MANY-BODY INTERACTIONS

Here along the line of [1], we derive how to create many-body interactions ($\exp i\phi\sigma_1^i\sigma_2^j\sigma_3^k$), where $i, j, k \in x, y, z$, based on CZ gate and single qubit rotations. This type of interaction term is widely required in QAOA and simulations of topological phases of matter.

First let's review two math theorems.

i) Exponential of collective spin operators.

Let A_i be a Pauli matrix acting on qubit i, which satisfies $A_i^2 = \mathbb{I}$. Since for different sites $[A_i, A_j] = 0$, so

$$e^{\frac{i\theta}{2}(A_1+A_2+\cdots+A_n)} = \prod_k^n e^{\frac{i\theta}{2}A_k} = \prod_k^n (\mathbb{I}\cos\frac{\theta}{2} + iA_k\sin\frac{\theta}{2}) \tag{1}$$

Specially when $\theta = \pi$, and observe i^n has a period of 4

$$e^{\frac{i\pi}{2}(A_1+A_2+\cdots+A_n)} = i^n\prod_k^n A_k = \begin{cases} \prod_k^n A_k, & \text{for} \quad n = 4k, k \in \mathbb{N} \\ -\prod_k^n A_k, & \text{for} \quad n = 4k-2, k \in \mathbb{N} \\ -i\prod_k^n A_k, & \text{for} \quad n = 4k-3, k \in \mathbb{N} \\ i\prod_k^n A_k, & \text{for} \quad n = 4k-1, k \in \mathbb{N} \end{cases} \tag{2}$$

Similarly, for $\theta = -\pi$

$$e^{\frac{-i\pi}{2}(A_1+A_2+\cdots+A_n)} = (-i)^n\prod_k^n A_k = \begin{cases} \prod_k^n A_k, & \text{for} \quad n = 4k, k \in \mathbb{N} \\ -\prod_k^n A_k, & \text{for} \quad n = 4k-2, k \in \mathbb{N} \\ i\prod_k^n A_k, & \text{for} \quad n = 4k-3, k \in \mathbb{N} \\ -i\prod_k^n A_k, & \text{for} \quad n = 4k-1, k \in \mathbb{N} \end{cases} \tag{3}$$

So,

$$\begin{aligned} \cos\left(\frac{\pi}{2}(A_1+A_2+\cdots+A_n)\right) &= \tfrac{1}{2}\left(e^{\frac{i\pi}{2}(A_1+A_2+\cdots+A_n)} + e^{-\frac{i\pi}{2}(A_1+A_2+\cdots+A_n)}\right) \\ &= \begin{cases} \prod_k^n A_k, & \text{for} \quad n = 4k, k \in \mathbb{N} \\ -\prod_k^n A_k, & \text{for} \quad n = 4k-2, k \in \mathbb{N} \\ 0, & \text{for} \quad n \text{ odd} \end{cases} \end{aligned} \tag{4}$$

$$\sin\left(\frac{\pi}{2}(A_1+A_2+\cdots+A_n)\right) = \begin{cases} \prod_k^n A_k, & \text{for} \quad n = 4k-3, k \in \mathbb{N} \\ -\prod_k^n A_k, & \text{for} \quad n = 4k-1, k \in \mathbb{N} \\ 0, & \text{for} \quad n \text{ even} \end{cases} \tag{5}$$

ii) Transform between Pauli operators, based on single qubit rotations.

For the cyclic rotations of the Pauli operators, similar to Eq. (1), we have

$$e^{\frac{i\phi\sigma_z}{2}}\sigma_x e^{\frac{-i\phi\sigma_z}{2}} = \sigma_x\cos\phi - \sigma_y\sin\phi \tag{6}$$

$$e^{\frac{i\phi\sigma_z}{2}}\sigma_y e^{\frac{-i\phi\sigma_z}{2}} = \sigma_y\cos\phi + \sigma_x\sin\phi \tag{7}$$

Second, to construct ($\exp i\phi\sigma_1^i\sigma_2^j\sigma_3^k$), we create a 3-body interaction as an example,

$$\begin{aligned} &\exp\left[\mathrm{i}\frac{\pi}{4}\sigma_1^z(\sigma_2^z+\sigma_3^z)\right]\exp\left[\mathrm{i}\phi\sigma_1^x\right]\exp\left[-\mathrm{i}\frac{\pi}{4}\sigma_1^z(\sigma_2^z+\sigma_3^z)\right] \\ &= \exp\left[\mathrm{i}\phi\left(\cos\left(\frac{\pi}{2}\hat{S}_z\right)\sigma_1^x - \sin\left(\frac{\pi}{2}\hat{S}_z\right)\sigma_1^y\right)\right] \\ &= \exp\left[-\mathrm{i}\phi\sigma_1^x\sigma_2^z\sigma_3^z\right] \end{aligned} \tag{8}$$

where we have defined $\hat{S}_z = \sum_{i=2}^3\sigma_i^z$. In the second line, we have used Eq. (7) and third line Eq. (4) and Eq. (5).

$$\begin{aligned} &\exp\left[\mathrm{i}\frac{\pi}{4}\sigma_1^z(\sigma_2^z+\sigma_3^z)\right]\exp\left[\mathrm{i}\phi\sigma_1^y\right]\exp\left[-\mathrm{i}\frac{\pi}{4}\sigma_1^z(\sigma_2^z+\sigma_3^z)\right] \\ &= \exp\left[\mathrm{i}\phi\left(\cos\left(\frac{\pi}{2}\hat{S}_z\right)\sigma_1^y + \sin\left(\frac{\pi}{2}\hat{S}_z\right)\sigma_1^x\right)\right] \\ &= \exp\left[-\mathrm{i}\phi\sigma_1^y\sigma_2^z\sigma_3^z\right] \end{aligned} \tag{9}$$

Note since $\sigma_1^z\sigma_2^z$ and $\sigma_1^z\sigma_{23}^z$ c ommutes, we can apply these two interactions either simultaneously or successively. And to combine single qubit rotations before and after this many-body interaction, we can create arbitrary type. For instance, to change $e^{-\mathrm{i}\phi\sigma_1^x\sigma_2^z\sigma_3^z}$ to $e^{-\mathrm{i}\phi\sigma_1^z\sigma_2^z\sigma_3^z}$, we apply single qubit rotations below

$$e^{\frac{i\pi\sigma_1^y}{4}}e^{-\mathrm{i}\phi\sigma_1^x\sigma_2^z\sigma_3^z}e^{-\frac{i\pi\sigma_1^y}{4}} = e^{-\mathrm{i}\phi\sigma_1^z\sigma_2^z\sigma_3^z} \tag{10}$$

where we have employ $e^{\frac{i\pi\sigma_y}{4}}\sigma_x e^{\frac{-i\pi\sigma_y}{4}} = \sigma_z$.

And similarly, we have

$$e^{\frac{i\pi\sigma_y}{4}}\sigma_x e^{\frac{-i\pi\sigma_y}{4}} = \sigma_z \tag{11}$$

$$e^{\frac{i\pi\sigma_y}{4}}\sigma_z e^{\frac{-i\pi\sigma_y}{4}} = -\sigma_x \tag{12}$$

$$e^{\frac{i\pi\sigma_x}{4}}\sigma_z e^{\frac{-i\pi\sigma_x}{4}} = \sigma_y \tag{13}$$

TOFFOLI GATE

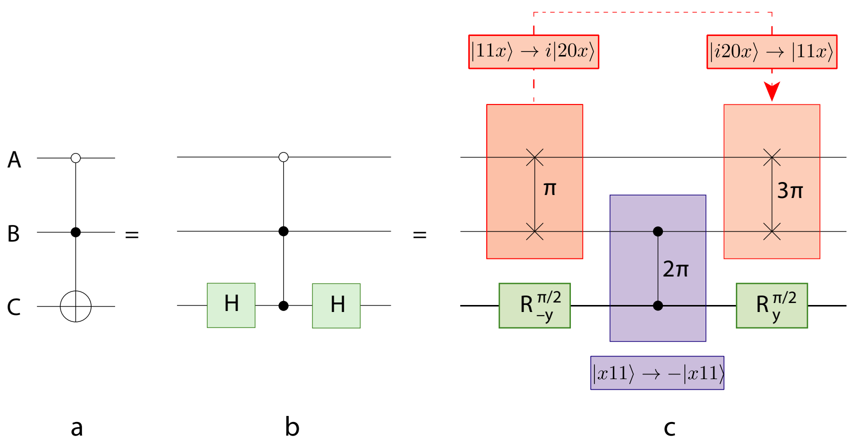

Figure 1. Toffoli gate [2]

Ref [2] constructs the Toffoli gate from a controlled-controlled phase (CCPHASE) gate sandwiched between two Hadamard gates as shown in Fig. 1. The target qubit C is inverted when the control qubits A and B are in the state $|01\rangle$ (whicdifferent from the conventional Toffoli gate control state $|11\rangle$).

Now in the CCPHASE gate, the only state change required is $|011\rangle \to -|011\rangle$, the other states remains the same. Since when a CZ gate operates on qubit B and C, both states $|011\rangle$ and $|111\rangle$ will get a phase. So to realize the CCPHASE gate, we need to separate or protect state $|111\rangle$ from getting the extra phase. This is done by hide $|111\rangle$ in the state $|200\rangle$ before the CZ operation on qubit B and C, and then swap back $|200\rangle$ to $|111\rangle$ after the CZ gate. This keeps $|111\rangle$ from going through the CZ process. Note, in this hiding process, we also hide state $|110\rangle$ from the CZ gate, however $|110\rangle$ remains unchanged after the CCPHASE gate.

FREDKIN GATE

Fredkin gate is a controlled-Swap gate, when the first qubit is 1, the gate swaps the states of the second and third qubits.

Fredkin gate quantum application reduce circuit depth

quantum finererprint iswap test

Initial state	Final state
$\|101\rangle$	$\|110\rangle$
$\|110\rangle$	$\|101\rangle$
$\|1xx\rangle$	$\|1xx\rangle$
$\|0xy\rangle$	$\|0xy\rangle$

The quantum optical Fredkin gate

In quantum optics there is a well-known construction of the Fredkin gate based on Kerr interaction sandwiched between beam-splitters[3, 4]. In analogy, we replace the Kerr interaction by a CZ gate, and the beam-splitters by a $\sqrt{i\text{SWAP}}$ gate and a reverse $\sqrt{i\text{SWAP}}$, that is $(\sqrt{i\text{SWAP}})^\dagger$, as shown in Fig. 2.

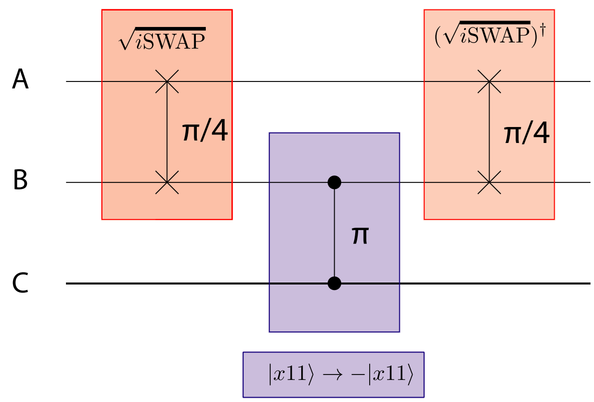

Figure 2. Fredkin gate

The exchange type interaction $H = g(e^{-i\varphi}\sigma_1^+\sigma_2^- + \text{H.C.})$ generate $\sqrt{i\text{SWAP}}$ and iSWAP gate, written in the basis 00,01,10,11, we have the following matrix form

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad i\text{SWAP} = e^{\frac{-i\pi H}{2}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -ie^{i\varphi} & 0 \\ 0 & -ie^{-i\varphi} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

$$\sqrt{i\text{SWAP}}{=}e^{\frac{-i\pi H}{4}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{ie^{i\varphi}}{\sqrt{2}} & 0 \\ 0 & -\frac{ie^{-i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad (\sqrt{i\text{SWAP}})^\dagger = e^{\frac{i\pi H}{4}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{ie^{i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{ie^{-i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \tag{15}$$

i) When the controlled qubit $c = 0$, then nothing changes $\sqrt{i\text{SWAP}}(\sqrt{i\text{SWAP}})^\dagger = I$.

ii) When the controlled qubit $c = 1$

input $|011\rangle$

$$
\begin{aligned}
&(\sqrt{i\mathrm{SWAP}})^{\dagger}_{\mathrm{AB}}(\mathrm{CZ})_{\mathrm{CB}}(\sqrt{i\mathrm{SWAP}})_{\mathrm{AB}}|011\rangle \\
&=(\sqrt{i\mathrm{SWAP}})^{\dagger}_{\mathrm{AB}}(\mathrm{CZ})_{\mathrm{CB}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{ie^{i\varphi}}{\sqrt{2}} & 0 \\ 0 & -\frac{ie^{-i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}|01\rangle|1\rangle \\
&=(\sqrt{i\mathrm{SWAP}})^{\dagger}_{\mathrm{AB}}(\mathrm{CZ})_{\mathrm{CB}}(|01\rangle - ie^{-i\varphi}|10\rangle)|1\rangle/\sqrt{2} \\
&=(\sqrt{i\mathrm{SWAP}})^{\dagger}_{\mathrm{AB}}(-|01\rangle - ie^{-i\varphi}|10\rangle)|1\rangle/\sqrt{2} \\
=&\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{ie^{i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{ie^{-i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(-|01\rangle - ie^{-i\varphi}|10\rangle)|1\rangle/\sqrt{2} \\
=&-ie^{-i\varphi}|101\rangle
\end{aligned}
$$

input $|101\rangle$

$$
\begin{aligned}
&(\sqrt{i\mathrm{SWAP}})^{\dagger}_{\mathrm{AB}}(\mathrm{CZ})_{\mathrm{CB}}(\sqrt{i\mathrm{SWAP}})_{\mathrm{AB}}|101\rangle \\
&=(\sqrt{i\mathrm{SWAP}})^{\dagger}_{\mathrm{AB}}(\mathrm{CZ})_{\mathrm{CB}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{ie^{i\varphi}}{\sqrt{2}} & 0 \\ 0 & -\frac{ie^{-i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}|10\rangle|1\rangle \\
&=(\sqrt{i\mathrm{SWAP}})^{\dagger}_{\mathrm{AB}}(\mathrm{CZ})_{\mathrm{CB}}(-ie^{i\varphi}|01\rangle + |10\rangle)|1\rangle/\sqrt{2} \\
&=(\sqrt{i\mathrm{SWAP}})^{\dagger}_{\mathrm{AB}}(ie^{i\varphi}|01\rangle + |10\rangle)|1\rangle/\sqrt{2} \\
=&\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{ie^{i\varphi}}{\sqrt{2}} & 0 \\ 0 & \frac{ie^{-i\varphi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(ie^{i\varphi}|01\rangle + |10\rangle)|1\rangle/\sqrt{2} \\
=&ie^{i\varphi}|101\rangle
\end{aligned}
$$

c)input $|111\rangle$, output $-|111\rangle$.
d)input $|001\rangle$, output $|001\rangle$.
Final result is a combined Toffoli and Fredkin gate. (single qubit rotations are not considered)

Fredkin gate by two-qubit gates interferencing effect

Apart from doing separate two-qubit gate subsequently, there is another way to implement the Fredkin gate by doing two-qubit CZ gates with the same gate strength simultaneously at $\frac{1}{\sqrt{2}}$ the CZ gate time. Observe in the Fredkin gate operation, the only transition occurred is between states $|110\rangle \Leftrightarrow |101\rangle$, the other output states remain the same as the input. Then the idea is to apply two-frequency drive to connect these transitions.

We construct a Λ system, spanned by $|110\rangle \Leftrightarrow |200\rangle \Leftrightarrow |101\rangle$, by applying two frequencies $\omega_1 - \omega_2 + \alpha_1$ (resonance frequencies between $|110\rangle$ and $|200\rangle$), and $\omega_1 - \omega_3 + \alpha_1$ (between $|101\rangle$ and $|200\rangle$). However, this two resonance frequencies also create a V system spanned by $|201\rangle \Leftrightarrow |111\rangle \Leftrightarrow |210\rangle$, as shown in Fig. 3.

In the subspace spanned by $|110\rangle = (1,0,0)^{\mathrm{T}}$,$|200\rangle = (0,1,0)^{\mathrm{T}}$,$|101\rangle = (0,0,1)^{\mathrm{T}}$, the effective Hamiltonian is $H_{\mathrm{eff}} = g(t)\hat{A}$, where

$$
\hat{A} = \begin{pmatrix} 0 & \lambda_1 & 0 \\ \lambda_1^* & 0 & \lambda_2 \\ 0 & \lambda_2^* & 0 \end{pmatrix} \tag{16}
$$

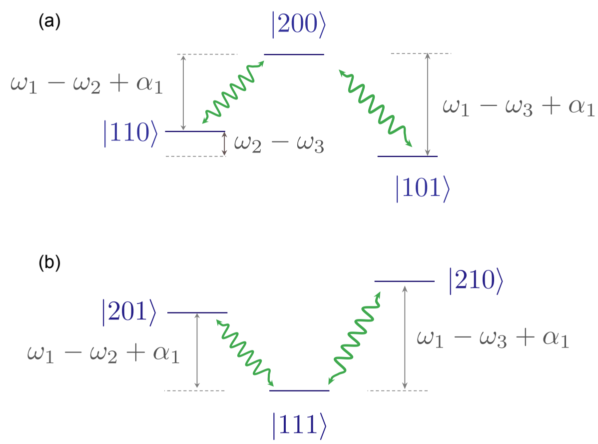

Figure 3. Fredkin gate

where we have assumed the CZ gate between qubit 0 and qubit 1 (2) has the gate strength $g(t)\lambda_1$ ($g(t)\lambda_2$).

Similarly, in the subspace spanned by $|201\rangle = (1,0,0)^{\mathrm{T}}$,$|111\rangle = (0,1,0)^{\mathrm{T}}$,$|210\rangle = (0,0,1)^{\mathrm{T}}$, we have the same Hamiltonian $H_{\text{eff}} = g(t)\hat{A}$. This is because $\omega_1-\omega_2+\alpha_1$ is the CZ gate between qubit 1 and qubit 2, then transitions $|110\rangle \Leftrightarrow |200\rangle$ and $|111\rangle \Leftrightarrow |201\rangle$ has the same Rabi frequency. Similarly, $\omega_1-\omega_3+\alpha_1$ is the CZ gate between qubit 1 and qubit 3, then transitions $|101\rangle \Leftrightarrow |200\rangle$ and $|111\rangle \Leftrightarrow |210\rangle$ has the same Rabi frequency. Then the time-evolution operator for both subspace is

$$\hat{U}(t) = \mathcal{T}\exp\left(-i\int_0^t \tilde{g}(t)\hat{A}dt\right) = \exp(-iG(t)\hat{A}) \tag{17}$$

$\mathcal{T}$ is the time-ordered operator. Under the condition $|\lambda_1|^2+|\lambda_2|^2=1$, $\hat{A}$ satisfies $\hat{A}^3=\hat{A}$, $\hat{A}^{2n+1}=\hat{A}$ and $\hat{A}^{2n}=\hat{A}^2$ for $n\geq 1$, the same with Holonomic operation [5], use the Taylor expansion for the exponential, the Eq. (17) becomes

$$\hat{U}(t) = I + \hat{A}^2\{\cos[G(t)]-1\} - i\hat{A}\sin[G(t)] \tag{18}$$

where

$$\hat{A}^2 = \begin{pmatrix} |\lambda_1|^2 & 0 & \lambda_1\lambda_2 \\ 0 & 1 & 0 \\ \lambda_1^*\lambda_2^* & 0 & |\lambda_2|^2 \end{pmatrix} \tag{19}$$

When $G(t)=\pi$,

$$\hat{U}_{G(t)=\pi} = I - 2\hat{A}^2 = \begin{pmatrix} 1-2|\lambda_1|^2 & 0 & -2\lambda_1\lambda_2 \\ 0 & -1 & 0 \\ -2\lambda_1^*\lambda_2^* & 0 & 1-2\,|\lambda_2|^2 \end{pmatrix} \tag{20}$$

From Eq. (20), $|200\rangle$ is decoupled from $|110\rangle$ and $|101\rangle$, then this evolution is further reduced to the subspace spanned by $|110\rangle = (1,0)^{\mathrm{T}}$,$|101\rangle = (0,1)^{\mathrm{T}}$, observe $\cos\theta = 1-2\sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2}-1$, set$\lambda_1=\sin\frac{\theta}{2}$,$\lambda_2=-e^{-i\phi}\cos\frac{\theta}{2}$

$$\hat{U} = \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & -\cos\theta \end{pmatrix} \tag{21}$$

where we have defined [5]

$$\begin{cases} -e^{i\phi}\tan\frac{\theta}{2} = \frac{\lambda_1}{\lambda_2} \\ |\lambda_1|^2+|\lambda_2|^2 = 1 \end{cases} \tag{22}$$

To create SWAP operation $\hat{U} = \sigma_x$, $\theta = \pi/2$, that is $\lambda_1 = -\lambda_2 = \frac{1}{\sqrt{2}}$.

In the V system, spanned by the basis $|201\rangle = (1,0,0)^{\mathrm{T}}$,$|111\rangle = (0,1,0)^{\mathrm{T}}$,$|210\rangle = (0,0,1)^{\mathrm{T}}$, the evolution creates SWAP operation between states $|201\rangle$ and $|210\rangle$, and $|111\rangle$ is decoupled, but it picks up a minus sign.

$$\hat{U}(t)|111\rangle = \begin{pmatrix} 1-2|\lambda_1|^2 & 0 & -2\lambda_1\lambda_2 \\ 0 & -1 & 0 \\ -2\lambda_1^*\lambda_2^* & 0 & 1-2|\lambda_2|^2 \end{pmatrix} |111\rangle = -|111\rangle \tag{23}$$

The other states, are far-off resonance, so the net effect is we implment a controlled SWAP-like gate, and the SWAP-like unitary is expressed as

In the full computational basis, the controlled SWAP-like gate is

$$U_{\mathrm{F}}(\theta,\phi) = \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& \cos\theta & e^{i\phi}\sin\theta & \\ &&&&& e^{-i\phi}\sin\theta & -\cos\theta & \\ &&&&&&& -1 \end{bmatrix}$$

note under the condition $\int_0^T g(t)dt = \pi$ and for $\lambda_1 = 1, \lambda_2 = 0$ ($\lambda_1 = 0, \lambda_2 = 1$) , we recover

, a single CZ gate on qubit 0 and 1 (2). to create SWAP operation in the space spanned by $|110\rangle$,$|101\rangle$

$\hat{U} = \sigma_x$, that is $\theta = \pi/2, \phi = 0$corresponding to $\lambda_1 = -1/\sqrt{2}, \lambda_2 = 1/\sqrt{2}$. compared with single CZ gate

$\lambda_1 = 1, \lambda_2 = 0$ or $\lambda_1 = 0, \lambda_2 = 1$, now the gate strength is reduced to $1/\sqrt{2}$. now if we use the same gate strength as a single CZ gate, then the requirement is

$$\int_0^T \sqrt{2}g(t)dt = \pi,$$

that is, the gate time is reduced to $1/\sqrt{2}$. But since $\lambda_1 = -1/\sqrt{2}$ we accturally reverse the pulse of the gate CZ_{12}

Interference effect !!! (compare STRIP, with symemetric transfer, can we relax the requirement of the same time dependance?)

By multiply a CCPHASE gate [2], we obtain a pure Fredkin gate.

$$U_{\mathrm{Fredkin}} = U_{\mathrm{F}}(\theta,\phi)U_{\mathrm{CCPHASE}}$$

To create a controlled iSwap gate or iFredkin gate, we can

apply two controlled-SWAP-like gate subsequently with tailored paremeters

$$U_{i\mathrm{Fredkin}} = U_{\mathrm{F}}(0,\frac{\pi}{2})U_{\mathrm{F}}(\frac{\pi}{2},\frac{\pi}{2})$$

with $\lambda_1 = 0, \lambda_2 = i\frac{\sqrt{2}}{2}$ and $\lambda_1 = \frac{\sqrt{2}}{2}, \lambda_2 = i\frac{\sqrt{2}}{2}$ for the first and second U_{F} gate

$$U_{\mathrm{F}}(\theta_1,\phi_1)U_{\mathrm{F}}(\theta_2,\phi_2)$$

$$\begin{bmatrix} 1 &&& \\ & \cos\theta_1 & e^{i\phi_1}\sin\theta_1 & \\ & e^{-i\phi_1}\sin\theta_1 & -\cos\theta_1 & \\ &&& -1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \cos\theta_2 & e^{i\phi_2}\sin\theta_2 & \\ & e^{-i\phi_2}\sin\theta_2 & -\cos\theta_2 & \\ &&& -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 &&& \\ & \cos\theta_1\cos\theta_2 + e^{i(\phi_1-\phi_2)}\sin\theta_1\sin\theta_2 & \cos\theta_1 e^{i\phi_2}\sin\theta_2 - e^{i\phi_1}\sin\theta_1\cos\theta_2 & \\ & e^{-i\phi_1}\sin\theta_1\cos\theta_2 - e^{-i\phi_2}\cos\theta_1\sin\theta_2 & e^{i(-\phi_1+\phi_2)}\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 & \\ &&& 1 \end{bmatrix}$$

$$=$$

assume

$$\phi_1 = \phi_2 = \phi,$$

simplify

$$\begin{bmatrix} 1 & & & \\ & \cos(\theta_1 - \theta_2) & e^{i\phi}\sin(\theta_2 - \theta_1) & \\ & -e^{-i\phi}\sin(\theta_2 - \theta_1) & \cos(\theta_1 - \theta_2) & \\ & & & 1 \end{bmatrix}$$

and if

$$\phi_1 = \phi_2 = \frac{\pi}{2}, \theta_2 = \theta_1 + \frac{\pi}{2}$$

we have

$$\begin{bmatrix} 1 & & & \\ & 0 & i & \\ & i & 0 & \\ & & & 1 \end{bmatrix}$$

Fredkin gate implemented by two simultaneous CZ gate with same strength at the same CZ gate time pi (not 2 pi). how to avoid being trapped in dark states and frequency crowding? numerical simulations in preparation

two-qubit gate decompostion

the interference gate can be decomposed into 3 two-qubit gates.

$$U_{\mathrm{F}}(\theta, \phi) = (I_0 \otimes \mathrm{XY}_{12}(\theta, \phi + \pi/2)) \cdot (\mathrm{CZ}_{01} \otimes I_2) \cdot \left(I_0 \otimes \mathrm{XY}^{\dagger}{}_{12}(\theta, \phi + \pi/2)\right) \tag{24}$$

where

$$\mathrm{XY}(\theta, \beta) = \begin{bmatrix} 1 & & & \\ & \cos\frac{\theta}{2} & i\sin\frac{\theta}{2}e^{i\beta} & \\ & i\sin\frac{\theta}{2}e^{-i\beta} & \cos\frac{\theta}{2} & \\ & & & 1 \end{bmatrix} \tag{25}$$

can be generated by exchange type interactions, for example, $\mathrm{XY}(\pi, 0) = i\mathrm{Swap}$.

$$\mathrm{CZ} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \tag{26}$$

$$
\begin{aligned}
&(I_0 \otimes \mathrm{XY}_{12}) \cdot (\mathrm{CZ}_{01} \otimes I_2) \cdot (I_0 \otimes \mathrm{XY}^{\dagger}{}_{12}) \\
=& \begin{bmatrix}
1 &&&&&&& \\
& \cos\frac{\theta}{2} & i\sin\frac{\theta}{2}e^{i\beta} &&&&& \\
& i\sin\frac{\theta}{2}e^{-i\beta} & \cos\frac{\theta}{2} &&&&& \\
&&& 1 &&&& \\
&&&& 1 &&& \\
&&&&& \cos\frac{\theta}{2} & i\sin\frac{\theta}{2}e^{i\beta} & \\
&&&&& i\sin\frac{\theta}{2}e^{-i\beta} & \cos\frac{\theta}{2} & \\
&&&&&&& 1
\end{bmatrix} \cdot \\
& \begin{bmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1 &&&& \\
&&&& 1 &&& \\
&&&&& 1 && \\
&&&&&& -1 & \\
&&&&&&& -1
\end{bmatrix} \cdot \\
& \begin{bmatrix}
1 &&&&&&& \\
& \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}e^{i\beta} &&&&& \\
& -i\sin\frac{\theta}{2}e^{-i\beta} & \cos\frac{\theta}{2} &&&&& \\
&&& 1 &&&& \\
&&&& 1 &&& \\
&&&&& \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}e^{i\beta} & \\
&&&&& -i\sin\frac{\theta}{2}e^{-i\beta} & \cos\frac{\theta}{2} & \\
&&&&&&& 1
\end{bmatrix} \\
=& \begin{bmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1 &&&& \\
&&&& 1 &&& \\
&&&&& \cos\theta & -i\sin\theta e^{i\beta} & \\
&&&&& i\sin\theta e^{-i\beta} & -\cos\theta & \\
&&&&&&& 1
\end{bmatrix}
\end{aligned} \tag{27}
$$

PARAMETRIC TWO-QUBIT GATES MEDIATED BY A TUNABLE COUPLER

The system is composed of a tunable coupler with frequency $\omega_c(t)$ connecting to n fixed-frequency qubits. Each qubit is modeled as a multi-level system, with $|i\rangle^{(k)}$ the eigenstates of qubit k, and $\omega_i^{(k)}$ the corresponding eigen-energy. The coupling strength to the coupler is $g_i^{(k)}$. Since the coupler has much higher frequency than the qubits, we assume the coupler will stay in the ground state during the gate operation. And for simplicity, the coupler is modeled as a harmonic oscillator, with annilation operator a. Then the Hamiltonian of the system takes the form

$$
H = \omega_c(t)a^{\dagger}a + \sum_k \sum_{i\in\mathcal{H}_M} \omega_i^{(k)} \mathit{\Pi}_{i,i}^{(k)} + \sum_k \sum_{i\in\mathcal{H}_{M-1}} g_i^{(k)} \left[\left(\mathit{\Pi}_{i,i+1}^{(k)} a^{\dagger} + \mathit{\Pi}_{i,i+1}^{(k)} a \right) + \mathrm{H.c.} \right], \tag{28}
$$

$$
\tag{29}
$$

where $\mathit{\Pi}_{i,j}^{(k)} = |i\rangle\langle j|^{(k)}$. The qubit-coupler interaction part is

$$
V = \sum_k \sum_{i\in\mathcal{H}_{M-1}} g_i^{(k)} [\left(\mathit{\Pi}_{i,i+1}^{(k)} a^{\dagger} + \mathit{\Pi}_{i,i+1}^{(k)} a \right) + \mathrm{H.c.}]
$$

, including both rotating and counter-rotating terms.

We decouple the ground state and the excited states of the coupler through the unitary Schrieffer Wolff transformation $U = e^{-K}$,where K is anti-Hermitian \[]

$$K(t) = \sum_{k=1,2} \sum_{i\in\mathcal{H}_{M-1}} [G_R^{(k)} + G_{CR}^{(k)}] \tag{30}$$

with

$$G_{\mathrm{R}}^{(k)} = \lambda_i^{(k)}(t)\Pi_{i,i+1}^{(k)} a^\dagger - \left(\lambda_i^{(k)}\right)^* (t)\Pi_{i+1,i}^{(k)} a$$
$$G_{\mathrm{CR}}^{(k)} = \Lambda_i^{(k)}(t)\Pi_{i,i+1}^{(k)} a - \left(\Lambda_i^{(k)}\right)^* (t)\Pi_{i+1,i}^{(k)} a^\dagger$$

both V,K are propotional to a small parameter $\sim g$, note to add higher-oder terms in the transformation K we can get many-body interactions terms as in Ref \cite},

Then then expand second order in g, we have

$$\begin{gathered}\tilde{H}_{\text{eff}} = UHU^\dagger + i\hbar\dot{U}U^\dagger \approx \\ \hat{H}_0 + V + [\hat{K}, \hat{H}_0 + V] + \tfrac{1}{2}[\hat{K}, [\hat{K}, \hat{H}_0]] - i\tfrac{\partial \hat{K}}{\partial t} + \tfrac{i}{2}\left[\hat{K}, \tfrac{\partial \hat{K}}{\partial t}\right]\end{gathered} \tag{31}$$

where we have used [6, 7]

$$\begin{aligned} e^{-\hat{K}}\hat{H}e^{\hat{K}} &= \hat{H} - [\hat{K}, \hat{H}] + \tfrac{1}{2}[\hat{K}, [\hat{K}, \hat{H}]] \\ &\quad -\tfrac{1}{6}[\hat{K}, [\hat{K}, [\hat{K}, \hat{H}]]] \ldots \\ \left(\tfrac{\partial e^{-\hat{K}}}{\partial t}\right) e^{\hat{K}} &= -\tfrac{\partial \hat{K}}{\partial t} + \tfrac{1}{2}\left[\hat{K}, \tfrac{\partial \hat{K}}{\partial t}\right] - \tfrac{1}{6}\left[\hat{K}, \left[\hat{K}, \tfrac{\partial \hat{K}}{\partial t}\right]\right] \cdots \end{aligned}$$

To eliminate first order transition in Eq. (31) we require

$$-[\hat{K}, \hat{H}_0] - i\frac{\partial \hat{K}}{\partial t} + V = 0. \tag{32}$$

Similar to Ref.[8, 9], we obtain the effective Hamiltonian

$$\begin{aligned} \tilde{H}_{\text{eff}} &= \hat{H}_0 - \frac{1}{2}[\hat{K}, V] \\ &= \omega_r a^\dagger a + \sum_{k=1,2}\sum_{i\in\mathcal{H}_M}\left[\tilde{\omega}_i^{(k)}\Pi_{i,i}^{(k)} + S_i^{(k)}\Pi_{i,i}^{(k)} a^\dagger a\right] \\ &\quad + \left[\sum_{i,j\in\mathcal{H}_{M-1}} J_{ij}\Pi_{i,i+1}^{(1)}\Pi_{j+1,j}^{(2)} + J'_{ij}\Pi_{i,i+1}^{(1)}\Pi_{j,j+1}^{(2)} + \text{H.c}\right] \end{aligned} \tag{33}$$

with the effective coupling strength between qubits

$$J_{ij} = \frac{1}{2}g_i^{(1)}\left(\lambda_j^{(2)} - \Lambda_j^{(2)}\right)^* + \frac{1}{2}g_i^{(2)}\left(\lambda_j^{(1)} - \Lambda_j^{(1)}\right) \tag{34}$$

$$J'_{ij} = \frac{1}{2}g_i^{(1)}\left(\lambda_j^{(2)} - \Lambda_j^{(2)}\right) + \frac{1}{2}g_i^{(2)}\left(\lambda_j^{(1)} - \Lambda_j^{(1)}\right), \tag{35}$$

where
$\tilde{\omega}_i^{(k)} = \omega_i^{(k)} + L_i^{(k)}$, $L_i^{(k)} = 2\Re[\chi_{i-1}^{(k)} - \mu_i^{(k)}]$, $S_i^{(k)} = 2\Re[\chi_{i-1}^{(k)} + \mu_{i-1}^{(k)} - \chi_i^{(k)} - \mu_i^{(k)}]$, $\chi_i^{(k)} = g_i^{(k)*}\lambda_i^{(k)}$, $\mu_i^{(k)} = g_i^{(k)*}\Lambda_i^{(k)}$.

$\lambda_i^{(k)}$, $\Lambda_i^{(k)}$ satisfy the following equations due to Eq. (32)

$$\Pi_{i,i+1}^{(k)} a^\dagger [g_i^{(k)} - i\dot{\lambda}_i^{(k)} - (\omega_{i+1}^{(k)} - \omega_i^{(k)} - \omega_r)\lambda_i^{(k)}] = 0 \tag{36}$$

$$\Pi_{i,i+1}^{(k)} a [g_i^{(k)} - i\dot{\Lambda}_i^{(k)} - (\omega_{i+1}^{(k)} - \omega_i^{(k)} + \omega_r)\Lambda_i^{(k)}] = 0 \tag{37}$$

with initial conditions $\lambda_i^{(k)}(0) = g_i^{(k)}/\Delta_i^{(k)}$, and $\Lambda_i^{(k)}(0) = g_i^{(k)}/\Sigma_i^{(k)}$.
here $\Delta_i^{(k)} = \omega_{i+1}^{(k)} - \omega_i^{(k)} - \omega_r$, $\Sigma_i^{(k)} = \omega_{i+1}^{(k)} - \omega_i^{(k)} + \omega_r$

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