Signal and Image Processing

Lab session 4

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1 Exercise

Let f be a discrete signal defined on 0,...,L-1, consider the signal $\tilde{f}$ built by symmetrizing f with respect to -1/2, namely

$$\tilde{f}_n = \begin{cases} f_n & \text{for } 0 \leq n < L \\ f_{-n-1} & \text{for } -L \leq n \leq -1 \end{cases}$$

We then assume $\tilde{f}$ is periodized with period 2L.

1.1 Using the properties of the DFT of $\tilde{f}$ of length 2L, show that f_n admits the following decomposition

$$f_n = \sum_{k=0}^{L-1} a_k \cos\left(\frac{k\pi}{L}(n+\frac{1}{2})\right)$$

for $0 \leq n \leq L-1$. Write the a_k with respect to the $\widehat{\tilde{f}}_k$ the coefficient of the DFT of $\tilde{f}$.

Using IDFT we can write $\tilde{f}_n$ as

$$\tilde{f}_n = \frac{1}{2L} \sum_{k=0}^{2L-1} \widehat{\tilde{f}}_k e^{\frac{i\pi kn}{L}}.$$

Then from the symetry of $\tilde{f}_n$ we get

$$f_n = \frac{1}{2}(\tilde{f}_n + \tilde{f}_{-n-1}) =$$

$$\frac{1}{2}\left(\frac{1}{2L}\sum_{k=0}^{2L-1} \widehat{\tilde{f}}_k e^{\frac{i\pi kn}{L}} + \frac{1}{2L}\sum_{k=0}^{2L-1} \widehat{\tilde{f}}_k e^{\frac{i\pi k(-n-1)}{L}}\right) =$$

$$\frac{1}{4L}\sum_{k=0}^{2L-1} \widehat{\tilde{f}}_k e^{\frac{-i\pi k}{L}} \left(e^{\frac{i\pi k(2n+1)}{2L}} + e^{\frac{-i\pi k(2n+1)}{2L}}\right) =$$

$$\frac{1}{2L}\sum_{k=0}^{2L-1} \widehat{\tilde{f}}_k e^{\frac{-i\pi k}{L}} \cos\left(\frac{k\pi}{L}(n+\frac{1}{2})\right) =$$

$$\frac{1}{2L}\widehat{\tilde{f}}_0 + \frac{1}{2L}\sum_{k=1}^{L-1}(\widehat{\tilde{f}}_k e^{\frac{-i\pi k}{L}} + \widehat{\tilde{f}_{2L-k}} e^{\frac{-i\pi(2L-k)}{L}}) \cos\left(\frac{k\pi}{L}(n+\frac{1}{2})\right)$$

So $a_0 = \frac{1}{2L}\widehat{\tilde{f}}_0$ and

$$a_k = \frac{1}{2L}(\widehat{\tilde{f}}_k e^{\frac{-i\pi k}{L}} + \widehat{\tilde{f}_{2L-k}} e^{\frac{-i\pi(2L-k)}{L}}) \qquad \text{for} \quad 1 \leq k \leq L-1$$

1.2 Show that

$$\lambda_k \sqrt{\frac{2}{L}} \cos(\frac{k\pi}{L}(n+1/2))$$

with $0 \leq k < L$ and

$$\lambda_k = \begin{cases} 2^{-\frac{1}{2}} & if k = 0 \\ 1 & else \end{cases}$$

is an orthonormal basis of $\mathbb{R}^L$.
Let

$$f_k(n) = \lambda_k \sqrt{\frac{2}{L}} \cos(\frac{k\pi}{L}(n+1/2))$$

Then to show that f_k forms an orthonormal basis, I will show that $< f_k, f_k >= 1$ and $< f_k, f_{l \neq k} >= 0$. The f_0 case is shown separately

$$f_0 = \frac{1}{\sqrt{2}} \sqrt{\frac{2}{L}} \cos(\frac{\pi * 0}{L}(n+1/2)) = \sqrt{\frac{1}{L}} \implies < f_0, f_0 > = \sum_{n=0}^{L-1} f_0^2 = \frac{L}{L} = 1$$

Now that $< f_0, f_0 > = 1$ is shown, $< f_k, f_k > = 1$ is shown for $k \neq 0$:

$$< f_k, f_k >= \sum_{n=0}^{L-1} \frac{2}{L} \cos(\frac{k\pi}{L}(n+1/2))^2 = \frac{2}{L} \sum_{n=0}^{L-1} \cos(\frac{k\pi}{2L}(2n+1))^2$$

and by the cosine double angle formula

$$= \frac{2}{L} \sum_{n=0}^{L-1} \cos(\frac{k\pi}{2L}(2n+1))^2 = \frac{2}{L} \sum_{n=0}^{L-1} (\frac{1}{2} + \frac{1}{2} \cos(\frac{k2\pi}{2L}(2n+1))$$

$$= \frac{2}{L} \frac{L}{2} + \sum_{n=0}^{L-1} \frac{1}{2} \cos(\frac{k2\pi}{2L}(2n+1)) = 1$$

Now that normality is shown, orthogonality $< f_k, f_{l \neq k} >= 0$ is shown:

$$< f_k, f_{l \neq k} > = \sum_{n=0}^{L-1} \sqrt{\frac{2}{L}} \cos(\frac{k\pi}{L}(n+1/2)) \sqrt{\frac{2}{L}} \cos(\frac{l\pi}{L}(n+1/2))$$

And then using the cosine product rules

$$= \frac{2}{L} \sum_{n=0}^{L-1} \cos(\alpha_k) \cos(\alpha_l) = \frac{2}{L} \frac{1}{2} \sum_{n=0}^{L-1} \cos(\alpha_k + \alpha_l) + \cos(\alpha_k - \alpha_l)$$

$$= \frac{1}{L} \sum_{n=0}^{L-1} \cos(\frac{\pi}{L}(n+1/2)(k+l)) + \cos(\frac{\pi}{L}(n+1/2)(k-l))$$

$$= \frac{1}{L} \sum_{n=0}^{L-1} Re(e^{i\frac{\pi}{L}(n+1/2)(k+l)}) + \sum_{n=0}^{L-1} Re(e^{i\frac{\pi}{L}(n+1/2)(k-l)})$$

$$= \frac{1}{L} Re(\sum_{n=0}^{L-1} e^{i\frac{\pi}{L}(n+1/2)(k+l)} + \sum_{n=0}^{L-1} e^{i\frac{\pi}{L}(n+1/2)(k-l)})$$

$$= \frac{1}{L} Re(e^{i\frac{\pi}{2L}(k+l)} \sum_{n=0}^{L-1} e^{i\frac{\pi}{L}n(k+l)} + e^{i\frac{\pi}{2L}(k-l)} \sum_{n=0}^{L-1} e^{i\frac{\pi}{L}n(k-l)})$$

$$= \frac{1}{L} Re(e^{i\frac{\pi}{2L}(k+l)} \frac{1 - e^{i\pi(k+l)}}{1 - e^{i\frac{\pi}{L}(k+l)}} + e^{i\frac{\pi}{2L}(k-l)} \frac{1 - e^{i\pi(k-l)}}{1 - e^{i\frac{\pi}{L}(k-l)}})$$

which equals 0 by

$$1 - e^{ig} = e^{\frac{ig}{2}}(e^{\frac{-ig}{2}} - e^{\frac{ig}{2}}) = -e^{\frac{ig}{2}} 2i \sin(\frac{g}{2})$$

which implies that the numerator in the formula above can be re-written as

$$= -\frac{1}{L} Re \left(e^{i\frac{\pi}{2L}(k+l)} \frac{e^{i\pi(k+l)/2} 2i \sin(\pi(k+l)/2)}{1 - e^{i\frac{\pi}{L}(k+l)}} + e^{i\frac{\pi}{2L}(k-l)} \frac{e^{i\pi(k-l)/2} 2i \sin(\pi(k-l)/2)}{1 - e^{i\frac{\pi}{L}(k-l)}} \right)$$

$$\frac{1}{L} Re \left(e^{\frac{i\pi}{2}(k+l)} \frac{2i \sin(\frac{\pi}{2}(k+l))}{2i \sin(\frac{\pi}{2L}(k+l))} + e^{\frac{i\pi}{2}(k-l)} \frac{2i \sin(\frac{\pi}{2}(k-l))}{2i \sin(\frac{\pi}{2L}(k-l))} \right) = 0$$

Which equals zero because if $(k + l)$ (or in that case $(k - l)$) is even then $\sin(\frac{\pi}{2}(k+l))$ is zero, if $(k+l)$ is odd, then $e^{\frac{i\pi}{2}(k+l)}$ is purely imaginary number, which is multiplied by a real number, therefore its real part will be zero.

1.3 How do you compute the coefficients of f_n in that basis?

Using results from 1.1 we just need to deal with the factors before basis functions. Let α_k be a k-th coordinate in this basis.

$$\alpha_0 = \frac{1}{2\sqrt{L}} \hat{\tilde{f}}_0$$

and

$$\alpha_k = \frac{1}{2\sqrt{2L}} (\hat{\tilde{f}}_k e^{\frac{-i\pi k}{L}} + \widehat{\tilde{f}_{2L-k}} e^{\frac{-i\pi(2L-k)}{L}}),$$

where we got $\hat{\tilde{f}}_k$ as DFT of $\tilde{f}$.

1.4 Write a program to decompose a signal f in the discrete cosine transform defined above.

For decomposition we use signal $f(t) = \sin\left(\frac{2\pi t}{L}\right)$ for $L = 40$ and $t = (0, \ldots, L-1)$. Signal f and $\tilde{f}$ shown in Figure 1. Decomposed signal is shown in Figure 2, and errors between it and the original signal in Figure 3

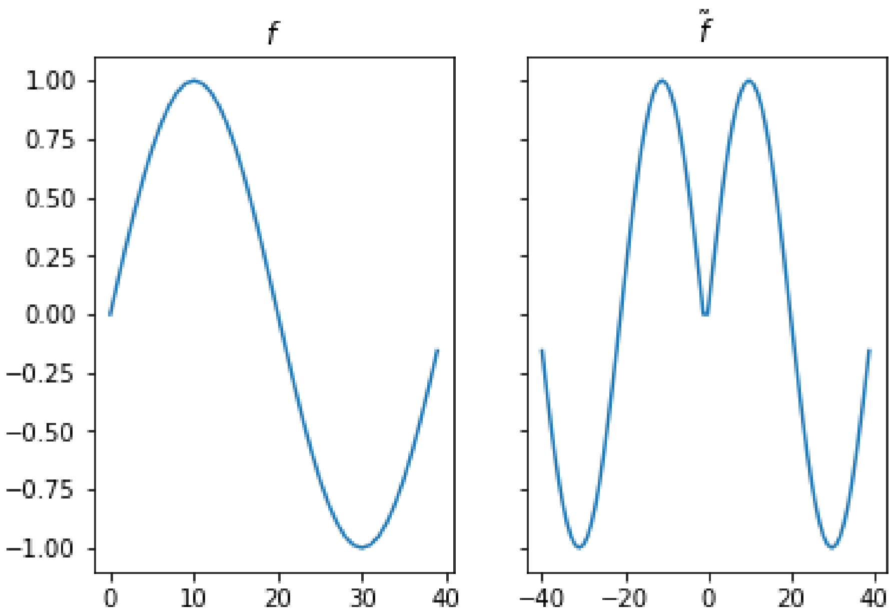

Figure 1: Used signal f and corresponding $\tilde{f}$.

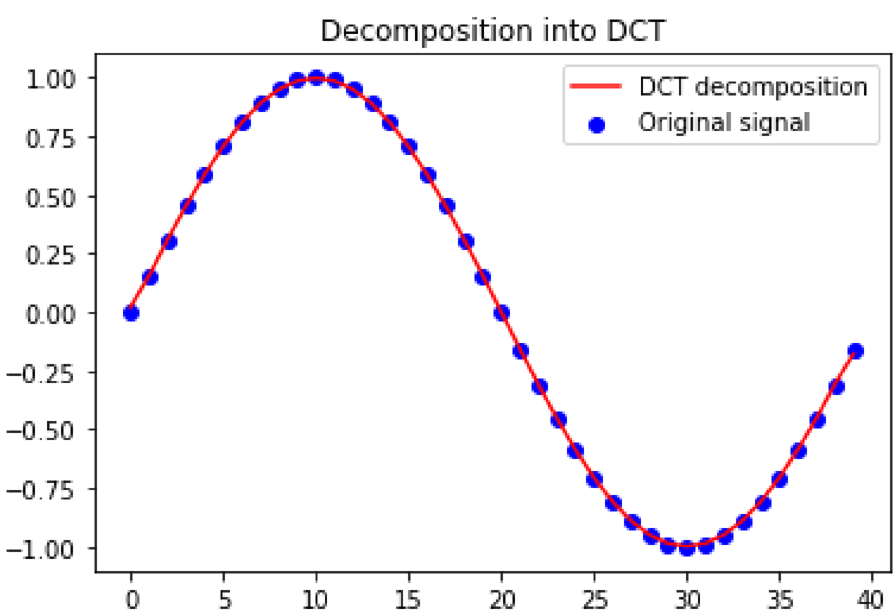

Figure 2: Comparison of discrete cosine transform of a signal and original signal

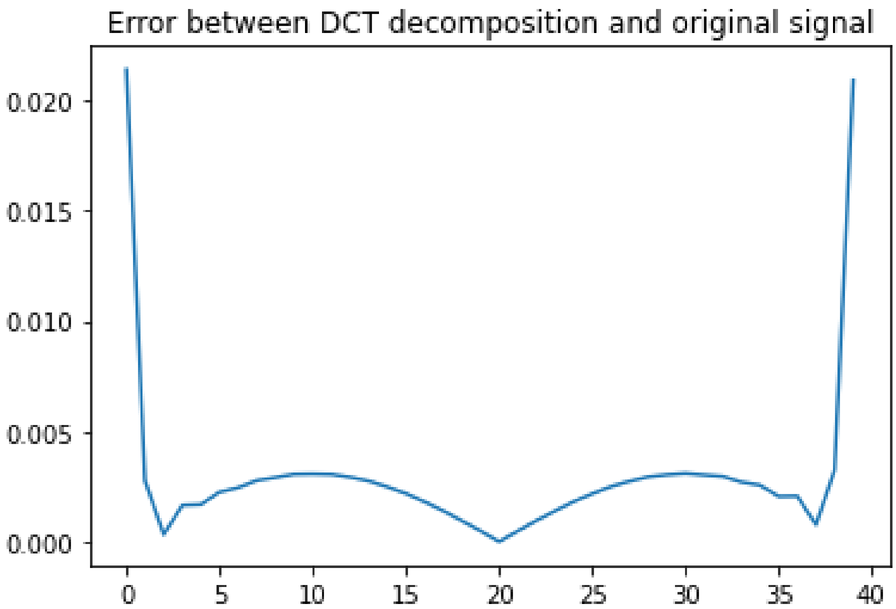

Figure 3: Errors between DCT and the original signal

1.5 Assuming that a basis of $\mathbb{R}^{LxL}$ is obtained using a tensor product of 1D basis, determine an orthonormal basis for images of size LxL.

The basis is given as

$$\left\{\lambda_j\sqrt{\frac{2}{L}}\cos\left(\frac{j\pi}{L}(n+\frac{1}{2})\right)\cdot\lambda_k\sqrt{\frac{2}{L}}\cos\left(\frac{k\pi}{L}(m+\frac{1}{2})\right)\right\}=\psi_{j,k}(n,m)$$

for $0 \leq j,k \leq L-1$.

2 Exercise

The standard JPEG algorithm decomposes an NxN image in blocks of size LxL and then applies a discrete cosine transform to each of the blocks of the image.

2.1 Record an image (np.imread command in python, use the cameraman gray scale image of size 256x256), decompose it in blocks of size 8x8 and carry out a discrete cosinus transform on each block.

The image chosen was the cameraman photo, shown in Figure 4.

Figure 4: Cameraman

2.2 How are the coefficients displayed in each block?

Within a block, the element at row i and column j gives the coefficient of the ith row's jth frequency element.

2.3 One then builds an 8x8 quantization matrix as follows:

$$Q(i,j) = 1 + q(1 + i + j)$$

for q = 5, one obtains the matrix provided in the assignment.

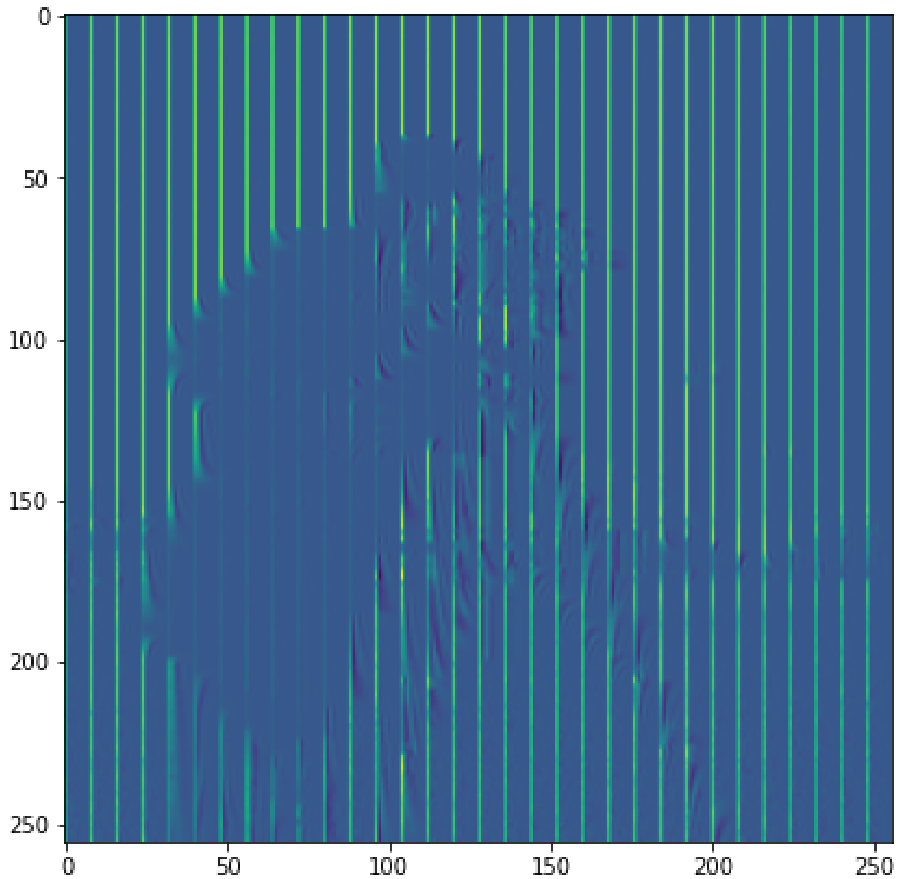

Figure 5: Discrete Cosine Transform of Cameraman. The first element of the frequency is proportional to the mean of the row, so the first column of each block appears in the image like the image.

2.4 Compute the proportion of non-zero coefficients in the quantized image as a function of q.

This is shown below in Figure 6. The precision of the rounding makes a significant difference in when the coefficients become identically 0. Because there are more finite represented numbers between 0 and 1, the precision for small numbers in this region is high.

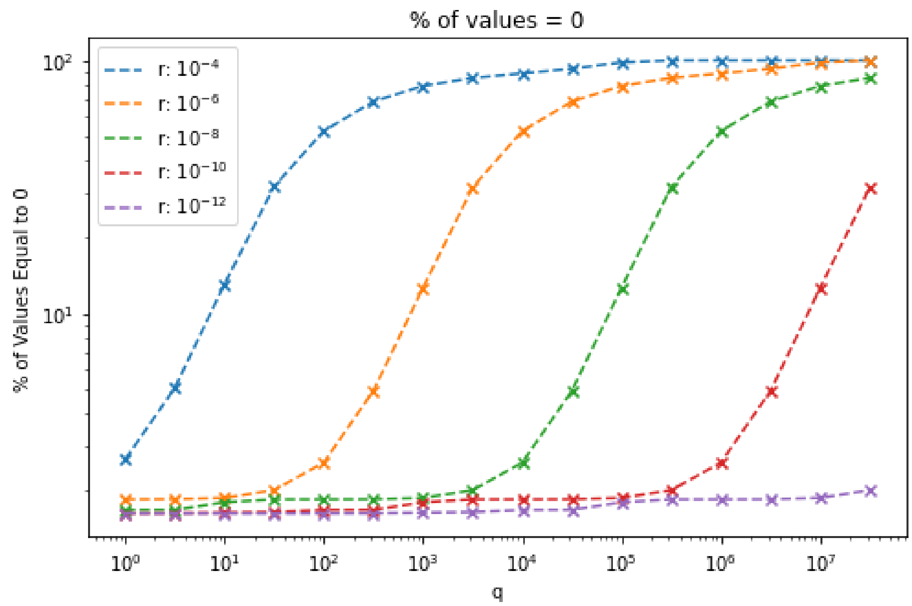

Figure 6: Percentage of values equal to 0. Different levels of rounding are plotted as r: $10^{-4}, 10^{-6}, 10^{-10}, 10^{-12}$, where r is the precision to which the values are rounded.

2.5 For final storage, the quantized coefficients in each blocks are browsed in zig-zag starting from the upper left corner, resulting in long sequences of zeros. Could you explain why it is intelligent to do so?

Because the values of Q(i,j) decrease as i and j increase, traversing in a zig-zag traverses in decreasing magnitude of Q(i,j). This means that as the path moves towards the bottom right, it is more likely for the values of the discrete cosine transform coefficients to get rounded to 0. When multiple consecutive numbers are rounded to 0, Huffman encoding allows for efficient lossless compression. The reason that Q is chosen to increase the likelihood of high frequency terms to be lost is that the human eye has difficulty distinguishing high resolution of an 8x8 pixel grid on an image of 256x256 pixels. So, each 8x8 block can afford to lose some high frequency components so that the storage is more efficient.