Math 103A Homework 1

James Holden

Winter 2019

Question 1. *Read sections 0 and 1 in the book.*

Question 2. *Let* ${F^x}_5$ *denote* $\{1, 2, 3, 4\}$ *together with the multiplication mod 5. Show that* ${F^x}_5$ *is a group whose identity element is 1.*

Proof. To show that a (set, binary operation) pair is a group, it is sufficient to show that:

(i) an identity element: $\exists e \in {F^x}_5$ such that $\forall a \in {F^x}_5 : a * e = e * a = a$

(ii) inverses exist for every element of the group: $\forall a \in {F^x}_5 : \exists a^{-1}$ such that $a * a^{-1} = a^{-1} * a = e$

(iii) group operation is closed: $\forall a, b \in {F^x}_5 : a * b \in {F^x}_5$

(iv) associativity: $\forall a, b, c \in {F^x}_5 : a * (b * c) = (a * b) * c$.

Showing these properties exist:

(i) an identity element: $\forall a \in {F^x}_5 : e = 1$ and $\forall a \in {F^x}_5, a * 1 = a \times 1 \pmod 5 = a$.

(ii) inverses exist for every element of the group: Since this group is only 4 elements, the pairs of inverses are $(2, 3)$, $(1, 1)$, and $(4, 4)$. If you multiply the elements of each of the pairs together and mod 5, you get 1, the identity.

(iii) group operation is closed: $\forall a, b \in {F^x}_5 : a * b = a \times b \pmod 5 \ \in {F^x}_5$

(iv) associativity: $\forall a, b, c \in {F^x}_5 : a * (b * c) = (a * b) * c$ holds because $(b * c) \in {F^x}_5$ and $(a * b) \in {F^x}_5$.

Therefore, ${F^x}_5$ is a group whose identity $= 1$.

□

Question 3. *Recall that* $\mathbb{Z}_4$ *denotes* $\{0, 1, 2, 3\}$ *together with addition mod 4. Show that* ${F^x}_5$ *and* $\mathbb{Z}_4$ *are isomorphic to each other.*

Proof. For two groups to be isomorphic to each other, they have to (1) be homomorphic to each other (2) have a bijective function. Let us denote $G = (\mathbb{Z}_4, +_4)$ and let $G' = ({F^x}_5, *_5)$.

Consider the function $\phi(x) = x + 1$. ϕ is a map from $G \to G'$. If $\phi(ab) = \phi(a)\phi(b)$ is true, ϕ is a homomorphism by the homomorphism property.

This means that if $\forall a, b \in G$ and $\phi(a), \phi(b) \in G'$, we have $\phi(a*b) = \phi(a)*\phi(b)$ where the group operation between a, b is the group operation of G and the group operation between $\phi(a), \phi(b)$ is the group operation of G'.

For example: let $a = 2$ and $b = 3$. This means $\phi(a * b) = \phi(2 +_5 3) = \phi(1) = 2$.

Likewise: $\phi(2) * \phi(3) = 3 * 4 = 3 \times_5 4 = 12 \bmod 5 \ = 2$.

It is also easy to see these two groups are bijective by ϕ. We have $\forall a \in G$, there is a corresponding $b \in G'$.

Hence $\phi : \mathbb{Z}_4 \cong {F^x}_5$.

□

Question 4. *Show that exp :* $(\mathbb{R}, +)$ *is isomorphic to* $((0, \infty), *)$. *(You may use without a proof properties of the exponential map from calculus.)*

Proof. Once again to show isomorphism, the function needs to satisfy (1) homorphism (2) bijection. Let $G = (\mathbb{R}, +)$ and let $G' = ((0, \infty), *)$. Let $a, b \in G$.

(1) homomorphism: $exp(a * b) = exp(a) * exp(b)$
$e^{a+b} = e^a \times e^b$

(2) bijection: We know that the exponential function is well defined. For every member of G, there is a corresponding member in G' which means exp is injective. Furthermore, exp is surjective because the exponential function is strictly increasing and $\forall b \in G', \exists a \in G$ such that $log(b) = a$. Therefore, exp is bijective between these to groups.

Hence $exp : (\mathbb{R}, +) \cong ((0, \infty), *)$. □

Question 5. *Exercise 1 page 19: 29, 30*

(a) *Problem 29: Find all solutions x of the given equation: $x +_{15} 7 = 3$ in $\mathbb{Z}_{15}$*

Proof. We define $\mathbb{Z}_{15} : \{0, 1, 2, \ldots, 14\}$ and let $x \in \mathbb{Z}_{15}$. It is easy to see that any $x \in \{0, 1, \ldots, 8\}$ is not a solution. This is because any x in such a range applied to this binary operation ($+_7$ mod 15) will result in an integer that is between 7 and 15. So we have to examine $x \in \{9, 10, \ldots, 14\}$. Solutions of this equation are $x \in \{11\}$. □

(b) *Problem 30: Find all solutions x of the given equation: $x +_{2\pi} \frac{3\pi}{2} = \frac{3\pi}{4}$ in $\mathbb{R}_{2\pi}$*

Proof. We define $\mathbb{R}_{2\pi} : \{0 \ldots 2\pi\}$ and let $x \in \mathbb{R}_{2\pi}$. We see that any $x \in \{0 \ldots \frac{\pi}{2}\}$ is not a solution. This is because any x in such a range applied to this binary operation ($+_{2\pi}$ mod $\frac{3\pi}{2}$) will result in an integer that is between $\frac{3\pi}{2}$ and 2π. So we have to examine $x \in \{\frac{\pi}{2} \ldots 2\pi\}$. Solutions of this equation are $x \in \{\frac{5\pi}{4}\}$. □