JBMO ShortLists 2002

$\boxed{1}$ A student is playing computer. Computer shows randomly 2002 positive numbers. Game's rules let do the following operations - to take 2 numbers from these, to double first one, to add the second one and to save the sum. - to take another 2 numbers from the remainder numbers, to double the first one, to add the second one, to multiply this sum with previous and to save the result. - to repeat this procedure, until all the 2002 numbers won't be used. Student wins the game if final product is maximum possible. Find the winning strategy and prove it.

$\boxed{2}$ Positive real numbers are arranged in the form: 1 3 6 10 15... 2 5 9 14... 4 8 13... 5 12... 11... Find the number of the line and column where the number 2002 stays.

$\boxed{3}$ Let a, b, c be positive real numbers such that $abc = \frac{9}{4}$. Prove the inequality: $a^3 + b^3 + c^3 > a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b}$

Jury's variant: Prove the same, but with $abc = 2$

$\boxed{5}$ Let a, b, c be positive real numbers. Prove the inequality: $\frac{a^3}{b^2} + \frac{b^3}{c^2} + \frac{c^3}{a^2} \geq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$

$\boxed{6}$ Let $a_1, a_2, ..., a_6$ be real numbers such that: $a_1 \neq 0, a_1a_6 + a_3 + a_4 = 2a_2a_5$ and $a_1a_3 \geq a_2^2$ Prove that $a_4a_6 \leq a_5^2$. When does equality holds?

$\boxed{7}$ Consider integers $a_i, i = \overline{1, 2002}$ such that $a_1^{-3} + a_2^{-3} + \ldots + a_{2002}^{-3} = \frac{1}{2}$ Prove that at least 3 of the numbers are equal.

$\boxed{8}$ Let ABC be a triangle with centroid G and A_1, B_1, C_1 midpoints of the sides BC, CA, AB. A paralel through A_1 to BB_1 intersects B_1C_1 at F. Prove that triangles ABC and FA_1A are similar if and only if quadrilateral AB_1GC_1 is cyclic.

$\boxed{9}$ In triangle ABC, H, I, O are orthocenter, incenter and circumcenter, respectively. CI cuts circumcircle at L. If $AB = IL$ and $AH = OH$, find angles of triangle ABC.

$\boxed{10}$ Let ABC be a triangle with area S and points D, E, F on the sides BC, CA, AB. Perpendiculars at points D, E, F to the BC, CA, AB cut circumcircle of the triangle ABC at points $(D_1, D_2), (E_1, E_2), (F_1, F_2)$. Prove that: $|D_1B \cdot D_1C - D_2B \cdot D_2C| + |E_1A \cdot E_1C - E_2A \cdot E_2C| + |F_1B \cdot F_1A - F_2B \cdot F_2A| > 4S$

$\boxed{11}$ Let ABC be an isosceles triangle with $AB = AC$ and $\angle A = 20^\circ$. On the side AC consider point D such that $AD = BC$. Find $\angle BDC$.

$\boxed{12}$ Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $BC = CD$. On the sides AB, BC, CD, DA we consider points K, L, L_1, K_1 such that quadrilateral KLL_1K_1 is rectangle. Then consider rectangles $MNPQ$ inscribed in the triangle BLK, where $M \in KB, N \in$

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$BL, P, Q \in LK$ and $M_1N_1P_1Q_1$ inscribed in triangle DK_1L_1 where P_1 and Q_1 are situated on the L_1K_1, M on the DK_1 and N_1 on the DL_1. Let S, S_1, S_2, S_3 be the areas of the $ABCD, KLL_1K_1, MNPQ, M_1N_1P_1Q_1$ respectively. Find the maximum possible value of the expression: $\dfrac{S_1 + S_2 + S_3}{S}$

13 Let $A_1, A_2, ..., A_{2002}$ be the arbitrary points in the plane. Prove that for every circle of the radius 1 and for every rectangle inscribed in this circle, exist 3 vertexes M, N, P of the rectangle such that: $MA_1 + MA_2 + ... + MA_{2002} + NA_1 + NA_2 + ... + NA_{2002} + PA_1 + PA_2 + ... + PA_{2002} \geq 6006$

JBMO ShortLists 2006

1. For an acute triangle ABC prove the inequality: $\sum_{cyclic} \frac{m_a^2}{-a^2+b^2+c^2} \geq \frac{9}{4}$ where m_a, m_b, m_c are lengths of corresponding medians.

2. Let x, y, z be positive real numbers such that $x + 2y + 3z = \frac{11}{12}$. Prove the inequality $6(3xy + 4xz + 2yz) + 6x + 3y + 4z + 72xyz \leq \frac{107}{18}$.

3. Let $n \geq 3$ be a natural number. A set of real numbers $\{x_1, x_2, \dots, x_n\}$ is called *summable* if $\sum_{i=1}^{n} \frac{1}{x_i} = 1$. Prove that for every $n \geq 3$ there always exists a *summable* set which consists of n elements such that the biggest element is: a) bigger than 2^{2n-2} b) smaller than n^2

4. Determine the biggest possible value of m for which the equation $2005x + 2007y = m$ has unique solution in natural numbers.

5. Determine all pairs (m, n) of natural numbers for which $m^2 = nk + 2$ where $k = \overline{n1}$.

6. Prove that for every composite number $n > 4$, numbers kn divides $(n-1)!$ for every integer k such that $1 \leq k \leq \lfloor \sqrt{n-1} \rfloor$.

7. Determine all numbers $\overline{abcd}$ such that $\overline{abcd} = 11(a + b + c + d)^2$.

8. Prove that there do not exist natural numbers $n \geq 10$ such that every n's digit is not zero, and all numbers which are obtained by permutating its digits are perfect squares.

9. Let $ABCD$ be a trapezoid with $AB \parallel CD, AB > CD$ and $\angle A + \angle B = 90^\circ$. Prove that the distance between the midpoints of the bases is equal to the semidifference of the bases.

10. Let $ABCD$ be a trapezoid inscribed in a circle $\mathcal{C}$ with $AB \parallel CD$, $AB = 2CD$. Let $\{Q\} = AD \cap BC$ and let P be the intersection of tangents to $\mathcal{C}$ at B and D. Calculate the area of the quadrilateral $ABPQ$ in terms of the area of the triangle PDQ.

11. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. Let $M \in AB$. A line through M (different from AB) cuts circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at Z, D, E, C respectively such that $D, E \in ZC$. Perpendiculars at B to the lines EB, ZB and AD respectively cut circle $\mathcal{C}_2$ in F, K and N. Prove that $KF = NC$.

12. Let ABC be an equilateral triangle of center O, and $M \in BC$. Let K, L be projections of M onto the sides AB and AC respectively. Prove that line OM passes through the midpoint of the segment KL.

13. Let A be a subset of the set $\{1, 2, \dots, 2006\}$, consisting of 1004 elements. Prove that there exist 3 distinct numbers $a, b, c \in A$ such that $gcd(a, b)$: a) divides c b) doesn't divide c

14 Let $n \geq 5$ be a positive integer. Prove that the set $\{1, 2, \ldots, n\}$ can be partitioned into two non-zero subsets S_n and P_n such that the sum of elements in S_n is equal to the product of elements in P_n.

Junior Balkan MO 1997

Belgrad, Yugoslavia

1 Show that given any 9 points inside a square of side 1 we can always find 3 which form a triangle with area less than $\frac{1}{8}$.

Bulgaria

2 Let $\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x^2-y^2}{x^2+y^2}=k$. Compute the following expression in terms of k:

$$E(x,y)=\frac{x^8+y^8}{x^8-y^8}-\frac{x^8-y^8}{x^8+y^8}.$$

Ciprus

3 Let ABC be a triangle and let I be the incenter. Let N, M be the midpoints of the sides AB and CA respectively. The lines BI and CI meet MN at K and L respectively. Prove that $AI+BI+CI>BC+KL$.

Greece

4 Determine the triangle with sides a,b,c and circumradius R for which $R(b+c)=a\sqrt{bc}$.

Romania

5 Let n_1, n_2, …, n_{1998} be positive integers such that

$$n_1^2+n_2^2+\cdots+n_{1997}^2=n_{1998}^2.$$

Show that at least two of the numbers are even.

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Junior Balkan MO 1998

Athens, Greece

$\boxed{1}$ Prove that the number $\underbrace{111\ldots11}_{1997}\underbrace{22\ldots22}_{1998}5$ (which has 1997 of 1-s and 1998 of 2-s) is a perfect square.

Yugoslavia

$\boxed{2}$ Let $ABCDE$ be a convex pentagon such that $AB = AE = CD = 1$, $\angle ABC = \angle DEA = 90^\circ$ and $BC + DE = 1$. Compute the area of the pentagon.

Greece

$\boxed{3}$ Find all pairs of positive integers (x, y) such that

$$x^y = y^{x-y}.$$

Albania

$\boxed{4}$ Do there exist 16 three digit numbers, using only three different digits in all, so that the all numbers give different residues when divided by 16?

Bulgaria

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Junior Balkan MO 1999

Plovdiv, Bulgaria

1 Let a, b, c, x, y be five real numbers such that $a^3+ax+y=0$, $b^3+bx+y=0$ and $c^3+cx+y=0$. If a, b, c are all distinct numbers prove that their sum is zero.

Ciprus

2 For each nonnegative integer n we define $A_n = 2^{3n}+3^{6n+2}+5^{6n+2}$. Find the greatest common divisor of the numbers $A_0, A_1, \ldots, A_{1999}$.

Romania

3 Let S be a square with the side length 20 and let M be the set of points formed with the vertices of S and another 1999 points lying inside S. Prove that there exists a triangle with vertices in M and with area at most equal with $\frac{1}{10}$.

Yugoslavia

4 Let ABC be a triangle with $AB = AC$. Also, let $D \in [BC]$ be a point such that $BC > BD > DC > 0$, and let $\mathcal{C}_1, \mathcal{C}_2$ be the circumcircles of the triangles ABD and ADC respectively. Let BB' and CC' be diameters in the two circles, and let M be the midpoint of $B'C'$. Prove that the area of the triangle MBC is constant (i.e. it does not depend on the choice of the point D).

Greece

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Junior Balkan MO 2000

Ohrid, Macedonia

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1. Let x and y be positive reals such that

$$x^3 + y^3 + (x+y)^3 + 30xy = 2000.$$

Show that $x + y = 10$.

2. Find all positive integers $n \geq 1$ such that $n^2 + 3^n$ is the square of an integer.

Bulgaria

3. A half-circle of diameter EF is placed on the side BC of a triangle ABC and it is tangent to the sides AB and AC in the points Q and P respectively. Prove that the intersection point K between the lines EP and FQ lies on the altitude from A of the triangle ABC.

Albania

4. At a tennis tournament there were $2n$ boys and n girls participating. Every player played every other player. The boys won $\frac{7}{5}$ times as many matches as the girls. It is knowns that there were no draws. Find n.

Serbia

Junior Balkan MO 2001

Nicosia, Ciprus

1 Solve the equation $a^3 + b^3 + c^3 = 2001$ in positive integers.

Mircea Becheanu, Romania

2 Let ABC be a triangle with $\angle C = 90^\circ$ and $CA \neq CB$. Let CH be an altitude and CL be an interior angle bisector. Show that for $X \neq C$ on the line CL, we have $\angle XAC \neq \angle XBC$. Also show that for $Y \neq C$ on the line CH we have $\angle YAC \neq \angle YBC$.

Bulgaria

3 Let ABC be an equilateral triangle and D, E points on the sides $[AB]$ and $[AC]$ respectively. If DF, EF (with $F \in AE$, $G \in AD$) are the interior angle bisectors of the angles of the triangle ADE, prove that the sum of the areas of the triangles DEF and DEG is at most equal with the area of the triangle ABC. When does the equality hold?

Greece

4 Let N be a convex polygon with 1415 vertices and perimeter 2001. Prove that we can find 3 vertices of N which form a triangle of area smaller than 1.

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Junior Balkan MO 2002

Targu Mures, Romania

1 The triangle ABC has $CA = CB$. P is a point on the circumcircle between A and B (and on the opposite side of the line AB to C). D is the foot of the perpendicular from C to PB. Show that $PA + PB = 2 \cdot PD$.

2 Two circles with centers O_1 and O_2 meet at two points A and B such that the centers of the circles are on opposite sides of the line AB. The lines BO_1 and BO_2 meet their respective circles again at B_1 and B_2. Let M be the midpoint of B_1B_2. Let M_1, M_2 be points on the circles of centers O_1 and O_2 respectively, such that $\angle AO_1M_1 = \angle AO_2M_2$, and B_1 lies on the minor arc AM_1 while B lies on the minor arc AM_2. Show that $\angle MM_1B = \angle MM_2B$.

Ciprus

3 Find all positive integers which have exactly 16 positive divisors $1 = d_1 < d_2 < \ldots < d_{16} = n$ such that the divisor d_k, where $k = d_5$, equals $(d_2 + d_4)d_6$.

4 Prove that for all positive real numbers a, b, c the following inequality takes place

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}.$$

Laurentiu Panaitopol, Romania

Junior Balkan MO 2003

Kusadasi, Turkey

$\boxed{1}$ Let n be a positive integer. A number A consists of $2n$ digits, each of which is 4; and a number B consists of n digits, each of which is 8. Prove that $A + 2B + 4$ is a perfect square.

$\boxed{2}$ Suppose there are n points in a plane no three of which are collinear with the property that if we label these points as $A_1, A_2, \ldots, A_n$ in any way whatsoever, the broken line $A_1A_2 \ldots A_n$ does not intersect itself. Find the maximum value of n.

Dinu Serbanescu, Romania

$\boxed{3}$ Let D, E, F be the midpoints of the arcs BC, CA, AB on the circumcircle of a triangle ABC not containing the points A, B, C, respectively. Let the line DE meets BC and CA at G and H, and let M be the midpoint of the segment GH. Let the line FD meet BC and AB at K and J, and let N be the midpoint of the segment KJ.

a) Find the angles of triangle DMN;

b) Prove that if P is the point of intersection of the lines AD and EF, then the circumcenter of triangle DMN lies on the circumcircle of triangle PMN.

$\boxed{4}$ Let $x, y, z > -1$. Prove that

$$\frac{1+x^2}{1+y+z^2} + \frac{1+y^2}{1+z+x^2} + \frac{1+z^2}{1+x+y^2} \geq 2.$$

Laurentiu Panaitopol

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Junior Balkan MO 2004

Novi Sad, Serbia and Montenegro

1. Prove that the inequality

$$\frac{x+y}{x^2-xy+y^2} \leq \frac{2\sqrt{2}}{\sqrt{x^2+y^2}}$$

holds for all real numbers x and y, not both equal to 0.

2. Let ABC be an isosceles triangle with $AC = BC$, let M be the midpoint of its side AC, and let Z be the line through C perpendicular to AB. The circle through the points B, C, and M intersects the line Z at the points C and Q. Find the radius of the circumcircle of the triangle ABC in terms of $m = CQ$.

3. If the positive integers x and y are such that $3x + 4y$ and $4x + 3y$ are both perfect squares, prove that both x and y are both divisible with 7.

4. Consider a convex polygon having n vertices, $n \geq 4$. We arbitrarily decompose the polygon into triangles having all the vertices among the vertices of the polygon, such that no two of the triangles have interior points in common. We paint in black the triangles that have two sides that are also sides of the polygon, in red if only one side of the triangle is also a side of the polygon and in white those triangles that have no sides that are sides of the polygon.

   Prove that there are two more black triangles that white ones.

Junior Balkan MO 2005

Veria, Greece

$\boxed{1}$ Find all positive integers x, y satisfying the equation

$$9(x^2 + y^2 + 1) + 2(3xy + 2) = 2005.$$

$\boxed{2}$ Let ABC be an acute-angled triangle inscribed in a circle k. It is given that the tangent from A to the circle meets the line BC at point P. Let M be the midpoint of the line segment AP and R be the second intersection point of the circle k with the line BM. The line PR meets again the circle k at point S different from R.

Prove that the lines AP and CS are parallel.

$\boxed{3}$ Prove that there exist

(a) 5 points in the plane so that among all the triangles with vertices among these points there are 8 right-angled ones;

(b) 64 points in the plane so that among all the triangles with vertices among these points there are at least 2005 right-angled ones.

$\boxed{4}$ Find all 3-digit positive integers $\overline{abc}$ such that

$$\overline{abc} = abc(a + b + c),$$

where $\overline{abc}$ is the decimal representation of the number.

Junior Balkan MO 2006

$\boxed{1}$ If $n > 4$ is a composite number, then $2n$ divides $(n-1)!$.

$\boxed{2}$ The triangle ABC is isosceles with $AB = AC$, and $\angle BAC < 60^\circ$. The points D and E are chosen on the side AC such that, $EB = ED$, and $\angle ABD \equiv \angle CBE$. Denote by O the intersection point between the internal bisectors of the angles $\angle BDC$ and $\angle ACB$. Compute $\angle COD$.

$\boxed{3}$ We call a number *perfect* if the sum of its positive integer divisors(including 1 and n) equals $2n$. Determine all *perfect* numbers n for which $n-1$ and $n+1$ are prime numbers.

$\boxed{4}$ Consider a $2n \times 2n$ board. From the ith line we remove the central $2(i-1)$ unit squares. What is the maximal number of rectangles 2×1 and 1×2 that can be placed on the obtained figure without overlapping or getting outside the board?

Junior Balkan MO 2007

Shumen, Bulgaria

1 Let a be positive real number such that $a^3 = 6(a+1)$. Prove that the equation $x^2+ax+a^2-6 = 0$ has no real solution.

2 Let $ABCD$ be a convex quadrilateral with $\angle DAC = \angle BDC = 36^\circ$, $\angle CBD = 18^\circ$ and $\angle BAC = 72^\circ$. The diagonals and intersect at point P . Determine the measure of $\angle APD$.

3 Given are 50 points in the plane, no three of them belonging to a same line. Each of these points is colored using one of four given colors. Prove that there is a color and at least 130 scalene triangles with vertices of that color.

4 Prove that if p is a prime number, then $7p + 3^p - 4$ is not a perfect square.

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$\boxed{1}$ Find all real numbers a, b, c, d such that

$$\begin{cases} a+b+c+d=20, \\ ab+ac+ad+bc+bd+cd=150. \end{cases}$$

$\boxed{2}$ The vertices A and B of an equilateral triangle ABC lie on a circle of radius 1, and the vertex C is in the interior of the circle k. A point D, different from B, lies on k so that $AD = AB$. The line DC intersects k for the second time at point E. Find the length of the line segment CE.

$\boxed{3}$ Find all prime numbers p, q, r, such that $\frac{p}{q} - \frac{4}{r+1} = 1$

$\boxed{4}$ A 4×4 table is divided into 16 white unit square cells. Two cells are called neighbors if they share a common side. A *move* consists in choosing a cell and the colors of neighbors from white to black or from black to white. After exactly n moves all the 16 cells were black. Find all possible values of n.

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