Math 103A Homework 2

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Question 1. *Let X be a non-empty set. Let $S_A = \{f : X \to X | f$ is a bijection$\}$. Show that S_A together with composition of functions forms a group.*

Proof. We want to show that $(S_A, \circ)$ is a group. Since f is bijective, we know that $f(a) = f(b)$ implies $a = b$ for all $a, b \in X$. To do so we verify the group properties:

1. identity element:
   The identity element is the identity function $id : X \to X$ given $x \in X$. Then we have $(f \circ id)(t) = f(id(t)) = f(t) = id(f(t)) = (id \circ f)(t)$
2. closed under group operation: for all $a, b \in X \to a \circ b \in X$
   $a \circ b \in X$
3. inverses exist for ever element in the group: for ever $a \in X, \exists a^{-1} \in X$ such that $a \circ a^{-1} = e$
   By the definition of f, every element in X must have a inverse because there exists a bijective mapping $f : X \to X$.
4. all elements are associative: for $a, b, c \in X$ the relation $a \circ (b \circ c) = (a \circ b) \circ c$ holds by associativity of composition of sets $X \to X$.

$\square$

Question 2. *Let $G = P(1, 2)$, the power set of $\{1, 2\}$. For any two sets $A, B \in G$ define $A * B = A \Delta B$. It was discussed in class that $(G, *)$ is a group. List all the subgroups of G.*

Proof. We define $G = (\{\{\}, \{1\}, \{2\}, \{1, 2\}\}, *)$. A subgroup of G is a subset of G that retains the group operation. It is clear that $(G, *)$ is a group.

Listing them out:
$\{\{\}\}$
$\{\{\}, \{1\}\}$
$\{\{\}, \{2\}\}$
$\{\{\}, \{1, 2\}\}$
$\{\{\}, \{1, 2\}, \{1, 2, 3\}\}$
are all subgroups.
$\{\{\}, \{1\}, \{1, 2\}\}$
$\{\{\}, \{2\}, \{1, 2\}\}$
are not subgroups because they are not closed under multiplication.

$\square$

Question 3. *Let $(G, *)$ be as in problem 2 (above). Show that $(G, *)$ is not isomorphic to $(Z_4, +_4)$. (Hint: suppose there exists an isomorphic from G onto Z_4, can the element $1 \in Z_4$ be in the image?)*

Proof. We define $G = (\{\{\}, \{1\}, \{2\}, \{1, 2\}\}, *)$ and $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Let $f : G \to \mathbb{Z}_4$. We want to show that f is not an isomorphism. According to the hint, $1 \in \mathbb{Z}_4$ is not in the image. This means that there does not exist $a \in G$ such that $f(a) = 1$.

Let $f : (G, *) \to (Z_4, +_4)$ be an isomorphism. Let f be the sum of the elements of the pairs of the power set, so :
$\{\} \in G \to 0 \in Z_4$
$\{1\} \in G \to 1 \in Z_4$
$\{2\} \in G \to 2 \in Z_4$

$\{1,2\} \in G \to 3 \in Z_4$
To me it seems like $1 \in Z_4 \in$ Image(f) because $f(1) = 1 \in Z_4$

$\square$

Question 4. *Let $(G, *)$ be a group and let H be a subgroup of G. Define the relation $g_1 \sim g_2 \Leftrightarrow {g_1}^{-1} * g_2 \in H$.*

(a) Show that $\sim$ is an equivalence relation.

Proof. $\sim$ is an equivalence relation if it

(1) is reflexive $a \sim a$

Check: For any $g \in G$, we have $g^{-1} * g = e \in H$ because H is a subgroup of G, hence $g \sim g$.

(2) is symmetric $a \sim b$ then $b \sim a$

Check: If $g_1 \sim g_2$ then $g_1^{-1} * g_2 \in H$. Since H is a subgroup, it is closed under inverses just like G. Therefore we have:

$g_1^{-1} * g_2 = (g_2^{-1} * g_1)^{-1} \in H$ and hence $g_2 \sim g_1$

(3) is transitive $a \sim b, b \sim c \to a \sim c$

Check: If $g_1 \sim g_2$ and $g_2 \sim g_3$ then $g_2^{-1} * g_1 \in H$ and $g_3^{-1} * g_2 \in H$.

Since H is closed under multiplication, $g_3^{-1} * g_1 = (g_3^{-1} * g_2)(g_2^{-1} * g_1) \in H$ therefore $g_1 \sim g_3$ $\square$

(b) Find the class of e, the identity element.

Proof. The class of e is just the subgroup H itself. Let $b \in$ (equivalence class of e). Let a be the equivalence class of e.

$b \sim e \Leftrightarrow b \sim a$

$\Leftrightarrow b^{-1} * a \in H$

$\Leftrightarrow b^{-1} * a = h,$ for some $h \in H$

$\Leftrightarrow b = ah^{-1},$ for some $h \in H$

$\Leftrightarrow b = ah,$ for some $h \in H$ because H is closed under inverses.

$\Leftrightarrow b \in eH$

$\Leftrightarrow b \in H$

$\square$

Question 5. *Let a be a positive real number. Note that for all $x, y \in R$ there exists a unique $k \in Z$ and a unique $0 \leq r \leq a$ so that:*
$x + y = r + ka$.
Denote $[0, a)$, the half open interval, by $\mathbb{R}_a$ and define the following addition on $\mathbb{R}_a$.
$x + ay = r$,
where $x + y = r + ka$ and $r \in [0, a)$.

(a) Show that $(\mathbb{R}_a, +_a)$ is a group.

Proof. Given $(\mathbb{R}_a, +_a)$ with addition $x + ay = r$, this is the group of positive real numbers including zero up to but not including a, under the specified addition.

Let $g, k \in \mathbb{R}_a$.

(1) identity: $e = 0$

For $t \in \mathbb{R}_a$, $t * e = x + e \times y = t$

(2) closed under operation

For $s, t \in \mathbb{R}_a$, $t * s = x + (x + sy)y = 2x + xy + sy^2 = r \in (\mathbb{R}_a, +_a)$

(3) inverse

For $a \in (\mathbb{R}_a, +_a), a^{-1} \in (\mathbb{R}_a, +_a)$ exists.

(4) associativity

For $a, b, c \in (\mathbb{R}_a, +_a)$, $a * (b * c) = (a * b) * c$

□

(b) *Show that $(R_1, +_1)$ is isomorphic to $(R_a, +_a)$ for any $a > 0$. (Therefore, $(R_a, +_a)$ and $(R_b, +_b)$ are isomorphic for any $a, b > 0$).*

Proof. Let $f : (R_1, +_1) \to (R_a, +_a)$. Let f be the tangent function. The tangent function bijectively maps the real numbers $[0, 1) \to (\infty, \infty)$. Therefore, f is an isomorphism and $(R_1, +_1) \cong (R_a, +_a)$.

□

(c) *(Bonus Problem) Prove or disprove: $(R_1, +_1)$ is isomorphic to $(R, +)$.*

Proof. $(R_1, +_1)$ cannot be isomorphic to the whole set of real numbers because of the tangent function. The tangent function is well-defined for all real numbers on the domain (-1, 1). Let $f : (R_1, +_1) \to (R, +)$. f is an injective homomorphism.

□

Question 6. *Exercise 4 page 45: 3, 12, 19*

Proof. (a) Problem 3: Let $*$ be defined on $\mathbb{R}^+$ by letting $a * b = \sqrt{ab}$.

Proof. (1) identity:

$a * e = \sqrt{a \times e} = \sqrt{a} \neq a$ for $a \in \mathbb{R}^+$

Therefore this fails the identity axiom. □

(b) Problem 12: All $n \times x$ diagonal matricies.

Proof. Let such a matrix be A.

We know $\det(A) \neq 0$ because for diagonal matrices, $det(A) = \prod_{i=1}^{n} a_{ii}$ and we know the diagonal elements are nonzero. Therefore, inverses exists for all such matrices A.

This set of matrices is also closed under matrix multiplication because for diagonal matrix A and diagonal matrix B, $\prod_{i=1}^{n} a_i b_i$.

It also inherits the properties of the group of all invertible $n \times n$ matrices, called the general linear group. □

(c) Problem 19 Let $S = \mathbb{R}$ without $\{-1\}$. * on S is

$a * b = a + b + ab$

Proof. * is a binary operation on S. This is because *: $S \times S \to S$.

$(S, *)$ is a group. This is because it is closed on group operation (ie. $a * b \in S$), identity exists (ie. $e = 0$), it is associative, and inverses exist (for every a there exists an a^-1).

$2 * x * 3 = 7 \in S$

$= (2 + x + 2x) * 3 = 7 \in S$

$= (2 + x + 2x) + 3 + 3 \times (2 + x + 2x) = 7 \in S$

$= 2 + 3x + 3 + 6 + 3x + 6x = 7 \in S$

$= 11 + 12x = 7 \in S$

$x = -\frac{1}{3}$, $x \in S$

□

$\square$