Balkan MO 1984

Athens, Greece

$\boxed{1}$ Let a, b, c be positive real numbers. Find all real solutions (x, y, z) of the sistem: $ax + by = (x - y)^2$ $by + cz = (y - z)^2$ $cz + ax = (z - x)^2$

$\boxed{2}$ Let $ABCD$ be a cyclic quadrilateral and let H_A, H_B, H_C, H_D be the orthocenters of the triangles BCD, CDA, DAB and ABC respectively. Show that the quadrilaterals $ABCD$ and $H_A H_B H_C H_D$ are congruent.

$\boxed{3}$ Show that for any positive integer m, there exists a positive integer n so that in the decimal representations of the numbers 5^m and 5^n, the representation of 5^n ends in the representation of 5^m.

$\boxed{4}$ Let a, b, c be positive real numbers. Find all real solutions (x, y, z) of the sistem: $ax + by = (x - y)^2$ $by + cz = (y - z)^2$ $cz + ax = (z - x)^2$

1 In a given triangle ABC, O is its circumcenter, D is the midpoint of AB and E is the centroid of the triangle ACD. Show that the lines CD and OE are perpendicular if and only if $AB = AC$.

2 Let $a, b, c, d \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ be real numbers such that $\sin a + \sin b + \sin c + \sin d = 1$ and $\cos 2a + \cos 2b + \cos 2c + \cos 2d \geq \frac{10}{3}$. Prove that $a, b, c, d \in [0, \frac{\pi}{6}]$

3 Let S be the set of all positive integers of the form $19a + 85b$, where a, b are arbitrary positive integers. On the real axis, the points of S are colored in red and the remaining integer numbers are colored in green. Find, with proof, whether or not there exists a point A on the real axis such that any two points with integer coordinates which are symmetrical with respect to A have necessarily distinct colors.

4 There are 1985 participants to an international meeting. In any group of three participants there are at least two who speak the same language. It is known that each participant speaks at most five languages. Prove that there exist at least 200 participans who speak the same language.

1 A line passing through the incenter I of the triangle ABC intersect its incircle at D and E and its circumcircle at F and G, in such a way that the point D lies between I and F. Prove that: $DF \cdot EG \geq r^2$.

2 Let $ABCD$ be a tetrahedron and let E, F, G, H, K, L be points lying on the edges AB, BC, CD, DA, DB, DC respectively, in such a way that $AE \cdot BE = BF \cdot CF = CG \cdot AG = DH \cdot AH = DK \cdot BK = DL \cdot CL$. Prove that the points E, F, G, H, K, L lie all on a sphere.

3 Let a, b, c be real numbers such that ab is not 0, $c > 0$ and let $(a_n)_{n\geq 1}$ be the sequence of real numbers defined by: $a_1 = a, a_2 = b$ and $a_{n+1} = \dfrac{a_n^2 + c}{a_{n-1}}$, $\forall n \geq 2$. Show that all the sequence's terms are integer numbers if and only if the numbers a, b and $\dfrac{a^2+b^2+c}{ab}$ are integers.

Remark: as Valentin mentions here [url]http://www.mathlinks.ro/Forum/viewtopic.php?p=492872search$_id = 51674358492872[/url], the 5-th romanian tst problem from 2006, follows immediatly from this bmo problem.$

Let ABC a triangle and P a point such that the triangles PAB, PBC, PCA have the same area and the same perimeter. Prove that if:

a) P is in the interior of the trinagle ABC then ABC is equilateral. b) P is in the exterior of the trinagle ABC then ABC is right angled triangle. ;)

Balkan MO 1987

$\boxed{1}$ Let a be a real number and let $f : R \to R$ be a function satisfying: $f(0) = \frac{1}{2}$ and $f(x+y) = f(x)f(a-y) + f(y)f(a-x)$, $\forall x, y \in \mathbb{R}$. Prove that f is constant.

$\boxed{2}$ Find all real numbers x, y greater than 1, satisfying the condition that the numbers $\sqrt{x-1} + \sqrt{y-1}$ and $\sqrt{x+1} + \sqrt{y+1}$ are nonconsecutive integers.

$\boxed{3}$ In the triangle ABC the following equality holds: $\sin^{23}\frac{A}{2}\cos^{48}\frac{B}{2} = \sin^{23}\frac{B}{2}\cos^{48}\frac{A}{2}$. Determine the value of $\frac{AC}{BC}$.

$\boxed{4}$ Two circles K_1 and K_2, centered at O_1 and O_2 with radii 1 and $\sqrt{2}$ respectively, intersect at A and B. Let C be a point on K_2 such that the midpoint of AC lies on K_1. Find the lenght of the segment AC if $O_1O_2 = 2$

Balkan MO 1988

$\boxed{1}$ Let ABC be a triangle and let M, N, P be points on the line BC such that AM, AN, AP are the altitude, the angle bisector and the median of the triangle, respectively. It is known that $\frac{[AMP]}{[ABC]} = \frac{1}{4}$ and $\frac{[ANP]}{[ABC]} = 1 - \frac{\sqrt{3}}{2}$. Find the angles of triangle ABC.

$\boxed{2}$ Find all polynomials of two variables $P(x, y)$ which satisfy $P(a, b)P(c, d) = P(ac + bd, ad + bc), \forall a, b, c, d \in R$

$\boxed{3}$ Let $ABCD$ be a tetrahedron and let d be the sum of squares of its edges' lengths. Prove that the tetrahedron can be included in a region bounded by two parallel planes, the distances between the planes being at most $\frac{\sqrt{d}}{2\sqrt{3}}$

$\boxed{4}$ Let $(a_n)_{n\geq 1}$ be a sequence defined by $a_n = 2^n + 49$. Find all values of n such that $a_n = pg, a_{n+1} = rs$, where p, q, r, s are prime numbers with $p < q, r < s$ and $q - p = s - r$.

Balkan MO 1989

1 Let n be a positive integer and let $d_1, ..., d_k$ be its divisors, such that $1 = d_1 < d_2 < ... < d_k = n$. Find all values of n for which $k \geq 4$ and $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

2 Let $\overline{a_n a_{n-1} ... a_1 a_0}$ be the decimal representation of a prime positive integer such that $n > 1$ and $a_n > 1$. Prove that the polynomial $P(x) = a_n x^n + ... + a_1 x + a_0$ cannot be written as a product of two nonconstant integer polynomials.

3 Let G be the centroid of a triangle ABC and let d be a line that intersects AB and AC at B_1 and C_1, respectively, such that the points A and G are not separated by d. Prove that: $[BB_1GC_1] + [CC_1GB_1] \geq \frac{4}{9}[ABC]$.

4 The elements of the set F are some subsets of $\{1, 2, ..., n\}$ and satisfy the conditions: i) if A belongs to F, then A has three elements; ii)if A and B are distinct elements of F , then A and B have at most one common element. Let $f(n)$ be the greatest possible number of elements of F. Prove that $\frac{n^2 - 4n}{6} \leq f(n) \leq \frac{n^2 - n}{6}$

Balkan MO 1990

1. The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 3$, and $a_{n+2} = (n+3)a_{n+1} - (n+2)a_n, \forall n \in \mathbb{N}$. Find all values of n for which a_n is divisible by 11.

2. The polynomial $P(X)$ is defined by $P(X) = (X + 2X^2 + ... + nX^{n)^2 = a_0 + a_1X + ... + a_{2n}X^{2n}}$. Prove that $a_{n+1} + a_{n+2} + ... + a_{2n} = \dfrac{n(n+1)(5n^2+5n+2)}{24}$.

3. Let ABC be an acute triangle and let A_1, B_1, C_1 be the feet of its altitudes. The incircle of the triangle $A_1B_1C_1$ touches its sides at the points A_2, B_2, C_2. Prove that the Euler lines of triangles ABC and $A_2B_2C_2$ coincide.

4. Find the least number of elements of a finite set A such that there exists a function $f : \{1, 2, 3, ...\} \to A$ with the property: if i and j are positive integers and $i - j$ is a prime number, then $f(i)$ and $f(j)$ are distinct elements of A.

Balkan MO 1991

1. Let ABC be an acute triangle inscribed in a circle centered at O. Let M be a point on the small arc AB of the triangle's circumcircle. The perpendicular dropped from M on the ray OA intersects the sides AB and AC at the points K and L, respectively. Similarly, the perpendicular dropped from M on the ray OB intersects the sides AB and BC at N and P, respectively. Assume that $KL = MN$. Find the size of the angle $\angle MLP$ in terms of the angles of the triangle ABC.

2. Show that there are infinitely many noncongruent triangles which satisfy the following conditions: i) the side lengths are relatively prime integers; ii)the area is an integer number; iii)the altitudes' lengths are not integer numbers.

3. A regular hexagon of area H is inscribed in a convex polygon of area P. Show that $P \leq \dfrac{3}{2}H$. When does the equality occur?

4. Prove that there is no bijective function $f : \{1, 2, 3, ...\} \to \{0, 1, 2, 3, ...\}$ such that: $f(mn) = f(m) + f(n) + 3f(m)f(n)$

Balkan MO 1992

Athens, Greece

1 For all positive integers m, n define $f(m,n) = m^{3^{4n}+6} - m^{3^{4n}+4} - m^5 + m^3$. Find all numbers n with the property that $f(m,n)$ is divisible by 1992 for every m.

Bulgaria

2 Prove that for all positive integers n the following inequality takes place

$$(2n^2+3n+1)^n \geq 6^n (n!)^2.$$

Cyprus

3 Let D, E, F be points on the sides BC, CA, AB respectively of a triangle ABC (distinct from the vertices). If the quadrilateral $AFDE$ is cyclic, prove that

$$\frac{4\mathcal{A}[DEF]}{\mathcal{A}[ABC]} \leq \left(\frac{EF}{AD}\right)^2.$$

Greece

4 For each integer $n \geq 3$, find the smallest natural number $f(n)$ having the following property:
$\star$ For every subset $A \subset \{1, 2, \ldots, n\}$ with $f(n)$ elements, there exist elements $x, y, z \in A$ that are pairwise coprime.

Balkan MO 1993

Nicosia, Cyprus

$\boxed{1}$ Let a, b, c, d, e, f be six real numbers with sum 10, such that

$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 + (e-1)^2 + (f-1)^2 = 6.$$

Find the maximum possible value of f.

Cyprus

$\boxed{2}$ A positive integer given in decimal representation $\overline{a_n a_{n-1} \dots a_1 a_0}$ is called *monotone* if $a_n \leq a_{n-1} \leq \dots \leq a_0$. Determine the number of monotone positive integers with at most 1993 digits.

$\boxed{3}$ Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers O_1 and O_2, respectively, are externally tangent at point λ. A circle $\mathcal{C}$ with center O touches $\mathcal{C}_1$ at A and $\mathcal{C}_2$ at B so that the centers O_1, O_2 lie inside C. The common tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ at λ intersects the circle $\mathcal{C}$ at K and L. If D is the midpoint of the segment KL, show that $\angle O_1OO_2 = \angle ADB$.

Greece

$\boxed{4}$ Let p be a prime and $m \geq 2$ be an integer. Prove that the equation

$$\frac{x^p + y^p}{2} = \left(\frac{x+y}{2}\right)^m$$

has a positive integer solution $(x, y) \neq (1, 1)$ if and only if $m = p$.

Romania

Balkan MO 1994

Novi Sad, Yugoslavia

1 An acute angle XAY and a point P inside the angle are given. Construct (using a ruler and a compass) a line that passes through P and intersects the rays AX and AY at B and C such that the area of the triangle ABC equals AP^2.

Greece

2 Let n be an integer. Prove that the polynomial $f(x)$ has at most one zero, where

$$f(x) = x^4 - 1994x^3 + (1993 + n)x^2 - 11x + n.$$

Greece

3 Let $a_1, a_2, \ldots, a_n$ be a permutation of the numbers $1, 2, \ldots, n$, with $n \geq 2$. Determine the largest possible value of the sum

$$S(n) = |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}|.$$

Romania

4 Find the smallest number $n \geq 5$ for which there can exist a set of n people, such that any two people who are acquainted have no common acquaintances, and any two people who are not acquainted have exactly two common acquaintances.

Bulgaria

Balkan MO 1995

Plovdiv, Bulgaria

$\boxed{1}$ For all real numbers x, y define $x \star y = \dfrac{x+y}{1+xy}$. Evaluate the expression

$$(\cdots(((2 \star 3) \star 4) \star 5) \star \cdots) \star 1995.$$

Macedonia

$\boxed{2}$ The circles $\mathcal{C}_1(O_1, r_1)$ and $\mathcal{C}_2(O_2, r_2)$, $r_2 > r_1$, intersect at A and B such that $\angle O_1AO_2 = 90^\circ$. The line O_1O_2 meets $\mathcal{C}_1$ at C and D, and $\mathcal{C}_2$ at E and F (in the order C, E, D, F). The line BE meets $\mathcal{C}_1$ at K and AC at M, and the line BD meets $\mathcal{C}_2$ at L and AF at N. Prove that

$$\frac{r_2}{r_1} = \frac{KE}{KM} \cdot \frac{LN}{LD}.$$

Greece

$\boxed{3}$ Let a and b be natural numbers with $a > b$ and having the same parity. Prove that the solutions of the equation

$$x^2 - (a^2 - a + 1)(x - b^2 - 1) - (b^2 + 1)^2 = 0$$

are natural numbers, none of which is a perfect square.

Albania

$\boxed{4}$ Let n be a positive integer and $\mathcal{S}$ be the set of points (x, y) with $x, y \in \{1, 2, \ldots, n\}$. Let $\mathcal{T}$ be the set of all squares with the verticesw in the set $\mathcal{S}$. We denote by a_k ($k \geq 0$) the number of (unordered) pairs of points for which there are exactly k squares in $\mathcal{T}$ having these two points as vertices. Prove that $a_0 = a_2 + 2a_3$.

Yugoslavia

Balkan MO 1996

Bacau, Romania

$\boxed{1}$ Let O be the circumcenter and G be the centroid of a triangle ABC. If R and r are the circumcenter and incenter of the triangle, respectively, prove that

$$OG \leq \sqrt{R(R-2r)}.$$

Greece

$\boxed{2}$ Let p be a prime number with $p > 5$. Consider the set $X = \{p - n^2 \mid n \in \mathbb{N},\ n^2 < p\}$. Prove that the set X has two distinct elements x and y such that $x \neq 1$ and $x \mid y$.

Albania

$\boxed{3}$ In a convex pentagon $ABCDE$, the points M, N, P, Q, R are the midpoints of the sides AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR and DM pass through a single point, prove that EN contains that point as well.

Yugoslavia

$\boxed{4}$ Suppse that $X = \{1, 2, \ldots, 2^{1996} - 1\}$, prove that there exist a subset A that satisfies these conditions:

a) $1 \in A$ and $2^{1996} - 1 \in A$;

b) Every element of A except 1 is equal to the sum of two (possibly equal) elements from A;

c) The maximum number of elements of A is 2012.

Romania

Balkan MO 1997

Kalabaka, Greece

$\boxed{1}$ Suppose that O is a point inside a convex quadrilateral $ABCD$ such that

$$OA^2 + OB^2 + OC^2 + OD^2 = 2\mathcal{A}[ABCD],$$

where by $\mathcal{A}[ABCD]$ we have denoted the area of $ABCD$. Prove that $ABCD$ is a square and O is its center.

Yugoslavia

$\boxed{2}$ Let $S = \{A_1, A_2, \ldots, A_k\}$ be a collection of subsets of an n-element set A. If for any two elements $x, y \in A$ there is a subset $A_i \in S$ containing exactly one of the two elements x, y, prove that $2^k \geq n$.

Yugoslavia

$\boxed{3}$ The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ touch each other externally at D, and touch a circle ω internally at B and C, respectively. Let A be an intersection point of ω and the common tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ at D. Lines AB and AC meet $\mathcal{C}_1$ and $\mathcal{C}_2$ again at K and L, respectively, and the line BC meets $\mathcal{C}_1$ again at M and $\mathcal{C}_2$ again at N. Prove that the lines AD, KM, LN are concurrent.

Greece

$\boxed{4}$ Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) = f^2(x) + y$$

for all $x, y \in \mathbb{R}$.

Balkan MO 1998

Nicosia, Cyprus

1 Consider the finite sequence $\left\lfloor \frac{k^2}{1998} \right\rfloor$, for $k = 1, 2, \ldots, 1997$. How many distinct terms are there in this sequence?

Greece

2 Let $n \geq 2$ be an integer, and let $0 < a_1 < a_2 < \cdots < a_{2n+1}$ be real numbers. Prove the inequality

$$\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \cdots + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_1 - a_2 + a_3 - \cdots + a_{2n+1}}.$$

Bogdan Enescu, Romania

3 Let $\mathcal{S}$ denote the set of points inside or on the border of a triangle ABC, without a fixed point T inside the triangle. Show that $\mathcal{S}$ can be partitioned into disjoint closed segemnts.

Yugoslavia

4 Prove that the following equation has no solution in integer numbers:

$$x^2 + 4 = y^5.$$

Bulgaria

Balkan MO 1999

Ohrid, Macedonia

1. Let O be the circumcenter of the triangle ABC. The segment XY is the diameter of the circumcircle perpendicular to BC and it meets BC at M. The point X is closer to M than Y and Z is the point on MY such that $MZ = MX$. The point W is the midpoint of AZ.

   a) Show that W lies on the circle through the midpoints of the sides of ABC;

   b) Show that MW is perpendicular to AY.

2. Let p be an odd prime congruent to 2 modulo 3. Prove that at most $p - 1$ members of the set $\{m^2 - n^3 - 1 \mid 0 < m,\ n < p\}$ are divisible by p.

3. Let ABC be an acute-angled triangle of area 1. Show that the triangle whose vertices are the feet of the perpendiculars from the centroid G to AB, BC, CA has area between $\frac{4}{27}$ and $\frac{1}{4}$.

4. Let $\{a_n\}_{n\geq 0}$ be a non-decreasing, unbounded sequence of non-negative integers with $a_0 = 0$. Let the number of members of the sequence not exceeding n be b_n. Prove that

$$(a_0 + a_1 + \cdots + a_m)(b_0 + b_1 + \cdots + b_n) \geq (m + 1)(n + 1).$$

Balkan MO 2000

Chisinau, Moldova

1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) = f^2(x) + y$$

for all $x, y \in \mathbb{R}$.

2 Let ABC be an acute-angled triangle and D the midpoint of BC. Let E be a point on segment AD and M its projection on BC. If N and P are the projections of M on AB and AC then the interior angule bisectors of $\angle NMP$ and $\angle NEP$ are parallel.

3 How many $1 \times 10\sqrt{2}$ rectangles can be cut from a 50×90 rectangle using cuts parallel to its edges?

4 Show that for any n we can find a set X of n distinct integers greater than 1, such that the average of the elements of any subset of X is a square, cube or higher power.

Balkan MO 2001

Belgrad, Yugoslavia

1 Let a, b, n be positive integers such that $2^n - 1 = ab$. Let $k \in \mathbb{N}$ such that $ab + a - b - 1 \equiv 0 \pmod{2^k}$ and $ab + a - b - 1 \not\equiv 0 \pmod{2^{k+1}}$. Prove that k is even.

2 A convex pentagon $ABCDE$ has rational sides and equal angles. Show that it is regular.

3 Let a, b, c be positive real numbers with $abc \leq a + b + c$. Show that

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

Cristinel Mortici, Romania

4 A cube side 3 is divided into 27 unit cubes. The unit cubes are arbitrarily labeled 1 to 27 (each cube is given a different number). A move consists of swapping the cube labeled 27 with one of its 6 neighbours. Is it possible to find a finite sequence of moves at the end of which cube 27 is in its original position, but cube n has moved to the position originally occupied by $27 - n$ (for each $n = 1, 2, \ldots, 26$)?

Balkan MO 2002

Antalya, Turkey

1. Consider n points $A_1, A_2, A_3, \dots, A_n$ ($n \geq 4$) in the plane, such that any three are not collinear. Some pairs of distinct points among $A_1, A_2, A_3, \dots, A_n$ are connected by segments, such that every point is connected with at least three different points. Prove that there exists $k > 1$ and the distinct points $X_1, X_2, \dots, X_{2k}$ in the set $\{A_1, A_2, A_3, \dots, A_n\}$, such that for every $i \in \overline{1, 2k-1}$ the point X_i is connected with X_{i+1}, and X_{2k} is connected with X_1.

2. Let the sequence $\{a_n\}_{n\geq 1}$ be defined by $a_1 = 20$, $a_2 = 30$ and $a_{n+2} = 3a_{n+1} - a_n$ for all $n \geq 1$. Find all positive integers n such that $1 + 5a_n a_{n+1}$ is a perfect square.

   {Moderator edit: Please discuss this problem [url=http://www.mathlinks.ro/viewtopic.php?t=321298]here[/

3. Two circles with different radii intersect in two points A and B. Let the common tangents of the two circles be MN and ST such that M, S lie on the first circle, and N, T on the second. Prove that the orthocenters of the triangles AMN, AST, BMN and BST are the four vertices of a rectangle.

4. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer n we have:

$$2n + 2001 \leq f(f(n)) + f(n) \leq 2n + 2002.$$

Balkan MO 2003

Tirana, Albania

$\boxed{1}$ Can one find 4004 positive integers such that the sum of any 2003 of them is not divisible by 2003?

$\boxed{2}$ Let ABC be a triangle, and let the tangent to the circumcircle of the triangle ABC at A meet the line BC at D. The perpendicular to BC at B meets the perpendicular bisector of AB at E. The perpendicular to BC at C meets the perpendicular bisector of AC at F. Prove that the points D, E and F are collinear.

Valentin Vornicu

$\boxed{3}$ Find all functions $f : \mathbb{Q} \to \mathbb{R}$ which fulfill the following conditions:

a) $f(1) + 1 > 0$;

b) $f(x + y) - xf(y) - yf(x) = f(x)f(y) - x - y + xy$, for all $x, y \in \mathbb{Q}$;

c) $f(x) = 2f(x + 1) + x + 2$, for every $x \in \mathbb{Q}$.

$\boxed{4}$ A rectangle $ABCD$ has side lengths $AB = m$, $AD = n$, with m and n relatively prime and both odd. It is divided into unit squares and the diagonal AC intersects the sides of the unit squares at the points $A_1 = A, A_2, A_3, \ldots, A_k = C$. Show that

$$A_1A_2 - A_2A_3 + A_3A_4 - \cdots + A_{k-1}A_k = \frac{\sqrt{m^2 + n^2}}{mn}.$$

1 The sequence $\{a_n\}_{n\geq 0}$ of real numbers satisfies the relation:

$$a_{m+n} + a_{m-n} - m + n - 1 = \frac{1}{2}(a_{2m} + a_{2n})$$

for all non-negative integers m and n, $m \geq n$. If $a_1 = 3$ find a_{2004}.

2 Solve in prime numbers the equation $x^y - y^x = xy^2 - 19$.

3 Let O be an interior point of an acute triangle ABC. The circles with centers the midpoints of its sides and passing through O mutually intersect the second time at the points K, L and M different from O. Prove that O is the incenter of the triangle KLM if and only if O is the circumcenter of the triangle ABC.

4 The plane is partitioned into regions by a finite number of lines no three of which are concurrent. Two regions are called "neighbors" if the intersection of their boundaries is a segment, or half-line or a line (a point is not a segment). An integer is to be assigned to each region in such a way that:

i) the product of the integers assigned to any two neighbors is less than their sum; ii) for each of the given lines, and each of the half-planes determined by it, the sum of the integers, assigned to all of the regions lying on this half-plane equal to zero.

Prove that this is possible if and only if not all of the lines are parallel.

Balkan MO 2005

$\boxed{1}$ Let ABC be an acute-angled triangle whose inscribed circle touches AB and AC at D and E respectively. Let X and Y be the points of intersection of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE and let Z be the midpoint of BC. Prove that the triangle XYZ is equilateral if and only if $\angle A = 60^\circ$.

$\boxed{2}$ Find all primes p such that $p^2 - p + 1$ is a perfect cube.

$\boxed{3}$ Let a, b, c be positive real numbers. Prove the inequality

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}.$$

When does equality occur?

$\boxed{4}$ Let $n \geq 2$ be an integer. Let S be a subset of $\{1, 2, \ldots, n\}$ such that S neither contains two elements one of which divides the other, nor contains two elements which are coprime. What is the maximal possible number of elements of such a set S?

$\boxed{1}$ Let a, b, c be positive real numbers. Prove the inequality

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1+abc}.$$

$\boxed{2}$ Let ABC be a triangle and m a line which intersects the sides AB and AC at interior points D and F, respectively, and intersects the line BC at a point E such that C lies between B and E. The parallel lines from the points A, B, C to the line m intersect the circumcircle of triangle ABC at the points A_1, B_1 and C_1, respectively (apart from A, B, C). Prove that the lines A_1E , B_1F and C_1D pass through the same point.

Greece

$\boxed{3}$ Find all triplets of positive rational numbers (m, n, p) such that the numbers $m+\frac{1}{np}$, $n+\frac{1}{pm}$, $p+\frac{1}{mn}$ are integers.

Valentin Vornicu, Romania

$\boxed{4}$ Let m be a positive integer and $\{a_n\}_{n\geq 0}$ be a sequence given by $a_0 = a \in \mathbb{N}$, and

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \equiv 0 \pmod 2, \\ a_n + m & \text{otherwise.} \end{cases}$$

Find all values of a such that the sequence is periodical (starting from the beginning).

Balkan MO 2007

Rhodos, Greece

$\boxed{1}$ Let $ABCD$ a convex quadrilateral with $AB = BC = CD$, with AC not equal to BD and E be the intersection point of it's diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120$.

$\boxed{2}$ Find all real functions f defined on IR, such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y,$$

for all real numbers x, y.

$\boxed{3}$ Find all positive integers n such that there exist a permutation σ on the set $\{1, 2, 3, \ldots, n\}$ for which

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\ldots + \sqrt{\sigma(n-1) + \sqrt{\sigma(n)}}}}}$$

is a rational number.

$\boxed{4}$ For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex $n-$ gons in the plane , such that $C_1 \cap C_2, C_2 \cap C_3, C_1 \cap C_3$ are finite. Find the maximum number of points of the sets $C_1 \cap C_2 \cap C_3$.

Balkan MO 2008

Macedonia

1. Given a scalene acute triangle ABC with $AC > BC$ let F be the foot of the altitude from C. Let P be a point on AB, different from A so that $AF = PF$. Let H, O, M be the orthocenter, circumcenter and midpoint of $[AC]$. Let X be the intersection point of BC and HP. Let Y be the intersection point of OM and FX and let OF intersect AC at Z. Prove that F, M, Y, Z are concyclic.

2. Is there a sequence $a_1, a_2, \ldots$ of positive reals satisfying simoultaneously the following inequalities for all positive integers n: a) $a_1 + a_2 + \ldots + a_n \le n^2$ b) $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_n} \le 2008$?

3. Let n be a positive integer. Consider a rectangle $(90n + 1) \times (90n + 5)$ consisting of unit squares. Let S be the set of the vertices of these squares. Prove that the number of distinct lines passing through at least two points of S is divisible by 4.

4. Let c be a positive integer. The sequence $a_1, a_2, \ldots$ is defined as follows $a_1 = c$, $a_{n+1} = a_n^2 + a_n + c^3$ for all positive integers n. Find all c so that there are integers $k \ge 1$ and $m \ge 2$ so that $a_k^2 + c^3$ is the mth power of some integer.

Balkan MO 2009

1. Solve the equation

$3^x - 5^y = z^2$

in positive integers.

2. Let MN be a line parallel to the side BC of a triangle ABC, with M on the side AB and N on the side AC. The lines BN and CM meet at point P. The circumcircles of triangles BMP and CNP meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.

3. A 9×12 rectangle is partitioned into unit squares. The centers of all the unit squares, except for the four corner squares and eight squares sharing a common side with one of them, are coloured red. Is it possible to label these red centres $C_1, C_2 ..., C_{96}$ in such way that the following to conditions are both fulfilled

(i) the distances $C_1C_2, ...C_{95}C_{96}, C_{96}C_1$ are all equal to $\sqrt{13}$

(ii) the closed broken line $C_1C_2...C_{96}C_1$ has a centre of symmetry?

{Bulgaria.

4. Denote by S the set of all positive integers. Find all functions $f : S \to S$ such that

$f\left(f^2(m) + 2f^2(n)\right) = m^2 + 2n^2$ for all $m, n \in S$.

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