

Lecture 5: Floating point

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601.229 Computer Systems Fundamentals



Floating point numbers

Numbers

- ▶ So far, we only dealt with integers
- ▶ But there are other types of numbers

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- ▶ But there are other types of numbers
- ▶ Rational numbers (from ratio \simeq fraction)
 - ▶ $3/4 = 0.75$
 - ▶ $10/3 = 3.33333333....$

Numbers

- ▶ So far, we only dealt with integers
- ▶ But there are other types of numbers
- ▶ Rational numbers (from ratio \simeq fraction)
 - ▶ $3/4 = 0.75$
 - ▶ $10/3 = 3.33333333....$
- ▶ Real numbers
 - ▶ $\pi = 3.14159265...$
 - ▶ $e = 2.71828182...$

Very Large Numbers

- ▶ Distance of sun and earth

150,000,000,000 meters

- ▶ Scientific notation

1.5×10^{11} meters

- ▶ Another example: number of atoms in 12 gram of carbon-12 (1 mol)

$6.022140857 \times 10^{23}$

Binary Numbers in Scientific Notation

- ▶ Example binary number (π again)

11.0010010001

- ▶ Scientific notation

1.10010010001×2^1

- ▶ General form

$1.x \times 2^y$

Representation

- ▶ IEEE 754 floating point standard
- ▶ Uses 4 bytes

31	30	29	...	24	23	22	21	...	1	0
s	exponent					fraction				
1 bit	8 bits					23 bits				

- ▶ Exponent is offset with a bias of 127
e.g. $2^{-6} \rightarrow \text{exponent} = -6 + 127 = 121$

Conversion into Binary

- ▶ $\pi = 3.14159265$
- ▶ Number before period: $3_{10} = 11_2$
- ▶ Conversion of fraction $.14159265$

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	$0.14159265 \times 2 \downarrow$
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0	0.2831853
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0	$0.2831853 \times 2 \downarrow$
0	0.5663706

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	$0.14159265 \times 2 \downarrow$
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0	$0.2831853 \times 2 \downarrow$
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0	$0.5663706 \times 2 \downarrow$
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1	0.1327412
---	-------------

Conversion into Binary

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- ▶ Number before period: $3_{10} = 11_2$
- ▶ Conversion of fraction .14159265

Digit	Calculation
	$0.14159265 \times 2 \downarrow$
0	$0.2831853 \times 2 \downarrow$
0	$0.5663706 \times 2 \downarrow$
1	$0.1327412 \times 2 \downarrow$
0	$0.2654824 \times 2 \downarrow$
0	$0.5309648 \times 2 \downarrow$
1	$0.0619296 \times 2 \downarrow$
0	$0.1238592 \times 2 \downarrow$
0	$0.2477184 \times 2 \downarrow$
0	$0.4954368 \times 2 \downarrow$
0	$0.9908736 \times 2 \rightarrow$

Digit	Calculation
1	$0.9817472 \times 2 \downarrow$
1	$0.9634944 \times 2 \downarrow$
1	$0.9269888 \times 2 \downarrow$
1	$0.8539776 \times 2 \downarrow$
1	$0.7079552 \times 2 \downarrow$
1	$0.4159104 \times 2 \downarrow$
0	$0.8318208 \times 2 \downarrow$
1	$0.6636416 \times 2 \downarrow$
1	$0.3272832 \times 2 \downarrow$
0	$0.6545664 \times 2 \downarrow$
1	0.3091328×2

- ▶ Binary: 11.00100100001111101101

Encoding into Representation

► π

$$1.1001001000011111101101 \times 2^1$$

► Encoding

Sign	Exponent	Fraction
0	10000000	1001001000011111101101

► Note: leading 1 in fraction is omitted

Special Cases

► Zero

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- ▶ Negative infinity ($-1/0$)
- ▶ Not a number ($0/0$ or $\infty - \infty$)

Encoding

Exponent	Fraction	Object
0	0	zero
0	>0	denormalized number
1-254	anything	floating point number
255	0	infinity
255	>0	NaN (not a number)

(denormalized number: $0.x \times 2^{-126}$)

Double Precision

- ▶ Single precision = 4 bytes

Sign	Exponent	Fraction
1 bit	8 bits	23 bits

- ▶ Double precision = 8 bytes

Sign	Exponent	Fraction
1 bit	11 bits	52 bits

Addition

Addition with Scientific Notation

- ▶ Decimal example, with 4 significant digits in encoding
- ▶ Example

$$0.1610 + 99.99$$

- ▶ In scientific notation

$$1.610 \times 10^{-1} + 9.999 \times 10^1$$

Addition with Scientific Notation

- ▶ Decimal example, with 4 significant digits in encoding
- ▶ Example

$$0.1610 + 99.99$$

- ▶ In scientific notation

$$1.610 \times 10^{-1} + 9.999 \times 10^1$$

- ▶ Bring lower number on same exponent as higher number

$$0.01610 \times 10^1 + 9.999 \times 10^1$$

Addition with Scientific Notation

- ▶ Round to 4 significant digits

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- ▶ Add fractions

$$0.016 + 9.999 = 10.015$$

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- ▶ Round to 4 significant digits

$$0.016 \times 10^1 + 9.999 \times 10^1$$

- ▶ Add fractions

$$0.016 + 9.999 = 10.015$$

- ▶ Adjust exponent

$$10.015 \times 10^1 = 1.0015 \times 10^2$$

Addition with Scientific Notation

- ▶ Round to 4 significant digits

$$0.016 \times 10^1 + 9.999 \times 10^1$$

- ▶ Add fractions

$$0.016 + 9.999 = 10.015$$

- ▶ Adjust exponent

$$10.015 \times 10^1 = 1.0015 \times 10^2$$

- ▶ Round to 4 significant digits

$$1.002 \times 10^2$$

Binary Floating Point Addition

► Numbers

$$0.5_{10} = \frac{1}{2}_{10}$$

Binary Floating Point Addition

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$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10}$$

Binary Floating Point Addition

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$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2$$

Binary Floating Point Addition

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$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2 = 1.000_2 \times 2^{-1}$$

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$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2 = 1.000_2 \times 2^{-1}$$

$$-0.4375_{10} = -\frac{7}{16}_{10}$$

Binary Floating Point Addition

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$$-0.4375_{10} = -\frac{7}{16}_{10} = -\frac{7}{2^4}_{10} = 0.0111_2 = -1.110_2 \times 2^{-2}$$

- Bring lower number on same exponent as higher number

$$-1.110 \times 2^{-2} = -0.111 \times 2^{-1}$$

Binary Floating Point Addition

- Numbers

$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2 = 1.000_2 \times 2^{-1}$$

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- Bring lower number on same exponent as higher number

$$-1.110 \times 2^{-2} = -0.111 \times 2^{-1}$$

- Add the fractions

$$1.000_2 \times 2^{-1} + (-0.111 \times 2^{-1}) = 0.001 \times 2^{-1}$$

Binary Floating Point Addition

- Numbers

$$0.5_{10} = \frac{1}{2}_{10} = \frac{1}{2^1}_{10} = 0.1_2 = 1.000_2 \times 2^{-1}$$

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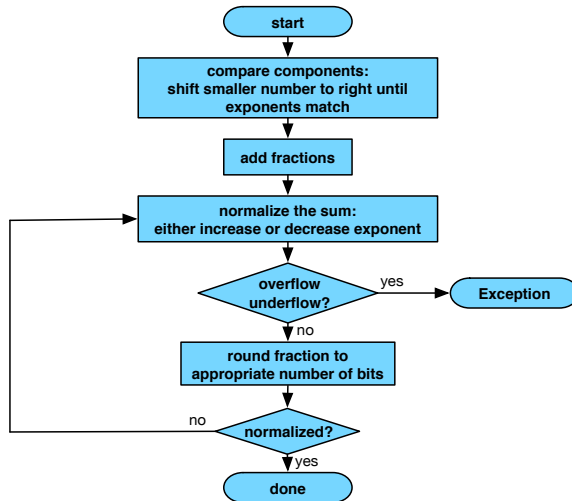
- Add the fractions

$$1.000_2 \times 2^{-1} + (-0.111 \times 2^{-1}) = 0.001 \times 2^{-1}$$

- Adjust exponent

$$0.001 \times 2^{-1} = 1.000 \times 2^{-4}$$

Flowchart



Multiplication

Multiplication with Scientific Notation

- ▶ Example: multiply 1.110×10^{10} and 9.200×10^{-5}

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$$1.110 \times 10^{10} \times 9.200 \times 10^{-5}$$

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Multiplication with Scientific Notation

- Example: multiply 1.110×10^{10} and 9.200×10^{-5}

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$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

Multiplication with Scientific Notation

- ▶ Example: multiply 1.110×10^{10} and 9.200×10^{-5}

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$$1.110 \times 9.200 \times 10^{-5+10}$$

- ▶ Add exponents

$$-5 + 10 = 5$$

Multiplication with Scientific Notation

- ▶ Example: multiply 1.110×10^{10} and 9.200×10^{-5}

$$1.110 \times 10^{10} \times 9.200 \times 10^{-5}$$

$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

- ▶ Add exponents

$$-5 + 10 = 5$$

- ▶ Multiply fractions

$$1.110 \times 9.200 = 10.212$$

Multiplication with Scientific Notation

- ▶ Example: multiply 1.110×10^{10} and 9.200×10^{-5}

$$1.110 \times 10^{10} \times 9.200 \times 10^{-5}$$

$$1.110 \times 9.200 \times 10^{-5} \times 10^{10}$$

$$1.110 \times 9.200 \times 10^{-5+10}$$

- ▶ Add exponents

$$-5 + 10 = 5$$

- ▶ Multiply fractions

$$1.110 \times 9.200 = 10.212$$

- ▶ Adjust exponent

$$10.212 \times 10^5 = 1.0212 \times 10^6$$

Binary Floating Point Multiplication

► Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

Binary Floating Point Multiplication

- ▶ Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

- ▶ Add exponents

$$-1 + (-2) = -3$$

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- ▶ Multiply fractions

$$1.000 \times -1.110 = -1.110$$

Binary Floating Point Multiplication

- ▶ Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

- ▶ Add exponents

$$-1 + (-2) = -3$$

- ▶ Multiply fractions

$$1.000 \times -1.110 = -1.110$$

$$1000 \times 1110 = 1110000$$

Binary Floating Point Multiplication

- ▶ Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

- ▶ Add exponents

$$-1 + (-2) = -3$$

- ▶ Multiply fractions

$$1.000 \times -1.110 = -1.110$$

$$1000 \times 1110 = 1110000$$

$$-1.110000$$

Binary Floating Point Multiplication

- ▶ Example

$$1.000 \times 2^{-1} \times -1.110 \times 2^{-2}$$

- ▶ Add exponents

$$-1 + (-2) = -3$$

- ▶ Multiply fractions

$$1.000 \times -1.110 = -1.110$$

$$1000 \times 1110 = 1110000$$

$$-1.110000$$

- ▶ Adjust exponent (not needed)

$$-1.110 \times 2^{-3}$$

Flowchart

