Hash tables
Shortest-paths

CS 146 - Spring 2017

Today

- Hash tables wrap-up
  - how to hash things well
  - how to handle collisions
- Shortest paths with negative weights
- Bellman-Ford algorithm

Hashing

- to reduce the universe U of all keys down to a reasonable size m for the table
- idea: m (size of table) should be about n, # keys
- hash function: h: U → {0, …m-1}
- **two keys ki, kj collide if h(ki) = h(kj)**
- problem: how do you minimize collisions? → create good hash functions
- problem: what to do in case of a collision? → collision resolution via chaining, open addressing, cuckoo hashing

Hashing and collisions

- problem: how do you minimize collisions? → create good hash functions
- problem: what to do in case of a collision? → collision resolution via chaining, open addressing, cuckoo hashing

Simple uniform hashing

assumption that

each key is equally likely to be hashed to any slot of the table,

independently of where other keys are hashed

(depends on having on having a good hash function, and/or keys already being random)

Good hash functions scramble things well

ie satisfy the condition of simple uniform hashing

- division method: does not satisfy uniform hashing
- multiplication method
- universal hashing

Multiplication method

- $h(k) = [(ak) \bmod 2^w] >> (w - r)$, where a is chosen at random, and k is w bits
- practical when is a is odd and $2^{w-1} < a < 2^w$, and not too close to either
- fast

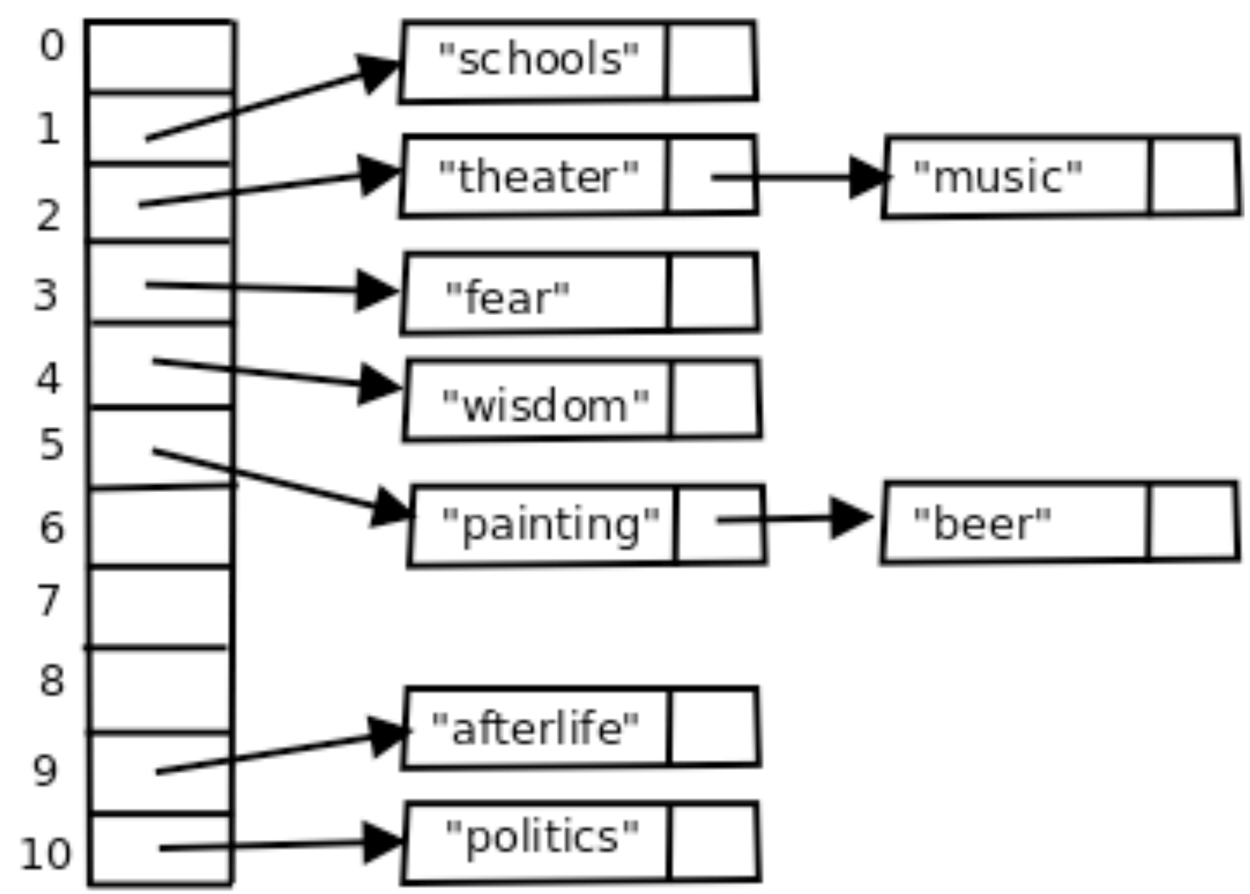
Universal hashing

- h(k) = [(ak + b) mod p] mod m, where a and b are chosen at random, p is a large prime > |U|

- lemma: can prove that for worst-case keys k1 != k2, Pr_{a,b}(h(k1) = h(k2)) = 1/m (proof relies on number theory)

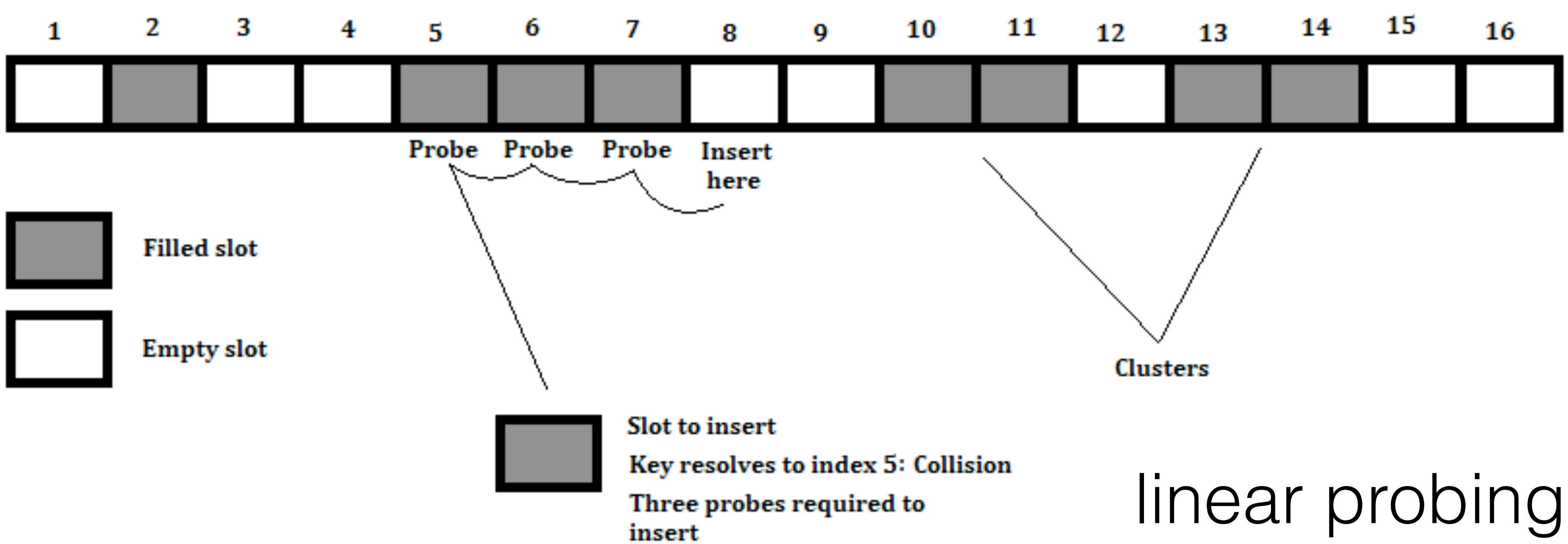
- consequence: E_{a,b}[# collisions with k1] = E[sum_k2 X_k1k2] = sum_k2 E[Xk1k2] = sum_k2 Pr[X_k1k2 = 1] = n/m = alpha

How do you resolve collisions?



hash chaining

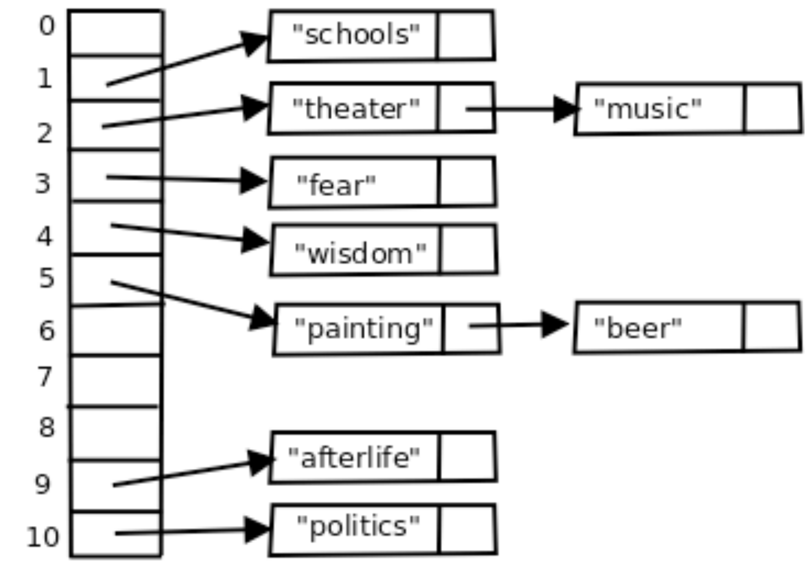
How do you resolve collisions?



linear probing

Open addressing: store values directly in the array

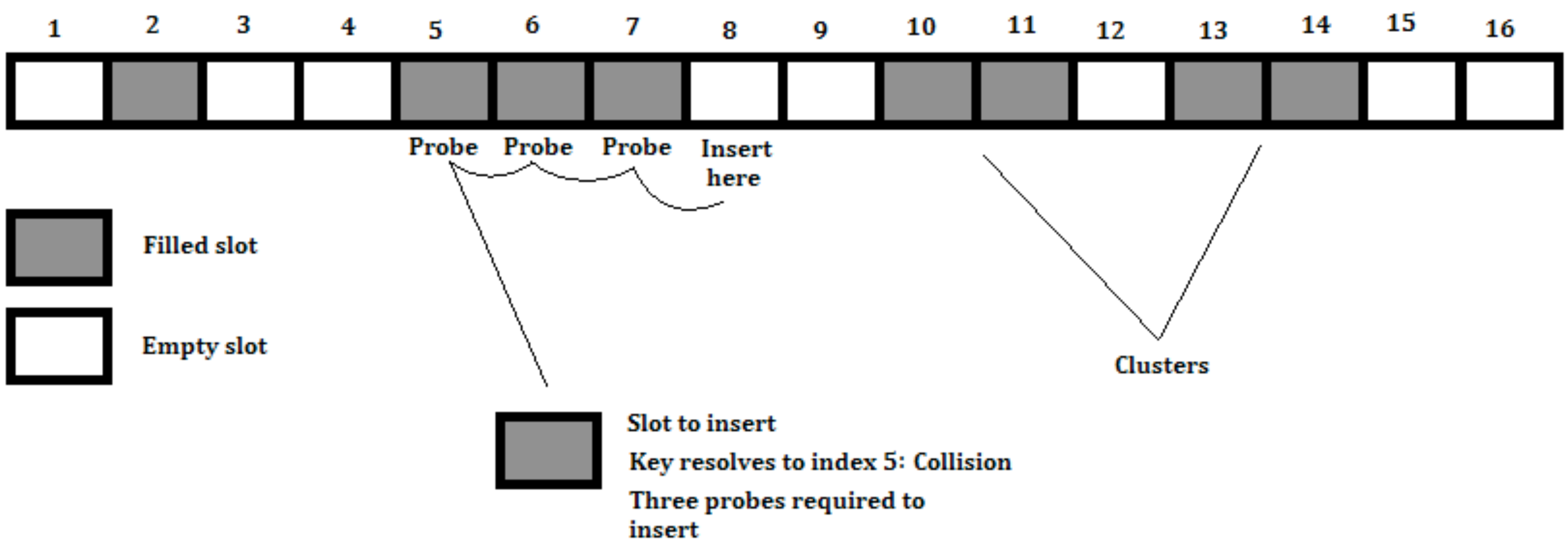
Hash chaining



- let n = # keys stored in table
- let m = # slots in table
- load factor alpha = n/m = expected # keys per slot = expected length of a chain
- expected running time for search is $\Theta(1+\alpha)$
- for applying hash function and random access to slot + search the list, which is O(1) if α is O(1), for example if m is $\Omega(n)$

Open addressing to resolve collisions

key value pairs are **stored inside the table**



a lot of different types

linear probing

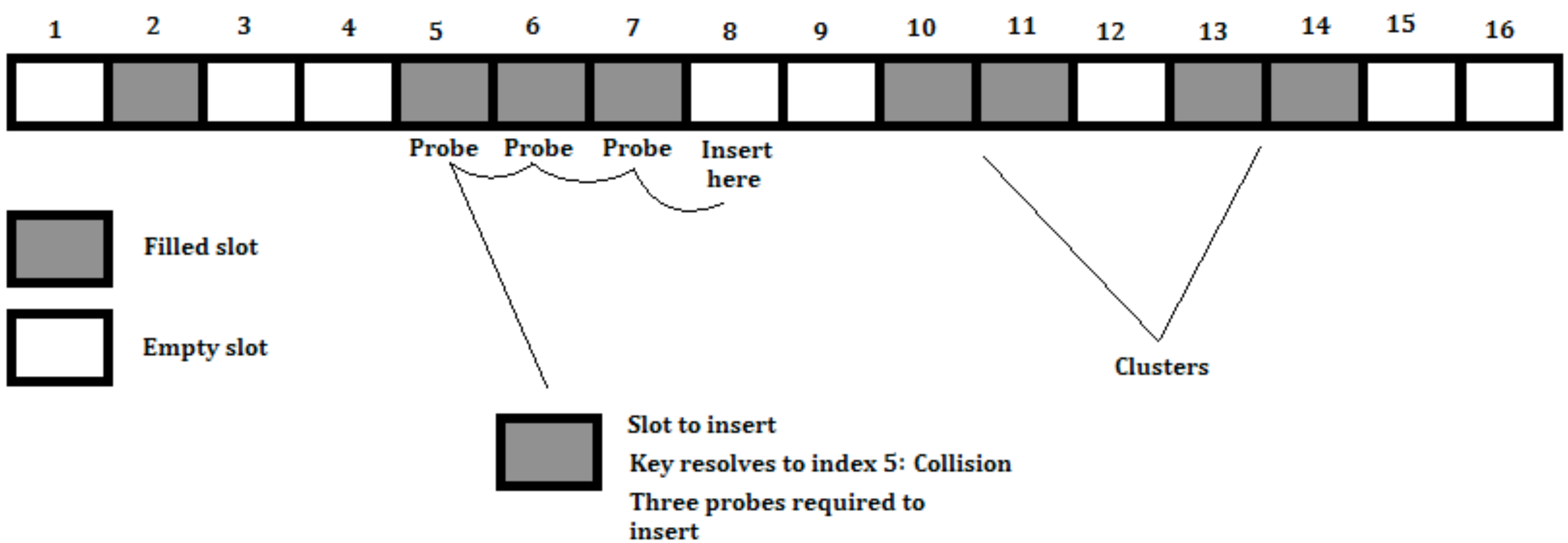
double hashing

quadratic probing

cuckoo hashing

Linear probing

- **to insert k,v**
- go to slot at h(k), if filled, go to slot at h(k) + 1, etc.
- store at first encountered slot that is empty

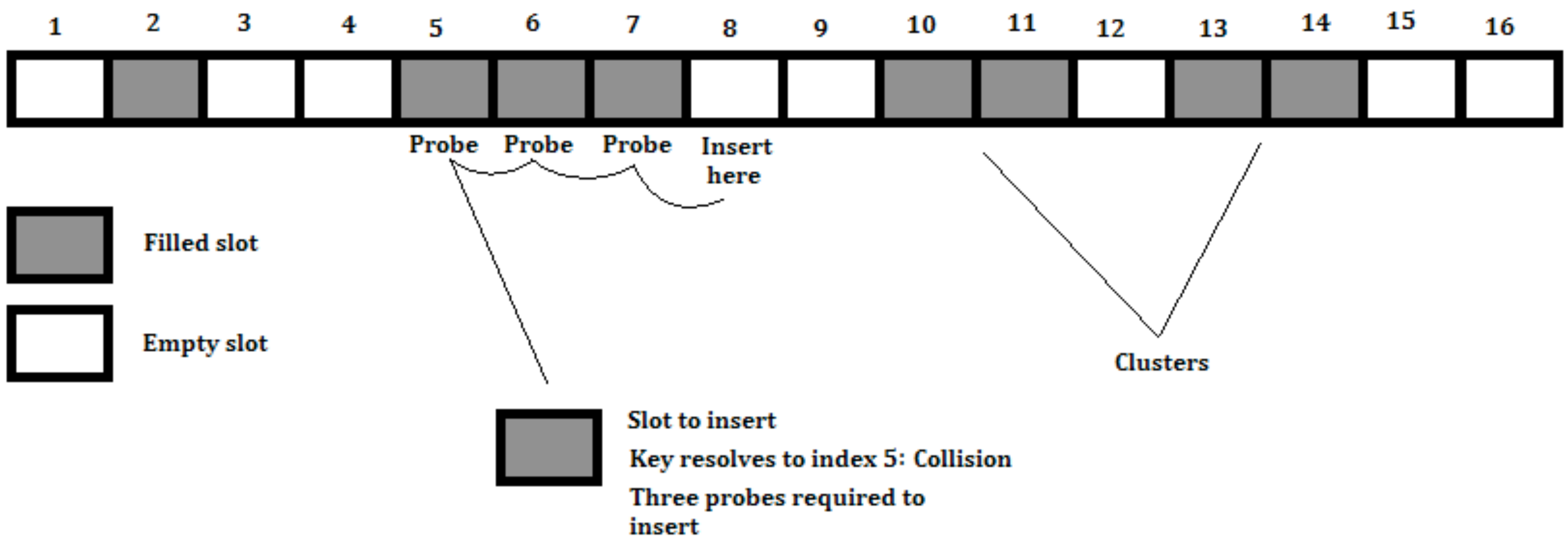


assume h(x) = x % 10

	Insert 18, 89, 21	Insert 58, 68	Insert 11
0		*A* 58	*A* 58
1	*A* 21	*A* 21	*A* 21
2		*A* 68	*A* 68
3			*A* 11
4			
5			
6			
7			
8	*A* 18	*A* 18	*A* 18
9	*A* 89	*A* 89	*A* 89

http://condor.depaul.edu/ntomuro/courses/417/notes/lecture4.html

Linear probing: search for value with key k



- scan each slot starting at h(k) (wrap around)
  - if empty, done, item not there
  - if key at that slot matches k, done, item there

Open addressing schemes in general

hash function specifies order of slots to probe for a key (for insert/search/delete)

$h(k, 0)$, $h(k, 1)$, $h(k, 2)$, …

- linear probing, uses an auxiliary hash function h'

h(k, i) = h'(k) + i mod m

- quadratic probing, uses an auxiliary hash function h'

h(k, i) = h'(k) + c1 i + c2 i^2 mod m

- double hashing, 2 auxiliary hash functions h1, h2

h(k, i) = h1(k) + i h2(k) mod m

How to delete key 73?

k	h(k)
3	6
8	2
16	4
20	3
34	2
52	6
73	2

hash function

assume linear probing

index	table
0	3, bob
1	20, alice
2	8, cat
3	73, doc
4	34, denis
5	16, dude
6	52, elf

Deletion

- don’t empty the slot
- mark it with special flag “delete me”
- when searching, treat “delete me” as a full slot
- when inserting, treat “delete me” as an empty slot

Open addressing analysis

- clustering when load factor is high

▪ under uniform hashing assumption, next op has
expected cost of <= 1/ (1 - alpha)

where alpha = n/m

▪ eg alpha = 90%, 10 expected probes

how large should the table be?

- want m = Theta(n) at all times
- don’t know how large n will be at creation itme
- m too small → slow, m too large → wasted space
- idea: start small (constant), grow and shrink as needed

What does it take to resize the table?

- changing the size of the table m
- changes the hash function (eg h(k) = ak mod m)
- must rebuild the hash table from scratch
- insert each item into new table at a new location
- takes Theta(n + m) time or
  Theta(n) time if m = Theta(n)

How often to grow table?

- if rebuild every time n = m, let m = m + 1. on every insert.
- cost is Theta(1 + 2 + … + n) = Theta(n^2)
- if rebuild every time n = m an let m = 2*m
- cost is Theta(1 + 2 + 4 + 8 + … + n) = Theta(n)
- a few inserts cost linear time, but Theta(1) “on average”

Amortized analysis

- operation has amortized cost T(n) if k operations cost <= k T(n)

- “T(n) amortized” roughly means T(n) “on average” but average is over all ops

- like paying rent: $1500/month in rent is $50/day

How often to grow table?

- if rebuild every time n = m an let m = 2*m
- maintain m = Theta(n)
- so alpha = Theta(1),
- **supports search in O(1) expected time, assuming simple uniform hashing or universal**

How often to shrink table?

- O(1) expected as is
- space can get too big with respect to n eg n inserts, n deletes
- solution: when n decreases to m/4, shrink to half m → 1/2m
- O(1) amortized cost for both inserts and deletes
- analysis is harder (see CLRS 17.4)

Open addressing vs chaining

- open addressing:
  - better cache performance (stored contiguously)
  - sensitive to hash functions: extra care to avoid clustering
  - sensitive to load factor: degrades above 70%
- chaining: less sensitive to hash functions
  - less efficient storage, pointers needed
  - less sensitive to load factors, still O(1)

Hash tables - the summary

- how to achieve O(1) search, insert, delete ?
- good hash functions
  - satisfy simple uniform hashing assumption
  - eg: multiplication method, universal hashing
- handling collisions
  - hash chaining or open addressing
  - performance is a function of load factor
  - double table size and rebuild to get O(1) amortized time for inserts

a hash function is **not** a random function

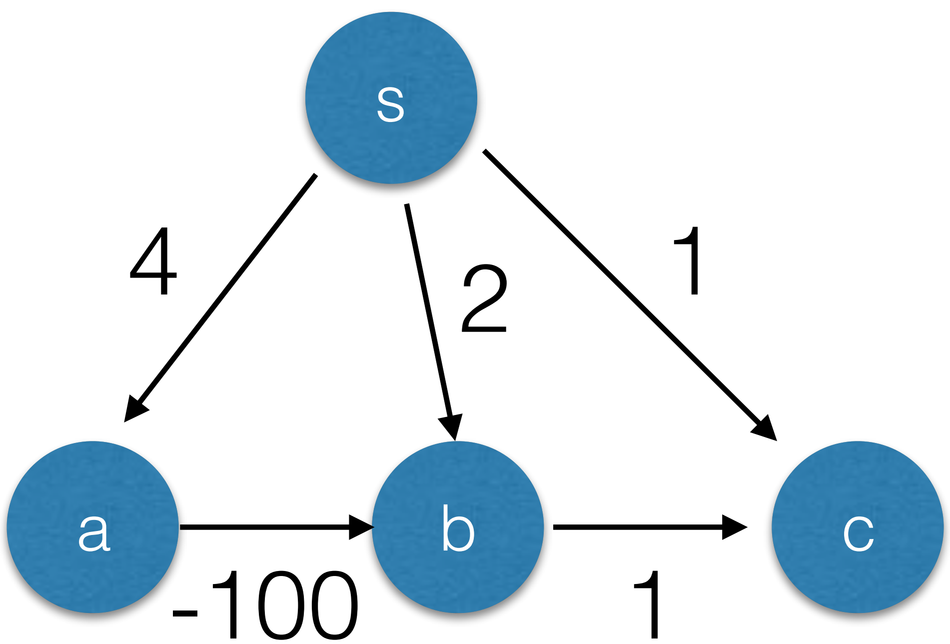
- it’s a function randomly chosen in a family of functions,

- eg choosing the parameter at random, but once chosen, it’s deterministic, running the same function again and again will yield the same value

Shortest paths

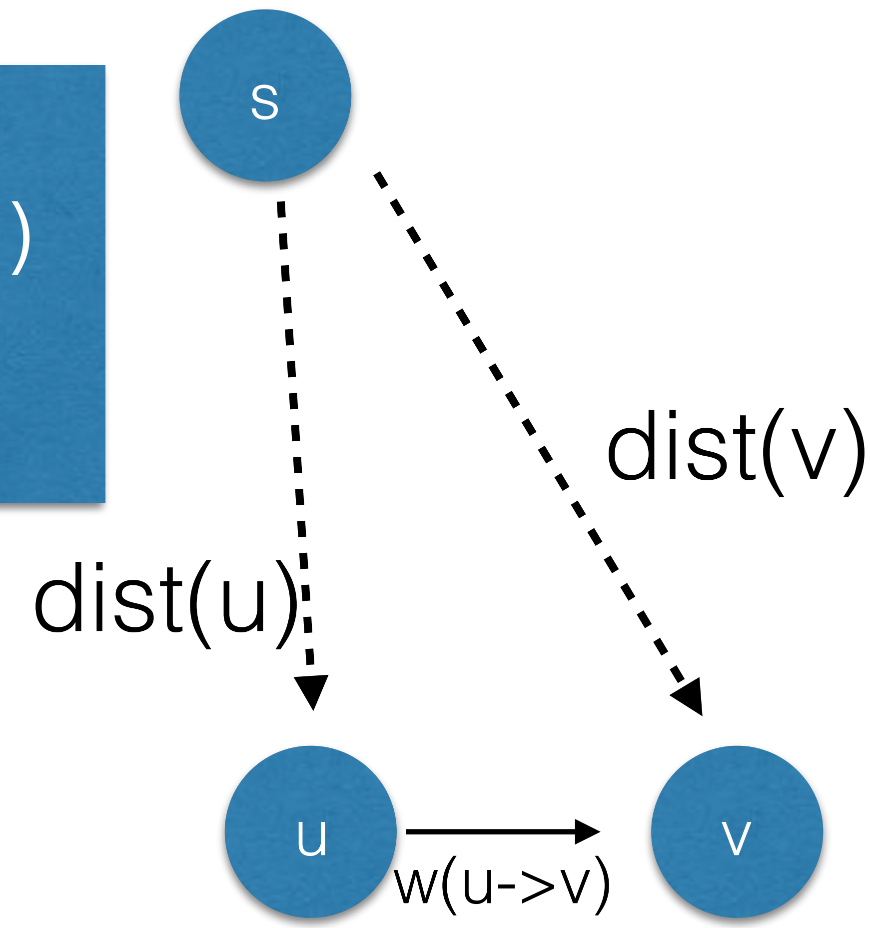
http://mathematica.stackexchange.com/questions/4084/finding-a-not-shortest-path-between-two-vertices

Recall from homework: Dijkstra’s algorithm does not always work on graphs containing negative cycles

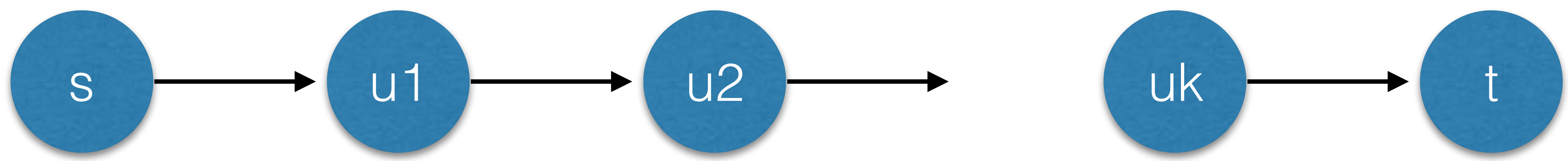


Why Dijkstra does not work if graph has negative edges?

```
void relax(u -> v) {
    if (dist(v) > dist(u) + w(u->v))
        dist(v) = dist(u) + w(u->v)
}
```



- gives correct distance for v if
    - dist(u) is correct AND
    - u is the second-to-last node in the shortest path to v
- relaxations are safe: too many updates of the form w→v will not make the value dist(v) too small



- the shortest path from s to t contains at most |V| - 1 edges
- b/c otherwise path loops through the same vertex, so the cycle (=loop) can be removed to get an even shorter path
- if relaxations are performed in order s->u1, u1->u2, … uk->t, distance to t will be set correctly
- true even if other relaxations performed in between the ones done in correct order

idea: relax ALL edges, V -1 times

```
bellmanFord(G, s) {
    Map dist = new Map();
    for each vertex v of G
        dist.put(v , infinity)
    dist.put(s, 0)

    repeat V - 1 times
        for each vertex v of G
            for neighbor w of v
                relax(v -> w)
    return dist;
}
```

initialize as before

loops through all edges

idea: relax ALL edges, V -1 times

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bellmanFord(G, s) {
    Map dist = new Map();
    for each vertex v of G
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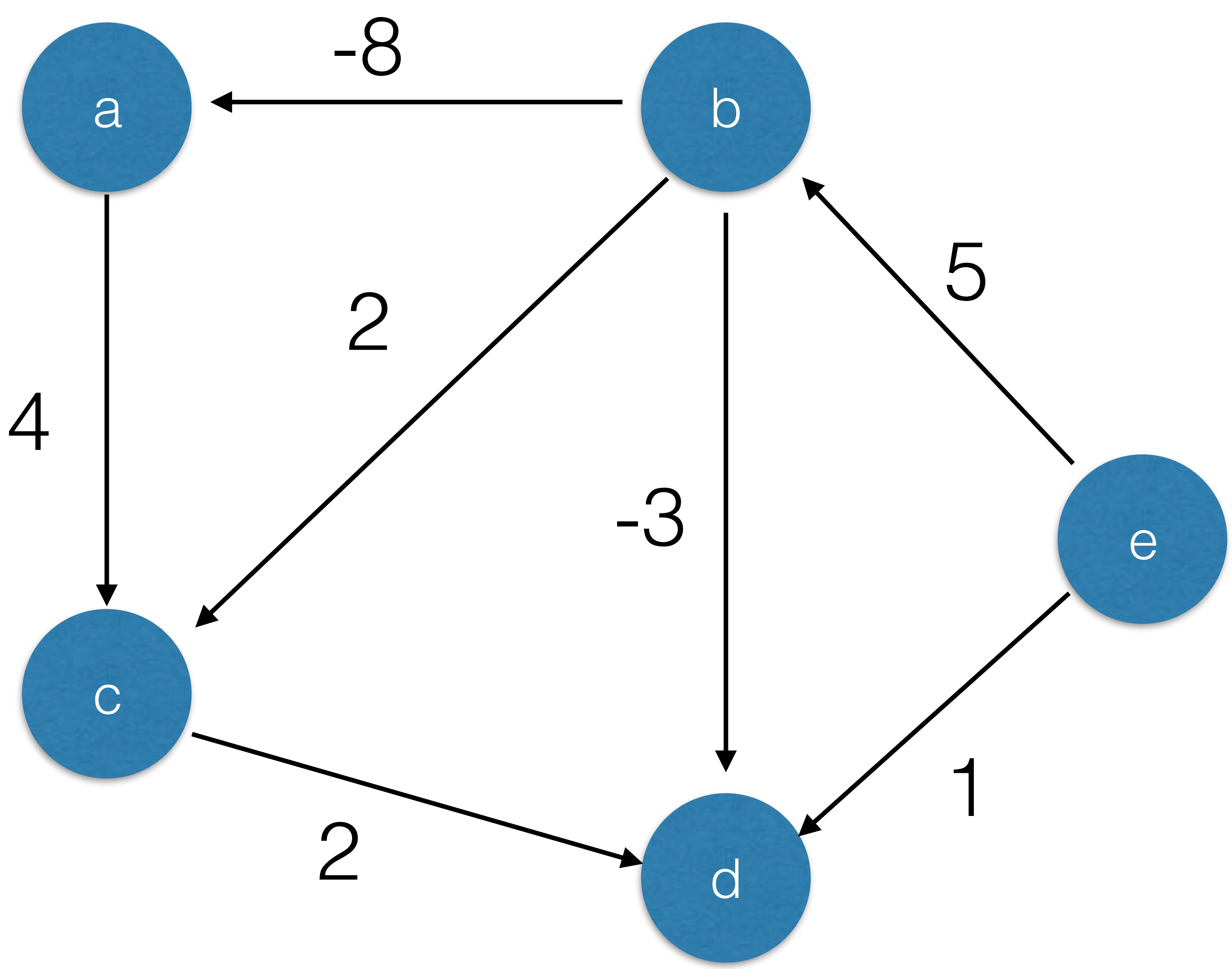
    repeat V - 1 times
        for each vertex v of G
            for neighbor w of v
                relax(v -> w)
    return dist;
}
```

initialize as before

one round of relaxation

time complexity: O(VE)

Example: find all single source shortest paths from e with the Bellman-Ford algorithm



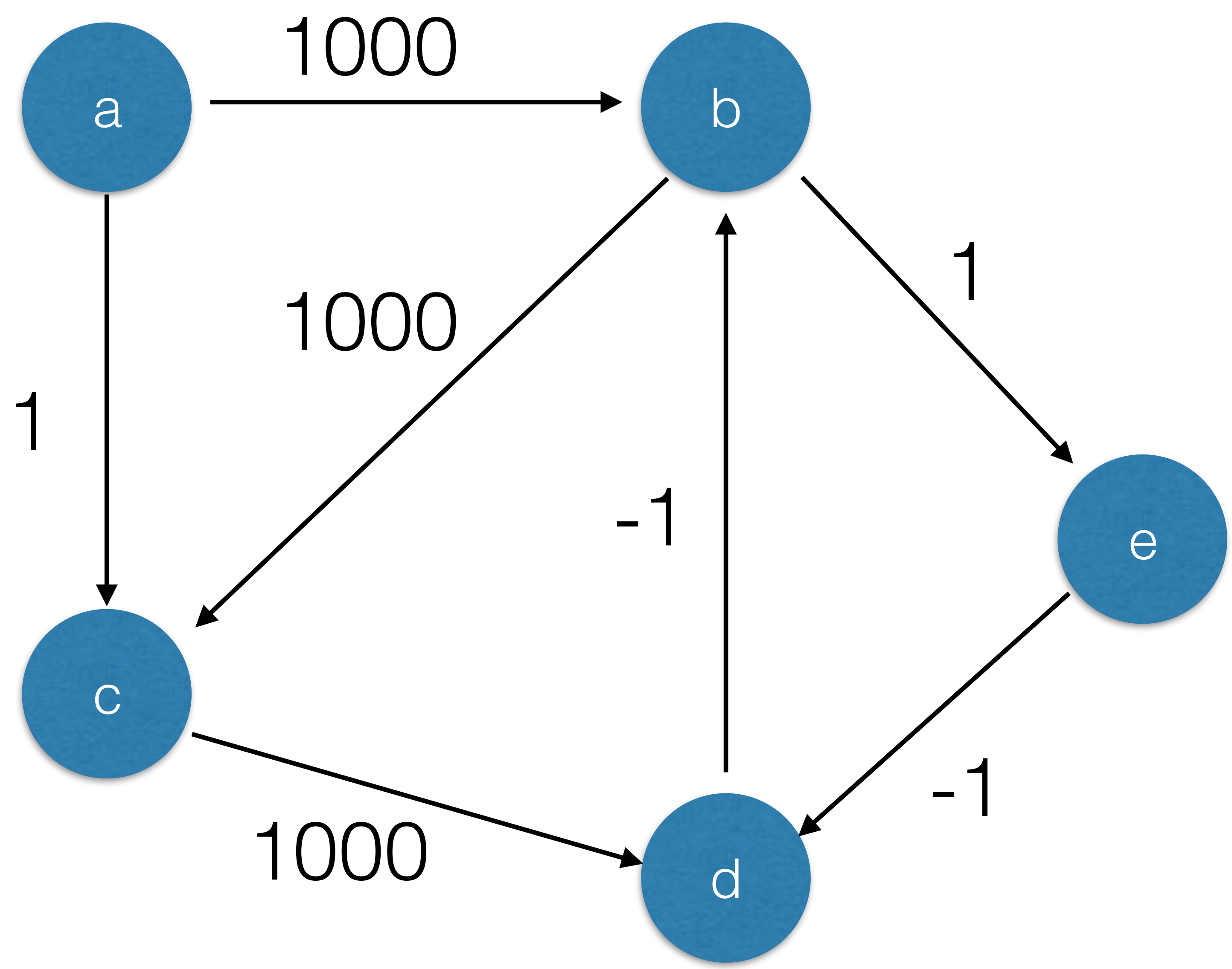
```
bellmanFord(G, s) {
    Map dist = new Map();
    for each vertex v of G
        dist.put(v , infinity)
    dist.put(s, 0)

    repeat V - 1 times
        changed = false
        for each vertex v of G
            for neighbor w of v
                if (dist(w) > dist(v) + w(v->w))
                    dist(w) = dist(v) + w(v->w)
                    changed = true
        if (!changed) break
    return dist;
}
```

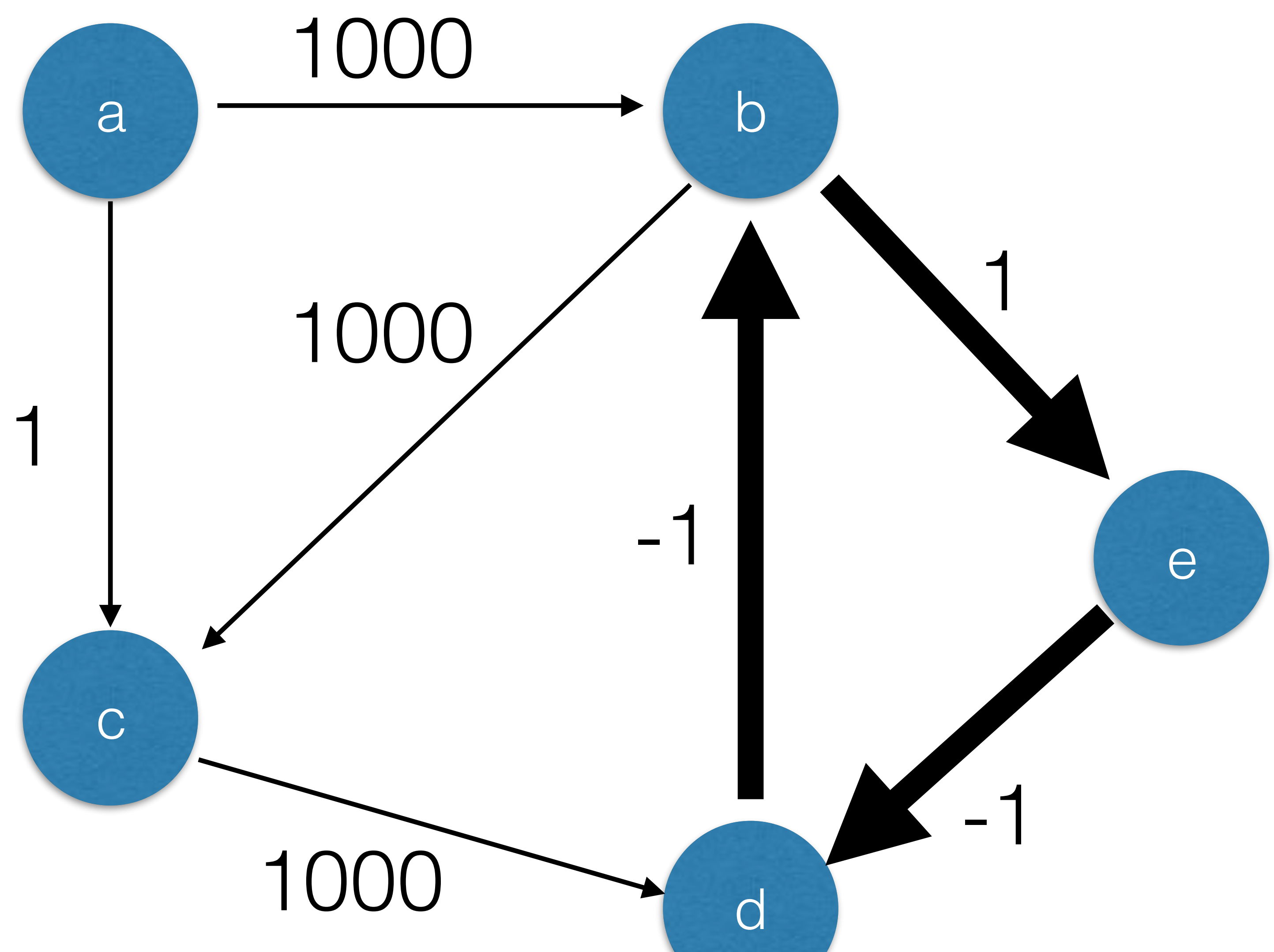
Observation:
if the shortest path tree has depth k, there are no more changes to the distances after k rounds of relaxation

Bellman-Ford with optimization

What is the shortest path from a to c?



Graphs with negative cycles don't have well-defined distances



negative (weight) cycle: cycle in which the sum of the edge weights is negative

```
bellmanFord(G, s) {
    Map dist = new Map();
    for each vertex v of G
        dist.put(v , infinity)
    dist.put(s, 0)

    repeat V - 1 times
        for each vertex v of G
            for neighbor w of v
                relax(v -> w)

    for each vertex v of G
        for neighbor w of v
            if (dist(v) > dist(w) + w(v->w)
                error("negative cycle detected");
    return dist;
}
```

one more round of relaxation
to see if there are
any more changes

Bellman-Ford with cycle detection

In summary

- single source shortest-path problem
- 2 relaxation-based algorithms
  - Dijkstra’s algorithm: O(E log V), edges non-neg
  - Bellman-Ford algorithm: O(VE), neg edges OK