Unemployment in a Production Network: Theory

Finn Schüle and Haoyu Sheng

February 6, 2023

1. Households and final goods production

We consider a closed, static economy model with no government spending. There is no saving mechanism in the economy, and real household consumption equates real GDP, denoted by Y. A final goods producer with constant returns to scale technology aggregates J sector outputs to produce Y

$$Y = \max_{\{c_i\}_{i=1}^J} \mathcal{D}\left(\{c_i\}_{i=1}^J\right)$$

Subject to the budget constraint

$$\sum_{i=1}^{J} p_i c_i = \sum_{i=1}^{J} w_i L_i.$$

$\mathcal{D}$ captures household preferences over final consumption goods, and w_i is the wage of sector i labor L_i.

The household's consumption decision can be computed using the first order condition:

$$\varepsilon_{c_i}^{\mathcal{D}} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}, \tag{1}$$

Brown University.

where $\varepsilon^{\mathcal{D}}_{c_i}$ denotes the households elasticity of utility with regards to the consumption of good i.

2. Sector level labor markets

We assume each sector has a separate labor market with a labor force of H_i possible workers, an exogenous separation rate s_i, and an exogenous recruiting cost r_i which measures the units of labor required to maintain each posted vacancy. When workers and firms meet there is a mutual gain from matching. There is no accepted theory for how wages are set in this context. For now we assume w_i follows a general wage schedule taken as given by both firms and workers. Hires are generated by a constant returns matching function in sector-level unemployment u_i and vacancies v_i

$$h_i = \phi_i m\left(u_i, v_i\right)$$

Let the sector-specific labor market tightness be $\theta_i = \frac{v_i}{u_i}$, the vacancy-filling rate $\mathcal{Q}_i(\theta_i) = \phi_i m\left(\frac{u_i}{v_i}, 1\right)$, and the job-finding rate $\mathcal{F}_i(\theta_i) = \phi_i m\left(1, \frac{v_i}{u_i}\right)$. We assume we start at a steady state featuring balanced flows, that is that the number of workers flowing into unemployment equals the number of workers flowing out of unemployment, labor supply satisfies

$$\begin{aligned} s_i L_i^s(\theta_i) &= \mathcal{F}_i(\theta_i) u_i \\ &= \mathcal{F}_i(\theta_i)(H_i - L_i^s(\theta_i)) \\ \Rightarrow L_i^s(\theta_i) &= \frac{\mathcal{F}_i(\theta_i)}{s_i + \mathcal{F}_i(\theta_i)} H_i \end{aligned} \tag{2}$$

Let N_i denote productive employees and R_i denote recruiters employed in sector i. Balanced flows implies the recruiter producer ratio $\tau_i(\theta_i) = \frac{R_i}{N_i}$ satisfies

$$\begin{aligned} s_i(N_i + R_i) &= \mathcal{Q}_i(\theta_i) v_i \\ \Rightarrow r_i s_i(N_i + R_i) &= \mathcal{Q}_i(\theta_i) R_i \\ \Rightarrow r_i s_i \tau_i(\theta_i)^{-1} &= \mathcal{Q}_i(\theta_i) - r_i s_i \\ \Rightarrow \tau_i(\theta_i) &= \frac{r_i s_i}{\mathcal{Q}_i(\theta_i) - r_i s_i} \end{aligned} \tag{3}$$

For a given target level of employment N_i, total required labor is $L_i^d(\theta_i) = \left(1 + \tau_i(\theta_i)\right) N_i$. We describe how labor demand, $L_i^d(\theta_i)$, is determined by firms' profit maximization in the next section.

3. Sector level firms

A representative firm in sector i uses labor N_i and intermediate inputs from sector j, x_{ij}, to produce output y_i using production technology f_i.

$$y_i = A_i f_i\left(N_i, \{x_{ij}\}_{j=1}^J\right)$$

Firms choose N_i and $\{x_{ij}\}_{j=1}^J$ to maximize profits, or equivalently to minimize costs. We assume firms are price takers in both input and output markets. Profits are given by

$$\pi_i = p_i A_i f_i\left(N_i, \{x_{ij}\}_{j=1}^J\right) - w_i(1 + \tau_i(\theta_i))N_i - \sum_{j=1}^J p_j x_{ij}$$

Firms choose inputs to solve

$$\max_{N_i, \left\{x_{ij}\right\}_{j=1}^J} \pi_i\left(N_i, \left\{x_{ij}\right\}_{j=1}^J\right)$$

Giving the first order conditions

$$\begin{aligned} p_i A_i f_{i,x_{ij}} &= p_j \\ p_i A_i f_{i,N_i} &= w_i\left(1 + \tau_i(\theta_i)\right) \end{aligned}$$

And labor demand is $L_i^d(\theta_i) = (1 + \tau_i(\theta_i))N_i$ for the optimal N_i. The equilibrium tightness equates labor demand and labor supply.

We can rewrite these expressions in terms of elasticities.

$$\varepsilon_{x_{ij}}^{f_i} = \frac{p_j x_{ij}}{p_i y_i} \tag{4}$$

$$\varepsilon_{N_i}^{f_i} = \left(1 + \tau_i(\theta_i)\right)\frac{w_i N_i}{p_i y_i} \tag{5}$$

From Equation 5, we can derive the labor demand equation:

$$L_i^d(\theta_i) = \varepsilon_{N_i}^{f_i} \frac{p_i y_i}{w_i} \tag{6}$$

4. Equilibrium

The equilibrium in this model can be characterized by a set of conditions guaranteeing labor market equilibrium and goods market equilibrium. The equilibrium is a collection of $10J + 2J^2$ endogenous variables $\left\{ p_i, y_i, \left\{x_{ij}, \varepsilon_{x_{ij}}^{f_i}\right\}_{j=1}^{J}, c_i, \varepsilon_{c_i}^{\mathcal{D}}, N_i, \varepsilon_{N_i}^{f_i}, \theta_i, w_i, L_i^d, L_i^s \right\}_{i=1}^{J}$ that satisfy equations 1 through 6, along with goods market clearing, labor market clearing, and constant returns, given exogenous variables $\left\{A_i, H_i\right\}_{i=1}^{J}$. We summarize the equilibrium conditions below for convenience.

4.1. Goods Market Equilibrium

In an equilibrium, firms intermediate input choices given prices and labor market chacteristics are profit maximizing:

(Intermediate input decision) $$\varepsilon_{x_{ij}}^{f_i} = \frac{p_j x_{ij}}{p_i y_i},$$

(Labor input decision) $$\varepsilon_{N_i}^{f_i} = \left(1 + \tau_i(\theta_i)\right) \frac{w_i N_i}{p_i y_i}.$$

Firms produce output via production technology f_i

(Production technology) $$y_i = A_i f_i \left(N_i, \left\{x_{ij}\right\}_{j=1}^{J} \right)$$

By constant returns to scale in production,

(Constant returns production) $$1 - \varepsilon_{N_i}^{f_i} = \sum_{j=1}^{J} \varepsilon_{x_{ij}}^{f_i}$$

In addition, the household maximizes their utility by choosing a consumption bundle that satisfies its first-order condition.

(Consumption decision) $$\varepsilon_{c_i}^{\mathcal{D}} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k}.$$

And by constant returns

$$\text{(Constant returns utility)} \qquad 1 = \sum_{i=1}^{J} \varepsilon_{c_i}^{\mathcal{D}}$$

Finally, the goods market has to clear, which means that, for each sector i, total production has to be equal to the sum of the household's consumption of good i and all other sectors' use of good i in their production:

$$\text{(Goods market clearing)} \qquad y_i = c_i + \sum_{j=1}^{J} x_{ji}.$$

In total, the goods market provides $5J + J^2 + 1$ restrictions.

4.2. Labor Market Equilibrium

From Equation 5, labor demand in sector i is defined as

$$\text{(Labor Demand)} \qquad L_i^d(\theta_i) = \varepsilon_{N_i}^{f_i} \frac{p_i y_i}{w_i}.$$

Recall, given sector level labor force participation H_i labor supply is

$$\text{(Labor Supply)} \qquad L_i^s(\theta_i) = \frac{f_i(\theta_i)}{s_i + f_i(\theta_i)} H_i.$$

Labor demand equals labor supply at an equilibrium in the labor market.

$$\text{(LM equilibrium)} \qquad L_i^d(\theta_i) = L_i^s(\theta_i).$$

These equilibrium conditions provide an additional $3J$ restrictions.

4.3. Summary

The equilibrium conditions outline above provide just $8J + J^2 + 1$ equations in $10J + 2J^2$ endogenous variables. The wage schedules taken as given by both households and firms provide another J restrictions. Nevertheless, as is typical in the literature, we need additional functional form assumptions on production and household preferences to close the model.

For instance, assuming Cobb-Douglas production and preferences fully parameterizes $\left\{ \left\{ \varepsilon^{f_i}_{x_{ij}} \right\}_{j=1}^{J}, \varepsilon^{f_i}_{N_i}, \varepsilon^{\mathcal{D}}_{c_i} \right\}_{i=1}^{J}$, giving us $2J + J^2$ additional restrictions, but removing $J + 1$ of the restrictions above.[1] Parametrizing production by assuming Cobb-Douglas therefore gives exactly the number of restrictions we need to close the model. Alternatively, assuming CES production indirectly provides restrictions to pin down the same set of elasticities.[2]

5. The Production Network

In this section, we first introduce notations that are key in understanding the production network. We then discuss three propagation mechanisms: prices, sales shares, and tightness.

5.1. Notation

We denote vectors and matrices by bold letters. For instance, $d\log \boldsymbol{x} = \begin{bmatrix} d\log x_1 & \cdots & d\log x_J \end{bmatrix}'$. We can conveniently capture many features of the production network through the following matrices

$$\boldsymbol{\Omega} = \begin{bmatrix} \varepsilon^{f_1}_{x_{11}} & \varepsilon^{f_1}_{x_{12}} & \cdots & \varepsilon^{f_1}_{x_{1J}} \\ \varepsilon^{f_2}_{x_{21}} & \varepsilon^{f_2}_{x_{22}} & \cdots & \varepsilon^{f_2}_{x_{2J}} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon^{f_J}_{x_{J1}} & \varepsilon^{f_J}_{x_{J2}} & \cdots & \varepsilon^{f_J}_{x_{JJ}} \end{bmatrix}, \quad \boldsymbol{\Psi} = (\boldsymbol{I} - \boldsymbol{\Omega})^{-1}.$$

In the standard production networks vocabulary, $\boldsymbol{\Omega}$ is the sales based input-output matrix and $\boldsymbol{\Psi}$ is the sales based Leontief inverse.

In addition define

$$\boldsymbol{\varepsilon}^{\mathcal{D}}_{\boldsymbol{c}} = \begin{bmatrix} \varepsilon^{\mathcal{D}}_{c_1} \\ \varepsilon^{\mathcal{D}}_{c_2} \\ \vdots \\ \varepsilon^{\mathcal{D}}_{c_J} \end{bmatrix}, \quad \boldsymbol{\varepsilon}^{\boldsymbol{f}}_{\boldsymbol{N}} = \begin{bmatrix} \varepsilon^{f_1}_{N_1} \\ \varepsilon^{f_2}_{N_2} \\ \vdots \\ \varepsilon^{f_J}_{N_J} \end{bmatrix}, \quad \boldsymbol{\varepsilon}^{\boldsymbol{\Omega}}_{\boldsymbol{\theta}} = \begin{bmatrix} \varepsilon^{q_1}_{\theta_1} \\ \varepsilon^{q_2}_{\theta_2} \\ \vdots \\ \varepsilon^{q_J}_{\theta_J} \end{bmatrix}, \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_1(\theta_1) \\ \tau_2(\theta_2) \\ \vdots \\ \tau_J(\theta_J) \end{bmatrix}$$

[1] Choosing elasticities directly makes the constant returns restrictions redundant.

[2] CES and Cobb-Douglas, a special case of CES, are the two most commonly assumed production technologies in the literature.

5.2. Price Propagation

Log-linearizing the production function, for each sector i, we have:

$$d\log y_i = \underbrace{\varepsilon^{f_i}_{A_i}}_{=1} d\log A_i + \varepsilon^{f_i}_{N_i} d\log N_i + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}} d\log x_{ij}$$

Plugging in Equation 5 and Equation 4, the first order conditions for optimal input usage, into the log-linearized production function gives

$$\begin{aligned}
d\log y_i &= \varepsilon^{f_i}_{N_i}\left[d\log \varepsilon^{f_i}_{N_i} + d\log y_i + d\log p_i - d\log w_i - d\log\left(1+\tau_i(\theta_i)\right)\right] \\
&\quad + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}}\left[d\log \varepsilon^{f_i}_{x_{ij}} + d\log y_i + d\log p_i - d\log p_j\right] + d\log A_i \\
&= \left[d\log y_i + d\log p_i\right] \underbrace{\left[\varepsilon^{f_i}_{N_i} + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}}\right]}_{=1 \text{ by crts}} + \underbrace{\left[d\varepsilon^{f_i}_{N_i} + \sum_{j=1}^{N} d\varepsilon^{f_i}_{x_{ij}}\right]}_{=0 \text{ by crts}} \\
&\quad - \varepsilon^{f_i}_{N_i}\left[d\log w_i + d\log(1+\tau_i(\theta_i))\right] - \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] + d\log A_i,
\end{aligned}$$

where the second inequality holds because the sum of elasticities equals one for constant returns to scale technology and $\varepsilon^{f_i}_{x_{ij}} d\log \varepsilon^{f_i}_{x_{ij}} = d\varepsilon^{f_i}_{x_{ij}}$.

Rearranging terms gives

$$\begin{aligned}
d\log p_i &= \varepsilon^{f_i}_{N_i}\left[d\log w_i + d\log(1+\tau_i(\theta_i))\right] + \sum_{j=1}^{N} \varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i \\
&= \varepsilon^{f_i}_{N_i}\left[d\log w_i + \varepsilon^{1+\tau_i}_{\theta_i} d\log \theta_i\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i \\
&= \varepsilon^{f_i}_{N_i}\left[d\log w_i - \tau_i(\theta_i)\varepsilon^{Q_i}_{\theta_i} d\log \theta_i\right] + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}}\left[d\log p_j\right] - d\log A_i \qquad (7)
\end{aligned}$$

By stacking equation (7) for each sector, we get the following expression for how prices change across the production network

$$d\log \boldsymbol{p} = \text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\left[d\log \boldsymbol{w} - \text{diag}\left(\boldsymbol{\tau}\right)\text{diag}\left(\varepsilon^{\Omega}_{\theta}\right)d\log\theta\right] + \boldsymbol{\Omega} d\log \boldsymbol{p} - d\log \boldsymbol{A}$$

(8)

$$\Rightarrow d\log \boldsymbol{p} = \boldsymbol{\Psi}(\underbrace{\text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)d\log \boldsymbol{w}}_{\text{factor prices}} - \underbrace{\text{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\text{diag}\left(\boldsymbol{\tau}\right)\text{diag}\left(\varepsilon^{\Omega}_{\theta}\right)d\log\theta}_{\text{searching and matching}} - \underbrace{d\log \boldsymbol{A}}_{\text{productivity}})$$

In other words, changes in prices comes from three sources - changes in wages, tightness, and productivity. The impact of changes in wages depends on labor elasticity of production, and the impact of changes in tightness depends additionally on the matching function and the recruiter-producer ratio. The impact of all three are amplified by the Leontief inverse $\boldsymbol{\Psi}$.

5.3. Sales Share Propgation

We can rewrite the goods market clearing condition in terms of Domar weights:

$$y_i = c_i + \sum_{j=1}^{J} x_{ji}$$

$$\Rightarrow \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k} = \frac{p_i c_i}{\sum_{k=1}^{J} p_k c_k} + \sum_{j=1}^{J} \frac{p_i x_{ji}}{p_j x_j} \frac{p_j x_j}{\sum_{k=1}^{J} p_k c_k}$$

(9)

$$\Rightarrow \lambda_i = \varepsilon^{\mathcal{D}}_{c_i} + \sum_{j=1}^{J} \varepsilon^{f_j}_{x_{ji}} \lambda_j,$$

where $\lambda_i = \frac{p_i y_i}{\sum_{k=1}^{J} p_k c_k}$ is the Domar weight of sector i.

By stacking (9) for each sector, we get the following expression for Domar weights across the production network.

$$\boldsymbol{\lambda}' = \varepsilon^{\mathcal{D}'}_{\boldsymbol{c}} + \boldsymbol{\lambda}'\boldsymbol{\Omega}$$

We can see how Domar weights change across the production network by totally differentiating

$$d\boldsymbol{\lambda}' = d\varepsilon^{\mathcal{D}'}_{\boldsymbol{c}} + d\boldsymbol{\lambda}'\boldsymbol{\Omega} + \boldsymbol{\lambda}' d\boldsymbol{\Omega}$$

$$\Rightarrow d\lambda' = \left[d\varepsilon_{\boldsymbol{c}}^{\mathcal{D}'} + \lambda' d\boldsymbol{\Omega} \right] \boldsymbol{\Psi} \tag{10}$$

5.4. Tightness Propagation

In response to shocks to either productivity or labor force, changes in labor demand has to equate changes in labor supply:

$$d \log L_i^s(\theta, \boldsymbol{H}) = d \log L_i^d(\theta, \boldsymbol{A}).$$

Let $\mathfrak{F} = \left[\frac{s_1}{s_1 + \mathcal{F}_1(\theta_1)} \varepsilon_{\theta_1}^{\mathcal{F}_1} \quad \cdots \quad \frac{s_J}{s_J + \mathcal{F}_J(\theta_J)} \varepsilon_{\theta_J}^{\mathcal{F}_J} \right]$, we can stack sector level Equation 2 and Equation 6 to get:

$$\begin{aligned}
\text{diag}\,(\mathfrak{F})\, d \log \theta + d \log \boldsymbol{H} &= d \log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}(\theta, \boldsymbol{A}) + d \log \ \boldsymbol{p}(\theta, \boldsymbol{A}) - d \log \boldsymbol{w}(\theta, \boldsymbol{A}) + d \log \ \boldsymbol{y}(\theta, \boldsymbol{A}) \\
&= d \log \varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}(\theta, \boldsymbol{A}) - d \log \boldsymbol{w}(\theta, \boldsymbol{A}) \\
&\quad + \boldsymbol{\Psi}(\text{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) d \log \boldsymbol{w}) \\
&\quad + \boldsymbol{\Psi}(\left(\boldsymbol{I} - \text{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) - \boldsymbol{\Omega}\right) d \log \boldsymbol{\lambda} + \text{diag}\left(\varepsilon_{\boldsymbol{N}}^{\boldsymbol{f}}\right) \left[\text{diag}\,(\mathfrak{F})\, d \log \theta + d \log \boldsymbol{H}\right])
\end{aligned}$$

NOTE: the non-invertibility issue appears again. Potential solution: wage schedule that relates to output or price changes. Alternatively, finding different ways to sub in $d \log \ p$ and $d \log \ y$. Key issue is that $d \log \ y$ derived based on labor supply condition.

6. Sector-level Response

Before aggregating, we are interested in exploring how sector-level economic variables respond to different shocks.

6.1. Output

First, we look at output. Log-linearizing Domar weights gives us:

$$d \log \lambda_i = d \log \ p_i + d \log \ y_i - d \log \sum_{k=1}^{J} p_k c_k.$$

Since this equation must hold for any i and j,

$$\begin{aligned} d\log\lambda_i - d\log\lambda_j &= d\log p_i - d\log p_j + d\log y_i - d\log y_j \\ &= d\log x_{ij} - d\log y_j \\ \Rightarrow d\log x_{ij} &= d\log y_j + d\log\lambda_i - d\log\lambda_j \end{aligned}$$

Using the sector level production function,

$$d\log y_i = d\log A_i + \varepsilon^{f_i}_{N_i} d\log N_i + \sum_{j=1}^{J} \varepsilon^{f_i}_{x_{ij}} \left(d\log y_j + d\log\lambda_i - d\log\lambda_j\right)$$

Since $(1+\tau_i(\theta_i))N_i = L_i$, $\delta\log N_i = d\log L_i - d\log(1+\tau_i(\theta_i))$, where $d\log L_i = d\log L^s_i = d\log L^d_i$. This implies that $d\log N_i = \left(\frac{s_i}{s_i+\mathcal{F}_i(\theta_i)}\varepsilon^{\mathcal{F}_i}_{\theta_i} + \varepsilon^{\Omega_i}_{\theta_i}\tau_i(\theta_i)\right) d\log\theta_i + d\log H_i$.

The sector-level log-linearized production function can be rewritten as:

$$\begin{aligned} d\log y_i = d\log A_i &+ \varepsilon^{f_i}_{N_i}\left[\left(\frac{s_i}{s_i+\mathcal{F}_i(\theta_i)}\varepsilon^{\mathcal{F}_i}_{\theta_i} + \varepsilon^{\Omega_i}_{\theta_i}\tau_i(\theta_i)\right) d\log\theta_i + d\log H_i\right] \\ &+ (1-\varepsilon^{f_i}_{N_i}) d\log\lambda_i + \sum_{j=1}^{J}\varepsilon^{f_i}_{x_{ij}}\left(d\log y_j - d\log\lambda_j\right) \end{aligned}$$

Stacking equations over sectors, we have:

$$\begin{aligned} d\log \boldsymbol{y} = d\log \boldsymbol{A} &+ \boldsymbol{\Omega} d\log \boldsymbol{y} + \left(\boldsymbol{I} - \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} \\ &+ \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\left[\left(\operatorname{diag}(\boldsymbol{\mathcal{F}}) + \operatorname{diag}(\boldsymbol{\tau})\operatorname{diag}\left(\varepsilon^{\boldsymbol{\Omega}}_{\boldsymbol{\theta}}\right)\right) d\log\boldsymbol{\theta} + d\log\boldsymbol{H}\right], \end{aligned}$$

which simplifies into:

$$\begin{aligned} d\log \boldsymbol{y} = \boldsymbol{\Psi}(d\log \boldsymbol{A} &+ \left(\boldsymbol{I} - \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right) - \boldsymbol{\Omega}\right) d\log\boldsymbol{\lambda} \\ &+ \operatorname{diag}\left(\varepsilon^{\boldsymbol{f}}_{\boldsymbol{N}}\right)\left[\left(\operatorname{diag}(\boldsymbol{\mathcal{F}}) + \operatorname{diag}(\boldsymbol{\tau})\operatorname{diag}\left(\varepsilon^{\boldsymbol{\Omega}}_{\boldsymbol{\theta}}\right)\right) d\log\boldsymbol{\theta} + d\log\boldsymbol{H}\right]). \end{aligned}$$

In general, sector level output behaves differently from the Cobb-Douglas case, but that difference is captured entirely by changes in Domar weights. This is a useful result, as discussed in the previous production networks literature, because this means we do not need to keep track of all intermediate input choices.

6.2. Unemployment

7. Aggregation

7.1. General Case

Using the first order condition,

$$d \log \varepsilon^{\mathcal{D}}_{c_i} = d \log p_i + d \log c_i - d \log \sum_{j=1}^{J} p_k c_k$$

along with the definition of the Domar weight,

$$d \log \sum_{k=1}^{J} p_k c_k = d \log p_i + d \log y_i - d \log \lambda_i$$

gives

$$d \log c_i = d \log \varepsilon^{\mathcal{D}}_{c_i} + d \log y_i - d \log \lambda_i$$

Which implies the log change in real GDP is

$$\begin{aligned}
d \log Y &= \boldsymbol{\varepsilon}^{\mathcal{D}\prime}_{\boldsymbol{c}} d \log \boldsymbol{c} \\
&= \boldsymbol{\varepsilon}^{\mathcal{D}\prime}_{\boldsymbol{c}} \left(d \log \boldsymbol{\varepsilon}^{\mathcal{D}}_{\boldsymbol{c}} + d \log \boldsymbol{y} - d \log \boldsymbol{\lambda} \right) \\
&= \boldsymbol{\varepsilon}^{\mathcal{D}\prime}_{\boldsymbol{c}} \left(d \log \boldsymbol{\varepsilon}^{\mathcal{D}}_{\boldsymbol{c}} - d \log \boldsymbol{p} + d \log \boldsymbol{\lambda} + \Xi_{\varepsilon} d \log \boldsymbol{\varepsilon}^{\boldsymbol{f}}_{\boldsymbol{N}} - d \log \boldsymbol{\lambda} \right) \\
&= -\boldsymbol{\varepsilon}^{\mathcal{D}\prime}_{\boldsymbol{c}} d \log \boldsymbol{p} + \boldsymbol{\varepsilon}^{\mathcal{D}\prime}_{\boldsymbol{c}} \left(d \log \boldsymbol{\varepsilon}^{\mathcal{D}}_{\boldsymbol{c}} + \Xi_{\varepsilon} d \log \boldsymbol{\varepsilon}^{\boldsymbol{f}}_{\boldsymbol{N}} \right)
\end{aligned}$$

7.2. Aggregate Output

7.3. Aggregate employment

References