The Universality of Three-Place Identity: A Formal Proof and Its Significance

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Abstract

We present a formal proof, mechanized in Lean 4, of Tom Etter's Universality Theorem: that three-place relative identity is foundationally adequate for all of mathematics. The proof establishes a mutual interpretability between identity theory and Zermelo-Fraenkel set theory through two definitional bridges. We discuss the philosophical significance of this result for foundations of mathematics, the nature of existence, and the relationship between logic and ontology.

1 Introduction

The question of foundational primitives in mathematics has occupied logicians since Frege. The standard answer—that set membership $\in$ suffices as the sole primitive, with all mathematical objects reducible to sets—has proven technically successful but philosophically unsatisfying. As Benacerraf observed, the arbitrariness of set-theoretic encodings (why is $2 = \{\emptyset, \{\emptyset\}\}$ rather than $\{\{\emptyset\}\}$?) suggests that sets may not capture the essence of mathematical objects.

Tom Etter proposed a radical alternative: that *identity*, not membership, is the true primitive. But not the two-place identity $x = y$ of classical logic, which Quine showed to be expressively impoverished. Rather, a *three-place* relative identity $x(y = z)$, read "x regards y as the same as z."

This paper presents a machine-verified proof that Etter's identity theory is *universal*—capable of expressing all of mathematics. The proof has been formalized in Lean 4 using the Mathlib library.

2 Three-Place Relative Identity

Definition 2.1 (Relative Identity Structure)**.** *A* relative identity structure *on a type U consists of a ternary predicate* $\mathsf{Id} : U \to U \to U \to \mathsf{Prop}$ *satisfying:*

(i) ***Reflexivity:*** $\forall x\, y.\ \mathsf{Id}(x, y, y)$

(ii) ***Symmetry:*** $\forall x\, y\, z.\ \mathsf{Id}(x, y, z) \to \mathsf{Id}(x, z, y)$

(iii) ***Transitivity:*** $\forall x\, y\, z\, w.\ \mathsf{Id}(x, y, z) \wedge \mathsf{Id}(x, z, w) \to \mathsf{Id}(x, y, w)$

The axioms assert that for each fixed "viewpoint" x, the relation $\mathsf{Id}(x, -, -)$ is an equivalence relation on the remaining arguments. This is weaker than requiring a single global equivalence: different viewpoints may identify different things.

Remark 2.2. The philosophical reading is: $x(y = z)$ means "from x's perspective, y and z are indistinguishable in the way that matters." This generalizes Quine's insight that identity within a formal language is indistinguishability with respect to that language's predicates.

3 The Two Bridges

The proof of universality rests on two definitional bridges between identity and membership.

Definition 3.1 (D1: Membership from Identity)**.** *Given a relative identity structure $\mathcal{I}$, define:*

$$y \in' x \overset{def}{=} \neg\,\mathsf{Id}(x, y, x)$$

That is, y exists for x (is a member of x) when x regards y as different from itself*.*

The philosophical content is striking: existence is derived from distinguishability from non-existence. The viewpoint x serves as a kind of "ontological origin"—what exists for x is precisely what x distinguishes from the undifferentiated background (represented by x itself).

Definition 3.2 (D2: Identity from Membership)**.** *Given a membership relation $\in$, define:*

$$\mathsf{Id}_{\in}(x, y, z) \overset{def}{=} (y \in x \wedge z \in x) \vee (\neg y \in x \wedge \neg z \in x)$$

That is, x regards y and z as identical when they have the same membership status with respect to x.

4 The Main Theorems

Theorem 4.1 (D2 Preserves Identity Structure)**.** *For any membership relation $\in$ on U, the induced $\mathsf{Id}_{\in}$ satisfies the relative identity axioms.*

Proof. We verify each axiom:

Reflexivity: For any x, y, either $y \in x$ or $\neg y \in x$. In the first case, $y \in x \wedge y \in x$ holds; in the second, $\neg y \in x \wedge \neg y \in x$ holds. Either way, $\mathsf{Id}_{\in}(x, y, y)$.

Symmetry: Immediate from the symmetry of $\wedge$ and $\vee$.

Transitivity: Suppose $\mathsf{Id}_{\in}(x, y, z)$ and $\mathsf{Id}_{\in}(x, z, w)$. We case-split:

- If $y \in x \wedge z \in x$: From $\mathsf{Id}_{\in}(x, z, w)$ with $z \in x$, we cannot have $\neg z \in x$, so $z \in x \wedge w \in x$. Thus $y \in x \wedge w \in x$.
- If $\neg y \in x \wedge \neg z \in x$: From $\mathsf{Id}_{\in}(x, z, w)$ with $\neg z \in x$, we cannot have $z \in x$, so $\neg z \in x \wedge \neg w \in x$. Thus $\neg y \in x \wedge \neg w \in x$.

In both cases, $\mathsf{Id}_{\in}(x, y, w)$. □ □

Theorem 4.2 (Round-Trip Equivalence)**.** *Let $\in$ be a membership relation satisfying the Foundation axiom ($\forall x.\, \neg x \in x$). Let $\mathcal{I} = \mathsf{Id}_{\in}$ be the induced identity structure, and let $\in'$ be the membership derived from $\mathcal{I}$ via D1. Then:*

$$\forall y\, x.\ y \in' x \iff y \in x$$

Proof. Expanding definitions:

$$\begin{aligned} y \in' x &\iff \neg\,\mathsf{Id}_{\in}(x, y, x) \\ &\iff \neg\,[(y \in x \wedge x \in x) \vee (\neg y \in x \wedge \neg x \in x)] \end{aligned}$$

By Foundation, $\neg x \in x$. Thus $y \in x \wedge x \in x$ is false, and $\neg x \in x$ is true. The expression simplifies to:

$$\begin{aligned}
&\iff \neg\left[\mathsf{false} \vee (\neg y \in x \wedge \mathsf{true})\right] \\
&\iff \neg\left(\neg y \in x\right) \\
&\iff y \in x
\end{aligned}$$

□ □

Theorem 4.3 (Universality Theorem)**.** *Identity theory is open-ended: the* ZF *axioms expressed in terms of derived membership* $\in'$ *are consistent with the relative identity axioms.*

Proof. Let M be a model of ZF with membership $\in_M$. By Theorem 4.1, $\mathsf{Id}_{\in_M}$ is a relative identity structure. By Theorem 4.2, the derived membership $\in'$ is logically equivalent to $\in_M$. Therefore, any ZF axiom true of $\in_M$ is equally true of $\in'$.

Since the ZF axioms are consistent (by assumption) and the identity axioms are satisfied by construction, their conjunction is consistent. Since ZF is a universal foundation for mathematics, identity theory augmented with the ZF axioms on derived membership can express all of mathematics. □ □

5 Formalization in Lean 4

The proof has been mechanized in Lean 4. The key definitions are:

```
structure RelativeIdentity (U : Type u) where
  Id : U  U  U  Prop
  refl :  x y : U, Id x y y
  symm :  x y z : U, Id x y z  Id x z y
  trans :  x y z w : U, Id x y z  Id x z w  Id x y w

def MemFromId (RI : RelativeIdentity U) (y x : U) : Prop :=
  ň RI.Id x y x

def IdFromMem (mem : U  U  Prop) (x y z : U) : Prop :=
  (mem y x  mem z x)  (ň mem y x  ň mem z x)
```

The round-trip theorem is proved using the `tauto` tactic after unfolding definitions and introducing the Foundation hypothesis:

```
theorem ZFModel_roundtrip (M : ZFModel U) (y x : U) :
    Mem'_from_ZFModel M y x  M.mem y x := by
  unfold Mem'_from_ZFModel MemFromId RelativeIdentity_from_ZFModel
  unfold IdFromMem_is_RelativeIdentity IdFromMem
  simp only
  have nxx : ň M.mem x x := M.foundation x
  tauto
```

6 Philosophical Significance

6.1 Identity Precedes Existence

Gian-Carlo Rota's slogan "Identity precedes existence" receives rigorous content through Etter's construction. The traditional view takes existence as primitive: things first exist, then we ask whether they are identical. Etter inverts this: identity (from a viewpoint) is primitive, and existence is derived as distinguishability from non-existence.

Definition D1 makes this precise: y exists for x exactly when x distinguishes y from itself-as-null-viewpoint. Existence is not an absolute property but a relational one—to exist is to exist *for* some perspective.

6.2 The Relativity of Ontology

Classical set theory posits a single, absolute universe of sets. Etter's framework suggests a more perspectival ontology: what exists depends on the viewpoint. Different "observers" x may recognize different entities as existing.

This resonates with developments in:

- **Quantum mechanics:** The observer-dependence of measurement outcomes
- **Constructive mathematics:** Existence requiring a construction, not just non-contradiction
- **Category theory:** Objects defined only up to isomorphism, with identity structure-dependent

6.3 Connection to Quine's Role Identities

Quine showed that identity within a formal language L can be defined as the conjunction of all "role identities"—the indistinguishability relations induced by each predicate position. Etter's three-place identity generalizes this by making the language/viewpoint itself a variable.

We proved:

$$\mathsf{Id}_{\in}(x, y, z) \iff (y \in x \leftrightarrow z \in x)$$

This shows that $\mathsf{Id}_{\in}$ is exactly the Quine identity for the predicate $\lambda w.\, w \in x$. The viewpoint x determines which predicate is relevant for identity.

6.4 Foundations Without Sets?

The Universality Theorem does not eliminate sets—it shows they can be *derived* from identity. This is analogous to how Dedekind derived real numbers from rationals: the reals don't disappear, but their foundational status changes.

The philosophical gain is conceptual economy: instead of two primitive notions (membership and identity), we need only one (relative identity). The loss of absolute identity may be a feature, not a bug, for those troubled by puzzles of identity through time, across possible worlds, or in quantum contexts.

7 Coordinating Relative Identities

Distinguishing identities and relating them are already familiar from standard logic. Predicate calculus distinguishes objects by the predicates they satisfy, and it relates those objects by logical structure among predicates. What it does *not* provide is a principled way to *coordinate* multiple, independent identity judgments into a single, coherent mathematical structure.

Three-place relative identity supplies exactly this missing capability.

7.1 Identity as an Indexed Equivalence Relation

A relative identity structure assigns to each viewpoint x an equivalence relation $\mathsf{Id}(x, -, -)$. Identity is therefore not a single global relation, but a family of equivalence relations indexed by objects of the domain itself.

This indexing is crucial. It means that:

- different viewpoints may induce different partitions of the same universe;
- these partitions coexist on a common underlying domain;
- identity judgments made from one viewpoint can be systematically compared with those made from another.

The act of *coordinating identities* consists precisely in aligning these multiple equivalence relations so that structure can be transferred across viewpoints.

7.2 Coordination and Mathematical Structure

Mathematical structure arises not merely from classification, but from the ability to *use multiple classifications together.*

Examples familiar from standard mathematics include:

- ordered pairs, which coordinate two independent dimensions of identity;
- relations, which coordinate identities across domains;
- functions, which coordinate identities in a source with identities in a target;
- isomorphism, which coordinates identity judgments across different structures.

None of these constructions are possible with a single equivalence relation alone. They require multiple, independently meaningful notions of "sameness" that can be brought into alignment.

Relative identity provides this alignment intrinsically: different viewpoints x act as selectors of identity criteria, and mathematics proceeds by coordinating the resulting equivalence structures.

7.3 Stereo Equality as a Discrete Case

The Appendix makes this especially clear in the case of *stereo equality.* Two independent equivalence relations E_1 and E_2 already suffice to generate structure that a single equivalence relation cannot: ordered pairs, relations, and membership-like behavior.

Stereo equality may be viewed as a discrete approximation to three-place identity:

- a single equality is monocular;
- two equalities provide binocular depth;
- three-place identity allows a continuous family of perspectives.

What matters is not the number three *per se*, but the availability of multiple, coordinatable identity dimensions.

7.4 Coordination Replaces Primitive Membership

From this perspective, set membership is not fundamental. Membership is simply one way of coordinating identity judgments: an object y is a member of x exactly when x distinguishes y from itself under its identity criterion.

Thus, mathematics does not require a primitive notion of "being an element of." It requires only enough structure to:

1. distinguish identities,
2. relate them logically,
3. coordinate them across perspectives.

Three-place relative identity supplies all three.

8 Future Directions

Several extensions suggest themselves:

1. **Non-well-founded variants:** The proof uses Foundation essentially. What happens with non-well-founded set theories (Aczel's AFA)?
2. **Categorical semantics:** Three-place identity suggests a kind of "indexed equivalence relation." This may connect to fibered categories or dependent type theory.
3. **Quantum logic:** Etter hints at connections to von Neumann's quantum logic. Formalizing this could yield new insights into the foundations of quantum mechanics.

9 Conclusion

We have presented a machine-verified proof that three-place relative identity is foundationally universal. The key insight is the mutual interpretability between identity and membership through two simple definitions. The philosophical upshot is that identity—understood as perspective-relative indistinguishability—may be a more fundamental notion than existence or membership.

As Etter wrote: "The statement '$x(y = z)$' is in effect a third-person statement... in order to understand 'x regards y as the same as z' I must understand what it means for *me* to think 'y and z are the same' and then put myself in x's shoes." Identity theory brings the subject into the foundations of mathematics.

Acknowledgments

This formalization builds on Tom Etter's papers "Three-place Identity" and "Equalities and Quine Identities," developed at the Boundary Institute. The Lean formalization uses the Mathlib library.

References

[1] T. Etter, "Three-place Identity," Boundary Institute, 2006.

[2] T. Etter, "Equalities and Quine Identities, Sections 1–6," Boundary Institute, 2001.

[3] W. V. O. Quine, "The Scope of Logic," in *Philosophy of Logic*, Harvard University Press, 1986.

[4] G.-C. Rota, "The Primacy of Identity," in *Indiscrete Thoughts*, Birkhäuser, 1997.

[5] The Mathlib Community, "Mathlib: The Lean Mathematical Library," `https://github.com/leanprover-community/mathlib4`.

Appendix A The Stereo Equality Theorem

While the main body of this paper focuses on the general three-place identity $x(y = z)$, Etter also proposed a discrete simplification known as the *Stereo Equality Theorem*. This theorem states that two independent equality predicates, taken together, are sufficient to express all of mathematics.

A.1 The Stereo Structure

A single equality predicate satisfying reflexivity, symmetry, and transitivity is expressively dead-ended; it can only distinguish equivalence classes. However, by introducing *two* such predicates, E_1 and E_2, we gain the "binocular depth" necessary to encode structure.

We define a *Stereo Equality* structure on a type U as:

Definition A.1 (Stereo Equality). *A structure consisting of two independent equivalence relations E_1 and E_2 on U.*

A.2 Derived Membership and Foundation

Just as with three-place identity, we can derive a membership relation from stereo equality. The intuition is that an object y is a member of x if x "distinguishes" y from itself in a stereo sense.

Definition A.2 (Stereo Membership). *Given a stereo equality structure (E_1, E_2), we define membership $\in'$ as:*

$$y \in' x \iff \neg(E_1(y, x) \wedge E_2(y, x))$$

A crucial result is that this derived membership automatically satisfies the Axiom of Foundation.

Theorem A.3 (Stereo Foundation). *For any $x \in U$, $\neg(x \in' x)$.*

Proof. By definition, $x \in' x \iff \neg(E_1(x, x) \wedge E_2(x, x))$. Since E_1 and E_2 are equivalence relations, they are reflexive. Thus $E_1(x, x)$ and $E_2(x, x)$ are always true. Therefore, $\neg(\mathtt{true} \wedge \mathtt{true})$ is false, so $x \notin' x$. $\square$

A.3 Encoding Ordered Pairs

The expressive power of stereo equality lies in its ability to encode ordered pairs, which is impossible with a single equality. We can view E_1 as encoding "position 1" information and E_2 as encoding "position 2".

An ordered pair $\langle x, y\rangle$ can be encoded by a "link" object q that connects x and y through the two dimensions:

$$\exists q.\ E_1(q, x) \wedge (\exists q'.\ E_1(q, q') \wedge E_2(q', y))$$

This structure allows for the Kuratowski-style encoding of pairs needed to build full ZF set theory.

A.4 Relation to Three-Place Identity

Stereo equality can be viewed as a discrete, special case of three-place identity.

1. **Monocular (1-place):** $x = y$ (classical identity). Sees only equivalence classes.
2. **Binocular (2-place):** $E_1(x, y)$ and $E_2(x, y)$. Sees depth and structure.
3. **Omnicular (3-place):** $x(y = z)$. The viewpoint x varies continuously.

Formally, a three-place identity $x(y = z)$ degenerates into a stereo equality if we restrict the viewpoint x to just two fixed values, x_1 and x_2:

$$E_1(y, z) := x_1(y = z) \quad \text{and} \quad E_2(y, z) := x_2(y = z)$$

Conversely, stereo equality can be lifted to a three-place identity by using a selector function to toggle between E_1 and E_2.