Huybrechts 1.2

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1.2.1 Let $(V, \langle \cdot, \cdot \rangle)$ be a $4-$dimensional Euclidean space and fix a vector $v \in V$ of unit length and an orientation of V. Say that I is a compatible almost-complex structure. Then we must have

$$\langle v, Iv \rangle = \langle Iv, I^2 v \rangle = -\langle Iv, v \rangle = -\langle v, Iv \rangle,$$

so Iv is orthogonal to v. Also, since I is orthogonal with respect to $\langle \cdot, \cdot \rangle$, we must have that $\|Iv\| = 1$.

Now the orthogonal complement to v is a three-dimensional Euclidean space, so the set of unit-length vectors in this complement is identified with S^2: $Iv \in S^2$. It is easily checked that I stabilizes $(v \oplus Iv)^\perp$, so I restricts to an almost-complex structure on this two-dimensional Euclidean space, and the restriction of I is compatible with the restriction of $\langle \cdot, \cdot \rangle$. By the discussion in Huybrechts (Example 1.2.12), the only remaining choice is of an orientation for this vector space. One choice gives a positive orientation and one a negative orientation to the basis $[v, Iv, w, Iw]$ for any $w \in (v \oplus Iv)^\perp$. (The choice of w does not matter, since if we had chosen $w' = aw + bIw$, then the transition map from $w, Iw \to w', Iw'$ is seen to have determinant $a^2 + b^2$.) Therefore we can characterize I by (1.) the choice of $Iv \in S^2$ for the fixed vector v and (2.) the sign of the orientation $+/-$. This gives a map from the set of almost complex structures to two copies of S^2.

On the other hand, any choice of $v' \in S^2 \subset v^\perp$ and orientation gives a compatible almost complex structure by defining $Iv = v'$, $Iv' = -v$, and then giving the orthogonal complement the almost complex structure with the chosen orientation. For it is easily checked that $I^2 z = -z$ for $z \in \{v, v', w, Iw\}$, which implies that $I^2 = -\mathrm{Id}$. It is easy to see also that the restriction of I is orthogonal on the two subspaces $v \oplus v'$ and its complement, and these subspaces are I-invariant, so I is orthogonal on V. Therefore I defined this way is a compatible almost-complex structure.

1.2.2 Take $\alpha = L^i \tilde{\alpha}$ for $\alpha \in P^{k-2i}$ and $\beta = L^j \tilde{\beta}$ for $\beta \in P^{k-2j}$. Say $i > j$. Now

$$(\alpha, \beta) = L^i \tilde{\alpha} \wedge L^j \tilde{\beta} \wedge \omega^{n-k} = \tilde{\alpha} \wedge \tilde{\beta} \wedge \omega^{n-k+i+j}.$$

Since $i > j$, $i + j \geq 2j + 1$. Therefore $n - k + i + j \geq n - (k - 2j) + 1$. But by Proposition 1.2.30, $\tilde{\beta} \in \mathrm{Ker}\ L^{n-(k-2j)+1}$. Therefore

$$(\alpha, \beta) = \tilde{\alpha} \wedge L^{n-k+i+j}(\tilde{\beta}) = 0.$$

This proves that the decomposition $\Lambda^k V^* = \oplus L^i P^{k-2i}$ is orthogonal with respect to the Hodge-Riemann pairing.

On the other hand, say $i \neq j$ and $p + q = k - 2i$ and $p' + q' = k - 2j$. Take $\gamma = L^i \tilde{\gamma}$ for $\tilde{\gamma} \in P^{p,q}$ and $\delta = L^i \tilde{\delta}$ for $\tilde{\delta} \in P^{p,q}$. If $i \neq j$, then we cannot have $(p, q) = (q', p')$, for this would imply $p + q = p' + q'$. But

$$(\gamma, \delta) = L^i \tilde{\gamma} \wedge L^j \tilde{\delta} \wedge \omega^{n-k} = \tilde{\gamma} \wedge \tilde{\delta} \wedge \omega^{n-k+i+j}.$$

This last term is zero by bidegree.

1.2.3 Let $z_1, \bar{z}_1, ..., z_n, \bar{z}_n$ be the ordered basis for $V_{\mathbb{C}}$ constructed in the discussion after Lemma 1.2.17. Let $i_1, ... i_p$ and $j_1, ..., j_q$ be ordered collections of indices for $p, q < n$, let $s_1, ..., s_{n-p}, t_1, ..., t_{n-q}$ be the complementary sets of indices, and let σ be the sign of the permutation

$$z_1, \bar{z}_1, ..., z_n, \bar{z}_n \to \bar{z}_{i_1}, ..., \bar{z}_{i_p}, z_{j_1}, ..., z_{j_q}, \bar{z}_{s_1}, ..., \bar{z}_{s_{n-p}}, z_{t_1}, ..., z_{t-n-q}.$$

Now $*$ is characterized by the relation

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle_{\mathbb{C}} \cdot \mathrm{Vol}.$$

Since powers of the $z_i, \bar{z}_i$ form an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_{\mathbb{C}}$, this relation implies that if $\beta = z_{i_1} \wedge ... \wedge z_{i_p} \wedge z_{j_1} \wedge ... \wedge z_{j_q}$, then $*\bar{z}_{i_1} \wedge ... \wedge \bar{z}_{i_p} \wedge z_{j_1} \wedge ... \wedge z_{j_q} = \sigma \bar{z}_{s_1} \wedge ... \wedge \bar{z}_{s_p} \wedge ... \wedge z_{t_1} \wedge ... \wedge z_{t_q}$. By the complex linearity of $*$, this implies that $*(\Lambda^{p,q} V^*) \subseteq \Lambda^{n-q,n-p} V^*$. This fact, along with linearity, gives the identity $* \Pi^{p,q} = \Pi^{n-q,n-p} *$.

It is easy to check the identity $[L, I] = 0$ on basis elements of this form, using the fact that ω has bidegree $(1,1)$. Finally, if $v = z_{i_1} \wedge ... \wedge z_{i_p} \wedge \bar{z}_{j_1} \wedge ... \wedge \bar{z}_{j_q}$, then

$$\begin{aligned}
[\Lambda, I]v &= (*^{-1} \circ L \circ * \circ I - I \circ *^{-1} \circ L \circ *)v \\
&= i^{p-q} *^{-1} \circ L(*v) - I \circ *^{-1} \circ L(*v) \\
&= i^{p-q} *^{-1} \omega \wedge (*v) - I *^{-1} \omega \wedge (*v) \\
&= i^{p-q} *^{-1} \omega \wedge (*v) - i^{(n-(n-p+1))-(n-(n-q+1))} *^{-1} \omega \wedge (*v) \\
&= 0.
\end{aligned}$$

Therefore, by linearity, $[\Lambda, I] = 0$.

1.2.4 The product of two primitive forms is not necessarily primitive. For choose a basis $z_1, \bar{z}_1, ..., z_n, \bar{z}_n$ for V as above. Then z_1 and $\bar{z}_1$ are both primitive. For $*z_1$ has degree $n-1$, so $L(*z_1)$ has degree $n+1$ and is therefore zero, which implies that $\Lambda(z_1) = *^{-1} \circ L \circ *(z_1) = 0$. A similar argument shows that $\Lambda(\bar{z}_1) = 0$.

On the other hand, $*(z_1 \wedge \bar{z}_1) = z_2 \wedge \bar{z}_2 \wedge ... \wedge z_n \wedge \bar{z}_n$, so

$$*(z_1 \wedge \bar{z}_1) = \frac{i}{2}(\sum_{i=1}^{n} z_1 \wedge \bar{z}_1) \wedge z_2 \wedge \bar{z}_2 \wedge ... \wedge z_n \wedge \bar{z}_n = \frac{i}{2} z_1 \wedge \bar{z}_1 \wedge ... \wedge z_n \wedge \bar{z}_n.$$

Therefore $*^{-1} \circ L \circ *(z_1 \wedge \bar{z}_1) \neq 0$, so $z_1 \wedge \bar{z}_1$ is not primitive.

1.2.5 We show that $\Omega = \omega_J + i\omega_K$ is a $(2,0)$ form by showing that if $\bar{z} \in V^{0,1}$, then $\Omega(\bar{z}, w)$ vanishes for all $w \in V_{\mathbb{C}}$. For

$$\omega_J(\bar{z}, w) + i\omega_K(\bar{z}, w) = \langle J\bar{z}, w \rangle + i \langle K\bar{z}, w \rangle = \langle J\bar{z}, w \rangle - i \langle JI\bar{z}, w \rangle = \langle J\bar{z}, w \rangle - i(-i) \langle J\bar{z}, w \rangle = 0.$$

1.2.7 Let x_i, y_i be a symplectic basis for V^*, so $y_i = Jx_i$. Let $a_{ij} = x^i \wedge x^j$, $b_{ij} = y^i \wedge y^j$, and $c_{ij} = x^i \wedge y^j$. Then a_{ij}, b_{ij} for $1 \leq i < j \leq n$ and a_{ij} for $1 \leq i, j \leq n$ form a basis for V^*. Let

$$T(\alpha, \beta) = \frac{\alpha \wedge \beta \wedge \omega^{n-2}}{\mathrm{Vol}}.$$

Then we can compute

$$T(a_{ij}, a_{kl}) = T(b_{ij}, b_{kl}) = T(a_{ij}, c_{kl}) = T(b_{ij}, c_{kl}) = 0,$$

$$T(a_{ij}, b_{kl}) = (n-2)!\, \delta_{ik}\delta_{jl},$$

$$T(c_{ij}, c_{kl}) = \begin{cases} (n-2)! & \text{if } i = l \text{ and } j = k \text{ or } i = j \neq k = l \\ 0 & \text{otherwise} \end{cases}$$

Therefore a matrix representation of T is block-diagonal with

1. $\binom{n}{2}$ blocks of the form $\begin{pmatrix} 0 & (n-2)! \\ (n-2)! & 0 \end{pmatrix}$ corresponding to c_{ij}, c_{ji} for $i \neq j$. Each block has signature $(1, 1)$.

2. one $n \times n$ block of the form $J - I$, where J is a matrix of all ones, corresponding to c_{ii}. This block has signature $(1, n-1)$.

3. n^2 blocks of the form $\begin{pmatrix} 0 & (n-2)! \\ (n-2)! & 0 \end{pmatrix}$ corresponding to the a_{ij}, b_{ij} for all i, j. Each block has signature $(1, 1)$.

Since these blocks account for $3n^2$ basis vectors, there are also n^2 zeros.

1.2.8 The formula is checked by induction on r. For $r = 0$, the result is trivial, since α is primitive ($\Lambda\alpha = 0$). For the inductive step

$$\begin{aligned}
\Lambda^s L^r \alpha &= \Lambda^{s-1}\Lambda L^r \alpha \\
&= \Lambda^{s-1} r(k-n+r-1)L^{r-1}\alpha \\
&= (r(k-n+r-1)) \\
&\quad \times (r(r-1)...((r-1)-(s-1)+1)(n-k-(r-1)+1)...(n-k-(r-1)+(s-1)))L^{(r-1)-(s-1)}\alpha \\
&= r(r-1)....(r-s+1)(n-k-r+1)...(n-k-r+s)L^{r-s}\alpha.
\end{aligned}$$

We have used Corollary 1.2.28 and the fact that $\Lambda\alpha = 0$.

1.2.10 With respect to the dual basis $x_i, ..., x_n, y_i, ..., y_n$, ω takes the coordinate form

$$\omega = \sum_{i=1}^{n} x_i \wedge y_i.$$

This is because the wedge products of pairs of these basis vectors form an orthonormal basis of $\Lambda^2 V^*$, and

$$\omega(x_i, Ix_j) = g(x_i, x_j) = \delta_{ij}, \quad \omega(x_i, x_j) = g(Ix_i, x_j) = 0, \quad \omega(y_i, y_j) = g(Ix_i, x_j) = 0.$$

Write

$$\alpha = \sum_{1 \leq i < j \leq n} a_{ij} x^i \wedge x^j + b_{ij} y^i \wedge y^j + \sum_{i,j=1}^{n} c_{ij} x^i \wedge y^j.$$

Then

$$\Lambda\alpha = \langle \Lambda\alpha, 1 \rangle = \langle \alpha, \omega \rangle = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \alpha(x_i, y_i),$$

as desired.