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Optimization of MIMO Channel

Subtitle of Master Thesis

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Abstract

Here comes the abstract...

Preface

This Master Thesis is part of the graduate study at the Department of Information Technology and Electrical Engineering (D-ITET) at the Swiss Federal Institute of Technology (ETH) Zurich.

The author certifies that this Master Thesis, and the research to which it refers, are the product of the author's own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referring practices of the discipline.

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Chapter 1

Information Theory

1.1 general Channel

//by book of Tse
We use $x(t)$ as the transmitted signal and $y(t)$ as the received signal. Since the signal is affected by different objects in its path the received signal can be written as a function of the resulting different paths

$$y(t) = \sum_i a_i(f,t)x(t-\tau_i(t)). \tag{1.1}$$

$a_i(f,t)$ describes the attenuation of each path which is mostly dependent on frequency and time. we can omit these dependencies if we can assume sufficient narrow bandwidth for a frequency flat channel and with relatively stationary objects a time flat channel, objects are transmitters, receivers, scatterers and so on. $\tau_i(t)$ is the delay in time of the path and is constant in a time flat channel.
To distinguish between these different channels we introduce some measures:

Delay Spread T_d describes the time difference from the first to the last arrival of the signal at the receiver.

Doppler Spread D_s describes the maximum frequency difference which is introduced by Doppler shifts.

coherence Bandwidth W_c is the inverse of the delay spread and a measure of the frequency flatness of the channel. It is the bandwidth at which we can assume that the channel is frequency independent.

coherence Time T_c is the inverse of the Doppler spread and a measure for the time flatness of the channel e.g. it is the time over which the channel is time invariant.

// image

1.1.1 The equivalent baseband representation

If we have a frequency-flat and time-flat channel we can represent our channel with a time varying channel gain h which leads to

$$y(t) = h \cdot x(t) \tag{1.2}$$

for our input output relation. But it is often simpler to use a complex baseband model. if we sample the system at a sufficiently high rate we get

$$y[m] = \sum_{l} h_l \cdot x[m] \tag{1.3}$$

where m describes discrete time $x[m]$ and $y[m]$ the transmitted respectively the received signal.

1.1.2 Channel Capacity

the channel capacity is a good figure of merit to describe the optimal performance of a communication system. It gives a maximum data rate at which we could transmit error-free when using appropriate coding.

AWGN-channel: in an AWGN channel we experience no fading and the received signal consists just the sent signal with additive white Gaussian noise $y = x + n$. The capacity depends only on the transmit P versus noise Power N_0:

$$C = log_2\left(1 + \frac{P}{N_0}\right) \text{ bits/s/Hz} \tag{1.4}$$

time- and frequency-flat fading channel: in a one tab fading channel the capacity is a function of the tab gain h:

$$C = log_2\left(1 + |h|^2 \frac{P}{N_0}\right) \text{ bits/s/Hz} \tag{1.5}$$

1.2 MIMO

IN the thrive for higher data rates the limits of the basic single Antenna communication channels became obvious. The bandwidth limitation due to scarcity and cost and power constraints due to physics and regulations called for different approaches. MIMO uses the spatial dimension of the channel to further increase reliability and speed.

1.2.1 from SISO to MIMO

We are still using the assumptions of frequency- and time-flat channels. In the SISO case the discrete complex baseband channel tab h characterized the channel sufficiently. If we now use multiple antennas an the transmitter and receiver side of the channel and make sure that they are sufficiently spaced it establishes multiple distinct channels due to small-scale fading processes. In this case our discrete complex baseband channel tab h is no longer a single coefficient but a matrix $\mathbf{H} \in \mathbb{C}^{N \times M}$ with M number of transmit antennas and N number of receive antennas.

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1M} \\ h_{21} & h_{22} & \dots & h_{2M} \\ \vdots & \vdots & \dots & \vdots \\ h_{N1} & h_{N2} & \dots & h_{NM} \end{bmatrix} \tag{1.6}$$

//figure
the input output relation of a MIMO channel is then written as

$$\mathbf{y}[m] = \mathbf{H}\mathbf{x}[m] + \mathbf{w}[m] \tag{1.7}$$

where $\mathbf{x} \in \mathbb{C}^M, \mathbf{y} \in \mathbb{C}^N$ and $\mathbf{w} \sim \mathcal{CN}(0, N_0\mathbf{I}_N)$

1.2.2 MIMO Capacity

The capacity of a MIMO channel without any precodeing can be written as:

$$C = \log_2 \det \left[\mathbf{I}_N + \frac{\text{SNR}}{M}\mathbf{H}\mathbf{H}^H\right] \tag{1.8}$$

with SNR as total transmit power P over noise power N_0.

1.3 Singular Value Decomposition

If we are in single user MIMO system and we can assume CSIT/R (Channel State Information at Transmitter and Receiver) we can fully diagonalize the channel by using the SVD of $\mathbf{H}$:

$$\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}^H \tag{1.9}$$

with $\mathbf{U}$ and $\mathbf{V}$ unitary and square and $\mathbf{S}$ a diagonal matrix consisting of all singular Values of $\mathbf{H}$.
With this knowledge we can multiply our transmit and received vector as follows: $\tilde{\mathbf{y}} = \mathbf{U}^H\mathbf{y}$ and $\tilde{\mathbf{x}} = \mathbf{V}\mathbf{x}$ and our system equation changes as follows:

$$\begin{aligned}
\tilde{\mathbf{y}} &= \mathbf{U}^H\mathbf{y} && (1.10)\\
&= \mathbf{U}^H(\mathbf{H}\tilde{\mathbf{x}} + \mathbf{w}) && (1.11)\\
&= \mathbf{U}^H\mathbf{U}\mathbf{S}\mathbf{V}^H\tilde{\mathbf{x}} + \tilde{\mathbf{w}} && (1.12)\\
&= \mathbf{S}\mathbf{V}^H\mathbf{V}\mathbf{x} + \tilde{\mathbf{w}} && (1.13)\\
&= \mathbf{S}\mathbf{x} + \tilde{\mathbf{w}}. && (1.14)
\end{aligned}$$

Now we have created min(N,M) parallel SISO systems out of our MIMO system. this brings us to the question of how to distribute the transmit power optimally over these systems to achieve the maximal possible ergodic rate.

1.4 the Receiver

In this project we were mainly looking into optimization on the transmitter side of the channel but the receiver structure has a huge impact at the performance of the system too. In this section we will use two different receiver structures:

1. Linear MMSE Receiver
   Linear receiver with the lowest possible error.

2. V-BLAST MMSE or SIC with MMSE (MMSE-SIC)
   Generally speaking the SIC receivers provide better Throughput but are more complicated.

1.4.1 Linear MMSE

The goal of the MMSE equalizer is to minimize the mean error at the filter output. We use $\mathbf{K}_y = \mathbf{H}\mathbf{K}_x\mathbf{H}^H + \mathbf{K}_n$ and equal power allocation at the transmitter $\mathbf{K}_x = (P/M)\mathbf{I}_N$. Then we can write the error covariance matrix as

$$\mathbf{K}_e = \mathbf{G}\mathbf{K}_y\mathbf{G}^H - \mathbf{G}\mathbf{H} - \mathbf{H}^H\mathbf{G}^H + \mathbf{I}_N \tag{1.15}$$

$$= \left(\mathbf{G}\mathbf{K}_y - \mathbf{H}^H\right)\mathbf{K}_y^{-1}\left(\mathbf{G}\mathbf{K}_y - \mathbf{H}^H\right)^H - \mathbf{H}^H\mathbf{K}_y^{-1}\mathbf{H} + \mathbf{I}_N \tag{1.16}$$

and since $\mathbf{K}_y$ is a non-negative definite matrix we need to minimize $\mathbf{G}\mathbf{K}_y - \mathbf{H}^H = \mathbf{0}$ with $\mathbf{K}_n = \sigma_n^2\mathbf{I}_N$. This leads us to the filter matrix $\mathbf{G}$ as

$$\mathbf{G} = \mathbf{H}^H\mathbf{K}_y^{-1} \tag{1.17}$$

$$= \mathbf{H}^H\left(\mathbf{H}\mathbf{H}^H + \sigma_n^2\mathbf{I}_N\right)^{-1} \tag{1.18}$$

$$= \left(\mathbf{H}\mathbf{H}^H + \sigma_n^2\mathbf{I}_N\right)^{-1}\mathbf{H}^H. \tag{1.19}$$

where the last equation uses the matrix inversion lemma.
Now we use $\tilde{\mathbf{x}} = \mathbf{G}\mathbf{y}$ to estimate the sent vector $\mathbf{x}$. This gives us

$$\mathbf{K}_{e,MMSE} = \mathbf{I}_N - \mathbf{H}^H\mathbf{K}_y^{-1}\mathbf{H} = \sigma_n^2\left(\mathbf{H}^H\mathbf{H} + \sigma_n^2\mathbf{I}_N\right)^{-1} \tag{1.20}$$

for the error covariance matrix and a channel capacity of

$$C = -\sum_{i=1}^{N}\log_2\left(\left[\mathbf{K}_{e,MMSE}\right]_{i,i}^{-1}\right). \tag{1.21}$$

The corresponding per-user SINR in dB is then calculated with

$$\mathrm{SINR}_i = \frac{1}{\left[\mathbf{K}_{e,MMSE}\right]_{i,i}^{-1}} - 1. \tag{1.22}$$

1.4.2 MMSE-SIC

We can use the result of one decorrelator filter to aid the others which results in a successive cancellation strategy: we decode $x_1[m]$ first and then subtract the decoded stream from the received vector. So the second decoder sees only the interference of the streams $x_3, \ldots, x_m$.But be aware of error propagation, this means that an error in decoder 2 leads to erroneous results in all subsequent decoders.
The V-BLAST gives us

$$R < \log_2\det\left(\mathbf{I}_N + \frac{1}{N_0}\mathbf{H}\mathbf{K}_{\mathbf{x}}\mathbf{H}^H\right)\ \text{bits/s/Hz} \tag{1.23}$$

with $\mathbf{K_x} = \operatorname{diag}\{P_1, \dots, P_M\}$ as the covariance matrix of the transmitted signal. Which leads to a capacity of

$$C = \max_{\operatorname{Tr}|\mathbf{K_x}| \le P} \log_2 \det\left(\mathbf{I}_N + \frac{1}{N_0}\mathbf{H}\mathbf{K_x}\mathbf{H}^H\right) \text{ bits/s/Hz} \tag{1.24}$$

which is the capacity of the fast fading MIMO channel.
In the case of CSIT we will discuss the optimal solution as the waterfilling solution later. But with no CSI on the transmitter side //tse proves that an equal Power allocation, in other words equal power on each transmit antenna, gives us the best performance

$$\mathbf{K_x} = \left(\frac{P}{M}\right)\mathbf{I}_N \tag{1.25}$$

which gives us a resulting capacity of

$$C = \log_2 \det\left(\mathbf{I}_N + \frac{\mathrm{SNR}}{M}\mathbf{H}\mathbf{H}^H\right) \text{ bits/s/Hz} \tag{1.26}$$

where $\mathrm{SNR} = P/N_0$ the transmit power over the noise power at the receiver. We see a power loss in comparison to full CSI of M which is achieved by using transmit beam-forming.
In general the decoding order of a this SIC receivers has no impact on the performance, but in our work we always decode the channel which provides the highest MMSE-SINR first. With this decoding order we can guarantee a solution for the later discussed max-min algorithm.
// picture
// code

Chapter 2

Optimization

The question now is, is it possible to increase the Throughput of this channel with knowledge of the channel Matrix $\mathbf{H}$. and to investigate this we use following Channel Representation and we are fully aware of its implications and imperfections.
// description of our channel

2.1 Waterfilling

If full CSI and full inter user coordination e.g. power and data can be freely distributed over all transmit antennas there is a easy way to calculate optimal solution; water-filling. Water filling is designed to diagonalize the channel with help of the SVD. One huge drawback to it is that we need the antenna coordination on the transmitter side, so it is only hardly feasible for a multiuser case.
As mentioned water-filling starts with the SVD of the channel $\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}^H$. It then uses $\tilde{\mathbf{x}} = \mathbf{V}\mathbf{x}$ and $\tilde{\mathbf{y}} = \mathbf{U}^H\mathbf{y}$ to rotate the transmit and receive vectors. $\tilde{\mathbf{w}} = \mathbf{U}^H\mathbf{w}$ is the rotate noise vector.

$$\tilde{\mathbf{y}} = \mathbf{U}^H\mathbf{U}\mathbf{S}\mathbf{V}^H\mathbf{V}\mathbf{x} + \mathbf{U}^H\mathbf{w} = \mathbf{S}\mathbf{x} + \tilde{\mathbf{w}}. \tag{2.1}$$

Now we have multiple parallel SISO "eigen-subchannel" systems. The waterfilling now says how to distribute the transmit power P_{tot} over the different subchannels. The idea is to give the "strongest" channel the most power and the "weakest" the least amount.

// picture
The question to answer how to distribute the power to maximize throughput C:

$$\begin{aligned} \underset{\mathbf{P}}{\text{maximize}} \quad & \sum_k \log_2\left(1 + \frac{P_k|\tilde{S}_k|^2}{N_0}\right) \\ \text{subject to} \quad & P_k \geq 0, \forall P_k \in \operatorname{diag}\{\mathbf{P}\}, \\ & \operatorname{trace}(\mathbf{P}) \leq P_{tot}. \end{aligned} \tag{2.2}$$

with noise Power N_0 and $\tilde{h}_k$ the complex channel gain of subchannel k We define $(\cdot)^+ := \max \cdot, 0$ the power allocation

$$P_k^* = \left(\frac{1}{\lambda} - \frac{N_0}{|\tilde{h}_k|^2}\right)^+ \tag{2.3}$$

with the water-level λ chosen such that the power constraint is met

$$\frac{1}{M}\sum_{k=1}^{M}\left(\frac{1}{\lambda}-\frac{N_0}{|\tilde{h}_k|^2}\right)^{+} \tag{2.4}$$

2.2 Multiuser Case

If antenna coordination is not possible for example in a multiuser environment the water filling solution is not reachable. Jindal proposes an algorithm hi is calling iterative waterfilling. The goal of this algorithm is to calculate the effective noise (Gaussian noise plus interference) for each user and then use water filling over these adapted channels. But since changing the power of one user changes the effective noise of all other users its mandatory to iterate until the algorithm converges. His algorithm optimizes with respect to an SIC receiver that achieves the maximum sum-rate. The papers goal is to solve the problem of beam-forming in a broadcast(BC) or downlink channel, so exactly the opposite direction than we are discussing in this report. But as //paper describes there is a duality of these two channels and we are able to use the optimized results of the multiple access channel (MAC) or uplink-channel to optimize the BC. His work is using this duality to solve the easier problem of the MAC and then transform it back to the BC problem, and thus solving the throughput maximization problem for the multiuser case.

Input: $\mathbf{Q^{(0)}} = \mathbf{0}$

repeat

Generate effective channels for each user

$$\mathbf{G}_k^{(n)} = \mathbf{H}_k\left(\mathbf{I}+\sum_{j\neq k}\mathbf{H}_j^H\mathbf{Q}_k^{(n-1)}\mathbf{H}_j\right)^{-1/2};$$

Treating these effective channels as parallel, noniterfering channels and obtain covariance matrices $\mathbf{S}_k^{(n)}$ by waterfilling with total power P_{ot}

$$\begin{aligned} \underset{\mathbf{S_k}}{\text{maximize}} \quad & \sum_k \log_2\left(\mathbf{I}+\left(\mathbf{G}_k^{(n)}\right)^H\mathbf{S}_k\mathbf{G}_k^{(n)}\right) \\ \text{subject to} \quad & \mathbf{S}_k \geq 0, \\ & \sum_k \operatorname{trace}(\mathbf{S}_k) \leq P_{tot} \end{aligned};$$

Compute the updated covariance matrices $\mathbf{Q}_k^{(n)}$ as

$$\mathbf{Q}_k^{(n)} = \frac{1}{M}\mathbf{S}_k^{(n)} + \frac{M-1}{M}\mathbf{Q}_k^{(n-1)};$$

until $\mathbf{Q}_k^{(n)} = \mathbf{Q}_k^{(n-1)}$;

// figure (compare wf to iwf to equal power allocation)

2.3 Optimizing for MMSE with K users

We have a channel with K users which all have one single antenna and one receiver with M antennas. Our system equation then is

$$\mathbf{y}^{(M\times 1)} = \mathbf{H}^{(M\times K)} \cdot \mathbf{x}^{(K\times 1)}. \tag{2.5}$$

As discussed before the receive covariance matrix is

$$\mathbf{K}_e = \left(\sqrt{\mathbf{P}}\mathbf{H}^H \frac{1}{\sigma^2}\mathbf{H}\sqrt{\mathbf{P}} + \mathbf{I}\right)^{-1} \tag{2.6}$$

with $\mathbf{P} = \operatorname{diag}\{P_1, \dots, P_K\}$ the power loading matrix where P_k is the power of user k.
We then calculate the signal to noise and interference ratio (SINR) and the rate R of each user as

$$\mathrm{SINR}_k = \frac{1}{(K_e)_{k,k}} \tag{2.7}$$

$$R_k = \log_2 1 + \mathrm{SINR}_k. \tag{2.8}$$

We further define a fixed power constraint P_{tot} so that $\sum_k P_k \leq P_{tgt}$ and $P_k \leq 0\,, \forall k$.

2.3.1 Gradient search

The idea of the steepest descent algorithm is to find the global maximum or minimum of a function by searching along the gradient of the function.

Input: $\mathbf{X}^{(\mathbf{0})}$, $f(\mathbf{X})$, $\nabla f(\mathbf{X})$, μ(step size)
Output: $\mathbf{X}^{(\mathrm{end})}$
1 **repeat**
2 $\quad \mathbf{X}^{(n+1)} = \mathbf{X}^{(n)} - \mu\nabla f(\mathbf{X}^{(n)})$;
3 **until** $\nabla f(\mathbf{X}^{(n)}) = 0$;

To optimize for the rates of a given channel $\mathbf{H}$ we define

$$f_p(\mathbf{P}) = \left\|(R(\mathbf{P}))\right\|_p = \left(\sum_k (R_k(\mathbf{P}))^p\right)^{1/p}. \tag{2.9}$$

as the p-norm of the user rates. and calculate its gradient $\nabla f_p(P)$ as

$$\nabla f_p(\mathbf{P}) = \begin{bmatrix} \frac{\partial f_p(\mathbf{P})}{\partial P_1} \\ \frac{\partial f_p(\mathbf{P})}{\partial P_2} \\ \vdots \\ \frac{\partial f_p(\mathbf{P})}{\partial P_K}. \end{bmatrix} \tag{2.10}$$

This gives us an iterative search algorithm

$$\mathbf{P}^{n+1} = \mathbf{P}^n + \mu\nabla\left\|(R(\mathbf{P}))\right\|_p \tag{2.11}$$

where $(\cdot)^n$ denotes the n-th iteration of the search algorithm and μ an arbitrary step size. In the following we use $\partial_{X_j}(f_p) = \frac{\partial f_p(\mathbf{X})}{\partial X_j}$ and $R = R(\mathbf{X})$.

We now calculate the gradient of

$$\left\|(R)\right\|_p = \left(\sum_k (R_k)^p \right)^{1/p}. \tag{2.12}$$

as follows

$$\nabla \left\|(R(\mathbf{P}))\right\|_p = \nabla \left[\left(\sum_k (R_k(\mathbf{P}))^p \right)^{1/p} \right] \tag{2.13}$$

$$= \begin{bmatrix} \partial_{p_1} \\ \partial_{p_2} \\ \vdots \\ \partial_{p_M} \end{bmatrix} \cdot \left(\sum_k (R_k(\mathbf{P}))^p \right)^{1/p}. \tag{2.14}$$

Due to the total power constraint P_{tot} and that no user can send with negative power we have a bounded optimization problem. So we need to convert it into an unbounded one:

1. to guarantee $P_k \geq 0 \, \forall k \in \operatorname{diag}\{\mathbf{P}\}$ we can introduce a diagonal matrix $\mathbf{X}$ as $X_k^2 = P_k$ with X_k is the k-th diagonal element of $\mathbf{X}$.

2. to guarantee $\operatorname{trace}(\mathbf{P}) \leq P_{tot}$ we tighten the condition to $\operatorname{trace}(\mathbf{P}) = P_{tot}$ and calculate the rate with:

$$R_k = -\log_2 \left(\mathbf{\Phi}^{-1}_{k,k} \right) \tag{2.15}$$

$$= -\log_2 \left(\left[\sqrt{\mathbf{P}} \mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H} \sqrt{\mathbf{P}} + \mathbf{I_M} \right]^{-1}_{k,k} \right) \tag{2.16}$$

$$= -\log_2 \left(\left[\sqrt{\mathbf{P} \frac{P_{tot}}{\operatorname{trace}(\mathbf{P})}} \mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H} \sqrt{\mathbf{P} \frac{P_{tot}}{\operatorname{trace}(\mathbf{P})}} + \mathbf{I_M} \right]^{-1}_{k,k} \right) \tag{2.17}$$

$$= -\log_2 \left(\left[\mathbf{X} \sqrt{\frac{P_{tot}}{\operatorname{trace}(\mathbf{X}^2)}} \mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H} \mathbf{X} \sqrt{\frac{P_{tot}}{\operatorname{trace}(\mathbf{X}^2)}} + \mathbf{I_M} \right]^{-1}_{k,k} \right) \tag{2.18}$$

$$= -\log_2 \left(\left[\frac{P_{tot}}{\operatorname{trace}(\mathbf{X}^2)} \mathbf{X} \mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H} \mathbf{X} + \mathbf{I_M} \right]^{-1}_{k,k} \right) \tag{2.19}$$

We now can calculate the partial derivatives of the Rates:

$$\partial_{x_j} \left\|(R(\mathbf{P}))\right\|_p = \partial_{x_j} \left[\left(\sum_k (R_k(\mathbf{P}))^p \right)^{1/p} \right] \tag{2.20}$$

$$= \frac{1}{p} \left(\sum_k (R_k(\mathbf{P}))^p \right)^{1/p-1} \cdot \sum_k \left[p \cdot (R_k(\mathbf{P}))^{p-1} \cdot \partial_{x_j} (R_k(\mathbf{P})) \right]. \tag{2.21}$$

with the partial derivative of the k-th users rate

$$\partial_{x_j}(R_k) = \partial_{x_j}\left(-\log_2 \mathbf{\Phi}^{-1}_{k.k}\right) \tag{2.22}$$

$$= \frac{-1}{\log 2}\frac{1}{\mathbf{\Phi}^{-1}_{k,k}} \cdot \partial_{x_j}\left(\mathbf{\Phi}^{-1}_{k,k}\right). \tag{2.23}$$

We use

$$\partial_{x_j}\left(\mathbf{\Phi}^{-1}_{k,k}\right) = \left(-\mathbf{\Phi}^{-1} \cdot \partial_{x_j}\left(\mathbf{\Phi}^{-1}\right) \cdot \mathbf{\Phi}^{-1}\right)_{k,k}. \tag{2.24}$$

to calculate the partial derivative of an inverse matrix, such that the derivative of the of the inverse matrix $\mathbf{\Phi}$ becomes

$$\partial_{x_j}\left(\mathbf{\Phi}^{-1}\right) = \partial_{x_j}\left(\frac{P_{tot}}{\text{trace}\left(\mathbf{X}^2\right)}\mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X} + \mathbf{I}_\mathbf{M}\right) \tag{2.25}$$

$$= \partial_{x_j}\left(\frac{P_{tot}}{\text{trace}\left(\mathbf{X}^2\right)}\mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X}\right) \tag{2.26}$$

$$= P_{tot} \cdot \partial_{x_j}\left(\frac{\mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X}}{\text{trace}\left(\mathbf{X}^2\right)}\right) \tag{2.27}$$

$$= P_{tot} \cdot \left(\frac{\partial_{x_j}\left(\mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X}\right)}{\text{trace}\left(\mathbf{X}^2\right)} - \frac{\mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X} \cdot \partial_{x_j}\left(\text{trace}\left(\mathbf{X}^2\right)\right)}{\left(\text{trace}\left(\mathbf{X}^2\right)\right)^2}\right). \tag{2.28}$$

The partial derivative of a matrix $\frac{\partial(\mathbf{X})}{\partial X_{i,j}} = E_{i,j}$ with $E_{i,j}$ the single entry matrix, 1 at (i,j) and zero elsewhere. Ant thus

$$\partial_{x_j}\left(\mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X}\right) = \mathbf{E}_{j,j}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X} + \mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{E}_{j,j} \tag{2.29}$$

$$\partial_{x_j}\left(\text{trace}\left(\mathbf{X}^2\right)\right) = 2\mathbf{X}\mathbf{E}_{j,j}. \tag{2.30}$$

Finally we combine it all together for the j-th partial derivative of the p-norm of the user rates:

$$\partial_{x_j}\left\|(R)\right\|_p = \left(\sum_k (R_k)^p\right)^{1/p-1} \cdot \sum_k \left[\frac{1}{\log 2}\frac{(R_k)^{p-1}}{\mathbf{\Phi}^{-1}_{k,k}} \cdot \left(\mathbf{\Phi}^{-1} \cdot P_{tot} \cdot \left(\frac{\mathbf{E}_{j,j}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X} + \mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{E}_{j,j}}{\text{trace}\left(\mathbf{X}^2\right)} - \frac{\mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X} \cdot 2\mathbf{X}\mathbf{E}_{j,j}}{\left(\text{trace}\left(\mathbf{X}^2\right)\right)^2}\right) \cdot \mathbf{\Phi}^{-1}\right)_{k,k}\right] \tag{2.31}$$

2.3.2 maximum throughput

We want to maximize the sum-rate for a given channel $\mathbf{H}$ and a fixed power constraint P_{tot}:

$$\begin{aligned} \underset{\mathbf{P}}{\text{maximize}} \quad & \sum_k R_k(\mathbf{P}) \\ \text{subject to} \quad & P_k \geq 0, \forall P_k \in \text{diag}\left\{\mathbf{P}\right\}, \\ & \text{trace}\left(\mathbf{P}\right) \leq P. \end{aligned} \tag{2.32}$$

this optimizes the diagonal precoding matrix $\mathbf{P}$ with $\mathbf{P} = \text{diag}\{P_1, \ldots, P_K\}, \forall P_k \geq 0$. Using the previously calculated gradient of $\left\|(R(\mathbf{P}))\right\|_p$ with $p = 1$ for $\left\|(R(\mathbf{P}))\right\|_1 =$

As sated before the p-norm gives us the possibility to use the same gradient for different applications; setting $p = 1$ gives us $\left\|(R(\mathbf{P}))\right\|_1 = \sum_k R_k$ to optimize for the sum-rate:

$$\partial_{x_j}\Big(\sum_k R_k\Big) = \sum_k \Bigg[\frac{1}{\log 2}\frac{1}{\mathbf{\Phi}^{-1}_{k,k}} \cdot \Bigg(\mathbf{\Phi}^{-1} \cdot P_{tot} \cdot \left(\frac{\mathbf{E}_{j,j}\mathbf{H}^H \frac{1}{\sigma^2}\mathbf{H}\mathbf{X} + \mathbf{X}\mathbf{H}^H \frac{1}{\sigma^2}\mathbf{H}\mathbf{E}_{j,j}}{\text{trace}\left(\mathbf{X}^2\right)} - \frac{\mathbf{X}\mathbf{H}^H \frac{1}{\sigma^2}\mathbf{H}\mathbf{X} \cdot 2\mathbf{X}\mathbf{E}_{j,j}}{\left(\text{trace}\left(\mathbf{X}^2\right)\right)^2}\right) \cdot \mathbf{\Phi}^{-1}\Bigg)_{k,k}\Bigg] \tag{2.33}$$

2.3.3 maximum min-rate

To maximize the min-rate we would use $p = -\infty$ for the$\|\cdot\|_{min}$ if there is no per user power constraint this gives us equal throughput for each user. But the min-norm is non differentiable and thus unusable for our gradient search algorithm. In search of a feasible solution we use the -30-norm, this gives us a good trade-of between stability and preciseness of the results. a p-Value of -100 could work to but the risk of getting non differentiable is too high. We thus optimize

$$\begin{aligned} \underset{\mathbf{P}}{\text{maximize}} \quad & \left\|R(\mathbf{P})\right\|_{-30} \\ \text{subject to} \quad & P_k \geq 0, \forall P_k \in \text{diag}\{\mathbf{P}\}, \\ & \text{trace}(\mathbf{P}) \leq P. \end{aligned} \tag{2.34}$$

which is the maxmin algorithm.

2.3.4 optimizing the gradient algorithm

One problem that occurred during the simulations was that at some point the gradient search got unstable, in our case it started to fluctuate and stopped moving towards the optimum. The reason of this bad and unwanted behavior lies in the step size of the function, and so there are two ways to fix it. The first and easy way is just to reduce the step size and increase the number of iterations accordingly, this way the problem occurs later and weaker and has less impact on the end result. The second much more complicated solution is to introduce adaptive step size.

Adaptive step size means, that for each iteration we search the optimal step size which allows us to take the biggest step towards our goal. So it should not be possible to choose a step size which gives us a worse result than we had on the iteration before. With this Algorithm we can prevent any fluctuation in the gradient and force it to go the fastest possible way. We do not have to calculate the gradient as often and can thus reduce the needed calculation time. Another nice side effect is, that we implicitly know that we reached the maximum or minimum when the step size is decreased to zero without finding any better points than the previous one.

2.4 optimizing VBlast

In this section we again try to optimize for the user rates of a given channel $\mathbf{H}$ with noise power σ^2. We use a system with K single-antenna users and one receiver with N antennas.

Input: $\mathbf{X}^{(0)}$, $f(\mathbf{X}^{(0)})$, $\nabla f(\mathbf{X}^{(0)})$, $\mu^{(0)}$, τ (step size multiplicator)

if $f(\mathbf{X}^{(n)}) \leq f(\mathbf{X}^{(n+1)})$ **then**
    repeat
        $\mu^{(n)} = \mu^{(n-1)} \cdot \tau$;
        $\mathbf{X}^{(n)} = \mathbf{X}^{(0)} - \mu^{(n)} \nabla f(\mathbf{X}^{(0)})$;
        calculate: $f(\mathbf{X}^{(n)})$;
    until $f(\mathbf{X}^{(n)}) \leq f(\mathbf{X}^{(n-1)})$;
    return $\mathbf{X}^{(n-1)}$
else
    repeat
        $\mu^{(n)} = \mu^{(n-1)} \cdot 1/\tau$;
        $\mathbf{X}^{(n)} = \mathbf{X}^{(0)} - \mu^{(n)} \nabla f(\mathbf{X}^{(0)})$;
        calculate: $f(\mathbf{X}^{(n)})$;
    until $f(\mathbf{X}^{(n)}) > f(\mathbf{X}^{(n-1)})$;
    return $\mathbf{X}^{(n)}$
end

This again gives us

$$\mathbf{y}^{(M\times 1)} = \mathbf{H}^{(M\times K)} \cdot \mathbf{x}^{(K\times 1)}. \tag{2.35}$$

as the system equation we use $\mathbf{P} = \operatorname{diag}\{P_1, \ldots, P_K\}$ as the diagonal power allocation matrix and P_{tot} the total power constraint, there will not be any per user power constraint. With a VBlast receiver we can calculate the SINR and rates per user as follows //formula and the sum-rate $\sum_k R_k$ becomes

$$\sum_k R_k = \log_2 \det\left(\sqrt{\mathbf{P}}\mathbf{H}^H \frac{1}{\sigma^2}\mathbf{H}\sqrt{\mathbf{P}} + \mathbf{I}\right). \tag{2.36}$$

2.4.1 maximizing the throughput of the system

We try to optimize

$$\begin{aligned} \underset{\mathbf{P}}{\text{maximize}} \quad & \sum_k R_k(\mathbf{P}) \\ \text{subject to} \quad & P_k \geq 0, \forall P_k \in \operatorname{diag}\{\mathbf{P}\}, \\ & \operatorname{trace}(\mathbf{P}) \leq P. \end{aligned} \tag{2.37}$$

One solution of this problem can be found by using the algorithm proposed by Jindal. The scenario Jindal uses in his paper is an extension of ours, he uses K users with M antennas and one receiver with N antennas, this gives him K $N \times M$ channel matrices. So to use his algorithm we only have to transform our $N \times K$ channel matrix into K $N \times K$ matrices and our channel becomes

$$\mathbf{y}_k = \mathbf{H}_k^{(N\times 1)} \cdot x_k \tag{2.38}$$

with $\mathbf{y} = \sum_k \mathbf{y}_k$ the received vector.

Another solution can be found by using a gradient search similar to the one for the LMMSE receiver. Only here we will use a numerical gradient which calculates the gradient numerically by increasing the input by a small value η for each dimension one after another. With a small enough value for η we will get a very good approximation for the gradient since

$$\frac{df(x)}{dx} = \lim_{\eta \to 0} \frac{f(x+\eta) - f(x)}{(x+\eta) - x}. \tag{2.39}$$

We will again use the norm instead of the sum to calculate the sum rate so we can use the

Input: $\mathbf{X}^{(\mathbf{0})}$, $f(\mathbf{X})$, η (interval), μ (step size)
Output: $\mathbf{X}^{(\mathrm{end})}$
repeat
  for $k = 1$ *to* M **do**
    $\mathbf{X}_e = \mathbf{X} + \eta \mathbf{E}_{k,k}$;
    $\partial_{X_k}(f(\mathbf{X}^{(n)}) = \frac{f(\mathbf{X}^{(n)} - f(\mathbf{X}_e)}{\eta}$;
  end
  $$\mathbf{X}^{(n+1)} = \mathbf{X}^{(n)} - \mu \begin{bmatrix} \partial_{X_1}(f(\mathbf{X}^{(n)})) \\ \partial_{X_2}(f(\mathbf{X}^{(n)})) \\ \vdots \\ \partial_{X_M}(f(\mathbf{X}^{(n)})) \end{bmatrix};$$
until $\nabla f(\mathbf{X}^{(n)}) = 0$;

code for the throughput maximizer with $p = 1$ and $p = -30$ gives us the maxmin optimizer. As shown in our results the numerical gradient yields the same result as Jindal's iterative water-filling algorithm.

2.5 minimize the total power for per user SINR targets

The idea is to fix SINR targets for each user SINR_{tgt_k} and then find the minimum required sum power $P_{tot} = \sum_k P_k$ to reach these targets. We no longer care about an upper bound for P_{tot} and as always will not introduce a per user power constraint.

$$\begin{aligned} \underset{\mathbf{P}}{\text{minimize}} \quad & \sum_k P_k \\ \text{subject to} \quad & \mathrm{SINR}_k = \mathrm{SINR}_{tgt_k},\ \forall k \in K, \\ & P_k \geq 0,\ \forall P_k \in \operatorname{diag}\{\mathbf{P}\}. \end{aligned} \tag{2.40}$$

The channel will again be

$$\mathbf{y}^{(M \times 1)} = \mathbf{H}^{(M \times K)} \cdot \mathbf{x}^{(K \times 1)}. \tag{2.41}$$

and we will only use the LMMSE receiver.

2.5.1 Fodor

Fodor extends the work of //paper which offer a closed form solution for the above optimization problem but only works for feasible SINR targets. He proposes following algorithm: In our single antenna per user case the precoding matrix $\mathbf{T}_k \in \mathbb{C}^{(N \times N)}$ with $\sum_{s=1}^{N} |\mathbf{T}_k^{(s,s)}|^2 = N$

Input: $t = 0$, $\epsilon^{(0)} = 1$, SINR targets $\mathbf{\Gamma} = \operatorname{diag}\left\{\gamma_k^{tgt}\right\}$ and transmission powers $\mathbf{P}^{(0)}$

repeat

calculate the effective interference for MMSE processing:

$$\zeta_{k,s} = \left\{\left(\left(d_{k,k}^{-\rho}\chi_{k,k}\mathbf{H}_{k,k}^{H}\left(\sum_{j\neq k} P_j d_{k,j}^{-\rho}\chi_{k,j}\mathbf{H}_{k,j}\mathbf{T}_j\mathbf{T}_j^{H}\mathbf{H}_{k,j}^{H} + N\sigma_n^2\mathbf{I}\right)^{-1}\mathbf{H}_{k,k} + \frac{1}{P_k}\mathbf{I}\right)^{-1}\right\}^{(s,s)};$$

calculate the optimal loading matrix

$$(\mathbf{T}_k)^{(s,s)} = \sqrt{\frac{\zeta_{k,s}N}{\sum_{j=1}^{N}\zeta_{k,j}}}\ \forall s \in [1, N];$$

calculate used Power

$$P_k = \frac{\zeta_{k,s}}{|(\mathbf{T}_k)^{(s,s)}|^2}(\gamma_{tgt} + 1)\ \forall k;$$

until $\mathbf{P}^{(n)} = \mathbf{P}^{(n-1)}$;

will always be one. $H_{k,k}$ denotes the channel matrix from user k to receiver k in our case we only have one base station and thus $\mathbf{H}_{k,j} \in \mathbb{C}^{(M\times 1)}$ with $j = 1$. In addition we set both the distance $d_{k,k}^{-\rho} = 1$ and the fading coefficient $\chi_{k,k} = 1$ since we ignore path losses.

2.5.2 analytical gradient

Again we like to compare the results with a analytical gradient search algorithm. So let us first calculate the gradient for the power minimization. The idea here was to minimize the difference between the target Rates and the calculated rate for the channel, only by changing the diagonal Values of $\mathbf{P}$. So our minimization problem is

$$\begin{aligned} \underset{\mathbf{P}}{\text{minimize}} \quad & \sum_k \left(R_k - R_{tgt}\right) \\ \text{subject to} \quad & P_k \geq 0,\ \forall P_k \in \operatorname{diag}\{\mathbf{P}\}. \end{aligned} \tag{2.42}$$

Again we use $\mathbf{X}^2 = \mathbf{P}$ to make sure to only get positive power Values but since we do not care about some power power budget P_{tot} we can removed the condition $\operatorname{trace}(\mathbf{P}) \leq P_{tot}$. $\mathbf{\Phi}$ then becomes

$$\mathbf{\Phi} = \mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X} + \mathbf{I}_\mathbf{N} \tag{2.43}$$

and the for the Rate of user k

$$R_k = -\log_2\left(\mathbf{\Phi}_{k,k}^{-1}\right) = -\log_2\left(\left[\mathbf{X}\mathbf{H}^H\frac{1}{\sigma^2}\mathbf{H}\mathbf{X} + \mathbf{I}_\mathbf{M}\right]_{k,k}^{-1}\right) \tag{2.44}$$

With $p = 2$ we calculate the distance such that the sign does not mess up the gradient and the norm to be able to reuse the code from the throughput maximization algorithm, we get:

$$\nabla \big\|(R(\mathbf{P}))\big\|_2 = \nabla \left[\left(\sum_k (R_k - R_{tgt})^2 \right)^{1/2} \right] \tag{2.45}$$

and for the partial derivative

$$\begin{aligned} \partial_{x_j} \big\|(R(\mathbf{P}))\big\|_2 &= \partial_{x_j} \left[\left(\sum_k (R_k - R_{tgt})^2 \right)^{1/2} \right] & (2.46) \\ &= \frac{1}{2} \left(\sum_k (R_k - R_{tgt})^2 \right)^{-1/2} \cdot \sum_k \left[2(R_k - R_{tgt}) \cdot \partial_{x_j} (R_k - R_{tgt}) \right]. & (2.47) \end{aligned}$$

where

$$\begin{aligned} \partial_{x_j} (R_k - R_{tgt}) &= \partial_{x_j} (R_k) & (2.48) \\ &= \partial_{x_j} \big(-\log_2 \mathbf{\Phi}^{-1}_{k.k} \big) & (2.49) \\ &= \frac{-1}{\log 2} \frac{1}{\mathbf{\Phi}^{-1}_{k,k}} \cdot \partial_{x_j} \big(\mathbf{\Phi}^{-1}_{k,k} \big) & (2.50) \end{aligned}$$

$$\partial_{x_j} \big(\mathbf{\Phi}^{-1}_{k,k} \big) = \left(-\mathbf{\Phi}^{-1} \cdot \partial_{x_j} \big(\mathbf{\Phi}^{-1} \big) \cdot \mathbf{\Phi}^{-1} \right)_{k,k}. \tag{2.51}$$

and the derivative of $\mathbf{\Phi}$ as

$$\begin{aligned} \partial_{x_j} \big(\mathbf{\Phi}^{-1} \big) &= \partial_{x_j} \left(\mathbf{X}\mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H}\mathbf{X} + \mathbf{I_M} \right) & (2.52) \\ &= \partial_{x_j} \left(\mathbf{X}\mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H}\mathbf{X} \right) & (2.53) \\ &= \mathbf{E}_{j,j} \mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H}\mathbf{X} + \mathbf{X}\mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H}\mathbf{E}_{j,j} & (2.54) \end{aligned}$$

and finally the partial derivative of R_k by the j-th element of $\mathbf{P}$

$$\begin{aligned} \partial_{X_j} (R_k) = &\left(\sum_k (R_k - R_{tgt})^2 \right)^{-1/2} \cdot \sum_k \left[\frac{R_k - R_{tgt}}{\log 2 \cdot \mathbf{\Phi}^{-1}_{k,k}} \cdot \right. \\ &\left. \cdot \left(\mathbf{\Phi}^{-1} \cdot \left(\mathbf{E}_{j,j} \mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H}\mathbf{X} + \mathbf{X}\mathbf{H}^H \frac{1}{\sigma^2} \mathbf{H}\mathbf{E}_{j,j} \right) \cdot \mathbf{\Phi}^{-1} \right)_{k,k} \right]. \end{aligned} \tag{2.55}$$

Chapter 3

results

Bibliography