PHYS 517 Project: Spherical Symmetric Numerical Relativity

Siyang Ling

April 27, 2020

1 Oppenheimer-Snyder Spherical Dust Collapse

Gravitational collapse is a well-studied phenomenon in general relativity. In particular, the case of *Oppenheimer-Synder spherical dust collapse* has been given special attention due to its simplicity. The two assumptions of this model are explained as follows:

"Spherical" Spherical symmetry implies that all physical variables (both metric and matter) should be functions of r (some radius parameter) alone. By Birkhoff's theorem, we know that the form of the metric is exactly the Schwarzschild metric in the vacuum exterior, and there is no radial dynamical degrees of freedom in the metric. This gives us easy-to-use boundary conditions for the metric, which is useful in numerical simulations.

"Dust" "Dust" refers to matter that does not interact with other particles. Namely, the equations of motion governing these particles are exactly their individual geodesic equations. In regard to a numerical scheme, this means that when we try to integrate for the motion of a particular particle, we don't need to consider its interaction with other particles, and the time derivatives of the particle parameters depend only on the local metric.

This model has been solved analytically by Oppenheimer and Snyder; Petrich, Shapiro and Teukolsky later computed some other relevant quantities in the case of polar time slicing. The metric can be divided in to two parts: the vacuum exterior and the interior where matter is present. The exterior has the usual static Schwarzschild metric, given by:

$$\mathrm{d}s^2 = -(1 - \frac{2M}{r_s})\,\mathrm{d}t^2 + (1 - \frac{2M}{r_s})^{-1}\,\mathrm{d}r_s^2 + r_s^2\,\mathrm{d}\Omega^2\,,$$

where r_s is the areal radius, and M is the ADM mass of the system. As the radius $R(\tau)$ of the collapsing star decreases, the Schwarzschild metric becomes valid over a wider range of r_s coordinates. (Namely, valid over $[R(\tau), \infty)$.)

The interior has the Friedman metric, given by:

$$\mathrm{d}s^2 = -\,\mathrm{d}\tau^2 + a^2(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\Omega^2).$$

Here, τ is the time coordinate, and χ is the comoving radial coordinate. Geodesic-following dust particles have constant χ throughout their world-lines.

My project is to numerically solve for the dynamics of gravitational collapse of a collection of dust particles, and extract relevant quantities (such as the time slicing function and the event horizon) from the solution.

2 The Numerical Scheme

The analytic solution given in Section 1 is only a qualitative description of the gravitational collapse. In order to solve for the dynamics of a concrete system, we use the famous ADM formulation of general relativity to solve for both the spacetime metric and the particle motion. More specifically, we will be using a *mean field, particle simulation scheme*, and the choice of coordinates will be *isotropic coordinates* with *polar slicing*.[1] In this section, we start by introducing the ingredients of the ADM formulation, the equation of motion for the dust particles, sketch the full numerical scheme, and finally go into the full details of linearization and discretization of the relevant partial differential equation.

2.1 ADM Equations in the Spherical Symmetric Case

The Einstein field equation, $G_{ab} = 8\pi T_{ab}$, is the fundamental postulate of general relativity. Given the matter distribution in all of spacetime, the Einstein field equation is a second-order nonlinear system of PDEs in the metric variables, which can be solved directly. However, for a concrete system such as a dust star, we can only specify the initial matter distribution, and the motion of the matter must be solved along with the evolution of the metric. The *ADM formulation* casts the covariant Einstein field equation into an initial value problem, such that metric variables in the "future" can be obtained by evolving the "current" metric variables.

For a spherically symmetric spacetime, we can simplify the canonical ADM metric to:

$$\mathrm{d}s^2 = -(\alpha^2 - A^2\beta^2)\,\mathrm{d}t^2 + 2A^2\beta\,\mathrm{d}r\,\mathrm{d}t + A^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin(\theta)^2\,\mathrm{d}\phi^2).$$

Here, $\alpha(r,t)$ is the *time lapse* function, $\beta(r,t) = \beta^r$ is the only non-vanishing component of the *shift vector*, and $A(r,t)$ is the radial metric component. By assuming the angular part of the metric to have component A^2r^2, we have taken what is called an *isotropic* choice of coordinates.

In order to fully describe the spacetime, it is also necessary to specify the *extrinsic curvature* K_{ij} of the time slice. Due to gauge freedom in our choice of coordiantes, we can use *polar slicing* and assume $K_T = K^\phi{}_\phi + K^\theta{}_\theta = 0$, so the only relevant variable is $K^r{}_r$.

In the ADM formulation, the Einstein equation breaks into 3 different equations: the Hamiltonian constraint, the momentum constraint, and the equation for metric evolution. The Hamiltonian and momentum constraints are merely conditions for the initial condition, only the equation for metric evolution is dynamical. Here, we list each of the 3 equations.

Hamiltonian constraint The Hamiltonian constraint is a second-order nonlinear PDE in $A(r)$:

$$\frac{1}{r^2}\partial_r(r^2\partial_r A^{1/2}) = -\frac{1}{4}A^{5/2}(8\pi\rho),$$

where ρ is the matter density. In the vacuum exterior we have $\rho = 0$, so A has a unique solution that takes the isotropic Schwarschild form:

$$A = (1 + \frac{M}{2r})^2.$$

This solution gives us the Robin boundary condition $\partial_r(rA^{1/2}) = 1$ at the outer limit $r = L$. Also, since A is a radial quantity, for smoothness we must also have $\partial_r A = 0$ at $r = 0$. Given the matter distribution at a particular time slice, this equation (along with the boundary conditions) fully determines the radial metric A.

Momentum constraint The momentum constraint is given by the following PDE:

$$D_i K^i{}_r - D_r K = 8\pi S_r,$$

where S_r is the radial momentum density. Given A and S_r, this equation has an analytic solution:

$$K^r{}_r = \frac{4\pi r S_r}{1 + r\partial_r A/A}.$$

We see that we don't need to solve a PDE for $K^r{}_r$ given A and S_r.

Metric evolution The evolution of the metric is given by the following PDE:

$$\partial_t A = \beta(\partial_r A + \frac{A}{r}).$$

Since A can be determined entirely from the Hamiltonian constraint, there is no need to solve this equation for A. This is a special phenomenon that only happens in spherically symmetric spacetime: due to Birkhoff's theorem, there is no dynamical degree of freedom in the metric. In systems without spherical symmetry, there may be dynamics in the metric alone (e.g. gravitational waves).

2.2 Gauge Functions

Here, we give the analytic solutions to the gauge functions α and β under isotropic coordinates and polar slicing, so that they may be determined given A and the matter distribution.

$$\begin{aligned}
\alpha =& [1-(\frac{M}{2r_{\max}})^2]\exp\left\{\frac{1}{2}\int_{r_{\max}}^{r}\frac{r(\partial_r A/A)^2+8\pi r S_{rr}}{1+r\partial_r A/A}\,\mathrm{d}r\right\}\\
\beta =& -r\int_r^\infty \alpha K^r{}_r\,\frac{\mathrm{d}r}{r},
\end{aligned}$$

where S_{rr} is the stress density.

2.3 Particle Profile

In the particle method, we explicitly store the masses, positions and velocities of a collection of particles, and evolve each of the particles using the geodesic equation. At a given time instant, the mass, position and velocity of a single particle can be specified (up to spherical symmetry) by the following variables:

- m The particle mass.
- r The radial position.
- u_r The radial velocity.
- u_ϕ The azimuthal velocity. This quantity is constant in time due to conservation of angular momentum.
- u_θ The polar velocity. We can take $u_\theta = 0$ due to symmetry.

In these variables, the geodesic equation is given by:

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} =& \frac{\alpha u_r}{A^2(\alpha u^0)} - \beta\\
\frac{\mathrm{d}u_r}{\mathrm{d}t} =& -(\alpha u^0)\partial_r\alpha + u_r\partial_r\beta + \frac{u_r^2}{u^0}\frac{\partial_r A}{A^3} + \frac{u_\phi^2}{u^0}(\frac{1}{r^3A^2}+\frac{\partial_r A}{r^2A^3})\\
u_\theta =& 0\\
\frac{\mathrm{d}u_\phi}{\mathrm{d}t} =& 0.
\end{aligned}$$

We can compute $\frac{\mathrm{d}r}{\mathrm{d}t}$ and $\frac{\mathrm{d}u_r}{\mathrm{d}t}$ from A and the gauge functions; this gives us the recipe to update a particle by time step Δt.

Note that normalization of the 4-velocity gives us:

$$W \equiv \alpha u^0 = (1+\frac{u_r^2}{A^2}+\frac{u_\phi^2}{r^2A^2})^{1/2}.$$

The stress-energy tensor of the entire particle profile is given by:

$$T^{ab} = \sum_i m_i n_i u_i^a u_i^b,$$

where n_i is the number density of a single particle. Assuming that the particle profile is isotropic, we can obtain the components of the stress-energy tensor:

$$\begin{aligned}
n_i &= \frac{1}{4\pi W A^3 r^2}\,\delta(r - r_i) \\
\rho &= \sum_i m_i n_i W^2 \\
S_r &= \sum_i m_i n_i W u_r^i \\
S_{rr} &= \sum_i m_i n_i u_r^i u_r^i \\
S &= \rho - \sum_i m_i n_i.
\end{aligned}$$

2.4 Sketch of the Numerical Scheme

We now have all the ingredients for solving the Einstein equation along with the matter motion. Here we sketch the numerical scheme step-by-step before discussing its details.

0 Choose an appropriate way to discretize the radial space grid $[0, L]$.

1 Create an initial collection of particles from some distribution. Now we have N_{particle} pairs of particle position and velocity, given by (r, θ, ϕ) and (u_r, u_θ, u_ϕ). Note that since $u_\theta = 0$ and $\frac{\mathrm{d}u_\phi}{\mathrm{d}t} = 0$, and that we are always averaging over shells parametrized by r, we only need to store r, u_r and constants u_ϕ, m for each particle.

2 Solve the Hamiltonian constraint equation for initial $A(r)$ with finite difference method. Since $\rho(r)$ is determined by $A(r)$ and the particle profile, we can plug $\rho(r)$ into the Hamiltonian constraint equation to obtain a single equation for $A(r)$.

3 Obtain the gauge functions $\alpha(r)$, $\beta(r)$ from $A(r)$ and the particle distribution using numerical integration. In order to get the quantities S_{rr} and ${K^r}_r$ in the integrands for α and β, use the formulas for stress density and analytic solution for ${K^r}_r$.

4 Update the particle states (r, u_r) by Δt using the geodesic equation. This can be done easily since $\frac{\mathrm{d}r}{\mathrm{d}t}$ and $\frac{\mathrm{d}u_r}{\mathrm{d}t}$ are manifestly functions of A and the gauge functions, which we know from step 2 and 3.

5 Repeat steps 2, 3 and 4. Stop until some number of time steps is reached.

6 Extract relevant quantities from the solution.

We can see that the hardest part of the scheme is step 2: solving for A. For the rest of the section, we will describe exactly how do we solve for A using finite difference method.

2.5 Linerization

Let $f = A^{1/2}$, then the Hamiltonian constraint equation can be written as:

$$\begin{aligned}
\frac{1}{r^2}\partial_r(r^2\partial_r f) &= -\frac{1}{4}f^5(8\pi\rho) \\
&= -2\pi f^5 \sum_i m_i n_i W_i^2 \\
&= -2\pi f^5 \sum_i m_i \frac{1}{4\pi r^2 f^6}(1 + (u_{r,i}^2 + \frac{u_{\phi,i}^2}{r_i^2})f^{-4})^{1/2}\,\delta(r - r_i) \\
&= -\frac{1}{2r^2}\sum_i m_i f^{-1}(1 + (u_{r,i}^2 + \frac{u_{\phi,i}^2}{r_i^2})f^{-4})^{1/2}\,\delta(r - r_i)
\end{aligned}$$

This is a non-linear elliptic PDE; it has the form $\nabla^2 f = F(f)$. In order to solve for f, we need to linearize the non-linear term in f. Writing $f^{N+1} = f^N + \delta f$, then $\nabla^2 f^{N+1} = F(f^{N+1})$ gives us:

$$\nabla^2\delta f = F(f^N + \delta f) - \nabla^2 f^N = F(f^N) + \frac{\delta F}{\delta f}\delta f - \nabla^2 f^N + \mathcal{O}\big(\delta f^2\big).$$

The above equation is linear in δf. $\frac{\delta F}{\delta f}$ is a functional of f, so for a function f^N close to the actual solution, we expect f^{N+1} to be even closer to the actual solution. Iterating this linearized equation should make f^N converge to the actual solution f as $N \to \infty$.

Now let's write it all out:

$$\begin{aligned}
&-\frac{1}{2r^2}\sum_i m_i (f^{N+1})^{-1}(1 + (u_{r,i}^2 + \frac{u_{\phi,i}^2}{r_i^2})(f^{N+1})^{-4})^{1/2}\,\delta(r - r_i) \\
=&-\frac{1}{2r^2}\sum_i m_i (f^{N})^{-1}(1 + (u_{r,i}^2 + \frac{u_{\phi,i}^2}{r_i^2})(f^{N})^{-4})^{1/2}\,\delta(r - r_i) \\
&[1 - ((f^N)^{-1} + 2(1 + (u_{r,i}^2 + \frac{u_{\phi,i}^2}{r_i^2})(f^N)^{-4})^{-1}(u_{r,i}^2 + \frac{u_{\phi,i}^2}{r_i^2})(f^N)^{-5})\delta f \\
&+ \mathcal{O}\big(\delta f^2\big)].
\end{aligned}$$

We denote:

$$F_i^N \equiv -\frac{1}{2r^2}m_i(f^N)^{-1}(1 + (u_{r,i}^2 + \frac{u_{\phi,i}^2}{r_i^2})(f^N)^{-4})^{1/2}\,\delta(r - r_i)\,.$$

Ignoring $\mathcal{O}(\delta f^2)$, we have our final linear elliptic equation of δf:

$$\begin{aligned}
&\frac{1}{r^2}\partial_r(r^2\partial_r\delta f)\\
&+\sum_i F_i^N((f^N)^{-1}+2(1+(u_{r,i}^2+\frac{u_{\phi,i}^2}{r_i^2})(f^N)^{-4})^{-1}(u_{r,i}^2+\frac{u_{\phi,i}^2}{r_i^2})(f^N)^{-5})\delta f\\
=&-\frac{1}{r^2}\partial_r(r^2\partial_r f^N)+\sum_i F_i^N
\end{aligned}$$

2.6 Space Discretization

The equation for f is a non-linear elliptic equation defined on $r \in [0, \infty)$. In order to solve it numerically, we discretize the interval $[0, L]$ into $N-1$ intervals with the following scheme:

$$\begin{aligned}
r_i =& ih - h/2 \ (i = 1, 2, ...N-1)\\
h =& \frac{L}{N-1}.
\end{aligned}$$

Denoting the discretized value of f at r_i to be f_i, the Laplacian operator is given by:

$$\begin{aligned}
(\partial_r f)_{n+1/2} \mapsto& \frac{f_{n+1}-f_n}{h}\\
(\frac{1}{r^2}\partial_r(r^2\partial_r f))_n \mapsto& \frac{1}{r_n^2}\frac{1}{h}((r^2\partial_r f)_{n+1/2}-(r^2\partial_r f)_{n-1/2})\\
=& \frac{1}{r_n^2}\frac{1}{h}(r_{n+1/2}^2\frac{f_{n+1}-f_n}{h}-r_{n-1/2}^2\frac{f_n-f_{n-1}}{h})\\
=& \frac{1}{h^2}(\frac{r_{n-1/2}^2}{r_n^2}f_{n-1}-(\frac{r_{n-1/2}^2}{r_n^2}+\frac{r_{n+1/2}^2}{r_n^2})f_n+\frac{r_{n+1/2}^2}{r_n^2}f_{n+1})\\
\delta(r-r_i) \mapsto& \frac{1}{h}\mathbb{1}_{[r_n,r_{n+1}]} \quad \text{where} \quad r_i \in [r_n, r_{n+1}]
\end{aligned}$$

To account for the boundary conditions $\partial_r f(0) = 0$ and $\partial_r(rf)(L) = 1$, we introduce "virtual" grid points $i = 0$ and $i = N$, then the boundary conditions are represented as:

$$\begin{aligned}
f_1 - f_0 = 0\\
r_N f_N - r_{N-1} f_{N-1} = h.
\end{aligned}$$

At the boundaries, the second-order derivatives $\frac{1}{r^2}\partial_r(r^2\partial_r f)$ are given by:

$$\begin{aligned}
\frac{1}{r^2}\partial_r(r^2\partial_r f)_1 =& \frac{1}{h^2}(\frac{r_{1/2}^2}{r_1^2}f_0 - (\frac{r_{1/2}^2}{r_1^2} + \frac{r_{3/2}^2}{r_1^2})f_1 + \frac{r_{3/2}^2}{r_1^2}f_2) = \frac{1}{h^2}\frac{r_{3/2}^2}{r_1^2}(f_2 - f_1) \\
\frac{1}{r^2}\partial_r(r^2\partial_r f)_{N-1} =& \frac{1}{h^2}(\frac{r_{N-3/2}^2}{r_{N-1}^2}f_{n-2} - (\frac{r_{N-3/2}^2}{r_{N-1}^2} + \frac{r_{N-1/2}^2}{r_{N-1}^2})f_{N-1} + \frac{r_{N-1/2}^2}{r_{N-1}^2}f_N) \\
=& \frac{1}{h^2}(\frac{r_{N-3/2}^2}{r_{N-1}^2}f_{n-2} - (\frac{r_{N-3/2}^2}{r_{N-1}^2} + \frac{r_{N-1/2}^2}{r_{N-1}^2})f_{N-1} + \frac{r_{N-1/2}^2}{r_{N-1}^2}\frac{h + r_{N-1}f_{N-1}}{r_N}) \\
=& \frac{1}{h^2}(\frac{r_{N-3/2}^2}{r_{N-1}^2}f_{n-2} - (\frac{r_{N-3/2}^2}{r_{N-1}^2} + \frac{r_{N-1/2}^2}{r_{N-1}^2}(1 - \frac{r_{N-1}}{r_N}))f_{N-1}) + \frac{r_{N-1/2}^2}{r_{N-1}^2}\frac{1}{r_N h}
\end{aligned}$$

In matrix form, the Laplacian matrix reads:

$$\frac{1}{h^2}\begin{bmatrix}
-\frac{r_{3/2}^2}{r_1^2} & \frac{r_{3/2}^2}{r_1^2} & 0 & 0 & \cdots \\
\frac{r_{3/2}^2}{r_2^2} & -\frac{r_{3/2}^2}{r_2^2} - \frac{r_{5/2}^2}{r_2^2} & \frac{r_{5/2}^2}{r_2^2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & \frac{r_{N-3/2}^2}{r_{N-1}^2} & -\frac{r_{N-3/2}^2}{r_{N-1}^2} - \frac{r_{N-1/2}^2}{r_N^2}(1 - \frac{r_{N-1}}{r_N})
\end{bmatrix}$$

This is a tridiagonal matrix; applying a tridiagonal matrix on a vector or solving a tridiagonal linear system can both be done in $\mathcal{O}(N)$ time, so operations involving this matrix will generally be fast.

To obtain the incremental solution δf, we need to solve the linear system:

$$(\frac{\delta F}{\delta f} - \nabla^2)\delta f = -F(f^N) + \nabla^2 f^N.$$

Discretizing, we know from above that $\frac{\delta F}{\delta f}$ is simply a diagonal matrix, so the equation for δf is still a tridiagonal system, which can be solved in $\mathcal{O}(N)$ time. In order to improve convergence, we add a λI term to $\frac{\delta F}{\delta f} - \nabla^2$, where I is the identity matrix. This is paramount to adding a $-\frac{\partial f}{\partial t}$ term in the original non-linear equation for f.

Note that the matrix gives the Laplacian for homogenous boundary conditions; in order to obtain the full Laplacian for f, a boundary term must be added after applying the Laplacian matrix. Also note that we expect the intermediate solutions δf to have homogeneous boundary conditions ($\partial_r \delta f(0) = 0$ and $\partial_r(r\delta f)(L) = 0$), so we don't need to worry about boundary terms for δf.

Finally, the evolution equation for f is:

$$\partial_t f = \beta(\partial_r f + \frac{f}{2r}).$$

Although this equation is redundant in principle, when solving for f given an updated matter distribution, it is helpful to use this evolution equation to obtain an initial guess for f, since this guess reduces the number of iterations required for the solution to converge.

3 Results

The numerical scheme described above was implemented in C++, and test runs were performed with satisfactory results. In my run, the space discretization scheme divides the radial interval $[0, L]$ into $N - 1$ equally spaced points, where $N = 10000$ and $L = 150$. The size of the particle profile is $N_{\text{particle}} = 200000$, and each particle has mass $m = 0.00001$. Initially, the particles are all at rest ($u_r = 0, u_\phi = 0$), and their spatial distribution is such that they have constant density (in isotropic coordinates) over a sphere of size $R_{\text{star}} = 10$. The time step is taken to be $\Delta t = 0.004$, and the evolution spans a total of $n_{\text{step}} = 40000$ time steps. It took around 10 minutes for the program to terminate. During the execution, information such as the metric function f, the time lapse α, matter tracers and light ray tracers were periodically written to files, which were used to produce graphical representations of the solution via independent Python scripts.

Note: in order to present the solutions in areal radius R instead of isotropic radius r (the one used in the previous sections), the spatial coordinates in all graphs below are scaled by $R = Ar = f^2 r$.

Figure 1 gives the metric A versus R/M, where $M = 1.3669$ is the ADM mass. Notice that the rest mass is given by $M_0 = N_{\text{particle}} m = 2$, and we have $M < M_0$. This is expected, as generally the negative gravitational potential energy contributes to the ADM mass, making it less than the rest mass. From the figure, we can see that the metric at $R/M > 10$ is roughly static, whereas the central value $A_c = A(R/M = 0)$ increases as time increases, and finally converge to roughly $A_c = 5.1$. This is not surprising: the metric solution at the vacuum exterior have the form $A = (1 + \frac{M}{2r})^2$; as the radius of the dust star decreases, the range over which this analytic solution is valid msut increase. Also, as more matter converge to the center, the central spacetime must become increasingly curved. Note that at late times $A(r)$ is almost vertical at $R/M = 2$: this is exactly the appearance of the event horizon, with a Schwarzschild radius predicted to be $r_s = 2M$.

Figure 2 gives the time lapse function α versus R/M at given time slices. We see that α is a bit lower than 1 for the majority of the simulation. At around $t = 120$, α begins decreasing drastically within the range $R/M < 2$; at the end of the simulation, the central α value becomes as low as 10^{-6}. This phenomenon is called α-freezing. Physically, it comes from the fact that all world lines within the event horizon $R/M = 2$ must reach the singularity in finite proper time, so the time lapse α, which describes the "speed" of the proper time relative to our coordinate time, must converge to 0 for the simulation to continue in our coordinate time t.

Figure 3 gives the fictitious matter tracers for the collapsing star, along with the identified event horizon. The dashed lines correspond to points in the dust star with fixed interior rest mass fractions. In our case, it simply means there are fixed number of particles within each dashed line. We can see the dust star collapsing into the center, and shortly after $t = 120$, the matter tracers cross the event horizon, and becomes stuck at around $R/M = 2$.

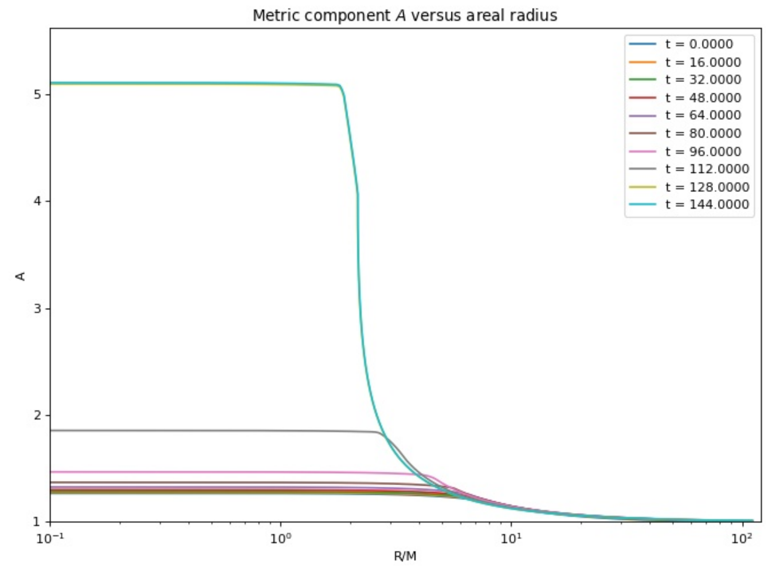

Figure 1: Solution for the radial metric A

The plot for the event horizon in Figure 3 was obtained from a separate method: by placing outward going light rays at various points in the spacetime, we can see whether the light ray stays in the event horizon forever or goes out to infinity. By identifying the boundaries where this transition happens, we find the growing event horizon at the center of the spacetime. The geodesic equation for light rays is given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\alpha}{A} - \beta.$$

From the plot, we see that a event horizon appears at around $t = 115$, and it rapidly grows to $R/M = 2$ shortly after $t = 120$.

Finally, I shall check my result against a previous paper by Petrich, Shapiro and Teukolsky[2], who employed a similar method to solve the problem of gravitational collapse of a dust star. We can see that the 3 figures I have obtained reproduce most of the qualitative results given in FIG.1, FIG.3, FIG.10 of the previous paper, although the gravitational collapse in the previous paper seem to be smoother than mine.

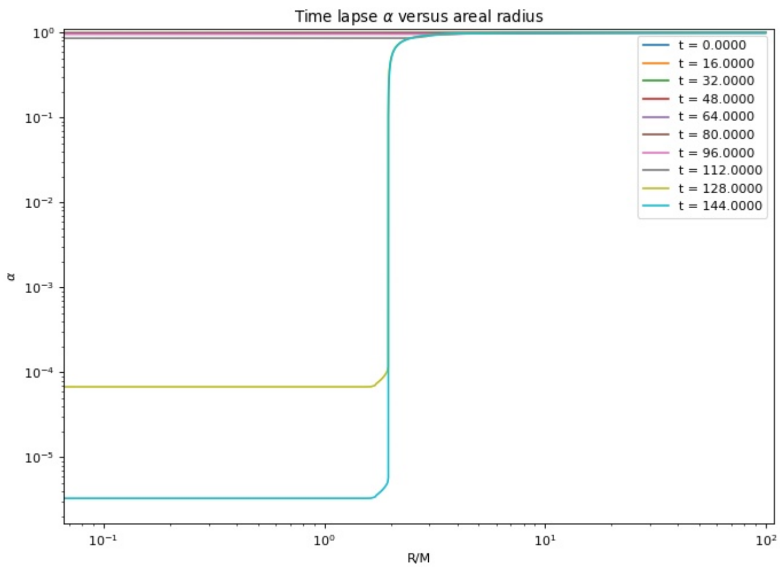

Figure 2: Solution for the time lapse α

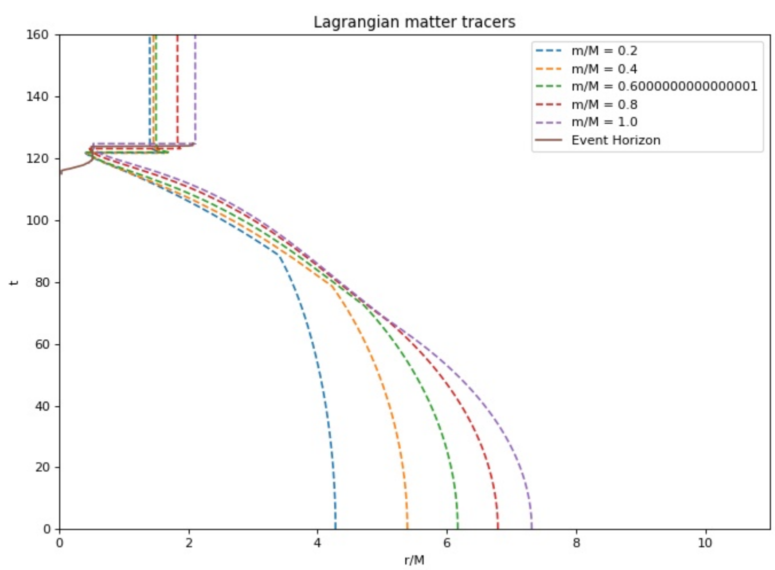

Figure 3: Matter tracers and event horizon

References

[1] Thomas W. Baumgarte and Stuart L. Shapiro. (2010) *Numerical Relativity: Solving Einstein's Equations on the Computer*. Cambridge University Press, Cambridge.

[2] Petrich, L. I., S. L. Shapiro, and S. A. Teukolsky (1986). Oppenheimer-Snyder collapse in polar time slicing. *Phys. Rev. D* **33**, 2100-2110.