How does Modern Portfolio Theory work?

Mathematics: Analysis and Approaches SL

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Introduction

Rationale

Finance and investment is a classic game of risk versus reward. The thrill of massive gains is only exceeded by the pains of disastrous losses. Mathematicians and statisticians alike have long attempted to “solve” the securities market by maximizing returns and minimizing risk. An early attempt at such a solution was Modern Capital Theory, more commonly known as Modern Portfolio Theory (MPT). MPT was developed in 1952 by Harry Markowitz, an American economist. Over time, his theory has gained followers as well as detractors. In order to form an opinion on MPT, I had to first understand what MPT is and how it works. Since finance and making money is of interest to everyone, there are many beginner financial resources online which attempt to explain MPT. Unfortunately, most of these resources are terrible, verbose, and mathematically incorrect. These suffice for the average individual who wants a quick and dirty answer, but as someone who wants to actually understand and critically think about everything I learn, almost to an obsession, this wasn’t enough. The only other option was to delve into academic papers, notably the original paper outlining MPT written by Markowitz. I figured I’d try to learn what all the fuss was about regarding MPT and truly understand it so that I can to form an educated opinion.

I have long had a fascination with finance and financial markets. I have written finance research papers in DECA, a business role-play club. These papers involve financial research about quantitative data as well as socio-economic factors. I have participated in numerous financial market games and competitions. In these, I primarily used a value investing approach to construct my portfolios, where I evaluated a company’s financial balance sheets and fundamentals. I was fairly lazy with portfolio weightings; I usually just divided the weightings evenly. A possibility for self-improvement would be to minimize risk and maximize returns using MPT. In this exploration, I intend to define and explain the math behind MPT. The mathematics involved will include optimization, variance, the expectation value and covariance. I include a sample calculation with one of my stock market game portfolios to see if and how I could have improved.

Assumptions

An assumption of MPT is that investors are rational and risk averse (i.e. they want consistent returns). The return is the change in value of the portfolio in a given amount of time; positive returns indicate an increase in value, and negative returns indicate a decrease in value. Given the same rate of return, it is assumed that it would be preferable to have a portfolio with a lower variance (small swings in portfolio value) versus a portfolio with higher variance (large swings in portfolio value). This assumption is the main basis behind MPT, as it is saying that humans are not robots. I have firsthand experience with this, as in my real life portfolio I had a small Bitcoin short position (betting against the value of Bitcoin) as well as some stock in a mining company. Overall, they eventually made about the same return, however the short Bitcoin position had significantly larger swings in value and I would have had a lot fewer sleepless nights if I had just bought more of the mining stock instead of the higher variance bitcoin position. This was one of the main achievements of MPT; helping determine the lowest risk possible in a given portfolio.

MPT assumes that asset returns follow a normal distribution, which means that the majority of returns will be close to the mean, with fewer returns at the extreme ends of the distribution. This never occurs in real life, especially in turbulent markets caused by economic or geopolitical factors, such as inflation, trade tariffs, and wars, none of which MPT takes into account.

Return

The total portfolio return equation.

The whole purpose of MPT is to create a relation between the expected value of the return of a portfolio and the variance of that value. We will then use optimization to find the minimum variance and the maximum expectation value. We will start by finding the return of a portfolio. A portfolio is a group of investments into financial securities. Securities are stocks, bonds, and real estate, and other similar investments. Securities are priced in a process similar to auctions; the price goes up if more people want to purchase the security, and the price goes down if there are more people who want to sell it.

N represents the number of securities in our portfolio. *t* represents the time invested per dollar and *i* represents a given security in the portfolio. Thus, r_{it} is the expected return of security *i* per dollar invested *i*. d_{it} represents the rate of return of *i* at time *t* using the time value of money. The time value of money is the concept that money today is worth more than money in the future, because of the potential earning power of money over time. Finally, X_i represents the amount of security *i* in the portfolio relative to the total amount in the portfolio (i.e., if half of our portfolio consisted of security *i,* then $X_i = 0.5$). X_i must be a value greater than zero, as we do not consider short sales, which is selling securities that are not owned by the investor as a bet against their value. To find the time-based return of a security, we can use the following equation.

$$R_i = \sum_{t=1}^{\infty} d_{it} * r_{it}$$

The summand (the term being added repeatedly) in this equation is just the expected return multiplied by the time-based rate of return. $d_{it} * r_{it}$ gives the rate of return per unit time. Notice that in this case time becomes discrete. *Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case* by *Merton (1969)* demonstrates the case of a continuous time scenario, which is an obvious improvement over MPT as in reality time is continuous. However, for a conceptual understanding of optimization in portfolio selection, the case of discrete time suffices. We then sum the summand over the period of time. Using equation for the return of one security, we can find the total return of the portfolio, or R_{total}.

$$R_{total} = \sum_{t=1}^{\infty} \sum_{i=1}^{N} d_{it} * r_{it} * X_i$$

Sum of sum is solved inside out. The first summation is similar to the first equation in that we are still multiplying d_{it} and r_{it}. As this is the instantaneous return, we just multiply it by it's weighting, which will give us the instantaneous weighted return. The summation is all the *i* values up to *N,* as we have *N* securities. This gives us the instantaneous return of the portfolio, and then by summing it up over time, we get the time-based return of the portfolio. As addition and multiplication are commutative, we can rewrite that as the following.

$$R_{total} = \sum_{i=1}^{N} [X_i(\sum_{t=1}^{\infty} d_{it} * r_{it})]$$

We are factoring out X_i, as it is not changing with respect to *t*. The summation within the brackets then becomes equivalent to R_i, so we can substitute $\sum_{t=1}^{\infty} d_{it} * r_{it}$ with R_i.

$$R_{total} = \sum_{i=1}^{N} R_i * X_i$$

This equation is saying that the sum of all the individual returns of each security will add up to the total return of the portfolio. This implies that X_i and R_i are independent of each other, which in the real world is true for the most part. As security prices move based on the level of demand, if X_i was high and the portfolio was large (in the hundreds of millions or billions of dollars), this will have an artificial effect on R_i. Unfortunately, I do not face this problem. Since all the X_i values are positive, and $\sum X_i = 1$, X_i is acting as a weighted average of R_i. Evidently, we will have some *i* values with higher R_i values and *i* values with lower R_i values. If it was that simple, then we should use the *i* value(s) with the largest R_i *values,* and ignore all the other *i* values (i.e. have a single security portfolio with the highest yield as that cannot be improved upon). Having a single security portfolio has a large risk value attached, and there is a level of variance, or uncertainty surrounding the R_i values that must be considered, hence the use of MPT.

Expectation value

As we do not know for certain what the R_i values are, we should replace that with the expectation value, $E(R_i)$. The expectation value is the expected, or most likely, value based on its probability distribution. For example, if we had an R_i value from a set named set 1, then R_i is just a value from set 1, but $E(R_1)$ is the most likely or expected R value from the set. Since the expectation value of a weighted sum is the sum of the expectation values, $E(R)$, can be expressed as follows, where X_i is the weight of $E(R_i)$.

$$E(R_{total}) = X_1E(R_1) + X_2E(R_2) \dots$$

That is all there is to finding the expected value of the return of a portfolio in MPT! The difficult part is determining $E(R_i)$. There are many mathematical approaches to this, such as using Brownian motion, Bayesian mathematics, and many others, however those go far beyond the scope of this exploration. In this exploration we will be using past results to attempt to model future returns. That is not a great way to invest, however all we need for a conceptual understanding is a ballpark figure, which past returns will provide.

Variance

The variance of a weighted sum is not so simple to find since the R_i values are correlated. We must first find the covariance of each pair of securities. The covariance of a pair of values can be expressed as the following. σ can indicate either variance or covariance, depending on the subscript (i.e., if it's one term, it is variance; if there's 2 terms, it is covariance).

$$\sigma_{R1\ and\ R2} = E\{[R_1 - E(R_1 - E(R_1)]\}[R_2 - E(R_2)]\}$$

The covariance of the values is how far the first R value is from its mean multiplied by how far the second R value is from its mean. This determines the correlation between the two sets. If the covariance is positive, then the two securities tend to move in the same direction; if it were opposite then they would tend to move in opposite directions. If it were zero, they would be independent. If we use the covariance to determine the correlation, then we can use the correlation to find the covariance. ρ is the standard symbol for the correlation coefficient.

$$\sigma_{R1\ and\ R2} = \sigma_{R1}\sigma_{R2}\rho_{R1\ and\ R2}$$

Determining the variance of a portfolio

This is the equation for the variance of a weighted sum. *i* and *j* are two different terms of the sum.

$$V = \sum_{i=1}^{N} X_i^2 V(X_i) + 2\sum_{i=1}^{N}\sum_{j>1}^{N} X_i X_j \sigma_{ij}$$

This appears complicated, but upon further inspection can be broken down quite simply. The summation on the left is the sum of the variance of each term. Squaring the weights makes the larger weights even larger relative to the smaller weights, which when squared increase less than the larger weights. The larger weights become more important in contributing to the spread of the weighted sum. Squaring also ensures that the resulting variance is always positive. A negative weight would imply that the variable contributes negatively to the spread of the weighted sum, which is incorrect in the context of variance. The summand on the right is the weighted covariance of a pair of terms. We take into account every term due to the fact that *i* starts at 1 and *j* is another non-*i* term. We multiply the term on the right by 2 because we are only taking into account the covariance of *i* on *j.* As the covariance of *i* on *j* is the same as the covariance of *j* on *i,* we need to multiply by 2 to take this into account. To further simplify the equation, we can take the case where *i=j=1*, where that will give us $V(R_i)= \sigma_{ii}$. This is not the covariance, despite having 2 letters, as they are the same term. The covariance of one term on itself is just the variance of that term, as the correlation is 1. In this case, we will get the variance of *R* to be simply the slightly modified second term in the first equation.

$$V(R) = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i \sigma_{ij}$$

Thus, the variance of a portfolio is the sum of the weighted covariances. Now we have all we need to create a relation between the expectation value and the variance of a portfolio, which was the entire point of MPT.

MPT uses variance as it’s measure of risk. While this is a somewhat accurate reflection of a portfolio’s volatility, or fluctuation in price (i.e. high volatility means large price fluctuations), it is not a good measure of risk. If the price of a security constantly kept going up, then it would have a high variance, but low risk for loss. An improvement upon MPT is the Post-Modern Portfolio Theory, which uses downside risk, tail risk, and liquidity risk to model more realistic risk factors in the real world. Downside risk only takes into account the risk of prices decreasing, without taking into account price increases. Tail risk is the risk of extreme or rare events that occur beyond what is predicted by standard statistical models, also known as black swan events. Liquidity risk refers to the risk of not being able to buy or sell an asset quickly enough without causing a significant change in its price. These reflect real world conditions and are better measures of risk than the variance used by MPT.

Graphing

We now have a set of E and V values for each set of X_i values that we have. As stated previously, we want the highest return possible for the lowest risk possible. Stated mathematically, we want the highest E value for the lowest V value. This is simply an approximation created on desmos to easily see the efficient frontier. In reality, a given portfolio the relation would not be a circle but rather an ellipse. As well, the E values will be positive as they have a positive rate of return (unless for some reason we project it to be negative) and all the V values must be positive as the variance is always a positive number. This can be expressed graphically as follows.

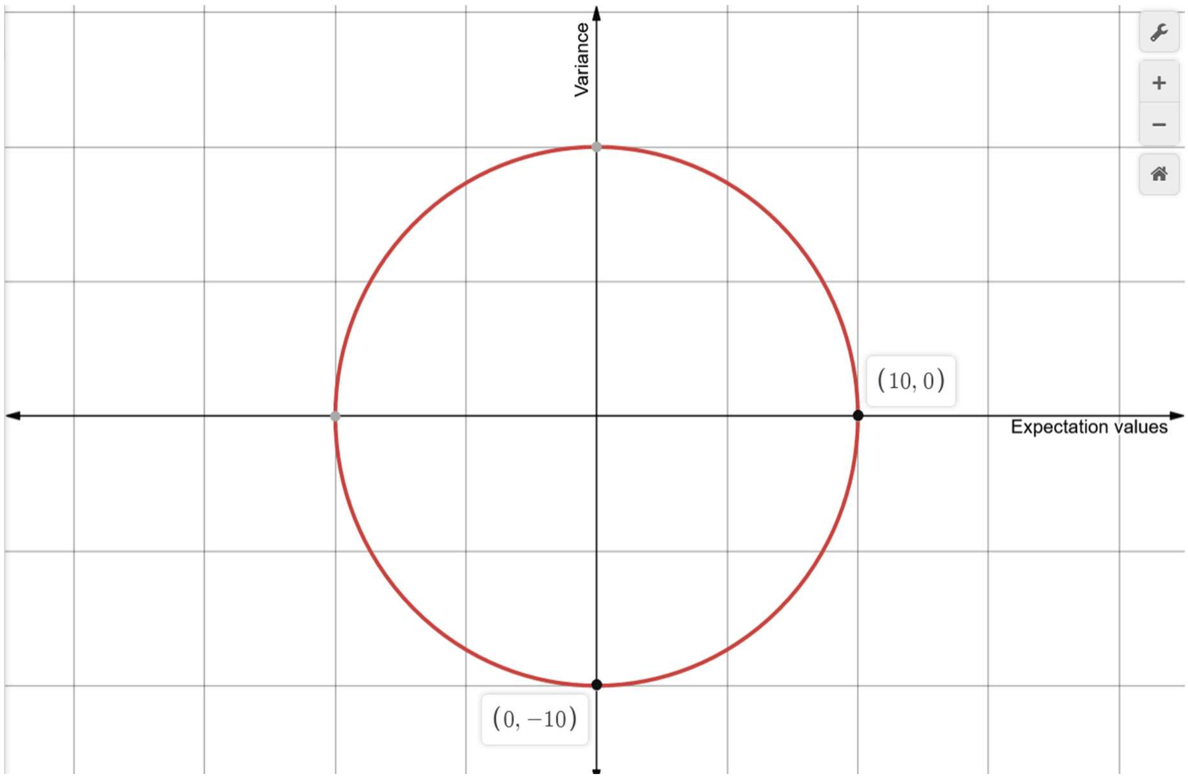

Figure 1 A sample (E,V) plot created on desmos.com.

Any point within this circle is a possible *(E,V)* combination for a portfolio with the same covariance and correlation values. Any point on the red line represents either an extreme *E* or *V* value or both. What we are looking for is the "efficient frontier", or the highest *E* value for the lowest *V* value possible. In this case, it is all the points on the red line in quadrant IV, or on the red line from the point (0, -10), to (10,0). Any point left of that line (i.e. with the same y value) will have a lower expectation value for the same variance and any point above that line (i.e.. with the same x value) will have a higher variance for the same expectation value. Of course, any point diagonally away from the efficient frontier will become less desirable for both of those reasons.

We can clearly see that the point (10,0) offers the highest return. This does not preclude the points in between from being on the efficient frontier. Recall that we can much more accurately determine variance with less uncertainty than the expectation value, as the uncertainty of the expectation value *is the variance*. There are many real-life scenarios where one would prefer a lesser variance, most of them having to do with there being a potential withdrawal need at any time. If the expectation value dropped even to the lowest amount predicted by the variance (if not more, as there is nothing stopping a security price from dropping below a random number we came up with), and there was a forced withdrawal, the portfolio forced would be selling at a low price, which is the opposite of the ideal, buying at a low price and selling at a higher price.

Application

I decided to next apply this to one of my own portfolios. In the fall of 2022, I participated in a stock market simulator game run on the Wharton Investment Simulator (WInS). I adopted my typical value investing approach, purchasing what I believed to be undervalued companies. However, not knowing about MPT, I didn't do anything about portfolio weightings. Instead, I weighed all 7 of my stocks almost equally. This meant that I left performance on the table. To see what was possible, I created the efficient frontier for my portfolio. I started by creating a table of the annual return on Excel of each of the 7 companies since 2010 (Appendix A), sourced from yahoo finance. Somehow this ended up being one of the more difficult parts as I had to manually go into each graph and measure the percentage change from January 1st to January 1st the next year. I thought there was an easier way but I couldn't find it, making this an extremely painstaking process. The average annual return was calculated by multiplying all the returns (technically the returns plus 1 in order for the math to work) together to get the total amount relative to 1 dollar of principal invested. I then took the 13th root to find the annual compounded growth rate. I decided to use that as my *E(R)* value as it would is a fairly decent approximation and as long at it is somewhat consistent, it doesn't matter too much if they are all consistently wrong by similar amounts because that won't affect the weights too much. For the sake of simplicity and ease of understanding, past returns suffice.

The variance was the more difficult value to find. I first constructed a covariance matrix, which was simple enough to do on Excel as it had its own function. Doing matrix multiplication on Excel was also fairly straightforward, but the real roadblock came when I tried to use Excel's built in optimizer, Solver. It took me quite a few attempts for it to produce an actual number, but it cheated a little by either creating a single security portfolio which ended up with the highest return or lowest variance. It took a few other tries to work out the constraints and missing brackets, but I eventually found the lowest variance possible of my portfolio: 0.001099441. The return value that came with this variance was 0.086129827, or 8.6%. I decided to increase the return by increments of 1% until I reached the maximum of 18.04%, which was the single security portfolio consisting of just Costco. These weightings and values can be found in Appendix B. This produced a function that looked similar to an exponential.

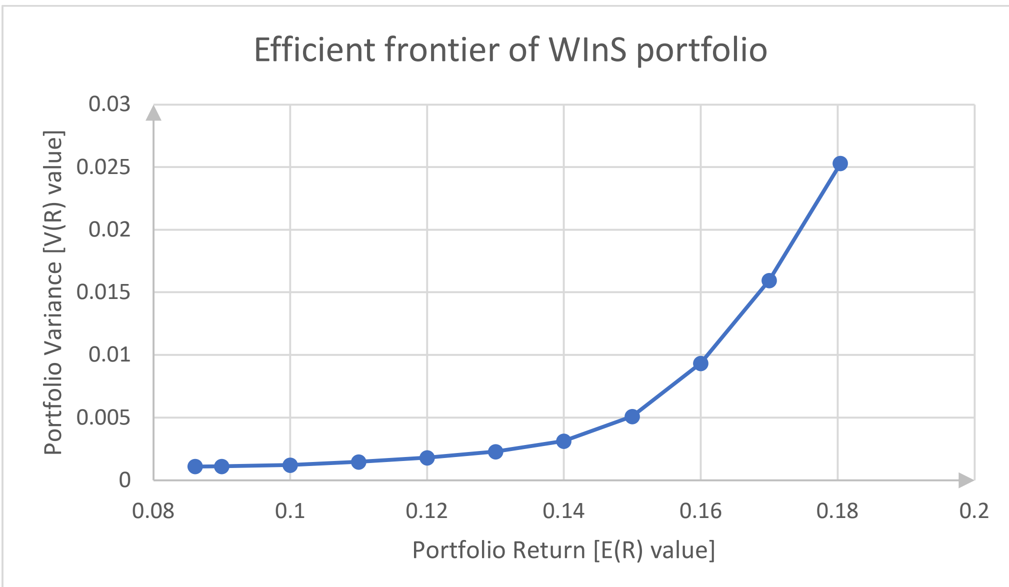

Figure 2 Efficient frontier of WInS portfolio created on Microsoft Excel

This is not an exponential, but one quarter of the perimeter of an ellipse. There was no point in attempting to find the rest of the possible *(E,V)* values, or even the others on the periphery, as they would not be as efficient as the ones on the efficient frontier. All the graphs that are found online of the efficient frontier show the inverse, with the return on the y-axis and the variance on the x-axis. I cannot seem to figure out why other than to call it "the efficient bullet" as the relation looks like a bullet. Personally, I find this much more intuitive and easier to read. We can see now that the minimum portfolio return would be about 8.6% per year, which comes with a very low variance, and the most we could get was 18.04%, albeit with a significantly higher variance. If I had a chance to redo that competition, I would have picked the portfolio with a 15% return as that is 4% higher than the 11% return I achieved in the competition, with a significantly lower variance. This was a very crude approximation, and there are zero guarantees that this actually works, however it does make portfolio selection fun.

Limitations

The most obvious limitation of MPT is the fact that there will always be a significant level of uncertainty surrounding the *E* and *V* values. There are mathematical methods of estimating these values, notably the Black-Scholes method of pricing options, however a reliance on these can lead to spectacular failure, such as in the case of Long Term Capital Management (LTCM). LTCM used a proprietary computer trading algorithm to make eye popping profits, but once LTCM started getting large negative returns, people started betting against LTCM itself, rather than valuing security prices independently, which eventually lead to their spectacular and almost-recession causing collapse. What LTCM did not and could not factor in was the human element. The securities market is a man-made creation that values securities accurately most of the time, but occasionally, they are subjected to the whims of man. The movement of security prices cannot be accurately predicted 100% of the time, which thus invalidates one of the key tenets of MPT, which is that we define the expectation value of a security. MPT can help us provide a good estimate but is not fail safe. By nature, this limitation cannot be overcome; the uncertainty can only be reduced but will never be zero. MPT assumes that the only source of risk in a portfolio is the variance of its returns. However, in reality, there are other sources of risk, such as credit risk, interest rate risk, and geopolitical risk. Incorporating these additional risk factors into the model could lead to more accurate portfolio optimization.

MPT cannot incorporate environmental, social, and governance (ESG) factors into the portfolio optimization process. ESG investing is basically investing as a social good, and purchasing securities of companies that are doing social good. There has been an increase in the relevance of ESG investing over the past few years with all of the world's issues, however as a math formula MPT does not allow the investor to take ESG investing into consideration.

MPT assumes that asset returns follow a normal distribution. However, in reality, asset returns often exhibit non-normal behavior, such as skewness and kurtosis. Skewness refers to the degree of asymmetry in the distribution of returns, while kurtosis refers to the degree of flatness in the distribution. A positive kurtosis value indicates that the ends of the distribution are more peaked and have more extreme values than a normal distribution. Accounting for these non-normalities could lead to more accurate estimates of risk and return. When asset returns exhibit non-normal behavior, MPT may not accurately capture the true risk and return characteristics of the assets. As a result, portfolios optimized using MPT may be suboptimal. To address this issue, researchers have developed alternative models such as the generalized autoregressive conditional heteroskedasticity (GARCH) model, which allows for more flexible and realistic distributions of asset returns. Using these models can lead to more accurate estimates of risk and return, which in turn can improve the efficiency of portfolio optimization.

Another limitation of MPT is that it is only perfectly correct at the moment it is created. As markets are dynamic and ever changing, the investor would be required to update their portfolio weightings repeatedly, a practical impossibility given the high level of fees and taxes that are associated with short term trading. MPT does not take into account trading fees and tax laws, as in the real world the more active a trader is the more costly their bills become.

Conclusion

I started this exploration attempting to break down the math behind MPT so that it became understandable. It nicely wraps everything we have learned so far in statistics and adds some spice with matrices and other non-syllabus material. I learned how to use MPT to create optimized portfolio weightings, which will help me achieve higher returns with lower variance in stock market competitions.

However, what I truly came away with was the understanding that despite humanity's best efforts, we still cannot solve financial markets. This is ironic, as we understand many natural phenomena, however financial markets, a man-made creation, still befuddle us. The sheer amount of assumptions and limitations made by MPT make it hard for it to actually be put into practice. There have been improvements to make it more accurate and less limited, but those new theories are still imperfect. As the case of LTCM shows, even 2 Nobel prize winners in economics can still spectacularly fail. Financial markets likely do have a solution, but we are unlikely to find it anytime soon. I eagerly await the solution and I hope it comes in my lifetime (maybe I'll be a part of it); until then, I will use MPT: an imperfect solution to an imperfect problem.

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Appendices

Appendix A – Stock prices for securities in the WInS portfolio.

	Citigroup	Merck	Shell	Enbridge	Toyota	US Steel	Costco
2010	42.90%	-1.40%	11.10%	22.03%	-6.57%	5.99%	22.04%
2011	-44.38%	4.60%	10.45%	30.60%	-12.87%	-54.71%	12.35%
2012	61.27%	11.33%	5.10%	16.84%	47.19%	-2.87%	22.61%
2013	25.85%	18.50%	2.00%	-0.90%	23.80%	16.34%	14.81%
2014	-1.61%	15.00%	-5.37%	18.61%	4.30%	-11.07%	20.73%
2015	-4.63%	-7.64%	-31.64%	-35.39%	-2.09%	-69.99%	14.05%
2016	14.84%	11.45%	21.84%	31.40%	-2.40%	313.66%	0.80%
2017	25.21%	-4.40%	26.96%	-3.30%	14.22%	6.60%	17.26%
2018	-25.91%	35.54%	-12.30%	-19.93%	-10.60%	-42.40%	9.85%
2019	44.57%	19.64%	-0.60%	21.52%	17.56%	-47.20%	41.45%
2020	22.63%	-10.36%	-41.64%	-19.30%	9.82%	56.15%	29.15%
2021	-2.06%	-1.81%	23.51%	22.16%	19.88%	41.98%	50.67%
2022	-25.10%	44.77%	31.91%	3.68%	-25.00%	5.21%	-14.94%
Average annual return	**3.56%**	**9.31%**	**-0.12%**	**4.50%**	**4.38%**	**-5.88%**	**18.04%**

Appendix B – Weightings for each data point on the efficient frontier

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	7.04%	0.25%	0.001099441	0.086129827
Merck	30.99%	2.89%		8.61%
Shell	10.58%	-0.01%		
Enbridge	7.70%	0.35%		
Toyota	12.50%	0.55%		
US Steel	4.31%	-0.25%		
Costco	26.88%	4.85%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	6.50%	0.23%	0.001109277	0.09
Merck	31.88%	2.97%		9.00%
Shell	9.60%	-0.01%		
Enbridge	7.25%	0.33%		
Toyota	11.39%	0.50%		
US Steel	4.33%	-0.25%		
Costco	29.05%	5.24%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return

Citigroup	4.96%	0.18%	0.001224273	0.1
Merck	34.46%	3.21%		10.00%
Shell	6.95%	-0.01%		
Enbridge	6.04%	0.27%		
Toyota	8.49%	0.37%		
US Steel	4.49%	-0.26%		
Costco	34.61%	6.24%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	3.12%	0.11%	0.001462203	0.11
Merck	37.38%	3.48%		11.00%
Shell	4.22%	-0.01%		
Enbridge	4.87%	0.22%		
Toyota	5.43%	0.24%		
US Steel	4.84%	-0.28%		
Costco	40.14%	7.24%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	0.21%	0.01%	0.001809611	0.12
Merck	40.34%	3.76%		12.00%
Shell	2.47%	0.00%		
Enbridge	3.79%	0.17%		
Toyota	1.64%	0.07%		
US Steel	5.44%	-0.32%		
Costco	46.10%	8.32%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	0.00%	0.00%	0.002277218	0.13
Merck	43.50%	4.05%		13.00%
Shell	0.00%	0.00%		
Enbridge	0.00%	0.00%		
Toyota	0.00%	0.00%		
US Steel	5.19%	-0.31%		
Costco	51.31%	9.26%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	0.00%	0.00%	0.003139256	0.140001
Merck	44.51%	4.14%		14.00%
Shell	0.00%	0.00%		
Enbridge	0.00%	0.00%		
Toyota	0.00%	0.00%		
US Steel	0.64%	-0.04%		
Costco	54.84%	9.89%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	0.00%	0.00%	0.005084795	0.15
Merck	34.82%	3.24%		15.00%
Shell	0.00%	0.00%		
Enbridge	0.00%	0.00%		
Toyota	0.00%	0.00%		
US Steel	0.00%	0.00%		
Costco	65.18%	11.76%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	0.00%	0.00%	0.009350293	0.16
Merck	23.37%	2.18%		16.00%
Shell	0.00%	0.00%		
Enbridge	0.00%	0.00%		
Toyota	0.00%	0.00%		
US Steel	0.00%	0.00%		
Costco	76.63%	13.82%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	0.00%	0.00%	0.015952069	0.17
Merck	11.91%	1.11%		17.00%
Shell	0.00%	0.00%		
Enbridge	0.00%	0.00%		
Toyota	0.00%	0.00%		
US Steel	0.00%	0.00%		
Costco	88.09%	15.89%		

Stock	Weighting	Weighted Return	Portfolio Variance	Portfolio Return
Citigroup	0.00%	0.00%	0.025296515	0.1804
Merck	0.00%	0.00%		18.04%
Shell	0.00%	0.00%		
Enbridge	0.00%	0.00%		
Toyota	0.00%	0.00%		
US Steel	0.00%	0.00%		
Costco	100.00%	18.04%		