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SPS Computing

Coin Games

Problem

Alice and Bob are plaing a game using a bunch of coins. The players pick several coins out of the bunch in turn. Each time a player is allowed to pick 1, 2 or 4 coins; the player that gets the last coin is the winner.

Assume that both players are very smart and that they will try their best to work out a strategy to win the game.

For example, if there are 2 coins and Alice is the first player to pick, she will definitely pick 2 coins and win. If there are 3 coins, Alice is still the first player to pick, but no matter if she picks 1 or 2 coins, Bob will get the last coin and win the game.

Given the number of coins and the order of players (i.e. the first and second players to pick the coins), you are required to write a program to calculate the winner of the game and to calculate how many different strategies there are for them to win the game.

You should use recursion to solve the problem. You can assume that there are no more than 30 coins.

Discuss the strategy employed by Alice and Bob.

Strategy

The aim of a player is to get the other player to have a number of coins divisible by 3. If a player has a number divisible by 3, they lose.

- If a player has 1 coin, they have one way to win.
- If a player has 2 coins, they have one way to win.
- If a player has 3 coins, they lose in two ways — because they can create situations where the other player has either 1 or 2 coins (see above).
- If a player has 4 coins, they can either win directly or by picking one coin, forcing the other player to have three.
- If a player starts with a quantity divisible by 3, they will lose eventually because no matter what they play, the other player can play strategically to keep them on a smaller quantity also divisible by 3 (1 then 2, 2 then 4 or 4 then 2), thus the starting player will eventually be reduced to 3; a losing position.
- If a player starts on a quantity not divisible by 3, its remainder could be either be 1 (play 1 or 4) or 2 (play 2); either way the other player can be left on a number divisible by 3.

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Thus, a function f can be considered that takes a number of coins and returns a number that represents the number of strategies for the winner to definitely win; a negative number indicates a loss for the player playing first:

$$
\begin{align}
f(1) &= 1 \tag{1}\\
f(2) &= 1 \tag{2}\\
f(3) &= -(f(2) + f(1)) \tag{3}\\
f(4) &= 1 + -f(3) \tag{4}\\
f(5) &= -f(3) \tag{5}\\
f(6) &= -(f(5) + f(4) + f(2)) \tag{6}\\
f(7) &= -(f(6) + f(3)) \tag{7}\\
f(8) &= -f(6) \tag{8}\\
&\dots \tag{9}\\
f(3k) &= -(f(3k-1) + f(3k-2) + f(3k-4)) \tag{10}\\
f(3k+1) &= -(f(3k) + f(3(k-1))) \tag{11}\\
f(3k+2) &= -f(3k) \tag{12}
\end{align}
$$

For $k \in \mathbb{N}, k \geq 6$.