Counting

Chapter 6

With Question/Answer Animations

Chapter Summary

- The Basics of Counting
- The Pigeonhole Principle
- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations
- Generating Permutations and Combinations (*not yet included in overheads*)

The Basics of Counting

Section 6.1

Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Examples, Examples, and Examples
- Tree Diagrams

Principles of Counting

- *Combinatorics*, the study of arrangements of objects, is an important part of discrete mathematics. This subject was studied as long ago as the seventeenth century.
- Enumeration, counting of objects with certain properties, is an important part of combinatorics.
- Counting is used to solve many different types of problems. For example, counting is used to determine the complexity of algorithms. Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand. Furthermore, counting techniques are used extensively when probabilities of events are computed.
- We will study here: Basic rules of counting, Pigeonhole principle, Permutations and Combinations, Analyzing gambling games, Computer Simulations etc.

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick a representative.

Generalized form of the Sum Rule

- Suppose that the tasks T_1, T_2,--------, T_m can be done in n_1, n_2,-----, n_m ways respectively and no two of these tasks can be done at the same time. Then the number of ways to do one of these tasks is $n_1 + n_2$ +------+ $n_{m.}$
- ***Example:*** A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from?
- ***Solution:*** The student can choose a project from the first list in 23 ways, from the second list in 15 ways, and from the third list in 19 ways. Hence, there are 23+15+19 = 57 projects to choose from.

The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

  $|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when $A_i \cap A_j = \emptyset$ for all i, j.

- The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

Example

- *Question*: The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chars that can be labeled differently?
- *Solution*: The procedure of labeling a chair consists of two tasks, namely, assigning one of the 26 letters and then assigning one of the 100 possible integers to the seat. The product rule shows that there are 26 . 100 = 2600 different ways that a chair can be labeled. Therefore the largest number of chairs that can be labeled differently is 2600.

Generalized form of the Product Rule:

- Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots\ldots, T_m$. If task T_i can be done in n_i ways after tasks $T_1, T_2, \ldots\ldots, T_{i-1}$ have been done, then there are $n_1.n_2.n_3\ldots\ldots n_m$ ways to carry out the procedure.

Example

- *Question*: In how many ways can an organization containing 26 members elect a president, treasurer, and secretary (assuming no person is elected to more than one position)?
- *Solution:* The president can be elected in 26 different ways; following this, the treasurer can be elected in 25 different ways (since the person chosen president is not eligible to be treasurer); and, following this, the secretary can be elected in 24 ways. Thus, by the product rule, there are 26 . 25 . 24 = 15600 different ways in which the organization can elect the officers.

The Product Rule

Example: Suppose a license plate contains two letters followed by three digits with the first digit not zero. How many different license plates can be printed?

Solution: Each letter can be printed in 26 different ways, the first digit in 9 ways and each of the other two digits in 10 ways. Hence 26 . 26 . 9. 10. 10 = 608400 different plates can be printed.

Product Rule in Terms of Sets

- If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2 , ..., and an element in A_m.
- By the product rule, it follows that:
  $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$

Combining the Sum and Product Rule

Many counting problems cannot be solved using just the sum rule or just the product rule. However, many complicated counting problems can be solved using both of these rules.

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution: Use the product rule.

$$26 + 26 \cdot 10 = 286$$

Example

- *Question:* In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. Moreover, a variable name must begin with a letter and must be different from five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

- *Solution*: Let V equal the number of different variable names in this version of BASIC. Let V_1 be the number of these that are one character long and V_2 be the number of these that are two characters long. Then, by the Sum rule, $V = V_1 + V_2$. Note that, $V_1 = 26$, since a one-character variable name must be a letter. Furthermore, by the product rule, there are $26 \cdot 36$ strings of length two that begin with a letter and end with an alphanumeric character. However, five of these are excluded, so that $V_2 = 26.36 - 5 = 931$. Hence, there are $V = 26 + 931 = 957$ different names for variables in this version of BASIC.

Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.

Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$$P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.$$

$$P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920.$$

$$P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880.$$

Consequently, $P = P_6 + P_7 + P_8 = 2,684,483,063,360$.

Subtraction Rule / Inclusion-Exclusion Principle

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

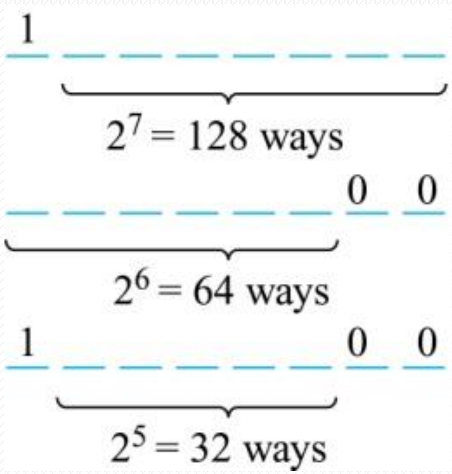
- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.



- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.

Example

- *Question*: How many positive integers not exceeding 1000 are divisible by 7 or 11?
- *Solution*: Let A be the set of positive integers not exceeding 1000 that are divisible by 7, and let B be the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of positive integers not exceeding 1000 that are divisible by either 7 or 11, and $A \cap B$ is the set of positive integers not exceeding 1000 that are divisible by both 7 and 11. From the principle of inclusion-exclusion we get,
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- $= \lfloor 1000/7 \rfloor + \lfloor 1000/11 \rfloor - \lfloor 1000/(7.11) \rfloor$
- $= 142 + 90 - 12$
- $= 220$

The Generalized principle of Inclusion-Exclusion:

- The principle of inclusion-exclusion can be generalized to find the number of ways to do one of n different tasks or, equivalently, to find the number of elements in the union of n sets, whenever n is a positive integer.
- Let $A_1, A_2, \ldots, A_n$ be finite sets. Then
- $| A_1 \cup A_2 \cup \ldots \cup A_n | = \sum | A_i | - \sum | A_i \cap A_j | + \sum | A_i \cap A_j \cap A_k | - \ldots + (-1)^{n+1} | A_1 \cap A_2 \cap \ldots \cap A_n |$
-
- *Example*: Give a formula for the number of elements in the union of four sets.
- *Solution*: The inclusion-exclusion principle shows that
- $| A_1 \cup A_2 \cup \ldots \cup A_n | = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + | A_1 \cap A_2 \cap A_3 | + | A_1 \cap A_2 \cap A4 | + | A_1 \cap A_3 \cap A_4 | + | A_2 \cap A_3 \cap A_4 | - | A_1 \cap A_2 \cap A_3 \cap A_4 |$.
- NB: Note that this formula contains 15 different terms i.e. $2^4 - 1$.

Applications of Inclusion-Exclusion

- **The sieve of Eratosthenes:**
- The sieve of Eratosthenes is used to find all primes not exceeding a specified positive integer. The following procedure is used to find the primes not exceeding 100.
- First the integers that are divisible by 2, other than 2, are deleted. Since 3 is the first integer greater than 2 that is left, all those integers divisible by 3, other than 3, are deleted. Since 5 is the next integer left after 3, those integers divisible by 5, other than 5, are deleted. The next integer left is 7, so those integers divisible by 7, other than 7, are deleted. Since all composite integers not exceeding 100 are divisible by 2, 3, 5 or 7, all remaining integers except 1 are prime. In Tabl-1 [Rosen, Page-363] is shown the above procedure. Here, the integers not underlined are the primes not exceeding 100.
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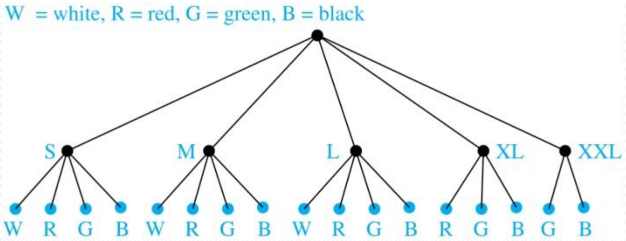
- Exercise: Find the number of primes less than 200 using the principle of inclusion-exclusion.

Applications of Inclusion-Exclusion

- **Derangement:**
- A *derangement* is a permutation of objects that leaves no object in its original position. For example, the permutation 21453 is a derangement of 12345 because no number is left in its original position. However, 21543 is not a derangement of 12345, because this permutation leaves 4 fixed.
- The number of derangements of a set with n elements is
- $D_n = n! [\, 1 - 1/1! + 1/2! - 1/3! + ... + (-1)^n 1/n! \,]$
- $e^{-1} = 1 - 1/1! + 1/2! - 1/3! + ... + (-1)^n 1/n! \cong 0.368$

Tree Diagrams

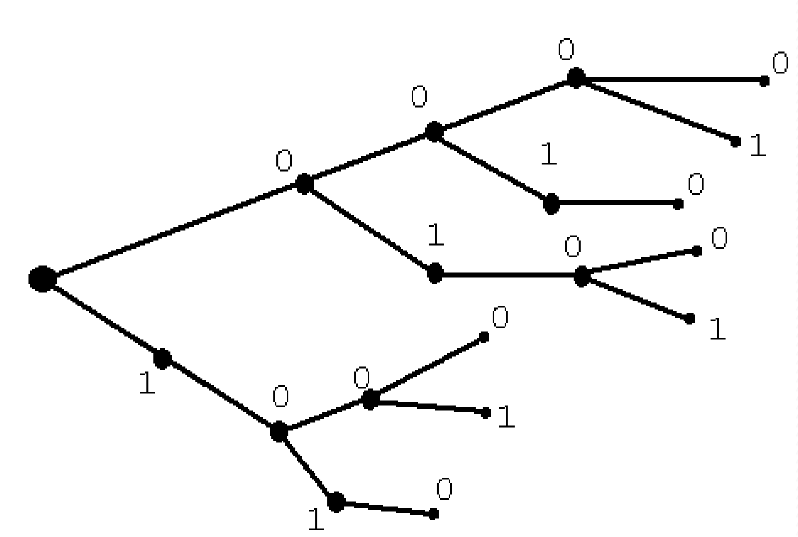
- **Tree Diagrams**: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example**: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?
- **Solution**: Draw the tree diagram.



- The store must stock 17 T-shirts.

Example

- *Question:* How many bit strings of length four do not have two consecutive 1s?
- *Solution:* The following tree diagram displays all bit strings if length four without two consecutive 1s.



The Pigeonhole Principle

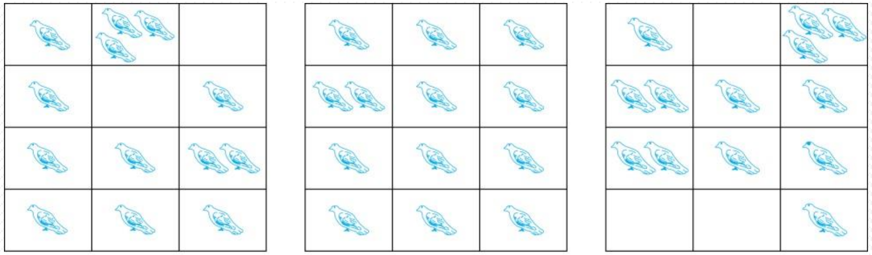
Section 6.2

Section Summary

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a) (b) (c)

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have $k + 1$ objects.

The Pigeonhole Principle

Corollary 1: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f .
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one. ◀

Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example (*optional*): Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Solution: Let n be a positive integer. Consider the $n + 1$ integers 1, 11, 111, …., 11…1 (where the last has $n + 1$ 1s). There are n possible remainders when an integer is divided by n. By the pigeonhole principle, when each of the $n + 1$ integers is divided by n, at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of n that has only 0s and 1s in its decimal expansion.

Pigeonhole Principle

- *Example*: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
- *Example*: Among any 27 English words, there must be at least two that begin with the same letter, since there are 26 letters in the English alphabet
- *Example*: How many students must be in a class to guarantee that at least two students born at the same day of a week?
- *Solution*: There are 7 days in a week. The pigeonhole principle shows that among any 8 students there must be at least two students who born at the same day of a week.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < \lceil N/k \rceil + 1$ has been used. This is a contradiction because there are a total of n objects. ◀

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

The Generalized Pigeonhole Principle

Example: a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

b) How many must be selected to guarantee that at least three hearts are selected?

Solution: a) We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards. At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$.

b) A deck contains 13 hearts and 39 cards which are not hearts. So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts. However, when we select 42 cards, we must have at least three hearts. (Note that the generalized pigeonhole principle is not used here.)

The Generalized Pigeonhole Principle

- *Example:* What is the minimum number of students required in a Discrete Mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, namely A, B, C, D and F.
- *Solution:* The minimum number of students needed to guarantee that at least six students receive the same grade is the smallest integer such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5.5 + 1 = 26$. Thus, 26 is the minimum number of students needed to be sure that at least 6 students will receive the same grade.

Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permuation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

- The number of r-permuatations of a set with n elements is denoted by $P(n,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2) \cdots (n-r+1)$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n-1$ ways, and so on until there are $(n-(r-1))$ ways to choose the last element.

- Note that $P(n,0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Permutations

- *Example*: How many different ways are there to select 4 different players from 10 players on a team to play four tennis matches, where the matches are ordered?
- *Solution:* P(10,4) = 10.9.8.7 = 5040

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition. ◀

Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\ C(n - n_1, n_2)\ \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!}=\frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

◀

Permutations with Indistinguishable Objects

- *Example*: How many seven-letter words can be formed using the word "BENZENE"?
- *Solution*: P (7; 3,2) = 7! / 3!2! = 20

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

The reasoning can be generalized to the following theorem. →

Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Objects into Boxes

- *Distinguishable objects* and *distinguishable boxes*.
  - There are $n!/(n_1!n_2!\cdots n_k!)$ ways to distribute n distinguishable objects into k distinguishable boxes.
  - (*See Exercises* 47 *and* 48 *for two different proofs.*)
  - Example: There are 52!/(5!5!5!5!32!) ways to distribute hands of 5 cards each to four players.
- *Indistinguishable objects* and *distinguishable boxes*.
  - There are $C(n + r - 1, n - 1)$ ways to place r indistinguishable objects into n distinguishable boxes.
  - Proof based on one-to-one correspondence between n-combinations from a set with k-elements when repetition is allowed and the ways to place n indistinguishable objects into k distinguishable boxes.
  - Example: There are $C(8 + 10 - 1, 10) = C(17,10) = 19{,}448$ ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distributing Objects into Boxes

- *Distinguishable objects* and *indistinguishable boxes*.
  - Example: There are 14 ways to put four employees into three indistinguishable offices (*see Example* 10).
  - There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes.
  - See the text for a formula involving *Stirling numbers of the second kind*.
- *Indistinguishable objects* and *indistinguishable boxes*.
  - Example: There are 9 ways to pack six copies of the same book into four identical boxes (*see Example* 11).
  - The number of ways of distributing n indistinguishable objects into k indistinguishable boxes equals $p_k(n)$, the number of ways to write n as the sum of at most k positive integers in increasing order.
  - No simple closed formula exists for this number.

Distributing Objects into Boxes

- ***Theorem***: The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i = 1, 2, …, k equals $n! / (n_1! \, n_2! \ldots n_k!)$
- *Example*: How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?
- *Solution*: First player can be dealt 5 cards in C(52,5) ways. The second player can be dealt 5 cards in C(47,5) ways, since only 47 cards are left. Similarly, the third player can be dealt 5 cards in C(42,5) and fourth players in C(37,5) ways. Hence, the total number of ways to deal four players 5 card each is
- C(52,5) C(47,5) C(42,5) C(37,5) = (52! / 47!5!) (47! / 42!5!) (42! / 37!5!) (37! / 32!5!)
- = 52! / (5!5!5!5!32!)

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

- The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section* 6.4.)

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$
$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

- *Example:* Find the number of combinations of the four objects a,b,c,d, taken three at a time.
- Solution: $C(4, 3) = 4! / (3! * 1!) = 4$. The combinations are: abc, abd, acd, bcd.
- The number of permutations for the above problem is: $P(4,3) = 24$.

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}\ .$$

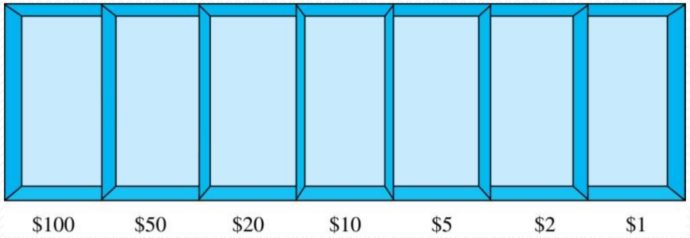
Hence, $C(n, r) = C(n, n - r)$.

- *Example:* How many committees of 3 can be formed 8 people?
- *Solution:* C (8,3) = (8.7.6) / (1.2.3) = 56

Combinations with Repetition

Example: How many ways are there to select five bills from a box containing at least five of each of the following denominations: $1, $2, $5, $10, $20, $50, and $100?

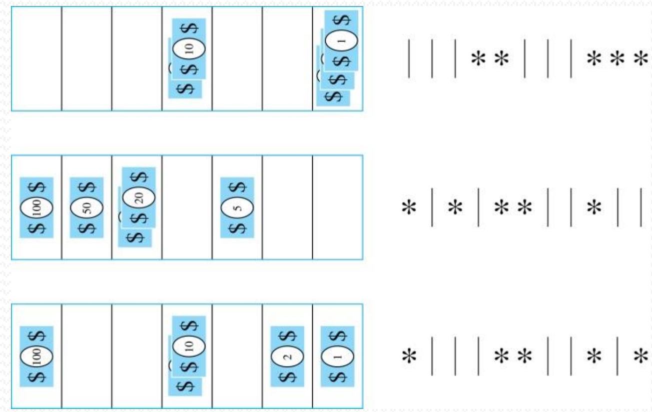
Solution: Place the selected bills in the appropriate position of a cash box illustrated below:



continued →

Combinations with Repetition

- Some possible ways of placing the five bills:



- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

- ***Theorem***: There are C (n+r-1, r) r-combinations from a set with n elements when repetition of elements is allowed.
- *Example*: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.
- *Solution*: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From the above theorem this equals C (4+6-1, 6) = C(9,6). since
- C (9,6) = C(9,3) = 9.8.7 / 1.2.3 = 84,
- There are 84 different ways to choose the six cookies.

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Binomial Coefficients and Identities

Section 6.4

Section Summary

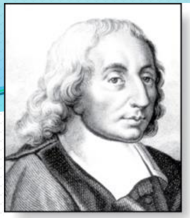
- The Binomial Theorem
- Pascal's Identity and Triangle
- Other Identities Involving Binomial Coefficients (*not currently included in overheads*)

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, $x y^2$, y^3 arise. The question is what are the coefficients?
    - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
    - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
    - To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
    - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3x y^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Blaise Pascal
(1623-1662)



Pascal's Identity

Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.

Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \quad ◀$$

See Exercise 19 *for an algebraic proof.*

Pascal's Triangle

- Pascal's Identity is the basis for a geometric arrangement of the binomial coefficients in a triangle, as shown in the following Figure.

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

$(a+b)^0 = 1$

$(a+b)^1 = a + b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3\,ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6\,a^2b^2 + 4ab^3 + b^4$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

……

Pascal's Triangle

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots.,n$.

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$
$$\cdots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
. . .

(b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Pascal’s Triangle

- The nth row in the triangle consists of the binomial coefficients
  - $\binom{n}{k}$, k = 0,1,......,n.
- This triangle is known as *Pascal’s Triangle*.
- The numbers in the Pascal's triangle has the following interesting properties:
- The first number and the last number in each row is 1.
- Every other number in the array can be obtained by adding the two numbers appearing directly above it. For example, 10 = 6 + 4, 15 = 10 + 5 etc. This

Pascal's Triangle

- ***Theorem***: Let n be a positive integer. Then
- $\sum C(n, k) = 2^n$
-
- ***Theorem***: Let m, n and r be nonnegative integers with r not exceeding either m or n. Then
- $C(m + n, r) = \sum C(m, r - k)\, C(n, k)$

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ◀

Binomial Theorem

- *Example*: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?
- *Solution*: $C(25, 13) = 25! / 13!12! = 5{,}200{,}300$
-
- *Example*: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?
- *Solution*: $C(25, 13)\ 2^{12} (-3)^{13} = -(25! / 13!12!)\ 2^{12}\ 3^{13}$
-
- Theorem: Let n be a positive integer. Then
- $\sum (-1)^k C(n, k) = 0$

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^{j}.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}.$$