THE PARABLE OF THE BOOKMAKER

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These notes are based on the chapter of the same name in *Financial Calculus* by Baxter and Rennie (Cambridge University Press, 1996).

1. Definition of odds

Suppose we have a race between horse A and horse B. A bookie is giving $n-m$ odds against horse A and $m-n$ odds against horse B. This means that:

- If someone bets \$1 on horse A and horse A wins, then the bookie pays them $\$(1+\frac{n}{m})$. So the person gets $\$\frac{n}{m}$ of return.
- If someone bets \$1 on horse A and horse B wins, then the bookie keeps the \$1. So the person looses \$1 .
- If someone bets \$1 on horse B and horse B wins, then the bookie pays them $\$(1+\frac{m}{n})$. So the person gets $\$\frac{m}{n}$ of return.
- If someone bets \$1 on horse B and horse A wins, then the bookie keeps the \$1. So the person looses \$1.

We observe that it is only the ratio $r = n/m$ that is relevant. (In practice the odds against the two horses may not be exact reciprocals; see below.)

2. Setting the odds

Question. Suppose the bookie has received $\$x$ and $\$y$ in bets on horse A and horse B, respectively. How should the bookie set the odds?

Probabilistic approach. Let p be the probability that horse A wins. Then the bookie's expected earnings as a function of the ratio $r = n/m$ are:

$$E(r) = x + y - px\,(1+r) - (1-p)y\left(1+\frac{1}{r}\right)$$

Hence, the expected earnings are positive when r lies between $\min((1-p)/p, y/x)$ and $\max((1-p)/p, y/x)$. If the bookie happens to know the probability p, then the bookie can choose $r = n/m$ appropriately in the interval $\min((1-p)/p, y/x)$ and $\max((1-p)/p, y/x)$ and make a profit.

Risk-free approach. On the other hand, suppose the bookie has no opinion about the probability p of horse A winning. Then the bookie may simply choose $r = n/m$ to be equal to y/x. There is no risk of loss in this case. This method requires no information about the actual probability p that horse A wins.

Practical comments. Often, the bookie needs to set the odds of the race before collecting the bets, i.e., the amounts x and y may depend on the values n and m, making the argument given above somewhat artificial. On the other hand, if the bookie manages to make the ratio r equal to y/x, then the bookie may make a profit from fees or fact that the odds for the two horses are not exact reciprocals of each other.

3. Relation to option pricing

Suppose a stock is valued at \$100. Suppose further that it is known that, in the next one day, the stock price will go up to \$110 with probability 0.7 and will fall to \$90 with probability 0.3. Given this information, what is the fair price of a one-day call option struck at \$100?

Probabilistic approach. One may say that the price of the call option should be \$7, which is the expected earnings of the option:

$$\max(0, 110 - 100) * 0.7 + \max(0, 90 - 100) * 0.3 = 7$$

Risk-free approach. A different approach involves replication. We can replicate the short call position by selling half a unit of the stock and holding \$45 in cash (assume the interest rate is zero). Indeed, if the stock rises to \$110, the value of this portfolio will be $-\$10$; if the stock falls to \$90, the value of this portfolio will be zero[1]. The fair price of the call option is therefore the current value of the replicating portfolio, namely $\frac{\$100}{2} - \$45 = \$5$.

Comments. Observe that the risk-free approach guarantees that there will be no loss, and does not depend on the probabilities of rising or falling. In general, suppose the current stock price is S_0, and the possible states at time $T = 1$ are S_A and S_B, where $S_A < S_0 < S_B$. Then, for $S_A < K < S_B$, the risk-neutral price of a call option with strike K and expiry $T = 1$ is:

$$\text{price of } K\text{-call} = \frac{(S_B - K)(S_0 - S_A)}{S_B - S_A}$$

Indeed, this is obtained by first obtaining a replicating portfolio by solving the system of equations:

$$\begin{cases} xS_A + y &= 0 \\ xS_B + y &= S_B - K \end{cases}$$

for x and y, and then substituting the solutions into $xS_0 + y$.

[1]One obtains the replicating portfolio by solving the pair of equations $110x + y = -10$ and $90x + y = 0$.