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7º Questionário - Controle Digital

1) a) $F(s) = \frac{Y(s)}{U(s)} = \frac{4}{s(s+2)} \Rightarrow Y(s)(s^2+2s) = 4U(s)$

$$\Updownarrow$$

$$\ddot{y}(t) + 2\dot{y}(t) = 4u(t)$$

Como $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ com $y(t) = x_1(t)$ e $\dot{x}_1(t) = \dot{y}(t) = x_2(t)$

$\dot{x}_1 = x_2$

Temos:

$$\dot{x}_2(t) + 2x_2(t) = 4u(t) \Rightarrow \dot{x}_2(t) = -2x_2(t) + 4u(t)$$

$$y(t) = x_1(t)$$

Então: $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(t)$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

b) $G = e^{AT}$ $\quad H = \left(\int_0^T e^{A\tau} d\tau \right) . B$

$A = \begin{bmatrix} 0 & 1T \\ 0 & -2T \end{bmatrix} \Rightarrow p(\lambda) = \det(A - \lambda I_2) = \det \begin{bmatrix} -\lambda & T \\ 0 & -2T-\lambda \end{bmatrix} = -\lambda.(-2-\lambda) = 2\lambda + \lambda^2 = 0$

$\lambda = 0$

$\lambda = -2T$

para $\lambda_1 = 0 \Rightarrow \begin{pmatrix} 0 & 1T \\ 0 & -2T \end{pmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = 0 \Rightarrow v_1 = \begin{bmatrix} T \\ 0 \end{bmatrix} = \begin{bmatrix} 1T \\ 0 \end{bmatrix}$

$-2x_2 = 0$

$\lambda_2 = -2T \Rightarrow \begin{pmatrix} +2T & 1 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow +2Tx_1 + x_2 = 0 \Rightarrow x_2 = -2Tx_1 \quad v_2 = x_1 \begin{bmatrix} 1T \\ -2T \end{bmatrix}$

$P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1T & 1T \\ 0 & -2T \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & -2T \end{bmatrix} \quad P^{-1} = \begin{bmatrix} \frac{1}{T} & \frac{1}{2T} \\ 0 & -\frac{1}{2T} \end{bmatrix}$

Diagonalizar: $AT = PDP^{-1} = \begin{bmatrix} T & T \\ 0 & -2T \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & -2T \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{T} & \frac{1}{2T} \\ 0 & -\frac{1}{2T} \end{bmatrix}$

$e^{AT} = Pe^{D}P^{-1} \Rightarrow e^{A} = \begin{bmatrix} T & T \\ 0 & -2T \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-2T} \end{bmatrix} \begin{bmatrix} \frac{1}{T} & \frac{1}{2T} \\ 0 & -\frac{1}{2T} \end{bmatrix} = \begin{bmatrix} T & Te^{-2T} \\ 0 & -2Te^{-2T} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{T} & \frac{1}{2T} \\ 0 & -\frac{1}{2T} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} - \frac{e^{-2T}}{2} \\ 0 & e^{-2T} \end{bmatrix}$

Considerando $e^{D} = e^{\begin{bmatrix} 0 & 0 \\ 0 & -2T \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2T} \end{bmatrix}$ por diagonal de $D = \begin{pmatrix} e^{0} & 0 \\ 0 & e^{-2T} \end{pmatrix}$

Chegamos em $e^{AT} = \begin{bmatrix} 1 & \frac{1}{2} - \frac{e^{-2T}}{2} \\ 0 & e^{-2T} \end{bmatrix} = G(T)$

$B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$H(T) = \left(\int_0^T \begin{bmatrix} 1 & \frac{1}{2} - \frac{e^{-2T}}{2} \\ 0 & e^{-2T} \end{bmatrix} \right) \cdot B$ $\quad H(T) = \left(\begin{bmatrix} \int_0^T 1\,dT & \int_0^T \frac{1}{2} - \frac{e^{-2T}}{2}\,dT \\ 0 & \int_0^T e^{-2T}\,dT \end{bmatrix} \right) \cdot B$

$\int_0^T G(T) = \begin{bmatrix} T & \frac{T}{2} - \frac{e^{-2T}}{-4} + \frac{1}{4} \\ 0 & \frac{e^{-2T}}{-2} - \frac{1}{2} \end{bmatrix} \xrightarrow{\times B} \underbrace{\begin{bmatrix} 2T + e^{-2T} - 1 \\ -2e^{-2T} + 2 \end{bmatrix}}_{H(T)} \xrightarrow{T=0,2} \begin{bmatrix} 0,4 + e^{-0,4} - 1 \\ -2e^{-0,4} + 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0,0703 \\ 0,6594 \end{bmatrix}}_{H(0,2)}$

$G(0,2) = \begin{bmatrix} 1 & 0,1648 \\ 0 & 0,6703 \end{bmatrix} \qquad H(0,2) = \begin{bmatrix} 0,0703 \\ 0,6594 \end{bmatrix}$

c) Logo $x[k+1] = \begin{bmatrix} 1 & 0,1648 \\ 0 & 0,6703 \end{bmatrix} x[k] + \begin{bmatrix} 0,0703 \\ 0,6594 \end{bmatrix} u[k]$ com $T = 0,2$

$y[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[k]$

Fazendo $k = 0 \Rightarrow x[1] = \begin{bmatrix} 1 & 0,1648 \\ 0 & 0,6703 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0,0703 \\ 0,6594 \end{bmatrix} u(0)$

$x[1] = \begin{bmatrix} 0,0703 \\ 0,6594 \end{bmatrix} \cdot u(0)$

Fazendo $k=1 \Rightarrow x(2) = \begin{bmatrix} 1 & 0{,}1648 \\ 0 & 0{,}6703 \end{bmatrix} \cdot \begin{bmatrix} 0{,}0703 \\ 0{,}6594 \end{bmatrix} \cdot v(0) + \begin{bmatrix} 0{,}0703 \\ 0{,}6594 \end{bmatrix} v(1)$

Como $x(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0{,}179 \\ 0{,}442 \end{bmatrix} v(0) + \begin{bmatrix} 0{,}0703 \\ 0{,}6594 \end{bmatrix} v(1)$

Resolvendo o sistema: $0{,}179\,v(0) + 0{,}0703\,v(1) = 1$ eq1

$0{,}442\,v(0) + 0{,}6594\,v(1) = 0$ eq2

eq2: $v(1) = -0{,}6703\,v(0)$, substituindo em eq1:

$0{,}1319\,v(0) = 1 \Rightarrow v(0) = 7{,}5828$

$v(1) = -5{,}0827$

Como $x[1] = \begin{bmatrix} 0{,}0703 \\ 0{,}6594 \end{bmatrix} \cdot v(0) = \begin{bmatrix} 0{,}0703 \\ 0{,}6594 \end{bmatrix} \cdot 7{,}5828 = \begin{bmatrix} 0{,}5332 \\ 5 \end{bmatrix}$

2) $u[k] \rightarrow \boxed{Q(z)} \xrightarrow{w[k]} \boxed{P(z)} \xrightarrow{y[k]}$

$$Q(z) = \frac{\beta z}{z-\alpha} \qquad P(z) = \frac{z+0{,}4}{z^2-z+0{,}5}$$

a) $\frac{w[z]}{U[z]} = \frac{\beta z}{z-\alpha} \Rightarrow z w(z) - \alpha w(z) = \beta z U(z) \Rightarrow w[k+1] - \alpha w[k] = \beta U[k+1]$

$\frac{y(z)}{w(z)} = \frac{z+0{,}4}{z^2-z+0{,}5} \Rightarrow y[k+2] - y[k+1] + 0{,}5\,y[k] = w[k+1] + 0{,}4\,w[k]$

Definimos $x_3[k] = w[k] \Rightarrow x_3[k+1] = \alpha x_3[k] + \beta U[k+1]$ $\quad w_3 = x_3$

$x_1[k] = y[k] \Rightarrow x_2[k+1] = -0{,}5x_1[k] + x_2[k] + w[k+1] + 0{,}4\,w[k]$

$x_2[k] = y[k+1] \qquad x_2[k+1] = -0{,}5x_1[k] + x_2[k] + \underbrace{x_3[k+1]}_{\alpha x_3[k] + \beta U[k+1]} + 0{,}4x_3[k]$

Temos $Q(k)$

$x_3[k+1] = \alpha x_3[k] + \beta U[k+1] \Rightarrow G_q = \alpha \quad H_q = \beta \quad C_q = 1 \quad D_q = 0$

$w_3[k] = x_3[k]$

$P(z)$ Temos:

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -0{,}5 & 1 \end{bmatrix}}_{G_p} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0{,}4 & 1 \end{bmatrix}}_{H_p} \cdot \begin{bmatrix} w[k] \\ w[k+1] \end{bmatrix}$$

$$y[k] = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_p} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} \qquad D_p = 0$$

b)

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -0{,}5 & 1 & 0{,}4+\alpha \\ 0 & 0 & \alpha \end{bmatrix}}_{G} \cdot \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \beta \\ \beta \end{bmatrix}}_{H} U[k+1]$$

$$y[k] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \cdot \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \end{bmatrix} \qquad D = 0$$

c) Controlável;

$$\mathcal{C} = [H \;\; GH \;\; G^2H] \Rightarrow \mathcal{C} = \begin{bmatrix} 0 & \beta & \beta+\beta(\alpha+0,4) \\ \beta & \beta+\beta(\alpha+0,4)-\frac{\beta}{2} & +\beta+\beta(\alpha+0,4)+\alpha\beta(0,4+\alpha) \\ \beta & \alpha\beta & \alpha^2\beta \end{bmatrix}$$

$n=3$

$\det(\mathcal{C}) = -\frac{(53\beta^3)}{50}$ logo rank $(\mathcal{C}) = 3 = n$ se $\beta \neq 0$

Observabilidade:

$$\mathcal{O} = \begin{bmatrix} C \\ CG \\ CG^2 \end{bmatrix}, n=3 \Rightarrow \mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0,5 & 1 & 0,4+\alpha \end{bmatrix} \Rightarrow \det(\mathcal{O}) = 0,4+\alpha$$

Logo rank $(\mathcal{O}) = n = 3$ se $\alpha \neq -0,4$

$$Q(z) \cdot P(z) = \frac{(z+0,4)\beta z}{(z^2-z+0,5)(z-\alpha)}$$

As condições para que seja controlável e observável são $\beta \neq 0$ e $\alpha \neq -0,4$

Quando não são atendidos, $\beta = 0$ o ganho fica 0 anulando toda a função de transferência

Quando $\alpha = -0,4$, o zero de $P(z)$ se cancelará com o polo de $Q(z)$ ficando:

$$Q(z) \cdot P(z) = \frac{\beta z}{z^2-z+0,5} \text{ quando } \alpha = -0,4.$$