb{Z}$ and $\mathrm{Pic}(X) \simeq H^2(X^k, \mathbb{Z})$.

4. $H^2(X, \mathcal{O}) = 0$ if the hypersurface is dimension 2?

5. Curves - especially elliptic ones! This seems cool.