Problem 1

$$f(x_1, x_2) = 1 + 2x_1 + 3(x_1^2 + x_2^2) + 4x_1 x_2$$

$$f(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\top \begin{bmatrix} 3 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 16 & 23 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \pi^2$$

$$f(x, y) = 3(x^2 + y^2) + 4xy + 5x + 6y + 7$$

1. Show that the functions are convex.

2. Obtain the optimal solution and optimal cost for each function by your hand

3. Use Python to design the gradient descent algorithm and find the optimal solution and optimal value. Also, obtain the convergence plot

4. Use Python to design the steepest (optimal step size) gradient descent and find the optimal solution and optimal value. Also, obtain the convergence plot

5. Let x^* be the optimal solution obtained by 2. Let $x^*_{Gradient}$ be the optimal solution obtained by 3. Let $x^*_{SteepestGradient}$ be the optimal solution obtained by 3. Plot the convergence plot of $\|x^* - x^*_{Gradient}\|, \ \|x^* - x^*_{SteepestGradient}\|$

6. Discuss the convergence speed in the convergence plots in 4 and 5

Problem 2

Consider the quadratic programming

$$f(x) = \frac{1}{2}x^\top Q x - b^\top x$$

$$\nabla f(x) = Qx - b$$

$$Q = Q^\top > 0,\ b \in \mathbb{R}^n$$

Use “randn” in numpy to generate a 1000 x 1000 matrix A. Then obtain Q by

$$Q = AA^\top + \rho I$$

Here, rho is any positive number such that Q is positive definite.
Use “randn” in numpy to generate 1000 x 1 vector b

1. Show that the function is convex.

2. Obtain the optimal solution and optimal cost using MATLAB “fmincon” (you can also use other optimization software package such as CVX)

3. Use Python to design the gradient descent algorithm and find the optimal solution and optimal value. Also, obtain the convergence plot

4. Use Python to design the steepest (optimal step size) gradient descent and find the optimal solution and optimal value. Also, obtain the convergence plot

5. Use Python to design the Neterov-2 algorithm and find the optimal solution and optimal value. Also, obtain the convergence plot

6. Let x^* be the optimal solution obtained by 2. Let $x^*_{Gradient}$ be the optimal solution obtained by 3. Let $x^*_{SteepestGradient}$ be the optimal solution obtained by 3. Let $x^*_{Nestrov-2}$be the optimal solution obtained by the Nesterov-2 in 5. Plot the convergence plot of

$$\|x^* - x^*_{Gradient}\|, \ \|x^* - x^*_{SteepestGradient}\| \quad \|x^* - x_{Nesterov-2}\|$$

7. Discuss the convergence speed in the convergence plots in 4 and 5

Problem 3

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

1. Find the optimal solution using the gradient descent method with the initial condition that you choose
2. Try the gradient descent with different initial conditions
   - For 1 and 2, you need to use Python
   - Discuss the convergence speeed with different initial conditions
   - Does the convergence depend on the initial condition?

Problem 4

$$\min_{x\in\mathbb{R}^n} c^\top x$$

$$\text{subject to } Ax \leq b$$

$$x_i \geq 0,\ i = 1, \ldots, n$$

note that

$$c \in \mathbb{R}^n,\ A \in \mathbb{R}^{p\times n}\ (p \leq n),\ b \in \mathbb{R}^p$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Problem 4

- Use the following MATLAB random generation code to generate random c, A, b data
  - c = rand(10,1), A = randn(8,10)*3+0.2, b = -5 + (5+5)*rand(8,1)
  - Use MATLAB “linprog” to find the optimal cost and optimal solution with the random data given above
- Use the following MATLAB random generation code to generate random c, A, b data
  - c = rand(100,1), A = randn(55,100)*5-1.2, b = -1 + (1+1)*rand(55,1)
  - Use MATLAB “linprog” to find the optimal cost and optimal solution with the random data given above
- Discuss the location of the optimal solution in terms of the constrains

$$Ax \leq b$$
$$x_i \geq 0,\ i = 1, \ldots, n$$

Problem 5

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{where } f(x) = \|Ax - b\|_Q^2 = (Ax - b)^\top Q(Ax - b)$$

$$A \in \mathbb{R}^{m \times n}$$

$$b \in \mathbb{R}^m$$

$$Q = Q^\top \in \mathbb{R}^{m \times n} : \text{ positive definite matrix}$$

Problem 5

1. Find the optimal solution to this problem by your hand

2. Let n=100, m=50, use Python to generate the Gaussian random data of A, b, Q. Then find the optimal solution via the gradient descent method and the solution of 1.

3. Let n=50, m=100 , use Python to generate the Gaussian random data of A, b, Q. Then find the optimal solution via the gradient descent method and the solution of 1.

4. Discuss the solutions of 2 and 3. Provide your discussion in terms of the matrix inverse and rank condition