Unemployment in a Production Network: Adding multiple occupations

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1. Households and final goods production

We consider a closed, static economy model with no government spending. There is no saving mechanism in the economy, and real household consumption equals real GDP, denoted by Y. A final goods producer with constant returns to scale technology aggregates J sector outputs to produce Y, the final consumption good

$$Y = \max_{\{c_i\}_{i=1}^J} \mathcal{D}\left(\{c_i\}_{i=1}^J\right)$$

Subject to the budget constraint

$$\sum_{i=1}^{J} p_i c_i = \sum_{i=1}^{J} w_o L_o^s(\theta_o).$$

$\mathcal{D}$ captures household preferences over final consumption goods, and w_o is the wage of labor supplied to occupation o, $L_o^s(\theta_o)$.

The household's consumption decision can be computed using the first order condition:

$$\varepsilon_{c_i}^{\mathcal{D}} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}, \tag{1}$$

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where $\varepsilon^{\mathcal{D}}_{c_i}$ denotes the households elasticity of utility with regards to the consumption of good i.

2. Occupation level labor markets

We assume there are $\mathcal{O}$ occupations with separate labor markets, a labor force of H_o possible workers, who all start out unemployed, and an exogenous recruiting cost r_o, which measures the units of labor required for a firm to maintain each posted vacancy in occupation o. When workers and firms meet there is a mutual gain from matching. There is no accepted theory for how wages are set in this context. For now we assume the nominal wage in occupation o, w_o, follows a general wage schedule that depends on productivity and the size of the labor force, and is taken as given by both firms and workers. Hires are generated by a constant returns matching function in occupation-level unemployment U_o and aggregate vacancies V_o, which measure all vacancy postings for occupation o,

$$h_o = \phi_o m\left(U_o, V_o\right)$$

The household supplies H_o searching workers to each sector. Let the sector-specific labor market tightness be $\theta_o = \frac{V_o}{H_o}$, the vacancy-filling rate $\mathcal{Q}_o(\theta_o) = \phi_o m\left(\frac{H_o}{V_o}, 1\right)$, and the job-finding rate $\mathcal{F}_o(\theta_o) = \phi_o m\left(1, \frac{v_o}{H_o}\right)$. Therefore, a fraction $\mathcal{F}_o(\theta_o)$ of H_o household finds a job, and labor supply satisfies

$$L^o_i(\theta_o) = \mathcal{F}_o(\theta_o)H_o \tag{2}$$

We assume firms take the occupation level tightness as given. One way of justifying this is with the assumption that each sector is populated by many identical competitive firms so that each firm only has an infinitesimal impact on aggregate vacancies, and therefore on aggregate tightness. Let N_{io} denote productive employees in occupation o working for sector i firms and let r_o be the cost of each vacancy for a firm. In order to hire N_{io} productive employees, the number of vacancies posted v_{io} has to satisfy $\mathcal{Q}_o(\theta_o)v_{io} = N_{io}+r_o v_{io}$, where $r_o v_{io}$ denotes the cost of posting the vacancies. Rearranging yields $v_{io} = \frac{N_{io}}{\mathcal{Q}_o(\theta_o)-r_o}$. Thus, hiring one unit of productive labor requires $\frac{1}{\mathcal{Q}_o(\theta_o)-r_o}$ vacancy postings, and requires $1+\tau_o(\theta_o)$ units of total labor, where

$$\tau_o(\theta_o) \equiv \frac{r_o}{\mathcal{Q}_o(\theta_o) - r_o}.$$

For a given target level of occupation o employment N_{io}, total required labor, or the labor demand, is $l^d_{io}(\theta_o) = (1 + \tau_o(\theta_o))\, N_{io}$. We describe how labor demand, $l^d_{io}(\theta_{io})$, is determined by firms' profit maximization in the next section.

Finally, we define aggregate occupation o labor demand as the sum of sectoral labor demands and aggregate vacancy postings as the sum of sectoral vacancy postings.

$$L^d_o(\theta_o) = \sum_{i=1}^{J} l^d_{io}(\theta_o)$$

$$V_o = \sum_{i=1}^{J} v_{io}$$

Market clearing in the labor market requires labor demand equal labor supply and that the vacancy posting choices of firms in each sector are consistent with aggregate tightness.

$$L^d_o = L^s_o$$

$$\theta_o = \frac{\sum_{i=1}^{J} v_{io}}{H_o}$$

3. Sector level firms

A representative firm in sector i uses workers in occupation o N_{io} and intermediate inputs from sector j, x_{ij}, to produce output y_i using constant returns production technology f_i.

$$y_i = A_i f_i\left(\left\{N_{io}\right\}_{o=1}^{\mathcal{O}}, \{x_{ij}\}_{j=1}^{J}\right)$$

Firms choose $\left\{N_{io}\right\}_{o=1}^{\mathcal{O}}$ and $\left\{x_{ij}\right\}_{j=1}^{J}$ to maximize profits, or equivalently to minimize costs. We assume firms are price takers in both input and output markets. Profits are given by

$$\pi_i = p_i f_i\left(\left\{N_{io}\right\}_{o=1}^{\mathcal{O}}, \{x_{ij}\}_{j=1}^{J}\right) - \sum_{o=1}^{\mathcal{O}} w_o(1 + \tau_o(\theta_o))N_{io} - \sum_{j=1}^{J} p_j x_{ij}$$

Firms choose inputs to solve

$$\max_{\{N_{io}\}_{o=1}^{\mathcal{O}},\left\{x_{ij}\right\}_{j=1}^{J}} \pi_i \left(\{N_{io}\}_{o=1}^{\mathcal{O}}, \left\{x_{ij}\right\}_{j=1}^{J} \right)$$

Giving the first order conditions

$$p_i f_{i,x_{ij}} = p_j$$
$$p_i f_{i,N_{io}} = w_o \left(1 + \tau_o(\theta_o)\right)$$

Labor demand is $l_{io}^d(\theta_o) = (1+\tau_o(\theta_o))N_{io}$ for the optimal N_{io}. The aggregate labor demand is therefore

$$L_o^d(\theta_o) = \sum_{i=1}^{J} \left(1 + \tau_o(\theta_o)\right) N_{io}$$

The equilibrium tightness equates aggregate labor demand and labor supply.

We can rewrite these expressions in terms of elasticities.

$$\varepsilon_{x_{ij}}^{f_i} = \frac{p_j x_{ij}}{p_i y_i} \tag{3}$$

$$\varepsilon_{N_{io}}^{f_i} = \left(1 + \tau_o(\theta_o)\right) \frac{w_o N_{io}}{p_i y_i} \tag{4}$$

From Equation 4, we can derive the an alternative expression for labor demand:

$$l_{io}^d(\theta_o) = \varepsilon_{N_{io}}^{f_i} \frac{p_i}{w_o} y_i \tag{5}$$

$$L_o^d(\theta_o) = \sum_{i=1}^{J} \varepsilon_{N_{io}}^{f_i} \frac{p_i}{w_o} y_i$$

4. Equilibrium

The equilibrium in this model can be characterized by a set of conditions guaranteeing labor market equilibrium and goods market equilibrium. The equilibrium is a collection of $4J + 2J^2 + 3\mathcal{O}J + 4\mathcal{O}$ endogenous variables

$$\left\{\left\{p_i, y_i, \left\{x_{ij}, \varepsilon^{f_i}_{x_{ij}}\right\}_{j=1}^{J}, c_i, \varepsilon^{\mathcal{D}}_{c_i}, \left\{N_{io}, \varepsilon^{f_i}_{N_{io}}, l^{d}_{io}\right\}_{o=1}^{\mathcal{O}}\right\}_{i=1}^{J}, \left\{\theta_o, w_o, L^d_o, L^s_o\right\}_{o=1}^{\mathcal{O}},\right\}$$

that satisfy equations 1 through 5, along with goods market clearing, labor market clearing, and constant returns, given exogenous variables $\left\{\{A_i\}_{i=1}^{J}, \{H_o\}_{o=1}^{\mathcal{O}}\right\}$. We summarize the equilibrium conditions below for convenience.

4.1. Goods Market Equilibrium

In an equilibrium, firms intermediate input choices given prices and labor market chacteristics are profit maximizing:

(Intermediate input decision) $$\varepsilon^{f_i}_{x_{ij}} = \frac{p_j x_{ij}}{p_i y_i},$$

(Labor input decision) $$\varepsilon^{f_i}_{N_{io}} = (1 + \tau_o(\theta_o))\,\frac{w_o N_{io}}{p_i y_i}.$$

Firms produce output via production technology f_i

(Production technology) $$y_i = A_i f_i\left(\{N_{io}\}_{o=1}^{\mathcal{O}}, \left\{x_{ij}\right\}_{j=1}^{J}\right)$$

By constant returns to scale in production,

(Constant returns production) $$1 = \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} + \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}}$$

In addition, the household maximizes their utility by choosing a consumption bundle that satisfies its first-order condition.

(Consumption decision) $$\varepsilon^{\mathcal{D}}_{c_i} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}.$$

And by constant returns

(Constant returns utility) $$1 = \sum_{i=1}^{J} \varepsilon^{\mathcal{D}}_{c_i}$$

Finally, the goods market has to clear, which means that, for each sector i, total production has to be equal to the sum of the household's consumption of good i and all other sectors' use of good i in their production:

$$\text{(Goods market clearing)} \qquad y_i = c_i + \sum_{j=1}^{J} x_{ji}.$$

In total, the goods market provides $4J + J^2 + J\mathcal{O} + 1$ restrictions.

4.2. Labor Market Equilibrium

From Equation 4, labor demand in sector i is defined as

$$\text{(Labor Demand)} \qquad l^d_{io}(\theta_o) = \varepsilon^{f_i}_{N_{io}} \frac{p_i y_i}{w_o}.$$

$$\text{(6)} \qquad L^d_o(\theta_o) = \sum_{j=1}^{J} l^d_{io}(\theta_o)$$

Given sector level labor force participation H_i labor supply is

$$\text{(Labor Supply)} \qquad L^s_o(\theta_o) = \mathcal{F}_o(\theta_o) H_o.$$

Labor demand equals labor supply at an equilibrium in the labor market.

$$\text{(LM equilibrium)} \qquad L^d_o(\theta_o) = L^s_o(\theta_o).$$

These equilibrium conditions provide an additional $3\mathcal{O} + J\mathcal{O}$ restrictions.

4.3. Summary

The equilibrium conditions outline above provide just $4J + J^2 + 2J\mathcal{O} + 3\mathcal{O} + 1$ equations in $4J + 2J^2 + 3\mathcal{O}J + 4\mathcal{O}$ endogenous variables. The wage schedules taken as given by both households and firms provide another $\mathcal{O}$ restrictions. Nevertheless, as is typical in the literature, we need additional functional form assumptions on production and household preferences to close the model.

For instance, assuming Cobb-Douglas production and preferences fully parameterizes $\left\{ \left\{ \varepsilon^{f_i}_{x_{ij}} \right\}_{j=1}^{J}, \left\{ \varepsilon^{f_i}_{N_i} \right\}_{o=1}^{\mathcal{O}}, \varepsilon^{\mathcal{D}}_{c_i} \right\}_{i=1}^{J}$, giving us $J + J^2 + J\mathcal{O}$ additional restrictions, but removing $J + 1$ of the restrictions above.[1] Parametrizing production by assuming Cobb-Douglas therefore gives exactly the number of restrictions we need to close the model. Alternatively, assuming CES production indirectly provides restrictions to pin down the same set of elasticities.[2]

5. The Production Network

In this section, we introduce key production network notation. We then discuss three propagation mechanisms: prices, sales shares, and tightness. For now, we proceed assuming a general production function. In subsequent applications we will parameterize the production functions to allow us to calibrate and estimate the model.

5.1. Notation

We denote vectors and matrices by bold letters. For instance, $d\log \boldsymbol{x} = \begin{bmatrix} d\log x_1 & \cdots & d\log x_J \end{bmatrix}'$. We can conveniently capture many features of the production network through the following matrices

$$\boldsymbol{\Omega} = \begin{bmatrix} \varepsilon^{f_1}_{x_{11}} & \varepsilon^{f_1}_{x_{12}} & \cdots & \varepsilon^{f_1}_{x_{1J}} \\ \varepsilon^{f_2}_{x_{21}} & \varepsilon^{f_2}_{x_{22}} & \cdots & \varepsilon^{f_2}_{x_{2J}} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon^{f_J}_{x_{J1}} & \varepsilon^{f_J}_{x_{J2}} & \cdots & \varepsilon^{f_J}_{x_{JJ}} \end{bmatrix}, \ \boldsymbol{\Psi} = (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}.$$

In the standard production networks vocabulary, $\boldsymbol{\Omega}$ is the sales based input-output matrix and $\boldsymbol{\Psi}$ is the sales based Leontief inverse.[3]

[1] Choosing elasticities directly makes the constant returns restrictions redundant.

[2] CES and Cobb-Douglas, a special case of CES, are the two most commonly assumed production technologies in the literature. In principle, assuming any production technology will do. An explicit functional form will allow us to characterize the required elasticities. We therefore continue below with general formulas that hold for any production technology. In applied work, we will need to either specify the functional form or be able to directly estimate the elasticities and how they change in response to shocks.

[3] Since our model abstracts from markup wedges on prices, the sales based and cost based input-output matrices will coincide.

In addition define

$$\boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{D}}} = \begin{bmatrix} \varepsilon_{c_1}^{\mathcal{D}} \\ \varepsilon_{c_2}^{\mathcal{D}} \\ \vdots \\ \varepsilon_{c_J}^{\mathcal{D}} \end{bmatrix}_{J\times 1}, \quad \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} = \begin{bmatrix} \varepsilon_{N_{11}}^{f_1} & \varepsilon_{N_{12}}^{f_1} & \cdots & \varepsilon_{N_{1\mathcal{O}}}^{f_1} \\ \varepsilon_{N_{21}}^{f_2} & \varepsilon_{N_{22}}^{f_2} & \cdots & \varepsilon_{N_{2\mathcal{O}}}^{f_2} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{N_{J1}}^{f_J} & \varepsilon_{N_{J2}}^{f_J} & \cdots & \varepsilon_{N_{J1=\mathcal{O}}}^{f_J} \end{bmatrix}_{J\times\mathcal{O}},$$

$$\boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\mathcal{Q}}} = \begin{bmatrix} \varepsilon_{\theta_1}^{\mathcal{Q}_1} \\ \varepsilon_{\theta_2}^{\mathcal{Q}_2} \\ \vdots \\ \varepsilon_{\theta_{\mathcal{O}}}^{\mathcal{Q}_{\mathcal{O}}} \end{bmatrix}_{\mathcal{O}\times 1}, \quad \boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\mathcal{F}}} = \begin{bmatrix} \varepsilon_{\theta_1}^{\mathcal{F}_1} \\ \varepsilon_{\theta_2}^{\mathcal{F}_2} \\ \vdots \\ \varepsilon_{\theta_{\mathcal{O}}}^{\mathcal{F}_{\mathcal{O}}} \end{bmatrix}_{\mathcal{O}\times 1}, \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_1(\theta_1) \\ \tau_2(\theta_2) \\ \vdots \\ \tau_{\mathcal{O}}(\theta_{\mathcal{O}}) \end{bmatrix}_{\mathcal{O}\times 1}$$

Furthermore, let

$$\boldsymbol{\mathcal{Q}} = \operatorname{diag}\left(\boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\mathcal{Q}}}\right)_{\mathcal{O}\times\mathcal{O}}, \ \boldsymbol{\mathcal{F}} = \operatorname{diag}\left(\boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\mathcal{F}}}\right)_{\mathcal{O}\times\mathcal{O}}, \ \boldsymbol{\mathcal{T}} = \operatorname{diag}(\boldsymbol{\tau})_{\mathcal{O}\times\mathcal{O}}$$

5.2. Price Propagation

Log-linearizing the production function, for each sector i, we have:

$$d\log y_i = \underbrace{\varepsilon_{A_i}^{f_i}}_{=1} d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} d\log N_{io} + \sum_{j=1}^{N} \varepsilon_{x_{ij}}^{f_i} d\log x_{ij}$$

Plugging in Equation 4 and Equation 3, the first order conditions for optimal input usage, into the log-linearized production function gives

$$\begin{aligned} d\log y_i &= \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} \left[d\log \varepsilon_{N_{io}}^{f_i} + d\log y_i + d\log p_i - d\log w_o - d\log\left(1+\tau_o(\theta_o)\right) \right] \\ &\quad + \sum_{j=1}^{N} \varepsilon_{x_{ij}}^{f_i} \left[d\log \varepsilon_{x_{ij}}^{f_i} + d\log y_i + d\log p_i - d\log p_j \right] + d\log A_i \\ &= \left[d\log y_i + d\log p_i \right] \underbrace{\left[\sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} + \sum_{j=1}^{N} \varepsilon_{x_{ij}}^{f_i} \right]}_{=1 \text{ by crts}} + \underbrace{\left[\sum_{o=1}^{\mathcal{O}} d\varepsilon_{N_{io}}^{f_i} + \sum_{j=1}^{N} d\varepsilon_{x_{ij}}^{f_i} \right]}_{=0 \text{ by crts}} \end{aligned}$$

$$
- \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d \log w_o + d \log(1 + \tau_o(\theta_o))\right] - \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} \left[d \log p_j\right] + d \log A_i,
$$

where the second equality holds because the sum of elasticities equals one for constant returns to scale technology and $\varepsilon^{f_i}_{x_{ij}} d \log \varepsilon^{f_i}_{x_{ij}} = d\varepsilon^{f_i}_{x_{ij}}$.

Rearranging terms gives

$$
\begin{aligned}
d \log p_i &= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d \log w_o + d \log(1 + \tau_o(\theta_o))\right] + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} \left[d \log p_j\right] - d \log A_i \\
&= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d \log w_o + \varepsilon^{1+\tau_o}_{\theta_o} d \log \theta_o\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left[d \log p_j\right] - d \log A_i \\
&= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d \log w_o - \tau_o(\theta_o) \varepsilon^{\mathcal{Q}_o}_{\theta_o} d \log \theta_o\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left[d \log p_j\right] - d \log A_i
\end{aligned}
$$

Stacing equations over sectors, we can write

$$
d \log \boldsymbol{p} = \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} \left[d \log \boldsymbol{w} - \mathcal{Q}\mathfrak{T} d \log \theta\right] + \boldsymbol{\Omega} d \log \boldsymbol{p} - d \log \boldsymbol{A}
$$

Which implies

$$
d \log \boldsymbol{p} = \boldsymbol{\Psi} \left[\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} \left[d \log \boldsymbol{w} - \mathcal{Q}\mathfrak{T} d \log \theta\right] - d \log \boldsymbol{A}\right]
$$

5.3. Sales Share Propgation

We can rewrite the goods market clearing condition in terms of Domar weights:

$$
\begin{aligned}
y_i &= c_i + \sum_{j=1}^{J} x_{ji} \\
\Rightarrow \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k} &= \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k} + \sum_{j=1}^{J} \frac{p_i x_{ji}}{p_j x_j} \frac{p_j x_j}{\sum_{k=1}^{J} p_k c_k} \\
\Rightarrow \lambda_i &= \varepsilon^{\mathcal{D}}_{c_i} + \sum_{j=1}^{J} \varepsilon^{f_j}_{x_{ji}} \lambda_j, \qquad (7)
\end{aligned}
$$

where $\lambda_i = \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k}$ is the Domar weight of sector i.

By stacking (7) for each sector, we get the following expression for Domar weights across the production network.

$$\lambda' = \varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} + \lambda'\boldsymbol{\Omega}$$

We can see how Domar weights change across the production network by totally differentiating

$$\begin{aligned} d\lambda' &= d\varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} + d\lambda'\boldsymbol{\Omega} + \lambda' d\boldsymbol{\Omega} \\ \Rightarrow d\lambda' &= \left[d\varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} + \lambda' d\boldsymbol{\Omega}\right]\boldsymbol{\Psi} \end{aligned} \tag{8}$$

The Domar weights will help us express how shocks propagate to output.

5.4. Output Propagation

Since the log-linearized expression for the Domar weight must hold for every sector, we can write

$$\begin{aligned} d\log\lambda_i - d\log\lambda_j &= d\log p_i - d\log p_j + d\log y_i - d\log y_j \\ &= d\log x_{ij} - d\log\varepsilon_{x_{ij}}^{f_i} - d\log y_j \\ \Rightarrow d\log x_{ij} &= d\log\lambda_i - d\log\lambda_j + d\log y_j + d\log\varepsilon_{x_{ij}}^{f_i} \end{aligned}$$

Plugging back into the production function,

$$\begin{aligned} d\log y_i &= d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} d\log N_{io} + \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i}\left(d\log\varepsilon_{x_{ij}}^{f_i} + d\log y_j + d\log\lambda_i - d\log\lambda_j\right) \\ &= d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} d\log N_{io} - \varepsilon_{N_i}^{f_i} d\log\varepsilon_{N_i}^{f_i} + \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i}\left(d\log y_j + d\log\lambda_i - d\log\lambda_j\right) \end{aligned}$$

Using the definition of labor demand,

$$\begin{aligned} d\log N_{io} &= d\log l_{io}^{d} - d\log(1+\tau_o(\theta_o)) \\ &= d\log l_{io}^{d} + \tau_o(\theta_o)\varepsilon_{\theta_o}^{\mathcal{Q}_o} d\log\theta_o \end{aligned}$$

Using the labor market clearing condition, and the definition of labor supply,

$$d\log N_{io} = \left(\varepsilon_{\theta_o}^{\mathcal{F}_o} + \tau_o(\theta_o)\varepsilon_{\theta_o}^{\mathcal{Q}_o}\right) d\log\theta_o + d\log H_o$$

Which means

$$\begin{aligned} d\log y_i = d\log A_i + \sum_{o=1}^{\mathcal{O}} \left[\varepsilon_{N_{io}}^{f_i}\left(\varepsilon_{\theta_o}^{\mathcal{F}_o} + \tau_o(\theta_o)\varepsilon_{\theta_o}^{\mathcal{Q}_o}\right) d\log\theta_o + \varepsilon_{N_{io}}^{f_i} d\log H_o - \varepsilon_{N_{io}}^{f_i} d\log \varepsilon_{N_{io}}^{f_i}\right] \\ + \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i}\left(d\log y_j + d\log\lambda_i - d\log\lambda_j\right) \end{aligned}$$

Stacking over sectors gives,

$$\begin{aligned} d\log \boldsymbol{y} = d\log \boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left(\mathfrak{F} + \mathfrak{Q}\mathfrak{T}\right) d\log\theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \boldsymbol{H} \\ - d\log\mathcal{E} + \boldsymbol{\Omega} d\log \boldsymbol{y} + \left(\operatorname{diag}\left(\boldsymbol{\Omega}\mathbf{1}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} \end{aligned}$$

Where $\mathbf{1}$ is a $J \times 1$ vector of ones and $d\log\mathcal{E}$ is the $J \times 1$ vector of diagonal elements of $\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}'}$. Which implies

$$\begin{aligned} d\log \boldsymbol{y} = \boldsymbol{\Psi}\left(d\log \boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left(\mathfrak{F} + \mathfrak{Q}\mathfrak{T}\right) d\log\theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \boldsymbol{H}\right) \\ - \boldsymbol{\Psi} d\log\mathcal{E} + \boldsymbol{\Psi}\left(\operatorname{diag}\left(\boldsymbol{\Omega}\mathbf{1}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} \end{aligned}$$

5.5. Tightness Propagation

Labor market clearing implies that changes in labor demand have to equal changes in labor supply:

$$d\log L_o^s(\theta, \boldsymbol{H}) = d\log L_o^d(\theta, \boldsymbol{A}).$$

$$\varepsilon_{\theta_o}^{\mathcal{F}_o} d\log\theta_o + d\log H_o = \sum_{i=1}^{J} \frac{l_{io}}{L_o^d} d\log l_{io}(\theta_o)$$

Where $\frac{l_{io}}{L_o^d} = \frac{\varepsilon_{N_{io}}^{f_i} p_i y_i}{\sum_{j=1}^{J} \varepsilon_{N_{jo}}^{f_j} p_j y_j}$. For every sector i other than the numerair we have

$$d\log l_{io}(\theta_o) = d\log \varepsilon_{N_{io}}^{f_i} - d\log w_o + d\log p_i + d\log y_i$$

$$
\begin{aligned}
&= d\log \varepsilon^{f_i}_{N_{io}} - d\log w_o + \boldsymbol{\Psi_i}\left[\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left[d\log \boldsymbol{w} - \mathfrak{Q}\mathfrak{T} d\log\theta\right] - d\log \boldsymbol{A}\right] \\
&+ \boldsymbol{\Psi_i}\left(d\log \boldsymbol{A} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left(\mathfrak{F} + \mathfrak{Q}\mathfrak{T}\right) d\log\theta + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} d\log \boldsymbol{H}\right) \\
&- \boldsymbol{\Psi_i} d\log \mathcal{E} + \boldsymbol{\Psi_i}\left(\mathrm{diag}\left(\boldsymbol{\Omega 1}\right) - \boldsymbol{\Omega}\right) d\log\lambda \\
&= d\log \varepsilon^{f_i}_{N_{io}} - d\log w_o + \boldsymbol{\Psi_i}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left[d\log \boldsymbol{w} + \mathfrak{F} d\log\theta + d\log \boldsymbol{H}\right] \\
&+ \boldsymbol{\Psi_i}\left[\left(\mathrm{diag}\left(\boldsymbol{\Omega 1}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} - d\log \mathcal{E}\right]
\end{aligned}
$$

Where $\boldsymbol{\Psi_i}$ is the ith row of $\boldsymbol{\Psi}$. Stacking over occupations gives

$$
\begin{aligned}
d\log \boldsymbol{l_i}(\theta) &= d\log \varepsilon^{\boldsymbol{f_i}'}_{\boldsymbol{N_i}} - d\log \boldsymbol{w} + \Xi_{\boldsymbol{i}}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left[d\log \boldsymbol{w} + \mathfrak{F} d\log\theta + d\log \boldsymbol{H}\right] \\
&+ \Xi_{\boldsymbol{i}}\left[\left(\mathrm{diag}\left(\boldsymbol{\Omega 1}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} - d\log \mathcal{E}\right]
\end{aligned}
$$

Where $d\log \varepsilon^{\boldsymbol{f_i}}_{\boldsymbol{N_i}}$ is the ith row of $d\log \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}$, and $\underbrace{\Xi_{\boldsymbol{i}}}_{\mathcal{O}\times J} = \begin{bmatrix}\Psi'_{\boldsymbol{i}} & \cdots & \Psi'_{\boldsymbol{i}}\end{bmatrix}'$. If sector j is the numerair, then in j we instead have

$$
\begin{aligned}
d\log l_{jo}(\theta_o) &= d\log \varepsilon^{f_j}_{N_{jo}} - d\log w_o + d\log y_j \\
&= d\log \varepsilon^{f_i}_{N_{io}} - d\log w_o + \boldsymbol{\Psi_i}\left[d\log \boldsymbol{A} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left(\mathfrak{F} + \mathfrak{Q}\mathfrak{T}\right) d\log\theta + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} d\log \boldsymbol{H}\right] \\
&+ \boldsymbol{\Psi_i}\left[\left(\mathrm{diag}\left(\boldsymbol{\Omega 1}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} - d\log \mathcal{E}\right]
\end{aligned}
$$

Stacking over occupation gives

$$
\begin{aligned}
d\log \boldsymbol{l_j}(\boldsymbol{\theta}) &= d\log \varepsilon^{\boldsymbol{f_j}'}_{\boldsymbol{N_j}} - d\log \boldsymbol{w} + \Xi_{\boldsymbol{j}}\left[d\log \boldsymbol{A} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left(\mathfrak{F} + \mathfrak{Q}\mathfrak{T}\right) d\log\theta + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} d\log \boldsymbol{H}\right] \\
&+ \Xi_{\boldsymbol{j}}\left[\left(\mathrm{diag}\left(\boldsymbol{\Omega 1}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} - d\log \mathcal{E}\right]
\end{aligned}
$$

Finally, summing over all sectors gives

$$
\begin{aligned}
d\log \boldsymbol{L^d}(\boldsymbol{\theta}) &= \mathcal{L} d\log \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} - \mathrm{diag}\left(\mathcal{L}\boldsymbol{1}\right) d\log \boldsymbol{w} + \mathcal{L}\Psi\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left[d\log \boldsymbol{w} + \mathfrak{F} d\log\theta + d\log \boldsymbol{H}\right] \\
&+ \mathcal{L}\Psi\left[\left(\mathrm{diag}\left(\boldsymbol{\Omega 1}\right) - \boldsymbol{\Omega}\right) d\log\lambda - d\log \mathcal{E}\right] + \mathcal{L}_{\boldsymbol{j}}\Psi\left[d\log \boldsymbol{A} - \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left[d\log \boldsymbol{w} - \mathfrak{Q}\mathfrak{T} d\log\theta\right]\right] \\
&= \mathcal{L} d\log \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} + \left[\left[\mathcal{L} - \mathcal{L}_{\boldsymbol{j}}\right]\Psi\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} - \mathrm{diag}\left(\mathcal{L}\boldsymbol{1}\right)\right] d\log \boldsymbol{w} + \mathcal{L}\Psi\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} d\log \boldsymbol{H} + \mathcal{L}_{\boldsymbol{j}}\Psi d\log \boldsymbol{A} \\
&+ \left[\mathcal{L}\Psi\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\mathfrak{F} + \mathcal{L}_{\boldsymbol{j}}\Psi\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\mathfrak{Q}\mathfrak{T}\right] d\log\theta + \mathcal{L}\Psi\left[\left(\mathrm{diag}\left(\boldsymbol{\Omega 1}\right) - \boldsymbol{\Omega}\right) d\log\lambda - d\log \mathcal{E}\right]
\end{aligned}
$$

Where $\mathcal{L}_{\mathcal{O}\times J}$ and $\mathcal{L}_{\boldsymbol{j}_{\mathcal{O}\times J}}$ are given by

$$\mathcal{L} = \begin{bmatrix} \frac{l_{11}}{L_1^d} & \frac{l_{21}}{L_1^d} & \cdots & \frac{l_{J1}}{L_1^d} \\ \frac{l_{12}}{L_2^d} & \frac{l_{22}}{L_2^d} & \cdots & \frac{l_{J2}}{L_2^d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{l_{1\mathcal{O}}}{L_{\mathcal{O}}^d} & \frac{l_{2\mathcal{O}}}{L_{\mathcal{O}}^d} & \cdots & \frac{l_{J\mathcal{O}}}{L_{\mathcal{O}}^d} \end{bmatrix}, \; \mathcal{L}_{\boldsymbol{j}} = \begin{bmatrix} 0 & \cdots & \frac{l_{j1}}{L_1^d} & \cdots & 0 \\ 0 & \cdots & \frac{l_{j2}}{L_2^d} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{l_{j\mathcal{O}}}{L_{\mathcal{O}}^d} & \cdots & 0 \end{bmatrix}$$

Let $\Xi_\theta = \left[\mathcal{L}\boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{F} + \mathcal{L}_{\boldsymbol{j}}\boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\Omega\mathcal{T}\right]$, then labor market clearing implies

$$\begin{aligned} d\log\theta = {} & \left[\mathcal{F} - \Xi_\theta\right]^{-1}\left[\left[\left[\mathcal{L} - \mathcal{L}_{\boldsymbol{j}}\right]\boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \text{diag}\left(\mathcal{L}\mathbf{1}\right)\right] d\log\boldsymbol{w} + \left[\mathcal{L}\boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log\boldsymbol{H} + \mathcal{L}_{\boldsymbol{j}}\boldsymbol{\Psi} d\log\boldsymbol{A}\right] \\ & + \left[\mathcal{F} - \Xi_\theta\right]^{-1}\left[\mathcal{L} d\log\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} + \mathcal{L}\boldsymbol{\Psi}\left[\left(\text{diag}\left(\boldsymbol{\Omega}\mathbf{1}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} - d\log\mathcal{E}\right]\right] \end{aligned}$$

5.5.1. Two Special Cases

We look at how tightness propagates for two special cases. First, we examine how tightness responds when there is only one occupation in the economy. In this case, firms only use one type of labor, and labor is fully mobile across all sectors. Second, we examine how tightness responds when each sector hires only sector-specific labor. This case represents when labor markets are rigid across sectors.

One occupation. WLOG, assume occupation 1 is the only occupation that we have. As a result, the μ and μ_i's can be rewritten as scalar variables.

$$\begin{aligned} \mu_i &= \frac{l_{i1}}{L_1^d} \\ \Xi &= \sum_{i=1}^{J} \boldsymbol{\mu_i}\boldsymbol{\Xi_i} = \sum_{i=1}^{J} \frac{l_{i1}}{L_1^d}\boldsymbol{\Psi}_i' \\ \mu &= 1 \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} &= \begin{bmatrix} \varepsilon_{N_{11}}^{f_1} \\ \varepsilon_{N_{21}}^{f_2} \\ \vdots \\ \varepsilon_{N_{J1}}^{f_J} \end{bmatrix} \\
\Xi_\theta &= \left[\boldsymbol{\Xi} \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} \boldsymbol{\mathcal{F}} + \boldsymbol{\mu_j} \boldsymbol{\Xi_j} \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} \boldsymbol{\mathcal{Q}} \boldsymbol{\mathcal{T}} \right] \\
&= \varepsilon_\theta^{\mathcal{F}} \sum_{k=1}^{J} \sum_{i=1}^{J} \frac{l_{i1}}{L_1^d} \Psi_{ik} \varepsilon_{N_{k1}}^{f_k} + \tau(\theta) \varepsilon_\theta^{\mathcal{Q}} \frac{l_{j1}}{L_1^d} \sum_{k=1}^{J} \Psi_{jk} \varepsilon_{N_{k1}}^{f_k} \\
d \log \boldsymbol{\mathcal{E}} &= \operatorname{diag}(\boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} d \log \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}'})' \\
&= \left[\varepsilon_{N_{11}}^{f_1} d \log \varepsilon_{N_{11}}^{f_1} \quad \varepsilon_{N_{21}}^{f_2} d \log \varepsilon_{N_{21}}^{f_2} \quad \dots \quad \varepsilon_{N_{J1}}^{f_J} d \log \varepsilon_{N_{J1}}^{f_J} \right]'
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d \log \theta = {} & \left[\varepsilon_\theta^{\mathcal{F}} - \Xi_\theta \right]^{-1} \left[\left[\sum_{k \neq j} \sum_{i=1}^{J} \frac{l_{i1}}{L_1^d} \Psi_{ik} \varepsilon_{N_{k1}}^{f_k} - 1 \right] d \log w + \left[\sum_{k=1}^{J} \sum_{i=1}^{J} \frac{l_{i1}}{L_1^d} \Psi_{ik} \varepsilon_{N_{k1}}^{f_k} - 1 \right] d \log H + \boldsymbol{\Psi_j} d \log \boldsymbol{A} \right] \\
& + \left[\varepsilon_\theta^{\mathcal{F}} - \Xi_\theta \right]^{-1} \left[\sum_{i=1}^{J} \frac{l_{i1}}{L_1^d} d \log \varepsilon_{N_{i1}}^{f_i} + \sum_{i=1}^{J} \frac{l_{i1}}{L_1^d} \boldsymbol{\Psi}_i \left(\operatorname{diag} \left(\boldsymbol{\Omega} \mathbf{1} \right) - \boldsymbol{\Omega} \right) d \log \boldsymbol{\lambda} \right] \\
& - \left[\varepsilon_\theta^{\mathcal{F}} - \Xi_\theta \right]^{-1} \sum_{k=1}^{J} \sum_{i=1}^{J} \frac{l_{i1}}{L_1^d} \Psi_{ik} \varepsilon_{N_{k1}}^{f_k} d \log \varepsilon_{N_{k1}}^{f_k}
\end{aligned}
$$

Sector-specific Occupation. With this setup, we have that $\frac{l_{ij}}{L_i^d} = 0$ for $i \neq j$ and $\frac{l_{ij}}{L_i^d} = 1$ for $i = j$. Then $\boldsymbol{\Xi} = \boldsymbol{\Psi}$ and $\boldsymbol{\mu} = \boldsymbol{I}$. In addition, $\boldsymbol{\mu_j} \boldsymbol{\xi_j} = [\mathbf{0} \dots \boldsymbol{\Psi_j} \dots \mathbf{0}]'$ with $\boldsymbol{\Psi_j}$ at the j-th row. We also have $\sum_{i=1}^{J} \mu_i d \log \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}'} = d \log \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}}$ and $d \log \boldsymbol{\mathcal{E}} = \operatorname{diag} \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} d \log \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}}$. As a result, $d \log \theta$ reduces to what we have in the rigid sectoral labor market case.

5.6. Wage changes

Since wages are not uniquely pinned down in models featuring matching frictions, we need to assume a wage schedule to close the model. In particular, we assume

$$
d \log \boldsymbol{w} = \boldsymbol{\Lambda_A} d \log \boldsymbol{A} + \boldsymbol{\Lambda_H} d \log \boldsymbol{H}
$$

Where $\boldsymbol{\Lambda_A}$ contains wage elasticities to productivity changes and $\boldsymbol{\Lambda_H}$ contains wage elasticities to labor force changes.

5.7. Unemployment

Unemployment in sector j is $U_j = H_j - L_j$. The log change in Unemployment is

$$d\log \boldsymbol{U} = \operatorname{diag}(\boldsymbol{U})^{-1}\left(\operatorname{diag}(\boldsymbol{H})\, d\log \boldsymbol{H} - \operatorname{diag}(\boldsymbol{L})\, d\log \boldsymbol{L}\right) \tag{6}$$

The unemployment rate in sector j is $u_j = \frac{H_j - L_j}{H_j}$. The log change in the unemployment rate is

$$d\log \boldsymbol{u} = (\boldsymbol{I} - \operatorname{diag}(\boldsymbol{u}))\operatorname{diag}(\boldsymbol{u})^{-1}(d\log \boldsymbol{H} - d\log \boldsymbol{L}) \tag{7}$$

6. Aggregation

6.1. General Case

Using the first order condition,

$$d\log \varepsilon^{\mathcal{D}}_{c_i} = d\log p_i + d\log c_i - d\log \sum_{j=1}^{J} p_k c_k$$

along with the definition of the Domar weight,

$$d\log \sum_{k=1}^{J} p_k c_k = d\log p_i + d\log y_i - d\log \lambda_i$$

gives

$$d\log c_i = d\log \varepsilon^{\mathcal{D}}_{c_i} + d\log y_i - d\log \lambda_i$$

Which implies the log change in real GDP is

$$\begin{aligned}
d\log Y &= \varepsilon^{\mathcal{D}\prime}_{\boldsymbol{c}}\, d\log \boldsymbol{c} \\
&= \varepsilon^{\mathcal{D}\prime}_{\boldsymbol{c}}\left(d\log \varepsilon^{\mathcal{D}}_{\boldsymbol{c}} + d\log \boldsymbol{y} - d\log \boldsymbol{\lambda}\right) \\
&= \varepsilon^{\mathcal{D}\prime}_{\boldsymbol{c}}\left(d\log \varepsilon^{\mathcal{D}}_{\boldsymbol{c}} - d\log \boldsymbol{p} + d\log \boldsymbol{\lambda} + \Xi_{\varepsilon} d\log \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} - d\log \boldsymbol{\lambda}\right)
\end{aligned}$$

$$= -\varepsilon_{\boldsymbol{c}}^{\mathcal{D}\prime} d\log \boldsymbol{p} + \varepsilon_{\boldsymbol{c}}^{\mathcal{D}\prime} \left(d\log \varepsilon_{\boldsymbol{c}}^{\mathcal{D}} + \Xi_{\varepsilon} d\log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \right)$$

6.2. Aggregate Output

6.3. Aggregate employment

7. A Cobb-Douglas Example

In this section, we examine the network propagation and aggregation results under Cobb-Douglas utility, matching function, and production function. We focus on only technology shocks. Specifically, this implies that:

$$\begin{aligned} d\log \varepsilon_{\boldsymbol{c}}^{\mathcal{D}} &= \mathbf{0} \\ d\log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} &= \mathbf{0} \\ d\boldsymbol{\Omega} &= \mathbf{0} \\ d\boldsymbol{\lambda} &= \mathbf{0} \\ d\log \boldsymbol{H} &= \mathbf{0} \end{aligned}$$

In addition, I assume rigid nominal wages, which means $d\log \boldsymbol{w} = 0$.

7.1. Propagation

Therefore, the output and tightness propagation are:

$$\begin{aligned} d\log \boldsymbol{y} &= \boldsymbol{\Psi} \left(d\log \boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} (\mathcal{F} + \mathcal{Q}\mathcal{T})\, d\log \theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \boldsymbol{H} \right) \\ d\log \boldsymbol{p} &= \boldsymbol{\Psi} \left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} \left[-\mathcal{Q}\mathcal{T} d\log \theta \right] - d\log \boldsymbol{A} \right] \end{aligned}$$

We see that sectoral outputs are impacted by technology directly through the production channel, and indirectly through the labor market channel.

In the absence of labor supply shocks, we have that

$$d\log \boldsymbol{y} + d\log \boldsymbol{p} = \boldsymbol{\Psi}\mathcal{F}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \theta$$

Thus, we have:

$$
\begin{aligned}
d\log l_{io}(\theta_o) &= d\log p_i + d\log y_i \\
\Rightarrow d\log L_o(\theta_o) &= \sum_{i=1}^{J} \frac{l_{io}}{L_o^D} d\log l_{io}(\theta_o) \\
&= \frac{1}{L_o^D}\begin{bmatrix} l_{1o} & l_{2o} & \dots l_{Jo} \end{bmatrix} (d\log \boldsymbol{p} + d\log \boldsymbol{y}) \\
&= \frac{1}{L_o^D}\begin{bmatrix} l_{1o} & l_{2o} & \dots l_{Jo} \end{bmatrix} \boldsymbol{\Psi}\mathcal{F}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log\theta \\
\Rightarrow d\log \boldsymbol{L^D}(\theta) &= \mathcal{L}\boldsymbol{\Psi}\mathcal{F}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log\theta,
\end{aligned}
$$

where $\mathcal{L}$ is the occupation-share matrix:

$$
\mathcal{L} = \begin{bmatrix} \frac{l_{11}}{L_1^D} & \frac{l_{21}}{L_1^D} & \dots & \frac{l_{J1}}{L_1^D} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{l_{1O}}{L_O^D} & \frac{l_{2O}}{L_O^D} & \dots & \frac{l_{JO}}{L_O^D} \end{bmatrix}.
$$

Thus

$$
\mathcal{F}d\log\theta + d\log \boldsymbol{H} = \mathcal{L}\boldsymbol{\Psi}\mathcal{F}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log\theta
$$

This implies that with the absence of labor supply shocks, $d\log\theta = \mathbf{0}$.

7.2. Aggregation

To draw comparison with Hulten's theorem, we consider only the aggregate response to technology shocks. The aggregation formula can be rewritten as:

$$
d\log Y = \boldsymbol{\Lambda_A} d\log \boldsymbol{A},
$$

where

$$
\boldsymbol{\Lambda_A} = \varepsilon_{\boldsymbol{c}}^{\mathcal{D}'}\boldsymbol{\Psi} d\log \boldsymbol{A},
$$

which is the same as Hulten's Theorem.

8. Comparison with no search frictions model.

Suppose that there are no search frictions. Instead, we treat labor from the $\mathcal{O}$ occupations as fixed supply factors of production. In equilibrium, wages adjust so that all workers are employed. Absent search and matching frictions, wages are therefore pinned down by firms' first order conditions.

References