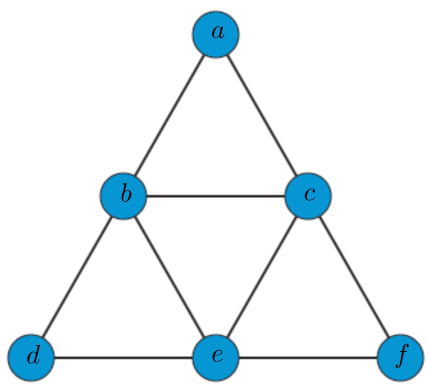
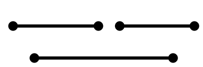
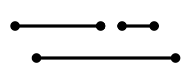
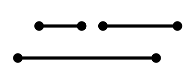
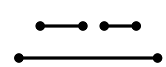
1. Recall that a chordal graph is a graph in which there is a perfect elimination ordering (an ordering of the vertices such that, for each vertex v, the neighbors of v that are earlier in the ordering form a clique). Give pseudocode for finding a maximum clique in a chordal graph. You may assume that a perfect elimination ordering of the graph has already been constructed. (Hint: look for the vertex in the clique that comes last in the ordering.)

Solution:
max-clique-size = 0
for each vertex v in perfect elimination ordering:
    if the number of earlier neighbors of v + 1 > max-clique-size:
        max-clique-size = number of earlier neighbors of v + 1
        max-clique-vertex = v
return max-clique-vertex and all its earlier neighbors

2. Let G be a graph shown below. Prove that G is not an interval graph. That is, show that it is not possible to find a set of six intervals in a line whose intersection pattern is this graph.

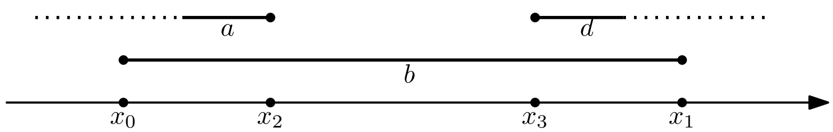


Solution: Suppose that G is an interval graph. Then in the interval drawing, both a and d's intervals must intersect b's but not each other's, so the interval drawing must look like

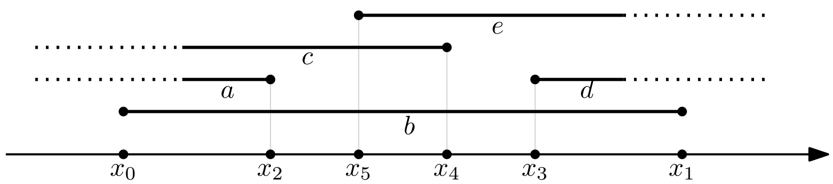
 or  or  or 

where the bottom interval represents b and the other two represent a and d. Without loss of generality, we can assume that a is to the left of d (because we could swap a and d's labels in the graph, and we would still have the same graph). For the sake of this argument, we only care that a and d each have an endpoint inside of b's interval.

So we will consider a partial drawing of these intervals (which represents any of the 4 cases above) as follows:



We know that $x_0 < x_2 < x_3 < x_1$. Now c's interval must overlap a and b's but not d's. So its right endpoint x_4 must be between x_0 and x_3. Similarly, e's interval must overlap b's and d's but not a's, so its left endpoint x_5 must be between x_2 and x_1. Moreover, c and e's intervals must overlap each other, so it must be that $x_2 < x_5 < x_4 < x_3$.

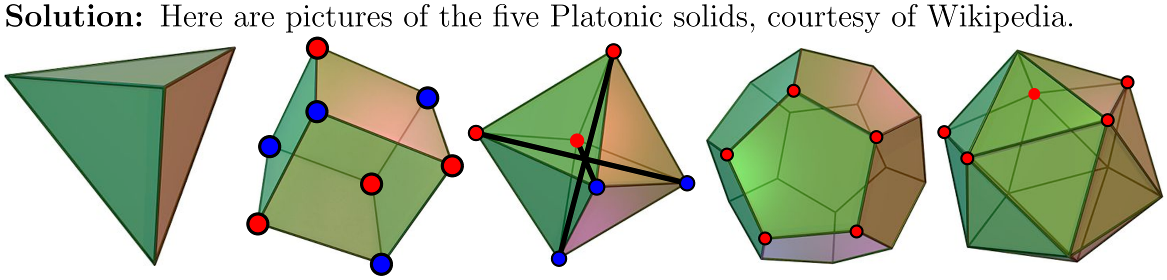


Finally f's interval must overlap with e's and c's, which means that it must contain a point between x_5 and x_4. So this point must be between x_2 and x_3 (because $x_2 < x_5 < x_4 < x_3$), so it must also be between x_0 and x_1 (because $x_0 < x_2 < x_3 < x_1$). This means that f's interval intersects b's interval, which brings us to a contradiction.

3. For each of the five Platonic solids (the tetrahedron, cube, octahedron, icosahedron, and dodecahedron) determine whether the corresponding graph is a perfect graph. For each graph that is not perfect, describe how to find an induced subgraph that is either an odd cycle of length greater than four, or the complement of an odd cycle of length greater than four. For each graph that is perfect, explain why.

   (You may find the following facts helpful: every bipartite graph is perfect, every complete graph is perfect, and the complement of a cycle of length exactly five is another cycle of length five.)

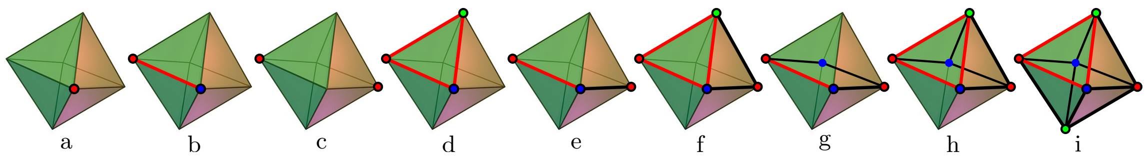
Solution: Here are pictures of the five Platonic solids, courtesy of Wikipedia.



tetrahedron cube octahedron dodecahedron icosahedron

The tetrahedron is perfect because it is the complete graph on four vertices. The cube is perfect because it is bipartite—a coloring is provided in the illustration. The octahedron is perfect because the complement of the octahedron is bipartite and therefore perfect. So by the perfect graph theorem, which states that a graph is perfect if and only if its complement is perfect, the octahedron is also perfect. The dodecahedron and icosahedron are not perfect as they have induced subgraphs that are 5-cycles, as depicted by the red vertices in the illustration.

Note that although the octahedron contains a 5-cycle as a subgraph, it does not contain a 5-cycle as an induced subgraph. So that argument won't work to prove that the octahedron is not perfect.



We can also prove that the octahedron is perfect by applying the definition of perfect graph directly. The octahedron has 9 induced subgraphs, modulo its symmetries. In each case, we need to check that the size of the largest clique in that subgraph (or clique number) is equal to the "coloring" number.

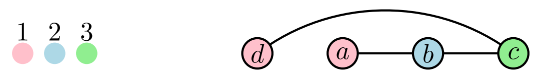
a. clique number = 1, "coloring" number = 1
b. clique number = 2, "coloring" number = 2
c. clique number = 1, "coloring" number = 1
d. clique number = 3, "coloring" number = 3
e. clique number = 2, "coloring" number = 2
f. clique number = 3, "coloring" number = 3
g. clique number = 2, "coloring" number = 2
h. clique number = 3, "coloring" number = 3
i. clique number = 3, "coloring" number = 3

Since in every induced subgraph, the clique number equals the coloring number of that subgraph, the octahedron is perfect.

4. 163 only: Find an example of a tree, and a degeneracy ordering on your tree, such that using the greedy coloring algorithm with this ordering colors the tree with more than the optimal number of colors.

Solution: The definition of degeneracy ordering that we use is the following: "a degeneracy ordering of a d-degenerate graph is an ordering of the vertices such that each vertex has at most d *later* neighbors."

The graph a—b—c—d has degeneracy 1. So a valid degeneracy ordering for it is d, a, b, c. With this ordering, the greedy coloring algorithm will produce a coloring that uses 3 colors.



This coloring is not optimal since the tree can be colored with only 2 colors.

263 only: Suppose that T is a tree with a degeneracy ordering σ such that using the greedy coloring algorithm with this ordering colors T with c colors. Show how to modify T and σ to produce a tree T' with a degeneracy ordering σ' such that using the greedy coloring algorithm with this ordering colors T' with $c + 1$ colors.

Solution: Construct T' from k copies of the tree T and an additional vertex v that connects to one vertex of every tree, each of a different color. Construct the degeneracy ordering σ' by concatenating k copies of σ followed by v. Because v has c earlier neighbors, all with a different color, v must use a new color.