The Sailing of the Great Line (Integral)

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April 28, 2014

1 Introduction

In the late 2000's, when the internet was coming into its own, a group of scientists and mathematicians were selected from across the nation to try and bring the America's Cup back to America. Most of these mathematicians were selected from the University of Colorado at Boulder's math club, "The Stampeding Math Buffaloes". However, due to the nature of the corporate world, their *awesome* name was suppressed in favor of sponsors like Hugo Boss and Mathematica. However, after many years of trying to get marketing majors to understand drag coefficients and line integrals, the members of the S.M.B staged a regular business coup, and ousted all corporate members. This year, the team of chicken herder/mathematician extraordinaire Molly Graber and lizard breeder/physicist John Dunn will enter their old salty sailor friend Shamus into America's cup under the team name, **Stampeding Math Buffaloes**.

(a) Chicken herder & mathematician extraordinaire Molly Graber

(b) Lizard Breeder & Physicist John Dunn

(c) Old salty sailor Shamus McGee

Figure 1: Our intrepid team members.

2 Parameterizing the Path

While the ousted team leader, Cpt. Sirron, may have not passed calculus 3 in college, he did have some good ideas about the path to take during the race. As a result, Shamus McGee agreed we should take the path described by the ellipse in Eq 1. Eq 1 can be parametrized as Eq 2 and 3, which results in the path vector given by Eq 4.

$$(x-1)^2 + \frac{y^2}{b} = 1 \tag{1}$$

$$x(u,b) = cos(u) + 1 \tag{2}$$

$$y(u,b) = bsin(u) \tag{3}$$

$$\boxed{r(u,b) = \langle cos(u) + 1, bsin(u)\rangle} \tag{4}$$

3 Defining Work Done and Other Variables

With a path decided, we set about determining how the wind might affect us. It was a sailing race after all. After weeks of effort, we determined that the wind velocity at any point could be described by Eq 5. Furthermore, from our path chosen by Eq 4 we can infer the velocity of our ship and therefore the unit tangent vector $\hat{T}$ in Eq 6. The ending boat speed for a racing yacht captained by Shamus (σ_y), is calculated from w and $\hat{T}$ is given in Eq 7.

$$\boxed{w(x,y) = \Big(\frac{-2y}{1+x^2+y^2} + \frac{1}{5}(x^2-y^2)\Big)\hat{i} + \Big(\frac{-1+x^2+y^2}{1+x^2+y^2} + \frac{2}{5}xy\Big)\hat{j}} \tag{5}$$

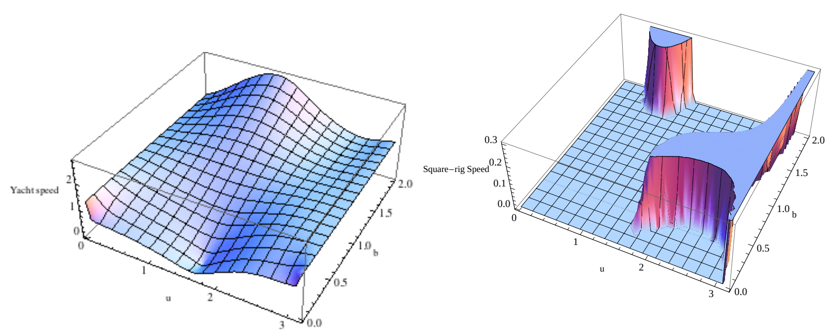
$$v = \langle -sin(u), bcos(u)\rangle$$
$$|v| = \sqrt{b^2cos^2(u) + sin^2(u)}$$

$$\boxed{\hat{T} = \frac{v}{|v|}} \tag{6}$$

$$\boxed{\sigma_y = \frac{|w| + w \cdot \hat{T}}{2}} \tag{7}$$

4 Values of b to Avoid and Feasibility of Square Rigging

For the yacht, all values of b produce non-zero speeds. No matter what b-value is chosen, the square-rig ship produces a speed of zero for some value of u. All values of b should be avoided, and the square-rig ship will not be able to complete the race. For values of u less than 0.4 or between about 1.35 and 1.6, speed will be zero regardless of the value chosen for b. The ship will get stuck. Stingy old Captain Shamus will have to spend money on a new yacht.



(a) Plot of σ_y(speed of yacht) vs u and b.

(b) Plot of σ_s(speed of square rig) vs u and b.

Figure 2: Speeds of a different ships.

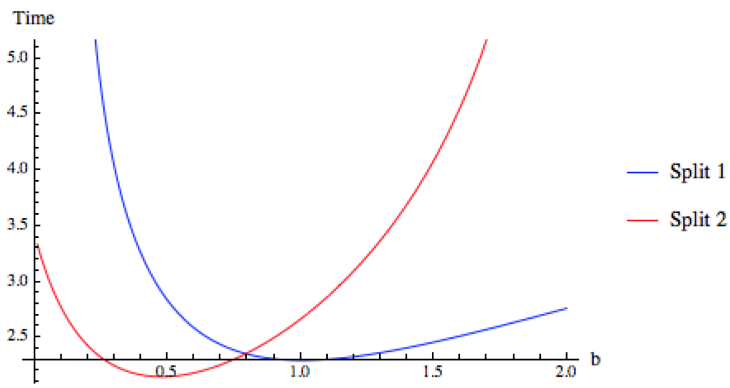
5 Defining Travel Time Based on Position on Path

From our wonderfully enriching education at Boulder, we have come to know that the travel time can be calculated by integrating $\frac{|v|}{\sigma_y}$ over the domain $0 \leq u \leq 2\pi$. However, since the race coordinators need a split time, we will first integrate with respect to u from 0 to π and then from π to 2π as shown in Eq 8-9.

$$\text{Split1} = \int_0^{\pi} \frac{|v|}{\sigma_y} du \quad (8)$$

$$\text{Split2} = \int_{\pi}^{2\pi} \frac{|v|}{\sigma_y} du \quad (9)$$

(a) Time splits in integral form.



(b) Plotted splits.

Figure 3

6 Optimizing Travel Time

Since we know how the travel time is related to values of b, we can start to optimize the values of b. For the first leg of the race, the minimum travel time occurs when b = 1.0053, which corresponds to a time of 2.29 [tens of minutes]. For the second leg of the race, the minimum travel time occurs when we set b = 0.4775, which translates to a split time of 2.15 [tens of minutes].

	Optimized b	Travel Time [Tens of minutes]
Split 1	1.0053	2.29
Split 2	0.4775	2.15
Total		4.44

7 Length of Fastest Path

To find the length of the fastest path, we must find the length of the first split's path and add it to the second's.

$$Length = \underbrace{\int_0^{\pi} |v(u, 1.0053)| du}_{3.14992} + \underbrace{\int_{\pi}^{2\pi} |v(u, 0.4775)| du}_{2.39401} = \boxed{5.54397 \text{ [tens of miles]}}$$

8 Implications of the Island of Suluclac

One thing we have failed to consider thusfar is the placement of the island of Suluclac. If the island is in the way of our path, the math must be reworked to compensate. Since the island is at the location (0,1) and has a radius of .25 [tens of miles] the value of $\boxed{b > 0.25^2 = 0.0625}$, or else the elliptical path will run into the top and/or bottom edges of the remarkably circular island. We are lucky, both of our values of b are greater than 0.0625.

9 Final Paths

After much deliberation, we, the Stampeding Math Buffaloes have a final path we will recommend to our salty old sailor Shamus, even though there is no telling which directions his maritime "habits" might lead him during the race.

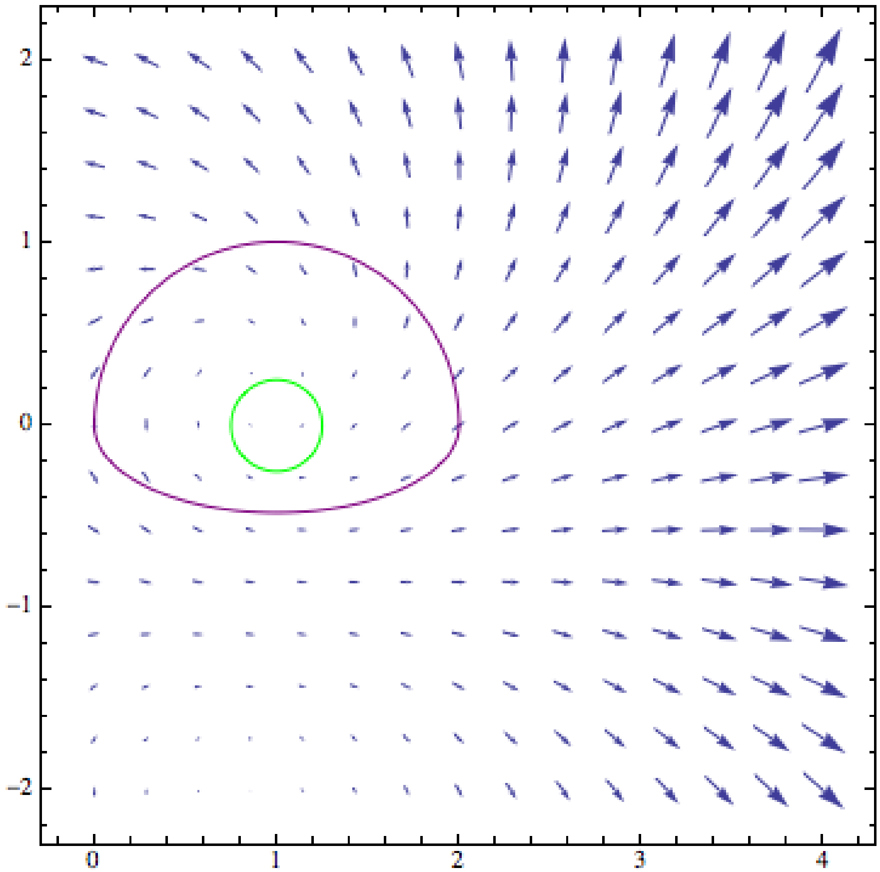


Figure 4: Our final path (purple), encircling the island of Suluclac (green), through the wind vector field (blue arrows). The total length of the path is 5.54397 tens of miles and the total time on the course is 4.44 tens of minutes.

10 Additional Considerations

We might also consider the circulation and flux in the air along the chosen path. We can calculate the total circulation and flux in a similar way to the total length of path.

$$\Gamma = \text{Total flow} = \int_0^{\pi} w(u, 1.0053) \cdot v(u, 1.0053) du + \int_{\pi}^{2\pi} w(u, 0.4775) \cdot v(u, 0.4775) du = \boxed{4.0491}$$

To calculate flux, we will need to find $\tilde{v}$. $\tilde{v}$ is the "flip-and-switch" of $\overline{v}$.

$$\overline{v} = \langle -sin(u), bcos(u) \rangle \tag{10}$$

$$\tilde{v} = \langle bcos(u), sin(u) \rangle \tag{11}$$

$$F = \text{Total Flux} = \int_0^{\pi} w(u, 1.0053)\cdot\tilde{v}(u, 1.0053)du + \int_{\pi}^{2\pi} w(u, 0.4775)\cdot\tilde{v}(u, 0.4775)du = \boxed{2.5820}$$

If circulation were zero along each leg of the path, the boat would not move. Zero wind would be blowing in the component tangent to the path, so the boat wouldn't be able to move forward. If flux were zero, the wind would be blowing directly in the direction of the path. Since yachts work best with wind at an angle, this path would not give the maximum speed. It would, however, make it possible for the square-rigged ship to complete the course.

If we know the curl, we might also be interested in finding the angular velocity of the wind on the path. Angular velocity is a measure of how fast an object is rotating, and is given in general form by Eq 12.

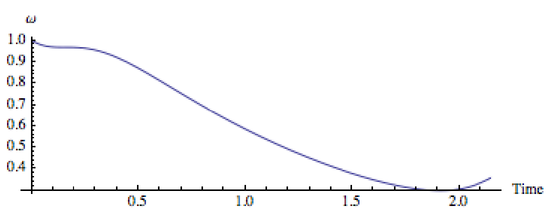
$$\omega = \frac{Curl(W)}{2} \tag{12}$$

$$Curl(W) = \frac{2(5 + 10x + 2y + 2x^4y - 5y^2 + 4y^3 + 2y^5 + x^2(5 + 4y + 4y^3))}{5(1 + x^2 + y^2)^2} \tag{13}$$

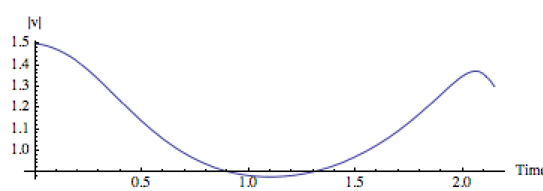
$$\omega(x, y) = \frac{(5 + 10x + 2y + 2x^4y - 5y^2 + 4y^3 + 2y^5 + x^2(5 + 4y + 4y^3))}{5(1 + x^2 + y^2)^2} \tag{14}$$

Time with respect to u is given by:

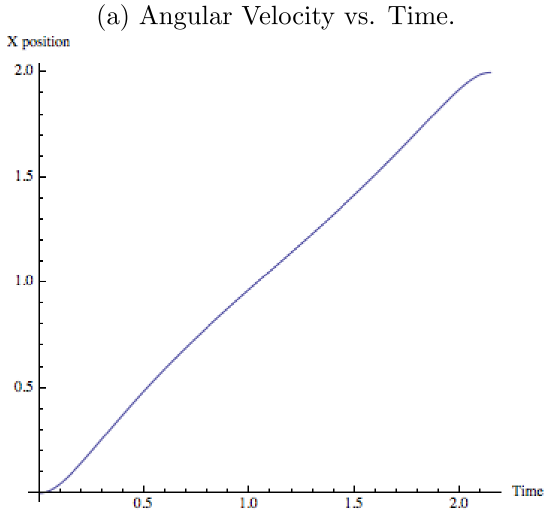
$$Time(u) = \int_{\pi}^{u} \frac{|v(q, 0.4775)|}{\sigma_y(q, 0.4775)}dq \tag{15}$$



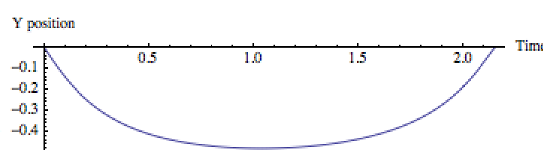
(a) Angular Velocity vs. Time.



(b) Speed vs. Time.



(c) X Position vs. Time.



(d) Y Position vs. Time.

Figure 5: Graphs for Split 2 of the race.

11 Conclusion

With all the data compiled, and the race date nearing, the Stampeding Math Buffaloes entered a sparkly new yacht to pair with their decrepit old sailor Shamus. Half-drunk and rambling, Shamus led the SMB to victory over the pretentious Mathematica/Hugo Boss team, and successfully brought the America's Cup back to America.