APPM 2360 Lab #1: Mortgage 101

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1 Introduction

In the wake of the financial crisis of 2008, American economists and mathematicians had an extensive amount of data to unravel. Speculation as to the cause of the crisis abounded and the entire US economy was examined in a quest for an answer. It appeared that the housing market was to blame. After further analysis, experts now believe that the housing crash was caused by a lack of understanding from the general public about the cost and load of debts in the form of mortgages that they were signing up for. In this lab, we aim to determine and explore the benefits and draw backs of different payment options and mortgage structures as these greatly affect the total amount paid over the lifetime of a mortgage.

2 Compounding Interest Mortgages

2.1 Finite Compounding

Let us examine some practical examples. Assume a person took out a mortgage with a rate of 5% and an initial balance of $300,000. We will calculate the total cost of the loan after 5 years, compounded once, twice, four times, and twelve times per year without any payments. To compute the total cost of the lone after 5 years, we can use the equation stated below:

$y(t) = (1 + \frac{r}{n})^{t*n} y(0)$

where y(t) is the outstanding balance after t years, n is the number of times compounded per year and r is the annual interest rate.

r = 0.05

t = 5.0

y(0) = 300000

when compounded once per year **$382,884**

when compounded twice per year **$384,025**

when compounded four times per year **$384,611**

when compounded twelve times per year **$385,008**

Often the general public is only interested in the rate of the loan, not seeing that the compounding value is a valuable factor as well. Unfortunately for them, it is. In the scenario given above, the difference over 5 years between compounding monthly and annually is $2124. This is not a small number, but is still something loan seekers overlooked.

2.2 Continuously Compounding

As the amount of times interest is compounded per year approaches infinity, the loan becomes continuously compounding. Following the equation $y(1) = y(0)e^{rt}$ if a person makes a payment of p every month, then the total cost of the mortgage can be determined by $y'(t, y) = ry(t) - 12 * p$,where $y(0) = y0$.

These equations take the form of differentials. Due to of the nature of such equations we can solve for the equilibrium solutions. These solutions will be all of the times that the mortgage has been paid off, because $y'(t) = 0$.

For the first equation, when any of the following are true, the loan has been paid off. $y(t) = \frac{12p}{r}$ or $r = \frac{12p}{y}(t)$ or $p = \frac{ry(t)}{12}$

For the second equation,
$y(t) = \frac{12p(t)}{r}$ or $r = \frac{12p(t)}{y(t)}$ or $p(t) = \frac{ry(t)}{12}$.

Using the differential equation 1 in reference section, we can expand upon our initial differential in order to figure out what the required payment per month will be. To do this we solved equation 1 using separation of variables, setting y(0) = 300,000, r = 0.05.

$$y'(t,y) = ry(t) - 12*p, \quad r = 0.05, \; y(0) = 300{,}000$$

$$\frac{dy}{dt} = ry - 12\,p$$

$$\frac{dy}{ry-12\,p} = dt$$

$$\int \frac{dy}{ry-12\,p} = \int dt$$

Here we will do a U-Sub with u = ry-12p and du = rdy. So dy = $\frac{du}{r}$

$$\frac{\ln(ry-12\,p)}{r} = t + C_1$$

$$\ln(ry-12p) = rt + C_2$$

$$ry-12p = e^{rt+C_2} = C_3\, e^{rt}$$

$$y(t) = \frac{C_3\, e^{rt}+12\,p}{r}, \text{ and } y(0) = 300{,}000$$

$$300{,}000 = \frac{C_3\, e^{r(0)}+12\,p}{r} = \frac{C_3\, e^{0}+12\,p}{r}$$

$$300{,}000 = \frac{C_3+12\,p}{r}$$

$$(300{,}000 * r) - 12p = C_3$$

$$C_3 = 300\,000\, r - 12\,p$$

$$y(t) = \frac{(300\,000\, r-12\,p)\, e^{rt}+12\,p}{r}$$

$$y(t) = \frac{(300\,000\,(0.05)-12\,p)\, e^{0.05\,t}+12\,p}{0.05}, \text{ when } r = 0.05$$

Generalizing this solution, we find the equation for the total value of the loan as a function of time and payment:

$$y(t) = \frac{(Lr - 12 * p)E^{rt} + 12p}{r}$$

Then we solve for P, the amount required to be paid each month.

$$0 * r = (Lr - 12p)E^{rt} + 12p$$

$$0 = LrE^{r}t - 12pE^{rt} + 12p$$

$$-LrE^{r}t = 12p(1 - e^{rt})$$

$$p(L,r,t) = \frac{-LrE^{rt}}{12(1-E^{rt})}$$

Where L is the initial value of the mortgage, r is the interest rate, t is time in years, and p is the monthly payment.

Using this function p(L,r,t) we can calculate the payment required for any continuously compounding mortgage.

3 Adjustable (ARM) vs. Fixed Rate Mortgages

3.1 Fixed Rate Mortgages

Using the function p(L,r,t) as defined above, we can solve for the required monthly payment for a mortgage of initial value L, rate r, and time t in years.

Test

3.1.1 Matlab Euler's Method Approximation

First we created both r(t) and p(t) where r(t) returns the rate as a function of time, and p(t) returns the payment as a function of time. Then we created the differential function using the equation above and the two subfunctions r(t) and p(t). Finally, we implemented Euler's method with a step size of 0.01 in-order to estimate the amount of time it would take for a loan to be paid off as a function of time and payment size.

3.2 Adjustable Rate Mortgages

Suppose that a bank offers an adjustable rate mortgage, with the rate fixed at 3% for the first 3 years, and then the rate follows a function which behaves as

$$r(t) = 0.05 + 0.001t + 0.003 * sin(\frac{\pi}{10}t)$$

This is an example of a adjustable rate mortgage, where after some amount of time, the interest rate begins to vary as a function of time. To get a better idea of what is the best payment method for this type of loan, we will assume two different scenarios. With each we will calculate the amount of time it will take to be paid off and the amount of interest paid. First, suppose the borrower pays \$2,300 per month on a loan with an initial value of \$300,000. Second, suppose the borrower pays \$3,00 per month on the same loan.

To solve for the amount of time it will take for the borrower to pay off the loan, we used the same Matlab code as above in section 3.1.1, which implements Euler's method in order to solve the differential equation $y'(t, y) = r(t)y(t) - 12 * p(t)$.

In order to calculate the amount of interest the borrower will be required to pay pay, we will simply take the found value of the amount of time needed to pay off the loan in years, multiply it by 12 months, then multiply again by the amount paid each month.

In the first case, we will set $p(t) = \$2,300$ and the rate (r(t)) equal to the adjustable rate mentioned above. In this case the borrower ends up spending between 15.28 and 15.29 years paying off the loan. We cannot be any more exact on the time taken to pay off the loan than this because of our finite step size of n = 0.01. At T=15.28 years, there is a balance of \$77.82, which would mean, the final payment would be \$77.82 for the last month.

The interest paid ends up at \$121,810 (Over $\frac{1}{3}$ of the value of the loan!).

In the second case, we will set $p(t) = \$3,000$ and the rate (r(t)) equal to the adjustable rate mentioned above. In this case the borrower ends up spending between 10.34 and 10.35 years paying off the loan. We cannot be any more exact on the time taken to pay off the loan than this because of our finite step size of n = 0.01. At T=10.34 years, there is a balance of \$101.97, which would mean, the final payment would be \$101.97 for the last month.

The interest paid ends up at \$72,342.

Incredibly, payments of only \$700 more per month result in a decrease in interest paid from \$121,810 to \$72,342, which comes out to a savings of \$49,468. On top of the incredible amount of money saved in the higher payments, the amount of time decreases from 15.29 years to 10.35 years. In order to better visualize the incredible difference between the two payment options both have been graphed in Figure 1 below.

Figure 1: Value of Loan vs Time With Monthly Payments

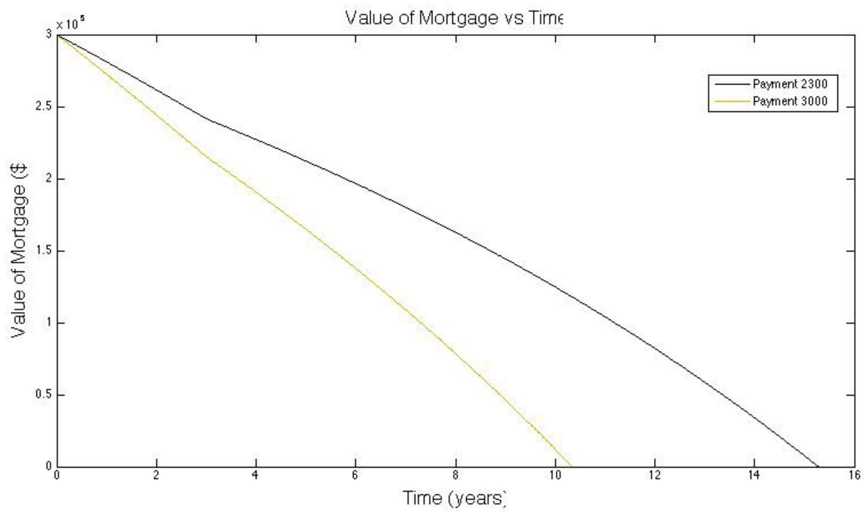

Figure 1 primarily shows that the monthly payment plan of \$2300 a month will result in a longer payment period than a payment plan for \$3000 a month. We also found that as the interest rate was changed, the curve after 3 years went up or down proportionally. This is to be expected since the interest rate is increasing as the monthly payment stays the same. This means that you are going to be making payments for a longer time. Conversely, as the interest rate drops, you don't have to make as many payments. Essentially, after 3 years your monthly payment "battles" the interest rate. If the rate is accruing more interest per month than you are putting in with monthly payments, you will never pay off the mortgage. However, if the interest accrued each month is less than your payment methods, you will end up paying off your mortgage.

4 Interest Only (IO) mortgages

5 Conclusions

We believe the increased popularity of adjustable rate mortgages over fixed rate has to do with the fact that adjustable rate mortgages are a function of time. If times are good, the interest rate over the course of the mortgage could possibly be much lower than a competitive fixed rate mortgage. But if times are bad, you could end up paying much more than a fixed rate mortgage. This is because the rate can begin inflating while people may not be able to afford the increasing payments. For instance, a case were fixed rate would be more advantageous than adjustable rates would be if one were considering buying a home and housing prices had been rapidly on the rise for the last decade and were projected to continue to rise. The best thing to do here would be to get into a fixed rate mortgage. Though there are downsides to this scenario. With the housing costs projected to continue rising, one may consider buying a house that is outside of their price range, depending on inflating house prices to be able to sell and make a profit. In this case if someone locked into a fixed rate loan, and then housing prices suddenly dropped, the owner would be stuck paying for a house they couldn't afford at a rate now less then market rate.

6 References

6.1 Equations

$$y'(t,y) = ry(t) - 12 * p \tag{1}$$

$$y(0) = y_0$$

$$y'(t,y) = r(t)y(t) - 12 * p(t) \tag{2}$$

$$y(t) = (1 + \frac{r}{n})^{nt} * y_0 \tag{3}$$

7 Apendix

7.1 code

Please see the code files included in the ZIP (or upload) of the report.