On the complexity and completeness of static constraints for breaking row and column symmetry, CP 2010

George Katsirelos, Nina Narodytska, and Toby Walsh.

December 7, 2010

Presented by Jenny Lam

Overview of the Paper

Title	On the Complexity and Completeness of Static Constraints for Breaking Row and Column Symmetry
Authors	George Katsirelos, Nina Narodytska, and Toby Walsh
Submitted	arXiv: July 5, 2010
To appear	Proceedings of the 16th International Conference on Principles and Practice of Constraint Programming (CP 2010)

What is Symmetry Breaking?

Motivations

- prune the search space
- eliminate redundant solutions
- retain one solution per symmetry class

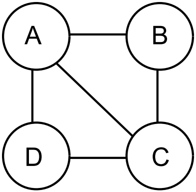
Concerns

- checking satisfaction can be NP-hard
- enforcing domain-consistency (DC) can be NP-hard
- more than one solution may be retained

Variable Symmetry: Example

4-color problem

- Domains: r, b, g, y
- Constraints: different color



Symmetric solutions

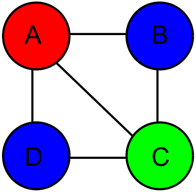
Variable Symmetry: Example

4-color problem

- Domains: r, b, g, y
- Constraints: different color

Symmetric solutions

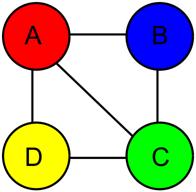
- (r, b, g, b),



Variable Symmetry: Example

4-color problem

- Domains: r, b, g, y
- Constraints: different color



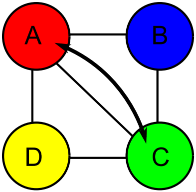
Symmetric solutions

- (r, b, g, b),
- (r, b, g, y)

Variable Symmetry: Example

4-color problem

- Domains: r, b, g, y
- Constraints: different color



Symmetric solutions

- (r, b, g, b),
- (r, b, g, y)
- (AC):

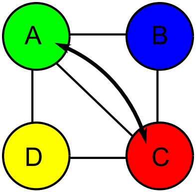
Variable Symmetry: Example

4-color problem

- Domains: r, b, g, y
- Constraints: different color

Symmetric solutions

- (r, b, g, b),
- (r, b, g, y)
- (AC): (g, b, r, y)



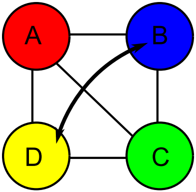
Variable Symmetry: Example

4-color problem

- Domains: r, b, g, y
- Constraints: different color

Symmetric solutions

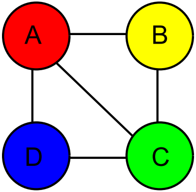
- (r, b, g, b),
- (r, b, g, y)
- (AC): (g, b, r, y)
- (BD):



Variable Symmetry: Example

4-color problem

- Domains: r, b, g, y
- Constraints: different color



Symmetric solutions

- (r, b, g, b),
- (r, b, g, y)
- (AC): (g, b, r, y)
- (BD): (r, y, g, b)

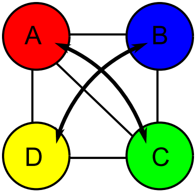
Variable Symmetry: Example

4-color problem

- Domains: r, b, g, y
- Constraints: different color

Symmetric solutions

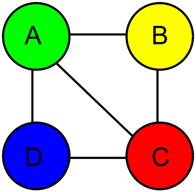
- (r, b, g, b),
- (r, b, g, y)
- (AC): (g, b, r, y)
- (BD): (r, y, g, b)
- $(AC)(BD)$:



Variable Symmetry: Example

4-color problem

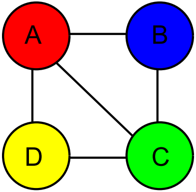
- Domains: r, b, g, y
- Constraints: different color



Symmetric solutions

- (r, b, g, b),
- (r, b, g, y)
- (AC): (g, b, r, y)
- (BD): (r, y, g, b)
- $(AC)(BD)$: (g, y, r, b)

Value Symmetry: Example



Symmetric solutions

Any permutation of the 4 colors is symmetric.

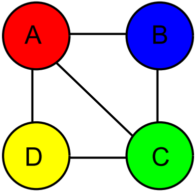
- (r, b, g, y)
- (rb):

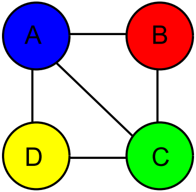
Value Symmetry: Example

Symmetric solutions

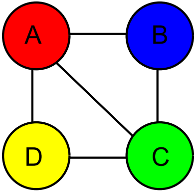
Any permutation of the 4 colors is symmetric.

- (r, b, g, y)
- (rb): (b, r, g, y)





Value Symmetry: Example

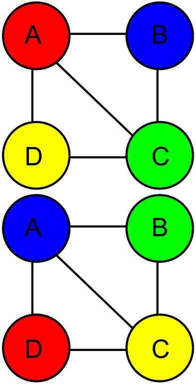


Symmetric solutions

Any permutation of the 4 colors is symmetric.

- (r, b, g, y)
- (rb): (b, r, g, y)
- $(rbgy)$:

Value Symmetry: Example

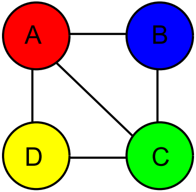


Symmetric solutions

Any permutation of the 4 colors is symmetric.

- (r, b, g, y)
- (rb): (b, r, g, y)
- $(rbgy)$: (b, g, y, r)

Value Symmetry: Example



Symmetric solutions

Any permutation of the 4 colors is symmetric.

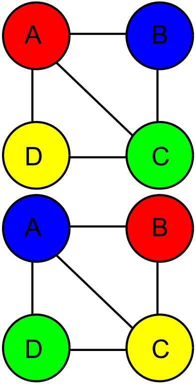
- (r, b, g, y)
- (rb): (b, r, g, y)
- $(rbgy)$: (b, g, y, r)
- $(rb)(gy)$:

Value Symmetry: Example

Symmetric solutions

Any permutation of the 4 colors is symmetric.

- (r, b, g, y)
- (rb): (b, r, g, y)
- $(rbgy)$: (b, g, y, r)
- $(rb)(gy)$: (b, r, y, g)



Variable and Value Symmetry: Definitions

Let (X, D, C) be a CSP such that
each variable $x \in X$ has the same domain D.

Variable Symmetry

a bijection $\sigma : X \to X$ of the variables that preserves solutions

$$\{X_i = a_i \mid i \in [1, n]\} \implies \left\{X_i = a_{\sigma(i)} \;\middle|\; i \in [1, n]\right\}$$

Value Symmetry

a bijection $\tau : D \to D$ of the values that preserves solutions

$$\{X_i = a_i \mid i \in [1, n]\} \implies \{X_i = \tau(a_i) \mid i \in [1, n]\}$$

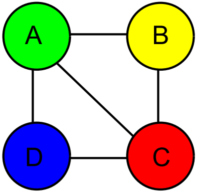
The LexLeader constraint

A solution satisfies LexLeader if

- of all solutions obtained by symmetry
- it is the smallest lexicographically.

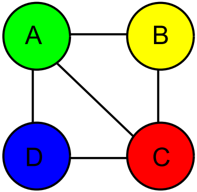
Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?



Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?

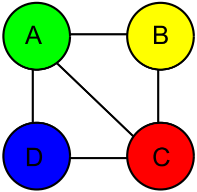


fix the variable ordering

- Order: (A, B, C, D)

Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?

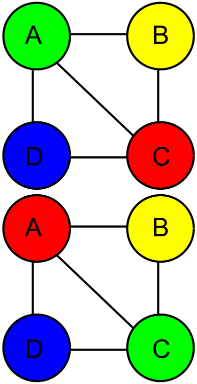


find all symmetric solutions

- Order: (A, B, C, D)

Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?



find all symmetric solutions

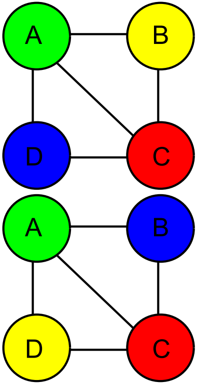
- Order: (A, B, C, D)
- $(AC) \Rightarrow (r, y, g, b)$

Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?

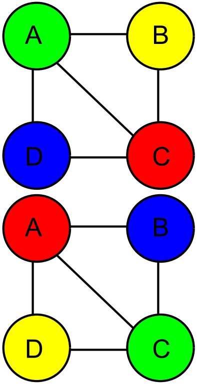
find all symmetric solutions

- Order: (A, B, C, D)
- $(AC) \Rightarrow (r, y, g, b)$
- $(BD) \Rightarrow (g, b, r, y)$



Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?

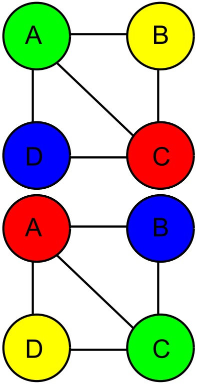


find all symmetric solutions

- Order: (A, B, C, D)
- $(AC) \Rightarrow (r, y, g, b)$
- $(BD) \Rightarrow (g, b, r, y)$
- $(AC)(BD) \Rightarrow (r, b, g, y)$

Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?

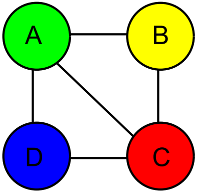


check ordering

- Order: (A, B, C, D)
- $(AC) \Rightarrow (r, y, g, b)$
- $(BD) \Rightarrow (g, b, r, y)$
- $(AC)(BD) \Rightarrow (r, b, g, y)$

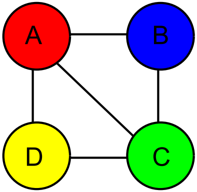
Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?



check ordering

- Order: (A, B, C, D)
- $(g, y, r, b) \leq_{\text{lex}} (r, y, g, b)$ ✓
- $(BD) \Rightarrow (g, b, r, y)$
- $(AC)(BD) \Rightarrow (r, b, g, y)$

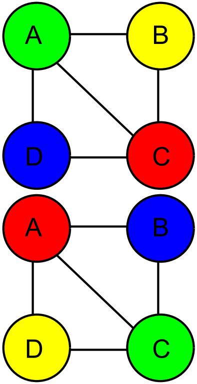


Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?

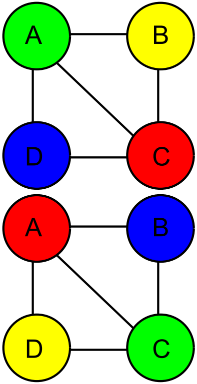
check ordering

- Order: (A, B, C, D)
- $(g, y, r, b) \leq_{\text{lex}} (r, y, g, b)$ ✓
- $(g, y, r, b) \leq_{\text{lex}} (g, b, r, y)$ $\times$
- $(AC)(BD) \Rightarrow (r, b, g, y)$



Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?

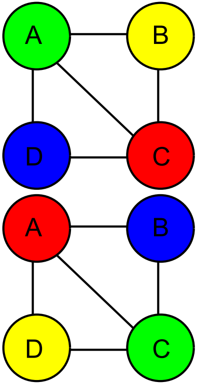


check ordering

- Order: (A, B, C, D)
- $(g, y, r, b) \leq_{\text{lex}} (r, y, g, b)$ ✓
- $(g, y, r, b) \leq_{\text{lex}} (g, b, r, y)$ $\times$
- $(g, y, r, b) \leq_{\text{lex}} (r, b, g, y)$ ✓

Breaking Symmetry with LexLeader

Does (g, y, r, b) satisfy LexLeader?



check ordering

- Order: (A, B, C, D)
- $(g, y, r, b) \leq_{\text{lex}} (r, y, g, b)$ ✓
- $(g, y, r, b) \leq_{\text{lex}} (g, b, r, y)$ ×
- $(g, y, r, b) \leq_{\text{lex}} (r, b, g, y)$ ✓

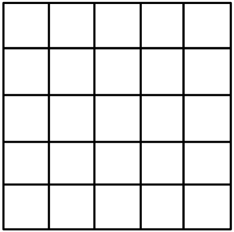
(g, y, r, b) does not satisfy LexLeader

Row and Column Symmetry: Definitions

A matrix of variables has

row symmetry if any row permutation preserves solutions

column symmetry if any column permutation preserves solutions

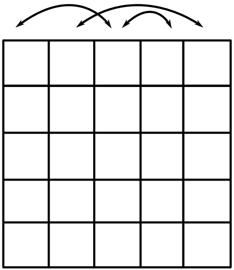


Row and Column Symmetry: Definitions

A matrix of variables has

row symmetry if any row permutation preserves solutions

column symmetry if any column permutation preserves solutions

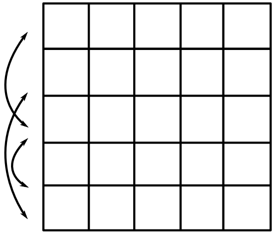


Row and Column Symmetry: Definitions

A matrix of variables has

row symmetry if any row permutation preserves solutions

column symmetry if any column permutation preserves solutions



Row and Column Symmetry: Example

The EFPA problem

(Equidistant Frequency Permutation Array)

- find v codewords
- each of length $q\lambda$
- each containing λ copies of symbols 0 to $q-1$
- each pair is Hamming distance d

Row and Column Symmetry: Example

$$\boxed{\begin{array}{cccccc} 0 & 2 & 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{array}}$$

The EFPA problem

- $v = 4$

Row and Column Symmetry: Example

$$\boxed{\begin{matrix} 0 & 2 & 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{matrix}}$$

The EFPA problem

- $v = 4$
- $\lambda = 2$

Row and Column Symmetry: Example

$$\boxed{\begin{array}{cccccc} 0 & 2 & 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{array}}$$

The EFPA problem

- $v = 4$
- $\lambda = 2$
- $q = 3$

Row and Column Symmetry: Example

$$\boxed{\begin{array}{cccccc} 0 & 2 & 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{array}}$$

The EFPA problem

- $v = 4$
- $\lambda = 2$
- $q = 3$
- $d = 4$

The DoubleLex constraint

A matrix solution satisfies DoubleLex if

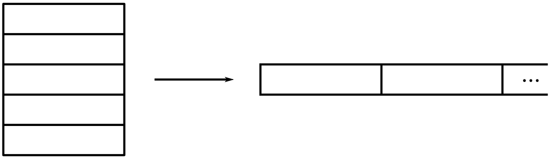
- the rows are lexicographically ordered, and
- the columns are lexicographically ordered.

Breaking Symmetry with DoubleLex

$$\underset{\text{(b)}}{\boxed{\begin{matrix} 0&0&1&1&2&2\\ 0&1&0&2&1&2\\ 0&2&1&2&0&1\\ 0&2&2&1&1&0 \end{matrix}}} \xleftarrow{\text{order rows}} \underset{\text{(a)}}{\boxed{\begin{matrix} 0&2&1&2&0&1\\ 0&2&2&1&1&0\\ 0&1&0&2&1&2\\ 0&0&1&1&2&2 \end{matrix}}} \xrightarrow{\text{order columns}} \underset{\text{(c)}}{\boxed{\begin{matrix} 0&0&1&1&2&2\\ 0&1&0&2&1&2\\ 0&1&2&0&2&1\\ 0&2&2&1&1&0 \end{matrix}}}$$

- (b) and (c) satisfy DoubleLex
- DoubleLex does not break all symmetries.

The RowWiseLexLeader constraint



A matrix satisfies RowWiseLexLeader if

- its rowwise linearization satisfies LexLeader.

Breaking Symmetry with RowWiseLexLeader

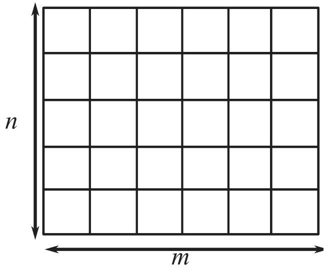
$$
\underset{\text{(b)}}{\boxed{\begin{matrix} 0&0&1&1&2&2\\ 0&1&0&2&1&2\\ 0&2&1&2&0&1\\ 0&2&2&1&1&0 \end{matrix}}}
\xleftarrow{\text{order rows}}
\underset{\text{(a)}}{\boxed{\begin{matrix} 0&2&1&2&0&1\\ 0&2&2&1&1&0\\ 0&1&0&2&1&2\\ 0&0&1&1&2&2 \end{matrix}}}
\xrightarrow{\text{order columns}}
\underset{\text{(c)}}{\boxed{\begin{matrix} 0&0&1&1&2&2\\ 0&1&0&2&1&2\\ 0&1&2&0&2&1\\ 0&2&2&1&1&0 \end{matrix}}}
$$

- Only (c) satisfies RowWiseLexLeader.
- RowWiseLexLeader breaks all row and column symmetries.
- Checking satisfaction of RowWiseLexLeader is NP-hard, but . . .

Complexity of RowWiseLexLeader

Theorem

Checking satisfaction of RowWiseLexLeader *is* $O(n!nm\log m)$ *for an* $n \times m$ *matrix.*



Proof.

- Given one of the $n!$ row permutations
- Sort the columns lexicographically in $O(nm\log m)$.

DoubleLex: a Worse-Case Example

Theorem

There is a $2n \times 2n$ *0/1 matrix class on which* DoubleLex *leaves* $n!$ *symmetric solutions.*

Proof.

$\square$

DoubleLex: a Worse-Case Example

Theorem

There is a $2n \times 2n$ 0/1 matrix class on which DoubleLex *leaves $n!$ symmetric solutions.*

Proof.

Consider a CSP with constraints:

- $3n$ 1-entries
- each row and column contains one or two 1-entries

$\square$

DoubleLex: a Worse-Case Example

Theorem

There is a $2n \times 2n$ 0/1 matrix class on which DoubleLex *leaves $n!$ symmetric solutions.*

Proof.

O	I^R
I^R	P

- O is a zero matrix
- I^R is the identity matrix vertically flipped
- P is a permutation matrix

□

DoubleLex: a Worse-Case Example

Theorem
There is a $2n \times 2n$ 0/1 matrix class on which DoubleLex *leaves $n!$ symmetric solutions.*

Proof.

O	I^R
I^R	P

- $3n$ 1-entries ✓

- O is a zero matrix
- I^R is the identity matrix vertically flipped
- P is a permutation matrix

□

DoubleLex: a Worse-Case Example

Theorem

There is a $2n \times 2n$ 0/1 matrix class on which DoubleLex *leaves $n!$ symmetric solutions.*

Proof.

O	I^R
I^R	P

- O is a zero matrix
- I^R is the identity matrix vertically flipped
- P is a permutation matrix

- $3n$ 1-entries ✓
- each row and column contains one or two 1-entries ✓

□

Special case: All-Different

In a All-Different CSP,

RowWiseLexLeader is equivalent to Order1stRowCol:

- the top-left entry is the smallest
- the first row and column are ordered

Special case: All-different

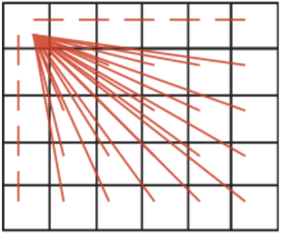
Theorem

DC can be enforced on Order1stRowCol *in polynomial time.*

Proof.

Constraints:

- $X_{1,1} < X_{i,j} \quad 1 < i \leq m$, $1 < j \leq n$
- $X_{i,1} < X_{i+1,1}$ $1 \leq i < m$
- $X_{1,j} < X_{1,j+1} \quad 1 \leq j < n$

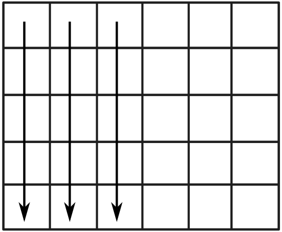


Each constraint only needs to be checked twice and bounds are enforced in constant time. □

The SnakeLex constraint

Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2,\ c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3},\ \overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4,\ c_3 \leq_{\text{lex}} c_5$
- ...



Row symmetry breaking

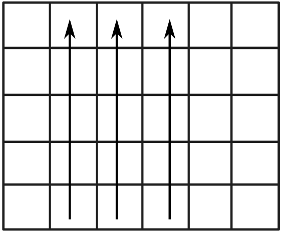
Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

The SnakeLex constraint

Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2$, $c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3}$, $\overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4$, $c_3 \leq_{\text{lex}} c_5$
- $\ldots$

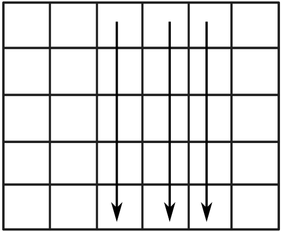


Row symmetry breaking

Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

The SNAKELEX constraint



Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2,\ c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3},\ \overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4,\ c_3 \leq_{\text{lex}} c_5$
- $\ldots$

Row symmetry breaking

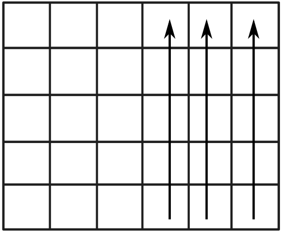
Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

The SNAKELEX constraint

Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2, \; c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3}, \; \overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4, \; c_3 \leq_{\text{lex}} c_5$
- $\ldots$

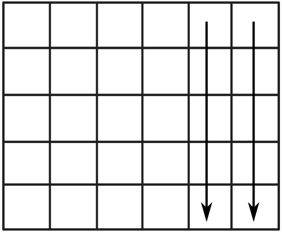


Row symmetry breaking

Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

The SnakeLex constraint



Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2$, $c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3}$, $\overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4$, $c_3 \leq_{\text{lex}} c_5$
- ...

Row symmetry breaking

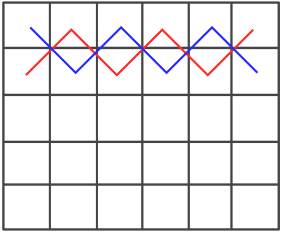
Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

The SnakeLex constraint

Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2$, $c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3}$, $\overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4$, $c_3 \leq_{\text{lex}} c_5$
- $\ldots$

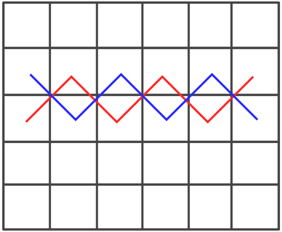


Row symmetry breaking

Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

The SnakeLex constraint



Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2$, $c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3}$, $\overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4$, $c_3 \leq_{\text{lex}} c_5$
- $\ldots$

Row symmetry breaking

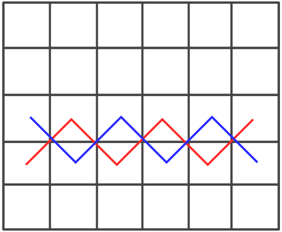
Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

The SnakeLex constraint

Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2, \; c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3}, \; \overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4, \; c_3 \leq_{\text{lex}} c_5$
- $\ldots$



Row symmetry breaking

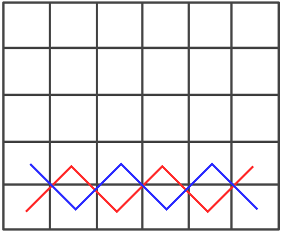
Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

The SNAKELEX constraint

Column symmetry breaking

- $c_1 \leq_{\text{lex}} c_2,\ c_1 \leq_{\text{lex}} c_3$
- $\overline{c_2} \leq_{\text{lex}} \overline{c_3},\ \overline{c_2} \leq_{\text{lex}} \overline{c_4}$
- $c_3 \leq_{\text{lex}} c_4,\ c_3 \leq_{\text{lex}} c_5$
- $\ldots$



Row symmetry breaking

Adjacent rows satisfy the entwined lex ordering

- $(x_{i,1}, x_{i+1,2}, x_{i,3}, \ldots) \leq_{\text{lex}} (x_{i+1,1}, x_{i,2}, x_{i+1,3}, \ldots)$

SnakeLex: a Worse-Case Example

Theorem

There is a $2n \times 2n+1$ 0/1 matrix class on which SnakeLex *leaves $O(4^n/\sqrt{n})$ symmetric solutions.*

Proof.

$$\boxed{\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}}$$

$\square$

SnakeLex: a Worse-Case Example

Theorem

There is a $2n \times 2n+1$ 0/1 matrix class on which SnakeLex *leaves $O(4^n/\sqrt{n})$ symmetric solutions.*

Proof.

$$\boxed{\begin{matrix} 0&1&0&0&1 \\ 0&0&0&1&1 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \end{matrix}} \longleftarrow \boxed{\begin{matrix} 0&1&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 1&0&0&0 \end{matrix}} \longrightarrow \boxed{\begin{matrix} 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&1 \\ 1&0&0&0&1 \end{matrix}}$$

$\square$

SnakeLex: a Worse-Case Example

Theorem

There is a $2n \times 2n+1$ 0/1 matrix class on which SnakeLex *leaves $O(4^n/\sqrt{n})$ symmetric solutions.*

Proof.

$$
\begin{array}{l}
0\,1\,0\,0\,0\,0 \ldots 0\,0\,0 \\
0\,0\,0\,1\,0\,0 \ldots 0\,0\,0 \\
0\,0\,0\,0\,0\,0 \ldots 0\,0\,0 \\
0\,0\,0\,0\,0\,1 \ldots 0\,0\,0 \\
\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots \quad \vdots\,\vdots\,\vdots \\
0\,0\,0\,0\,0\,0 \ldots 1\,0\,0 \\
0\,0\,0\,0\,0\,0 \ldots 0\,0\,1 \\
0\,0\,0\,0\,0\,0 \ldots 0\,1\,0 \\
\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots \quad \vdots\,\vdots\,\vdots \\
0\,0\,0\,0\,1\,0 \ldots 0\,0\,0 \\
0\,0\,1\,0\,0\,0 \ldots 0\,0\,0 \\
1\,0\,0\,0\,0\,0 \ldots 0\,0\,0
\end{array}
$$

In general, any column vector with exactly n 1-entries can be appended to this matrix.
The matrix will still satisfy SnakeLex.

$\square$

SnakeLex: a Worse-Case Example

Theorem

There is a $2n \times 2n + 1$ 0/1 matrix class on which SnakeLex *leaves $O(4^n/\sqrt{n})$ symmetric solutions.*

Proof.

By Sterling's formula

$$\binom{2n}{n} = O\left(\frac{4^n}{\sqrt{n}}\right).$$

$\square$

Summary

	RowWiseLex	DoubleLex	SnakeLex
Completeness	Yes	No	No
Check Satisfaction	$O(n!nm\log m)$	Polynomial	Polynomial
DC	?	NP-hard	?

Empirical evidence suggest SnakeLex is superior to DoubleLex.

What we missed

- Relationship to breaking value symmetry
- Dynamic symmetry breaking methods