Exercise 2.2-3

Jack Maney

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Assuming that we're searching a list l for an element v, and assume that the probability of any element in l equaling v is $\frac{1}{n}$ (where l has n elements). Let X be the random variable denoting the number of elements of l that needs to be searched to find v via linear search. Then, we have

$$E(X) = \sum_{i=1}^{n} iP(X = i) = 1 \cdot \frac{1}{n} + \sum_{i=2}^{n} iP(\text{the first } i-1 \text{ elements of } l \text{ are not } v)$$

$$= \frac{1}{n} + \sum_{i=2}^{n} i\frac{1}{n}\left(1 - \frac{1}{n}\right)^{i-1} = \frac{1}{n}\left(\sum_{i=1}^{n} i\left(1 - \frac{1}{n}\right)^{i-1}\right).$$

Now, consider the polynomial $f(x) = \sum_{i=1}^{n} i(1-x)^{i-1}$. We have,

$$\int f(x)dx = C - \sum_{i=1}^{n}(1-x)^i = C + 1 - \sum_{i=0}^{n}(1-x)^i$$

$$= C + 1 - \frac{1-(1-x)^{n+1}}{1-(1-x)} = C + 1 - \frac{1-(1-x)^{n+1}}{x}$$

(for $x \neq 0$, obviously).

Thus,

$$f(x) = \frac{d}{dx}\left(1 - \frac{1-(1-x)^{n+1}}{x}\right) = \frac{(n+1)x(1-x)^n + (1-x)^{n+1} - 1}{x^2}$$

So, letting $x = \frac{1}{n}$, we have

$$E(X) = \frac{1}{n}\left(\frac{(n+1)(1/n)(1-1/n)^n + (1-1/n)^{n+1} - 1}{1/n^2}\right)$$

$$= n\left(\frac{n+1}{n}\left(1 - \frac{1}{n}\right)^n + \left(1 - \frac{1}{n}\right)^{n+1} - 1\right) = \left(1 - \frac{1}{n}\right)^n (n + 1 + n - 1 - n)$$

$$= \left(1 - \frac{1}{n}\right)^n n$$

Note that this is asymptotic to $\frac{n}{e}$.

Of course, the best case is when v is the first element of l, in which case our running time is $\Theta(1)$. Our worst case scenario is when v is either the last element of l or is not present in l (i.e. we have to search all of l to find v), in which case our running time is $\Theta(n)$.