1. Introduction

My research lies at the intersection of representation theory, algebraic geometry, and quantum algebra. My work contributes to the central quest of geometric representation theory, namely, the search for unifying geometric perspectives as tools to solve long-standing algebraic problems in representation theory.

One of the most inspirational results in my field is the **Beilinson–Bernstein localization** theorem, which describes representations of a Lie algebra in terms of differential equations on the corresponding flag variety [B]. As a consequence, powerful geometric techniques can be imported into the study of Lie algebras. One concrete application of this theorem is the establishment of the Kazhdan–Lusztig conjecture for the count of multiplicities of simple modules for a Lie algebra inside standard modules. The philosophy of the Beilinson–Bernstein localization theorem lies behind many of the research questions I address, either directly or indirectly. In particular, I am contributing to a new perspective on this theorem via the geometry of the so-called **wonderful compactification** of an algebraic group, which was introduced by de Concini and Procesi. The wonderful compactification is a projective variety which captures the equivariant degenerations of the group and links the geometry of the group to the geometry of its partial flag varieties and Levi subgroups.

Another source of inspiration and new perspectives comes from the rich theory of **quantum groups**, which are q-deformations of universal enveloping algebras of Lie algebras. Quantum groups were discovered around 1980 in the study of inverse scattering methods and solutions of quantum integral systems; surprisingly, they exhibit intricate connections to diverse areas of mathematics. For example, there are rich parallels between the representation theory of quantum groups when q is a root of unity and Lie algebras in positive characteristic. In both settings, various relevant algebras exhibit large centers and define matrix bundles (i.e. Azumaya algebras) over classical geometric objects. While the deep reasons for this connection remain somewhat mysterious, this program has been advanced in several settings by Lusztig, Bezrukavnikov, and others [ABG, BFG].

Moreover, just as the Beilinson–Bernstein localization theorem provides a bridge between Lie algebras and algebraic geometry, versions of this theorem connect quantum groups to noncommutative algebraic geometry [BK, T]. From this perspective, quantum groups appear as symmetries in noncommutative algebraic geometry. A guiding theme behind part of my research is the construction of quantum group versions of geometric objects in representation theory in order to understand categories of representations via analogues of the Beilinson–Bernstein localization theorem. The constructions often take the form of q-deformations of algebras (i.e. as global rather than formal quantizations) and pivot on the structure of the quantum group as a q-deformation of the universal enveloping algebra of a Lie algebra. One instance of this construction is in the setting of **multiplicative quiver varieties** which is the subject of some of my ongoing work. Another instance is that of hypertoric varieties, and has been treated in previous work of mine and of Stadnik [G3, S].

In future work, I plan to extend and generalize some of my current projects, and to explore new settings where geometric techniques can illuminate algebraic phenomena. For example, I aim to reinterpret the **quantum Beilinson–Bernstein localization** theorem using the construction of my thesis on the wonderful compactification for quantum groups [G2]. I further plan to further exploit the construction of my thesis in the development of **quantum character sheaves** and investigate conjectural connections with certain topological quantum field theories.

A new setting in which I plan to apply the techniques I've developed is the study of finite reductive groups, also known as **finite groups of Lie type**. These are groups that arise as the set of fixed points of (generalized) Frobenius maps on connected reductive groups defined over the algebraic closure of a finite field, e.g. the general linear group $\mathrm{GL}_n \mathbb{F}_q$ over a finite field. Finite reductive groups feature in the classification of finite simple groups, and exhibit intricate representation theory, which has been studied extensively by Deligne and Lusztig using powerful techniques of algebraic geometry [C1, DL]. The irreducible characters of these groups have been understood in some sense, but many of the underlying phenomena are mysterious. Methods from topological field theory, higher category theory, and derived algebraic geometry provide re-interpretations and cleaner results in the subject [G1, Z].

2. Previous and ongoing work

2.1. **The wonderful compactification for quantum groups.** A key tenet of algebraic geometry, due to Grothendieck, asserts that a space can be completely understood through its category of sheaves. Furthermore, q-deformations of the category of sheaves can be viewed as categories of sheaves on a (nonexistent) quantum version of the original space. This philosophy is particularly well-suited for constructing quantizations of projective varieties (such as the wonderful compactification) where it is not enough to consider q-deformations of the algebra of global functions. Quasicoherent sheaves on quantum projective varieties are described by Proj categories of graded rings [AZ].

In [G2], I compactify quantum groups by quantizing the wonderful compactification. Specifically, I define a q-deformation of the category of quasicoherent sheaves on the wonderful compactification $\overline{G}$ of a semisimple adjoint group G. My approach relies on the quantum Peter–Weyl theorem, which describes the quantum coordinate algebra $\mathcal{O}_q(G)$ in terms of matrix coefficients, and leads to the definition of a certain multi-filtration on $\mathcal{O}_q(G)$.

Theorem 1 ([G2])**.** *The Rees algebra* $\mathrm{Rees}_q(G)$ *of* $\mathcal{O}_q(G)$ *with the Peter–Weyl filtration is a q-deformation of the coordinate ring of the Vinberg semigroup for G.*

The Vinberg semigroup is an affine monoid that can be regarded as a simpler, linear version of the wonderful compactification $\overline{G}$. The algebra $\mathrm{Rees}_q(G)$ quantizes a natural Poisson structure on the Vinberg semigroup, and carries an action of the maximal torus T of G. Fix λ to be a regular dominant weight, and write $\mathrm{Rees}_q(G)_{\lambda^n}$ for the λ^n-weightspace. Set

$$\mathsf{QCoh}_q(\overline{G}) = \mathsf{Proj}\left(\bigoplus_{n\geq 0} \mathrm{Rees}_q(G)_{\lambda^n}\right).$$

Theorem 2 ([G2])**.** *The category* $\mathsf{QCoh}_q(\overline{G})$ *is the category of quasicoherent sheaves on the quantum wonderful compactification. More precisely, it constitutes a flat q-deformation of* $\mathsf{QCoh}(\overline{G})$*, compatible with the structure of $\overline{G}$ as a Poisson variety with a $G \times G$-action.*

2.2. **Beilinson–Bernstein localization via asymptotics.** A fundamental result of geometric representation theory is the Beilinson–Bernstein localization theorem, which describes representations of any semisimple Lie algebra in terms of $\mathcal{D}$-modules, i.e. modules for the algebra of differential operators, on the associated flag variety. In joint work with D. Ben-Zvi, we place this theorem within the framework of the wonderful compactification and asymptotics of matrix coefficients [BG1]. Precursors to our work appear in [BN2, ENV].

Our approach stems from the fact that the flag variety for a group governs the geometry of the group at infinity. More precisely, the closed stratum in the wonderful compactification $\overline{G}$ is the square of the flag variety $\mathcal{B} \times \mathcal{B}$, and, taking limits of $\mathcal{D}$-modules on G to infinity produces families of sheaves on $\mathcal{B} \times \mathcal{B}$. We define a functor

$$\mathrm{Asymp} : D(G) \to D_{\mathrm{mon}}(\mathcal{Y})\text{-mod},$$

from the category of (filtered) $\mathcal{D}$-modules on G to the category of (monodromic) $\mathcal{D}$-modules on a punctured tubular neighborhood[1] $\mathcal{Y}$ of $\mathcal{B}\times\mathcal{B}$ in $\overline{G}$. This functor is compatible with the localization functors:

$$\begin{tikzcd} & U\mathfrak{g}\otimes U\mathfrak{g}\text{-mod} \arrow[dl, "\mathsf{Loc}"'] \arrow[dr, "\mathsf{Loc}"] & \\ D_{\mathrm{mon}}(\mathcal{Y}) & & D(G) \arrow[ll, "\mathrm{Asymp}"] \end{tikzcd}$$

Here, the localization functor $\mathsf{Loc} : U\mathfrak{g}\otimes U\mathfrak{g}\text{-mod} \to D_{\mathrm{mon}}(\mathcal{Y})$ is a doubled version of the Beilinson–Bernstein functor, in families, while the localization functor $\mathsf{Loc} : U\mathfrak{g}\otimes U\mathfrak{g}\text{-mod} \to D(G)$ is a version of the matrix coefficients map.

We interpret the asymptotics functor Asymp through the Vinberg semigroup, and it is closely related to Verdier specialization (an avatar of nearby cycles). The asymptotics functor has a 'partial' version for each conjugacy class of parabolic subgroup, and these form a geometric realization of parabolic restriction. As an application, our results produce an algebraic interpretation of aspects of harmonic analysis on real reductive groups.

2.3. **Multiplicative hypertoric and quiver varieties.** Multiplicative quiver varieties are a class of moduli spaces of quiver representations introduced by Crawley-Boevey and Shaw [CS] in their solution to the Deligne–Simpson problem. In joint work with D. Jordan, we study quantized multiplicative quiver varieties (as well as quantized character varieties) when the quantization parameter is a root of unity. We describe these quantized moduli spaces as Azumaya algebras over the classical multiplicative quiver variety.

Quantized quiver varieties are constructed via the process of quantum Hamiltonian reduction starting from an algebra $\mathcal{D}_q(\mathrm{Mat}(Q,\mathbf{d}))$ of q-difference operators on a quiver Q equipped with a dimension vector $\mathbf{d}$, which is built from quantum Weyl algebras [J, GZ]. In studying these algebras for q a root of unity, we exploit a remarkable compatibility between Lusztig's quantum Frobenius homomorphism on the quantum group with the quantum Hamiltonian reduction construction applied to Azumaya algebras. We call such compatible data a Frobenius quantum moment map, and show – echoing [BFG] and [VV] – that "Azumaya algebras descend through Frobenius quantum Hamiltonian reductions". A second important tool in our work is the method of Poisson orders as developed in [BG2].

Theorem 3. *Let $\ell > 2$ be an integer, and let q be a primitive ℓ-th root of unity.*

1. *The algebra $\mathcal{D}_q(\mathrm{Mat}(Q,\mathbf{d}))$ is finitely generated over a central subalgebra isomorphic as a Poisson algebra to the coordinate ring of the classical framed multiplicative quiver variety.*
2. *The resulting sheaf of algebras over the classical framed multiplicative quiver variety $\mathcal{M}$ is Azumaya over the preimage by the group-valued moment map of the big Bruhat cell $G^* \subset G$.*
3. *The quantized multiplicative quiver variety of* [J] *is the global sections of a coherent sheaf of Azumaya algebras over $\mathcal{M}$ obtained by fiber-wise quantum Hamiltonian reduction of $\mathcal{D}_q()$.*

This joint work with D. Jordan grew out of my previous work on hypertoric varieties. Hypertoric varieties are hyperkähler analogues of toric varieties and retain many of their combinatorial and geometric structures. Multiplicative hypertoric varieties are group-like versions of ordinary hypertoric varieties constructed via group-valued Hamiltonian reduction. In my paper [G3], I construct quantizations of multiplicative hypertoric varieties and describe root-of-unity behavior. Similar results were obtained independently by N. Cooney [C2].

[1] Explicitly, we take $\mathcal{Y}$ to be the so-called asymptotic cone $\frac{G/N\times N^-\backslash G}{T}$, where N is the unipotent radical of a Borel subgroup containing the fixed maximal torus T. Thus, $\mathcal{Y}$ defines a T-bundle $\mathcal{Y}\to\mathcal{B}\times\mathcal{B}$. We consider only weakly T-equivariant (i.e. monodromic) $\mathcal{D}$-modules on $\mathcal{Y}$.

3. Research Objectives

In this section, I discuss several problems which will constitute my future work.

3.1. Beilinson–Bernstein asymptotics for quantum groups.

3.1. Beilinson–Bernstein asymptotics for quantum groups. Recall that $\mathcal{Y} = \frac{G/N \times N^- \backslash G}{T}$.

Problem 4. Define the appropriate q-deformation $D_{q,\mathrm{mon}}(\mathcal{Y})$ of the category $D_{\mathrm{mon}}(\mathcal{Y})$, together with a functor $\mathrm{Asymp}_q : D_q(G) \to D_{q,\mathrm{mon}}(\mathcal{Y})$ making the following diagram commute

$$\begin{tikzcd} & U_q\mathfrak{g} \otimes U_q\mathfrak{g}\text{-mod} \arrow[dl, "\mathsf{Loc}"'] \arrow[dr, "\mathsf{Loc}"] & \\ D_{q,\mathrm{mon}}(\mathcal{Y}) & & D_q(G) \arrow[ll, "\mathrm{Asymp}_q"] \end{tikzcd}$$

One difficulty in developing the correct q-deformations is that standard geometric constructions often rely on notions of semi-stability whose analogues in the quantum world are often difficult to define. In my thesis [G2], I define the wonderful compactification for quantum groups as a certain non-commutative projective scheme. One anticipated application of this construction is a new understanding of the quantum Beilinson–Bernstein theorem. While the category $D_q(G)$ of quantum D-modules on G has been defined in [BK] and is somewhat well-understood, the category of quantum D-modules on the wonderful compactification has not; we propose to study this:

Problem 5. Define the appropriate quantum analogue of the category of D-modules on the wonderful compactification, and interpret Problem 4 through Verdier specialization on the quantum wonderful compactification.

3.2. Quantum character sheaves. Character sheaves were invented by Lusztig in order to adapt classical constructions from the theory of finite groups to the setting of algebraic groups [L]. A character sheaf on a reductive group G is a certain $\mathcal{D}$-module on G that is equivariant for the conjugation action of G. (When working over the algebraic closure of a finite field, character sheaves give rise the characters of finite reductive groups.) The category of character sheaves[2] is the image of the horocycle correspondence functor

$$F : \mathcal{H} \to (\mathcal{D}(G)\text{-mod})^G$$

from the Hecke category $\mathcal{H}$ of Borel equivariant $\mathcal{D}$-modules on the flag variety to the category of conjugation equivariant $\mathcal{D}$-modules on G. The Hecke category is a categorical analogue of the group algebra of G, and character sheaves form the categorical center of $\mathcal{H}$ [BN1, BFO].

I propose to develop the theory of quantum character sheaves as a contribution to the study of harmonic analysis on quantum groups. The first step is formulating the quantum Hecke category $\mathcal{H}_q$. This is accomplished by starting with the category $\mathcal{D}_q(G/B)$ of quantum $\mathcal{D}$-modules on the flag variety, as defined in [BK], and imposing additional B-equivariance, or more precisely, a compatible $\mathcal{O}_q(B)$-coaction. Next, the category $(\mathcal{D}_q(G)\text{-mod})^G$ of equivariant quantum $\mathcal{D}$-modules on G appears in [BBJ].

Problem 6. Construct a q-deformation of the horocycle transform $F_q : \mathcal{H}_q \to (\mathcal{D}_q(G)\text{-mod})^G$, and define the category of quantum character sheaves as the image of F_q.

The functor of F_q involves the formulation of certain functoriality properties for quantum $\mathcal{D}$-modules, and is expected to be adjoint to Verdier specialization on the wonderful compactification for the quantum group, thus elevating the work of [BFO] to the quantum level.

[2] For the purposes of this summary, 'character sheaf' will mean 'unipotent character sheaf'.

The theory of quantum character sheaves is expected to align with the quantum geometric Langlands program. There is a 4-dimensional topological field theory Z that forms a natural setting for the quantum (Betti) geometric Langlands correspondence, defined recently in [BBJ]. The value of Z on a point is the category $\mathsf{Rep}(U_q\mathfrak{g})$. Its value on a circle is essentially $\mathcal{D}_q(G)$-mod, while the value on a torus is $(\mathcal{D}_q(G)\text{-mod})^G$. The following problem is inspired by results of [BN1]:

Problem 7. Realize the quantum Hecke category $\mathcal{H}_q$ as a 2-dualizable subcategory of $\mathcal{D}_q(G)$-mod and identify the categorical center of $\mathcal{H}_q$ with the category of quantum character sheaves.

3.3. **Lusztig's non-abelian Fourier transform.** Let Γ be a finite group, $\mathbb{C}[\Gamma]$ the group algebra, and $Z_\Gamma \subseteq \mathbb{C}[\Gamma]$ be its center, i.e. the space of class functions. The set of two-class functions consists of functions on the set of commuting pairs of elements of Γ, up to simultaneous conjugation:

$$Z_\Gamma^{(2)} = \mathbb{C}\left[\{(x,y) \in \Gamma \times \Gamma \mid [x,y] = 1\}/\Gamma\right].$$

Choosing one coordinate over the other, we obtain two bases for $Z_\Gamma^{(2)}$, both indexed by pairs (g,σ) where g is a conjugacy class representative of Γ and σ is an irreducible character of the centralizer $C_\Gamma(g)$. The change-of-basis matrix is given by the following formula, known as Lusztig's non-abelian Fourier transform:

$$\{(x,\sigma),\ (x',\sigma')\} = \frac{1}{|C_\Gamma(x)||C_\Gamma(x')|} \sum_{\substack{g\in\Gamma \\ gxg^{-1}\in C_\Gamma(x')}} \sigma(gx'g^{-1})\overline{\sigma'(g^{-1}xg)}. \tag{1}$$

In fact, $Z_\Gamma^{(2)}$ is the space of functions on the set of (equivalence classes of) G-local systems on the topological torus $S^1 \times S^1$, and is the algebra assigned to $S^1 \times S^1$ by the 3-dimensional fully extended TQFT assigning the category $\mathsf{Rep}(\Gamma)$ of finite-dimensional representations of Γ to a point. The above formula can be interpreted as arising from an $\mathrm{SL}_2\,\mathbb{Z}$ via this TQFT.

Lusztig's non-abelian Fourier transform appears in the representation theory of finite reductive groups. More specifically, for a given finite reductive group G its (unipotent) characters are divided into families $\hat{G}_i \subseteq \hat{G}$. To each family $\hat{G}_i$, one can attach is a finite group $\Gamma = \Gamma(\hat{G}_i)$. The members of the family $\hat{G}_i$ are in bijection with pairs (g,σ) where g is a conjugacy class representative of Γ and σ is an irreducible character of the centralizer $C_\Gamma(g)$. These are also in bijection with certain easily-constructed generalized characters arising from the Weyl group corresponding to G, and the change-of-basis matrix is exactly Lusztig's non-abelian Fourier transform (Equation 1) for the finite group $\Gamma = \Gamma(\hat{G}_i)$. The reason for the appearance of this formula is still quite mysterious and worthy of further study; especially in connection to TQFTs.

3.4. **The Hecke category.** Let G be a reductive group over $\overline{\mathbb{F}}_p$ with Frobenius endomorphism F and fixed F-stable Borel subgroup B. Let $\mathcal{H} := \mathcal{D}_B^b(G/B \times G/B)$ denote the Hecke category. The Frobenius endomorphism descends to the flag variety G/B. Let $\mathcal{H}^{\mathrm{Weil}}$ denote the category of $\mathcal{F} \in \mathcal{H}$ with a fixed Weil structure, i.e. an isomorphism $F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$.

Problem 8. Find a geometric description of $HH_0(\mathcal{H}^{\mathrm{Weil}})$.

One approach is to construct a trace functor $\mathcal{H}^{\mathrm{Weil}} \to \mathcal{D}^b(\frac{G}{G})^{\mathrm{Weil}}$ whose image is the Hochschild homology $HH_0(\mathcal{H}^{\mathrm{Weil}})$ of $\mathcal{H}^{\mathrm{Weil}}$. In characteristic zero, this approach identifies the Hochschild homology of the Hecke category with the category of character sheaves, i.e. the image of $\mathcal{H}$ in $\mathcal{D}^b(\frac{G}{G})$ under the horocycle correspondence. This research direction is closely related to Zhu's recent categorical trace constructions [Z], and Gaitsgory's explanations on how to recover Deligne–Lusztig representations and characters via higher traces [G1].

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