Floating Point Numbers

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7 November 2016

Numbers

- So far, we only dealt with integers
- But there are other types of numbers

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- Rational numbers (from ratio $\simeq$ fraction)
  - 3/4 = 0.75
  - 10/3 = 3.33333333....

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- Rational numbers (from ratio $\simeq$ fraction)
  - 3/4 = 0.75
  - 10/3 = 3.33333333....

- Real numbers
  - π = 3.14159265...
  - e = 2.71828182...

Very Large Numbers

- Distance of sun and earth

$$150{,}000{,}000{,}000 \text{ meters}$$

- Scientific notation

$$1.5 \times 10^{11} \text{ meters}$$

- Another example: number of atoms in 12 gram of carbon-12 (1 mol)

$$6.022140857 \times 10^{23}$$

Binary Numbers in Scientific Notation

- Example binary number (π again)

$$11.0010010001$$

- Scientific notation

$$1.10010010001 \times 2^1$$

- General form

$$1.\mathrm{x} \times 2^{\mathrm{y}}$$

Representation

- IEEE 754 floating point standard

- Uses 4 bytes

31	30	29	28	27	26	25	24	23	22	21	20	...	2	1	0
s	exponent								fraction						
1 bit	8 bits								23 bits						

- Exponent is offset with a bias of 127

  e.g. $2^{-6} \rightarrow$ exponent = -6 + 127 = 121

Conversion into Binary

- $\pi = 3.14159265$
- Number before period: $3_{10} = 11_2$
- Conversion of fraction $.14159265$

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	$0.14159265 \times 2 \downarrow$

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0	0.2831853

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0	0.5663706

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- Number before period: $3_{10} = 11_2$
- Conversion of fraction $.14159265$

Digit	**Calculation**
	$0.14159265 \times 2 \downarrow$
0	$0.2831853 \times 2 \downarrow$
0	$0.5663706 \times 2 \downarrow$
1	0.1327412

Conversion into Binary

- $\pi = 3.14159265$
- Number before period: $3_{10} = 11_2$
- Conversion of fraction .14159265

Digit	Calculation	Digit	Calculation
	$0.14159265 \times 2 \downarrow$	1	$0.9817472 \times 2 \downarrow$
0	$0.2831853 \times 2 \downarrow$	1	$0.9634944 \times 2 \downarrow$
0	$0.5663706 \times 2 \downarrow$	1	$0.9269888 \times 2 \downarrow$
1	$0.1327412 \times 2 \downarrow$	1	$0.8539776 \times 2 \downarrow$
0	$0.2654824 \times 2 \downarrow$	1	$0.7079552 \times 2 \downarrow$
0	$0.5309648 \times 2 \downarrow$	1	$0.4159104 \times 2 \downarrow$
1	$0.0619296 \times 2 \downarrow$	0	$0.8318208 \times 2 \downarrow$
0	$0.1238592 \times 2 \downarrow$	1	$0.6636416 \times 2 \downarrow$
0	$0.2477184 \times 2 \downarrow$	1	$0.3272832 \times 2 \downarrow$
0	$0.4954368 \times 2 \downarrow$	0	$0.6545664 \times 2 \downarrow$
0	$0.9908736 \times 2 \rightarrow$	1	0.3091328×2

- Binary: 11.001001000011111101101

Encoding into Representation

- π

$$1.10010010000111111101101 \times 2^1$$

- Encoding

Sign	Exponent	Fraction
0	10000000	10010010000111111101101

- Note: leading 1 in fraction is omitted

Special Cases

- Zero

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- Negative infinity (-1/0)
- Not a number (0/0 or $\infty - \infty$)

Encoding

Exponent	Fraction	Object
0	0	zero
0	>0	denormalized number
1-254	anything	floating point number
255	0	infinity
255	>0	NaN (not a number)

(denormalized number: $0.x \times 2^{-126}$)

Double Precision

- Single precision = 4 bytes
- Double precision = 8 bytes

Sign	Exponent	Fraction
1 bit	8 bits	23 bits
1 bit	11 bits	52 bits

addition

Addition with Scientific Notation

- Decimal example, with 4 significant digits in encoding

- Example

$$0.1610 + 99.99$$

- In scientific notation

$$1.610 \times 10^{-1} + 9.999 \times 10^{1}$$

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$$0.1610 + 99.99$$

- In scientific notation

$$1.610 \times 10^{-1} + 9.999 \times 10^{1}$$

- Bring lower number on same exponent as higher number

$$0.01610 \times 10^{1} + 9.999 \times 10^{1}$$

Addition with Scientific Notation

- Round to 4 significant digits

$$0.016 \times 10^1 + 9.999 \times 10^1$$

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$$0.016 \times 10^1 + 9.999 \times 10^1$$

- Add fractions

$$0.016 + 9.999 = 10.015$$

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- Adjust exponent

$$10.015 \times 10^1 = 1.0015 \times 10^2$$

Addition with Scientific Notation

- Round to 4 significant digits

$$0.016 \times 10^1 + 9.999 \times 10^1$$

- Add fractions

$$0.016 + 9.999 = 10.015$$

- Adjust exponent

$$10.015 \times 10^1 = 1.0015 \times 10^2$$

- Round to 4 significant digits

$$1.002 \times 10^2$$

Binary Floating Point Addition

- Numbers $\quad 0.5_{10} = \frac{1}{2}_{10}$

Binary Floating Point Addition

- Numbers $0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10}$

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- Numbers $\quad 0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2$

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- Bring lower number on same exponent as higher number

$$-1.110 \times 2^{-2} = -0.111 \times 2^{-1}$$

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$$-1.110 \times 2^{-2} = -0.111 \times 2^{-1}$$

- Add the fractions

$$1.000_2 \times 2^{-1} + (-0.111 \times 2^{-1}) = 0.001 \times 2^{-1}$$

Binary Floating Point Addition

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- Bring lower number on same exponent as higher number

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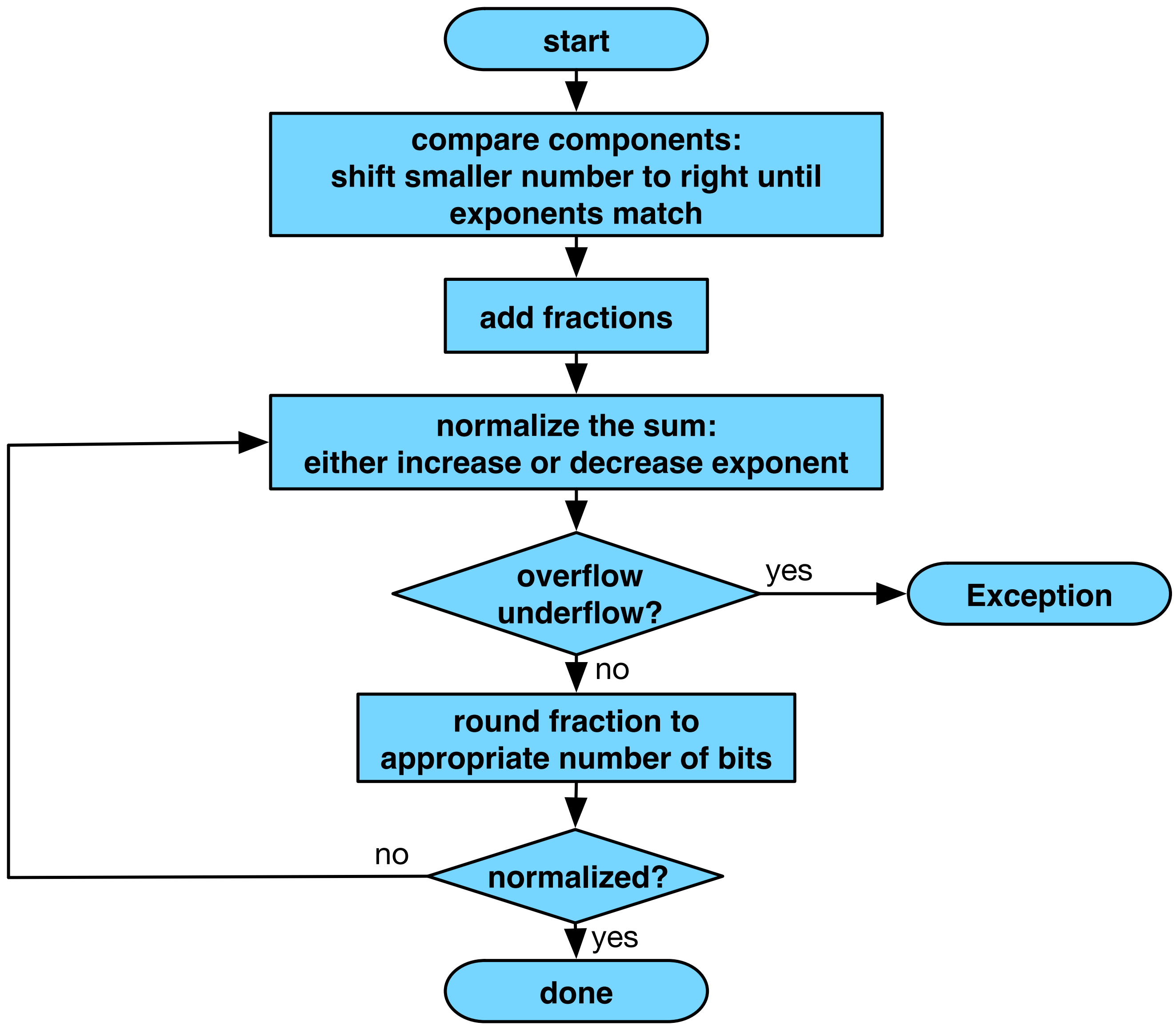
- Add the fractions

$$1.000_2 \times 2^{-1} + (-0.111 \times 2^{-1}) = 0.001 \times 2^{-1}$$

- Adjust exponent

$$0.001 \times 2^{-1} = 1.000 \times 2^{-4}$$

Flowchart


start
compare components:
shift smaller number to right until
exponents match
add fractions
normalize the sum:
either increase or decrease exponent
overflow
underflow?
yes
Exception
no
round fraction to
appropriate number of bits
no
normalized?
yes
done

multiplication

Multiplication with Scientific Notation

- Example: multiply 1.110×10^{10} and 9.200×10^{-5}

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$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

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$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

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- Example: multiply 1.110×10^{10} and 9.200×10^{-5}

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$$1.110 \times 9.200 \times 10^{-5+10}$$

- Add exponents

$$-5 + 10 = 5$$

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- Example: multiply 1.110×10^{10} and 9.200×10^{-5}

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$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

- Add exponents

$$-5 + 10 = 5$$

- Multiply fractions

$$1.110 \times 9.200 = 10.212$$

Multiplication with Scientific Notation

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$$1.110 \times 10^{10} \;\times\; 9.200 \times 10^{-5}$$

$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

- Add exponents

$$-5 + 10 = 5$$

- Multiply fractions

$$1.110 \times 9.200 = 10.212$$

- Adjust exponent

$$10.212 \times 10^{5} = 1.0212 \times 10^{6}$$

Binary Floating Point Multiplication

- Example

$$1.000 \times 2^{-1} \ \times \ -1.110 \times 2^{-2}$$

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- Example

$$1.000 \times 2^{-1} \; \times \; -1.110 \times 2^{-2}$$

- Add exponents

$$-1 + (-2) = -3$$

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$$1.000 \times 2^{-1} \ \times \ -1.110 \times 2^{-2}$$

- Add exponents

$$-1 + (-2) = -3$$

- Multiply fractions

$$1.000 \times -1.110 = -1.110$$

Binary Floating Point Multiplication

- Example

$$1.000 \times 2^{-1} \ \times \ -1.110 \times 2^{-2}$$

- Add exponents

$$-1 + (-2) = -3$$

- Multiply fractions

$$1.000 \times -1.110 = -1.110$$

$$1000 \times 1110 = 1110000$$

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- Example

$$1.000 \times 2^{-1} \; \times \; -1.110 \times 2^{-2}$$

- Add exponents

$$-1 + (-2) = -3$$

- Multiply fractions

$$1.000 \times -1.110 = -1.110$$

$$1000 \times 1110 = 1110000$$

$$-1.110000$$

Binary Floating Point Multiplication

- Example

$$1.000 \times 2^{-1} \;\times\; -1.110 \times 2^{-2}$$

- Add exponents

$$-1 + (-2) = -3$$

- Multiply fractions

$$1.000 \times -1.110 = -1.110$$

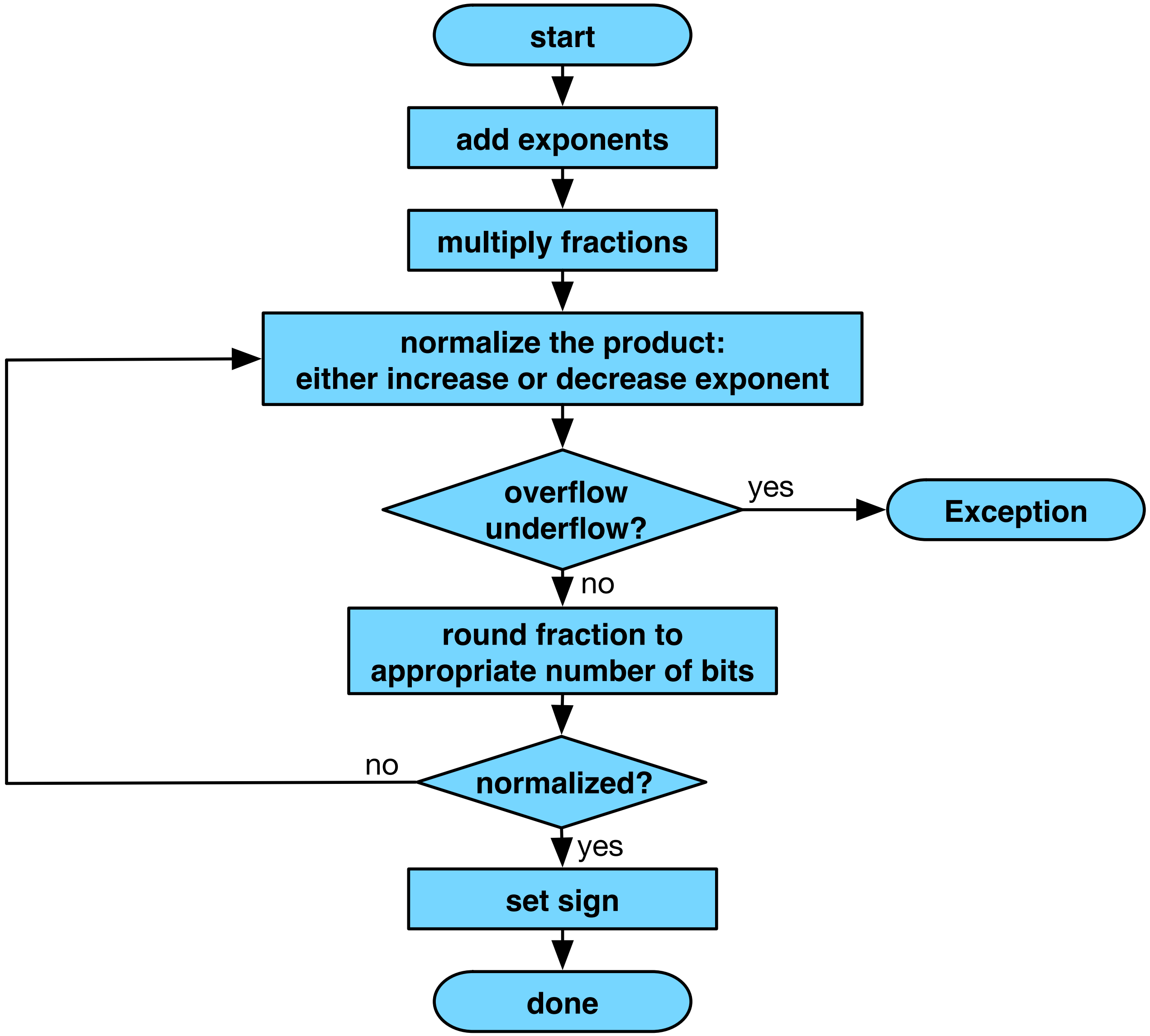
$$1000 \times 1110 = 1110000$$

$$-1.110000$$

- Adjust exponent (not needed)

$$-1.110 \times 2^{-3}$$

Flowchart


start
add exponents
multiply fractions
normalize the product:
either increase or decrease exponent
overflow
underflow?
yes
Exception
no
round fraction to
appropriate number of bits
no
normalized?
yes
set sign
done

mips instructions

Instructions

- Both single precision (s) and double precision (d)
- Addition (add.s / add.d)
- Subtraction (sub.s / sub.d)
- Multiplication (mul.s / mul.d)
- Division (div.s / div.d)
- Comparison (c.x.s / c.x.d)
  - equality (x = eq), inequality (x = neq)
  - less than (x = lt), less than or equal (x = le)
  - greater than (x = gt), greater than or equal (x = ge)
- Floating point branch on true (bclt) or fals (bclf)

Floating Point Registers

- MIPS has a separate set of registers for floating point numbers

- Little overhead, since used for different instructions
  - no need to specify in add, subtract, etc. instruction codes
  - different wiring for floating point / integer registers
  - much more limited use for floating point registers
    (e.g., never an address)

- Double precision = 2 registers used

Example

- Conversion Fahrenheit to Celsius (5.0/9.0 $\times$ (x - 32.0))
- Input value x stored in register $f12, constant in offsets to $gp
- Code

```
lwcl   $f16, const5($gp)    ; load 5.0
lwcl   $f18, const9($gp)    ; load 9.0
div.s  $f16, $f16, $f18     ; $f16 = 5.0/9.0
lwcl   $f18, const32($gp)   ; load 32.0
sub.s  $f18, $f12, $f18     ; $f18 = x-32.0
mul.s  $f0, $f16, $f18      ; $f0 = result
```