ALGEBRAIC GROUPS: WEEK 12 HOMEWORK

Let G be a connected linear algebraic group over an algebraically closed field k of characteristic zero.

(1) Show that if G admits a faithful finite-dimensional irreducible representation, then G is reductive. (Hint: Use 2.4.15 in Springer.)

(2) (a) Prove that the radical $R([G,G])$ of the commutator subgroup $[G,G]$ is contained in the radical $R(G)$ of G.

(b) Suppose G is reductive. Prove that $[G,G]$ is semisimple. (Hint: Use a lemma from class.)

(c) Suppose G is reductive. Consider the product of the normal subgroups $R(G)$ and $[G,G]$:

$$R(G).[G,G] = \{xy \mid x \in R(G),\ y \in [G,G]\},$$

which is itself a subgroup of G. Show that this product is all of G. (You can use the fact that any semisimple group is equal to its own commutator.)

(3) (a) Let U (resp. U^-) be the subgroup of upper-triangular (resp. lower-triangular) unipotent matrices in SL_2. Show that U and U^- together generate all of SL_2.

(b) Prove that any element in U (resp. U^-) can be expressed as a commutator of a diagonal matrix in SL_2 and an element in U (resp. U^-). Conclude that $[\mathrm{SL}_2, \mathrm{SL}_2] = \mathrm{SL}_2$.

(c) Argue that $[\mathrm{PSL}_2, \mathrm{PSL}_2] = \mathrm{PSL}_2$.