Método de relajación para la ecuación de Laplace y de Poisson

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Credits: Computational Physics - Landau and Paez, Deitel, cplusplus tutorial

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Partial Differential Equations

$$A\frac{\partial^2 U}{\partial x^2} + 2B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} + D\frac{\partial U}{\partial x} + E\frac{\partial U}{\partial y} = F$$

Elliptic	*Parabolic*	*Hyperbolic*
$d = AC - B^2 > 0$	$d = AC - B^2 = 0$	$d = AC - B^2 < 0$
$\nabla^2 U(x) = -4\pi\rho(x)$	$\nabla^2 U(\mathbf{x}, t) = a\partial U/\partial t$	$\nabla^2 U(\mathbf{x}, t) = c^{-2}\partial^2 U/\partial t^2$
Poisson's	Heat	Wave

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Partial Differential Equations

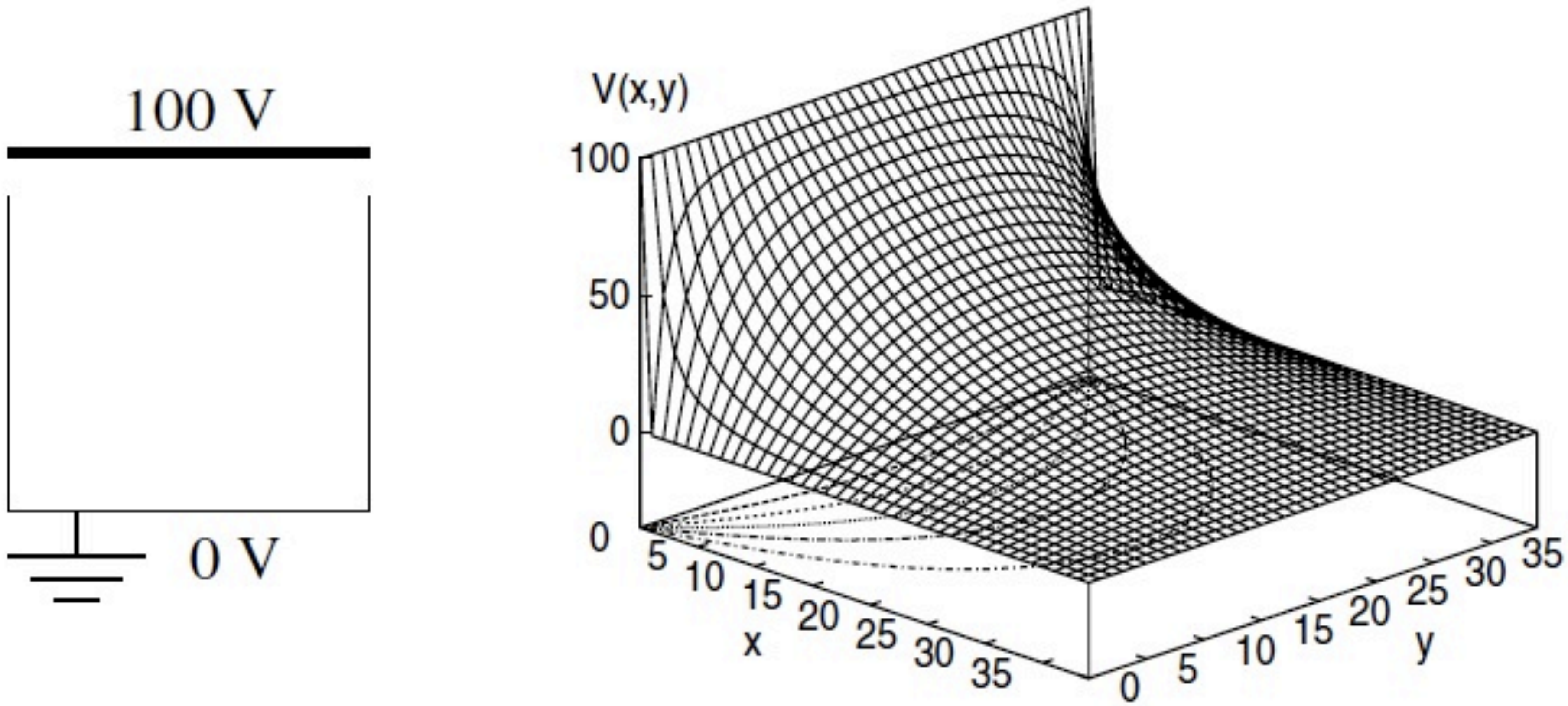

Fig. 23.1 *Left:* The region of space within a square in which we want to determine the electric potential. There is a wire at the top kept at a constant 100 V and a grounded wire at the sides and bottom. *Right*: The electric potential as a function of x and y. The projections onto the xy plane are equipotential surfaces or lines.

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Laplace and Poisson Differential Equations

$$\frac{\partial^2 U(x,y)}{\partial x^2} + \frac{\partial^2 U(x,y)}{\partial y^2} = \begin{cases} 0 & \text{Laplace's equation} \\ -4\pi\rho(\mathbf{x}) & \text{Poisson's equation} \end{cases}$$

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Finite Difference Method for Laplace and Poisson Equations

$$
\begin{aligned}
U(x+\Delta x, y) &= U(x,y) + \frac{\partial U}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(\Delta x)^2 + \cdots \\
U(x-\Delta x, y) &= U(x,y) - \frac{\partial U}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(\Delta x)^2 - \cdots
\end{aligned}
$$

$$\frac{\partial^2 U(x,y)}{\partial x^2} \simeq \frac{U(x+\Delta x,y) + U(x-\Delta x,y) - 2U(x,y)}{(\Delta x)^2} + \mathcal{O}(\Delta x^4)$$

$$\frac{\partial^2 U(x,y)}{\partial y^2} \simeq \frac{U(x,y+\Delta y) + U(x,y-\Delta y) - 2U(x,y)}{(\Delta y)^2} + \mathcal{O}(\Delta y^4)$$

$$
\begin{aligned}
&\frac{U(x+\Delta x,y) + U(x-\Delta x,y) - 2U(x,y)}{(\Delta x)^2} \\
&\quad + \frac{U(x,y+\Delta y) + U(x,y-\Delta y) - 2U(x,y)}{(\Delta y)^2} = -4\pi\rho
\end{aligned}
$$

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Finite Difference Method for Laplace and Poisson Equations

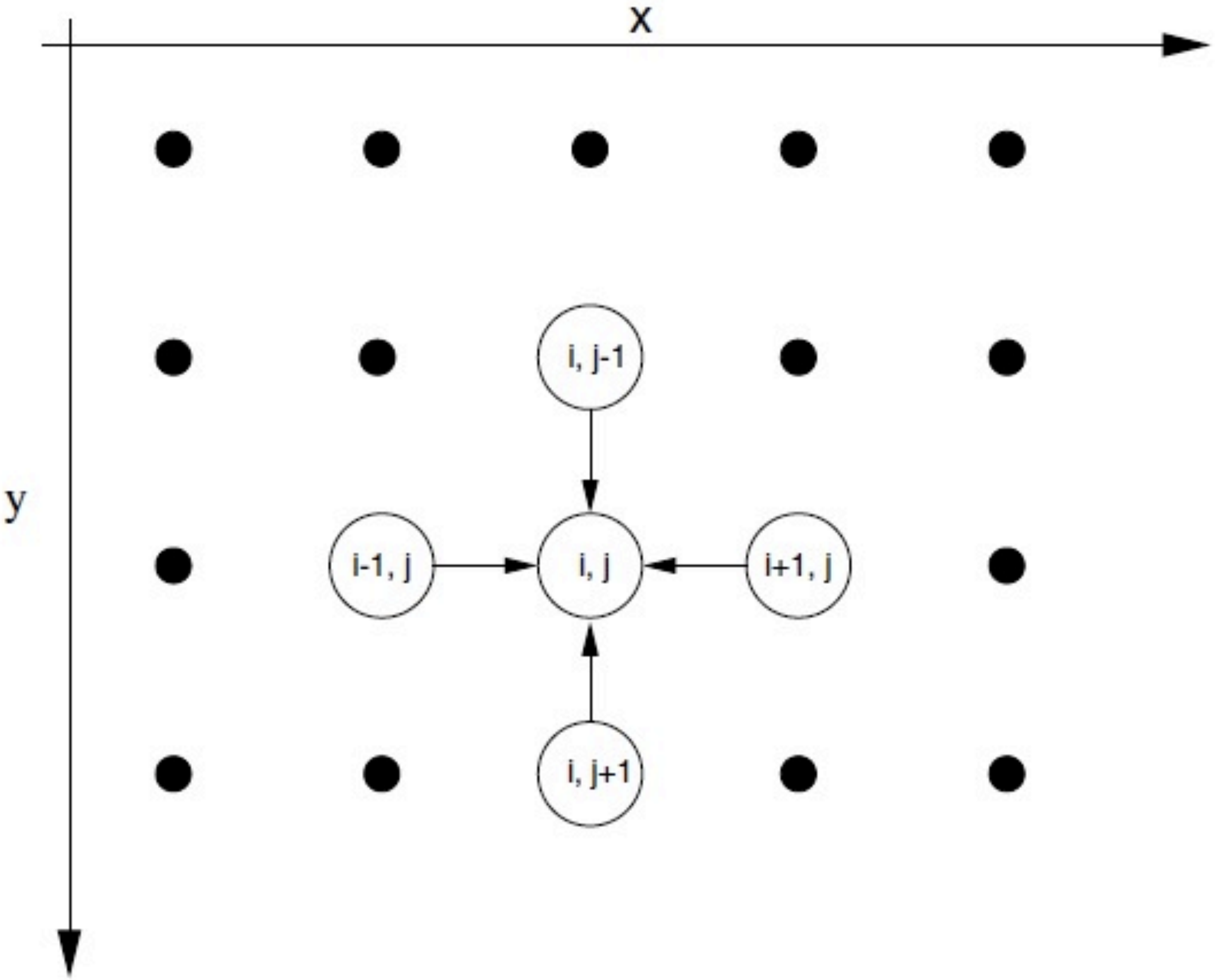

Fig. 23.3 The algorithm for Laplace's equation in which the potential at the point $(x, y) = (i, j)\Delta$ equals the average of the potential values at the four nearest-neighbor points. The nodes with white centers correspond to fixed values of the potential along the boundaries.

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Finite Difference Method for Laplace and Poisson Equations

For equal spacing in x and y

$$U(x+\Delta x,y)+U(x-\Delta x,y)+U(x,y+\Delta y)+U(x,y-\Delta y)-4U(x,y) = -4\pi\rho$$

$$U(x,y) \simeq \frac{1}{4}\left[U(x+\Delta,y)+U(x-\Delta,y)+U(x,y+\Delta)+U(x,y-\Delta)\right] + \pi\rho(x,y)\Delta^2 \qquad (23.25)$$

$$x = x_0 + i\Delta, \qquad y = y_0 + j\Delta \qquad i,j = 0,\ldots,N_{\text{max-1}}$$
$$\Delta x = \Delta y = \Delta = L/(N_{\max}-1)$$

$$U_{i,j} = \frac{1}{4}\left[U_{i+1,j}+U_{i-1,j}+U_{i,j+1}+U_{i,j-1}\right] + \pi\rho(i\Delta, j\Delta)\Delta^2$$

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Finite Difference Method for Laplace and Poisson Equations

$$U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right] + \pi\rho(i\Delta, j\Delta)\Delta^2$$

- This is a relaxation method: You have to pass over many times in order to reach an equilibrium state.
- How many times should you pass over?
- How to improve? Gauss-Seidel.
- What happens with time dependent potentials: Search for Crank-Nicholson method.

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Finite Difference Method for Laplace and Poisson Equations

$$U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right] + \pi\rho(i\Delta, j\Delta)\Delta^2$$

- Implement the relaxation finite difference method to solve the Laplace equation.

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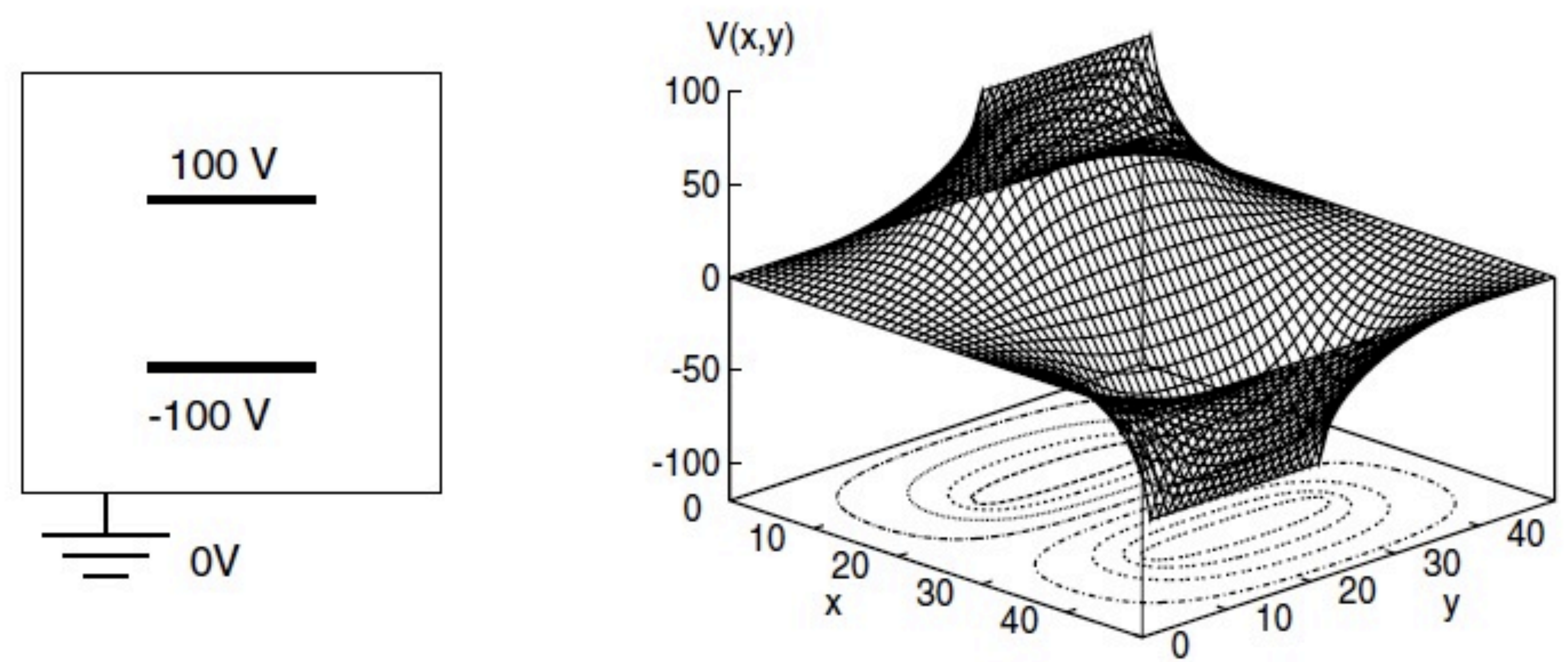

Fig. 23.4 *Left*: A simple model of a parallel-plate capacitor (or of a vacuum-tube diode). A realistic model would have the plates closer together, in order to condense the field, and the enclosing, grounded box so large that it has no effect on the field near the capacitor. *Right*: A numerical solution for the electric potential for this geometry. The projection on the xy plane gives the equipotential lines.

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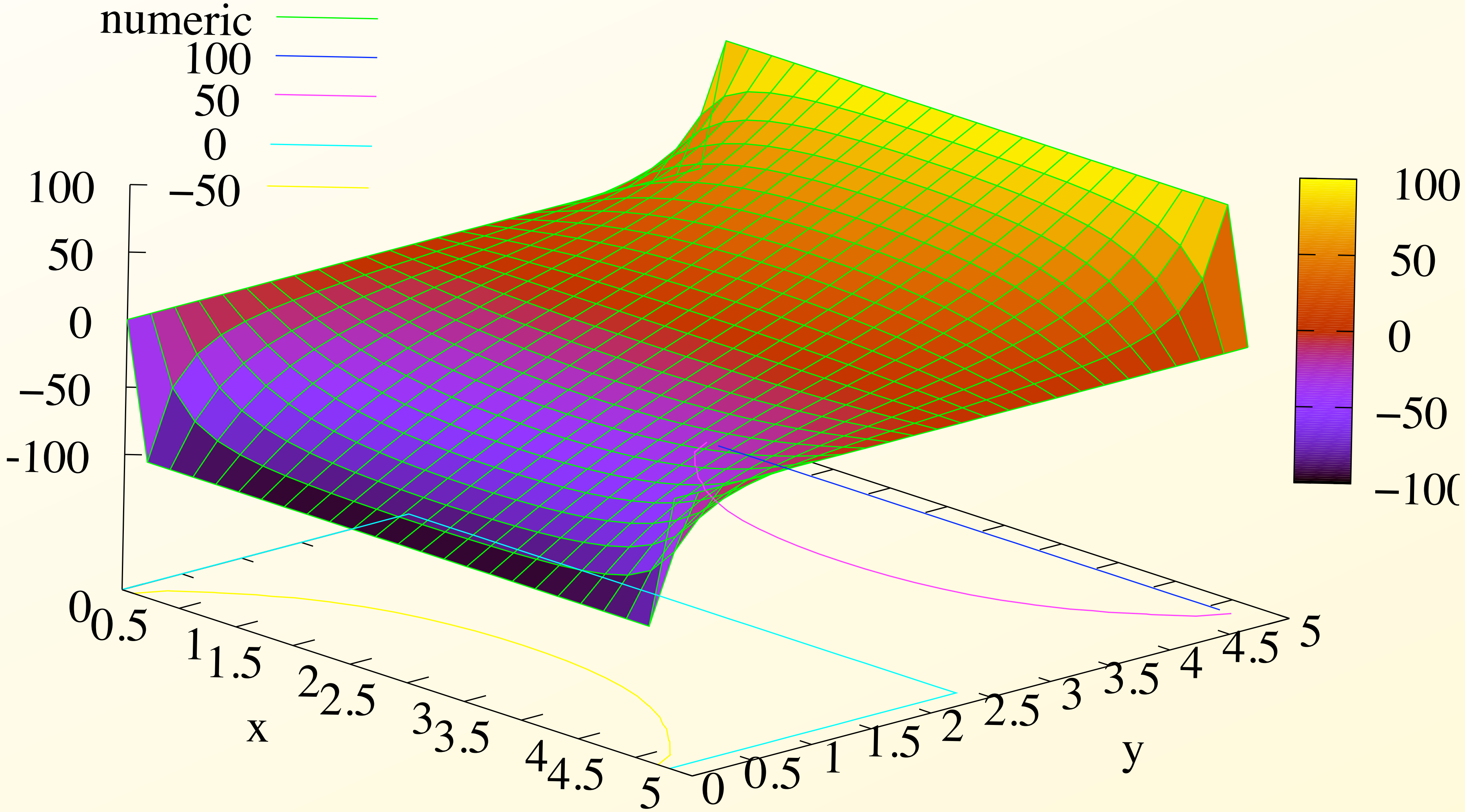
Solution Examples



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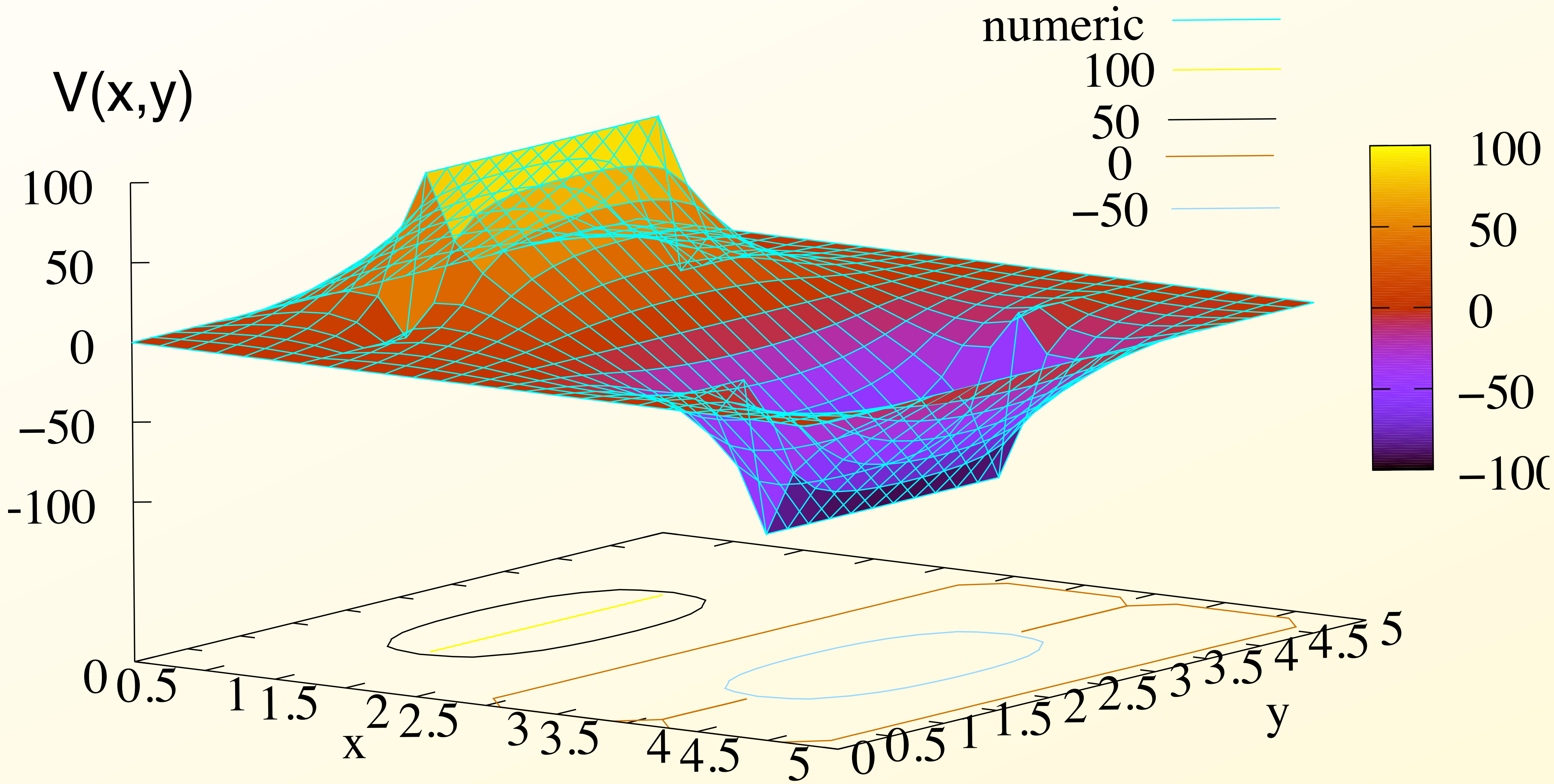

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