Unemployment in a Production Network: Theory

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1. Households and final goods production

We consider a closed, static economy model with no government spending. There is no saving mechanism in the economy, and real household consumption equals real GDP, denoted by Y. A final goods producer with constant returns to scale technology aggregates J sector outputs to produce Y

$$Y = \max_{\{c_i\}_{i=1}^J} \mathcal{D}\left(\{c_i\}_{i=1}^J\right)$$

Subject to the budget constraint

$$\sum_{i=1}^{J} p_i c_i = \sum_{i=1}^{J} w_i L_i^s(\theta_i).$$

$\mathcal{D}$ captures household preferences over final consumption goods, and w_i is the wage of labor supplied to sector i, $L_i^s(\theta_i)$.

The household's consumption decision can be computed using the first order condition:

$$\varepsilon_{c_i}^{\mathcal{D}} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}, \tag{1}$$

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where $\varepsilon^{\mathcal{D}}_{c_i}$ denotes the households elasticity of utility with regards to the consumption of good i.

2. Sector level labor markets

We assume each sector has a separate labor market with a labor force of H_i possible workers, who all start out unemployed, and an exogenous recruiting cost r_i, which measures the units of labor required to maintain each posted vacancy. When workers and firms meet there is a mutual gain from matching. There is no accepted theory for how wages are set in this context. For now we assume w_i follows a general wage schedule taken as given by both firms and workers. Hires are generated by a constant returns matching function in sector-level unemployment u_i and vacancies v_i

$$h_i = \phi_i m\left(u_i, v_i\right)$$

The household supplies H_i searching workers to each sector. Let the sector-specific labor market tightness be $\theta_i = \frac{v_i}{H_i}$, the vacancy-filling rate $\mathcal{Q}_i(\theta_i) = \phi_i m\left(\frac{u_i}{H_i}, 1\right)$, and the job-finding rate $\mathcal{F}_i(\theta_i) = \phi_i m\left(1, \frac{v_i}{H_i}\right)$. Therefore, a fraction $\mathcal{F}_i(\theta_i)$ of H_i household finds a job, and labor supply satisfies

$$L^s_i(\theta_i) = \mathcal{F}_i(\theta_i) H_i \tag{2}$$

Let N_i denote productive employees and r_i the cost of each vacancy. In order to hire N_i productive employees, the number of vacancies posted v_i has to satisfy $\mathcal{Q}_i(\theta_i)v_i = N_i + r_i v_i$, where $r_i v_i$ denotes the cost of posting the vacancies. Rearranging yields $v_i = \frac{N_i}{\mathcal{Q}_i(\theta_i) - r_i}$. Thus, hiring one unit of productive labor requires $\frac{1}{\mathcal{Q}_i(\theta_i) - r_i}$ vacancy postings, and requires $1 + \tau_i(\theta_i)$ units of total labor, where

$$\tau_i(\theta_i) \equiv \frac{r_i}{\mathcal{Q}_i(\theta_i) - r_i}.$$

For a given target level of employment N_i, total required labor, or the labor demand, is $L^d_i(\theta_i) = \left(1 + \tau_i(\theta_i)\right) N_i$. We describe how labor demand, $L^d_i(\theta_i)$, is determined by firms' profit maximization in the next section.

3. Sector level firms

A representative firm in sector i uses labor N_i and intermediate inputs from sector j, x_{ij}, to produce output y_i using constant returns production technology f_i.

$$y_i = A_i f_i \left(N_i, \{x_{ij}\}_{j=1}^{J}\right)$$

Firms choose N_i and $\{x_{ij}\}_{j=1}^{J}$ to maximize profits, or equivalently to minimize costs. We assume firms are price takers in both input and output markets. Profits are given by

$$\pi_i = p_i f_i \left(N_i, \{x_{ij}\}_{j=1}^{J}\right) - w_i(1+\tau_i(\theta_i))N_i - \sum_{j=1}^{J} p_j x_{ij}$$

Firms choose inputs to solve

$$\max_{N_i, \left\{x_{ij}\right\}_{j=1}^{J}} \pi_i \left(N_i, \left\{x_{ij}\right\}_{j=1}^{J}\right)$$

Giving the first order conditions

$$\begin{aligned} p_i f_{i,x_{ij}} &= p_j \\ p_i f_{i,N_i} &= w_i \left(1+\tau_i(\theta_i)\right) \end{aligned}$$

Labor demand is $L_i^d(\theta_i) = (1+\tau_i(\theta_i))N_i$ for the optimal N_i. The equilibrium tightness equates labor demand and labor supply.

We can rewrite these expressions in terms of elasticities.

$$\varepsilon_{x_{ij}}^{f_i} = \frac{p_j x_{ij}}{p_i y_i} \tag{3}$$

$$\varepsilon_{N_i}^{f_i} = \left(1+\tau_i(\theta_i)\right) \frac{w_i N_i}{p_i y_i} \tag{4}$$

From Equation 4, we can derive the labor demand equation:

$$L_i^d(\theta_i) = \varepsilon_{N_i}^{f_i} \frac{p_i y_i}{w_i} \tag{5}$$

4. Equilibrium

The equilibrium in this model can be characterized by a set of conditions guaranteeing labor market equilibrium and goods market equilibrium. The equilibrium is a collection of $10J + 2J^2$ endogenous variables $\left\{ p_i, y_i, \left\{x_{ij}, \varepsilon^{f_i}_{x_{ij}}\right\}_{j=1}^{J}, c_i, \varepsilon^{\mathcal{D}}_{c_i}, N_i, \varepsilon^{f_i}_{N_i}, \theta_i, w_i, L^d_i, L^s_i \right\}_{i=1}^{J}$ that satisfy equations 1 through 5, along with goods market clearing, labor market clearing, and constant returns, given exogenous variables $\{A_i, H_i\}_{i=1}^{J}$. We summarize the equilibrium conditions below for convenience.

4.1. Goods Market Equilibrium

In an equilibrium, firms intermediate input choices given prices and labor market chacteristics are profit maximizing:

(Intermediate input decision) $$\varepsilon^{f_i}_{x_{ij}} = \frac{p_j x_{ij}}{p_i y_i},$$

(Labor input decision) $$\varepsilon^{f_i}_{N_i} = \left(1 + \tau_i(\theta_i)\right) \frac{w_i N_i}{p_i y_i}.$$

Firms produce output via production technology f_i

(Production technology) $$y_i = A_i f_i \left(N_i, \left\{x_{ij}\right\}_{j=1}^{J} \right)$$

By constant returns to scale in production,

(Constant returns production) $$1 - \varepsilon^{f_i}_{N_i} = \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}}$$

In addition, the household maximizes their utility by choosing a consumption bundle that satisfies its first-order condition.

(Consumption decision) $$\varepsilon^{\mathcal{D}}_{c_i} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}.$$

And by constant returns

(Constant returns utility) $$1 = \sum_{i=1}^{J} \varepsilon^{\mathcal{D}}_{c_i}$$

Finally, the goods market has to clear, which means that, for each sector i, total production has to be equal to the sum of the household's consumption of good i and all other sectors' use of good i in their production:

(Goods market clearing) $$y_i = c_i + \sum_{j=1}^{J} x_{ji}.$$

In total, the goods market provides $5J + J^2 + 1$ restrictions.

4.2. Labor Market Equilibrium

From Equation 4, labor demand in sector i is defined as

(Labor Demand) $$L_i^d(\theta_i) = \varepsilon_{N_i}^{f_i} \frac{p_i y_i}{w_i}.$$

Given sector level labor force participation H_i labor supply is

(Labor Supply) $$L_i^s(\theta_i) = \mathcal{F}_i(\theta_i) H_i.$$

Labor demand equals labor supply at an equilibrium in the labor market.

(LM equilibrium) $$L_i^d(\theta_i) = L_i^s(\theta_i).$$

These equilibrium conditions provide an additional $3J$ restrictions.

4.3. Summary

The equilibrium conditions outline above provide just $8J + J^2 + 1$ equations in $10J + 2J^2$ endogenous variables. The wage schedules taken as given by both households and firms provide another J restrictions. Nevertheless, as is typical in the literature, we need additional functional form assumptions on production and household preferences to close the model.

For instance, assuming Cobb-Douglas production and preferences fully parameterizes $\left\{ \left\{ \varepsilon_{x_{ij}}^{f_i} \right\}_{j=1}^{J}, \varepsilon_{N_i}^{f_i}, \varepsilon_{c_i}^{\mathcal{D}} \right\}_{i=1}^{J}$, giving us $2J + J^2$ additional restrictions, but removing $J + 1$ of the restrictions above.[1] Parametrizing production by assuming Cobb-Douglas there-

[1] Choosing elasticities directly makes the constant returns restrictions redundant.

fore gives exactly the number of restrictions we need to close the model. Alternatively, assuming CES production indirectly provides restrictions to pin down the same set of elasticities.[2]

5. The Production Network

In this section, we introduce key production network notation. We then discuss three propagation mechanisms: prices, sales shares, and tightness. For now, we proceed assuming a general production function. In subsequent applications we will parameterize the production functions to allow us to calibrate and estimate the model.

5.1. Notation

We denote vectors and matrices by bold letters. For instance, $d\log \boldsymbol{x} = \begin{bmatrix} d\log x_1 & \cdots & d\log x_J \end{bmatrix}'$. We can conveniently capture many features of the production network through the following matrices

$$\boldsymbol{\Omega} = \begin{bmatrix} \varepsilon^{f_1}_{x_{11}} & \varepsilon^{f_1}_{x_{12}} & \cdots & \varepsilon^{f_1}_{x_{1J}} \\ \varepsilon^{f_2}_{x_{21}} & \varepsilon^{f_2}_{x_{22}} & \cdots & \varepsilon^{f_2}_{x_{2J}} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon^{f_J}_{x_{J1}} & \varepsilon^{f_J}_{x_{J2}} & \cdots & \varepsilon^{f_J}_{x_{JJ}} \end{bmatrix}, \ \boldsymbol{\Psi} = (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}.$$

In the standard production networks vocabulary, $\boldsymbol{\Omega}$ is the sales based input-output matrix and $\boldsymbol{\Psi}$ is the sales based Leontief inverse.[3]

In addition define

$$\boldsymbol{\varepsilon}^{\boldsymbol{\mathcal{D}}}_{\boldsymbol{c}} = \begin{bmatrix} \varepsilon^{\mathcal{D}}_{c_1} \\ \varepsilon^{\mathcal{D}}_{c_2} \\ \vdots \\ \varepsilon^{\mathcal{D}}_{c_J} \end{bmatrix}, \ \boldsymbol{\varepsilon}^{\boldsymbol{f}}_{\boldsymbol{N}} = \begin{bmatrix} \varepsilon^{f_1}_{N_1} \\ \varepsilon^{f_2}_{N_2} \\ \vdots \\ \varepsilon^{f_J}_{N_J} \end{bmatrix}, \ \boldsymbol{\varepsilon}^{\boldsymbol{Q}}_{\boldsymbol{\theta}} = \begin{bmatrix} \varepsilon^{q_1}_{\theta_1} \\ \varepsilon^{q_2}_{\theta_2} \\ \vdots \\ \varepsilon^{q_J}_{\theta_J} \end{bmatrix}, \ \boldsymbol{\tau} = \begin{bmatrix} \tau_1(\theta_1) \\ \tau_2(\theta_2) \\ \vdots \\ \tau_J(\theta_J) \end{bmatrix}$$

[2] CES and Cobb-Douglas, a special case of CES, are the two most commonly assumed production technologies in the literature. In principle, assuming any production technology will do. An explicit functional form will allow us to characterize the required elasticities.

[3] Since our model abstracts from markup wedges on prices, the sales based and cost based input-output matrices will coincide.

5.2. Price Propagation

Log-linearizing the production function, for each sector i, we have:

$$d\log y_i = \underbrace{\varepsilon^{f_i}_{A_i}}_{=1} d\log A_i + \varepsilon^{f_i}_{N_i} d\log N_i + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} d\log x_{ij}$$

Plugging in Equation 4 and Equation 3, the first order conditions for optimal input usage, into the log-linearized production function gives

$$\begin{aligned}
d\log y_i &= \varepsilon^{f_i}_{N_i}\left[d\log\varepsilon^{f_i}_{N_i} + d\log y_i + d\log p_i - d\log w_i - d\log\left(1+\tau_i(\theta_i)\right)\right] \\
&\quad + \sum_{j=1}^{N}\varepsilon^{f_i}_{x_{ij}}\left[d\log\varepsilon^{f_i}_{x_{ij}} + d\log y_i + d\log p_i - d\log p_j\right] + d\log A_i \\
&= \left[d\log y_i + d\log p_i\right]\underbrace{\left[\varepsilon^{f_i}_{N_i} + \sum_{j=1}^{N}\varepsilon^{f_i}_{x_{ij}}\right]}_{=1 \text{ by crts}} + \underbrace{\left[d\varepsilon^{f_i}_{N_i} + \sum_{j=1}^{N} d\varepsilon^{f_i}_{x_{ij}}\right]}_{=0 \text{ by crts}} \\
&\quad - \varepsilon^{f_i}_{N_i}\left[d\log w_i + d\log(1+\tau_i(\theta_i))\right] - \sum_{j=1}^{N}\varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] + d\log A_i,
\end{aligned}$$

where the second equality holds because the sum of elasticities equals one for constant returns to scale technology and $\varepsilon^{f_i}_{x_{ij}} d\log\varepsilon^{f_i}_{x_{ij}} = d\varepsilon^{f_i}_{x_{ij}}$.

Rearranging terms gives

$$\begin{aligned}
d\log p_i &= \varepsilon^{f_i}_{N_i}\left[d\log w_i + d\log(1+\tau_i(\theta_i))\right] + \sum_{j=1}^{N}\varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i \\
&= \varepsilon^{f_i}_{N_i}\left[d\log w_i + \varepsilon^{1+\tau_i}_{\theta_i} d\log\theta_i\right] + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i \\
&= \varepsilon^{f_i}_{N_i}\left[d\log w_i - \tau_i(\theta_i)\varepsilon^{Q_i}_{\theta_i} d\log\theta_i\right] + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i \qquad (6)
\end{aligned}$$

By stacking equation (6) for each sector, we get the following expression for how prices change across the production network

$$d \log \boldsymbol{p} = \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\left[d \log \boldsymbol{w} - \operatorname{diag}(\boldsymbol{\tau}) \operatorname{diag}\left(\varepsilon^{\Omega}_{\theta}\right) d \log \theta\right] + \boldsymbol{\Omega} d \log \boldsymbol{p} - d \log \boldsymbol{A}$$

(7)

$$\Rightarrow d \log \boldsymbol{p} = \boldsymbol{\Psi}(\underbrace{\operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) d \log \boldsymbol{w}}_{\text{factor prices}} - \underbrace{\operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) \operatorname{diag}(\boldsymbol{\tau}) \operatorname{diag}\left(\varepsilon^{\Omega}_{\theta}\right) d \log \theta}_{\text{searching and matching}} - \underbrace{d \log \boldsymbol{A}}_{\text{productivity}})$$

In other words, price changes come from three sources - changes in wages, changes in tightness, and changes in productivity. The impact of changes in wages depends on labor elasticity of production, and the impact of changes in tightness depends additionally on the matching function and the recruiter-producer ratio. The impact of all three is mediated by the Leontief inverse $\boldsymbol{\Psi}$.

5.3. Sales Share Propgation

We can rewrite the goods market clearing condition in terms of Domar weights:

$$y_i = c_i + \sum_{j=1}^{J} x_{ji}$$

$$\Rightarrow \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k} + \sum_{j=1}^{J} \frac{p_i x_{ji}}{p_j x_j} \frac{p_j x_j}{\sum_{k=1}^{J} p_k c_k}$$

(8)

$$\Rightarrow \lambda_i = \varepsilon^{\mathcal{D}}_{c_i} + \sum_{j=1}^{J} \varepsilon^{f_j}_{x_{ji}} \lambda_j,$$

where $\lambda_i = \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k}$ is the Domar weight of sector i.

By stacking (8) for each sector, we get the following expression for Domar weights across the production network.

$$\boldsymbol{\lambda}' = \varepsilon^{\mathcal{D}'}_{\boldsymbol{c}} + \boldsymbol{\lambda}' \boldsymbol{\Omega}$$

We can see how Domar weights change across the production network by totally differentiating

$$d\boldsymbol{\lambda}' = d\varepsilon^{\mathcal{D}'}_{\boldsymbol{c}} + d\boldsymbol{\lambda}' \boldsymbol{\Omega} + \boldsymbol{\lambda}' d\boldsymbol{\Omega}$$

$$\Rightarrow d\boldsymbol{\lambda}' = \left[d\varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} + \boldsymbol{\lambda}' d\boldsymbol{\Omega}\right] \boldsymbol{\Psi} \tag{9}$$

The Domar weights will help us express how shocks propagate to output.

5.4. Output Propagation

Log-linearizing Domar weights gives us:

$$d\log \lambda_i = d\log\, p_i + d\log\, y_i - d\log \sum_{k=1}^{J} p_k c_k.$$

$$\begin{aligned} d\lambda_i &= \lambda_i d\log\, p_i + \lambda_i d\log\, y_i - \lambda_i d\log \sum_{k=1}^{J} w_k L_k^s \\ &= \lambda_i d\log\, p_i + \lambda_i d\log\, y_i + \; p_i y_i d\frac{1}{\sum_{k=1}^{J} w_k L_k^s} \\ &= \lambda_i d\log\, p_i + \lambda_i d\log\, y_i + \; p_i y_i \left[-\sum_{j=1}^{J} \frac{L_j^s}{\left(\sum_{k=1}^{J} w_k L_k^s\right)^2} dw_j - \sum_{j=1}^{J} \frac{w_k}{\left(\sum_{k=1}^{J} w_k L_k^s\right)^2} dL_j^s \right] \end{aligned}$$

Let $\upsilon_i = \frac{w_i L_i}{\sum_{k=1}^{J} w_k L_k}$ denote the income share of sector i in total labor income. Then, we can write

$$d\lambda_i = \lambda_i d\log\, p_i + \lambda_i d\log\, y_i - \lambda_i \sum_{j=1}^{J} \upsilon_j \left[d\log w_j + d\log L_j^s\right]$$

Which implies,

$$d\log\, y_i = \frac{d\lambda_i}{\lambda_i} - d\log\, p_i + \sum_{j=1}^{J} \upsilon_j \left[d\log w_j + d\log L_j^s\right]$$

Stacking equations over sectors, gives

$$d\log\, \boldsymbol{y} = \mathrm{diag}\,(\lambda)^{-1}\, d\boldsymbol{\lambda} - d\log\, \boldsymbol{p} + \boldsymbol{\Upsilon} \left[d\log \boldsymbol{w} + d\log \boldsymbol{L}^s\right]$$

Since the log-linearized expression for the Domar weight must hold for every sector, we can also write

$$
\begin{aligned}
d\log\lambda_i - d\log\lambda_j &= d\log p_i - d\log p_j + d\log y_i - d\log y_j \\
&= d\log x_{ij} - d\log\varepsilon^{f_i}_{x_{ij}} - d\log y_j \\
\Rightarrow d\log x_{ij} &= d\log\lambda_i - d\log\lambda_j + d\log y_j + d\log\varepsilon^{f_i}_{x_{ij}}
\end{aligned}
$$

Plugging back into the production function,

$$
\begin{aligned}
d\log y_i &= d\log A_i + \varepsilon^{f_i}_{N_i} d\log N_i + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left(d\log\varepsilon^{f_i}_{x_{ij}} + d\log y_j + d\log\lambda_i - d\log\lambda_j\right) \\
&= d\log A_i + \varepsilon^{f_i}_{N_i} d\log N_i - \varepsilon^{f_i}_{N_i} d\log\varepsilon^{f_i}_{N_i} + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left(d\log y_j + d\log\lambda_i - d\log\lambda_j\right)
\end{aligned}
$$

Using the definition of labor demand,

$$
\begin{aligned}
d\log N_i &= d\log L^d_i - d\log(1+\tau_i(\theta_i)) \\
&= d\log L^d_i + \tau_i(\theta_i)\varepsilon^{\mathcal{Q}_i}_{\theta_i} d\log\theta_i
\end{aligned}
$$

Using the labor market clearing condition, and the definition of labor supply,

$$
d\log N_i = \left(\varepsilon^{\mathcal{F}_i}_{\theta_i} + \tau_i(\theta_i)\varepsilon^{\mathcal{Q}_i}_{\theta_i}\right) d\log\theta_i + d\log H_i
$$

Which means

$$
\begin{aligned}
d\log y_i = {} & d\log A_i + \varepsilon^{f_i}_{N_i}\left(\varepsilon^{\mathcal{F}_i}_{\theta_i} + \tau_i(\theta_i)\varepsilon^{\mathcal{Q}_i}_{\theta_i}\right) d\log\theta_i + \varepsilon^{f_i}_{N_i} d\log H_i - \varepsilon^{f_i}_{N_i} d\log\varepsilon^{f_i}_{N_i} \\
& + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left(d\log y_j + d\log\lambda_i - d\log\lambda_j\right)
\end{aligned}
$$

Stacking over sectors gives,

$$
\begin{aligned}
d\log\boldsymbol{y} = {} & d\log\boldsymbol{A} + \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\left(\operatorname{diag}(\boldsymbol{\mathcal{F}}) + \operatorname{diag}(\boldsymbol{\tau})\operatorname{diag}\left(\varepsilon^{\boldsymbol{\mathcal{Q}}}_{\theta}\right)\right) d\log\theta + \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) d\log\boldsymbol{H} \\
& - \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) d\log\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} + \boldsymbol{\Omega} d\log\boldsymbol{y} + \left(\boldsymbol{I} - \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda}
\end{aligned}
$$

Which implies

$$d\log \boldsymbol{y} = \boldsymbol{\Psi}\left(d\log \boldsymbol{A} + \operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right)\left(\operatorname{diag}(\mathcal{F}) + \operatorname{diag}(\boldsymbol{\tau})\operatorname{diag}\left(\varepsilon_{\theta}^{\Omega}\right)\right) d\log\theta + \operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) d\log \boldsymbol{H}\right)$$
$$- \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) d\log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} + \boldsymbol{\Psi}\left(\boldsymbol{I} - \operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda}$$

5.5. Tightness Propagation

In response to shocks to either productivity or labor force, changes in labor demand has to equate changes in labor supply:

$$d\log L_i^s(\theta, \boldsymbol{H}) = d\log L_i^d(\theta, \boldsymbol{A}).$$

Let $\mathcal{F} = \begin{bmatrix}\varepsilon_{\theta_1}^{\mathcal{F}_1} & \cdots & \varepsilon_{\theta_J}^{\mathcal{F}_J}\end{bmatrix}$, we can stack sector level Equation 2 and Equation 5 to get:

$$\begin{aligned}
\operatorname{diag}(\mathcal{F})\, d\log\theta + d\log\boldsymbol{H} &= d\log\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}(\theta, \boldsymbol{A}) + d\log\,\boldsymbol{p}(\theta,\boldsymbol{A}) - d\log\boldsymbol{w}(\theta,\boldsymbol{A}) + d\log\,\boldsymbol{y}(\theta,\boldsymbol{A}) \\
&= d\log\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - d\log\boldsymbol{w} \\
&\quad + \boldsymbol{\Psi} d\log\boldsymbol{A} + \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right)\left(\operatorname{diag}(\mathcal{F}) + \operatorname{diag}(\boldsymbol{\tau})\operatorname{diag}\left(\varepsilon_{\theta}^{\Omega}\right)\right) d\log\theta \\
&\quad + \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) d\log\boldsymbol{H} - \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) d\log\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \\
&\quad + \boldsymbol{\Psi}\left(\boldsymbol{I} - \operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} + \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) d\log\boldsymbol{w} \\
&\quad - \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right)\operatorname{diag}(\boldsymbol{\tau})\operatorname{diag}\left(\varepsilon_{\theta}^{\Omega}\right) d\log\theta - \boldsymbol{\Psi} d\log\boldsymbol{A} \\
&= \left(\boldsymbol{I} - \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right)\right)\left(d\log\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - d\log\boldsymbol{w}\right) \\
&\quad + \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right)\left(\operatorname{diag}(\mathcal{F})\, d\log\theta + d\log\boldsymbol{H}\right) \\
&\quad + \boldsymbol{\Psi}\left(\boldsymbol{I} - \operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda}
\end{aligned}$$

Which implies

(10)
$$\left(\boldsymbol{I} - \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right)\right)\operatorname{diag}(\mathcal{F})\, d\log\theta = \left(\boldsymbol{I} - \boldsymbol{\Psi}\operatorname{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right)\right)\left(d\log\varepsilon_{\boldsymbol{n}}^{\boldsymbol{f}} - d\log\boldsymbol{w} + d\log\boldsymbol{H} + d\log\boldsymbol{\lambda}\right)$$

Clearly, one solution to this system of equations is

$$d\log\theta = \operatorname{diag}(\mathcal{F})^{-1}\left(d\log\varepsilon_{\boldsymbol{n}}^{\boldsymbol{f}} - d\log\boldsymbol{w} + d\log\boldsymbol{H} + d\log\boldsymbol{\lambda}\right) \tag{11}$$

But, it is not the only possible solution since $\left(\boldsymbol{I} - \boldsymbol{\Psi}\text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\right)$ is not invertible since

$$\left(\boldsymbol{I} - \boldsymbol{\Psi}\text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\right) = \boldsymbol{\Psi}\left(\boldsymbol{I} - \boldsymbol{\Omega} - \text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\right)$$

By constant returns production, the columns of the matrix $\left(\boldsymbol{I} - \boldsymbol{\Omega} - \text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\right)$ sum to zero. As a result, letting $\mathbf{1}$ be a column vector of ones, we can write

$$\left(\boldsymbol{I} - \boldsymbol{\Omega} - \text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\right)\mathbf{1} = \mu\mathbf{1}$$

For $\mu = 0$. In words, 0 is a right eigenvalue of $\left(\boldsymbol{I} - \boldsymbol{\Omega} - \text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\right)$. Nevertheless, (10) demonstrates that in equilibrium changes in tightness are determined by a linear map from changes in labor elasticities, wages, labor force participation, and sales shares. In particular, if there exists a linear map $\mathcal{M}$ satisfying $\left(\boldsymbol{I} - \boldsymbol{\Psi}\text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\right)\mathcal{M} = \boldsymbol{I}$, then (11) is the solution to (10) for $\mathcal{M}$.

5.5.1. Alternative Approach

By definition, $\theta_i = \frac{v_i}{H_i}$. Log-linearizing yields $d\log\theta_i = d\log v_i - d\log H_i$.

Since $v_i = \frac{N_i}{\mathcal{Q}_i(\theta_i)-r_i}$, we have

$$\begin{aligned} d\log v_i = d\log N_i - d\log(\mathcal{Q}_i(\theta_i) - r_i) &= d\log N_i - \frac{\mathcal{Q}_i(\theta_i)}{\mathcal{Q}_i(\theta_i) - r_i} d\log \mathcal{Q}_i(\theta_i) \\ &= d\log N_i - \frac{\mathcal{Q}_i(\theta_i)}{\mathcal{Q}_i(\theta_i) - r_i}\varepsilon^{\mathcal{Q}_i}_{\theta_i} d\log\theta \\ &= d\log N_i + (1 - \tau_i(\theta_i))\varepsilon^{\mathcal{Q}_i}_{\theta_i} d\log\theta \end{aligned}$$

6. Sector-level Response

Before aggregating, we are interested in exploring how sector-level economic variables respond to different shocks.

6.1. Output

First, we look at output. Log-linearizing Domar weights gives us:

$$d\log\lambda_i = d\log p_i + d\log y_i - d\log\sum_{k=1}^{J} p_k c_k.$$

Since this equation must hold for any i and j,

$$\begin{aligned} d\log\lambda_i - d\log\lambda_j &= d\log p_i - d\log p_j + d\log y_i - d\log y_j \\ &= d\log x_{ij} - d\log y_j \\ \Rightarrow d\log x_{ij} &= d\log y_j + d\log\lambda_i - d\log\lambda_j \end{aligned}$$

Using the sector level production function,

$$d\log y_i = d\log A_i + \varepsilon^{f_i}_{N_i} d\log N_i + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left(d\log y_j + d\log\lambda_i - d\log\lambda_j\right)$$

Since $(1+\tau_i(\theta_i))N_i = L_i$, $\delta\log N_i = d\log L_i - d\log(1+\tau_i(\theta_i))$, where $d\log L_i = d\log L_i^s = d\log L_i^d$. This implies that $d\log N_i = \left(\frac{s_i}{s_i+\mathcal{F}_i(\theta_i)}\varepsilon^{\mathcal{F}_i}_{\theta_i} + \varepsilon^{\Omega_i}_{\theta_i}\tau_i(\theta_i)\right) d\log\theta_i + d\log H_i$.

The sector-level log-linearized production function can be rewritten as:

$$\begin{aligned} d\log y_i = d\log A_i &+ \varepsilon^{f_i}_{N_i}\left[\left(\frac{s_i}{s_i+\mathcal{F}_i(\theta_i)}\varepsilon^{\mathcal{F}_i}_{\theta_i} + \varepsilon^{\Omega_i}_{\theta_i}\tau_i(\theta_i)\right) d\log\theta_i + d\log H_i\right] \\ &+ (1-\varepsilon^{f_i}_{N_i}) d\log\lambda_i + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left(d\log y_j - d\log\lambda_j\right) \end{aligned}$$

Stacking equations over sectors, we have:

$$\begin{aligned} d\log \boldsymbol{y} = d\log\boldsymbol{A} + \boldsymbol{\Omega} d\log\boldsymbol{y} &+ \left(\boldsymbol{I} - \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} \\ &+ \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\left[\left(\operatorname{diag}(\mathcal{F}) + \operatorname{diag}(\boldsymbol{\tau})\operatorname{diag}\left(\varepsilon^{\Omega}_{\theta}\right)\right) d\log\theta + d\log\boldsymbol{H}\right], \end{aligned}$$

which simplifies into:

$$\begin{aligned} d\log\boldsymbol{y} = \boldsymbol{\Psi}(d\log\boldsymbol{A} &+ \left(\boldsymbol{I} - \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} \\ &+ \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\left[\left(\operatorname{diag}(\mathcal{F}) + \operatorname{diag}(\boldsymbol{\tau})\operatorname{diag}\left(\varepsilon^{\Omega}_{\theta}\right)\right) d\log\theta + d\log\boldsymbol{H}\right]). \end{aligned}$$

In general, sector level output behaves differently from the Cobb-Douglas case, but that difference is captured entirely by changes in Domar weights. This is a useful result, as discussed in the previous production networks literature, because this means we do not need to keep track of all intermediate input choices.

6.2. Unemployment

7. Aggregation

7.1. General Case

Using the first order condition,

$$d\log \varepsilon_{c_i}^{\mathcal{D}} = d\log\, p_i + d\log c_i - d\log \sum_{j=1}^{J} p_k c_k$$

along with the definition of the Domar weight,

$$d\log \sum_{k=1}^{J} p_k c_k = d\log\, p_i + d\log\, y_i - d\log \lambda_i$$

gives

$$d\log c_i = d\log \varepsilon_{c_i}^{\mathcal{D}} + d\log\, y_i - d\log \lambda_i$$

Which implies the log change in real GDP is

$$\begin{aligned} d\log Y &= \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} d\log \boldsymbol{c} \\ &= \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} \left(d\log \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}} + d\log\, \boldsymbol{y} - d\log \boldsymbol{\lambda} \right) \\ &= \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} \left(d\log \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}} - d\log\, \boldsymbol{p} + d\log \boldsymbol{\lambda} + \Xi_{\varepsilon} d\log \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} - d\log \boldsymbol{\lambda} \right) \\ &= -\boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} d\log\, \boldsymbol{p} + \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} \left(d\log \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}} + \Xi_{\varepsilon} d\log \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} \right) \end{aligned}$$

7.2. Aggregate Output

7.3. Aggregate employment

References