Unemployment in a Production Network: Adding multiple occupations

Finn Schüle and Haoyu Sheng

May 8, 2023

1. Households and final goods production

We consider a closed, static economy model with no government spending. There is no saving mechanism in the economy, and real household consumption equals real GDP, denoted by Y. A final goods producer with constant returns to scale technology aggregates J sector outputs to produce Y, the final consumption good

$$Y = \max_{\{c_i\}_{i=1}^J} \mathcal{D}\left(\{c_i\}_{i=1}^J\right)$$

Subject to the budget constraint

$$\sum_{i=1}^{J} p_i c_i = \sum_{i=1}^{J} w_o L_o^s(\theta_o).$$

$\mathcal{D}$ captures household preferences over final consumption goods, and w_o is the wage of labor supplied to occupation o, $L_o^s(\theta_o)$.

The household's consumption decision can be computed using the first order condition:

$$\varepsilon_{c_i}^{\mathcal{D}} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}, \tag{1}$$

Brown University.

where $\varepsilon_{c_i}^{\mathcal{D}}$ denotes the households elasticity of utility with regards to the consumption of good i.

2. Occupation level labor markets

We assume there are $\mathcal{O}$ occupations with separate labor markets, a labor force of H_o possible workers, who all start out unemployed, and an exogenous recruiting cost r_o, which measures the units of labor required for a firm to maintain each posted vacancy in occupation o. When workers and firms meet there is a mutual gain from matching. There is no accepted theory for how wages are set in this context. For now we assume the nominal wage in occupation o, w_o, follows a general wage schedule that depends on productivity and the size of the labor force, and is taken as given by both firms and workers. Hires are generated by a constant returns matching function in occupation-level unemployment U_o and aggregate vacancies V_o, which measure all vacancy postings for occupation o,

$$h_o = \phi_o m\left(U_o, V_o\right)$$

The household supplies H_o searching workers to each sector. Let the sector-specific labor market tightness be $\theta_o = \frac{V_o}{H_o}$, the vacancy-filling rate $\mathcal{Q}_o(\theta_o) = \phi_o m\left(\frac{H_o}{V_o}, 1\right)$, and the job-finding rate $\mathcal{F}_o(\theta_o) = \phi_o m\left(1, \frac{v_o}{H_o}\right)$. Therefore, a fraction $\mathcal{F}_o(\theta_o)$ of H_o household finds a job, and labor supply satisfies

$$L_i^o(\theta_o) = \mathcal{F}_o(\theta_o)H_o \tag{2}$$

We assume firms take the occupation level tightness as given. One way of justifying this is with the assumption that each sector is populated by many identical competitive firms so that each firm only has an infinitesimal impact on aggregate vacancies, and therefore on aggregate tightness. Let N_{io} denote productive employees in occupation o working for sector i firms and let r_o be the cost of each vacancy for a firm. In order to hire N_{io} productive employees, the number of vacancies posted v_{io} has to satisfy $\mathcal{Q}_o(\theta_o)v_{io} = N_{io}+r_o v_{io}$, where $r_o v_{io}$ denotes the cost of posting the vacancies. Rearranging yields $v_{io} = \frac{N_{io}}{\mathcal{Q}_o(\theta_o)-r_o}$. Thus, hiring one unit of productive labor requires $\frac{1}{\mathcal{Q}_o(\theta_o)-r_o}$ vacancy postings, and requires $1+\tau_o(\theta_o)$ units of total labor, where

$$\tau_o(\theta_o) \equiv \frac{r_o}{\mathcal{Q}_o(\theta_o) - r_o}.$$

For a given target level of occupation o employment N_{io}, total required labor, or the labor demand, is $l_{io}^{d}(\theta_o) = (1 + \tau_o(\theta_o))\, N_{io}$. We describe how labor demand, $l_{io}^{d}(\theta_{io})$, is determined by firms' profit maximization in the next section.

Finally, we define aggregate occupation o labor demand as the sum of sectoral labor demands and aggregate vacancy postings as the sum of sectoral vacancy postings.

$$L_o^d(\theta_o) = \sum_{i=1}^{J} l_{io}^{d}(\theta_o)$$
$$V_o = \sum_{i=1}^{J} v_{io}$$

Market clearing in the labor market requires labor demand equal labor supply and that the vacancy posting choices of firms in each sector are consistent with aggregate tightness.

$$L_o^d = L_o^s$$
$$\theta_o = \frac{\sum_{i=1}^{J} v_{io}}{H_o}$$

3. Sector level firms

A representative firm in sector i uses workers in occupation o N_{io} and intermediate inputs from sector j, x_{ij}, to produce output y_i using constant returns production technology f_i.

$$y_i = A_i f_i\left(\left\{N_{io}\right\}_{o=1}^{\mathcal{O}}, \{x_{ij}\}_{j=1}^{J}\right)$$

Firms choose $\left\{N_{io}\right\}_{o=1}^{\mathcal{O}}$ and $\left\{x_{ij}\right\}_{j=1}^{J}$ to maximize profits, or equivalently to minimize costs. We assume firms are price takers in both input and output markets. Profits are given by

$$\pi_i = p_i f_i\left(\left\{N_{io}\right\}_{o=1}^{\mathcal{O}}, \{x_{ij}\}_{j=1}^{J}\right) - \sum_{o=1}^{\mathcal{O}} w_o(1 + \tau_o(\theta_o))N_{io} - \sum_{j=1}^{J} p_j x_{ij}$$

Firms choose inputs to solve

$$\max_{\{N_{io}\}_{o=1}^{\mathcal{O}}, \left\{x_{ij}\right\}_{j=1}^{J}} \pi_i \left(\{N_{io}\}_{o=1}^{\mathcal{O}}, \left\{x_{ij}\right\}_{j=1}^{J} \right)$$

Giving the first order conditions

$$\begin{aligned} p_i f_{i,x_{ij}} &= p_j \\ p_i f_{i,N_{io}} &= w_o \left(1 + \tau_o(\theta_o)\right) \end{aligned}$$

Labor demand is $l_{io}^{d}(\theta_o) = (1+\tau_o(\theta_o))N_{io}$ for the optimal N_{io}. The aggregate labor demand is therefore

$$L_o^d(\theta_o) = \sum_{i=1}^{J} \left(1 + \tau_o(\theta_o)\right) N_{io}$$

The equilibrium tightness equates aggregate labor demand and labor supply.

We can rewrite these expressions in terms of elasticities.

$$\varepsilon_{x_{ij}}^{f_i} = \frac{p_j x_{ij}}{p_i y_i} \tag{3}$$

$$\varepsilon_{N_{io}}^{f_i} = \left(1 + \tau_o(\theta_o)\right) \frac{w_o N_{io}}{p_i y_i} \tag{4}$$

From Equation 4, we can derive the an alternative expression for labor demand:

$$\begin{aligned} l_{io}^{d}(\theta_o) &= \varepsilon_{N_{io}}^{f_i} \frac{p_i}{w_o} y_i \\ L_o^d(\theta_o) &= \sum_{i=1}^{J} \varepsilon_{N_{io}}^{f_i} \frac{p_i}{w_o} y_i \end{aligned} \tag{5}$$

4. Equilibrium

The equilibrium in this model can be characterized by a set of conditions guaranteeing labor market equilibrium and goods market equilibrium. The equilibrium is a collection of $4J + 2J^2 + 3\mathcal{O}J + 4\mathcal{O}$ endogenous variables

$$\left\{ \left\{ p_i, y_i, \left\{ x_{ij}, \varepsilon^{f_i}_{x_{ij}} \right\}_{j=1}^{J}, c_i, \varepsilon^{\mathcal{D}}_{c_i}, \left\{ N_{io}, \varepsilon^{f_i}_{N_{io}}, l^{d}_{io} \right\}_{o=1}^{\mathcal{O}} \right\}_{i=1}^{J}, \left\{ \theta_o, w_o, L^d_o, L^s_o \right\}_{o=1}^{\mathcal{O}}, \right\}$$

that satisfy equations 1 through 5, along with goods market clearing, labor market clearing, and constant returns, given exogenous variables $\left\{ \{A_i\}_{i=1}^{J}, \{H_o\}_{o=1}^{\mathcal{O}} \right\}$. We summarize the equilibrium conditions below for convenience.

4.1. Goods Market Equilibrium

In an equilibrium, firms intermediate input choices given prices and labor market chacteristics are profit maximizing:

(Intermediate input decision) $$\varepsilon^{f_i}_{x_{ij}} = \frac{p_j x_{ij}}{p_i y_i},$$

(Labor input decision) $$\varepsilon^{f_i}_{N_{io}} = (1 + \tau_o(\theta_o)) \frac{w_o N_{io}}{p_i y_i}.$$

Firms produce output via production technology f_i

(Production technology) $$y_i = A_i f_i \left(\left\{ N_{io} \right\}_{o=1}^{\mathcal{O}}, \left\{ x_{ij} \right\}_{j=1}^{J} \right)$$

By constant returns to scale in production,

(Constant returns production) $$1 = \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} + \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}}$$

In addition, the household maximizes their utility by choosing a consumption bundle that satisfies its first-order condition.

(Consumption decision) $$\varepsilon^{\mathcal{D}}_{c_i} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}.$$

And by constant returns

(Constant returns utility) $$1 = \sum_{i=1}^{J} \varepsilon^{\mathcal{D}}_{c_i}$$

Finally, the goods market has to clear, which means that, for each sector i, total production has to be equal to the sum of the household's consumption of good i and all other sectors' use of good i in their production:

(Goods market clearing) $$y_i = c_i + \sum_{j=1}^{J} x_{ji}.$$

In total, the goods market provides $4J + J^2 + J\mathcal{O} + 1$ restrictions.

4.2. Labor Market Equilibrium

From Equation 4, labor demand in sector i is defined as

(Labor Demand) $$l^d_{io}(\theta_o) = \varepsilon^{f_i}_{N_{io}} \frac{p_i y_i}{w_o}.$$

(6) $$L^d_o(\theta_o) = \sum_{j=1}^{J} l^d_{io}(\theta_o)$$

Given sector level labor force participation H_i labor supply is

(Labor Supply) $$L^s_o(\theta_o) = \mathcal{F}_o(\theta_o)H_o.$$

Labor demand equals labor supply at an equilibrium in the labor market.

(LM equilibrium) $$L^d_o(\theta_o) = L^s_o(\theta_o).$$

These equilibrium conditions provide an additional $3\mathcal{O} + J\mathcal{O}$ restrictions.

4.3. Summary

The equilibrium conditions outline above provide just $4J + J^2 + 2J\mathcal{O} + 3\mathcal{O} + 1$ equations in $4J + 2J^2 + 3\mathcal{O}J + 4\mathcal{O}$ endogenous variables. The wage schedules taken as given by both households and firms provide another $\mathcal{O}$ restrictions. Nevertheless, as is typical in the literature, we need additional functional form assumptions on production and household preferences to close the model.

For instance, assuming Cobb-Douglas production and preferences fully parameterizes $\left\{ \left\{ \varepsilon^{f_i}_{x_{ij}} \right\}_{j=1}^{J}, \left\{ \varepsilon^{f_i}_{N_i} \right\}_{o=1}^{\mathcal{O}}, \varepsilon^{\mathcal{D}}_{c_i} \right\}_{i=1}^{J}$, giving us $J + J^2 + J\mathcal{O}$ additional restrictions, but removing $J+1$ of the restrictions above.[1] Parametrizing production by assuming Cobb-Douglas therefore gives exactly the number of restrictions we need to close the model. Alternatively, assuming CES production indirectly provides restrictions to pin down the same set of elasticities.[2]

5. The Production Network

In this section, we introduce key production network notation. We then discuss three propagation mechanisms: prices, sales shares, and tightness. For now, we proceed assuming a general production function. In subsequent applications we will parameterize the production functions to allow us to calibrate and estimate the model.

5.1. Notation

We denote vectors and matrices by bold letters. For instance, $d\log \boldsymbol{x} = \begin{bmatrix} d\log x_1 & \cdots & d\log x_J \end{bmatrix}'$. We can conveniently capture many features of the production network through the following matrices

$$\boldsymbol{\Omega} = \begin{bmatrix} \varepsilon^{f_1}_{x_{11}} & \varepsilon^{f_1}_{x_{12}} & \cdots & \varepsilon^{f_1}_{x_{1J}} \\ \varepsilon^{f_2}_{x_{21}} & \varepsilon^{f_2}_{x_{22}} & \cdots & \varepsilon^{f_2}_{x_{2J}} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon^{f_J}_{x_{J1}} & \varepsilon^{f_J}_{x_{J2}} & \cdots & \varepsilon^{f_J}_{x_{JJ}} \end{bmatrix}, \; \boldsymbol{\Psi} = (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}.$$

In the standard production networks vocabulary, $\boldsymbol{\Omega}$ is the sales based input-output matrix and $\boldsymbol{\Psi}$ is the sales based Leontief inverse.[3]

[1] Choosing elasticities directly makes the constant returns restrictions redundant.

[2] CES and Cobb-Douglas, a special case of CES, are the two most commonly assumed production technologies in the literature. In principle, assuming any production technology will do. An explicit functional form will allow us to characterize the required elasticities. We therefore continue below with general formulas that hold for any production technology. In applied work, we will need to either specify the functional form or be able to directly estimate the elasticities and how they change in response to shocks.

[3] Since our model abstracts from markup wedges on prices, the sales based and cost based input-output matrices will coincide.

In addition define

$$\boldsymbol{\varepsilon}^{\mathcal{D}}_{\boldsymbol{c}} = \begin{bmatrix} \varepsilon^{\mathcal{D}}_{c_1} \\ \varepsilon^{\mathcal{D}}_{c_2} \\ \vdots \\ \varepsilon^{\mathcal{D}}_{c_J} \end{bmatrix}_{J\times 1}, \quad \boldsymbol{\varepsilon}^{\boldsymbol{f}}_{\boldsymbol{N}} = \begin{bmatrix} \varepsilon^{f_1}_{N_{11}} & \varepsilon^{f_1}_{N_{12}} & \cdots & \varepsilon^{f_1}_{N_{1\mathcal{O}}} \\ \varepsilon^{f_2}_{N_{21}} & \varepsilon^{f_2}_{N_{22}} & \cdots & \varepsilon^{f_2}_{N_{2\mathcal{O}}} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon^{f_J}_{N_{J1}} & \varepsilon^{f_J}_{N_{J2}} & \cdots & \varepsilon^{f_J}_{N_{J\mathcal{O}}} \end{bmatrix}_{J\times\mathcal{O}},$$

$$\boldsymbol{\varepsilon}^{\mathcal{Q}}_{\boldsymbol{\theta}} = \begin{bmatrix} \varepsilon^{\mathcal{Q}_1}_{\theta_1} \\ \varepsilon^{\mathcal{Q}_2}_{\theta_2} \\ \vdots \\ \varepsilon^{\mathcal{Q}_\mathcal{O}}_{\theta_\mathcal{O}} \end{bmatrix}_{\mathcal{O}\times 1}, \quad \boldsymbol{\varepsilon}^{\mathcal{F}}_{\boldsymbol{\theta}} = \begin{bmatrix} \varepsilon^{\mathcal{F}_1}_{\theta_1} \\ \varepsilon^{\mathcal{F}_2}_{\theta_2} \\ \vdots \\ \varepsilon^{\mathcal{F}_\mathcal{O}}_{\theta_\mathcal{O}} \end{bmatrix}_{\mathcal{O}\times 1}, \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_1(\theta_1) \\ \tau_2(\theta_2) \\ \vdots \\ \tau_\mathcal{O}(\theta_\mathcal{O}) \end{bmatrix}_{\mathcal{O}\times 1}$$

Furthermore, let

$$\boldsymbol{\mathcal{Q}} = \operatorname{diag}\left(\boldsymbol{\varepsilon}^{\mathcal{Q}}_{\boldsymbol{\theta}}\right)_{\mathcal{O}\times\mathcal{O}}, \; \boldsymbol{\mathcal{F}} = \operatorname{diag}\left(\boldsymbol{\varepsilon}^{\mathcal{F}}_{\boldsymbol{\theta}}\right)_{\mathcal{O}\times\mathcal{O}}, \; \boldsymbol{\mathcal{T}} = \operatorname{diag}\left(\boldsymbol{\tau}\right)_{\mathcal{O}\times\mathcal{O}}$$

And let $\boldsymbol{\mathcal{L}}_{\mathcal{O}\times J}$ be

$$\boldsymbol{\mathcal{L}} = \begin{bmatrix} \frac{l_{11}}{L^d_1} & \frac{l_{21}}{L^d_1} & \cdots & \frac{l_{J1}}{L^d_1} \\ \frac{l_{12}}{L^d_2} & \frac{l_{22}}{L^d_2} & \cdots & \frac{l_{J2}}{L^d_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{l_{1\mathcal{O}}}{L^d_\mathcal{O}} & \frac{l_{2\mathcal{O}}}{L^d_\mathcal{O}} & \cdots & \frac{l_{J\mathcal{O}}}{L^d_\mathcal{O}} \end{bmatrix}$$

5.2. Wage changes

Since wages are not uniquely pinned down in models featuring matching frictions, we need to assume a wage schedule to close the model. In particular, it will be convenient to assume

$$d\log \boldsymbol{w} - \boldsymbol{\mathcal{L}} d\log \boldsymbol{p} = \boldsymbol{\Lambda_A} d\log \boldsymbol{A} + \boldsymbol{\Lambda_H} d\log \boldsymbol{H}$$

We can think of $d\log \boldsymbol{w} - \boldsymbol{\mathcal{L}} d\log \boldsymbol{p}$ as akin to a real wage. It is the wage net of occupational employment share weighted sectoral prices. $\boldsymbol{\Lambda_A}$ contains wage elasticities to productivity changes and $\boldsymbol{\Lambda_H}$ contains wage elasticities to labor force changes.

5.3. Price Propagation

Log-linearizing the production function, for each sector i, we have:

$$d\log y_i = \underbrace{\varepsilon^{f_i}_{A_i}}_{=1} d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} d\log N_{io} + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} d\log x_{ij}$$

Plugging in Equation 4 and Equation 3, the first order conditions for optimal input usage, into the log-linearized production function gives

$$\begin{aligned} d\log y_i &= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log \varepsilon^{f_i}_{N_{io}} + d\log y_i + d\log p_i - d\log w_o - d\log\left(1+\tau_o(\theta_o)\right)\right] \\ &\quad + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} \left[d\log \varepsilon^{f_i}_{x_{ij}} + d\log y_i + d\log p_i - d\log p_j\right] + d\log A_i \\ &= \left[d\log y_i + d\log p_i\right] \underbrace{\left[\sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}}\right]}_{=1 \text{ by crts}} + \underbrace{\left[\sum_{o=1}^{\mathcal{O}} d\varepsilon^{f_i}_{N_{io}} + \sum_{j=1}^{N} d\varepsilon^{f_i}_{x_{ij}}\right]}_{=0 \text{ by crts}} \\ &\quad - \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log w_o + d\log(1+\tau_o(\theta_o))\right] - \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} \left[d\log p_j\right] + d\log A_i, \end{aligned}$$

where the second equality holds because the sum of elasticities equals one for constant returns to scale technology and $\varepsilon^{f_i}_{x_{ij}} d\log \varepsilon^{f_i}_{x_{ij}} = d\varepsilon^{f_i}_{x_{ij}}$.

Rearranging terms gives

$$\begin{aligned} d\log p_i &= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log w_o + d\log(1+\tau_o(\theta_o))\right] + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} \left[d\log p_j\right] - d\log A_i \\ &= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log w_o + \varepsilon^{1+\tau_o}_{\theta_o} d\log \theta_o\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left[d\log p_j\right] - d\log A_i \\ &= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log w_o - \tau_o(\theta_o) \varepsilon^{\mathcal{Q}_o}_{\theta_o} d\log \theta_o\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left[d\log p_j\right] - d\log A_i \end{aligned}$$

Stacing equations over sectors, we can write

$$d\log \boldsymbol{p} = \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \left[d\log \boldsymbol{w} - \mathfrak{Q}\mathfrak{T} d\log \theta\right] + \boldsymbol{\Omega} d\log \boldsymbol{p} - d\log \boldsymbol{A}$$

Which implies

$$d\log \boldsymbol{p} = \boldsymbol{\Psi} \left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \left[d\log \boldsymbol{w} - \mathfrak{Q}\mathfrak{T} d\log \theta\right] - d\log \boldsymbol{A}\right]$$

Or equivalently

$$\left(\boldsymbol{I} - \boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right) d\log \boldsymbol{p} = \boldsymbol{\Psi} \left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \left[d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p} - \mathfrak{Q}\mathfrak{T} d\log \theta\right] - d\log \boldsymbol{A}\right]$$

5.4. Sales Share Propagation

We can rewrite the goods market clearing condition in terms of Domar weights:

$$y_i = c_i + \sum_{j=1}^{J} x_{ji}$$

$$\Rightarrow \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k} + \sum_{j=1}^{J} \frac{p_i x_{ji}}{p_j x_j} \frac{p_j x_j}{\sum_{k=1}^{J} p_k c_k}$$

$$\Rightarrow \lambda_i = \varepsilon_{c_i}^{\mathcal{D}} + \sum_{j=1}^{J} \varepsilon_{x_{ji}}^{f_j} \lambda_j, \tag{7}$$

where $\lambda_i = \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k}$ is the Domar weight of sector i.

By stacking (7) for each sector, we get the following expression for Domar weights across the production network.

$$\lambda' = \varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} + \lambda' \boldsymbol{\Omega}$$

We can see how Domar weights change across the production network by totally differentiating

$$d\lambda' = d\varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} + d\lambda' \boldsymbol{\Omega} + \lambda' d\boldsymbol{\Omega}$$

$$\Rightarrow d\lambda' = \left[d\varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} + \lambda' d\boldsymbol{\Omega}\right] \boldsymbol{\Psi} \tag{8}$$

The Domar weights will help us express how shocks propagate to output.

5.5. Output Propagation

Since the log-linearized expression for the Domar weight must hold for every sector, we can write

$$\begin{aligned} d\log\lambda_i - d\log\lambda_j &= d\log p_i - d\log p_j + d\log y_i - d\log y_j \\ &= d\log x_{ij} - d\log\varepsilon^{f_i}_{x_{ij}} - d\log y_j \\ \Rightarrow d\log x_{ij} &= d\log\lambda_i - d\log\lambda_j + d\log y_j + d\log\varepsilon^{f_i}_{x_{ij}} \end{aligned}$$

Plugging back into the production function,

$$\begin{aligned} d\log y_i &= d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} d\log N_{io} + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left(d\log\varepsilon^{f_i}_{x_{ij}} + d\log y_j + d\log\lambda_i - d\log\lambda_j\right) \\ &= d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log N_{io} - d\log\varepsilon^{f_i}_{N_i}\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left(d\log y_j + d\log\lambda_i - d\log\lambda_j\right) \end{aligned}$$

Using the definition of labor demand,

$$\begin{aligned} \sum_i \frac{l_{io}}{L_o} d\log N_{io} &= d\log L^d_o - d\log(1+\tau_o(\theta_o)) \\ &= d\log L^d_o + \tau_o(\theta_o)\varepsilon^{\mathcal{Q}_o}_{\theta_o} d\log\theta_o \end{aligned}$$

From output labor usage, we have that labor usage ratio for an occupation by two different sectors as:

$$\frac{l_{io}}{l_{jo}} = \frac{\varepsilon^{f}_{N_{io}}\lambda_i}{\varepsilon^{f}_{N_{jo}}\lambda_j}$$

for any $l_{io}, l_{jo} > 0$

Log-linearizing it, assuming Cobb-Douglas preferences, yields:

$$d\log l_{io} = d\log l_{jo}$$

Also since, $d\log l_{io} = d\log N_{io} + d\log(1+\tau_o(\theta_o)) = d\log l_{jo} = d\log N_{jo} + d\log(1+\tau_o(\theta_o))$, we have that $d\log N_{io} = d\log N_{jo}$

Using the labor market clearing condition, and the definition of labor supply,

$$\sum_k \frac{l_{ko}}{L_o} d\log N_{ko} = \left(\varepsilon_{\theta_o}^{\mathcal{F}_o} + \tau_o(\theta_o)\varepsilon_{\theta_o}^{\mathcal{Q}_o}\right) d\log\theta_o + d\log H_o$$
$$\Rightarrow d\log N_{io} \underbrace{\sum_k \frac{l_{ko}}{L_o}}_{=1} = \left(\varepsilon_{\theta_o}^{\mathcal{F}_o} + \tau_o(\theta_o)\varepsilon_{\theta_o}^{\mathcal{Q}_o}\right) d\log\theta_o + d\log H_o$$

Plugging this back into the linearized production function gives:

$$d\log y_i = d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} \left[\left(\varepsilon_{\theta_o}^{\mathcal{F}_o} + \tau_o(\theta_o)\varepsilon_{\theta_o}^{\mathcal{Q}_o}\right) d\log\theta_o + d\log H_o - d\log\varepsilon_{N_i}^{f_i}\right]$$
$$+ \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i} \left(d\log y_j + d\log\lambda_i - d\log\lambda_j\right)$$

Stacking over sectors gives,

$$d\log \boldsymbol{y} = d\log\boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} (\boldsymbol{\mathcal{F}} + \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{T}})\, d\log\theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log\boldsymbol{H}$$
$$- d\log\boldsymbol{\mathcal{E}} + \boldsymbol{\Omega} d\log\boldsymbol{y} + (\mathrm{diag}(\boldsymbol{\Omega}\mathbf{1}) - \boldsymbol{\Omega})\, d\log\boldsymbol{\lambda}$$

Where $\mathbf{1}$ is a $J \times 1$ vector of ones and $d\log\boldsymbol{\mathcal{E}}$ is the $J \times 1$ vector of diagonal elements of $\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}'}$. Which implies

$$d\log\boldsymbol{y} = \boldsymbol{\Psi}\left(d\log\boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} (\boldsymbol{\mathcal{F}} + \boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{T}})\, d\log\theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log\boldsymbol{H}\right)$$
$$- \boldsymbol{\Psi} d\log\boldsymbol{\mathcal{E}} + \boldsymbol{\Psi}(\mathrm{diag}(\boldsymbol{\Omega}\mathbf{1}) - \boldsymbol{\Omega})\, d\log\boldsymbol{\lambda}$$

5.6. Tightness Propagation

Labor market clearing implies that changes in labor demand have to equal changes in labor supply:

$$d\log L_o^s(\theta, \boldsymbol{H}) = d\log L_o^d(\theta, \boldsymbol{A}).$$
$$\varepsilon_{\theta_o}^{\mathcal{F}_o} d\log\theta_o + d\log H_o = \sum_{i=1}^{J} \frac{l_{io}}{L_o^d} d\log l_{io}(\theta_o)$$

Where $\frac{l_i o}{L_o^d} = \frac{\varepsilon_{N_{io}}^{f_i} p_i y_i}{\sum_{j=1}^{J} \varepsilon_{N_{jo}}^{f_j} p_j y_j}$[4]. For every sector i we have

$$d \log l_{io}(\theta_o) = d \log \varepsilon_{N_{io}}^{f_i} - d \log w_o + d \log p_i + d \log y_i$$

Which implies that

$$d \log \boldsymbol{L^d}(\theta) = \operatorname{diag}\left(\mathcal{L} d \log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) - \left[d \log \boldsymbol{w} - \mathcal{L} d \log \boldsymbol{p}\right] + \mathcal{L} d \log \boldsymbol{y}$$

since $\sum_{i=1}^{J} \frac{l_{io}}{L_o^d} = 1$ for all o. Plugging in for $d \log \boldsymbol{y}$ gives

$$\begin{aligned} d \log \boldsymbol{L^d}(\theta) = \operatorname{diag}\left(\mathcal{L} d \log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) - \left[d \log \boldsymbol{w} - \mathcal{L} d \log \boldsymbol{p}\right] \\ + \mathcal{L} \boldsymbol{\Psi} \left[d \log \boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} (\mathcal{F} + \mathfrak{Q}\mathfrak{T})\, d \log \theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d \log \boldsymbol{H}\right] \\ + \mathcal{L} \boldsymbol{\Psi} \left[(\operatorname{diag}(\boldsymbol{\Omega} \boldsymbol{1}) - \boldsymbol{\Omega})\, d \log \boldsymbol{\lambda} - d \log \mathcal{E}\right] \end{aligned}$$

Labor market clearing implies

$$\begin{aligned} \mathcal{F} d \log \theta + d \log \boldsymbol{H} = \operatorname{diag}\left(\mathcal{L} d \log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) - \left[d \log \boldsymbol{w} - \mathcal{L} d \log \boldsymbol{p}\right] \\ + \mathcal{L} \boldsymbol{\Psi} \left[d \log \boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} (\mathcal{F} + \mathfrak{Q}\mathfrak{T})\, d \log \theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d \log \boldsymbol{H}\right] \\ + \mathcal{L} \boldsymbol{\Psi} \left[(\operatorname{diag}(\boldsymbol{\Omega} \boldsymbol{1}) - \boldsymbol{\Omega})\, d \log \boldsymbol{\lambda} - d \log \mathcal{E}\right] \end{aligned}$$

Which pins down first order changes in log tightness as

$$\begin{aligned} d \log \theta = \left[\mathcal{F} - \Xi_\theta\right]^{-1} \left[\mathcal{L} \boldsymbol{\Psi} d \log \boldsymbol{A} - \left[d \log \boldsymbol{w} - \mathcal{L} d \log \boldsymbol{p}\right] + \left[\mathcal{L} \boldsymbol{\Psi} \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d \log \boldsymbol{H}\right] \\ + \left[\mathcal{F} - \Xi_\theta\right]^{-1} \left[\operatorname{diag}\left(\mathcal{L} d \log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) + \mathcal{L} \boldsymbol{\Psi} \left[(\operatorname{diag}(\boldsymbol{\Omega} \boldsymbol{1}) - \boldsymbol{\Omega})\, d \log \boldsymbol{\lambda} - d \log \mathcal{E}\right]\right] \end{aligned}$$

Where $\Xi_\theta = \mathcal{L} \boldsymbol{\Psi} \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} [\mathcal{F} + \mathfrak{Q}\mathfrak{T}]$. Notice, all terms in the second line are zero assuming Cobb-Douglas production technology.

[4] One implication of this formula is that I think we should be able to check whether the elasticities $\left\{\left\{\varepsilon_{N_{io}}^{f}\right\}_{o=1}^{\mathcal{O}}\right\}_{i=1}^{J}$ are consistent with the Domar weights.

5.6.1. Two Special Cases

We look at how tightness propagates for two special cases. First, we examine how tightness responds when there is only one occupation in the economy. In this case, firms only use one type of labor, and labor is fully mobile across all sectors. Second, we examine how tightness responds when each sector hires only sector-specific labor. This case represents when labor markets are rigid across sectors.

One occupation. With one occupation, we have

$$\mathcal{L} = \begin{bmatrix} \frac{l_{11}}{L_1^d} & \frac{l_{21}}{L_1^d} & \cdots & \frac{l_{J1}}{L_1^d} \end{bmatrix}$$

$$\mathcal{L}\boldsymbol{\Psi} = \begin{bmatrix} \sum_{j=1}^{J} \frac{l_{j1}}{L_1^d}\Psi_{j1} & \sum_{j=1}^{J} \frac{l_{j1}}{L_1^d}\Psi_{j2} & \cdots & \sum_{j=1}^{J} \frac{l_{j1}}{L_1^d}\Psi_{jJ} \end{bmatrix}$$

$$\mathcal{L}\boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{F}} = \sum_{i=1}^{J}\sum_{j=1}^{J} \varepsilon_{N_{1i}}^{f} \frac{l_{j1}}{L_1^d}\Psi_{ji}$$

Sector-specific Occupation. With this setup, we have that $\mathcal{L} = \boldsymbol{I}$, and we can rewrite $d\log\theta$ as

$$\begin{aligned} d\log\theta = {}& \left[\mathcal{F} - \Xi_\theta\right]^{-1}\left[\boldsymbol{\Psi} d\log\boldsymbol{A} - \left[d\log\boldsymbol{w} - d\log\boldsymbol{p}\right] - \left[\boldsymbol{I} - \boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right] d\log\boldsymbol{H}\right] \\ & + \left[\mathcal{F} - \Xi_\theta\right]^{-1}\left[\operatorname{diag}\left(d\log\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) + \boldsymbol{\Psi}\left[\left(\operatorname{diag}\left(\boldsymbol{\Omega}\mathbf{1}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} - d\log\mathcal{E}\right]\right] \end{aligned}$$

Where $\Xi_\theta = \boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left[\mathcal{F} + \mathcal{Q}\mathcal{T}\right]$

5.7. Unemployment

Unemployment in occupation o is $U_o = H_o - L_o$. The log change in Unemployment is

$$d\log\boldsymbol{U} = d\log\boldsymbol{H} - d\log\boldsymbol{L} \tag{9}$$

Note though, that this equation would be different in a model that allows for an existing initial stock of workers. In that case we would need to re-weight by changes in the relative stock.

6. Aggregation

6.1. General Case

Using the first order condition,

$$d\log \varepsilon^{\mathcal{D}}_{c_i} = d\log p_i + d\log c_i - d\log \sum_{j=1}^{J} p_k c_k$$

along with the definition of the Domar weight,

$$d\log \sum_{k=1}^{J} p_k c_k = d\log p_i + d\log y_i - d\log \lambda_i$$

gives

$$d\log c_i = d\log \varepsilon^{\mathcal{D}}_{c_i} + d\log y_i - d\log \lambda_i$$

Which implies the log change in real GDP is

$$\begin{aligned} d\log Y &= \boldsymbol{\varepsilon}^{\mathcal{D}\prime}_{\boldsymbol{c}} d\log \boldsymbol{c} \\ &= \boldsymbol{\varepsilon}^{\mathcal{D}\prime}_{\boldsymbol{c}} \left(d\log \boldsymbol{\varepsilon}^{\mathcal{D}}_{\boldsymbol{c}} + d\log \boldsymbol{y} - d\log \boldsymbol{\lambda} \right) \end{aligned}$$

6.2. Aggregate employment

The aggregate labor force, employment, and unemployment are $H^{agg} = \sum_{o=1}^{\mathcal{O}} H_o$, $L^{agg} = \sum_{o=1}^{\mathcal{O}} L_o$, and $U^{agg} = \sum_{o=1}^{\mathcal{O}} U_o$. Changes in aggregates are therefore given by

$$\begin{aligned} dH^{agg} &= \sum_{o=1}^{\mathcal{O}} dH_o \\ dL^{agg} &= \sum_{o=1}^{\mathcal{O}} dL_o \\ dU^{agg} &= \sum_{o=1}^{\mathcal{O}} dU_o \end{aligned}$$

Or in terms of log changes

$$
\begin{aligned}
d\log H^{agg} &= \frac{1}{H^{agg}} \sum_{o=1}^{\mathcal{O}} H_o d\log H_o \\
d\log L^{agg} &= \frac{1}{L^{agg}} \sum_{o=1}^{\mathcal{O}} L_o d\log L_o \\
d\log U^{agg} &= \frac{1}{U^{agg}} \sum_{o=1}^{\mathcal{O}} U_o d\log U_o
\end{aligned}
$$

In matrix notation

$$
\begin{aligned}
d\log H^{agg} &= \frac{1}{H^{agg}} \boldsymbol{H}' d\log \boldsymbol{H} \\
d\log L^{agg} &= \frac{1}{L^{agg}} \boldsymbol{L}' d\log \boldsymbol{L} \\
d\log U^{agg} &= \frac{1}{U^{agg}} \boldsymbol{U}' d\log \boldsymbol{U}
\end{aligned}
$$

Substituting in for $d\log \boldsymbol{L}$

$$
\begin{aligned}
d\log L^{agg} &= \frac{1}{L^{agg}} \boldsymbol{L'} \left[\mathcal{L} d\log \boldsymbol{y} - \left[d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p} \right] \right] \\
&= \frac{1}{L^{agg}} \boldsymbol{L'} \left[\mathcal{L} \left[\boldsymbol{\Pi}_{y,A} d\log \boldsymbol{A} + \boldsymbol{\Pi}_{y,H} d\log \boldsymbol{H} - \boldsymbol{\Lambda}_A d\log \boldsymbol{A} - \boldsymbol{\Lambda}_H d\log \boldsymbol{H} \right] \right] \\
&= \boldsymbol{\Pi}_{L^{agg},A} d\log \boldsymbol{A} + \boldsymbol{\Pi}_{L^{agg},H} d\log \boldsymbol{H}
\end{aligned}
$$

Where

$$
\begin{aligned}
\boldsymbol{\Pi}_{L^{agg},A} &= \frac{1}{L^{agg}} \boldsymbol{L}' \left[\mathcal{L} \boldsymbol{\Pi}_{y,A} - \boldsymbol{\Lambda}_A \right] \\
\boldsymbol{\Pi}_{L^{agg},H} &= \frac{1}{L^{agg}} \boldsymbol{L}' \left[\mathcal{L} \boldsymbol{\Pi}_{y,H} - \boldsymbol{\Lambda}_H \right]
\end{aligned}
$$

And

$$
\begin{aligned}
d\log U^{agg} &= \frac{1}{U^{agg}} \boldsymbol{U'} \left[d\log \boldsymbol{H} - d\log \boldsymbol{L} \right] \\
&= \frac{1}{U^{agg}} \boldsymbol{U'} \left[d\log \boldsymbol{H} - \left[\mathcal{L} \left[\boldsymbol{\Pi}_{y,A} d\log \boldsymbol{A} + \boldsymbol{\Pi}_{y,H} d\log \boldsymbol{H} - \boldsymbol{\Lambda}_A d\log \boldsymbol{A} - \boldsymbol{\Lambda}_H d\log \boldsymbol{H} \right] \right] \right] \\
&= \boldsymbol{\Pi}_{U^{agg},A} d\log \boldsymbol{A} + \boldsymbol{\Pi}_{U^{agg},H} d\log \boldsymbol{H}
\end{aligned}
$$

Where

$$\Pi_{U^{agg},A} = \frac{1}{U^{agg}} \boldsymbol{U}' \left[\boldsymbol{\Lambda}_A - \mathcal{L}\Pi_{y,A}\right]$$
$$\Pi_{U^{agg},H} = \frac{1}{U^{agg}} \boldsymbol{U}' \left[\boldsymbol{I} + \boldsymbol{\Lambda}_H - \mathcal{L}\Pi_{y,H}\right]$$

7. An Cobb-Douglas Example

In this section, we examine the network propagation and aggregation results under Cobb-Douglas utility, matching function, and production function. We focus on only technology shocks. Specifically, this implies that:

$$\begin{aligned} d\log \varepsilon_{\boldsymbol{c}}^{\mathcal{D}} &= \boldsymbol{0} \\ d\log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} &= \boldsymbol{0} \\ d\boldsymbol{\Omega} &= \boldsymbol{0} \\ d\boldsymbol{\lambda} &= \boldsymbol{0} \\ d\log \boldsymbol{H} &= \boldsymbol{0} \end{aligned}$$

In addition, I assume rigid nominal wages, which means $d\log \boldsymbol{w} = 0$.

7.1. Propagation

Therefore, the output and tightness propagation are:

$$\begin{aligned} d\log \boldsymbol{y} &= \Psi \left(d\log \boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} (\mathcal{F} + \mathfrak{Q}\mathfrak{T})\, d\log\theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \boldsymbol{H}\right) \\ \left(\boldsymbol{I} - \Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right) d\log \boldsymbol{p} &= \Psi \left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \left[d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p} - \mathfrak{Q}\mathfrak{T} d\log\theta\right] - d\log \boldsymbol{A}\right] \\ d\log\theta &= \left[\mathcal{F} - \Xi_\theta\right]^{-1} \left[\mathcal{L}\Psi d\log \boldsymbol{A} - \left[d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p}\right] + \left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log \boldsymbol{H}\right] \end{aligned}$$

where $\Xi_\theta = \mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} [\mathcal{F} + \mathfrak{Q}\mathfrak{T}]$.

We see that sectoral outputs are impacted by technology and labor supply shocks directly through the production channel, and indirectly through the labor market channel.

7.2. Propagation in terms of supply and demand side shocks

First, we can rewrite $d\log\theta$:

$$\begin{aligned} d\log\theta &= \left[\mathcal{F}-\Xi_\theta\right]^{-1}\left[\mathcal{L}\Psi d\log \boldsymbol{A}-\left[d\log \boldsymbol{w}-\mathcal{L}d\log\ \boldsymbol{p}\right]+\left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}-\boldsymbol{I}\right]d\log \boldsymbol{H}\right] \\ &= \left[\mathcal{F}-\Xi_\theta\right]^{-1}\left[\mathcal{L}\Psi d\log \boldsymbol{A}-\left[\Lambda_A d\log \boldsymbol{A}+\Lambda_{\boldsymbol{H}} d\log \boldsymbol{H}\right]+\left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}-\boldsymbol{I}\right]d\log \boldsymbol{H}\right] \\ &= \left[\mathcal{F}-\Xi_\theta\right]^{-1}\left[(\mathcal{L}\Psi-\Lambda_A)\,d\log \boldsymbol{A}+\left(\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}-\boldsymbol{I}-\Lambda_{\boldsymbol{H}}\right)d\log \boldsymbol{H}\right], \end{aligned}$$

where $\Xi_\theta=\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left[\mathcal{F}+\mathfrak{Q}\mathfrak{T}\right]$.

Let us now write out $d\log\ \boldsymbol{p}$ by substituting in the relative wages and the change in tightnesses.

$$\begin{aligned} \left(\boldsymbol{I}-\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right)d\log\ \boldsymbol{p} &= \Psi\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left[d\log \boldsymbol{w}-\mathcal{L}d\log\ \boldsymbol{p}-\mathfrak{Q}\mathfrak{T}d\log\theta\right]-d\log \boldsymbol{A}\right] \\ &= \Psi\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left(\Lambda_A-\mathfrak{Q}\mathfrak{T}\left[\mathcal{F}-\Xi_\theta\right]^{-1}(\mathcal{L}\Psi-\Lambda_A)\right)-\boldsymbol{I}\right]d\log \boldsymbol{A} \\ &\quad+\Psi\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left(\Lambda_H-\mathfrak{Q}\mathfrak{T}\left[\mathcal{F}-\Xi_\theta\right]^{-1}\left(\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}-\boldsymbol{I}-\Lambda_{\boldsymbol{H}}\right)\right)\right]d\log \boldsymbol{H} \end{aligned}$$

7.3. Aggregation

The log change in real GDP is

$$\begin{aligned} d\log Y &= \varepsilon_{\boldsymbol{c}}^{\mathcal{D}\prime}d\log \boldsymbol{c} \\ &= \varepsilon_{\boldsymbol{c}}^{\mathcal{D}\prime}\left(d\log\varepsilon_{\boldsymbol{c}}^{\mathcal{D}}+d\log\ \boldsymbol{y}-d\log\lambda\right) \\ &= \varepsilon_{\boldsymbol{c}}^{\mathcal{D}\prime}d\log\ \boldsymbol{y} \\ &= \varepsilon_{\boldsymbol{c}}^{\mathcal{D}\prime}\Psi\left(d\log \boldsymbol{A}+\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}(\mathcal{F}+\mathfrak{Q}\mathfrak{T})\,d\log\theta+\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}d\log \boldsymbol{H}\right) \\ &= \Pi_{\boldsymbol{A}}d\log \boldsymbol{A}+\Pi_{\boldsymbol{H}}d\log \boldsymbol{H}, \end{aligned}$$

where

$$\begin{aligned} \Pi_{\boldsymbol{A}} &= \varepsilon_{\boldsymbol{c}}^{\mathcal{D}\prime}\Psi\left(\boldsymbol{I}+\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}(\mathcal{F}+\mathfrak{Q}\mathfrak{T})\left[\mathcal{F}-\Xi_\theta\right]^{-1}(\mathcal{L}\Psi-\Lambda_{\boldsymbol{A}})\right) \\ \Pi_{\boldsymbol{H}} &= \varepsilon_{\boldsymbol{c}}^{\mathcal{D}\prime}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left(\boldsymbol{I}+(\mathcal{F}+\mathfrak{Q}\mathfrak{T})\left[\mathcal{F}-\Xi_\theta\right]^{-1}\left(\left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}-\boldsymbol{I}\right]-\Lambda_{\boldsymbol{H}}\right)\right) \end{aligned}$$

7.4. Comparing with Hulten's Theorem

To draw a fair comparison with Hulten's theorem, we only consider technology shocks. Since under Cobb-Douglas preferences, sales share are simply $\varepsilon_{\boldsymbol{c}}^{\mathcal{D}'}\boldsymbol{\Psi}$, for Hulten's theorem to hold, we only need to shut off the propagation channel for tightness. This requires that $\boldsymbol{\Lambda_A} = \mathcal{L}\boldsymbol{\Psi}$. The price propagation equation now becomes:

$$\left(\boldsymbol{I} - \boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right) d\log \boldsymbol{p} = \boldsymbol{\Psi}\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\boldsymbol{\Psi} d\log \boldsymbol{A} - d\log \boldsymbol{A}\right]$$

One such case is when $d\log \boldsymbol{w} = 0$, which gives that $d\log \boldsymbol{p} = -\boldsymbol{\Psi} d\log \boldsymbol{A}$. One can easily verify that this is one of the solutions to the system above. Note that directly solving the system above is not feasible, since $\left(\boldsymbol{I} - \boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right)$ is not invertible. However, one can pick an arbitrary numeraire to pin down the system of prices and wages.

8. Comparison with no search frictions model.

Suppose that there are no search frictions. Instead, we treat labor from the $\mathcal{O}$ occupations as fixed supply factors of production. In equilibrium, wages adjust so that all workers are employed. Absent search and matching frictions, wages are therefore pinned down by firms' first order conditions, which now read

$$\varepsilon_{x_{ij}}^{f_i} = \frac{p_j x_{ij}}{p_i y_i}$$
$$\varepsilon_{N_{io}}^{f_i} = \frac{w_o N_{io}}{p_i y_i}$$

Labor market clearing now requires

$$H_o = \sum_{i=1}^{J} N_{io}$$

All other conditions remain unchanged. An equilibrium in the frictionless economy is a collection of prices $\{p_i\}_{i=1}^{J}$, wages, $\{w_o\}_{o=1}^{\mathcal{O}}$, input choices $\left\{\left\{x_{ij}\right\}_{j=1}^{J}, \left\{N_{io}\right\}_{o=1}^{\mathcal{O}}\right\}_{i=1}^{J}$, sectoral output $\{y_i\}_{i=1}^{J}$, and elasticities, $\left\{\left\{\varepsilon_{x_{ij}}^{f_i}\right\}_{j=1}^{J}, \left\{\varepsilon_{N_{io}}^{f_i}\right\}_{o=1}^{\mathcal{O}}, \varepsilon_{c_i}^{\mathcal{D}}\right\}_{i=1}^{J}$ such that

(i) Firms choose inputs to maximize profits.

(ii) Households choose final consumption to maximize utility subject to the budget constraint.

(iii) All goods and labor markets clear.

First order log price changes satisfy

$$d\log \boldsymbol{p} = \boldsymbol{\Psi}\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \boldsymbol{w} - d\log \boldsymbol{A}\right]$$

First order log wage changes satisfy

$$d\log \boldsymbol{w} = \mathcal{L}\left(d\log \boldsymbol{p} + d\log \boldsymbol{y}\right) - d\log \boldsymbol{H}$$

To compute the change in sectoral outputs, we can use the linearized production function, with rewriting change in intermediate input as output in other sectors and changes in sales shares. This gives us:

$$\begin{aligned} d\log y_i &= d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} d\log N_{io} + \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i}\left(d\log \varepsilon_{x_{ij}}^{f_i} + d\log y_j + d\log \lambda_i - d\log \lambda_j\right) \\ &= d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i}\left[d\log N_{io} - d\log \varepsilon_{N_i}^{f_i}\right] + \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i}\left(d\log y_j + d\log \lambda_i - d\log \lambda_j\right) \end{aligned}$$

Note that from labor clearing and occupation labor use ratio,

$$d\log N_{io} = d\log H_o,$$

Under Cobb-Douglas preferences and production functions, this becomes:

$$d\log y_i = d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} d\log H_o + \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i} d\log y_j$$

Stacking the equations yields that:

$$d\log \boldsymbol{y} = d\log \boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \boldsymbol{H} + \boldsymbol{\Omega} d\log \boldsymbol{y}$$

9. Comparison with no production network model.

Suppose that there are no production networks. Then,

$$\boldsymbol{\Psi} = (\boldsymbol{I} - \mathbf{0})^{-1} = \boldsymbol{I}$$

Assuming each sector has a linear production function:

$$y_i = A_i N_i.$$

We now have:

$$w_i(1 + \tau(\theta_i)) = p_i A_i$$

Labor market clearing requires:

$$\mathcal{F} d\log\theta + d\log H = d\log \boldsymbol{y} - (d\log \boldsymbol{w} - d\log \boldsymbol{p})$$

Using the definition of labor demand and labor market clearing,

$$\begin{aligned} d\log N_i &= d\log L_i^d - d\log(1 + \tau_i(\theta_i)) \\ &= d\log L_i^d + \tau_i(\theta_i)\varepsilon_{\theta_i}^{\Omega_i} d\log\theta_i \end{aligned}$$

Log-linearizing production yields:

$$d\log \boldsymbol{y} = d\log \boldsymbol{A} + (\boldsymbol{\mathcal{F}} + \boldsymbol{\mathfrak{Q}\mathfrak{T}})\, d\log\theta + d\log \boldsymbol{H}$$

Plugging this back into labor market clearing, we have:

$$\begin{aligned} &\mathcal{F} d\log\theta + d\log H = d\log \boldsymbol{A} + (\boldsymbol{\mathcal{F}} + \boldsymbol{\mathfrak{Q}\mathfrak{T}})\, d\log\theta + d\log \boldsymbol{H} - \Lambda_A d\log \boldsymbol{A} - \Lambda_H d\log \boldsymbol{H} \\ &\qquad \Rightarrow d\log\theta = -\boldsymbol{\mathfrak{T}}^{-1}\boldsymbol{\mathfrak{Q}}^{-1}\left((I - \Lambda_A)\, d\log \boldsymbol{A} - \Lambda_H d\log \boldsymbol{H}\right) \end{aligned}$$

We can then compute output changes:

$$\begin{aligned} d\log \boldsymbol{y} &= d\log \boldsymbol{A} + d\log \boldsymbol{H} + (\boldsymbol{\mathcal{F}} + \boldsymbol{\mathfrak{Q}\mathfrak{T}})\, d\log\theta \\ &= d\log \boldsymbol{A} + d\log \boldsymbol{H} - (\boldsymbol{\mathcal{F}} + \boldsymbol{\mathfrak{Q}\mathfrak{T}})\, \mathfrak{T}^{-1}\mathfrak{Q}^{-1}\left((I - \Lambda_A)\, d\log \boldsymbol{A} - \Lambda_H d\log \boldsymbol{H}\right) \\ &= \left(\boldsymbol{\Lambda_A} d\log \boldsymbol{A} - \boldsymbol{\mathcal{F}\mathfrak{T}}^{-1}\boldsymbol{\mathfrak{Q}}^{-1}(\boldsymbol{I} - \boldsymbol{\Lambda_A})\right) d\log \boldsymbol{A} + \left((\boldsymbol{I} + \boldsymbol{\Lambda_H}) + \boldsymbol{\mathcal{F}\mathfrak{T}}^{-1}\boldsymbol{\mathfrak{Q}}^{-1}\boldsymbol{\Lambda_H}\right) d\log \boldsymbol{H} \end{aligned}$$

10. The role of wages

In this section we consider how changing the assumption about wage adjustments changes the propagation of shocks. In particular, we consider the class of wage schedules given by

$$d\log \boldsymbol{w} - \mathcal{L}d\log \boldsymbol{p} = \gamma_A \mathcal{L}\boldsymbol{\Psi} d\log A + \gamma_H \left[\mathcal{L}\boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log \boldsymbol{H}$$

Hulten's theorem holds as special case of this assumption where production and preferences are Cobb-Douglas, $\gamma_A = 1$, and $d\log \boldsymbol{H} = 0$. Therefore, this assumption allows us to explore the quantitative implications of deviations from the Hulten knife edge case by varying γ_A and γ_H. Clearly, even allowing for general γ_A and γ_H, this is a very restrictive assumption about how wages change. We view it as a convenient expositional tool.

Given the assumption about wages, our model defines a mapping from shocks to aggregate output changes. As shown above, assuming Cobb-Douglas production and preferences, we can write this mapping as

$$d\log Y = \boldsymbol{\Pi}_{\boldsymbol{A}} d\log \boldsymbol{A} + \boldsymbol{\Pi}_{\boldsymbol{H}} d\log \boldsymbol{H},$$

The for the class of wage schedules above,

$$\begin{aligned}
\boldsymbol{\Pi}_{\boldsymbol{A}} &= \varepsilon_{\boldsymbol{c}}^{\mathcal{D}'}\boldsymbol{\Psi}\left(\boldsymbol{I} + (1-\gamma_A)\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}(\mathcal{F} + \mathfrak{Q}\mathfrak{T})\left[\mathcal{F} - \Xi_\theta\right]^{-1}\mathcal{L}\boldsymbol{\Psi}\right)\\
\boldsymbol{\Pi}_{\boldsymbol{H}} &= \varepsilon_{\boldsymbol{c}}^{\mathcal{D}'}\boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left(\boldsymbol{I} + (1-\gamma_H)(\mathcal{F} + \mathfrak{Q}\mathfrak{T})\left[\mathcal{F} - \Xi_\theta\right]^{-1}\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right]\right)
\end{aligned}$$

11. Marginal Product of Labor

The marginal product of labor can be separated into two cases. When $\varepsilon_{N_{io}}^{f_i} = 0$, $MP_{N_{io}} = 0$.

Otherwise,

$$d\log MP_{N_{io}} = d\log y_i - d\log N_{io}$$

From labor market clearing, we have:

$$d \log N_{io} = \left(\varepsilon^{\mathcal{F}_o}_{\theta_o} + \tau_o(\theta_o) \varepsilon^{\mathcal{Q}_o}_{\theta_o} \right) d \log \theta_o + d \log H_o$$

In the case with one occupation per sector, we have that:

$$\begin{aligned} d \log \boldsymbol{MP} &= d \log \boldsymbol{y} - (\mathcal{F} + \mathfrak{T}\mathfrak{Q})\, d \log \theta - d \log \boldsymbol{H} \\ &= \Pi_{\boldsymbol{MP,A}} d \log \boldsymbol{A} + \Pi_{\boldsymbol{MP,H}} d \log \boldsymbol{H} \end{aligned}$$

where

$$\begin{aligned} \Pi_{\boldsymbol{MP,A}} &= \boldsymbol{\Psi} - \left(\boldsymbol{I} - \boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} \right) (\mathcal{F} + \mathfrak{Q}\mathfrak{T}) \left[\mathcal{F} - \Xi_\theta \right]^{-1} (\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}) , \\ \Pi_{\boldsymbol{MP,H}} &= -@\boldsymbol{I} + \boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} - \left(\boldsymbol{I} - \boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} \right) (\mathcal{F} + \mathfrak{Q}\mathfrak{T}) \left[\mathcal{F} - \Xi_\theta \right]^{-1} \left(\boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} - \boldsymbol{I} - \boldsymbol{\Lambda_H} \right) \end{aligned}$$

[KEEPING THIS HERE FOR REFERENCE]

$$\begin{aligned} \Pi_{\theta,A} &= \left[\mathcal{F} - \Xi_\theta \right]^{-1} (\mathcal{L}\boldsymbol{\Psi} - \Lambda_A) , \\ \Pi_{\theta,H} &= \left[\mathcal{F} - \Xi_\theta \right]^{-1} \left(\mathcal{L}\boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} - \boldsymbol{I} - \boldsymbol{\Lambda_H} \right) , \\ \Pi_{y,A} &= \boldsymbol{\Psi} \left[\boldsymbol{I} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} (\mathcal{F} + \mathfrak{Q}\mathfrak{T}) \left[\mathcal{F} - \Xi_\theta \right]^{-1} [\mathcal{L}\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}] \right] , \\ \Pi_{y,H} &= \boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} \left[\boldsymbol{I} + (\mathcal{F} + \mathfrak{Q}\mathfrak{T}) \left[\mathcal{F} - \Xi_\theta \right]^{-1} \left[\mathcal{L}\boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} - \boldsymbol{I} - \boldsymbol{\Lambda_H} \right] \right] , \\ \Xi_\theta &= \mathcal{L}\boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} [\mathcal{F} + \mathfrak{Q}\mathfrak{T}] \end{aligned}$$

$$\begin{aligned} \Pi_{y,A} &= \boldsymbol{\Psi} \left[\boldsymbol{I} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} (\mathcal{F} + \mathfrak{Q}\mathfrak{T}) \left[\mathcal{F} - \Xi_\theta \right]^{-1} [\mathcal{L}\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}] \right] \\ &= \boldsymbol{\Psi} + \mathcal{L}^{-1}\mathcal{F} \left[\mathcal{F} - \Xi_\theta \right]^{-1} [\mathcal{L}\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}] - \mathcal{L}^{-1} \left(\mathcal{F} - \mathcal{L}\boldsymbol{\Psi} \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} (\mathcal{F} + \mathfrak{Q}\mathfrak{T}) \right) \left[\mathcal{F} - \Xi_\theta \right]^{-1} [\mathcal{L}\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}] \\ &= \boldsymbol{\Psi} + \mathcal{L}^{-1}\mathcal{F} \left[\mathcal{F} - \Xi_\theta \right]^{-1} [\mathcal{L}\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}] - \mathcal{L}^{-1} [\mathcal{L}\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}] \\ &= \boldsymbol{\Psi} + \mathcal{L}^{-1} \left(\left[\boldsymbol{I} - \Xi_\theta \mathcal{F}^{-1} \right]^{-1} - \boldsymbol{I} \right) [\mathcal{L}\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}] \end{aligned}$$

Can simplify even more into:

$$\Pi_{y,A} = \boldsymbol{\Psi} + \mathcal{L}^{-1} \left(\left[\boldsymbol{I} - \Xi_\theta \mathcal{F}^{-1} \right]^{-1} - \boldsymbol{I} \right) [\mathcal{L}\boldsymbol{\Psi} - \boldsymbol{\Lambda_A}]$$

$$= \mathcal{L}^{-1}\Lambda_{\boldsymbol{A}} + \left[\mathcal{L} - \Xi_\theta \mathcal{F}^{-1}\mathcal{L}\right]^{-1} [\mathcal{L}\Psi - \Lambda_{\boldsymbol{A}}]$$

12. Old Wage Stuff

12.1. A Cobb-Douglas Special Case

Assuming Cobb-Douglas production technologies and preferences yields a simpler price adjustment equation

$$\left(\boldsymbol{I} - \gamma\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L} - (\gamma-1)\Xi_{\boldsymbol{p}}\mathcal{L}\right) d\log \boldsymbol{p} = -\left(\boldsymbol{I} + \Xi_{p}\mathcal{L}\right)\Psi d\log \boldsymbol{A} - \Xi_{\boldsymbol{p}}\left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log \boldsymbol{H}$$

Note when $\gamma = 0$, $\Xi_{\boldsymbol{w}} = \left(\boldsymbol{I} + \Xi_{\boldsymbol{p}}\mathcal{L}\right)$, and the expression above reduces to the Hulten example outlined above. Therefore, if $d\log \boldsymbol{H} = 0$ Hulten's theorem holds in a knife edge case of a much broader class of wage schedules. Violations in Hulten's theorem result whenever nominal wages respond to technology shocks and whenever $d\log \boldsymbol{H} \neq 0$, for instance because of an endogenous response of workers to wage and tightness changes.

12.2. Alternative Sub

$$\begin{aligned}\left(\boldsymbol{I} - \gamma\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right) d\log \boldsymbol{p} &= -\Psi\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathfrak{Q}\mathfrak{T} d\log\theta + d\log \boldsymbol{A}\right]\\ \Rightarrow d\log \boldsymbol{p} &= -\left(\boldsymbol{I} - \gamma\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right)^{-1}\Psi\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathfrak{Q}\mathfrak{T} d\log\theta + d\log \boldsymbol{A}\right]\end{aligned}$$

Tightness is now:

$$\begin{aligned}\left[\mathcal{F} - \Xi_\theta\right] d\log\theta = &\left[\mathcal{L}\Psi d\log \boldsymbol{A} - (1-\gamma)\mathcal{L}\left(\boldsymbol{I} - \gamma\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right)^{-1}\Psi\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathfrak{Q}\mathfrak{T} d\log\theta + d\log \boldsymbol{A}\right]\right]\\ &+ \left[\left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log \boldsymbol{H}\right]\\ d\log\theta = \Xi^{-1}&\left[\mathcal{L}\Psi d\log \boldsymbol{A} - (1-\gamma)\mathcal{L}\left(\boldsymbol{I} - \gamma\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right)^{-1}\Psi d\log \boldsymbol{A} + \left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log \boldsymbol{H}\right]\end{aligned}$$

where

$$\begin{aligned}\Xi &= \mathcal{F} - \Xi_\theta + (1-\gamma)\mathcal{L}\left(\boldsymbol{I} - \gamma\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right)^{-1}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathfrak{Q}\mathfrak{T}\\ &= \mathcal{F} - \mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left[\mathcal{F} + \mathfrak{Q}\mathfrak{T}\right] + (1-\gamma)\left(\boldsymbol{I} - \gamma\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right)^{-1}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathfrak{Q}\mathfrak{T}\end{aligned}$$

13. Beveridge Curve Formulation

In order to examine how the Beveridge curve responds to technology and labor supply shocks, we first need to write u and v in terms of primitives.

Unemployment at the end of the period for occupation o is:

$$U_o = (1 - \mathcal{F}(\theta_o))H_o.$$

Log-linearizing it yields:

$$\begin{aligned} d\log U_o &= d\log(1 - \mathcal{F}(\theta_o)) + d\log H_o \\ &= -\frac{\mathcal{F}(\theta_o)}{1 - \mathcal{F}(\theta_o)} \varepsilon_{\theta_o}^{\mathcal{F}_o} d\log\theta_o + d\log H_o \end{aligned}$$

Let $\boldsymbol{F} = \mathrm{diag}(\mathcal{F}(\theta_1), \mathcal{F}(\theta_2) \ldots \mathcal{F}(\theta_O))$, stacking the equations yields:

$$d\log \boldsymbol{U} = -\boldsymbol{F}\,(\boldsymbol{I} - \boldsymbol{F})^{-1}\, \boldsymbol{\mathcal{F}} d\log\theta + d\log \boldsymbol{H}$$

Alternatively, if we want to think about beginning period unemployment, we then have $U_o = H_o$, implying $d\log U_o = d\log H_o$.

Vacancy postings, on the other hand, satisfy:

$$V_o = \sum_{i=1}^{J} v_{io} = \frac{\sum_{i=1}^{J} N_{io}}{\mathcal{Q}_o(\theta_o) - r_o},$$

and log-linearizing yields:

$$\begin{aligned} d\log V_o &= d\log \sum_{i=1}^{J} N_{io} - d\log(\mathcal{Q}_o(\theta_o) - r_o) \\ &= \sum_{i=1}^{J} \frac{N_{io}}{\sum_{j=1}^{J} N_{jo}} d\log N_{io} - d\log(\mathcal{Q}_o(\theta_o) - r_o) \\ &= \sum_{i=1}^{J} \frac{N_{io}}{\sum_{j=1}^{J} N_{jo}} \left[d\log p_i + d\log y_i - d\log(1 + \tau_o(\theta_o)) - d\log w_o\right] - d\log(\mathcal{Q}_o(\theta_o) - r_o) \\ &= \sum_{i=1}^{J} \frac{L_{io}}{L_o} \left[d\log p_i + d\log y_i\right] - d\log(1 + \tau_o(\theta_o)) - d\log w_o - d\log(\mathcal{Q}_o(\theta_o) - r_o) \end{aligned}$$

$$= \sum_{i=1}^{J} \frac{L_{io}}{L_o} d\log y_i - \left(d\log w_o - \sum_{i=1}^{J} \frac{L_{io}}{L_o} d\log p_i \right) - d\log \mathcal{Q}_o(\theta_o)$$

$$= \sum_{i=1}^{J} \frac{L_{io}}{L_o} d\log y_i - \left(d\log w_o - \sum_{i=1}^{J} \frac{L_{io}}{L_o} d\log p_i \right) - \varepsilon_{\theta_o}^{\mathcal{Q}_o} d\log \theta_o$$

Stacking them yields:

$$d\log \boldsymbol{V} = \mathcal{L} d\log \boldsymbol{y} - (d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p}) - \boldsymbol{\mathcal{Q}} d\log \theta$$

14. Marginal product of labor

The marginal product of type o labor in sector i is

$$\frac{\partial y_i}{\partial N_{io}} = \begin{cases} \frac{\varepsilon_{N_{io}}^{f_i}}{N_{io}} y_i & \text{if } \varepsilon_{N_{io}}^{f_i} > 0 \\ 0 & \text{if } \varepsilon_{N_{io}}^{f_i} = 0 \end{cases} = \mathrm{mp}_{io}$$

So, assuming Cobb-Douglas production

$$d\log \mathrm{mp}_{io} = \begin{cases} d\log y_i - d\log N_{io} & \text{if } \varepsilon_{N_{io}}^{f_i} > 0 \\ 0 & \text{if } \varepsilon_{N_{io}}^{f_i} = 0 \end{cases}$$

Note, if $\varepsilon_{N_{io}}^{f_i} = 0$, then $\frac{l_{io}}{L_o} = 0$. So we can write,

$$\frac{l_{io}}{L_o} d\log \mathrm{mp}_{io} = \frac{l_{io}}{L_o} \left(d\log y_i - d\log N_{io} \right)$$

Substituting in for $d\log N_{io}$ from the first order condition of firms gives

$$\frac{l_{io}}{L_o} d\log \mathrm{mp}_{io} = \frac{l_{io}}{L_o} \left(d\log(1 + \tau_o(\theta_o)) + d\log w_o - d\log p_i \right)$$

$$= \frac{l_{io}}{L_o} \left(-\tau_o(\theta_o) \varepsilon_{\theta_o}^{Q_o} d\log \theta_o + d\log w_o - d\log p_i \right)$$

Then summing over sectors, and using $\sum_{i=1}^{J} \frac{l_{io}}{L_o} = 1$, we can write

$$\mathcal{L}_o' d\log \mathrm{MP}_o = -\tau_o(\theta_o) \varepsilon_{\theta_o}^{\mathcal{Q}_o} d\log \theta_o + d\log w_o - \mathcal{L}_o' d\log \boldsymbol{p}$$

Where

$$\mathcal{L}_o = \begin{bmatrix} \frac{l_{1o}}{L_o} \\ \frac{l_{2o}}{L_o} \\ \vdots \\ \frac{l_{Jo}}{\Lambda_o} \end{bmatrix}, \; d\log \mathrm{MP}_o = \begin{bmatrix} d\log \mathrm{mp}_{1o} \\ d\log \mathrm{mp}_{2o} \\ \vdots \\ d\log \mathrm{mp}_{Jo} \end{bmatrix}$$

Now, stacking over occupations gives

$$\mathcal{L} d\log \mathrm{MP} = -\mathfrak{T}\mathfrak{Q} d\log\theta + d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p}$$

Now, suppose that price adjusted wages respond exactly proportionally to the network adjusted marginal product of labor. Then

$$\begin{aligned} \mathcal{L} d\log \mathrm{MP} &= -\mathfrak{T}\mathfrak{Q} d\log\theta + \mathcal{L} d\log \mathrm{MP} \\ \Rightarrow -\mathfrak{T}\mathfrak{Q} d\log\theta &= 0 \\ \Rightarrow d\log\theta &= 0 \end{aligned}$$

Suppose instead that wages change proportionally to changes in the marginal product, so that $d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p} = \gamma d\log \mathrm{MP}$. Then,

$$d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p} = -\frac{\gamma}{1-\gamma}\mathfrak{Q}\mathfrak{T} d\log\theta$$

Prices now satisfy

$$\left(\boldsymbol{I} - \boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\mathcal{L}\right) d\log \boldsymbol{p} = -\boldsymbol{\Psi}\left[\frac{1}{1-\gamma}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\mathfrak{Q}\mathfrak{T} d\log\theta + d\log \boldsymbol{A}\right]$$

And changes in tightness satisfy,

$$d\log\theta = [\mathcal{F} - \Xi_{MP}]^{-1}\mathcal{L}\boldsymbol{\Psi} d\log \boldsymbol{A} + [\mathcal{F} - \Xi_{MP}]^{-1}\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} - \boldsymbol{I}\right] d\log \boldsymbol{H}$$

Where

$$\Xi_{MP} = \frac{\gamma}{1-\gamma}\mathfrak{Q}\mathfrak{T} + \mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}(\mathcal{F} + \mathfrak{Q}\mathfrak{T})$$

Which means we can write this scenario in the same terms as above,

$$d \log \boldsymbol{w} - \mathcal{L} d \log \boldsymbol{p} = \boldsymbol{\Lambda}_A d \log \boldsymbol{A} + \boldsymbol{\Lambda}_H d \log \boldsymbol{H}$$

Where

$$\begin{aligned} \boldsymbol{\Lambda}_A &= -\frac{\gamma}{1-\gamma} \mathfrak{Q} \mathfrak{T} \left[\mathcal{F} - \Xi_{MP}\right]^{-1} \boldsymbol{\mathcal{L}} \boldsymbol{\Psi} \\ \boldsymbol{\Lambda}_H &= -\frac{\gamma}{1-\gamma} \mathfrak{Q} \mathfrak{T} \left[\mathcal{F} - \Xi_{MP}\right]^{-1} \left[\boldsymbol{\mathcal{L}} \boldsymbol{\Psi} \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] \end{aligned}$$