Signal and Image Processing

Lab session 3

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1 Excercise 1 - Approximation of a spectrum filter analysis

1.1 Give the expression $S_N(\lambda)$ with respect to f_0.

$$S_N(\lambda) = \sum_{n=-N}^{N-1} f(na)e^{-2i\pi na\lambda} = \sum_{n=-N}^{N-1} \left(e^{2i\pi a(F_0-\lambda)}\right)^n = \sum_{n=0}^{2N-1} \left(e^{2i\pi a(F_0-\lambda)}\right)^{(n-N)} =$$

$$= e^{-2i\pi Na(F_0-\lambda)} \sum_{n=0}^{2N-1} \left(e^{2i\pi a(F_0-\lambda)}\right)^n = e^{-2i\pi Na(F_0-\lambda)} \left(\frac{1-e^{2i\pi a(F_0-\lambda)2N}}{1-e^{2i\pi a(F_0-\lambda)}}\right) =$$

$$= e^{-i\pi a(F_0-\lambda)} \left(\frac{e^{-2i\pi Na(F_0-\lambda)} - e^{2i\pi Na(F_0-\lambda)}}{e^{-i\pi a(F_0-\lambda)} - e^{i\pi a(F_0-\lambda)}}\right) = e^{-i\pi a(F_0-\lambda)} \frac{\sin\left(2\pi Na(F_0-\lambda)\right)}{\sin\left(\pi a(F_0-\lambda)\right)} =$$

$$= e^{-i\pi(f_0-\frac{\lambda}{F_e})} \frac{\sin\left(2\pi N(f_0-\frac{\lambda}{F_e})\right)}{\sin\left(\pi(f_0-\frac{\lambda}{F_e})\right)}$$

1.2 $F_e = \frac{1}{32}$. Compute $S_N(\frac{k}{T})$ when $F_0 = 7$ and when $f_0 = 0.2$. Compare with $S_N(\lambda)$.

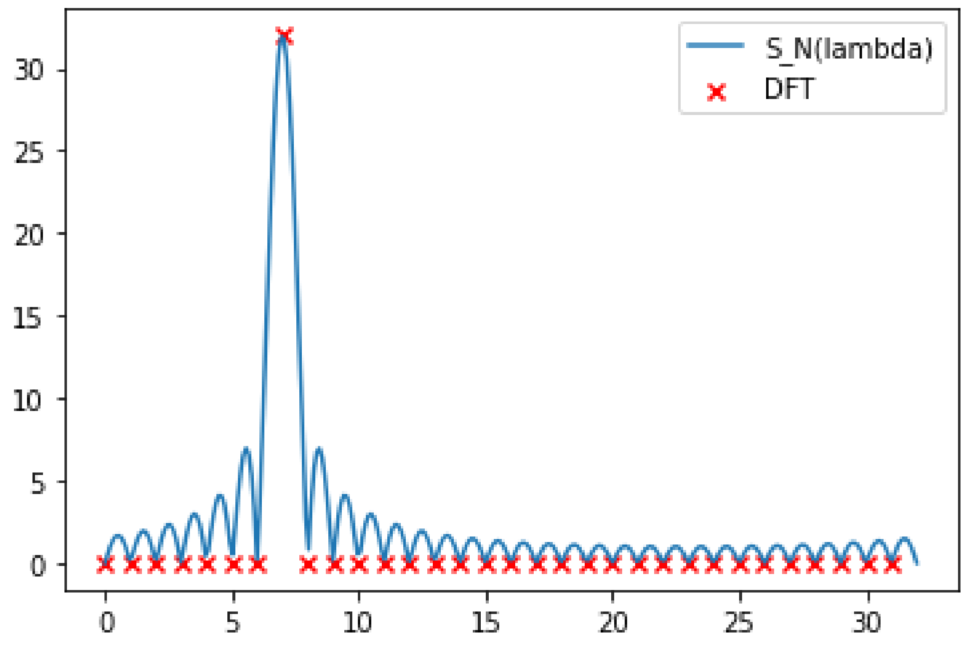

Figure 1: Comparison of $S_N(\frac{k}{T})$ and $S_N(\lambda)$ when $F_e = \frac{1}{32}$ and $F_0 = 7$

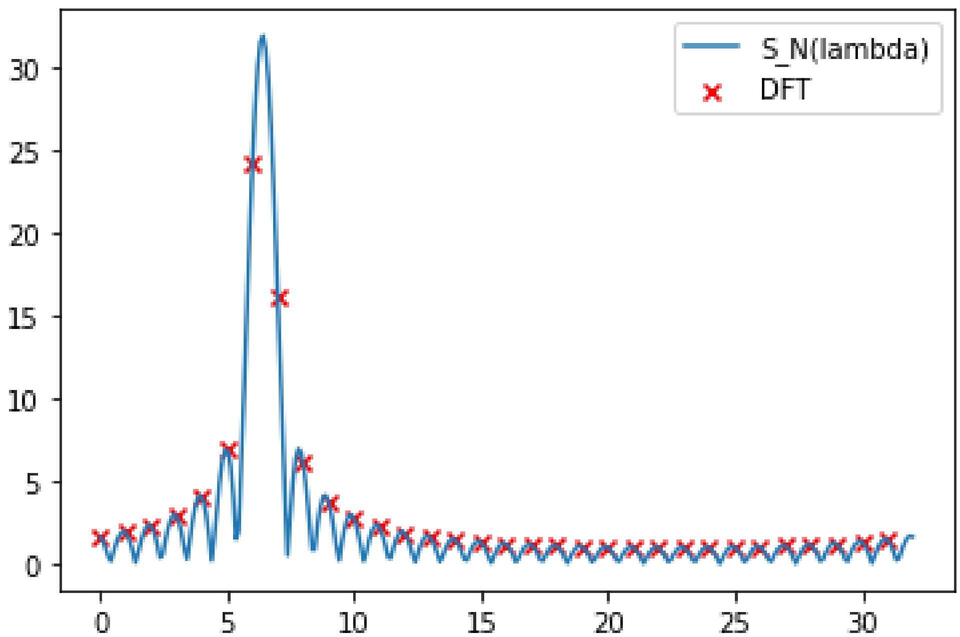

Figure 2: Comparison of $S_N(\frac{k}{T})$ and $S_N(\lambda)$ when $F_e = \frac{1}{32}$ and $f_0 = 0.2$

2 Excercise 2 - On the use of DFT for filtering

2.1 Give the expression for g_n and show that g is periodic with period N

$$f = \sum_{n \in \mathbb{Z}} f_n \delta_{na}$$

$$d = \sum_{n=0}^{3} \frac{1}{4} \delta_{na}$$

$$g = d * f = \sum_{m \in \mathbb{Z}} g_m \delta_{ma}$$

$$g_m = \sum_{n \in \mathbb{Z}} f_n \delta_{na} g_{m-n} \delta_{m-n}$$

$$g_m = \frac{1}{4}(f_m + f_{m+1} + f_{m+2} + f_{m+3})$$

The function f is periodic with period N, so g is periodic with period N as well since it is a linear sum of f.

2.2 Explain how to compute g using the DFT and its inverse. Write the program.

$$g_n = \text{IDFT}(\text{DFT}(d)\text{DFT}(f))$$

where "DFT" means "Discrete Fourier Transform" as implemented by numpy.fft.fft and "IDFT" means "Inverse Discrete Fourier Transform" as implemented by numpy.fft.ifft. d and f are points d_n and f_n.

2.3 Compute g and plot its coefficient for specified values.

The function g and the absolute value of its DFT are shown below in Figures 3 and 4.

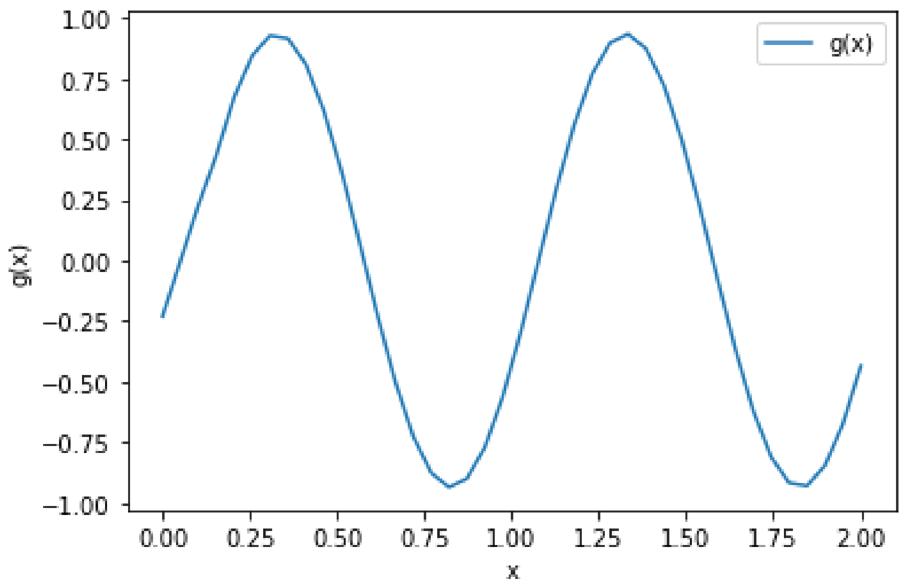

Figure 3: Function g Computed using FFT of f and d.

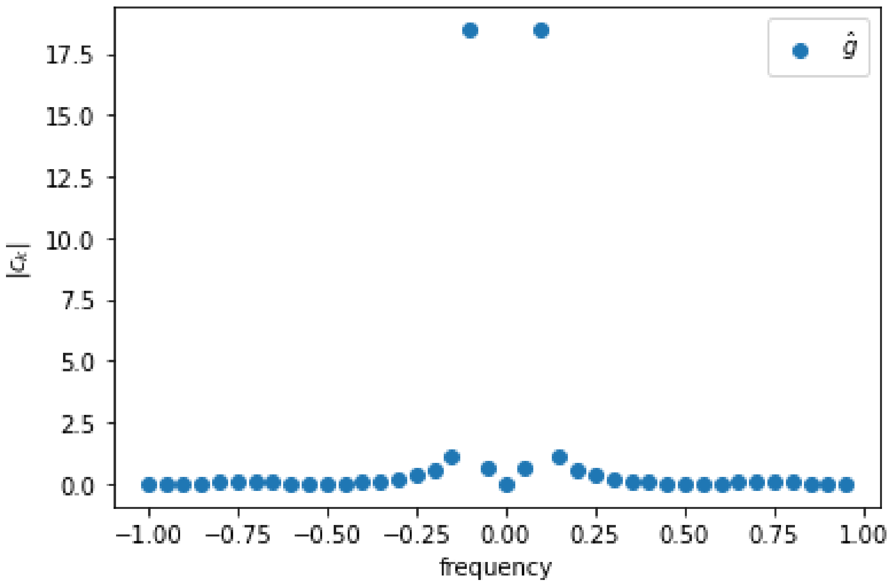

Figure 4: Function g Computed using FFT of f and d.

2.4 Explain how one can retrieve f from g (deconvolution) using DFT and its inverse. Write the programme. What if DFT of d vanishes for some indices?

$$f_n = IDFT(\frac{DFT(g)}{DFT(d)})$$

with the same notation as above.

2.5 If the inverse filter is modified into

$$InvD_k = \begin{cases} \frac{1}{D_K} & if|D_k| > \epsilon \\ \frac{1}{\epsilon} & otherwise. \end{cases}$$

Use this filter to recover f, plot the obtained signals for variable ϵ

Some values of FFT(d) are so small that division by them causes numerical problems. This can be fixed by setting all values of D_k below some ϵ to ϵ. The effect of this on the recovery of f is shown for an unmodified D_k and different values of ϵ. The normed error is also shown with a comparison against unmodified D_k. From the plot of the normed error, as long as ϵ is not set too large (≈ 10) or too small ($\approx 10^{-20}$) the modified inverse filter is much closer to the true signal. This is shown shown below in Figures 5 and 6.

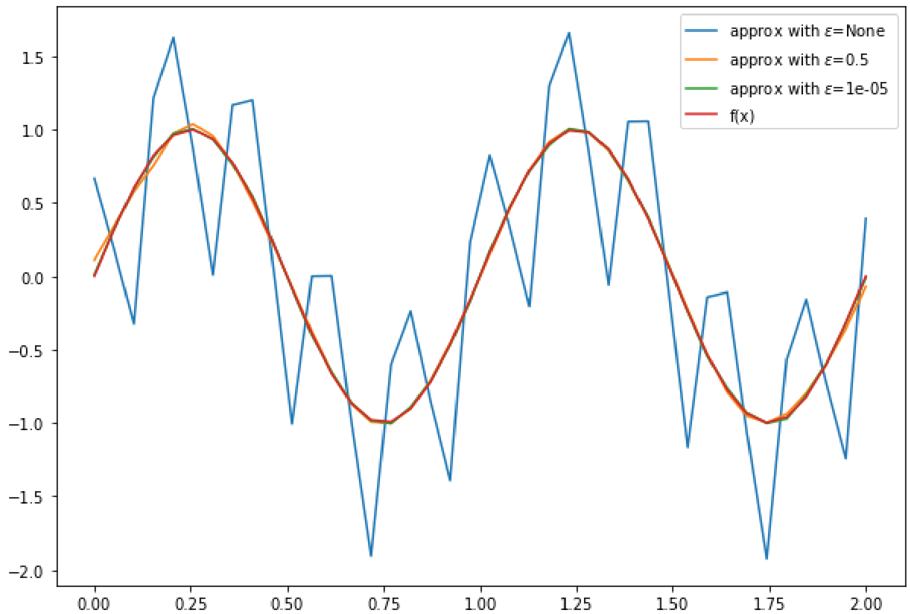

Figure 5: Function f Computed using Inverse Method

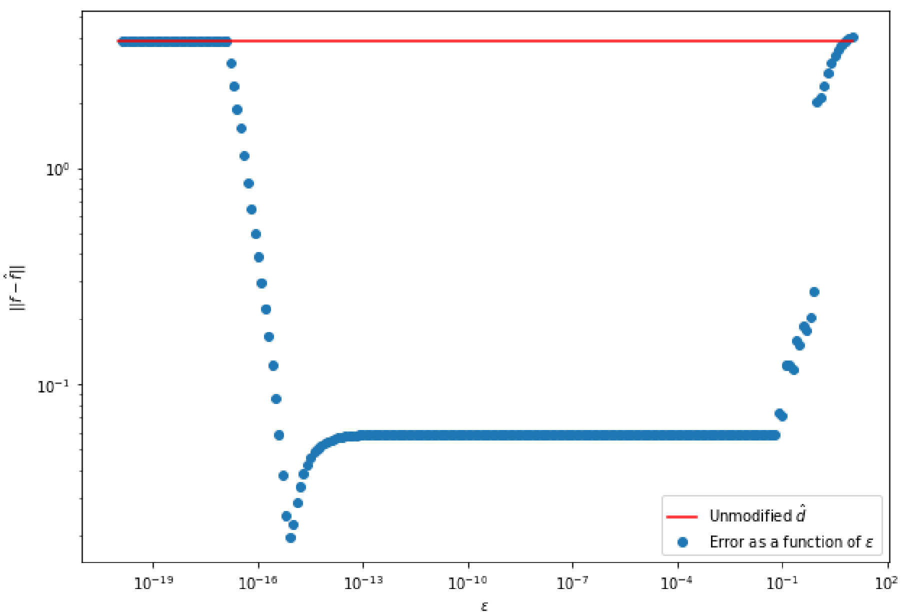

Figure 6: Normed Errors for Function f Computed using Inverse Method with Different ϵ. It is important to note that the machine precision for doubles is on the order of 10^{-15}.

3 Excercise 3 - Implementation of a finite impulse response filter

3.1 Compute the specified h_n.

Let

$$H(\lambda) = \chi_{[-\frac{1}{4},\frac{1}{4}[} \quad \text{for} \quad \lambda \in [-\frac{1}{2},\frac{1}{2}[$$

then

$$h_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} H(\lambda)e^{2i\pi n\lambda}\mathrm{d}\lambda = \int_{-\frac{1}{4}}^{\frac{1}{4}} e^{2i\pi n\lambda}\mathrm{d}\lambda = \frac{1}{2i\pi n}\left(e^{\frac{i\pi n}{2}} - e^{-\frac{i\pi n}{2}}\right) = \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}$$

3.2

One would like to keep only N coefficients h_n. If N is odd, one keeps the indices $-(N-1)/2 \leq n \leq (N-1)/2$. Give the corresponding values for h_n when $N = 15$, propose a causal version of the filter by using an appropriate translation. What is the effect of this translation in the frequency domain? Compute the phase of this filter (argument of its Fourier transform). Show that this phase is pieces linear, plot the modulus of the Fourier transform of the filter thus obtained and

compare it with H. Note that to compute a filter with piece-wise linear phase is essential to avoid phase distortion.
For $N = 15$, the $h(n)$ values are shown along with the continuous valued function $sin(n\pi/2)/(\pi n)$.

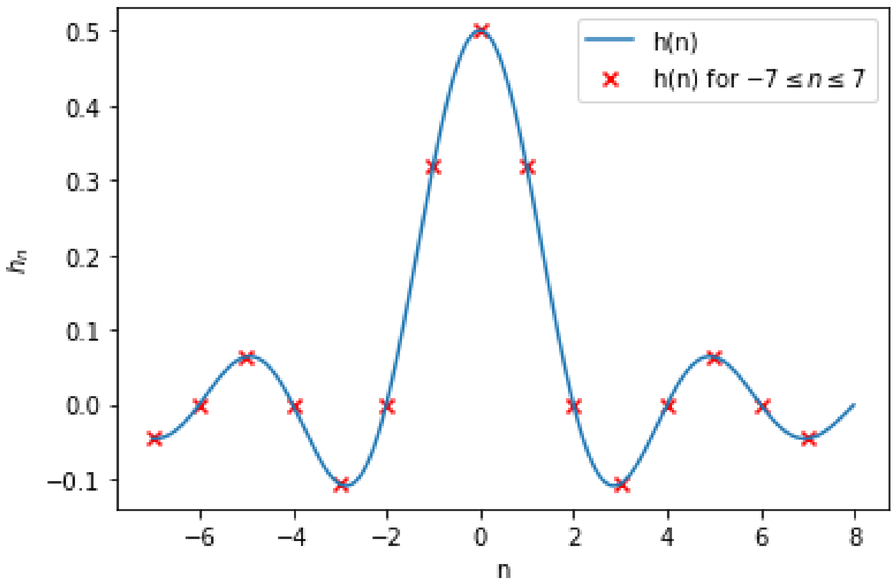

Figure 7: $sin(n\pi/2)/(\pi n)$ for$-(N-1)/2 \leq n \leq (N-1)/2$.

The definition of causality for a filter h is that

$$\forall n < 0 \qquad h_n = 0$$

Shifting a signal follows:

$$\hat{x}(n-\Delta)_k = e^{-2i\pi k\Delta/N}\hat{x}_k$$

For this to be true, the filter will be shifted by N so that $h_{n<0} = 0$. This can be done by adding the factor $e^{-2i\pi k}$ for every $\hat{x}_k$. The phase is the argument of the exponential, so the phase shift is linear (k).

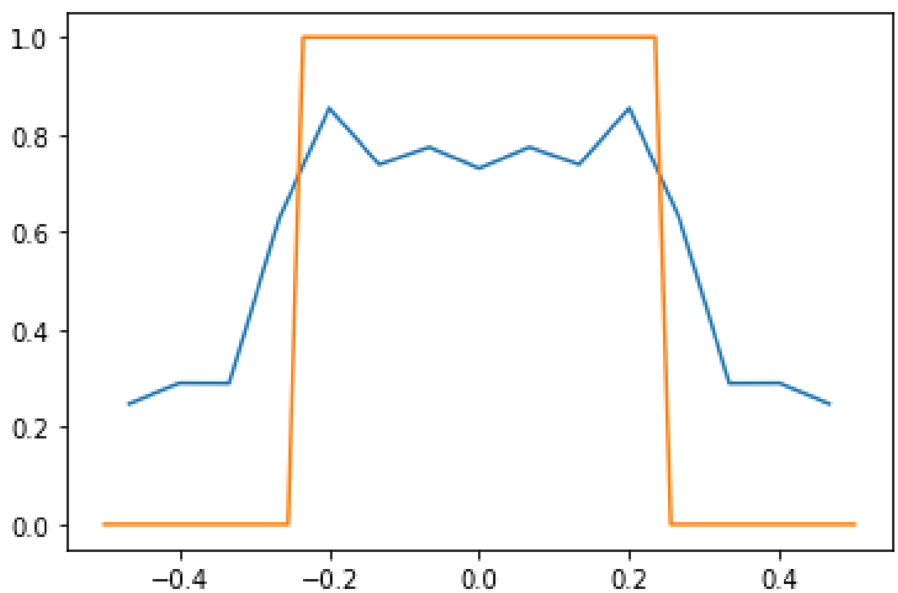

Figure 8: Comparison of Fourrier transform of $h_n \quad -7 \leq n \leq 7$ with the original signal

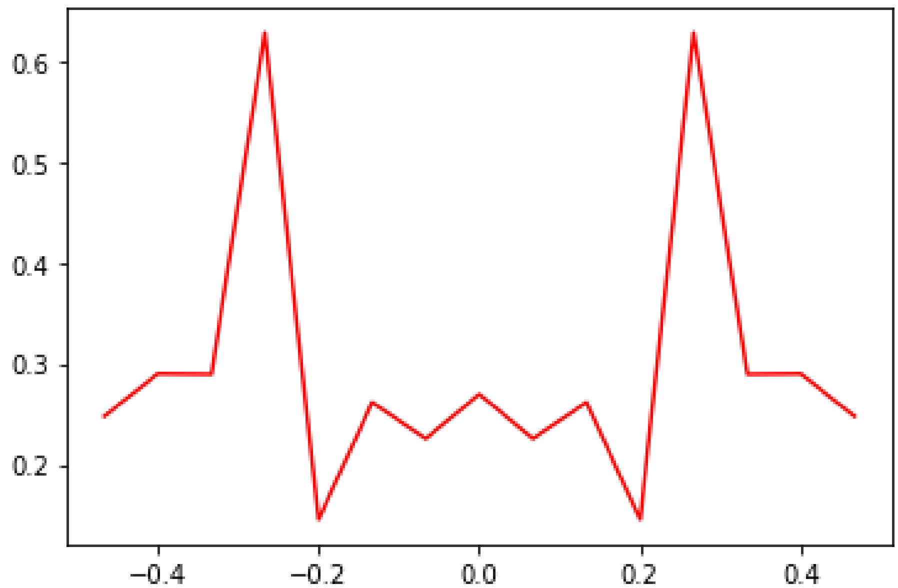

Figure 9: Error between Fourrier transform of $h_n \quad -7 \leq n \leq 7$ and the original signal

3.3

$$\hat{h}_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} H(\lambda) e^{i\pi(2n-1)\lambda} \mathrm{d}\lambda = \int_{-\frac{1}{4}}^{\frac{1}{4}} e^{i\pi(2n-1)\lambda} \mathrm{d}\lambda = \frac{1}{i\pi(2n-1)} \left(e^{\frac{i\pi(2n-1)}{4}} - e^{-\frac{i\pi(2n-1)}{4}} \right) = \frac{2}{\pi(2n-1)} \sin\left(\frac{\pi(2n-1)}{4}\right)$$

For $N = 6$ values of $\hat{h}_n$ are shown below, along with continuously valued function $\frac{2}{\pi(2n-1)} \sin\left(\frac{\pi(2n-1)}{4}\right)$

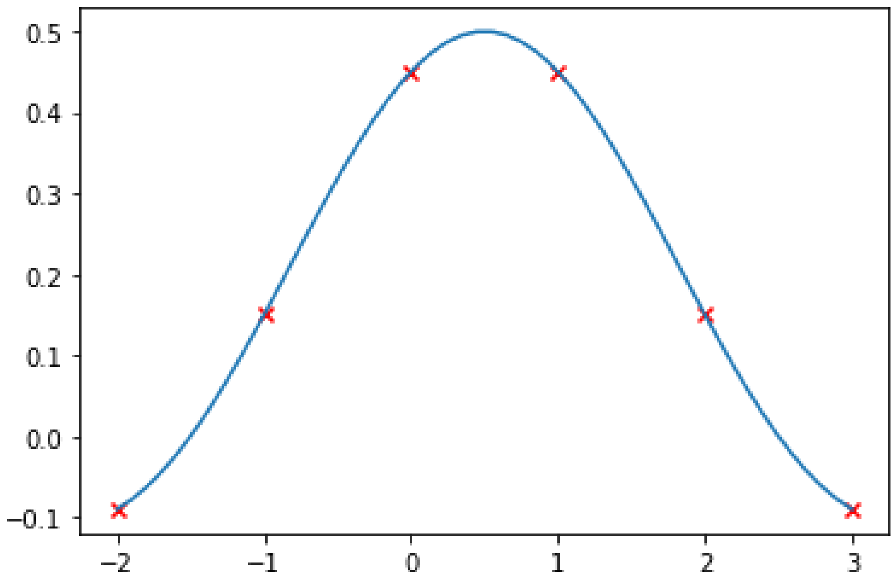

Figure 10: Values of $\hat{h}_n$, $-2 \leq n \leq 3$

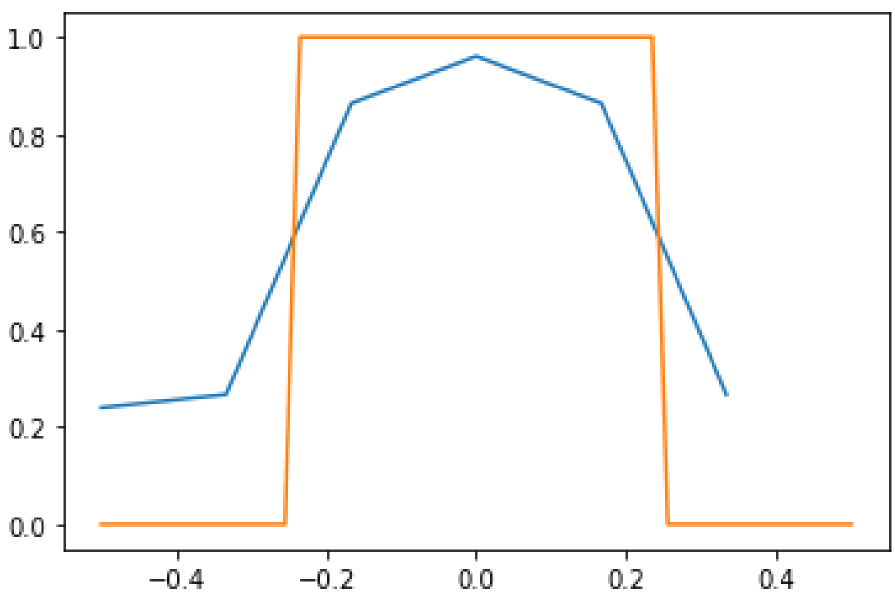

Figure 11: Comparison of Fourrier transform of $\hat{h}_n \quad -2 \leq n \leq 3$ with the original signal

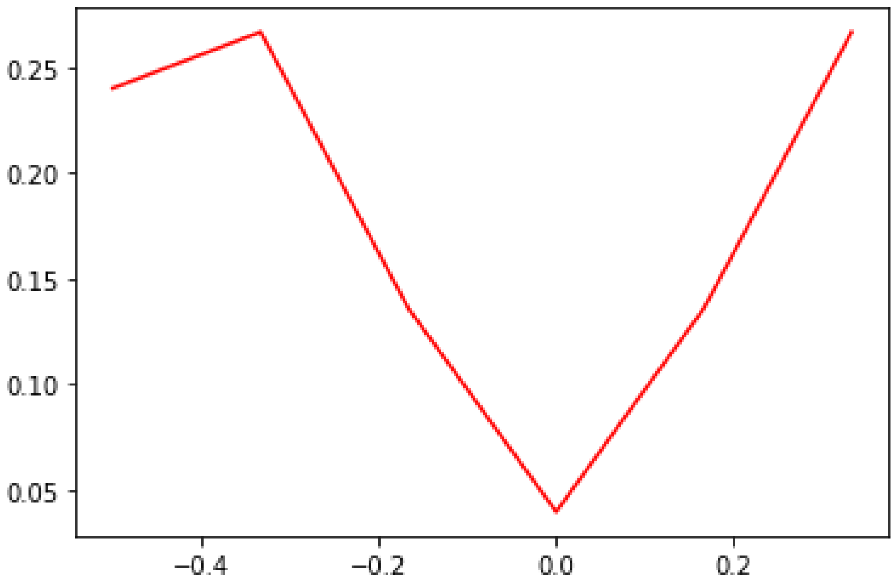

Figure 12: Error between Fourrier transform of $\hat{h}_n \quad -2 \leq n \leq 3$ and the original signal

3.4

Show more generally that if h_n defined for $0 \leq n \leq P$ is such that $h_{P-n} = h_n$, then the Fourier transform of the filter h is with linear phase.
If $h_{P-n} = h_n$ is true, then h can be shifted to be centered on the origin in time. Then the Fourier transform of a symmetric real signal is symmetric and real. Then by the equation

$$\hat{x}(n-\Delta)_k = e^{-2i\pi k\Delta/N}\hat{x}_k$$

, the output signal is a real, symmetric signal times a linear phase shift

$$\hat{x}_k = Xe^{2i\pi k(\theta-\Delta)/N}$$

where X and θ come from $\hat{x}_k = Xe^{2i\pi\theta k/N}$.