A model with endogenous vacancies

Finn Schüle and Haoyu Sheng

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1. Introduction

So far we have assumed that there is an existing stock E_i of workers in each sector. This assumption makes sense in the short run if firms are unable fire workers quickly. However, in the medium term this seems unlikely. Most of the production networks literature, by assuming constant returns to scale production technology and flexible prices, seems to be more consistent with tracking medium term outcomes, when most prices have had time to adjust and there are no truly fixed factors of production. In this note I adjust our model of the labor market to be more consistent with medium term outcomes by assuming what we observe at any given point in time is close to a steady state in the matching framework: close to what we have previously seen in Pascal's class.

When working with the model in this formulation, it makes sense to think of data exercises as thinking about the medium run effects of shocks on both output and labor markets. We can start from a snapshot of employment, unemployment, and vacancies. Introduce shocks and see where the model settles. If we choose to introduce a dynamic aspect, we should do this in a way fully consistent with the matching model in any case, rather than just assuming a some exogenous number of existing employees E_i with no description of how this changes over time.

Brown University.

2. Households and final goods production

We consider a closed economy model with no government spending. Real household consumption is real GDP, denoted by Y. A final goods producer with constant returns to scale technology aggregates J sector outputs to produce Y

$$Y = \max_{\{c_i\}_{i=1}^J} \mathcal{D}\left(\{c_i\}_{i=1}^J\right)$$

Subject to the budget constraint

$$\sum_{i=1}^J p_i c_i = \sum_{i=1}^J \left[w_i L_i + \pi_i\right].$$

$\mathcal{D}$ captures household preferences over final consumption goods, w_i is the wage of sector i labor L_i, and π_i is the profit in sector i.

Let f_x denote the first derivative of the function f with respect to the argument x. The first order conditions for final production imply

$$\mathcal{D}_{c_i} = p_i \mu \tag{1}$$

$$\sum_{i=1}^J p_i c_i = \sum_{i=1}^J \left[w_i L_i + \pi_i\right] \tag{2}$$

Where μ is the Lagrangian multiplier on the budget constraint. Alternatively, we can write (1) in terms of the elasticity of final goods production to sector i inputs

$$\varepsilon_{c_i}^{\mathcal{D}} = p_i c_i \frac{\mu}{Y}$$

We can rearrange for c_i, multiply by p_i, and sum over all industries to get

$$\begin{aligned} \frac{\varepsilon_{c_i}^{\mathcal{D}}}{p_i} \frac{Y}{\mu} &= c_i \\ \Rightarrow \varepsilon_{c_i}^{D} \frac{Y}{\mu} &= p_i c_i \\ \Rightarrow \frac{Y}{\mu} \underbrace{\sum_{j=1}^J \varepsilon_{c_j}^{\mathcal{D}}}_{=1 \text{ by crts}} &= \sum_{j=1}^J p_j c_j \end{aligned}$$

Giving,

$$\varepsilon_{c_i}^{\mathcal{D}} = \frac{p_i c_i}{\sum_{j=1}^{J} p_j c_j}$$

Finally, using $d\log f(x, y) = \varepsilon_x^f d\log x + \varepsilon_y^f d\log y$

$$d\log Y = \sum_{i=1}^{N} \varepsilon_{c_i}^{\mathcal{D}} d\log c_i$$

[I DONT THINK THIS IS QUITE RIGHT SINCE THERE IS A MAX SO $Y \neq \mathcal{D}$]

[DOES ENVELOPE CONDITION APPLY TO ANY c_i HERE SINCE THEY ARE ALL IN THE SET OF THINGS BEING CHOSEN AND NOT PARAMETERS]

3. Sector level labor markets

We assume each sector has a separate labor market with a labor force of H_i possible workers, an exogenous separation rate s_i, and an exogenous recruiting cost r_i which measures the in units of labor required to maintain each posted vacancy. When workers and firms meet there is a mutual gain from matching. There is no accepted theory for how wages are set in this context. For now we assume w_i follows a general waging schedule taken as given by both firms and workers. Hires are are generated by a constant returns matching function in unemployment u_i and vacancies v_i

$$h_i = \phi_i m\left(u_i, v_i\right)$$

Let $\theta_i = \frac{v_i}{u_i}$, $\mathcal{Q}_i(\theta_i) = \phi_i m\left(\frac{u_i}{v_i}, 1\right)$, and $\mathcal{F}_i(\theta_i) = \phi_i m\left(1, \frac{v_i}{u_i}\right)$. We assume we start at a steady state featuring balanced flows, that is that the number of workers flowing into unemployment equals the number of workers flowing out of unemployment, labor supply satisfies

$$\begin{aligned}
s_i L_i^s(\theta_i) &= \mathcal{F}_i(\theta_i) u_i \\
&= \mathcal{F}_i(\theta_i)(H_i - L_i^s(\theta_i)) \\
\Rightarrow L_i^s(\theta_i) &= \frac{\mathcal{F}_i(\theta_i)}{s_i + \mathcal{F}_i(\theta_i)} H_i
\end{aligned}$$

Let N_i denote productive employees and R_i denote recruiters employed in sector i. Balanced flows implies the recruiter producer ratio $\tau_i(\theta_i) = \frac{R_i}{N_i}$ satisfies

$$\begin{aligned} s_i(N_i + R_i) &= \mathcal{Q}_i(\theta_i)v_i \\ \Rightarrow r_i s_i(N_i + R_i) &= \mathcal{Q}_i(\theta_i)R_i \\ \Rightarrow r_i s_i \tau_i(\theta_i)^{-1} &= \mathcal{Q}_i(\theta_i) - r_i s_i \\ \Rightarrow \tau_i(\theta_i) &= \frac{r_i s_i}{\mathcal{Q}_i(\theta_i) - r_i s_i} \end{aligned}$$

For a given target level of employment N_i, total required labor is $L_i^d(\theta_i) = \left(1 + \tau_i(\theta_i)\right) N_i$. We describe how labor demand, $L_i^d(\theta_i)$, is determined by firms' profit maximization in the next section.

4. Sector level firms

Firms in sector i use labor N_i and intermediate inputs from sector j, x_{ij}, to produce output y_i using production technology f_i.

$$y_i = A_i f_i\left(N_i, \{x_{ij}\}_{j=1}^J\right)$$

Firms choose N_i and $\{x_{ij}\}_{j=1}^J$ to maximize profits, or equivalently to minimize costs. We assume firms are price takers in both input and output markets. Profits are given by

$$\pi_i = p_i f_i\left(N_i, \{x_{ij}\}_{j=1}^J\right) - w_i(1 + \tau_i(\theta_i))N_i - \sum_{j=1}^J p_j x_{ij}$$

Firms choose inputs to solve

$$\max_{N_i, \left\{x_{ij}\right\}_{j=1}^J} \pi_i\left(N_i, \left\{x_{ij}\right\}_{j=1}^J\right)$$

Giving the first order conditions

(FOC x_{ij}) $$p_i f_{i,x_{ij}} = p_j$$

(FOC N_i) $$p_i f_{i,N_i} = w_i\left(1 + \tau_i(\theta_i)\right)$$

And labor demand is $L_i^d(\theta_i) = (1+\tau_i(\theta_i))N_i$ for the optimal N_i. The equilibrium tightness equates labor demand and labor supply.

As was the case in the final production sector, we can rewrite these expressions in terms of elasticities.

$$\varepsilon^{f_i}_{x_{ij}} = \frac{p_j x_{ij}}{p_i y_i}$$
$$\varepsilon^{f_i}_{N_i} = \left(1+\tau_i(\theta_i)\right)\frac{w_i N_i}{p_i y_i}$$

Using $d\log f(x,y) = \varepsilon^f_x d\log x + \varepsilon^f_y d\log y$, for each sector i

$$d\log y_i = \underbrace{\varepsilon^{f_i}_{A_i}}_{=1} d\log A_i + \varepsilon^{f_i}_{N_i} d\log N_i + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} d\log x_{ij}$$

Plugging in the first order conditions gives

$$\begin{aligned}
d\log y_i &= \varepsilon^{f_i}_{N_i}\left[d\log\varepsilon^{f_i}_{N_i} + d\log y_i + d\log p_i - d\log w_i - d\log\left(1+\tau_i(\theta_i)\right)\right] \\
&\quad + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left[d\log\varepsilon^{f_i}_{x_{ij}} + d\log y_i + d\log p_i - d\log p_j\right] + d\log A_i \\
&= \left[d\log y_i + d\log p_i\right]\underbrace{\left[\varepsilon^{f_i}_{N_i} + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\right]}_{=1\text{ by crts}} + \underbrace{\left[d\varepsilon^{f_i}_{N_i} + \sum_{j=1}^{J} d\varepsilon^{f_i}_{x_{ij}}\right]}_{=0\text{ by crts}} \\
&\quad - \varepsilon^{f_i}_{N_i}\left[d\log w_i + d\log(1+\tau_i(\theta_i))\right] - \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] + d\log A_i
\end{aligned}$$

Where the second inequality holds because the sum of elasticities equals one for constant returns to scale technology and $\varepsilon^{f_i}_{x_{ij}} d\log\varepsilon^{f_i}_{x_{ij}} = d\varepsilon^{f_i}_{x_{ij}}$. Rearranging terms gives

$$\begin{aligned}
d\log p_i &= \varepsilon^{f_i}_{N_i}\left[d\log w_i + d\log(1+\tau_i(\theta_i))\right] + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i \\
&= \varepsilon^{f_i}_{N_i}\left[d\log w_i + \varepsilon^{1+\tau_i}_{\theta_i} d\log\theta_i\right] + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i
\end{aligned}$$

$$(3) \qquad = \varepsilon^{f_i}_{N_i}\left[d\log w_i - \tau_i(\theta_i)\varepsilon^{\mathcal{Q}_i}_{\theta_i} d\log\theta_i\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i$$

Note that the labor share of the economy is given by:

$$\begin{aligned}\Lambda = \lambda_L &= \frac{w\sum_i L_i}{\sum_i p_i c_i} \\ &= \frac{w\sum_i (1+\tau_i(\theta_i))N_i}{\sum_i p_i c_i}\end{aligned}$$

From optimal labor choice, we have that:

$$(1+\tau_i(\theta_i))N_i = w\, p_i\, y_i \epsilon^{f_i}_{N_i},$$

which gives us that

$$\Lambda = \frac{w^2\sum_i \epsilon^{f_i}_{N_1} p_i y_i}{\sum_i p_i c_i} = w^2\sum_i \epsilon^{f_i}_{N_i}\lambda_i$$

Since λ_i is fixed, as shown below in the market clearing condition:

$$d\log w = \frac{d\log\Lambda}{2}$$

Finally, market clearing in sector i is given by

$$\begin{aligned} y_i &= c_i + \sum_{j=1}^{J} x_{ji} \\ \Rightarrow \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k} &= \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k} + \sum_{j=1}^{J}\frac{p_i x_{ji}}{p_j x_j}\frac{p_j x_j}{\sum_{k=1}^{J} p_k c_k}\end{aligned}$$

Which we can rewrite as

$$(4) \qquad \lambda_i = \varepsilon^{\mathcal{D}}_{c_i} + \sum_{j=1}^{J}\varepsilon^{f_j}_{x_{ji}}\lambda_j$$

Where $\lambda_i = \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k}$ is the Domar weight of sector i.

This implies that $\boldsymbol{\lambda} = \boldsymbol{\Psi}\boldsymbol{\epsilon}_{\boldsymbol{c}}^{\mathcal{D}}$

5. Equilibrium in the labor market

From (FOC N_i) labor demand in sector i is

$$L_i^d(\theta_i) = \varepsilon_{N_i}^{f_i} \frac{p_i y_i}{w_i} \tag{Labor Demand}$$

Recall, labor supply is

$$L_i^s(\theta_i) = \frac{f_i(\theta_i)}{s_i + f_i(\theta_i)} H_i \tag{Labor Supply}$$

Labor demand equals labor supply at an equilibrium in the labor market.

$$L_i^d(\theta_i) = L_i^s(\theta_i). \tag{LM equilibrium}$$

This equilibrium conditions implicitly pins down θ_i, and therefore $d\log\theta_i$.

6. The production network

We denote vectors and matrices by bold letters. For instance, $d\log \boldsymbol{x} = \begin{bmatrix} d\log x_1 & \cdots & d\log x_J \end{bmatrix}'$. We can conveniently capture many features of the production network through the following matrices

$$\boldsymbol{\Omega} = \begin{bmatrix} \varepsilon_{x_{11}}^{f_1} & \varepsilon_{x_{12}}^{f_1} & \cdots & \varepsilon_{x_{1J}}^{f_1} \\ \varepsilon_{x_{21}}^{f_2} & \varepsilon_{x_{22}}^{f_2} & \cdots & \varepsilon_{x_{2J}}^{f_2} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{x_{J1}}^{f_J} & \varepsilon_{x_{J2}}^{f_J} & \cdots & \varepsilon_{x_{JJ}}^{f_J} \end{bmatrix}, \ \boldsymbol{\Psi} = (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}.$$

In the standard production networks vocabulary, $\boldsymbol{\Omega}$ is the sales based input-output matrix and $\boldsymbol{\Psi}$ is the sales based Leontief inverse.

In addition define

$$\boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}} = \begin{bmatrix} \varepsilon_{c_1}^{\mathcal{D}} \\ \varepsilon_{c_2}^{\mathcal{D}} \\ \vdots \\ \varepsilon_{c_J}^{\mathcal{D}} \end{bmatrix}, \ \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} = \begin{bmatrix} \varepsilon_{N_1}^{f_1} \\ \varepsilon_{N_2}^{f_2} \\ \vdots \\ \varepsilon_{N_J}^{f_J} \end{bmatrix}, \ \boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\Omega}} = \begin{bmatrix} \varepsilon_{\theta_1}^{q_1} \\ \varepsilon_{\theta_2}^{q_2} \\ \vdots \\ \varepsilon_{\theta_J}^{q_J} \end{bmatrix}, \ \boldsymbol{\tau} = \begin{bmatrix} \tau_1(\theta_1) \\ \tau_2(\theta_2) \\ \vdots \\ \tau_J(\theta_J) \end{bmatrix}$$

By stacking equation (3) for each sector, we get the following expression for how prices change across the production network

$$\begin{aligned} d\log \boldsymbol{p} &= \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} \odot \left[d\log \boldsymbol{w} - \left(\boldsymbol{\tau} \odot \boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\Omega}}\right) \odot d\log \boldsymbol{\theta}\right] + \boldsymbol{\Omega} d\log \boldsymbol{p} - d\log \boldsymbol{A} \\ \Rightarrow d\log \boldsymbol{p} &= \boldsymbol{\Psi}\left(\boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} \odot \left[d\log \boldsymbol{w} - \left(\boldsymbol{\tau} \odot \boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\Omega}}\right) \odot d\log \boldsymbol{\theta}\right]\right) - \boldsymbol{\Psi} d\log \boldsymbol{A} \end{aligned} \tag{5}$$

Where $\odot$ denotes the Hadamard (element wise) product.

Similarly, by stacking (4) for each sector, we get the following expression for Domar weights across the production network.

$$\boldsymbol{\lambda}' = \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} + \boldsymbol{\lambda}'\boldsymbol{\Omega}$$

We can see how Domar weights change across the production network by totally differentiating

$$\begin{aligned} d\boldsymbol{\lambda}' &= d\boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} + d\boldsymbol{\lambda}'\boldsymbol{\Omega} + \boldsymbol{\lambda}' d\boldsymbol{\Omega} \\ \Rightarrow d\boldsymbol{\lambda}' &= \left[d\boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} + \boldsymbol{\lambda}' d\boldsymbol{\Omega}\right] \boldsymbol{\Psi} \end{aligned} \tag{6}$$

6.1. Final production sector as another network producer

Baqaee and Rubbo (2022) point out the notational convenience of including the final goods producer in the set of sectoral producers. Treating final goods production as just another sector in the production network allows us to simplify some of the expressions above. We can recast the final producer as a profit maximizing, or equivalently cost minimizing, firm that assembles the consumption good using inputs from other sectors and sells this consumption good to households. Let P denote the aggregate price level, that is, the price level such that $PY = \sum_{i=1}^{J} p_i c_i$ where c_i maximizes household utility subject to the household budget constraint.

The we assume a perfectly competitive final production sector whose representative firm solves

$$\max_{\{c_i\}_{i=1}^N} P\mathcal{D}\left(\{c_i\}_{j=1}^N\right) - \sum_{i=1}^N p_i c_i$$

The first order conditions of this problem imply

$$\begin{aligned} P\mathcal{D}_{c_i} &= p_i \\ \Rightarrow \varepsilon_{c_i}^{\mathcal{D}} &= \frac{p_i c_i}{P\mathcal{D}} \end{aligned}$$

Where $P\mathcal{D} = \sum_{i=1}^N p_i c_i$ by perfect competition and constant returns to scale, so this condition is in fact exactly what we had above.

We can amend the network matrices as follows

$$\widehat{\boldsymbol{\Omega}} = \begin{bmatrix} 0 & \varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} \\ \mathbf{0} & \boldsymbol{\Omega} \end{bmatrix}, \ \widehat{\boldsymbol{\Psi}} = \left(\boldsymbol{I} - \widehat{\boldsymbol{\Omega}}\right)^{-1}, \ d\log \widehat{\boldsymbol{p}} = \begin{bmatrix} d\log P \\ d\log \boldsymbol{p} \end{bmatrix}, \ \boldsymbol{\lambda} = \begin{bmatrix} 1 \\ \boldsymbol{\lambda} \end{bmatrix}$$

We can then rewrite the pricing equation as

$$d\log \widehat{\boldsymbol{p}} = \widehat{\boldsymbol{\Psi}}\left(\widehat{\boldsymbol{\varepsilon}}_{\boldsymbol{N}}^{\boldsymbol{f}} \odot \left[d\log \widehat{\boldsymbol{w}} - \left(\widehat{\boldsymbol{\tau}} \odot \widehat{\boldsymbol{\varepsilon}}_{\theta}^{\Omega}\right) \odot d\log \widehat{\theta}\right]\right) - \widehat{\boldsymbol{\Psi}} d\log \widehat{\boldsymbol{A}} \tag{7}$$

And the change in shares satisfies

$$d\widehat{\boldsymbol{\lambda}} = \widehat{\boldsymbol{\lambda}}' d\widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Psi}} \tag{8}$$

7. Aggregation

7.1. Cobb-Douglas Case

With Cobb-Douglas production in all sectors, including the final production sector

$$d\widehat{\boldsymbol{\Omega}} = \mathbf{0}$$

Which implies by (8)

$$d\widehat{\lambda}' = \mathbf{0}.$$

In words, all the the information we need is captured by the pricing equation. For a given $d\log \widehat{\boldsymbol{p}}$, the first order conditions from profit maximization imply that for every i, j

$$d\log p_j - d\log p_i = d\log y_i - d\log x_{ij}$$

Plugging in the production function gives

$$d\log p_j - d\log p_i = d\log A_i + \varepsilon^{f_i}_{N_i} d\log N_i + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} d\log x_{ij} - d\log x_{ij}$$

Stacking the equations for each x_{ij} gives

$$\begin{aligned}
&\left(\boldsymbol{I} - \boldsymbol{E}_i\right) d\log \boldsymbol{p} = \boldsymbol{E}_i d\log \boldsymbol{A} + \varepsilon^{f_i}_{N_i} \boldsymbol{E}_i d\log \boldsymbol{N} + \left(\Gamma_i - \boldsymbol{I}\right) d\log \boldsymbol{x}_i \\
&\quad \Rightarrow d\log \boldsymbol{x}_i = \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \left(\boldsymbol{I} - \boldsymbol{E}_i\right) d\log \boldsymbol{p} - \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \boldsymbol{E}_i d\log \boldsymbol{A} - \varepsilon^{f_i}_{N_i} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \boldsymbol{E}_i d\log \boldsymbol{N}
\end{aligned}$$

Where, if $\boldsymbol{e}_i$ is a vector with a one in the ith position and zeros elsewhere,

$$\varepsilon^{\boldsymbol{f_i}}_{\boldsymbol{x_i}} = \begin{bmatrix} \varepsilon^{f_i}_{x_{i1}} \\ \vdots \\ \varepsilon^{f_i}_{x_{iJ}} \end{bmatrix}_{J\times 1}, \quad \Gamma_i = \begin{bmatrix} \varepsilon^{\boldsymbol{f_i}'}_{\boldsymbol{x_i}} \\ \vdots \\ \varepsilon^{\boldsymbol{f_i}'}_{\boldsymbol{x_i}} \end{bmatrix}_{J\times J}, \quad \boldsymbol{E}_i = \begin{bmatrix} \boldsymbol{e}_i' \\ \vdots \\ \boldsymbol{e}_i' \end{bmatrix}_{J\times J}$$

Multiplying both sides by $\varepsilon^{\boldsymbol{f_i}'}_{\boldsymbol{x_i}}$ and adding $\varepsilon^{f_i}_{N_i} d\log N_i + d\log A_i$ to both sides gives

$$\begin{aligned}
d\log y_i = {}& \varepsilon^{\boldsymbol{f_i}'}_{\boldsymbol{x_i}} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \left(\boldsymbol{I} - \boldsymbol{E}_i\right) d\log \boldsymbol{p} - \varepsilon^{\boldsymbol{f_i}'}_{\boldsymbol{x_i}} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \boldsymbol{E}_i d\log \boldsymbol{A} \\
& - \varepsilon^{f_i}_{N_i} \varepsilon^{\boldsymbol{f_i}'}_{\boldsymbol{x_i}} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \boldsymbol{E}_i d\log \boldsymbol{N} + \varepsilon^{f_i}_{N_i} d\log N_i + d\log A_i
\end{aligned}$$

Which we can rewrite as

$$d\log y_i = \varepsilon^{\boldsymbol{f_i}'}_{\boldsymbol{x_i}} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \left(\boldsymbol{I} - \boldsymbol{E}_i\right) d\log \boldsymbol{p}$$

$$- \left(\varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}\prime} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \boldsymbol{E}_i - \boldsymbol{e}_i' \boldsymbol{E}_i \right) d \log \boldsymbol{A}$$
$$- \varepsilon_{N_i}^{f_i} \left(\varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}\prime} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \boldsymbol{E}_i - \boldsymbol{e}_i' \boldsymbol{E}_i \right) d \log \boldsymbol{N}$$

And for the labor input

$$d \log(1 + \tau_i(\theta_i)) + d \log w_i - d \log p_i = d \log y_i - d \log N_i$$
$$\Rightarrow d \log w_i - \tau_i(\theta_i) \varepsilon_{\theta_i}^{Q_i} d \log \theta_i - d \log p_i = d \log A_i + \varepsilon_{N_i}^{f_i} d \log N_i + \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i} d \log x_{ij} - d \log N_i$$

Stacking these equations yields

$$\begin{bmatrix} d \log \boldsymbol{p} \\ d \log w_i - \tau_i(\theta_i) \varepsilon_{\theta_i}^{Q_i} d \log \theta_i \end{bmatrix} - d \log p_i - d \log A_i = \left(\Gamma_i - \boldsymbol{I}\right) \begin{bmatrix} d \log \boldsymbol{x}_i \\ d \log N_i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} d \log \boldsymbol{x}_i \\ d \log N_i \end{bmatrix} = \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \left(\begin{bmatrix} d \log \boldsymbol{p} \\ d \log w_i - \tau_i(\theta_i) \varepsilon_{\theta_i}^{Q_i} d \log \theta_i \end{bmatrix} - d \log p_i - d \log A_i \right)$$

Which, in turn, implies

$$d \log y_i = \varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}\prime} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \left(\begin{bmatrix} d \log \boldsymbol{p} \\ d \log w_i - \tau_i(\theta_i) \varepsilon_{\theta_i}^{Q_i} d \log \theta_i \end{bmatrix} - d \log p_i - d \log A_i \right)$$

Where

$$\varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}} = \begin{bmatrix} \varepsilon_{x_{i1}}^{f_i} \\ \vdots \\ \varepsilon_{x_{iJ}}^{f_i} \\ \varepsilon_{N_i}^{f_i} \end{bmatrix}_{J+1 \times 1}, \quad \Gamma_i = \begin{bmatrix} \varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}\prime} \\ \vdots \\ \varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}\prime} \end{bmatrix}_{J+1 \times J+1}$$

Define

$$\boldsymbol{E}_i = \begin{bmatrix} \boldsymbol{e}_i' \\ \vdots \\ \boldsymbol{e}_i' \end{bmatrix}$$

Where $\boldsymbol{e}_i$ is a $J \times 1$ vector with 1 at the ith position and 0 elsewhere. Then we can write

$$d\log y_i = \varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}\,\prime} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \begin{bmatrix} \left(\boldsymbol{I} - \boldsymbol{E}_i\right) d\log \boldsymbol{p} - \boldsymbol{E}_i d\log \boldsymbol{A} \\ d\log w_i - \tau_i(\theta_i)\varepsilon_{\theta_i}^{Q_i} d\log\theta_i - d\log p_i - d\log A_i \end{bmatrix}$$

Plugging in for $d\log \boldsymbol{p}$ using the pricing equation yields an expression for sector level production given price changes, changes in tightness, and changes in productivity.

$$\begin{aligned} d\log y_i = {} & \varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}\,\prime} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \begin{bmatrix} \boldsymbol{\Psi}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \odot \left[d\log \boldsymbol{w} - \left(\boldsymbol{\tau} \odot \varepsilon_{\boldsymbol{\theta}}^{\boldsymbol{Q}}\right) \odot d\log \boldsymbol{\theta}\right]\right) - \boldsymbol{\Psi} d\log \boldsymbol{A} \\ d\log w_i - \tau_i(\theta_i)\varepsilon_{\theta_i}^{Q_i} d\log\theta_i \end{bmatrix} \\ & - \varepsilon_{\boldsymbol{x_i}}^{\boldsymbol{f_i}\,\prime} \left(\Gamma_i - \boldsymbol{I}\right)^{-1} \left[\boldsymbol{1} \odot \left(\boldsymbol{\Psi}_i \left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \odot \left[d\log \boldsymbol{w} - \left(\boldsymbol{\tau} \odot \varepsilon_{\boldsymbol{\theta}}^{\boldsymbol{Q}}\right) \odot d\log \boldsymbol{\theta}\right]\right) - \boldsymbol{\Psi}_i d\log \boldsymbol{A} + d\log A_i\right)\right] \end{aligned} \tag{9}$$

Where $\boldsymbol{\Psi}_i$ is the ith row of $\boldsymbol{\Psi}$. [Would be good to have something here that makes it easier to isolate As and θs.]

7.2. CES Case

Suppose firms in each sector, including the final production sector, have CES technology

$$y_i = A_i \left(\omega_{iN}^{\frac{1}{\sigma_i}} N_i^{\frac{\sigma_i - 1}{\sigma_i}} + \sum_{j=1}^{J} \omega_{ij}^{\frac{1}{\sigma_i}} x_{ij}^{\frac{\sigma_i - 1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i - 1}}$$

Where ω_{ij} captures the importance of sector j's output in sector i production, and σ captures the elasticity of substitution across sectors.

The firm's first order condition now implies

$$p_j = p_i A_i \left(\omega_{iN}^{\frac{1}{\sigma_i}} N_i^{\frac{\sigma_i - 1}{\sigma_i}} + \sum_{j=1}^{J} \omega_{ij}^{\frac{1}{\sigma_i}} x_{ij}^{\frac{\sigma_i - 1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i - 1} - 1} \omega_{ij}^{\frac{1}{\sigma_i}} x_{ij}^{\frac{\sigma_i - 1}{\sigma_i} - 1}$$

Multiplying both sides by $\frac{x_{ij}}{y_i}$ gives

$$\varepsilon_{x_{ij}}^{f_i} = \frac{p_j x_{ij}}{p_i y_i} = \omega_{ij}^{\frac{1}{\sigma_i}} x_{ij}^{\frac{\sigma_i - 1}{\sigma_i}} \left(\omega_{iN}^{\frac{1}{\sigma_i}} N_i^{\frac{\sigma_i - 1}{\sigma_i}} + \sum_{j=1}^{J} \omega_{ij}^{\frac{1}{\sigma_i}} x_{ij}^{\frac{\sigma_i - 1}{\sigma_i}} \right)^{-1}$$

Which is not constant in general.

[Not sure where to go from here.]

7.3. General Case

[Not sure how this case works.]

8. Applications

8.1. Response to productivity shocks

Suppose there is a shock to productivity in one sector, for instance the energy sector. We are interested in how unemployment and output change across the production network in response to such a shock. Knowing how unemployment responds at the sector level and the aggregate level is crucial for a well designed policy response. Where should we focus resources for retraining and relocating workers? How much should we expect unemployment claims to increases? What policies are most likely to be effective in reducing unemployment and increasing output? In this section we explore the response to productivity shocks $d \log \boldsymbol{A}$ to answer questions like these.

8.1.1. Cobb-Douglas Case

An equilibrium in the sector i labor market requires

$$\varepsilon_{N_i}^{f_i} \frac{p_i(A_i, \theta_i)}{w_i(A_i, \theta_i)} y_i(A_i, \theta_i) = \frac{\mathcal{F}_i(\theta_i)}{s_i + \mathcal{F}_i(\theta_i)} H_i$$

Where we explicitly include the dependence of p_i and y_i on productivity A_i and tightness θ_i for expositional clarity. For now, we allow the wage schedule to be an arbitrary function of A_i and θ_i. With Cobb-Douglas technology $\varepsilon^{f_i}_{N_i}$ is a fixed parameter of the model.

The equilibrium condition implies,

$$d \log L_i^d = d \log L_i^s$$

On the left hand side we have

$$d \log p_i(\boldsymbol{A}, \theta) - d \log w_i(\boldsymbol{A}, \theta) + d \log y_i(\boldsymbol{A}, \theta) = \frac{s_i}{s_i + \mathcal{F}_i(\theta_i)} \varepsilon^{\mathcal{F}_i}_{\theta_i} d \log \theta_i$$

8.1.2. CES Case

8.2. Response to changes in the size of the labor force

8.2.1. Cobb-Douglas Case

8.2.2. CES Case

8.3. Does network structure matter for efficient aggregate level of unemployment?

8.4. What is the efficient distribution the aggregate labor force across sectors?

References

Baqaee, D. and E. Rubbo (2022). Micro Propagation and Macro Aggregation.