Resolução da Prova 2 (2018) de Probabilidades B

1. Sejam X e Y variáveis aleatórias iid $exp(1)$. Defina $U = 2X + Y$ e $V = 2X - Y$. Verifique se U e V são independentes.

$U + V = (2X + Y) + (2X - Y) = 4X \;\Rightarrow X = \frac{U+V}{4}$

$U - V = (2X + Y) - (2X - Y) = 2Y \;\Rightarrow Y = \frac{U-V}{2}$

$J(x,y) = \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = \big|-2 - (2)\big| = 4$

$f_{X,Y}(x,y) = f_X(x) f_Y(y) = (e^{-x})(e^{-y}) = e^{-(x+y)}$

$f_{U,V}(u,v) = f_{X,Y}\big(\frac{u+v}{4}, \frac{u-v}{2}\big)\big(J(x,y)\big)^{-1} = \frac{1}{4}e^{-(\frac{u+v}{4} + \frac{u-v}{2})} = \frac{1}{4}e^{-(\frac{3u-v}{4})}$

$X \geq 0$ e $Y \geq 0 \;\Rightarrow \frac{U+V}{4} \geq 0 \; e \; \frac{U-V}{2} \geq 0 \Rightarrow U + V \geq 0$ e $U \geq V$

$V \leq U$ e $-U \leq V \Rightarrow \boxed{-U \leq V \leq U}$

$$f_U(u) = \int_{-u}^{u} (\frac{1}{4}e^{-(\frac{3u-v}{4})})dv = \Big[e^{-(\frac{3u-v}{4})}\Big]_{-u}^{u} = e^{-\frac{u}{2}} - e^{-u}$$

$$f_V(v) = \int_{-v}^{\infty} (\frac{1}{4}e^{-(\frac{3u-v}{4})})du = \Big[-\frac{1}{3}e^{-(\frac{3u-v}{4})}\Big]_{-v}^{\infty} = \frac{1}{3}e^{v} \qquad \text{se } \; v < 0$$

$$f_V(v) = \int_{v}^{\infty} (\frac{1}{4}e^{-(\frac{3u-v}{4})})du = \Big[-\frac{1}{3}e^{-(\frac{3u-v}{4})}\Big]_{v}^{\infty} = \frac{1}{3}e^{-\frac{v}{2}} \qquad \text{se } v \geq 0$$

$\boxed{f_U(u) f_V(v) \neq f_{U,V}(u,v) \;\Rightarrow\; U \not\perp\!\!\!\perp V}$

2. a) Seja $X \sim Bernoulli(\frac{1}{3})$ e defina uma variável aleatória $Y = \frac{1}{x+2}$. Quando valem $E(Y)$ e $var(Y)$?

b) Sejam X_1, X_2 e X_3 variáveis aleatórias independentes com distribuição comum Bernoulli(p). Seja $Y = X_{(3)} = \max(X_1, X_2, X_3)$.
Encontre $F_Y(0.5) = P(X_{(3)} \leq 0.5)$.

$$E(Y) = E(\tfrac{1}{X+2}) = \sum_{x:p(x)>0} \left(\frac{1}{x+2}\right) p(x) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \boxed{\frac{4}{9}}$$

$$Var(Y) = E[(Y-\mu)^2] = E\left[\left(\tfrac{1}{X+2} - \tfrac{4}{9}\right)^2\right] = \left(\tfrac{1}{324}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{81}\right)\left(\tfrac{1}{3}\right) = \boxed{\frac{1}{162}}$$

Se $X_{(3)}$ é menor que $\frac{1}{2}$, então $X_{(2)}$ e $X_{(1)}$ tambem serão.

$$\begin{aligned}
P\left(X_{(3)} \leq \tfrac{1}{2}\right) &= P\left(X_{(1)} \leq \tfrac{1}{2}, X_{(2)} \leq \tfrac{1}{2}, X_{(3)} \leq \tfrac{1}{2}\right) \\
&= P\left(X_{(1)} \leq \tfrac{1}{2}\right) P\left(X_{(2)} \leq \tfrac{1}{2}\right) P\left(X_{(3)} \leq \tfrac{1}{2}\right) \\
&= \left[P\left(X_{(3)} \leq \tfrac{1}{2}\right)\right]^3 \\
&= \left[(1-p)\right]^3 \\
&= \boxed{(1-p)^3}
\end{aligned}$$

3. Considere o vetor $(X,\ Y)$ com a seguinte densidade conjunta

$$f(x,y) = 2,\ 0 \leq x \leq 1, 0 \leq y \leq 1-x.$$

a) X e Y são independentes e identicamente distribuídas?

b) Encontre a distribuição de $Z = X+Y$ utilizando:

i) A função distribuição (acumulada).

ii) A fórmula direta para distribuição da soma.

iii) O métododo do Jacobiano.

c) Encontre a densidade condicional $f(x/y)$.

$$f_X(x) = \int_0^{1-x} 2dy = [2y]_0^{1-x} = 2(1-x) \qquad x \in [0,1]$$

$$f_Y(y) = \int_0^{1-y} 2dx = [2x]_0^{1-y} = 2(1-y) \qquad y \in [0,1]$$

$$f_X(x)f_Y(y) \ = \ (2(1-x))(2(1-y)) \ = \ 4(1-x)(1-y) \ \neq \ f_{X,Y}(x,y) \ \Rightarrow \ \boxed{X \not\perp Y}$$

- $F_Z(z) = P(Z \leq z) = P(X+Y \leq z) = \int_0^z \int_0^z 2dxdz = \left[2\frac{z^2}{2}\right]_0^z = z^2, \qquad 0 \leq z \leq 1$

- $F_Z(z) = \int_0^z 2dx = [2x]_0^z = 2z \qquad z \in [0,1]$

- $Z = X+Y$ e $W = X-Y \ \Rightarrow \ X = \frac{Z+W}{2}$ e $Y = \frac{Z-W}{2}$

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \big|(-1)-(1)\big| = 2$$

$$f_{Z,W}(z,w) = f_{X,Y}\left(\tfrac{Z+W}{2}, \tfrac{Z-W}{2}\right)(J(x,y))^{-1} = (2)(\tfrac{1}{2}) \qquad D_{Z,W} = \begin{cases} -1 \leq Z \leq 1 \\ |Z| \leq W \leq 1 \end{cases}$$

$$F_Z(z) = \int_{-1}^z \int_{-z}^1 dwdz = \left[z + \frac{z^2}{2}\right]_{-1}^z = z + \frac{z^2}{2} + \frac{1}{2}, \qquad \text{se } -1 \leq z \leq 0$$

$$F_Z(z) = F_Z(0) + \int_0^z \int_z^1 dwdz = \frac{1}{2} + \left[z - \frac{z^2}{2}\right]_0^z = \frac{1}{2} + z - \frac{z^2}{2}, \qquad \text{se } 0 \leq z \leq 1$$

$$\boxed{F_Z(z) = \begin{cases} 0, & \text{se} \quad z < -1 \\ \frac{1+2z+z^2}{2}, & \text{se} \quad -1 \leq z < 0 \\ \frac{1+2z-z^2}{2}, & \text{se} \quad 0 \leq z < 1 \\ 1, & \text{se} \quad 1 \leq z \end{cases}}$$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}$

$$\boxed{f_{X|Y}(x|y) = \frac{1}{1-y}, \qquad 0 \leq x \leq 1-y}$$