FILTRATIONS AND TORSION PAIRS IN ABRAMOVICH-POLISHCHUK'S HEART

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ABSTRACT. We study abelian subcategories and torsion pairs in Abramovich-Polishchuk's heart. And we apply the construction from [Liu21] on a full triangulated subcategory $\mathcal{D}_{\bar{S}}^{\leq 1}$ in $D(X \times S)$, for an arbitrary smooth projective variety S. We also define a notion of l-th level stability, which is a generalization of the slope stability and the Gieseker stability. We show that for any object E in Abramovich-Polishchuk's heart, there is a unique filtration whose factors are l-th level semistable, and the phase vectors are decreasing in a lexicographic order.

1. INTRODUCTION

The theory of Bridgeland stability conditions was introduced by Bridgeland in [Bri07], motivated by Douglas's work on D-branes and Π-stability (see [Dou02]). This theory was further studied by Kontsevich and Soilbelman (see [KS08]), and it turns out to be related to many different branches of mathematics including algebraic geometry ([BM14b], [BM14a]), symplectic geometry ([Joy15], [Smi18]), representation theory ([KS08], [Bri17]), and curve counting theories ([Tod10], [Tod13]).

Abramovich-Polishchuk's construction of sheaves of t-structures ([AP06],[Pol07]) becomes a fundamental tool in the study of Bridgeland stability conditions (see [BLM$^+$21]) and moduli problems of complexes ([BM14b]). More explicitly, given a t-structure with Noetherian heart on the bounded derived category of coherent sheaves $D(X)$, with X a smooth variety over a field k of characteristic 0, Abramovich and Polishchuk ([AP06]) constructed a t-structure on $D(X \times S)$, where S is another smooth variety over the same base field k. Their construction was refined by Polishchuk in his paper [Pol07], where he removed the characteristic 0 and smoothness assumption.

Their construction has important applications on the study of Bridgeland stability conditions. For instance, Bayer and Macrì used it to construct a nef divisor on the moduli space of Bridgeland semistable objects (see [BM14a] and [BM14b]); the author used their construction to construct Bridgeland stability conditions on $X \times S$ (see [Liu21]), where S is a smooth projective curve.

In this paper, we will further study the relation between Abramovich-Polishchuk's heart (AP heart for short) and stability conditions. As in [Liu21], suppose that we have a stability condition $\sigma = (\mathcal{A}, Z)$ on $D(X)$ (see Definition 2.3 below for the definition of

stability conditions), with $\mathcal{A}$ the Noetherian heart of a bounded t-structure on $D(X)$ and $Z : K_0(\mathcal{A}) \to \mathbb{C}$ a central charge with discrete image. We denote the corresponding AP heart on $D(X \times S)$ by $\mathcal{A}_S$. Then for any object $E \in \mathcal{A}_S$, there is a complexified Hilbert polynomial

$$L_E(n) = a_r(E)n^r + a_{r-1}(E)n^{r-1} + \cdots + a_0(E) + i(b_r(E)n^r + b_{r-1}(E)n^{r-1} \cdots + b_0(E))$$

where $r = dim(S)$ and a_i, b_i are linear maps from the Grothendieck group $K_0(D(X \times S))$ to $\mathbb{R}$ for any $0 \le i \le r$.

For any integer $0 \le j \le r$, one can define the full subcategory $\mathcal{A}_S^{\le j}$ in $\mathcal{A}_S$ consisting of objects whose complexified Hilbert polynomial is of degree $\le j$. It is easy to show that $\mathcal{A}_S^{\le j}$ is an abelian subcategory. And there is a unique filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

for any object $E \in \mathcal{A}_S$, such that E_j is the maximal subobject of E in $\mathcal{A}_S^{\le j}$. If we use $\mathcal{D}_S^{\le 1}$ to denote the minimal triangulated subcategory generated by $\mathcal{A}_S^{\le 1}$ in $D(X \times S)$, we can apply the construction in [Liu21] on $\mathcal{D}_S^{\le 1}$. Thus we get the following result.

Theorem 1.1. *The triangulated category $\mathcal{D}_S^{\le 1}$ admits stability conditions.*

In the meantime, we prove a sequence of positivity results on the coefficients of $L_E(n)$ (see also [Liu20, Lemma 4.1]). These positivity results help us to construct weak stability functions on $\mathcal{A}_S$ and its abelian subcategories. This lead us to the notion of l-th level stability, which can be viewed as a generalization of both the slope stability and the Gieseker stability. However, the phase of an object with respect to such stability is no longer a single real number, but a vector of real numbers. Then we show the existence and uniqueness of the filtration of an object $E \in \mathcal{A}_S$ with respect to such stability, which we call the lexicographic order filtration of E.

Theorem 1.2. *Assume $l \le j$, given $t_1, t_2, \cdots, t_l \in \mathbb{Q}_{>0}$, for any object $E \in \mathcal{A}_S^{\le j}$, there exists a unique filtration*

$$0 = E_0 \subset E_1 \subset E_2 \cdots \subset E_n = E,$$

such that each quotient factor E_i/E_{i-1} is l-th level semistable of phase $\vec{\phi}_i$ and

$$\vec{\phi}_1 > \vec{\phi}_2 > \cdots > \vec{\phi}_n.$$

Given $\vec{t} = (t_1, \cdots, t_{r+1}) \in \mathbb{Q}_{>0}^{r+1}$ and a phase vector $\vec{\phi} = (\phi_1, \cdots, \phi_{r+1}) \in (0,1)^{r+1}$, using this theorem, we can construct torsion pairs on $\mathcal{A}_S$:

$$\mathcal{T}_{\vec{\phi}}^{\vec{t}} := \{E \in \mathcal{A}_S | all\ the\ factors\ in\ theorem\ 1.2\ have\ phases\ > \vec{\phi}\},$$

$$\mathcal{F}_{\vec{\phi}}^{\vec{t}} := \{E \in \mathcal{A}_S | all\ the\ factors\ in\ theorem\ 1.2\ have\ phases\ \le \vec{\phi}\}.$$

In the end of this paper, we state some positivity results on the tilted heart $\mathcal{A}_{S,\vec{\phi}}^{\vec{t}} := \langle \mathcal{F}_{\vec{\phi}}^{\vec{t}}[1], \mathcal{T}_{\vec{\phi}}^{\vec{t}} \rangle$ (see Theorem 5.12).

Outline of the paper. In Section 2, we review some basic definitions and results in the theory of stability conditions. In Section 3, we review some necessary results from [AP06], [Pol07] and [Liu21]. In Section 4, we study some abelian subcategories on $\mathcal{A}_S$ and state some general results on weak stability conditions. In Section 5, we apply the results from previous section on $\mathcal{A}_S$. As a result, we define the notion of l-th level stability and prove the existence and uniqueness of lexicographic order filtration. Moreover, we use the filtration to construct torsion pairs of $\mathcal{A}_S$ and show some positivity results on the tilted hearts.

Notation and Conventions. In this paper, we work over an algebraically closed field in arbitrary characteristic. All varieties are integral separated algebraic schemes of finite type over such a field k. We will use $D(X)$ rather than the usual notation $D^b(cohX)$ to denote the bounded derived category of coherent sheaves on X. We use $\mathbb{H}$ to denote the upper half plane, i.e., the set of complex numbers with positive imaginary part. All categories are assumed to be essentially small in this paper.

2. Bridgeland Stability conditions

In this section, we review the definition and some basic results on (weak) stability conditions (See [BBD82], [Pol07], [Bri08], [KS08] and [BMT14]).

Definition 2.1. Let $\mathcal{D}$ be an triangulated category. A t-structure on $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the conditions (i)-(iii) below. We denote $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ for every $n \in \mathbb{Z}$. Then the conditions are:
(i) $Hom(X, Y) = 0$ for every $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$;
(ii) $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$.
(iii) every object $X \in \mathcal{D}$ fits into an exact triangle

$$\tau^{\leq 0} X \to X \to \tau^{\geq 1} X \to \cdots$$

with $\tau^{\leq 0} X \in \mathcal{D}^{\leq 0}$, $\tau^{\geq 1} X \in \mathcal{D}^{\geq 1}$.

It is not hard to see that the objects $\tau^{\leq 0} X$ and $\tau^{\geq 0} X$ are defined functorially. The functors $\tau^{\leq 0}$ and $\tau^{\geq 1} := \tau^{\geq 0}[-1]$ are called truncation functors (see e.g. [HT07, Proposition 8.1.8] for the basic properties of truncation functors). The heart of a t-structure is $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. It is an abelian category (see e.g. [HT07, Theorem 8.1.9]). We have the associated cohomology functors defined by $H^0 = \tau^{\leq 0} \tau^{\geq 0} : \mathcal{D} \to \mathcal{A}$, $H^i(X) := H^0(X[i])$.

Motivated by the work from String Theory (see e.g. [Dou02]), the notion of t-structures was refined by Bridgeland to the following one in [Bri07].

Definition 2.2. A slicing on a triangulated category $\mathcal{D}$ consists of full subcategories $\mathcal{P}(\phi) \in \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:
(a) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

(b) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $Hom_{\mathcal{D}}(A_1, A_2) = 0$,

(c) for every $0 \neq E \in \mathcal{D}$ there is a sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_m$$

and a sequence of morphisms

$$0 = E_0 \xrightarrow{f_1} E_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} E_m = E$$

such that the cone of f_j is in $\mathcal{P}(\phi_j)$ for all j.

This definition of slicings can be viewed as a refinement of t-structures in triangulated categories.

Definition 2.3. A stability condition on $\mathcal{D}$ consists of a pair $(\mathcal{P}, Z)$, where $\mathcal{P}$ is a slicing and $Z : K_0(D) \to \mathbb{C}$ is a group homomorphism such that the following conditions are satisfied:

(a) If $0 \neq E \in \mathcal{P}(\phi)$ then $Z(E) = m(E)exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$.

(b) (Support property) The central charge Z factors as $K_0(\mathcal{D}) \xrightarrow{v} \Lambda \xrightarrow{g} \mathbb{C}$, where Λ is a finite rank lattice, v is a group homomorphism and g is a group homomorphism, and there exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{ker(g)}$ is negative definite, and $Q(v(E)) \geq 0$, for any object $E \in \mathcal{P}(\phi)$.

Remark 2.4. If we only require $m(E)$ to be nonnegative in (a), then the pair $(\mathcal{P}, Z)$ is called a weak stability condition (see e.g. [BLM+21, Definition 14.1]). By [Bri08, Lemma 2.2], there is a S^1 action on the space of stability conditions. Specifically, for any element $e^{i\theta} \in S^1$, $e^{i\theta} \cdot (Z, \mathcal{P}) = (Z', \mathcal{P}')$ by setting $Z' = e^{i\theta}Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(\phi - \theta)$.

There is an equivalent way of defining a stability condition, which will be more frequently used in this paper. Firstly, we need to define what is a (weak) stability function Z on an abelian category $\mathcal{A}$.

Definition 2.5. Let $\mathcal{A}$ be an abelian category. We call a group homomorphism $Z : K_0(\mathcal{A}) \to \mathbb{C}$ a weak stability function on $\mathcal{A}$ if, for $E \in \mathcal{A}$, we have $Im(Z(E)) \geq 0$, with $Im(Z(E)) = 0 \implies Re(Z(E)) \leq 0$. If moreover, for $E \neq 0$, $Im(Z(E)) = 0 \implies Re(Z(E)) < 0$, we say that Z is a stability function.

Definition 2.6. A (weak) stability condition on $\mathcal{D}$ is a pair $\sigma = (\mathcal{A}, Z)$ consisting of the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$ and a (weak) stability function $Z : K_0(A) \to \mathbb{C}$ such that (a) and (b) below are satisfied:

(a) (HN-filtration) The function Z allows us to define a slope for any object E in the heart $\mathcal{A}$ by

$$\mu_\sigma(E) := \begin{cases} -\frac{Re(Z(E))}{Im(Z(E))} & \text{if } Im(Z(E)) > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The slope function gives a notion of stability: A nonzero object $E \in \mathcal{A}$ is σ semistable if for every proper subobject F, we have $\mu_\sigma(F) \leq \mu_\sigma(E)$. Moreover, if $\mu_\sigma(F) < \mu_\sigma(E)$ holds for any proper subobject $F \subset E$, we say that E is σ stable.

We require any nonzero object E of $\mathcal{A}$ to have a Harder–Narasimhan filtration in σ semistable ones, i.e., there exists a unique filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{m-1} \subset E_m = E$$

such that E_i/E_{i-1} is σ semistable and $\mu_\sigma(E_i/E_{i-1}) > \mu_\sigma(E_{i+1}/E_i)$ for any $1 \leq i \leq m$. The quotient E_i/E_{i-1} is called the i-th HN factor of E.

(b) (Support property) Equivalently as in Definition 2.3, the central charge Z factors as $K_0(\mathcal{D}) \xrightarrow{v} \Lambda \xrightarrow{g} \mathbb{C}$. And there exists a quadratic form Q on $\Lambda_\mathbb{R}$ such that $Q|_{ker(g)}$ is negative definite and $Q(v(E)) \geq 0$ for any σ semistable object $E \in \mathcal{A}$.

Remark 2.7. The equivalence of these two definitions is given by setting $\mathcal{P}(\phi)$ to be the full subcategory consisting of σ semistable objects in $\mathcal{A}$.

Sometimes in this paper (especially in Section 5), a pair $(\mathcal{A}, Z)$ of an abelian category $\mathcal{A}$ and a (weak) stability function Z which only satisfies (a) will also be called a (weak) stability condition.

Although the following result is well known, the author could not find a reference including the proof in full generality. So we include a proof here.

Proposition 2.8. *If we have a pair $(\mathcal{A}, Z)$, where $\mathcal{A}$ is an abelian category, and $Z : K_0(\mathcal{A}) \to \mathbb{C}$ is a weak stability function. Assume that*

- $\mathcal{A}$ *is Noetherian,*
- *the image of $Im(Z)$ is discrete in $\mathbb{R}$.*

Then Harder–Narasimhan filtrations exist in $\mathcal{A}$ with respect to Z.

Proof. The proof is essentially the same as the proof of [MS17, Proposition 4.10]. The only difference is that we have to deal with the more general case when Z is a weak stability function.

Let $E \in \mathcal{A}$ be a nonzero object. By Noetherianity of $\mathcal{A}$ and Lemma 4.11, one can find a maximal subobject $E_0 \subset E$ among the subobjects $\{F \subset E | Z(F) = 0\}$. If $E_0 = E$, the statement is trivial as E is semistable. Assume that the quotient object $Q := E/E_0 \in \mathcal{A}$ is nonzero. By $\mathcal{H}'(Q)$ we denote the convex hull of the set of all $Z(F)$ for $F \subset Q$. By [MS17, Lemma 4.9] (one can easily go through the proof to see that this lemma also holds even if Z is just a weak stability condition), this set is bounded from the left. By $\mathcal{H}_l$ we denote the half plane to the left of the line between $Z(Q)$ and 0. If Q is semistable, the short exact sequence

$$0 \to E_0 \to E \to Q \to 0$$

provides us the Harder–Narasimhan filtration of E when $\mu(Q) < \infty$, and E itself is semistable when $\mu(Q) = \infty$. Otherwise, the set $\mathcal{H}(Q) = \mathcal{H}'(Q) \cap \mathcal{H}_l$ is a convex polygon.

Let $v_0 = 0, v_1, \cdots, v_n = Z(Q)$ be the extremal vertices of $\mathcal{H}(Q)$, we claim that there exist subobjects $Q_i \subset Q$ with $Z(Q_i) = v_i$ for $i = 1, \cdots, n-1$. Indeed, by definition of $\mathcal{H}(Q)$ there is a sequence of objects $Q_{j,i} \subset Q$ such that

$$\varinjlim_{j \to \infty} Z(Q_{j,i}) = v_i.$$

Since $Im(Z)$ has discrete image, we can assume $Im(Z(Q_{j,i})) = Im(v_i)$ for all j. For any $\epsilon > 0$, we can find $N \gg 0$ such that $|Z(Q_{j,i}) - v_i| < \epsilon$ for any $j \geq N$. Consider the following short exact sequence

$$0 \to Q_{j_1,i} \cap Q_{j_2,i} \to Q_{j_1,i} \oplus Q_{j_2,i} \to Q_{j_1,i} + Q_{j_2,i} \to 0,$$

where $Q_{j_1,i} \cap Q_{j_2,i}$ and $Q_{j_1,i} + Q_{j_2,i}$ are the intersection and sum of $Q_{j_1,i}, Q_{j_2,i}$ inside Q. By the definition of weak stability functions, we have $Im(Z(Q_{j_1,i} + Q_{j_2,i})) \geq Im(v_i)$ as Q_{j_1}, Q_{j_2} are subobjects of $Q_{j_1,i} + Q_{j_2,i}$. Assume that $Im(Z(Q_{j_1,i} + Q_{j_2,i})) > Im(v_i)$ for contrary, we get that $Im(Z(Q_{j_1,i} \cap Q_{j_2,i})) < Im(v_i)$. If we take ϵ to be small enough, the segment connecting $Z(Q_{j_1,i} \cap Q_{j_2,i})$ and $Z(Q_{j_1,i} + Q_{j_2,i})$ would be very close to the point v_i, and one of the points $Z(Q_{j_1,i} \cap Q_{j_2,i})$, $Z(Q_{j_1,i} + Q_{j_2,i})$ would lie outside the convex hull $\mathcal{H}'(Q)$, contradicting the definition of $\mathcal{H}'(Q)$.

Hence, we can assume that $Im(Z(Q_{j_1,i} + Q_{j_2,i})) = Im(v_i)$ for all $j_1, j_2 \gg 0$. Moreover, as $Q_{j_1,i}, Q_{j_2,i}$ are subobjects of $Q_{j_1,i} + Q_{j_2,i}$, we have that

$$Re(v_i) \leq Re(Z(Q_{j_1,i} + Q_{j_2,i}) \leq min\{Re(Z(Q_{j_1,i})), Re(Z(Q_{j_2,i}))\}$$

by Definition 2.5. For a given $N \gg 0$, we replace $Q_{j,i}$ by $Q_{j,i} + \cdots + Q_{N+1,i} + Q_{N,i}$ for any $j > N$. After this replacement, we can assume that $Q_{N,i} \subset Q_{N+1,i} \subset \cdots \subset Q$, and

$$\varinjlim_{j \to \infty} Z(Q_{j,i}) = v_i.$$

By Noetherianity of $\mathcal{A}$, we get a subobject $Q_i \subset Q$ with $Z(Q_i) = v_i$. Hence, the claim is proved.

Next, we will modify Q_i a little bit so that the following conditions hold.

(1) The inclusion $Q_{i-1} \subset Q_i$ holds for all $i = 1, \cdots, n$.
(2) The object $G_i \coloneqq Q_i / Q_{i-1}$ is semistable for all $i = 1, \cdots, n$.

Same as the proof of [MS17, Proposition 4.10], one can show that $Z(Q_{i-1} \cap Q_i) = Z(Q_{i-1})$ and $Z(Q_{i-1} + Q_i) = Z(Q_i)$. Hence, we can replace Q_i by $Q_1 + \cdots + Q_i$ and get the condition (1).

Similar to the proof of [MS17, Proposition 4.10], one can show that the quotient object G_i is quasi-semistable (see Definition 4.16) for all $i = 1, \cdots n$, i.e., if we take K_i to be the maximal subobject of G_i among the sobobjects with trivial central charge, the quotient object G_i / K_i will be semistable. We can modify Q_i in the following way. We take $\widetilde{Q_i}$ to be the pullback of K_{i+1} along the surjection $Q_{i+1} \to G_{i+1}$, as in the following commutative

diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q_i & \longrightarrow & \widetilde{Q}_i & \longrightarrow & K_{i+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_i & \longrightarrow & Q_{i+1} & \longrightarrow & G_{i+1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_{i+1}/K_{i+1} & \xrightarrow{\ id\ } & G_{i+1}/K_{i+1} & &
\end{array}
$$

Each row and column in this diagram is a short exact sequence. If we modify

$$Q_{n-1}, Q_{n-2}, \cdots Q_1$$

inductively using this construction, we get a sequence of $Q_1 \subset Q_2 \subset \cdots \subset Q_{n-1} \subset Q$ with condition (2) holds. Hence, if we take E_i to be the pullback of Q_i along the surjection $E \to Q$ for any $1 \le i \le n$, the sequence $E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E$ is a Harder–Narasimhan filtration of E if $\mu(E_1) < +\infty$. In the case $\mu(E_1) = +\infty$, the sequence $E_1 \subset \cdots \subset E_{n-1} \subset E$ is a Harder–Narasimhan filtration of E.   □

Remark 2.9. Although we do not ask for the uniqueness of Harder–Narasimhan filtration in this proposition, the uniqueness follows automatically from the definition of semistable objects. In fact, the proof of [HN75, Lemma 1.3.7] can be easily generalized to our case.

There is an important operation called tilting with respect to a torsion pair, which is very useful for constructing stability conditions.

Definition 2.10. A torsion pair in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ which satisfy $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$, and such that every object $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \to T \to E \to F \to 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Remark 2.11. In this paper, most torsion pairs are coming from weak stability conditions $\sigma = (\mathcal{A}, Z)$. In fact, let

$$\mathcal{T} = \{E \in \mathcal{A} \mid \mu_{\sigma,min}(E) > 0\} \ and \ \mathcal{F} = \{E \in \mathcal{A} \mid \mu_{\sigma,max}(E) \le 0\}$$

be a pair of full subcategories, where $\mu_{\sigma,min}(E)$ is the slope of the last HN-factor of E and $\mu_{\sigma,max}(E)$ is the slope of the first HN-factor of E. It is easy to see that this is a torsion pair.

Lemma 2.12 ([HRS96, Proposition 2.1]). *Suppose $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$. Then $\mathcal{A}^{\#} = \langle \mathcal{F}[1], \mathcal{T} \rangle$ is a heart of a bounded t-structure on $\mathcal{D}$.*

In this paper, we are interested in the case when $\mathcal{D}$ is the bounded derived category of coherent sheaves on an algebraic variety X or an admissible component of it. From now on, X will be a smooth projective variety over an algebraically closed field k, and $D(X)$ will be the bounded derived category of coherent sheaves on X.

3. AP HEART AND COMPLEXIFIED HILBERT POLYNOMIAL

In this section, we will recall some results from [AP06], [Pol07], [BM14a] and [Liu21]. We will work under the following setup in the rest of this paper.

Setup: Suppose X and S are smooth projective varieties, and $\sigma = (\mathcal{A}, Z)$ is a stability condition on $D(X)$, where $\mathcal{A}$ is Noetherian and the image of Z is discrete. The heart $\mathcal{A}$ corresponds to a bounded t-structure $(D^{\leq 0}(X), D^{\geq 0}(X))$ on $D(X)$.

For any t-structure $(D^{\leq 0}(X), D^{\geq 0}(X))$ on $D(X)$, we have the following theorem.

Theorem 3.1 ([Pol07, Theorem 3.3.6])**.** *Suppose S is a projective variety of dimension r, and $\mathcal{O}(1)$ is an ample line bundle on S. There exists a global t-structure on $D(X \times S)$ defined as*

$$D^{[a,b]}(X \times S) = \{E \in D(X \times S) \mid \boldsymbol{R}p_*(E \otimes q^*(\mathcal{O}(n))) \in D^{[a,b]}(X) \; for \; all \; n \gg 0\}.$$

Here p, q are the projections from $X \times S$ to X and S respectively, $a < b$ are integers and can be infinite. Moreover, the global heart

$$\mathcal{A}_S = D^{\leq 0}(X \times S) \cap D^{\geq 0}(X \times S)$$

is Noetherian and independent of the choice of $\mathcal{O}(1)$.

Remark 3.2. This theorem can be viewed as a generalization of Serre's vanishing. In fact, if $\mathcal{A} = coh(X)$ is the abelian category of coherent sheaves on X, then $\mathcal{A}_S = coh(X \times S)$ is the abelian category of coherent sheaves on $X \times S$.

Remark 3.3. In [Liu21], we observed that $Z(\boldsymbol{R}p_*(E \otimes q^*\mathcal{O}(n)))$ is a polynomial of degree no more than $dim(S) = r$, whose leading coefficient is a weak stability function on $\mathcal{A}_S$. We denote this polynomial by $L_E(n)$ for any object E in $\mathcal{A}_S$.

We also have the following theorem.

Theorem 3.4 ([Liu21, Theorem 3.3])**.** *Assume S is a smooth projective variety of dimension r, we define $(\mathcal{A}_S, Z_S)$ as below:*

$$\mathcal{A}_S = \{E \in D(X \times S) \mid \boldsymbol{R}p_*(E \otimes q^*(\mathcal{O}(n))) \in \mathcal{A} \; for \; all \; n \gg 0\}$$

$$Z_S(E) = \lim_{n \to +\infty} \frac{Z(\boldsymbol{R}p_*(E \otimes q^*(\mathcal{O}(n))))r!}{n^r vol(\mathcal{O}(1))},$$

where $vol(\mathcal{O}(1))$ is the volume of $\mathcal{O}(1)$ (see [Laz04, Definition 2.2.31] for the definition of volume of line bundle). Then this pair is a weak stability condition on $D(X \times S)$.

Example 3.5. *If we take $X = Spec(\mathbb{C})$, $\mathcal{A}$ is the category of $\mathbb{C}$-vector spaces and $Z(V) = z \cdot dim(V)$ for any finite dimensional $\mathbb{C}$ vector space, where $z \in \mathbb{H} \cup \mathbb{R}_{\leq 0}$. Then the global heart is the category of coherent sheaves on S, and $L_E(n) = z \cdot Hilb_E(n)$ for $n \gg 0$. Therefore, we call $L_E(n)$ the complexified Hilbert polynomial.*

We get the following slope function μ_1 from the pair $(\mathcal{A}_S, Z_S)$:

$$\mu_1(E) := \begin{cases} -\frac{Re(Z_S(E))}{Im(Z_S(E))} & \text{if } Im(Z_S(E)) > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

This weak stability condition is closely related to the global slicing constructed in [BM14a, Section 4]. We include the explicit construction of the global slicing in the following paragraph.

Given a stability condition $\sigma = (Z, \mathcal{P})$ on $D(X)$ and a phase $\phi \in \mathbb{R}$, we have its associated t-structure

$$(\mathcal{P}(> \phi) = \mathcal{D}_\phi^{\leq -1}(X), \mathcal{P}(\leq \phi) = \mathcal{D}_\phi^{\geq 0}(X))$$

on $D(X)$. By Abramovich and Polishchuk's construction, we get a global t-structure

$$(\mathcal{P}_S(> \phi) := \mathcal{D}_\phi^{\leq -1}(X \times S), \mathcal{P}_S(\leq \phi) := \mathcal{D}_\phi^{\geq 0}(X \times S))$$

on $D(X \times S)$. Then we have the following lemma in [BM14a, Section 4].

Lemma 3.6 ([BM14a, Lemma 4.6]). *Assume $\sigma = (Z, \mathcal{P})$ is a weak stability condition as in our setup, and $\mathcal{P}_S(> \phi)$, $\mathcal{P}_S(\leq \phi)$ defined as above. There is a slicing $\mathcal{P}_S$ on $D^b(X \times S)$ defined by*

$$\mathcal{P}_S(\phi) = \mathcal{P}_S(\leq \phi) \cap \bigcap_{\epsilon > 0} \mathcal{P}_S(> \phi - \epsilon).$$

By Lemma 3.6, one can describe the relation between μ_1 semistable objects and the global slicing $\mathcal{P}_S$.

Proposition 3.7 ([Liu21, Proposition 3.14]). *If S is a smooth projective variety, and $E \in \mathcal{A}_S$ is semistable with respect to μ_1 of phase ϕ and $Z_S(E) \neq 0$, then there exists a short exact sequence*

$$0 \to K \to E \to Q \to 0$$

such that $K \in \mathcal{P}_S(\phi)$, $Q \in \mathcal{P}_S(< \phi)$ and $Z_S(Q) = 0$.

4. Abelian subcategories in $\mathcal{A}_S$

For the simplicity of our statements and arguments, we introduce the following definition.

Definition 4.1. If $(\mathcal{A}, Z)$ is a stability condition, and the image of Z lies in $\mathbb{Q} \oplus \mathbb{Q}i$, we call $(\mathcal{A}, Z)$ a rational stability condition. We use $RStab(X)$ to denote the set of rational stability conditions on $D(X)$.

Remark 4.2. By [AP06, Proposition 5.0.1], we know the heart $\mathcal{A}$ of a rational stability condition is Noetherian. We focus on the rational stability conditions just for the simplicity of statements and arguments. All results and proofs in the rest of this paper can be easily adapted to the stability conditions whose heart is Noetherian and central charge has discrete image in its imaginary part.

From now on, unless otherwise specified, we assume that our original stability condition σ on X is a rational stability condition. As in Remark 3.3, for any $E \in \mathcal{A}_S$, we can write

$$L_E(n) := \Sigma_{k=0}^r (a_k(E) + ib_k(E))n^k,$$

where $a_i, b_i : K_0(\mathcal{A}_S) \to \mathbb{Q}$ are group homomorphisms for $0 \le i \le r$.

Then we have the following positivity of coefficients a_k and b_k.

Proposition 4.3. *For any $E \in \mathcal{A}_S$, we have the following inequalities.*

(1) $b_r(E) \ge 0$.

(2) If $b_r(E) = 0$, then $a_r(E) \le 0$ and $b_{r-1}(E) \ge 0$.

(3) In general, if

$$b_r(E) = a_r(E) = b_{r-1}(E) = \cdots = a_i(E) = b_{i-1}(E) = 0,$$

then $a_{i-1}(E) \le 0$ and $b_{i-2}(E) \ge 0$ for any $2 \le i \le r$.

(4) Moreover, if E is a nonzero object and

$$b_r(E) = a_r(E) = b_{r-1}(E) = \cdots = a_1(E) = b_0(E) = 0,$$

then $a_0(E) < 0$.

Proof. See the proof in [Liu20, Lemma 4.1]. □

There are some full subcategories of $\mathcal{A}_S$.

Definition 4.4. For any integer $1 \le j \le r$, we can define a full subcategory $\mathcal{A}_S^{\le j}$ in $\mathcal{A}_S$ in the following way:

$$obj(\mathcal{A}_S^{\le j}) = \{E \in \mathcal{A}_S \,|deg(L_E(n)) \le j\}.$$

Lemma 4.5. *If E is an object in $\mathcal{A}_S^{\le j}$, and Q is a subquotient of E in $\mathcal{A}_S$, then $Q \in \mathcal{A}_S^{\le j}$.*

Proof. Assume E is an object in $\mathcal{A}_S^{\le j}$, let

$$0 \to K \to E \to Q \to 0$$

be a short exact sequence in $\mathcal{A}_S$.

Suppose k, q are the maximal integers such that $a_k(K) + ib_k(K) \ne 0$ and $a_q(Q) + ib_q(Q) \ne 0$ respectively. If $k > j$, then we get

$$a_k(K) + ib_k(K) = -a_k(Q) - ib_k(Q).$$

By Proposition 4.3, we get $b_k(K) \ge 0$, and if $b_k(K) = 0$, we have $a_k(K) < 0$ by the definition of k. Then $b_k(Q) \le 0$, and if $b_k(Q) = 0$, we have $a_k(Q) > 0$. Therefore, we have $q > k$ by Proposition 4.3. Similarly, we can prove that $k > q$, a contradiction.

Therefore, k, q can not exceed j, which implies $K, Q \in \mathcal{A}_S^{\le j}$. □

Corollary 4.6. $\mathcal{A}_S^{\le j}$ *is an abelian category for any $0 \le j \le r$.*

Proposition 4.7. *There exists subcategory $\mathcal{F}^{>j}$ in $\mathcal{A}_S$ for any $1 \le j \le r$, such that the pair $(\mathcal{A}_S^{\le j}, \mathcal{F}^{>j})$ is a torsion pair of $\mathcal{A}_S$.*

Proof. We claim that for any object $E \in \mathcal{A}_S$, there exists a maximal subobject $E_j \in \mathcal{A}_S^{\leq j}$ of E.

To prove the claim, we firstly show that for any two subobjects $E_1, E_2 \subset E$ and $E_1, E_2 \in \mathcal{A}_S^{\leq j}$, there exists a third subobject $E_3 \in \mathcal{A}_S^{\leq j}$ such that $E_1, E_2 \subset E_3 \subset E$.

Suppose we have two injections

$$E_1 \xhookrightarrow{f} E \quad and \quad E_2 \xhookrightarrow{g} E.$$

Then we have a morphism

$$E_1 \oplus E_2 \xrightarrow{(f,g)} E.$$

Let E_3 be the image of this morphism, by Lemma 4.5, we know that $E_3 \in \mathcal{A}_S^{\leq j}$. And $E_1, E_2 \subset E_3 \subset E$ by construction.

Hence it suffices to prove that the increasing sequence

$$E_1 \subset E_2 \subset \cdots \subset E$$

of subobjects $E_k \in \mathcal{A}_S^{\leq j}$ stabilizes after finite step. This follows directly from the fact $\mathcal{A}_S$ is Noetherian.

Let $F^{>j}$ be the full subcategory consisting of the objects whose maximal subobjects in $\mathcal{A}_S^{\leq j}$ is 0. Then it is easy to see that $(\mathcal{A}_S^{\leq j}, \mathcal{F}^{>j})$ is a torsion pair. $\square$

Corollary 4.8. *There exists a unique filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

for any object $E \in \mathcal{A}_S$, such that E_j is the maximal subobject of E in $\mathcal{A}_S^{\leq j}$.

Proof. By the proof of Proposition 4.7, we can take E_j to be the maximal subobject in $\mathcal{A}_S^{\leq j}$. Since $\mathcal{A}_S^{\leq j}$ is a full subcategory of $\mathcal{A}_S^{\leq k}$ for any integers $1 \leq j < k \leq r$, we get $E_j \subset E_k$, hence we get the unique filtration. $\square$

Remark 4.9. One can view this as a generalization of the torsion filtration of a sheaf (see [HL10, Definition 1.1.4]).

If we denote the minimal triangulated subcategory generated by $\mathcal{A}_S^{\leq 1}$ in $D(X \times S)$ by $\mathcal{D}_S^{\leq 1}$, then we can apply the construction in [Liu21] on $\mathcal{D}_S^{\leq 1}$.

Corollary 4.10. *The triangulated category $\mathcal{D}_S^{\leq 1}$ admits stability conditions.*

Proof. By Proposition 4.3, we know that for any $t \in \mathbb{Q}_{>0}$, the central charge Z_t defined by

$$Z_t(E) = a_1(E)t - b_0(E) + i \cdot b_1(E)t$$

is a weak stability function on $\mathcal{A}_S^{\leq 1}$. And the pair $\sigma_t := (\mathcal{A}_S^{\leq 1}, Z_t)$ is a weak stability condition by the rationality of the image of Z_t and Noetherianity of $\mathcal{A}_S^{\leq 1}$. Hence this gives us a torsion pair as in Remark 2.11, and we tilt the heart $\mathcal{A}_S^{\leq 1}$ in $\mathcal{D}_S^{\leq 1}$ with respect to such a torsion pair as in Lemma 2.12. We use $\mathcal{A}_S^{\leq 1,t}$ to denote the tilted heart.

By [Liu20, Theorem 4.5], we know that if $E \in \mathcal{A}_S^{\leq 1}$ is σ_t semistable, we have that

$$b_1(E)a_0(E) - b_0(E)a_1(E) \geq 0.$$

Therefore, we can use the results in [Liu21] to show that the pair $(\mathcal{A}_S^{\leq 1,t}, Z_S^{s,t})$, where $Z_S^{s,t}(E) = b_1(E)s + a_0(E) + i(-a_1(E)t + b_0(E))$, is a stability condition on $\mathcal{D}^{\leq 1}$ for any $s, t \in \mathbb{Q}_{>0}$. We omit the proof as it is exact the same as in [Liu21, Sections 4-5]. $\qquad\square$

4.1. **Abelian subcategories from a weak stability condition.** In this subsection, we consider a more general situation. Let $\sigma = (\mathcal{A}, Z)$ be an arbitrary weak stability condition on a triangulated category $\mathcal{D}$.

Lemma 4.11. *Let $\mathcal{A}_0$ denote the full subcategory in $\mathcal{A}$ consisting of objects E whose central charge is 0, then $\mathcal{A}_0$ is an abelian subcatgory.*

Proof. It is very easy to see that $\mathcal{A}_0$ is closed under subobjects and quotient objects. In particular, $\mathcal{A}_0$ is an abelian subcategory. $\qquad\square$

Remark 4.12. Corollary 4.6 can also be viewed as a direct consequence of Lemma 4.11. Indeed, for any $0 \leq j \leq r$, $(\mathcal{A}_S^{\leq j}, a_j + ib_j)$ is a weak stability condition, hence we can apply Lemma 4.11 inductively.

Let $\mathcal{P}_\sigma(\phi)$ be the subcategory of σ semistable objects whose phase is ϕ. It is easy to see that $\mathcal{A}_0 \subset \mathcal{P}_\sigma(1)$.

Lemma 4.13. *If we have a short exact sequence in $\mathcal{A}$*

$$0 \to K \to E \xrightarrow{f} Q \to 0,$$

where $K \in \mathcal{P}_\sigma(\phi)$ and $Q \in \mathcal{A}_0$, then we have the following short exact sequence in $\mathcal{A}$

$$0 \to Q' \to E \to K' \to 0,$$

where $Q' \in \mathcal{A}_0$ and $K' \in \mathcal{P}_\sigma(\phi)$.

Proof. If $K \in \mathcal{P}_\sigma(1)$, we know that E is in $\mathcal{P}_\sigma(1)$ as well. Hence we can take $K' = E$ and $Q' = 0$.

Now we can assume that $0 < \phi < 1$. By the HN property of σ, there exists a maximal subobject $Q' \in \mathcal{P}_\sigma(1)$ with a natural inclusion $i : Q' \hookrightarrow E$. Hence we have the following short exact sequence

$$0 \to Q' \xrightarrow{i} E \xrightarrow{g} K' \to 0.$$

We claim that $Z(Q') = 0$ and $K' \in \mathcal{P}_\sigma(\phi)$.

If $Z(Q') \neq 0$, then we have $Z(Q') \in \mathbb{R}_{<0}$. Let $G := ker(f \circ i) : Q' \to Q$, then we have $Z(G) = Z(Q') \in \mathbb{R}_{<0}$ and $G \hookrightarrow K$. This contradicts the facts $K \in \mathcal{P}_\sigma(\phi)$ and $\phi < 1$. Hence, we get $Z(Q') = 0$.

Now we know that $Z(E) = Z(K')$. Suppose that K' is not semistable. Let F' be a destabilizing subobject of K' and $F = g^{-1}(F')$. Then we have $Z(F) = Z(F')$ as $Z(Q') = 0$ and a morphism $f' : F \to Q$ which is the composition of f and the inclusion from F to E.

Let $H := ker(f')$. We know that $Z(H) = Z(F)$ and H is a subobject of K. Since $Z(K) = Z(K') = Z(E)$, H destabilizes K as $\mu(H) = \mu(F) = \mu(F') < \mu(K') = \mu(K)$, which contradicts the assumption that K is semistable.

Therefore, the claim is proved, and we get the short exact sequence.   □

Remark 4.14. In the proof of this lemma, we also proved that Q' is a subobject of Q.

Definition 4.15. We call $\mathcal{A}_0$ the abelianizer of the weak stability condition $\sigma = (\mathcal{A}, Z)$. If there exists a short exact sequence

$$0 \to Q' \to E \to K' \to 0,$$

where $Q' \in \mathcal{A}_0$ and $K' \in \mathcal{P}_\sigma(\phi)$, we say that E is quasi-semistable.

The following Proposition is the justification of naming $\mathcal{A}_0$ the abelianizer of σ.

Proposition 4.16. *Let*

$$\mathcal{A}_\sigma(\phi) := \{E \in \mathcal{A} \mid E \text{ is } quasi-semistable \text{ and } Z(E) \in exp(i\pi\phi) \cdot \mathbb{R}_{\geq 0}\}.$$

Then $\mathcal{A}_\sigma(\phi)$ is an abelian subcategory.

Proof. If $\phi = 1$, then $\mathcal{A}_\sigma(1) = \mathcal{P}_\sigma(1)$. Then $\mathcal{A}_\sigma(1)$ is an abelian category since it is closed under subobjects and quotient objects.

Now we can assume $0 < \phi < 1$. If we have a morphism $f : E_1 \to E_2$ in $\mathcal{A}$ between two objects $E_1, E_2 \in \mathcal{A}_\sigma(\phi)$. There exists the following diagram in $\mathcal{A}$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q_1 & \longrightarrow & E_1 & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f} & & & & \\
0 & \longrightarrow & Q_2 & \longrightarrow & E_2 & \longrightarrow & K_2 & \longrightarrow & 0
\end{array}
$$

where $Q_i \in \mathcal{A}_0$ and $K_i \in \mathcal{P}_\sigma(\phi)$ for $i = 1, 2$.

Since $0 < \phi < 1$ and $Q_i \in \mathcal{A}_0$, there is no nontrivial morphism from Q_1 to K_2. Hence the diagram can be completed by the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q_1 & \longrightarrow & E_1 & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & Q_2 & \longrightarrow & E_2 & \longrightarrow & K_2 & \longrightarrow & 0.
\end{array}
$$

It suffices to show that $ker(f), im(f)$ and $coker(f)$ are quasi-semistable.

By snake lemma, we have the following long exact sequence

$$0 \to ker(h) \to ker(f) \to ker(g) \xrightarrow{\delta} coker(h) \to coker(f) \to coker(g) \to 0.$$

This can be decomposed into two short exact sequences

$$0 \to ker(h) \to ker(f) \to ker(\delta) \to 0$$

and

$$0 \to coker(\delta) \to coker(f) \to coker(g) \to 0.$$

We know that $im(g)$ is a subobject of K_2 and a quotient object of K_1, and $K_1, K_2 \in \mathcal{P}_\sigma(\phi)$. This implies that $im(g) \in \mathcal{P}_\sigma(\phi)$, hence $ker(g) \in \mathcal{P}_\sigma(\phi)$ as well. Since $coker(h)$ is a quotient object of Q_2, it is an object in $\mathcal{A}_0$ by Lemma 4.11. Therefore, $Z(ker(\delta)) = Z(ker(g))$ and $ker(\delta) \subset ker(g)$ imply that $ker(\delta) \in \mathcal{P}_\sigma(\phi)$. Thus we get that $ker(f)$ is quasi-semistable of phase ϕ.

Now we want to show that $coker(f) \in \mathcal{A}_\sigma(\phi)$. It is easy to see that $coker(g) \in \mathcal{A}_\sigma(\phi)$. Indeed, suppose F is a subobject of $coker(g)$ that destabilizes $coker(g)$. Let $H := coker(g)/F$, then if $Z(F) \neq 0$, we have $Z(H) \neq 0$ and $\mu_\sigma(H) < \mu_\sigma(coker(g)) = \mu_\sigma(K_2)$. This contradicts to the assumption $K_2 \in \mathcal{P}_\sigma(\phi)$ as H is a quotient object of K_2. Hence we proved that any destabilizing subobject of $coker(g)$ is in $\mathcal{A}_0$. This is equivalent to $coker(g) \in \mathcal{A}_\sigma(\phi)$.

By Definition 4.15, we have $Q' \in \mathcal{A}_0$ and $K \in \mathcal{P}_\sigma(\phi)$ and the following short exact sequence

$$0 \to Q' \to coker(g) \to K \to 0.$$

Furthermore, we have the following commutative diagram by snake's lemma.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & coker(\delta) & \longrightarrow & Q & \longrightarrow & Q' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & coker(\delta) & \longrightarrow & coker(f) & \longrightarrow & coker(g) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & K & \xrightarrow{\ id\ } & K & \longrightarrow & 0
\end{array}
$$

where Q is the kernel of the composed morphism $coker(f) \to coker(g) \to K$, and every row and every column is a short exact sequence. Notice that $coker(\delta)$ is a quotient object of $coker(h) \in \mathcal{A}_0$, hence $Q \in \mathcal{A}_0$, and $coker(f) \in \mathcal{A}_\sigma(\phi)$.

It is left to prove that $im(f) \in \mathcal{A}_\sigma(\phi)$. By snake lemma, we have the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & ker(h) & \longrightarrow & ker(f) & \longrightarrow & ker(\delta) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_1 & \longrightarrow & E_1 & \longrightarrow & K_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & im(h) & \longrightarrow & im(f) & \longrightarrow & J & \longrightarrow & 0
\end{array}
$$

where $J := K_1/ker(\delta)$, $im(h) \in \mathcal{A}_0$ and every column is a short exact sequence. Hence it suffices to prove that $J \in \mathcal{A}_\sigma(\phi)$. This follows from the following commutative diagram.

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & im(\delta) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ker(\delta) & \longrightarrow & K_1 & \longrightarrow & J & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \\
0 & \longrightarrow & ker(g) & \longrightarrow & K_1 & \longrightarrow & im(g) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & im(\delta) & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

where $im(\delta) \in \mathcal{A}_0$ and $im(g) \in \mathcal{P}_\sigma(\phi)$, and every column is a short exact sequence. Hence $J \in \mathcal{A}_\sigma(\phi)$, the proof is complete. $\square$

Remark 4.17. When $\sigma = (\mathcal{A}, Z)$ is a stability condition, we know that $\mathcal{A}_0$ is trivial. In this case, Proposition 4.16 is [Bri07, Lemma 5.2].

5. LEXICOGRAPHIC ORDER FILTRATIONS AND TORSION PAIRS

In this section, we further study the possible applications of Abramovich-Polishchuk's construction on stability conditions. More explicitly, we want to generalize the construction in [Liu21] to the case when S is a higher dimensional projective variety. The idea is to do multiple times of tilts on $\mathcal{A}_S$ by induction, and prove suitable inequalities at each inductive step. However, the usual tilted heart (in Remark 2.11) does not satisfy the inequalities we need. In this section, we come up with a finer cut of $\mathcal{A}_S$ into torsion pairs (Corollary 5.11), whose tilted heart satisfies the inequalities we need (see Theorem 5.12). This leads us to the notion of l-th level semistable objects (Definition 5.6), which is a generalization of the slope stability and the Gieseker stability. We expect this to be the first step towards the generalization of the construction in [Liu21].

Let us focus on the abelian subcategories $\mathcal{A}_S^{\leq j}$, Proposition 4.3 and Proposition 4.16 give us lots of weak stability conditions and abelian subcategories in $\mathcal{A}_S$. As in Remark 2.7, we call a pair $(\mathcal{A}, Z)$ a stability condition if this pair admits HN property (without the reference to any triangulated categories and t-structures).

Lemma 5.1. *Let* $Z_j^t(E) = a_j(E)t - b_{j-1}(E) + ib_j(E)$, *then* $\sigma^t := (\mathcal{A}_S^{\leq j}, Z_j^t)$ *is a weak stability condition for any* $t \in \mathbb{Q}_{>0}$ *and any integer* $0 \leq j \leq r$.

Proof. By Proposition 4.3, Z_j^t is a weak stability function on $\mathcal{A}_S^{\leq j}$. Then by the rationality of b_j, a_j and the Noetherianity of $\mathcal{A}_S^{\leq j}$, we see that σ^t admits HN property. $\square$

Apply Proposition 4.16 on the weak stability condition σ^t in Lemma 5.1, we have the following abelian subcategories in $\mathcal{A}_S$.

Definition 5.2. For the simplicity of notation, let $\mathcal{A}^t(\phi)$ denote $\mathcal{A}_{\sigma^t}(\phi)$. Then $\mathcal{A}^t(\phi)$ is an abelian subcategory by Proposition 4.16.

Lemma 5.3. *We have the following stability conditions.*
(1) The pair
$$\sigma_1^{t,t'} = (\mathcal{A}^t(1), a_{j-1}t' - b_{j-2} + i(b_{j-1} - a_j t))$$
is a weak stability condition for any $t, t' \in \mathbb{Q}_{>0}$.
(2) The pair
$$\sigma_\phi^{t,t'} = (\mathcal{A}^t(\phi), a_{j-1}t' - b_{j-2} + ib_j)$$
is a weak stability condition for any $t, t' \in \mathbb{Q}_{>0}$ and $\phi \in (0,1)$.

Proof. For (1), assume that $E \in \mathcal{A}^t(1)$ and $b_{j-1}(E) - a_j(E)t = 0$. Then we get $b_j(E) = a_j(E) = b_{j-1}(E) = 0$. By Proposition 4.3, this implies that $a_{j-1}(E) \leq 0$ and $b_{j-2}(E) \geq 0$. Therefore, we know that $a_{j-1}t' - b_{j-2} + ib_j$ is a weak stability function on $\mathcal{A}^t(\phi)$.

For (2), assume that $E \in \mathcal{A}^t(\phi)$ and $b_j(E) = 0$. Then since $\phi \in (0,1)$, we have that $Z_j^t(E) = 0$, hence $b_j(E) = a_j(E) = b_{j-1}(E) = 0$. Therefore, by the same reason, we know that $a_{j-1}t' - b_{j-2} + ib_j$ is a weak stability function on $\mathcal{A}^t(\phi)$.

The HN property following from rationality of central charge and the Noetherianity of $\mathcal{A}^t(\phi)$. $\square$

We can construct weak stability conditions inductively by Proposition 4.3. For convention, we define $b_{-1} = 0$.

Lemma 5.4. *(1) Inductively, for any $1 \leq k \leq j$, let $\mathcal{A}_{\phi_1,\phi_2,\cdots,\phi_k}^{t_1,t_2,\cdots,t_k} := \mathcal{A}_{\sigma_{\phi_1,\phi_2,\cdots,\phi_{k-1}}^{t_1,t_2,\cdots,t_k}}(\phi_k)$,*
then the pair
$$\sigma_{\phi_1,\phi_2,\cdots,\phi_k}^{t_1,t_2,\cdots,t_{k+1}} = (\mathcal{A}_{\phi_1,\phi_2\cdots,\phi_k}^{t_1,t_2,\cdots,t_k}, a_{j-k}t_{k+1} - b_{j-k-1} + ib_j)$$
is a weak stability condition for any $t_1,\cdots,t_{k+1} \in \mathbb{Q}_{>0}$ and any $\phi_1\cdots,\phi_k \in (0,1)$.
(2) Given a sequence of real numbers $\phi_1,\cdots,\phi_k \in (0,1]$, let $M = \{1 \leq m \leq k | \phi_i = 1\}$.
Assume that M is a nonempty set, and let n be the maximal integer in M. Then the pair
$$\sigma_{\phi_1,\phi_2,\cdots,\phi_k}^{t_1,t_2,\cdots,t_{k+1}} = (\mathcal{A}_{\phi_1,\phi_2\cdots,\phi_k}^{t_1,t_2,\cdots,t_k}, a_{j-k}t_{k+1} - b_{j-k-1} - i(a_{j-n+1}t_n - b_{j-n}))$$
is a weak stability condition for any $t_1,\cdots,t_{k+1} \in \mathbb{Q}_{>0}$.

Proof. Firstly, we claim that for any $t_1, t_2, \cdots, t_{k+1} \in \mathbb{Q}_{>0}$ and $\phi_1, \cdots, \phi_k \in (0,1]$, the abelianizer (see Definition 4.15) of $\sigma_{\phi_1,\phi_2,\cdots,\phi_k}^{t_1,t_2,\cdots,t_{k+1}}$ is the full abelian subcategory consists of the following objects
$$\{E \in \mathcal{A}_S^{\leq j} | b_j(E) = a_j(E) = \cdots = b_{j-k}(E) = a_{j-k}(E) = b_{j-k-1}(E) = 0\}.$$

The claim and lemma can be proved by induction on k, the case $k = 1$ follows from Lemma 5.3 and Proposition 4.3.

Let (C_l) denote the proposition that the claim is true for all $1 \le k \le l$, and (L_l) denote the proposition that the lemma is true for all $1 \le k \le l$. Then we will prove that (C_{l-1}) implies (L_l) and (L_l) implies (C_l). Hence both propositions are true for any $1 \le k \le j$.

Indeed, assume that (C_{k-1}) is true. If $\phi_k < 1$, we assume that $E \in \mathcal{A}^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_k}$ and the imaginary part of E in $\sigma^{t_1,t_2,\cdots,t_{k+1}}_{\phi_1,\phi_2,\cdots,\phi_k}$ is 0. Since $\phi_k < 1$, the imaginary parts of $\sigma^{t_1,t_2,\cdots,t_{k+1}}_{\phi_1,\phi_2,\cdots,\phi_k}$ and $\sigma^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_{k-1}}$ are the same by construction. By definition of $\mathcal{A}^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2\cdots,\phi_k}$, we know that E is in the abelianizer of $\sigma^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_{k-1}}$. Hence by (C_{k-1}), we get that

$$b_j(E) = a_j(E) = \cdots = b_{j-k+1}(E) = a_{j-k+1}(E) = b_{j-k}(E) = 0.$$

Therefore, by Proposition 4.3, $a_{j-k}(E)t_{k+1} - b_{j-k-1}(E) \le 0$, hence (L_k) and (C_k) holds.

Now we consider the case when $\phi_k = 1$. We assume that $E \in \mathcal{A}^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_k}$ and the imaginary part of E in $\sigma^{t_1,t_2,\cdots,t_{k+1}}_{\phi_1,\phi_2,\cdots,\phi_k}$ is 0, i.e., $a_{j-k+1}(E)t_k - b_{j-k}(E) = 0$. Since $a_{j-k+1}t_k - b_{j-k}$ is the real part of $\sigma^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_{k-1}}$ and E is semistable of phase 1 with respect to $\sigma^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_{k-1}}$. We get that E is in the abelianizer of $\sigma^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_{k-1}}$. Hence (L_k) and (C_k) holds.

The HN property following from rationality of central charge and the Noetherianity of $\mathcal{A}^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_k}$. $\square$

Remark 5.5. In fact, t_i is not necessarily a fixed real number. It could be a continuous function $t_i : (0,1] \to \mathbb{R}_{>0}$, $\phi_{i+1} \mapsto t_i(\phi_{i+1})$, such that $t_i(1) \in \mathbb{Q}_{>0}$. We assume them to be fixed numbers for the simplicity of the notation.

Definition 5.6. Given $t_1, t_2, \cdots, t_l \in \mathbb{Q}_{>0}$, suppose E is σ^{t_1} semistable of phase ϕ_1 and $E \in \mathcal{A}^{t_1,t_2,\cdots,t_{k-1}}_{\phi_1,\phi_2\cdots,\phi_{k-1}}$ is semistable with respect to $\sigma^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_{k-1}}$ of phase ϕ_k for any $1 < k \le l$. Then we say that E is l-th level semistable of phase $\vec{\phi} = (\phi_1, \cdots, \phi_l)$, where $\phi_i \in (0, 1]$.

Example 5.7. *Note that in the trivial case $X = Spec(\mathbb{C})$, $\sigma = (\mathrm{Vect}, i \cdot dim)$, where Vect is the category of finite dimensional $\mathbb{C}$ vector spaces. And S is a smooth projective variety of dimension r. Then being 1-st level semistable is equivalent to being slope semistable, and being $(r+1)$-th level semitable is equivalent to being Gieseker semistable.*

We will define the lexicographic order of such vectors, i.e., we say that

$$(\phi_1, \phi_2, \cdots, \phi_l) > (\psi_1, \psi_2, \cdots, \psi_l)$$

if $\phi_1 > \psi_1$, or $\phi_1 = \psi_1, \phi_2 = \psi_2, \cdots, \phi_k = \psi_k$ and $\phi_{k+1} > \psi_{k+1}$ for some $1 \le k \le l - 1$.

Lemma 5.8. *Given $t_1, t_2, \cdots, t_k \in \mathbb{Q}_{>0}$, assume that E is $(k-1)$-th level semistable of phase $(\phi_1, \phi_2, \cdots, \phi_{k-1})$. Let*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

be its HN filtration with respect to $\sigma^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_{k-1}}$. Then every HN factor E_i/E_{i-1} is k-th level semistable of phase $(\phi_1, \cdots, \phi_{k-1}, \phi_{k,i})$, where $\phi_{k,1} > \phi_{k,2} > \cdots > \phi_{k,n}$ is a decreasing sequence of real numbers.

Proof. We let Q_i denote the HN factor E_i/E_{i-1}. Then Q_i is semistable with respect to $\sigma^{t_1,t_2,\cdots,t_k}_{\phi_1,\phi_2,\cdots,\phi_{k-1}}$ of phase $\phi_{k,i}$.

We claim that: for any object $F \in \mathcal{A}^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_k}$ which is semistable with respect to $\sigma^{t_1,\cdots,t_{k+1}}_{\phi_1,\cdots,\phi_k}$, F is also semistable with respect to $\sigma^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_{k-1}}$ of phase ϕ_k. The lemma follows naturally from the claim.

Indeed, by definition of $\mathcal{A}^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_k}$, we have the following short exact sequence

$$0 \to F_0 \to F \to F' \to 0$$

where F_0 is in the abelianizer of $\sigma^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_{k-1}}$ and F' is semistable with respect to $\sigma^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_{k-1}}$ of phase ϕ_k.

If $F_0 \simeq 0$, the claim is true. So now we assume that F_0 is a nonzero object. By the proof of Lemma 5.4, we know that

$$b_j(F_0) = a_j(F_0) = \cdots = b_{j-k+1}(F_0) = a_{j-k+1}(F_0) = b_{j-k}(F_0) = 0.$$

Hence the slope of F_0 with respect to $\sigma^{t_1,\cdots,t_{k+1}}_{\phi_1,\cdots,\phi_k}$ is $+\infty$. By the assumption $F \in \mathcal{A}^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_k}$ is semistable with respect to $\sigma^{t_1,\cdots,t_{k+1}}_{\phi_1,\cdots,\phi_k}$ of phase ϕ_k, we get that the imaginary part of F with respect to $\sigma^{t_1,\cdots,t_{k+1}}_{\phi_1,\cdots,\phi_k}$ should also be 0. By the proof of Lemma 5.4, we know that F is in the abelianizer of $\sigma^{t_1,\cdots,t_k}_{\phi_1,\cdots,\phi_{k-1}}$. Hence the claim is true and in this case we have $\phi_1 = \phi_2 = \cdots = \phi_{k+1} = 1$. □

Remark 5.9. From the proofs of Lemma 5.4 and Lemma 5.8, we know that if E is k-th level semistable of phase $(\phi_1, \cdots, \phi_k)$, then $\phi_i = 1$ implies that E is in the abelianizer of $\sigma^{t_1,\cdots,t_i}_{\phi_1,\cdots,\phi_{i-1}}$ and hence $\phi_h = 1$ for all $1 \le h \le i$.

Theorem 5.10. *Assume $l \le j$, given $t_1, t_2, \cdots, t_l \in \mathbb{Q}_{>0}$, for any object $E \in \mathcal{A}^{\le j}_S$, there exists a unique filtration*

$$0 = E_0 \subset E_1 \subset E_2 \cdots \subset E_n = E,$$

such that each quotient factor E_i/E_{i-1} is l-th level semistable of phase $\vec{\phi}_i$ and

$$\vec{\phi}_1 > \vec{\phi}_2 > \cdots > \vec{\phi}_n$$

.

Proof. For any $k < l$, we claim that if E is k-th level semistable of phase $\vec{\phi} = (\phi_1, \phi_2 \cdots, \phi_k)$. There exists a filtration for E as in the statement of theorem. Moreover, the first k coordinates of all the vectors $\vec{\phi}_1 > \vec{\phi}_2 > \cdots > \vec{\phi}_n$ are equal to $\vec{\phi}$.

The claim can be shown by induction. Firstly, if E is $(l-1)$-th level semistable of phase $(\phi_1, \phi_2, \cdots, \phi_{l-1})$. Then by Lemma 5.8, we get the claim.

Assume that the claim is true for any $(l-i)$-th level semistable E. The goal is to prove the claim for any $(l-i-1)$-th level semistable object E.

Now assume that E is $(l-i-1)$-th level semistable of phase $(\phi_1, \phi_2, \cdots, \phi_{l-i-1})$, take its HN filtration with respect to $\sigma^{t_1,t_2,\cdots,t_{l-i}}_{\phi_1,\phi_2,\cdots,\phi_{l-i-1}}$. By induction on the number of HN factors, we can reduce to the case when there are only two HN factors.

Indeed, suppose we have the following short exact sequence

$$0 \to K \to E \xrightarrow{f} Q \to 0,$$

where K, Q are $(l-i)$-th semistable of phases

$$(\phi_1, \phi_2, \cdots, \phi_{l-i-1}, \phi_{l-i,1}), \ (\phi_1, \phi_2, \cdots, \phi_{l-i-1}, \phi_{l-i,2})$$

respectively.

By induction, there exist filtrations

$$0 = K_0 \subset K_1 \subset \cdots \subset K_{n_1} = K$$

and

$$0 = Q_0 \subset Q_1 \subset \cdots \subset Q_{n_2} = Q$$

satisfying the requirements in the claim.

Now we take $E_i = K_i$ for $0 \le i \le n_1$ and $E_{n_1+h} = f^{-1}(Q_h)$ for $0 \le h \le n_2$, where $f^{-1}(Q_h)$ is the pullback of Q_h along the map f. Then it is easy to check the filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n_1+n_2} = E$$

satisfies the requirements of the claim. Hence we proved the existence of the filtration.

The uniqueness follows from the uniqueness of HN filtration. In fact, if we have a filtration

$$0 = E_0 \subset E_1 \subset E_2 \cdots \subset E_n = E,$$

such that each quotient factor E_i/E_{i-1} is l-th level semistable of phase $\vec{\phi}_i$ and

$$\vec{\phi}_1 > \vec{\phi}_2 > \cdots > \vec{\phi}_n,$$

then there is a partition

$$S_1 = \{\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_{i_1}\}, S_2 = \{\vec{\phi}_{i_1+1}, \cdots, \vec{\phi}_{i_2}\}, \cdots, S_{k+1} = \{\vec{\phi}_{i_k+1}, \cdots, \vec{\phi}_n\}$$

of the set $\{\vec{\phi}_1, \cdots, \vec{\phi}_n\}$ such that all the vectors in S_i have the same first coordinate ψ_i and $\psi_1 > \psi_2 > \cdots > \psi_{k+1}$.

By the proof of Proposition 4.16, we know that

$$0 = E_0 \subset E_{i_1} \subset E_{i_2} \subset \cdots \subset E_{i_k} \subset E$$

is the HN filtration of E with respect to σ^{t_1}. Hence it is unique. Then by inductively partitioning the vectors according to their next coordinates, we get the uniqueness of the filtration. $\square$

We can use Theorem 5.10 to get finer torsion pairs of $\mathcal{A}_S$.

Corollary 5.11. *Given* $\vec{t} = (t_1, \cdots, t_{r+1}) \in \mathbb{Q}_{>0}^{r+1}$ *and* $\vec{\phi} = (\phi_1, \cdots, \phi_{r+1}) \in (0,1)^{r+1}$. *Let*

$$\mathcal{T}_{\vec{\phi}}^{\vec{t}} := \{E \in \mathcal{A}_S | \text{all the factors in theorem 5.10 have phases} > \vec{\phi}\},$$

$$\mathcal{F}_{\vec{\phi}}^{\vec{t}} := \{E \in \mathcal{A}_S | \text{all the factors in theorem 5.10 have phases} \le \vec{\phi}\}.$$

This pair is a torsion pair of $\mathcal{A}_S$.

Proof. This follows directly from Theorem 5.10.   □

Hence we have a tilted heart $\mathcal{A}_{S,\vec{\phi}}^{\vec{t}} := \langle \mathcal{F}_{\vec{\phi}}^{\vec{t}}[1], \mathcal{T}_{\vec{\phi}}^{\vec{t}} \rangle$. Take $\vec{\phi} = (\frac{1}{2}, \cdots, \frac{1}{2})$, and denote the tilted heart by $\mathcal{A}_S^{t_1, \cdots, t_{r+1}}$.

Theorem 5.12. *For any object E in $\mathcal{A}_S^{t_1, t_2, \cdots, t_{r+1}}$, we have the following inequalities.*
  (1) $-a_r(E)t_1 + b_{r-1}(E) \geq 0$.
  (2) For any positive integer $0 < l \leq r$, if

$$-a_r(E)t_1 + b_{r-1}(E) = \cdots = -a_{r+1-l}(E)t_l + b_{r-l}(E) = 0,$$

then we have $-a_{r-l}(E)t_{l+1} + b_{r-l-1}(E) \geq 0$. [1]

Proof. By the definition of $\mathcal{A}_S^{t_1, t_2, \cdots, t_{r+1}}$, we have the following short exact sequence in $\mathcal{A}_S^{t_1, t_2, \cdots, t_{r+1}}$.

$$0 \to F[1] \to E \to T \to 0,$$

where $F \in \mathcal{F}_{\vec{\phi}}^{\vec{t}}$ and $T \in \mathcal{T}_{\vec{\phi}}^{\vec{t}}$, where $\vec{\phi} = (\frac{1}{2}, \cdots, \frac{1}{2})$ and $\vec{t} = (t_1, t_2, \cdots, t_{r+1})$.

By induction, we can easily show that if

$$-a_r(E)t_1 + b_{r-1}(E) = \cdots = -a_{r+1-l}(E)t_l + b_{r-l}(E) = 0,$$

then we have $T, F \in \mathcal{A}_{\frac{1}{2}, \cdots, \frac{1}{2}}^{t_1, t_2, \cdots, t_l}$. Hence by the definition of $\mathcal{F}_{\vec{\phi}}^{\vec{t}}$ and $\mathcal{T}_{\vec{\phi}}^{\vec{t}}$, we have

$$-a_{r-l}(T)t_{l+1} + b_{r-l-1}(T) \geq 0$$

and

$$-a_{r-l}(F)t_{l+1} + b_{r-l-1}(F) \leq 0.$$

This completes the proof.   □

Remark 5.13. We choose the vector $\vec{\phi}$ to be $(\frac{1}{2}, \cdots, \frac{1}{2})$ just for the simplicity of the statement. One can easily generalize the statement for arbitrary vector $\vec{\phi} \in (0, 1]^{r+1}$.

CONFLICT OF INTEREST

The author declares that there is no conflict of interest.

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[1] Recall the convention $b_{-1} = 0$.

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