Unemployment in a Production Network: Additional factor of production, assuming Cobb-Douglas

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1. Households and final goods production

We consider a closed, static economy model with no government spending. There is no saving mechanism in the economy, and real household consumption equals real GDP, denoted by Y. A final goods producer with constant returns to scale technology aggregates J sector outputs to produce Y, the final consumption good

$$Y = \max_{\{c_i\}_{i=1}^J} \mathcal{D}\left(\{c_i\}_{i=1}^J\right)$$

Subject to the budget constraint

$$\sum_{i=1}^J p_i c_i = \sum_{i=1}^J w_o L_o^s(\theta_o) + rK.$$

$\mathcal{D}$ captures household preferences over final consumption goods, and w_o is the wage of labor supplied to occupation o, $L_o^s(\theta_o)$. K is a fixed factor of production, which we can think of as capturing inputs not accounted for in labor or intermediates, like capital,

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imported inputs, energy etc. r is the price of the additional fixed factor, which we assume is paid to the household.

The household's consumption decision can be computed using the first order condition:

$$\varepsilon^{\mathcal{D}}_{c_i} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}, \tag{1}$$

where $\varepsilon^{\mathcal{D}}_{c_i}$ denotes the households elasticity of utility with regards to the consumption of good i.

2. Occupation level labor markets

We assume there are $\mathcal{O}$ occupations with separate labor markets, a labor force of H_o possible workers, who all start out unemployed, and an exogenous recruiting cost r_o, which measures the units of labor required for a firm to maintain each posted vacancy in occupation o. When workers and firms meet there is a mutual gain from matching. There is no accepted theory for how wages are set in this context. For now we assume the nominal wage in occupation o, w_o, follows a general wage schedule that depends on productivity and the size of the labor force, and is taken as given by both firms and workers. Hires are generated by a constant returns matching function in occupation-level unemployment U_o and aggregate vacancies V_o, which measure all vacancy postings for occupation o,

$$h_o = \phi_o m\left(U_o, V_o\right)$$

The household supplies H_o searching workers to each sector. Let the sector-specific labor market tightness be $\theta_o = \frac{V_o}{H_o}$, the vacancy-filling rate $\mathcal{Q}_o(\theta_o) = \phi_o m\left(\frac{H_o}{V_o}, 1\right)$, and the job-finding rate $\mathcal{F}_o(\theta_o) = \phi_o m\left(1, \frac{v_o}{H_o}\right)$. Therefore, a fraction $\mathcal{F}_o(\theta_o)$ of H_o household finds a job, and labor supply satisfies

$$L^o_i(\theta_o) = \mathcal{F}_o(\theta_o) H_o \tag{2}$$

We assume firms take the occupation level tightness as given. One way of justifying this is with the assumption that each sector is populated by many identical competitive firms so that each firm only has an infinitesimal impact on aggregate vacancies, and therefore on aggregate tightness. Let N_{io} denote productive employees in occupation o working for sector i firms and let r_o be the cost of each vacancy for a firm. In order to hire N_{io} productive employees, the number of vacancies posted v_{io} has to satisfy $\mathcal{Q}_o(\theta_o)v_{io} = N_{io} + r_o v_{io}$, where $r_o v_{io}$ denotes the cost of posting the vacancies. Rearranging yields $v_{io} = \frac{N_{io}}{\mathcal{Q}_o(\theta_o) - r_o}$. Thus, hiring one unit of productive labor requires $\frac{1}{\mathcal{Q}_o(\theta_o) - r_o}$ vacancy postings, and requires $1 + \tau_o(\theta_o)$ units of total labor, where

$$\tau_o(\theta_o) \equiv \frac{r_o}{\mathcal{Q}_o(\theta_o) - r_o}.$$

For a given target level of occupation o employment N_{io}, total required labor, or the labor demand, is $l^d_{io}(\theta_o) = (1 + \tau_o(\theta_o))\, N_{io}$. We describe how labor demand, $l^d_{io}(\theta_{io})$, is determined by firms' profit maximization in the next section.

Finally, we define aggregate occupation o labor demand as the sum of sectoral labor demands and aggregate vacancy postings as the sum of sectoral vacancy postings.

$$L^d_o(\theta_o) = \sum_{i=1}^{J} l^d_{io}(\theta_o)$$
$$V_o = \sum_{i=1}^{J} v_{io}$$

Market clearing in the labor market requires labor demand equal labor supply and that the vacancy posting choices of firms in each sector are consistent with aggregate tightness.

$$L^d_o = L^s_o$$
$$\theta_o = \frac{\sum_{i=1}^{J} v_{io}}{H_o}$$

3. Sector level firms

A representative firm in sector i uses workers in occupation o, N_{io}, intermediate inputs from sector j, x_{ij}, and $\mathcal{K}$ additional factors K_{ik}, to produce output y_i using constant returns production technology f_i.

$$y_i = A_i f_i\left(\{N_{io}\}_{o=1}^{\mathcal{O}}, \{x_{ij}\}_{j=1}^{J}, \{K_{ik}\}_{k=1}^{\mathcal{K}}\right)$$

Firms choose $\{N_{io}\}_{o=1}^{\mathcal{O}}$, $\left\{x_{ij}\right\}_{j=1}^{J}$, and $\{K_{ik}\}_{k=1}^{\mathcal{K}}$ to maximize profits, or equivalently to minimize costs. We assume firms are price takers in both input and output markets. Profits are given by

$$\pi_i = p_i A_i f_i\left(\{N_{io}\}_{o=1}^{\mathcal{O}}, \{x_{ij}\}_{j=1}^{J}, \{K_{ik}\}_{k=1}^{\mathcal{K}}\right) - \sum_{o=1}^{\mathcal{O}} w_o(1+\tau_o(\theta_o))N_{io} - \sum_{j=1}^{J} p_j x_{ij} - \sum_{k=1}^{\mathcal{K}} r_k K_{ik}$$

Firms choose inputs to solve

$$\max_{\{N_{io}\}_{o=1}^{\mathcal{O}}, \left\{x_{ij}\right\}_{j=1}^{J}, \{K_{ik}\}_{k=1}^{\mathcal{K}}} \pi_i\left(\{N_{io}\}_{o=1}^{\mathcal{O}}, \left\{x_{ij}\right\}_{j=1}^{J}, \{K_{ik}\}_{k=1}^{\mathcal{K}}\right)$$

Giving the first order conditions

$$\begin{aligned}
p_i f_{i,x_{ij}} &= p_j \\
p_i f_{i,N_{io}} &= w_o\left(1+\tau_o(\theta_o)\right) \\
p_i f_{i,K_{ik}} &= r_k
\end{aligned}$$

Labor demand is $l_{io}^{d}(\theta_o) = (1{+}\tau_o(\theta_o))N_{io}$ for the optimal N_{io}. The aggregate labor demand is therefore

$$L_o^d(\theta_o) = \sum_{i=1}^{J}\left(1+\tau_o(\theta_o)\right)N_{io}$$

The equilibrium tightness equates aggregate labor demand and labor supply. Similarly, for the additional factor of production K

$$K_k^s = \sum_{i=1}^{J} K_{ik}$$

Where we assume aggregate factor supply K is determined endogenously.

We can rewrite these expressions in terms of elasticities.

$$\varepsilon_{x_{ij}}^{f_i} = \frac{p_j x_{ij}}{p_i y_i} \tag{3}$$

$$\varepsilon_{N_{io}}^{f_i} = \left(1+\tau_o(\theta_o)\right)\frac{w_o N_{io}}{p_i y_i} \tag{4}$$

$$\varepsilon_{K_{ik}}^{f_i} = \frac{r_k K_{ik}}{p_i y_i} \tag{5}$$

From Equation 4, we can derive the an alternative expression for labor demand:

$$\begin{aligned} l_{io}^{d}(\theta_o) &= \varepsilon_{N_{io}}^{f_i} \frac{p_i}{w_o} y_i \\ L_o^d(\theta_o) &= \sum_{i=1}^{J} \varepsilon_{N_{io}}^{f_i} \frac{p_i}{w_o} y_i \end{aligned} \tag{6}$$

4. The Production Network

In this section, we introduce key production network notation. We then discuss three propagation mechanisms: prices, sales shares, and tightness. For now, we proceed assuming a general production function. In subsequent applications we will parameterize the production functions to allow us to calibrate and estimate the model.

4.1. Notation

We denote vectors and matrices by bold letters. For instance, $d\log \boldsymbol{x} = \begin{bmatrix} d\log x_1 & \cdots & d\log x_J \end{bmatrix}'$. We can conveniently capture many features of the production network through the following matrices

$$\boldsymbol{\Omega} = \begin{bmatrix} \varepsilon_{x_{11}}^{f_1} & \varepsilon_{x_{12}}^{f_1} & \cdots & \varepsilon_{x_{1J}}^{f_1} \\ \varepsilon_{x_{21}}^{f_2} & \varepsilon_{x_{22}}^{f_2} & \cdots & \varepsilon_{x_{2J}}^{f_2} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{x_{J1}}^{f_J} & \varepsilon_{x_{J2}}^{f_J} & \cdots & \varepsilon_{x_{JJ}}^{f_J} \end{bmatrix}, \ \boldsymbol{\Psi} = (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}.$$

In the standard production networks vocabulary, $\boldsymbol{\Omega}$ is the sales based input-output matrix and $\boldsymbol{\Psi}$ is the sales based Leontief inverse.[1]

[1] Since our model abstracts from markup wedges on prices, the sales based and cost based input-output matrices will coincide.

In addition define

$$\boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\boldsymbol{\mathcal{D}}} = \begin{bmatrix} \varepsilon_{c_1}^{\mathcal{D}} \\ \varepsilon_{c_2}^{\mathcal{D}} \\ \vdots \\ \varepsilon_{c_J}^{\mathcal{D}} \end{bmatrix}_{J\times 1}, \quad \boldsymbol{\varepsilon}_{\boldsymbol{N}}^{\boldsymbol{f}} = \begin{bmatrix} \varepsilon_{N_{11}}^{f_1} & \varepsilon_{N_{12}}^{f_1} & \cdots & \varepsilon_{N_{1\mathcal{O}}}^{f_1} \\ \varepsilon_{N_{21}}^{f_2} & \varepsilon_{N_{22}}^{f_2} & \cdots & \varepsilon_{N_{2\mathcal{O}}}^{f_2} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{N_{J1}}^{f_J} & \varepsilon_{N_{J2}}^{f_J} & \cdots & \varepsilon_{N_{J\mathcal{O}}}^{f_J} \end{bmatrix}_{J\times \mathcal{O}}, \quad \boldsymbol{\varepsilon}_{\boldsymbol{K}}^{\boldsymbol{f}} = \begin{bmatrix} \varepsilon_{K_{11}}^{f_1} & \varepsilon_{K_{12}}^{f_1} & \cdots & \varepsilon_{K_{1\mathcal{K}}}^{f_1} \\ \varepsilon_{K_{21}}^{f_2} & \varepsilon_{K_{22}}^{f_2} & \cdots & \varepsilon_{K_{2\mathcal{K}}}^{f_2} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{K_{J1}}^{f_J} & \varepsilon_{K_{J2}}^{f_J} & \cdots & \varepsilon_{K_{J\mathcal{K}}}^{f_J} \end{bmatrix}_{J\times \mathcal{K}}$$

$$\boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\mathcal{Q}}} = \begin{bmatrix} \varepsilon_{\theta_1}^{\mathcal{Q}_1} \\ \varepsilon_{\theta_2}^{\mathcal{Q}_2} \\ \vdots \\ \varepsilon_{\theta_{\mathcal{O}}}^{\mathcal{Q}_{\mathcal{O}}} \end{bmatrix}_{\mathcal{O}\times 1}, \quad \boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\mathcal{F}}} = \begin{bmatrix} \varepsilon_{\theta_1}^{\mathcal{F}_1} \\ \varepsilon_{\theta_2}^{\mathcal{F}_2} \\ \vdots \\ \varepsilon_{\theta_{\mathcal{O}}}^{\mathcal{F}_{\mathcal{O}}} \end{bmatrix}_{\mathcal{O}\times 1}, \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_1(\theta_1) \\ \tau_2(\theta_2) \\ \vdots \\ \tau_{\mathcal{O}}(\theta_{\mathcal{O}}) \end{bmatrix}_{\mathcal{O}\times 1}$$

Furthermore, let

$$\boldsymbol{\mathcal{Q}} = \operatorname{diag}\left(\boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\mathcal{Q}}}\right)_{\mathcal{O}\times\mathcal{O}}, \ \boldsymbol{\mathcal{F}} = \operatorname{diag}\left(\boldsymbol{\varepsilon}_{\boldsymbol{\theta}}^{\boldsymbol{\mathcal{F}}}\right)_{\mathcal{O}\times\mathcal{O}}, \ \boldsymbol{\mathcal{T}} = \operatorname{diag}\left(\boldsymbol{\tau}\right)_{\mathcal{O}\times\mathcal{O}}$$

And let $\boldsymbol{\mathcal{L}}_{\mathcal{O}\times J}$ be

$$\boldsymbol{\mathcal{L}} = \begin{bmatrix} \frac{l_{11}}{L_1^d} & \frac{l_{21}}{L_1^d} & \cdots & \frac{l_{J1}}{L_1^d} \\ \frac{l_{12}}{L_2^d} & \frac{l_{22}}{L_2^d} & \cdots & \frac{l_{J2}}{L_2^d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{l_{1\mathcal{O}}}{L_{\mathcal{O}}^d} & \frac{l_{2\mathcal{O}}}{L_{\mathcal{O}}^d} & \cdots & \frac{l_{J\mathcal{O}}}{L_{\mathcal{O}}^d} \end{bmatrix}$$

4.2. Wage changes

Since wages are not uniquely pinned down in models featuring matching frictions, we need to assume a wage schedule to close the model. In particular, it will be convenient to assume

$$d\log \boldsymbol{w} - \boldsymbol{\mathcal{L}}d\log \boldsymbol{p} = \boldsymbol{\Lambda_A} d\log \boldsymbol{A} + \boldsymbol{\Lambda_H} d\log \boldsymbol{H} + \boldsymbol{\Lambda_K} d\log \boldsymbol{K}$$

We can think of $d\log \boldsymbol{w} - \boldsymbol{\mathcal{L}}d\log \boldsymbol{p}$ as akin to a real wage. It is the wage net of occupational employment share weighted sectoral prices. $\boldsymbol{\Lambda_A}$ contains wage elasticities to productivity changes and $\boldsymbol{\Lambda_H}$ contains wage elasticities to labor force changes.

4.3. Price Propagation

Log-linearizing the production function, for each sector i, we have:

$$d\log y_i = \underbrace{\varepsilon^{f_i}_{A_i}}_{=1} d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} d\log N_{io} + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} d\log x_{ij} + \sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} d\log K_{ik}$$

Plugging in Equation 4, Equation 3, and Equation 5, the first order conditions for optimal input usage, into the log-linearized production function gives

$$\begin{aligned}
d\log y_i &= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log \varepsilon^{f_i}_{N_{io}} + d\log y_i + d\log p_i - d\log w_o - d\log\left(1+\tau_o(\theta_o)\right)\right] \\
&+ \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} \left[d\log \varepsilon^{f_i}_{x_{ij}} + d\log y_i + d\log p_i - d\log p_j\right] \\
&+ \sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} \left[d\log \varepsilon^{f_i}_{K_{ik}} + d\log y_i + d\log p_i - d\log r_k\right] + d\log A_i \\
&= \left[d\log y_i + d\log p_i\right] \underbrace{\left[\sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} + \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}}\right]}_{=1 \text{ by crts}} + \underbrace{\left[\sum_{k=1}^{\mathcal{K}} d\varepsilon^{f_i}_{k_{ik}} + \sum_{o=1}^{\mathcal{O}} d\varepsilon^{f_i}_{N_{io}} + \sum_{j=1}^{N} d\varepsilon^{f_i}_{x_{ij}}\right]}_{=0 \text{ by crts}} \\
&- \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log w_o + d\log(1+\tau_o(\theta_o))\right] - \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} \left[d\log p_j\right] - \sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} d\log r_k + d\log A_i,
\end{aligned}$$

where the second equality holds because the sum of elasticities equals one for constant returns to scale technology and $\varepsilon^{f_i}_{x_{ij}} d\log \varepsilon^{f_i}_{x_{ij}} = d\varepsilon^{f_i}_{x_{ij}}$.

Rearranging terms gives

$$\begin{aligned}
d\log p_i &= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log w_o + d\log(1+\tau_o(\theta_o))\right] + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} \left[d\log p_j\right] + \sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} d\log r_k - d\log A_i \\
&= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log w_o + \varepsilon^{1+\tau_o}_{\theta_o} d\log \theta_o\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left[d\log p_j\right] + \sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} d\log r_k - d\log A_i \\
&= \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[d\log w_o - \tau_o(\theta_o) \varepsilon^{\mathcal{Q}_o}_{\theta_o} d\log \theta_o\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left[d\log p_j\right] + \sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} d\log r_k - d\log A_i
\end{aligned}$$

Stacking equations over sectors, we can write

$$d\log \boldsymbol{p} = \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left[d\log \boldsymbol{w} - \mathfrak{Q}\mathfrak{T} d\log\theta\right] + \boldsymbol{\Omega} d\log \boldsymbol{p} + \varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log \boldsymbol{r} - d\log \boldsymbol{A}$$

Which implies

$$d\log \boldsymbol{p} = \boldsymbol{\Psi}\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left[d\log \boldsymbol{w} - \mathfrak{Q}\mathfrak{T} d\log\theta\right] + \varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log \boldsymbol{r} - d\log \boldsymbol{A}\right]$$

Or equivalently

$$\left(\boldsymbol{I} - \boldsymbol{\Psi}\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\mathcal{L}\right) d\log \boldsymbol{p} = \boldsymbol{\Psi}\left[\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left[d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p} - \mathfrak{Q}\mathfrak{T} d\log\theta\right] + \varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log \boldsymbol{r} - d\log \boldsymbol{A}\right]$$

Where $d\log \boldsymbol{r}$ is pinned down by market clearing in the additional factor market for K_k. Assuming Cobb-Douglas production, for any i

$$d\log r_k = d\log y_i + d\log p_i - d\log K_{ik}$$

Which means that $d\log k_{ik} = d\log K_{jk}$ for all i, j. Furthermore, this means that $d\log K_{ik} = d\log K_k^s$ for all i. So we can write,

$$d\log r_k = d\log y_i + d\log p_i - d\log K_k^s$$

Which pins down $d\log r_k$ given a numeraire sector.

4.4. Output Propagation

Since the log-linearized expression for the Domar weight must hold for every sector, we can write

$$\begin{aligned} d\log\lambda_i - d\log\lambda_j &= d\log p_i - d\log p_j + d\log y_i - d\log y_j \\ &= d\log x_{ij} - d\log \varepsilon_{x_{ij}}^{f_i} - d\log y_j \\ \Rightarrow d\log x_{ij} &= d\log\lambda_i - d\log\lambda_j + d\log y_j + d\log \varepsilon_{x_{ij}}^{f_i} \end{aligned}$$

Plugging back into the production function,

$$d\log y_i = d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon_{N_{io}}^{f_i} d\log N_{io} + \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i} + d\log y_j + \sum_{k=1}^{\mathcal{K}} \varepsilon_{K_{ik}}^{f_i} d\log K_{ik}$$

$$= d\log A_i + \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} d\log N_{io} + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} d\log y_j + \sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} d\log K^s_K$$

Using the definition of labor demand,

$$\begin{aligned}\sum_i \frac{l_{io}}{L_o} d\log N_{io} &= d\log L^d_o - d\log(1+\tau_o(\theta_o)) \\ &= d\log L^d_o + \tau_o(\theta_o)\varepsilon^{\mathcal{Q}_o}_{\theta_o} d\log\theta_o\end{aligned}$$

From output labor usage, we have that labor usage ratio for an occupation by two different sectors as:

$$\frac{l_{io}}{l_{jo}} = \frac{\varepsilon^{f}_{N_{io}}\lambda_i}{\varepsilon^{f}_{N_{jo}}\lambda_j}$$

for any $l_{io}, l_{jo} > 0$

Log-linearizing it, assuming Cobb-Douglas preferences, yields:

$$d\log l_{io} = d\log l_{jo}$$

Also since, $d\log l_{io} = d\log N_{io} + d\log(1+\tau_o(\theta_o)) = d\log l_{jo} = d\log N_{jo} + d\log(1+\tau_o(\theta_o))$, we have that $d\log N_{io} = d\log N_{jo}$

Using the labor market clearing condition, and the definition of labor supply,

$$\begin{aligned}\sum_k \frac{l_{ko}}{L_o} d\log N_{ko} &= \left(\varepsilon^{\mathcal{F}_o}_{\theta_o} + \tau_o(\theta_o)\varepsilon^{\mathcal{Q}_o}_{\theta_o}\right) d\log\theta_o + d\log H_o \\ \Rightarrow d\log N_{io} \underbrace{\sum_k \frac{l_{ko}}{L_o}}_{=1} &= \left(\varepsilon^{\mathcal{F}_o}_{\theta_o} + \tau_o(\theta_o)\varepsilon^{\mathcal{Q}_o}_{\theta_o}\right) d\log\theta_o + d\log H_o\end{aligned}$$

Plugging this back into the linearized production function gives:

$$\begin{aligned}d\log y_i = d\log A_i &+ \sum_{o=1}^{\mathcal{O}} \varepsilon^{f_i}_{N_{io}} \left[\left(\varepsilon^{\mathcal{F}_o}_{\theta_o} + \tau_o(\theta_o)\varepsilon^{\mathcal{Q}_o}_{\theta_o}\right) d\log\theta_o + d\log H_o\right] \\ &+ \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} d\log y_j + \sum_{k=1}^{\mathcal{K}} \varepsilon^{f_i}_{K_{ik}} d\log K^s_k\end{aligned}$$

Stacking over sectors gives,

$$d\log \boldsymbol{y} = d\log \boldsymbol{A} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} (\mathfrak{F} + \mathfrak{Q}\mathfrak{T})\, d\log\theta + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} d\log \boldsymbol{H} + \boldsymbol{\Omega} d\log \boldsymbol{y} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}} d\log \boldsymbol{K}^{s}$$

Which implies

$$d\log \boldsymbol{y} = \boldsymbol{\Psi}\left(d\log \boldsymbol{A} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} (\mathfrak{F} + \mathfrak{Q}\mathfrak{T})\, d\log\theta + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} d\log \boldsymbol{H} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}} d\log \boldsymbol{K}^{s}\right)$$

4.5. Tightness Propagation

Labor market clearing implies that changes in labor demand have to equal changes in labor supply:

$$d\log L^{s}_{o}(\theta, \boldsymbol{H}) = d\log L^{d}_{o}(\theta, \boldsymbol{A}).$$

$$\varepsilon^{\mathcal{F}_o}_{\theta_o} d\log\theta_o + d\log H_o = \sum_{i=1}^{J} \frac{l_{io}}{L^{d}_{o}} d\log l_{io}(\theta_o)$$

Where $\frac{l_{io}}{L^{d}_{o}} = \frac{\varepsilon^{f_i}_{N_{io}} p_i y_i}{\sum_{j=1}^{J} \varepsilon^{f_j}_{N_{jo}} p_j y_j}$[2]. For every sector i we have

$$d\log l_{io}(\theta_o) = d\log \varepsilon^{f_i}_{N_{io}} - d\log w_o + d\log p_i + d\log y_i$$

Which implies that

$$d\log \boldsymbol{L^{d}}(\theta) = \operatorname{diag}\left(\mathcal{L} d\log \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) - \left[d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p}\right] + \mathcal{L} d\log \boldsymbol{y}$$

since $\sum_{i=1}^{J} \frac{l_{io}}{L^{d}_{o}} = 1$ for all o. Plugging in for $d\log \boldsymbol{y}$ gives

$$\begin{aligned} d\log \boldsymbol{L^{d}}(\theta) &= \operatorname{diag}\left(\mathcal{L} d\log \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) - \left[d\log \boldsymbol{w} - \mathcal{L} d\log \boldsymbol{p}\right] \\ &\quad + \mathcal{L}\boldsymbol{\Psi}\left[d\log \boldsymbol{A} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} (\mathfrak{F} + \mathfrak{Q}\mathfrak{T})\, d\log\theta + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}} d\log \boldsymbol{H} + \varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}} d\log \boldsymbol{K}^{s}\right] \end{aligned}$$

[2] One implication of this formula is that I think we should be able to check whether the elasticities $\left\{\left\{\varepsilon^{f}_{N_{io}}\right\}_{o=1}^{\mathcal{O}}\right\}_{i=1}^{J}$ are consistent with the Domar weights.

Labor market clearing implies

$$\mathfrak{F} d\log\theta + d\log \boldsymbol{H} = \mathcal{L}\Psi\left[d\log \boldsymbol{A} + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left(\mathfrak{F} + \mathfrak{Q}\mathfrak{T}\right)d\log\theta + \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} d\log \boldsymbol{H} + \varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log \boldsymbol{K}^{s}\right] - \left[d\log \boldsymbol{w} - \mathcal{L} d\log\ \boldsymbol{p}\right]$$

Which pins down first order changes in log tightness as

$$d\log\theta = \left[\mathfrak{F} - \Xi_{\theta}\right]^{-1}\left[\mathcal{L}\Psi d\log \boldsymbol{A} - \left[d\log \boldsymbol{w} - \mathcal{L} d\log\ \boldsymbol{p}\right] + \left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log \boldsymbol{H} + \mathcal{L}\Psi\varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log \boldsymbol{K}^{s}\right]$$

Where $\Xi_{\theta} = \mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\left[\mathfrak{F} + \mathfrak{Q}\mathfrak{T}\right]$.

4.6. Unemployment

Unemployment in occupation o is $U_o = H_o - L_o$. The log change in Unemployment is

$$d\log \boldsymbol{U} = d\log \boldsymbol{H} - d\log \boldsymbol{L} \tag{7}$$

Note though, that this equation would be different in a model that allows for an existing initial stock of workers. In that case we would need to re-weight by changes in the relative stock.

5. Aggregation

5.1. General Case

Using the first order condition,

$$d\log\varepsilon_{c_i}^{\mathcal{D}} = d\log\ p_i + d\log c_i - d\log\sum_{j=1}^{J} p_k c_k$$

along with the definition of the Domar weight,

$$d\log\sum_{k=1}^{J} p_k c_k = d\log\ p_i + d\log\ y_i - d\log\lambda_i$$

gives

$$d\log c_i = d\log\varepsilon_{c_i}^{\mathcal{D}} + d\log\ y_i - d\log\lambda_i$$

Which implies the log change in real GDP is

$$\begin{aligned} d\log Y &= \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} d\log \boldsymbol{c} \\ &= \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}\prime} \left(d\log \boldsymbol{\varepsilon}_{\boldsymbol{c}}^{\mathcal{D}} + d\log \boldsymbol{y} \right) \end{aligned}$$

5.2. Aggregate employment

The aggregate labor force, employment, and unemployment are $H^{agg} = \sum_{o=1}^{\mathcal{O}} H_o$, $L^{agg} = \sum_{o=1}^{\mathcal{O}} L_o$, and $U^{agg} = \sum_{o=1}^{\mathcal{O}} U_o$. Changes in aggregates are therefore given by

$$\begin{aligned} dH^{agg} &= \sum_{o=1}^{\mathcal{O}} dH_o \\ dL^{agg} &= \sum_{o=1}^{\mathcal{O}} dL_o \\ dU^{agg} &= \sum_{o=1}^{\mathcal{O}} dU_o \end{aligned}$$

Or in terms of log changes

$$\begin{aligned} d\log H^{agg} &= \frac{1}{H^{agg}} \sum_{o=1}^{\mathcal{O}} H_o d\log H_o \\ d\log L^{agg} &= \frac{1}{L^{agg}} \sum_{o=1}^{\mathcal{O}} L_o d\log L_o \\ d\log U^{agg} &= \frac{1}{U^{agg}} \sum_{o=1}^{\mathcal{O}} U_o d\log U_o \end{aligned}$$

In matrix notation

$$\begin{aligned} d\log H^{agg} &= \frac{1}{H^{agg}} \boldsymbol{H}' d\log \boldsymbol{H} \\ d\log L^{agg} &= \frac{1}{L^{agg}} \boldsymbol{L}' d\log \boldsymbol{L} \\ d\log U^{agg} &= \frac{1}{U^{agg}} \boldsymbol{U}' d\log \boldsymbol{U} \end{aligned}$$

Substituting in for $d\log \boldsymbol{L}$

$$
\begin{aligned}
d\log L^{agg} &= \frac{1}{L^{agg}}\boldsymbol{L'}\left[\mathcal{L}d\log \boldsymbol{y} - \left[d\log \boldsymbol{w} - \mathcal{L}d\log \boldsymbol{p}\right]\right] \\
&= \frac{1}{L^{agg}}\boldsymbol{L'}\left[\mathcal{L}\left[\boldsymbol{\Pi}_{y,A}d\log \boldsymbol{A} + \boldsymbol{\Pi}_{y,H}d\log \boldsymbol{H} - \boldsymbol{\Lambda}_A d\log \boldsymbol{A} - \boldsymbol{\Lambda}_H d\log \boldsymbol{H}\right]\right] \\
&= \boldsymbol{\Pi}_{L^{agg},A}d\log \boldsymbol{A} + \boldsymbol{\Pi}_{L^{agg},H}d\log \boldsymbol{H}
\end{aligned}
$$

Where

$$
\begin{aligned}
\boldsymbol{\Pi}_{L^{agg},A} &= \frac{1}{L^{agg}}\boldsymbol{L}'\left[\mathcal{L}\boldsymbol{\Pi}_{y,A} - \boldsymbol{\Lambda}_A\right] \\
\boldsymbol{\Pi}_{L^{agg},H} &= \frac{1}{L^{agg}}\boldsymbol{L}'\left[\mathcal{L}\boldsymbol{\Pi}_{y,H} - \boldsymbol{\Lambda}_H\right]
\end{aligned}
$$

And

$$
\begin{aligned}
d\log U^{agg} &= \frac{1}{U^{agg}}\boldsymbol{U}'\left[d\log \boldsymbol{H} - d\log \boldsymbol{L}\right] \\
&= \frac{1}{U^{agg}}\boldsymbol{U}'\left[d\log \boldsymbol{H} - \left[\mathcal{L}\left[\boldsymbol{\Pi}_{y,A}d\log \boldsymbol{A} + \boldsymbol{\Pi}_{y,H}d\log \boldsymbol{H} - \boldsymbol{\Lambda}_A d\log \boldsymbol{A} - \boldsymbol{\Lambda}_H d\log \boldsymbol{H}\right]\right]\right] \\
&= \boldsymbol{\Pi}_{U^{agg},A}d\log \boldsymbol{A} + \boldsymbol{\Pi}_{U^{agg},H}d\log \boldsymbol{H}
\end{aligned}
$$

Where

$$
\begin{aligned}
\boldsymbol{\Pi}_{U^{agg},A} &= \frac{1}{U^{agg}}\boldsymbol{U}'\left[\boldsymbol{\Lambda}_A - \mathcal{L}\boldsymbol{\Pi}_{y,A}\right] \\
\boldsymbol{\Pi}_{U^{agg},H} &= \frac{1}{U^{agg}}\boldsymbol{U}'\left[\boldsymbol{I} + \boldsymbol{\Lambda}_H - \mathcal{L}\boldsymbol{\Pi}_{y,H}\right]
\end{aligned}
$$

6. Marginal product of labor

The marginal product of type o labor in sector i is

$$
\frac{\partial y_i}{\partial N_{io}} = \begin{cases} \frac{\varepsilon^{f_i}_{N_{io}}}{N_{io}} y_i & \text{if } \varepsilon^{f_i}_{N_{io}} > 0 \\ 0 & \text{if } \varepsilon^{f_i}_{N_{io}} = 0 \end{cases} = \mathrm{mp}_{io}
$$

So, assuming Cobb-Douglas production

$$
d\log \mathrm{mp}_{io} = \begin{cases} d\log y_i - d\log N_{io} & \text{if } \varepsilon^{f_i}_{N_{io}} > 0 \\ 0 & \text{if } \varepsilon^{f_i}_{N_{io}} = 0 \end{cases}
$$

Note, if $\varepsilon^{f_i}_{N_{io}} = 0$, then $\frac{l_{io}}{L_o} = 0$. So we can write,

$$\frac{l_{io}}{L_o} d \log \mathrm{mp}_{io} = \frac{l_{io}}{L_o} \left(d \log y_i - d \log N_{io} \right)$$

Substituting in for $d \log N_{io}$ from the first order condition of firms gives

$$\begin{aligned} \frac{l_{io}}{L_o} d \log \mathrm{mp}_{io} &= \frac{l_{io}}{L_o} \left(d \log(1 + \tau_o(\theta_o)) + d \log w_o - d \log p_i \right) \\ &= \frac{l_{io}}{L_o} \left(-\tau_o(\theta_o) \varepsilon^{Q_o}_{\theta_o} d \log \theta_o + d \log w_o - d \log p_i \right) \end{aligned}$$

Then summing over sectors, and using $\sum_{i=1}^{J} \frac{l_{io}}{L_o} = 1$, we can write

$$\mathcal{L}'_o d \log \mathrm{MP}_o = -\tau_o(\theta_o) \varepsilon^{\Omega_o}_{\theta_o} d \log \theta_o + d \log w_o - \mathcal{L}'_o d \log \boldsymbol{p}$$

Where

$$\mathcal{L}_o = \begin{bmatrix} \frac{l_{1o}}{L_o} \\ \frac{l_{2o}}{L_o} \\ \vdots \\ \frac{l_{Jo}}{\Lambda_o} \end{bmatrix}, \; d \log \mathrm{MP}_o = \begin{bmatrix} d \log \mathrm{mp}_{1o} \\ d \log \mathrm{mp}_{2o} \\ \vdots \\ d \log \mathrm{mp}_{Jo} \end{bmatrix}$$

Now, stacking over occupations gives

$$\mathcal{L} d \log \mathrm{MP} = -\mathfrak{T} \mathfrak{Q} d \log \theta + d \log \boldsymbol{w} - \mathcal{L} d \log \boldsymbol{p}$$

Now, suppose that price adjusted wages respond exactly proportionally to the network adjusted marginal product of labor. Then

$$\begin{aligned} &\mathcal{L} d \log \mathrm{MP} = -\mathfrak{T} \mathfrak{Q} d \log \theta + \mathcal{L} d \log \mathrm{MP} \\ &\Rightarrow -\mathfrak{T} \mathfrak{Q} d \log \theta = 0 \\ &\Rightarrow d \log \theta = 0 \end{aligned}$$

Suppose instead that wages change proportionally to changes in the marginal product, so that $d \log \boldsymbol{w} - \mathcal{L} d \log \boldsymbol{p} = \gamma d \log \mathrm{MP}$. Then,

$$d \log \boldsymbol{w} - \mathcal{L} d \log \boldsymbol{p} = -\frac{\gamma}{1 - \gamma} \mathfrak{Q} \mathfrak{T} d \log \theta$$

Changes in tightness now satisfy,

$$d\log\theta = [\mathcal{F} - \Xi_{MP}]^{-1}\left[\mathcal{L}\Psi d\log\boldsymbol{A} + \left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log\boldsymbol{H} + \mathcal{L}\Psi\varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log\boldsymbol{K}^{s}\right]$$

Where

$$\Xi_{MP} = \frac{\gamma}{1-\gamma}\Omega\mathcal{T} + \mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}(\mathcal{F} + \Omega\mathcal{T})$$

Which means we can write this scenario in the same terms as above,

$$d\log\boldsymbol{w} - \mathcal{L}d\log\boldsymbol{p} = \boldsymbol{\Lambda}_{A}d\log\boldsymbol{A} + \boldsymbol{\Lambda}_{H}d\log\boldsymbol{H} + \boldsymbol{\Lambda}_{K}d\log\boldsymbol{K}^{s}$$

Where

$$\begin{aligned}
\boldsymbol{\Lambda}_{A} &= -\frac{\gamma}{1-\gamma}\Omega\mathcal{T}\,[\mathcal{F} - \Xi_{MP}]^{-1}\mathcal{L}\Psi \\
\boldsymbol{\Lambda}_{H} &= -\frac{\gamma}{1-\gamma}\Omega\mathcal{T}\,[\mathcal{F} - \Xi_{MP}]^{-1}\left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] \\
\boldsymbol{\Lambda}_{K} &= -\frac{\gamma}{1-\gamma}\Omega\mathcal{T}\,[\mathcal{F} - \Xi_{MP}]^{-1}\mathcal{L}\Psi\varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}}
\end{aligned}$$

7. Nominally Rigid Wages

Suppose instead that wages are fully nominally rigid, so that

$$d\log\boldsymbol{w} - \mathcal{L}d\log\boldsymbol{p} = -\mathcal{L}d\log\boldsymbol{p}$$

In this case

$$d\log\theta = \left[\mathcal{F} - \Xi_{\theta}\right]^{-1}\left[\mathcal{L}\Psi d\log\boldsymbol{A} + \mathcal{L}d\log\boldsymbol{p} + \left[\mathcal{L}\Psi\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}} - \boldsymbol{I}\right] d\log\boldsymbol{H} + \mathcal{L}\Psi\varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log\boldsymbol{K}^{s}\right]$$

And

$$d\log\boldsymbol{p} = \Psi\left[\varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log\boldsymbol{r} - d\log\boldsymbol{A} - \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\Omega\mathcal{T}d\log\theta\right]$$

Plugging into the expression for $d\log\theta$ gives

$$d\log\theta = \left[\mathcal{F} - \Xi_{\theta}\right]^{-1}\mathcal{L}\Psi\left[\varepsilon_{\boldsymbol{K}}^{\boldsymbol{f}} d\log\boldsymbol{r} - d\log\boldsymbol{A} - \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\Omega\mathcal{T}d\log\theta\right]$$

$$+\left[\mathcal{F}-\Xi_\theta\right]^{-1}\left[\mathcal{L}\boldsymbol{\Psi} d\log \boldsymbol{A}+\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}-\boldsymbol{I}\right]d\log \boldsymbol{H}+\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}}d\log \boldsymbol{K}^s\right]$$

$$\Rightarrow \left[\boldsymbol{I}+\left[\mathcal{F}-\Xi_\theta\right]^{-1}\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\mathfrak{Q}\mathfrak{T}\right]d\log\theta=\left[\mathcal{F}-\Xi_\theta\right]^{-1}\mathcal{L}\boldsymbol{\Psi}\left[\varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}}d\log \boldsymbol{r}-d\log \boldsymbol{A}\right]$$
$$+\left[\mathcal{F}-\Xi_\theta\right]^{-1}\left[\mathcal{L}\boldsymbol{\Psi} d\log \boldsymbol{A}+\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}-\boldsymbol{I}\right]d\log \boldsymbol{H}+\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}}d\log \boldsymbol{K}^s\right]$$

Let

$$\Xi_{nom}=\left[\boldsymbol{I}+\left[\mathcal{F}-\Xi_\theta\right]^{-1}\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\mathfrak{Q}\mathfrak{T}\right]^{-1}\left[\mathcal{F}-\Xi_\theta\right]^{-1}$$

Then

$$d\log\theta=\Xi_{nom}\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}}\left[d\log \boldsymbol{r}+d\log \boldsymbol{K}^s\right]-\left[\boldsymbol{I}-\mathcal{L}\boldsymbol{\Psi}\right]d\log \boldsymbol{A}+\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}-\boldsymbol{I}\right]d\log \boldsymbol{H}\right]$$

Using the result for $d\log \boldsymbol{r}+d\log \boldsymbol{K}^s=d\log y_i\mathbf{1}$ if i is the numeraire,

$$d\log\theta=\Xi_{nom}\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}}\mathbf{1}d\log y_i-\left[\boldsymbol{I}-\mathcal{L}\boldsymbol{\Psi}\right]d\log \boldsymbol{A}+\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}-\boldsymbol{I}\right]d\log \boldsymbol{H}\right]$$

Now, for output we have

$$\begin{aligned}
d\log \boldsymbol{y}&=\boldsymbol{\Psi}\left(d\log \boldsymbol{A}+\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left(\mathcal{F}+\mathfrak{Q}\mathfrak{T}\right)d\log\theta+\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}d\log \boldsymbol{H}+\varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}}d\log \boldsymbol{K}^s\right)\\
&=\boldsymbol{\Psi}\left[\boldsymbol{I}-\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left(\mathcal{F}+\mathfrak{Q}\mathfrak{T}\right)\psi_{nom}\left[\boldsymbol{I}-\mathcal{L}\boldsymbol{\Psi}\right]\right]d\log \boldsymbol{A}\\
&+\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left[\boldsymbol{I}+\left(\mathcal{F}+\mathfrak{Q}\mathfrak{T}\right)\Xi_{nom}\left[\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}-\boldsymbol{I}\right]\right]d\log \boldsymbol{H}\\
&+\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}}d\log \boldsymbol{K}^s+\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\left(\mathcal{F}+\mathfrak{Q}\mathfrak{T}\right)\Xi_{nom}\mathcal{L}\boldsymbol{\Psi}\varepsilon^{\boldsymbol{f}}_{\boldsymbol{K}}\mathbf{1}d\log y_i
\end{aligned}$$