January 2019

Chapter 2.6: Differential Calculus on Complex Manifolds

Huybrechts, *Complex Geometry*

2.6.1 Let (M, I) be an almost-complex manifold and say (U, ϕ) and (V, ψ) are two systems of charts (complex structures) on M inducing the same almost-complex structure. Take charts ϕ and ψ trivializing overlapping neighborhood of M. I claim that $\phi \circ \psi^{-1}$ is holomorphic. This is equivalent to the statement that $D(\phi \circ \psi^{-1})$ is $\mathbb{C}$-linear. But for any tangent vector v,

$$D(\phi \circ \psi^{-1})Iv = D\phi \circ D\psi^{-1}Iv = D\phi(ID\psi^{-1}v) = ID\phi(D\psi^{-1}v) = ID(\phi \circ \psi^{-1})v.$$

This is because both coordinate charts are holomorphic with respect to I.

2.6.2 Let S be a surface and $\langle \cdot, \cdot \rangle$ the metric on S. Let (U_i, ϕ_i) be a system of oriented trivializations. As described in Example 1.2.12, there is a unique almost complex structure on each $\phi_i(U_i)$ such that $\langle v, Iv \rangle = 0$, $\|I(v)\| = \|v\|$, and $v, I(v)$ is positively oriented. Since each of these three conditions is preserved by oriented isometries, these local choices of I glue to a unique almost complex structure on S. But I must be integrable, for the complex dimension of the tangent space to S is 1, so $\mathcal{A}^{0,2}(X) = 0$ (Prop.2.6.15). Therefore it follows from Newlander-Neirenberg that the resulting almost complex structure is complex.

2.6.3 The natural complex structure on $\mathbb{P}^1$ is the one given in each of the standard coordinate charts by multiplying by i. Call this J. It can be checked that for the charts $U_i \subseteq \mathbb{P}^1, i = 0, 1$, the flat metric on $\mathbb{C} \simeq U_i$ is conformal to the Fubini-Study metric. Therefore J satisfies $\langle v, Jv \rangle_{FS}$ and $\|Jv\|_{FS} = \|v\|_{FS}$, and $\{v, Jv\}$ is positively oriented. So this is the same complex structure as given in the previous exercise.

2.6.5 The $\bar{\partial}$-Poincare lemma on the star-shaped set $M = \mathbb{C}^n$ implies that $H^{0,1}_{\bar{\partial}}(M) = 0$. Now a hypersurface H defines an element of $H^1(M, \Omega^0)$ as follows: locally, H is the vanishing set of f_α defined on an open set U_α. Given the overlap $U_{\alpha\beta}$ of two such charts, the ratio f_α / f_β defines a Cech 1-cocycle $(U_{\alpha\beta}, g_{\alpha\beta})$. The question is then whether $g_{\alpha\beta}$ is the boundary of $(U_\alpha, h|_{U_\alpha})$ for a global holomorphic function h, that is, whether $(U_{\alpha\beta}, g_{\alpha\beta})$ is zero in $H^1(M, \Omega^0)$. But we have seen that $H^1(M, \Omega^0) \simeq H^{0,1}_{\bar{\partial}}(M) = 0$.

2.6.6 Calculation omitted.

2.6.7 To verify that this definition makes sense, we show that $\operatorname{Im} \partial\bar{\partial}$ is contained in $\operatorname{Ker} d$:

$$d\partial\bar{\partial}f = (\partial + \bar{\partial})\partial\bar{\partial}f = \partial^2\bar{\partial}f - \partial\bar{\partial}^2 f = 0$$

by Corollary 2.6.18. Exercise 1.3.4 shows that for a polydisc B, $\operatorname{Im} \partial\bar{\partial} = \operatorname{Ker} d$, so $H^{p,q}_{\mathrm{BC}}(B) = 0$.

For a complex manifold X, a natural map $H^{p,q}_{\mathrm{BC}}(X) \to H^{p,q}(X)$ is given as follows: represent $[\alpha]_{\mathrm{BC}} \in H^{p,q}_{\mathrm{BC}}(X)$ by a section $\alpha \in \mathcal{A}^{p,q}$. Then $\bar{\partial}\alpha = 0$ because $d\alpha = 0$ and the decomposition $d = \partial + \bar{\partial}$ is direct, so α defines an element $[\alpha] \in H^{p,q}(X)$. This map is well-defined because if $\alpha = \partial\bar{\partial} f = -\bar{\partial}\partial f \in \mathrm{Im}\ \bar{\partial}$, then $[\alpha] = 0 \in H^{p,q}(X)$.

There is also a natural map $H^{p,q}_{\mathrm{BC}}(X) \to H^{p+q}(X, \mathbb{C})$. As in the previous paragraph, the map is to send a form to "itself", and then to observe that Im $\partial\bar{\partial} f \subseteq$ Im d, since

$$d(\bar{\partial} f) = (\partial + \bar{\partial})\bar{\partial} f = \partial\bar{\partial} f + \bar{\partial}^2 f = \partial\bar{\partial} f.$$

2.6.8 Let f be the map $(M, I) \mapsto (M, -I)$ sending $[z_0 : z_1 : z_2 : z_3] \to [\bar{z_0} : \bar{z_1} : \bar{z_2} : \bar{z_3}]$. It is clear that f is a diffeomorphism. I claim that f is holomorphic. This is equivalent to the statement that Df is $\mathbb{C}$-linear.

Now
$$Df = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

I don't see how this argument uses this particular variety.