Linearity Property applied to convergent series

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1 Introduction

We want to prove that

$$\sum \Theta(f(n)) \in \Theta(\sum f(n))$$

2 Proof

2.1 Assumptions

$$\sum \Theta(f(n))$$
$$and$$
$$\Theta(\sum f(n))$$

could also imply that $f(n)$ is not the same function throughout each sum, but variable accordingly the iterator variable of the sum, so we assume that

$$\sum \Theta(f(n))$$

means

$$\Theta(h(n)) + \Theta(g(n)) + ...$$

(similarly for $\Theta(\sum f(n))$), so it's the most general way.

2.2 Proof

We prove the theorem by proving that $\Theta(h(n)) + \Theta(g(n)) \in \Theta(h(n) + g(n))$.

To simplify the proof, we'll prove that $O(h(n)) + O(g(n)) \in O(h(n) + g(n))$, since proving it for Ω is specular. The validity of these claims will eventually prove it for Θ.

So we want to prove

$$O(h(n)) + O(g(n)) \in O(h(n) + g(n))$$

By definition of $BigO$ we have that
$\exists c_1 > 0, n_1 > 0 \ : \ \forall n \geq n_1, \ h'(n) \leq c_1 h(n)$
$\exists c_2 > 0, n_2 > 0 \ : \ \forall n \geq n_2, \ g'(n) \leq c_2 g(n)$
Where $h'(n)$ and $g'(n)$ are the anonymous functions contained in $O(h(n))$ and $O(g(n))$.
Let $n_3 = MAX(n_1, n_2)$, then

$$h'(n) + g'(n) \leq c_1 h(n) + c_2 g(n) \ \forall n \geq n_3$$

Let $c_3 = MAX(c_1, c_2)$, then

$$h'(n) + g'(n) \leq c_1 h(n) + c_2 g(n) \leq c_3 (h(n) + g(n)) \ \forall n \geq n_3$$

$$\implies h'(n) + g'(n) \leq c_3 (h(n) + g(n)) \ \forall n \geq n_3$$

which means that $h'(n) + g'(n) \in O(h(n) + g(n))$
$\square$

3 Notes

Most books do an abuse of notation, reporting this theorem as

$$\sum \Theta(f(n)) = \Theta(\sum f(n))$$