CONTINUOUS BLACKJACK

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Contents

1. Continuous Blackjack

1.1. **The rules of the game.** There are M players, who are ordered. Each player in turn starts with score 0 and adds to their score by drawing successive independent random numbers uniformly distributed between 0 and 1, with the option to stop after any draw. A player's turn automatically ends if their cumulative score ever exceeds 1, in which case that player goes 'bust'. Each player sees the scores of the previous players (as well as whether or not they went bust). After all players have had a turn, the winner is the player with the highest score (not exceeding 1).

Question: For each $m \in \{1, \ldots, M\}$, what is the optimal strategy for the m-th player?

1.2. **Preliminary analysis.** Consider the following tentative strategy for any player, which, for now, ignores the scores of the other players.

- For some predetermined $t \in (0, 1]$, draw until your score exceeds t, and then stop.

Proposition 1.1. *For the above strategy, we have:*

(1) *The probability of having score at most t after n draws is $\frac{t^n}{n!}$.*

(2) *The probability of stopping after $n+1$ draws and not going bust is $\frac{t^n}{n!}(1-t)$.*

(3) *The probability of not going bust is* $e^t(1-t)$.

Proof. We introduce some notation. Let X_i be a uniform random variable on the interval $[0,1]$, for $i = 1, 2, \ldots$, and assume the X_i are independent. The number of draws made using the above strategy is a random variable N defined as:

$$N = \max\{n \ : \ \sum_{i=1}^{n} X_i \leq t\}.$$

To prove the first claim, we proceed by induction on n. The case $n = 1$ is immediate. Let $n \geq 1$. Since X_{n+1} is uniform on $[0,1]$, its probability density function $f_{X_{n+1}}$ satisfies $f_{X_{n+1}}(s) = 1$ for $s \in [0,1]$. The claim follows from the following computation invoking conditional probability and the induction hypothesis:

$$\begin{aligned} P\left(\sum_{i=1}^{n+1} X_i \leq t\right) &= \int_{s=0}^{t} P\left(\sum_{i=1}^{n+1} X_i \leq t \ \Big| \ X_{n+1} = s\right) f_{X_{n+1}}(s) ds \\ &= \int_{s=0}^{t} P\left(\sum_{i=1}^{n} X_i \leq t - s\right) ds = \int_{s=0}^{t} \frac{(t-s)^n}{n!} ds = \frac{t^{n+1}}{(n+1)!} \end{aligned}$$

(See Section **??** below for alternative proofs of the first claim.) For the second claim, observe that the probability of stopping after $n+1$ draws and not going bust is given by $P\left(\sum_{i=1}^{n} X_i \leq t \text{ and } \ \sum_{i=1}^{n+1} X_i \in (t,1]\right)$. Suppose $\sum_{i=1}^{n} X_i = s$ for some $s \leq t$. Then

$$P\left(\sum_{i=1}^{n+1} X_i \in (t,1]\right) = P\left(X_{n+1} \in (t-s, 1-s]\right) = 1 - t,$$

regardless of s. Since X_{n+1} is independent of the previous X_i, the desired probability becomes $P\left(\sum_{i=1}^{n} X_i \leq t\right)(1-t) = \frac{t^n}{n!}(1-t)$. Finally, to compute the probability in the third claim, we partition according to the number N of draws:

$$\begin{aligned} P(\text{didn't go bust}) &= \sum_{n=1}^{\infty} P(\text{stopped after } n+1 \text{ draws and didn't go bust}) \\ &= \sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n} X_i \leq t \text{ and } \ \sum_{i=1}^{n+1} X_i \in (t,1]\right) = \sum_{n=1}^{\infty} \frac{t^n}{n!}(1-t) \end{aligned}$$

The result follows from the Taylor expansion for e^t. □

1.3. **Assumption.** Observe that, if player A stops before their score exceeds that of a previous player, then player A would have no chance of winning. Hence we make the following assumption:

- Each player will draw until their score is at least that of every previous player.

In particular, no player follows the strategy "draw twice, then stop," which may have some merit in the case that the players do not see each others' scores.

1.4. **First player.** Suppose the first player follows the strategy of drawing until their score exceeds t, and then stopping.

Proposition 1.2. *The probability that that the first player wins using the above strategy is:*

$$e^t \int_{s=t}^{1} (1 - e^s - se^s)^{M-1} ds$$

Proof. Call the first player Alice. In order to win, Alice must not go bust. Given that Alice has not gone bust, her score S is a random variable uniformly distributed on the interval $(t, 1]$, and hence has probability density function $\frac{1}{1-t}$ on that interval. Using the assumption that each player will draw until their score is at least that of all the previous players, we compute the probability that Alice wins with the above strategy:

$$\begin{aligned}
P(\text{win}) &= P(\text{doesn't go bust})P(\text{all other players go bust or do not exceed Alice's score}) \\
&= e^t(1-t)\int_{s=1}^{1} P(\text{all other players either do not exceed } s \text{ or go bust}) f_S(s) ds \\
&= e^t(1-t)\int_{s=1}^{1} \left(\prod_{i=2}^{N} P(\text{player } i \text{ either does not exceed } s \text{ or goes bust})\right) \frac{1}{1-t} ds \\
&= e^t \int_{s=1}^{1} \prod_{i=2}^{M} (1 - e^s(1-s)) ds = e^t \int_{s=1}^{1} (1 - e^s(1-s))^{M-1} ds
\end{aligned}$$

The result follows. □

Hence, the optimal value of t would be the one that maximizes the quantity:

$$p_{M-1}(t) = p(t) = e^t \int_{s=t}^{1} (1 - e^s - se^s)^{M-1} ds$$

The derivative is:

$$p'(t) = e^t \int_{s=t}^{1} (1 - e^s - se^s)^{M-1} ds - e^t(1 - e^t - te^t)^{M-1}$$

Since $p'(0) > 0$ and $p'(1) = -e < 0$, we conclude that $p(t)$ has a maximum in $[0, 1]$. This maximum does not have a closed form, but can be computed numerically. Let t_{M-1} be this quantity. (If $M = 1$, then $t_1 = 0$.) The subscript indicates that $M - 1$ players must still play.

1.5. **General strategy.** Let $m \in \{1, \ldots, M\}$. By the assumption above, the m-th player will draw until their score exceeds that of all $m - 1$ previous players. On the other hand, the m-player desires to maximize their chances of beating the scores of the $M - m$ remaining players. By the analysis of the previous section, this is achieved by drawing until their score exceeds t_{M-m}, and then stopping.

- Draw until the score exceeds the maximum of the scores of all previous players and t_{M-m}, then stop.

The following code implements this algorithm. It uses a function `draw()` which draws a random number uniformly from $[0,1]$, and the function `optim(n)` which computes the maximum t_n of $p_n(t) = e^t \int_{s=t}^{1} (1 - e^s - se^s)^n ds$ on $[0,1]$.

```
def play(previous_scores: List[float], num_remaining: int):

    # Compute the maximum so far.
    max_so_far = max(previous_scores)

    # Compute the optimal stopping value based on the number of
        players remaining.
    t = optim(num_remaining)

    # Set the stopping value.
    stopping_value = max(max_so_far, t)

    # Initiate the score and perform the loop.
    score = 0
    while score <= stopping_value:
        score += draw()
        if score >= 1:
            print("Bust!")
            return
    print("Score:", score)
```

2. Discussion

For $n = 1, 2, \ldots$, let A_n be the following subset of $\mathbb{R}^n$:

$$A_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \text{ and } \sum_i x_i \leq 1 \right\}.$$

Set $a_n = \mathrm{Vol}(A_n)$. Observe that a_n is equal to the probability that the sum $X_1 + X_2 + \cdots + X_n$ of independent uniform random variables on $[0,1]$ is at most 1. We computed above that this probability is equal to $1/n!$. In this section, we discuss alternative ways to arrive at this result.

2.1. **Recursive approach.** It is immediate that $a_1 = 1$. We aim to find a recursive relation between a_{n+1} and a_n. To this end, let $t \in (0,1]$ and set $c_t : \mathbb{R}^n \to \mathbb{R}^n$ be the map that scales each coordinate by t, that is, $c_t(x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$. Observe that:

$$c_t(A_n) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \text{ and } \sum_i x_i \leq t \right\}.$$

Therefore,

$$\mathrm{Vol}(c_t(A_n)) = \det(c_t)\mathrm{Vol}(A_n) = t^n a_n.$$

Now,

$$\begin{aligned} a_{n+1} &= \mathrm{Vol}\left(A_{n+1}\right) \\ &= \int_{s=0}^{t} \mathrm{Vol}(\{(x_1,\ldots,x_n,x_{n+1}) \in [0,1]^{n+1} \mid \sum_{i=1}^{n+1} x_i \leq 1 \text{ and } x_{n+1} = s\})ds \\ &= \int_{s=0}^{t} \mathrm{Vol}(\{(x_1,\ldots,x_n) \in [0,1]^{n} \mid \sum_{i=1}^{n} x_i \leq 1 - s\})ds \\ &= \int_{s=0}^{t} \mathrm{Vol}(c_{1-s}(A_n))ds = a_n \int_{s=0}^{t} (1-s)^n ds = \frac{a_n}{(n+1)!} \end{aligned}$$

Together with the initial condition that $a_1 = 1$, we obtain that $a_n = \frac{1}{n!}$.

2.2. **Geometric approach.** For $n \geq 1$, set

$$\Delta_n = \{(s_1,\ldots,s_n) \in \mathbb{R}^n \mid 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq 1\}.$$

Lemma 2.1. $\mathrm{Vol}(\Delta_n) = \frac{1}{n!}$

Proof. Let $C_n = [0,1]^n$ be the unit hypercube in $\mathbb{R}^n$, and let

$$C_n^\circ = \{(s_1,\ldots,s_n) \in C_n \mid s_i \neq s_j \text{ for } i \neq j\}$$

be the unit hypercube with all diagonals removed. Observe that there is a free action of the symmetric group S_n on C_n°, and a fundamental domain is given by

$$\Delta_n^\circ = \{(s_1,\ldots,s_n) \in \mathbb{R}^n \mid 0 \leq s_1 < s_2 < \cdots < s_n \leq 1\}$$

Hence:

$$\mathrm{Vol}(\Delta_n) = \mathrm{Vol}(\Delta_n^\circ) = \frac{\mathrm{Vol}(C_n^\circ)}{|S_n|} = \frac{\mathrm{Vol}(C_n)}{n!} = \frac{1}{n!}$$

where the first equality follows from the fact that $\mathrm{Vol}(\Delta_n^\circ) = \mathrm{Vol}(\Delta_n)$, and the two equalities use from the fact that $\mathrm{Vol}(C_n) = \mathrm{Vol}(C_n^\circ) = 1$. □

Corollary 2.2. $a_n = \frac{1}{n!}$.

Proof. By the preceding lemma, it suffices to show that $\mathrm{Vol}(\Delta_n) = \mathrm{Vol}(A_n)$. Indeed, Δ_n is the image of A_n under the linear map $\mathbb{R}^n \to \mathbb{R}^n$ given by the following matrix with determinant 1:

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}$$

(That is, $M_{ij} = 1$ if $i \geq j$ and 0 otherwise.) □