Mismatch Unemployment in Production Networks: Baqaee-Rubbo Framework

Finn Schüle and Haoyu Sheng

October 10, 2022

1. Setup

An explicitly Cobb-Douglas version of other notes, where we assume no wedges and $a_{ij} = \Omega_{ij}$ are fixed Cobb-Douglas weights in the production function. We maintain the assumption that the consumption good is produced by a final goods producer, and is part of the matrix Ω.

$$\Omega = \begin{bmatrix} 0 & a_{C1} & \cdots & a_{CN} \\ 0 & a_{11} & \cdots & a_{1N} \\ \vdots & & \ddots & \vdots \\ 0 & a_{N1} & \cdots & a_{NN} \end{bmatrix}$$

Below I define $\mathcal{U} \equiv N \cup C$

2. Exogenous Vacancies

The primitives are $\boldsymbol{E} = \begin{bmatrix} E_1 & \cdots & E_N \end{bmatrix}'$ the sector level stock of existing employee employer relationships; $\boldsymbol{U} = \begin{bmatrix} U_1 & \cdots & U_N \end{bmatrix}'$, the distribution of unemployed workers

Brown University.

across sectors; $\boldsymbol{Z} = \begin{bmatrix} Z_1 & \cdots & Z_N \end{bmatrix}'$ a Hicks-neutral productivity shifter in each sector; and $\boldsymbol{V} = \begin{bmatrix} V_1 & \cdots & V_N \end{bmatrix}'$, the number of vacancies in each sector. In the terminology of Baqaee and Rubbo (2022) Assumption 2, technology is given by $A = (\boldsymbol{Z}, \boldsymbol{E}, \boldsymbol{U}, \boldsymbol{V})$.[1] In this case, there is no independent labor factor because firms are not free to pick their labor input. They are only free to pick their intermediate inputs.

2.1. Cost minimization and the price determination via the forward equation

Since there are no wedges

$$mc_i = p_i \tag{1}$$

It follows from (1) that

$$d\log p_i = d\log mc_i.$$

To determine how prices change with primitives, we therefore need to work out how marginal costs mc_i change with primitives. We assume firms take input prices as given. Given input prices $\boldsymbol{p}$ and w, the cost function of firm i solves

$$c\left(\boldsymbol{p}; w, L_i, \overline{x}\right) = \min_{\left\{x_{ij}\right\}_{j\in\mathcal{U}}} wL_i + \sum_{j=1}^{N} p_j x_{ij} \text{ s.t. } \overline{x} = Z_i f_i\left(L_i, \left\{x_{ij}\right\}_{j\in N}\right)$$

We can write the Lagrangian of the cost minimization as:

$$\mathcal{L} = wL_i + \sum_{j\in\mathcal{U}} p_j x_{ij} + \lambda\left(\overline{x} - Z_i f_i\left(L_i, \left\{x_{ij}\right\}_{j\in N}\right)\right)$$

The first order conditions are:

$$\frac{\partial\mathcal{L}}{\partial x_{ij}} = p_j - a_{ij}\lambda\frac{\bar{x}}{x_{ij}} = 0$$

[1]Since $(\boldsymbol{E}, \boldsymbol{U}, \boldsymbol{V})$ pins down $\boldsymbol{L}$, labor is effectively a fixed factor of production. Once we extend our analysis to endogenous vacancies, a part of the labor— the existing stock of workers—will remain a fixed factor of production. But, crucially, part of the labor income will become variable from the perspective of the firms.

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{x} - Z_i f_i \left(L_i, \left\{ x_{ij} \right\}_{j \in N} \right) = 0$$

which give that:

$$x_{ij} = \frac{a_{ij} \lambda \bar{x}}{p_j}$$

Plugging into the second first order condition, we have that:

$$\begin{aligned}
\overline{x} &= Z_i L_i^{a_{iL}} \prod_{j \in \mathcal{U}} \left(\frac{\lambda \overline{x} a_{ij}}{p_j} \right)^{a_{ij}} \\
&= Z_i L_i^{a_{iL}} \left(\lambda \overline{x} \right)^{\sum_{j \in \mathcal{U}} a_{ij}} \prod_{j \in \mathcal{U}} \left(\frac{a_{ij}}{p_j} \right)^{a_{ij}} \\
\Rightarrow \overline{x}^{1 - \sum_{j \in \mathcal{U}} a_{ij}} &= \lambda^{\sum_{j \in \mathcal{U}} a_{ij}} Z_i L_i^{a_{iL}} \prod_{j \in \mathcal{U}} \left(\frac{a_{ij}}{p_j} \right)^{a_{ij}}
\end{aligned}$$

Rearranging for $\lambda = mc_i$ gives

$$mc_i = \left[Z_i^{-1} L_i^{-a_{iL}} \overline{x}^{-\left(1 - \sum_{j \in \mathcal{U}} a_{ij}\right)} \prod_{j \in \mathcal{U}} \left(\frac{p_j}{a_{ij}} \right)^{a_{ij}} \right]^{\frac{1}{\sum_{j \in \mathcal{U}} a_{ij}}}$$

Taking logs gives and letting $\Theta_i = \left(\sum_{j \in \mathcal{U}} a_{ij} \right)^{-1}$

$$\log mc_i = \Theta_i \left[\sum_{j \in \mathcal{U}} a_{ij} \left(\log p_j - \log a_{ij} \right) - a_{iL} \log L_i - \log Z_i - \left(1 - \Theta_i^{-1} \right) \log \overline{x} \right]$$

[2] Which implies, where we are substituting $\overline{x} = x_i$ because we are evaluating changes from the current economy featuring realized output x_i. Any changes in desire output from x_i will have first order effects on marginal costs.

$$d \log mc_i = \Theta_i \left[\sum_{j \in \mathcal{U}} a_{ij} d \log p_j - a_{iL} d \log L_i - d \log Z_i - \left(1 - \Theta_i^{-1} \right) d \log x_i \right]$$

[2] We drop a μ_i in our derivation here when we differentiate, when in fact this sticks around.

Plugging back into the expression from the definition of μ_i gives

$$d\log p_i = \Theta_i \left[\sum_{j\in\mathcal{U}} a_{ij} d\log p_j - a_{iL} d\log L_i - d\log Z_i - \left(1-\Theta_i^{-1}\right) d\log x_i \right]$$

Since this condition holds for each sector i, price changes satisfy the following system of equations

$$d\log \boldsymbol{p} = \boldsymbol{\Phi} d\log \boldsymbol{p} - \Phi_L \odot d\log \boldsymbol{L} - \Theta \odot d\log \boldsymbol{Z} - (\Theta - \mathbf{1}) \odot d\log \boldsymbol{x}$$

Where $\odot$ denotes the Hadamard (element-wise) product

$$\boldsymbol{\Phi} = \begin{bmatrix} 0 & \Theta_C a_{C1} & \cdots & \Theta_C a_{CN} \\ 0 & \Theta_1 a_{11} & \cdots & \Theta_1 a_{1N} \\ \vdots & & \ddots & \vdots \\ 0 & \Theta_N a_{N1} & \cdots & \Theta_N a_{NN} \end{bmatrix}, \; \Phi_L = \begin{bmatrix} 0 \\ \Theta_1 a_{1L} \\ \vdots \\ \Theta_N a_{NL} \end{bmatrix}, \; \Theta = \begin{bmatrix} \Theta_C \\ \Theta_1 \\ \vdots \\ \Theta_N \end{bmatrix}$$

And $d\log \boldsymbol{x} = \begin{bmatrix} d\log x_c & d\log x_1 & \cdots & d\log x_N \end{bmatrix}'$ for any variable x_i. Price changes therefore satisfy

$$d\log \boldsymbol{p} = -(\boldsymbol{I} - \boldsymbol{\Phi})^{-1} \left[\Phi_L \odot d\log \boldsymbol{L} + \Theta \odot d\log \boldsymbol{Z} + (\Theta - \mathbf{1}) \odot d\log \boldsymbol{x} \right]$$

Where, implicitly, the final consumption good is the numeraire and $d\log p_C = 0$. The other terms in $d\log \boldsymbol{p}$ therefore capture price changes relative to the final consumption good.

$d\log \boldsymbol{L}$ is itself a function of fundamentals $\boldsymbol{E}$, $\boldsymbol{U}$, and $\boldsymbol{V}$.

$$L_i = E_i + \phi_i m(U_i, V_i)$$

Which implies, using $d\log f(x,y) = \varepsilon_x^f d\log x + \varepsilon_y^f d\log y$, where ε_x^f is the elasticity of f with respect to x

$$\begin{aligned} d\log L_i &= \varepsilon_E^{L_i} d\log E_i + \varepsilon_V^{L_i} d\log V_i + \varepsilon_U^{L_i} d\log U_i \\ &= \frac{E_i}{E_i + \phi_i m(U_i, V_i)} d\log E_i + \frac{\phi_i m(U_i, V_i)}{E_i + \phi_i m(U_i, V_i)} \varepsilon_V^{m_i} d\log V_i + \frac{\phi_i m(U_i, V_i)}{E_i + \phi_i m(U_i, V_i)} \varepsilon_U^{m_i} d\log U_i \end{aligned}$$

So, in terms of the fundamentals $\boldsymbol{E}$, $\boldsymbol{V}$, and $\boldsymbol{U}$, the system of pricing equations is

$$
\begin{aligned}
(2) \qquad d\log \boldsymbol{p} = &-(\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\left[\Phi_L \odot \left(\varepsilon_E^L \odot d\log \boldsymbol{E} + \varepsilon_V^L \odot d\log \boldsymbol{V} + \varepsilon_U^L \odot d\log \boldsymbol{U}\right)\right] \\
&-(\boldsymbol{I}-\boldsymbol{\Phi})^{-1}\left[\boldsymbol{\Theta} \odot d\log \boldsymbol{Z} + (\boldsymbol{\Theta}-\boldsymbol{1}) \odot d\log \boldsymbol{x}\right]
\end{aligned}
$$

Where the ε terms denote vectors of sector specific elasticities.

2.2. Sales Changes and the Backward Equation

The sector i market clearing condition is[3]

$$x_i = \sum_{j\in\mathcal{U}} x_{ji}$$

where x_{Ci} is final consumption of good i and is capture by the a_{Ci} terms in the expressions above. Differentiating both sides and using $d\log x_i = \frac{dx_i}{x_i}$ gives

$$x_i d\log x_i = x_{Ci} d\log x_{Ci} + \sum_{j=1}^{N} x_{ji} d\log x_{ji}$$

Multiplying both sides by $\frac{p_i}{GDP}$

$$
\begin{aligned}
\Rightarrow \lambda_i d\log x_i &= \frac{p_i x_{Ci}}{p_C x_C} d\log x_{Ci} + \sum_{j=1}^{N} \lambda_j \frac{p_i x_{ji}}{p_j x_j} d\log x_{ji} \\
&= \sum_{j\in\mathcal{U}} \lambda_j \frac{p_i x_{ji}}{p_j x_j} d\log x_{ji}
\end{aligned}
$$

Profit maximization/cost minimization and Cobb-Douglas technology implies[4]

$$a_{ji} = \frac{p_i x_{ji}}{mc_j x_j} = \frac{p_i x_{ji}}{p_j x_j}$$

[3] I think the problem is this market clearing condition does not hold for the final consumption good. Because, for the final consumption good, this market clearing condition implies $x_C = 0$ since the final consumption good does not get used by any of the other sectors as an intermediate input, including in its own sector. Does that mean this first order condition does not work in other sectors either?

[4] With imperfect competition, the condition would be $a_{ij} = \frac{p_i x_{ji}}{mc_j x_j} = \mu_j \frac{p_i x_{ji}}{p_j x_j}$.

So

$$\lambda_i d\log x_i = \sum_{j\in\mathcal{U}} \lambda_j a_{ji} d\log x_{ji}$$

Using the first order conditions

$$\log x_{ji} = \log a_{ji} + \log p_j + \log x_j - \log p_i$$

We can write

$$\begin{aligned}
\lambda_i d\log x_i &= \sum_{j\in\mathcal{U}} \lambda_j a_{ji} \left[d\log p_j - d\log p_i + d\log x_j\right] \\
d\log x_i &= \sum_{j\in\mathcal{U}} \frac{\lambda_j a_{ji}}{\lambda_i} \left[d\log p_j + d\log x_j\right] - d\log p_i \underbrace{\sum_{j\in\mathcal{U}} \frac{\lambda_j a_{ji}}{\lambda_i}}_{\gamma_i}
\end{aligned}$$

In matrix notation,

$$\begin{aligned}
d\log \boldsymbol{x} &= \boldsymbol{\Lambda} d\log \boldsymbol{p} + \boldsymbol{\Lambda} d\log \boldsymbol{x} - \boldsymbol{\gamma} \odot d\log \boldsymbol{p} \\
\Rightarrow d\log \boldsymbol{x} &= (\boldsymbol{I} - \boldsymbol{\Lambda})^{-1} \left[\boldsymbol{\Lambda} d\log \boldsymbol{p} - \boldsymbol{\gamma} \odot d\log \boldsymbol{p}\right] \qquad (3)
\end{aligned}$$

Where

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_N \\ \frac{1}{\lambda_1} & 1 & \cdots & \frac{\lambda_N}{\lambda_1} \\ \vdots & & \ddots & \vdots \\ \frac{1}{\lambda_N} & \frac{\lambda_1}{\lambda_N} & \cdots & 1 \end{bmatrix} \odot \Omega^T, \ \boldsymbol{\gamma} = \begin{bmatrix} \gamma_C \\ \gamma_1 \\ \cdots \\ \gamma_N \end{bmatrix}$$

2.3. Combining the forward and backward equations

Since (3) pins down output changes as a function of prices alone, plugging (3) in (2) gives a system of equations in $d\log \boldsymbol{p}$. The fixed point $d\log \overline{\boldsymbol{p}}$ for a given $d\log \boldsymbol{U}$ expresses how prices change when we reallocated unemployment across sectors. Plugging back into (3) then pins down how quantities change.

If we could get rid of the element-wise products somehow through algebra, then we could possibly solve for prices without needed to solve a fixed point equation. We can do this by appropriately stacking coefficient matrices and performing element wise multiplication on a matrix-to-matrix rather than vector to log change basis.

For example,

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \left(\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \odot \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So from (2), ignoring all terms that do not contain $d \log \boldsymbol{x}$,

$$d \log \boldsymbol{p} = - \left((\boldsymbol{I} - \boldsymbol{\Phi})^{-1} \odot \begin{bmatrix} (\Theta - \mathbf{1})' \\ \vdots \\ (\Theta - \mathbf{1})' \end{bmatrix} \right) d \log \boldsymbol{x} + \text{ Additional Terms}$$

Similarly, from (3)

$$\begin{aligned} d \log \boldsymbol{x} &= (\boldsymbol{I} - \boldsymbol{\Lambda})^{-1} \boldsymbol{\Lambda} d \log \boldsymbol{p} - \left((\boldsymbol{I} - \boldsymbol{\Lambda})^{-1} \odot \begin{bmatrix} \gamma' \\ \vdots \\ \gamma' \end{bmatrix} \right) d \log \boldsymbol{p} \\ &= \left((\boldsymbol{I} - \boldsymbol{\Lambda})^{-1} \boldsymbol{\Lambda} - (\boldsymbol{I} - \boldsymbol{\Lambda})^{-1} \odot \begin{bmatrix} \gamma' \\ \vdots \\ \gamma' \end{bmatrix} \right) d \log \boldsymbol{p} \end{aligned}$$

Plugging (3) into (2) allows us to solve for $d \log \boldsymbol{p}$.[5]

$$\begin{aligned} d \log \boldsymbol{p} &= -(\boldsymbol{I} + \Xi)^{-1} (\boldsymbol{I} - \boldsymbol{\Phi})^{-1} \left[\Phi_L \odot \left(\varepsilon_E^L \odot d \log \boldsymbol{E} + \varepsilon_V^L \odot d \log \boldsymbol{V} + \varepsilon_U^L \odot d \log \boldsymbol{U} \right) + \boldsymbol{\Theta} \odot d \log \boldsymbol{Z} \right] \\ &= -[(\boldsymbol{I} - \boldsymbol{\Phi}) (\boldsymbol{I} + \Xi)]^{-1} \left[\Phi_L \odot \left(\varepsilon_E^L \odot d \log \boldsymbol{E} + \varepsilon_V^L \odot d \log \boldsymbol{V} + \varepsilon_U^L \odot d \log \boldsymbol{U} \right) + \boldsymbol{\Theta} \odot d \log \boldsymbol{Z} \right] \end{aligned}$$

Where

$$\Xi = \left((\boldsymbol{I} - \boldsymbol{\Phi})^{-1} \odot \begin{bmatrix} (\Theta - \mathbf{1})' \\ \vdots \\ (\Theta - \mathbf{1})' \end{bmatrix} \right) \left((\boldsymbol{I} - \boldsymbol{\Lambda})^{-1} \boldsymbol{\Lambda} - (\boldsymbol{I} - \boldsymbol{\Lambda})^{-1} \odot \begin{bmatrix} \gamma' \\ \vdots \\ \gamma' \end{bmatrix} \right)$$

[5] Assuming the complicated matrix we need to invert is in fact invertible.

3. Endogenous Vacancies

With endogenous vacancies, firms are free to choose the number of new hires they want to employ by picking $\boldsymbol{V}$. Technology is now given by $A = (\boldsymbol{Z}, \boldsymbol{E}, \boldsymbol{U})$. One way to model this would be to assume there is an additional 'recruiting' sector for each $i \in N$. The intermediate input from this sector is additional labor above $\boldsymbol{E}$ and the price is $w+$ additional recruiting costs per additional hire.

4. Endogenous Vacancies and Separations

References

Baqaee, D. and E. Rubbo (2022). Micro Propagation and Macro Aggregation.