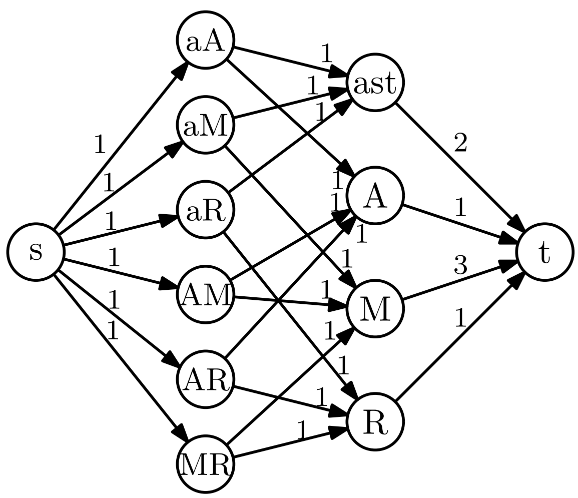
1. Draw the flow network representing a baseball elimination problem in which We wish to know whether the Angels have been eliminated from the American League West; the other teams in the division are the Astros, A's, Mariners, and Rangers. There are six games left for the other teams to play, one for each pair of teams. If they win all their remaining games, the Angels can afford to let the Astros win at most two games, the Mariners win at most three games, and the A's and Rangers win at most one game.

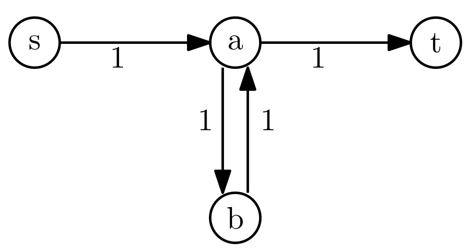
   (You do not need to find the maximum flow for your network; just show its vertices, edges, and edge capacities.)

Solution:



2. In the solution to a maximum flow problem, is it always true (for every possible network and every possible maximum flow on that network) that the subset of the edges that have nonzero flow amounts forms a directed acyclic graph? Explain why or find a counterexample.

Solution: It's not always true. In the following graph, the maximum flow value is 1, and the maximum flow may or may not include flow between a and b, which form a cycle.



3. For both of the following parts, you may use the fact that an n-vertex directed graph (without multiple edges between the same two vertices) has $O(n^2)$ edges.

   (a) Prove that, for some n-vertex inputs to the maximum flow problem, every augmenting path algorithm can be forced to use a number of augmenting paths that is proportional to n^2.

   Solution: Let one of the nodes be the source, one of them be the sink and evenly split the rest into two sets A and B. Connect the source to all the nodes in A, all the nodes in A to all those in B, and all those in B to the sink. Set the capacities of the edges connecting A and B to 1 and the rest to $(n-2)/2$. There are $(n-2)^2/4$ paths from the source to to the sink, none of which share an edge. So all of these paths must be used by any augmenting path algorithm.

   (b) (265 only): Prove that, for every n-vertex input to the maximum flow problem, there exists a sequence of augmenting paths with only $O(n^2)$ paths. (Hint: find the maximum flow first and use it to guide your choice of paths. Try to find paths that eliminate at least one edge from the remaining flow.)

   Solution: Start with the maximum flow. Select a path in which there is flow and that contains an edge in which there is the smallest amount of flow. Remove the flow through that path and all edges in which there is no longer flow. Repeat the path selection and flow removal until there is no flow left.

   The paths that are selected are a complete set of augmenting paths, and there can be at most n^2 of them.

4. Suppose we are holding a preference ballot with three candidates: Washington, Jefferson, and Hamilton. Each voter can choose one of the six permutations of those candidates, describing which one is their favorite, second favorite, and least favorite. There are 100 voters. Choose how many voters should vote for each of the six permutations, in such a way that that Washington is the Condorcet winner (he would win each one-on-one pairing of two candidates that he participates in) but gets the smallest number of first-place votes (so he would be eliminated first in an instant-runoff election).

   Solution: Let the breakdown of the 100 voters be as follows:

$$W > J > H : x_1$$
$$W > H > J : x_2$$
$$J > W > H : x_3$$
$$H > W > J : x_4$$
$$J > H > W : x_5$$
$$H > J > W : x_6$$

Since Washington is the Condorcet winner:

$$x_1 + x_2 + x_4 > x_3 + x_5 + x_6 \qquad \text{Washington beats Jefferson head-to-head.}$$
$$x_1 + x_2 + x_3 > x_4 + x_5 + x_6 \qquad \text{Washington beats Hamilton head-to-head.}$$

But since Washington has the smallest number of first-place votes:

$$x_1 + x_2 < x_3 + x_5 \qquad \text{Washington has fewer 1st-place votes than Jefferson.}$$
$$x_1 + x_2 < x_4 + x_6 \qquad \text{Washington has fewer 1st-place votes than Hamilton.}$$

Finally, the variables must add to 100. One possible set of values is

$$x_1 = x_2 = 1 \qquad x_3 = x_4 = 49 \qquad x_5 = x_6 = 0.$$