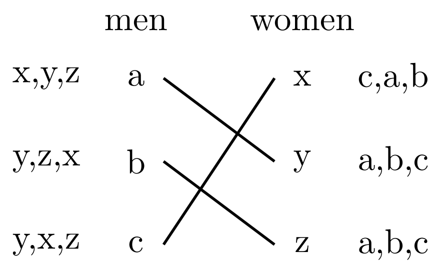
1. In the stable marriage algorithm described in class (the one in which the men propose to the women in preference order, and each woman accepts the best proposal she has seen so far) must there always exist at least one man who ends up married to his first choice? Either prove that such a man always exists, or find an example where all men settle for later preferences.

Solution: Consider the following set of people and their preference list:



The stable marriage algorithm will yield these matchings, in which no man marries his first choice.

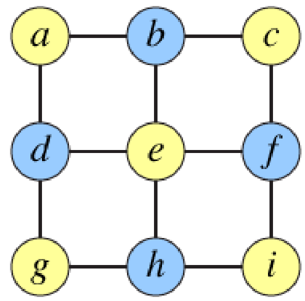
2. Let n be an even number, and form an undirected graph with n vertices numbered from 0 to $n-1$ by adding edges connecting each vertex numbered i to the three vertices numbered $i-1$, $i+1$, and $i+n/2 \pmod{n}$. For instance, when n is four, this graph can be drawn using the sides and diagonals of a square. For which values of n is this graph bipartite? Explain how to find a 2-coloring of the graph for the values of n that make this graph bipartite, and how to find an odd-length cycle for the values of n that make this graph non-bipartite.

Solution: If n is even, then it is either congruent to $0 \pmod 4$ (i.e. 4, 8, 12, etc.) or $2 \pmod 4$ (i.e. 2, 6, 10, etc.). The graphs described in this problem are cycles with additional cross-edges between vertices that are diametrically opposite.

If $n \equiv 0 \pmod 4$, $n/2$ is even, so the cycle 0, 1, 2, $\ldots$, $n/2$ and back to 0 is an odd cycle. So the graph is not bipartite.

If $n \equiv 2 \pmod 4$, then $n/2$ is an odd number, so the cross-edges (which connect i to $i + n/2 \pmod{n}$) always connect an even vertex to an odd vertex. Therefore, coloring all even vertices one color and all odd vertices another color produces a valid 2-coloring.

3. Let G be the bipartite graph shown below. Starting from an empty partial matching, find a sequence of alternating paths that leads to a maximum matching in G. Choose your alternating paths in such a way that the first one has one edge, the second one has three edges, the third one has five edges, and the fourth one has seven edges. (Hint: it may be easier to solve the problem by working backwards.)



Solution: In the following sequence of alternating paths, red edges represent matched vertices.

$$f - e$$
$$c - f - e - d$$
$$b - c - f - e - d - g$$
$$a - b - c - f - e - d - g - h$$

4. Let T be a tree with four leaves and two degree-three vertices. Draw a polygon P with axis-parallel sides, such that the vertices of T correspond one-for-one with axis-parallel slices through pairs of 270°-vertices of P, and such that two vertices of T are adjacent if and only if the corresponding two slices cross or touch each other. Find a partition of your polygon into a minimum number of rectangles, and describe which of the slices in your partition correspond to the vertices of a maximum independent set in T.

Solution: In this solution, the blue, purple, yellow and navy vertices form a maximum independent set in T. The minimum number of rectangles is 5.