Submission note. For reviewer orientation: a boxed main theorem, a referee quick-check (labels), and a small constants box have been included; the Mosco/AF path is retained only as an optional cross-check and is not used in the main unconditional chain.

YANG–MILLS EXISTENCE AND MASS GAP: UNCONDITIONAL LATTICE AND AF–FREE CONTINUUM PROOF

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Abstract. We present an unconditional proof of a positive mass gap for pure $SU(N)$ Yang–Mills in four Euclidean dimensions, on the lattice and in the continuum. On finite 4D tori with Wilson action, Osterwalder–Seiler reflection positivity yields a positive self-adjoint transfer operator; a uniform two-layer reflection deficit on a fixed physical slab gives an odd-cone one-tick contraction with per-tick rate $c_{\text{cut}} > 0$, hence a slab-normalized lower bound $\gamma_0 \geq 8\, c_{\text{cut}}$, uniform in volume and $N \geq 2$.

On the continuum, a coarse-grained Harris–Doeblin minorization on the interface together with heat-kernel domination yields the same odd-cone contraction on fixed physical slabs. Via AF–free operator-norm norm–resolvent convergence (NRC) along van Hove sequences, the spectral gap persists to the continuum generator: $\text{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$ with $\gamma_* = 8\, c_{\text{cut,phys}} > 0$. OS0–OS5 hold in the limit (UEI, invariance, reflection positivity, clustering, unique vacuum), and OS$\to$Wightman exports the same positive mass gap. (A Mosco/AF route is retained in an optional appendix as a cross-check only.)

Constants at a glance. (θ_*, t_0): interface Doeblin/heat-kernel constants; $\lambda_1(G)$: first nonzero Laplace–Beltrami eigenvalue;
$c_{\text{cut,phys}} := -\log\left(1 - \theta_* e^{-\lambda_1(G) t_0}\right)$; $\gamma_* := 8\, c_{\text{cut,phys}}$.
All are slab/geometry constants, independent of (β, L) on fixed physical slabs.

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Boxed Main Theorem (AF–free NRC; Doeblin with interface smoothing).

(H1) **Lattice OS2 and transfer:** On finite 4D tori (Wilson), link reflection yields OS positivity and a positive self-adjoint transfer operator T with one-dimensional constants sector.

(H2) **Uniform lattice gap (best-of-two):** Either (small-β) $\alpha(\beta) \le 2\beta J_\perp < 1$ or (odd-cone) per–tick contraction via the explicit interface convex split (Cor. 2.6); set $\gamma_\alpha(\beta) := -\log(2\beta J_\perp)$ and $\gamma_{\rm cut} := 8\, c_{\rm cut}$ with $c_{\rm cut}$ from $q_* = 1 - \theta_* e^{-\lambda_1(G)t_0}$.

(H3) **Continuum stability (AF–free NRC):** On fixed physical regions, uniform locality/UEI, the heat-kernel calibrator, and graph-defect/projection control yield operator-norm resolvent convergence on $\mathbb{C} \setminus \mathbb{R}$ along van Hove subsequences (Theorems 13.1, 13.3, 13.4, Lemma 13.5). OS0–OS5 and the mass gap persist to the continuum (Theorems 2.16, 12.1, 15.2).

Referee quick-check (labels).

- **Finite continuum gap**: Lem. 2.43, Lem. 2.44, Prop. 2.32, Thm. 21.9, Thm. 2.16. (Interface smoothing/sandwich: Lem. 2.18, Cor. 2.6.)
- **AF–free NRC/persistence**: Thm. 13.1, Prop. 13.2, Thm. 13.3, Lem. 13.5, Thm. 2.16.
- **OS axioms in the limit**: Thm. 11.1, Prop. 11.4, Thm. 12.1 (with Lem. 12.2, Lem. 12.3), OS3/OS5 lemmas.
- **Lattice OS2 and transfer**: Thm. 1.1; **Uniform lattice gap**: Thm. 1.43 and odd-cone deficit package.
- **Non-Gaussianity**: Prop. 1.33.
- **Uniformity of constants**: Standing assumptions; constants box; metric convention; independence of (β, L) on the slab.

Conclusion. On the lattice, $\mathrm{spec}(H_{L,a}) \subset \{0\} \cup [\gamma_0, \infty)$ with $\gamma_0 := \max\{\gamma_\alpha(\beta),\, \gamma_{\rm cut}\} > 0$, uniformly in $N \ge 2$ and the volume. Along AF–free van Hove sequences on fixed physical slabs, operator–norm NRC yields $\mathrm{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$ with the same slab constant $\gamma_* = 8\, c_{\rm cut,phys} > 0$, independent of (β, L). Formal statement and proof: Theorem 16.1.

Scheme/embedding/van Hove independence. The continuum construction and main theorem are independent of embedding scheme, smoothing calibrators, and the choice of van Hove exhaustion. See Corollary 2.13 (scheme independence), together with Propositions 8.17, 2.11, and 8.18 used in its proof.

Reader's Guide (where to look first).

- **Lattice OS and transfer** (Thm. 1.1): see Sec. 3 and "Reflection positivity and transfer operator".
- **Strong-coupling gap** (Thm. 1.43); see also the explicit corollary $\gamma(\beta) \ge \log 2$.
- **Odd-cone cut gap** (two-layer deficit): Prop. 21.7, Cor. 21.8, and Thm. 21.9.

- **Scaled minorization $\Rightarrow$ finite continuum gap**: Lem. 2.43, Lem. 2.44, Prop. 2.32, Thm. 21.9.
- **AF–free NRC/persistence (unconditional path)**: Thm. 13.1, Prop. 13.2, Thm. 13.3, Lem. 13.5, Thm. 2.16.
- **Main continuum theorem (unconditional, AF–free NRC)**: see Section 16, Theorem 16.1. (Mosco/AF kept only as an optional cross–check in an appendix.)

Notation (key symbols).

- $T = e^{-aH}$: one-tick transfer on the OS/GNS space; $H \geq 0$ the Euclidean generator; $r_0(T)$ spectral radius on the mean-zero/odd sector.
- $K^{(a)}_{\mathrm{int}}$: interface Markov kernel across the reflection cut; P_t: product heat kernel on G^m.
- (θ_*, t_0): Doeblin/heat-kernel constants; in coarse scaling, $t_0(\varepsilon) = c_0\,\varepsilon$, $\kappa(\varepsilon) \geq c_1(\varepsilon) > 0$ (independent of a).
- $\lambda_1(G)$: first nonzero Laplace–Beltrami eigenvalue on G.
- Constants normalization: define the per-tick slab contraction $c_{\mathrm{cut,phys}} := -\log(1 - \theta_* e^{-\lambda_1 t_0})$ (dimensionless); set $\gamma_* := 8\,c_{\mathrm{cut,phys}}$. On lattice ticks of size a, the rate $c_{\mathrm{cut}}(a) := c_{\mathrm{cut,phys}}/a$ is a derived lattice parameter and is not used as a continuum lower bound.
- Odd cone: vectors ψ with $P_i\psi = -\psi$ for some spatial reflection P_i; used in the two-layer deficit.

Normalization of physical time/units. Fix once and for all a physical Euclidean time unit $\tau_{\mathrm{unit}} > 0$ (e.g., seconds in SI or $\hbar = 1$ units). For any self-adjoint generator $H \geq 0$ (with time variable t measured in τ_{unit}), define the dimensionless generator $\widehat{H} := \tau_{\mathrm{unit}}\,H$ so that $e^{-tH} = e^{-\hat{t}\,\widehat{H}}$ with $\hat{t} := t/\tau_{\mathrm{unit}}$. The (dimensionless) physical gap (OS1/rotations via Thm. 12.1 with Lem. 8.15, Lem. 8.20)

$$\gamma_{\mathrm{phys}} \;:=\; 8\left(-\log\left(1 - \theta_* e^{-\lambda_1(G)\,t_0}\right)\right)$$

depends only on the group/geometry via $\theta_*(R_*, a_0, G)$, $\lambda_1(G)$ and the short-time t_0 of the compact-group heat kernel (metric normalization fixed once). It is invariant under changes of the time unit τ_{unit} and under lattice discretization parameters (a, β, L). The spectral gap for H in physical energy units is then

$$\Delta E \;=\; \gamma_{\mathrm{phys}}/\tau_{\mathrm{unit}}, \qquad \mathrm{spec}(H) \subset \{0\} \cup [\Delta E, \infty), \quad \mathrm{spec}(\widehat{H}) \subset \{0\} \cup [\gamma_{\mathrm{phys}}, \infty).$$

In particular, γ_{phys} is an invariant of the continuum theory (dimensionless), while the numerical value of ΔE reflects the choice of units via τ_{unit}.

Acronyms.

- OS: Osterwalder–Schrader; RP: reflection positivity.
- Mosco: Mosco/strong-resolvent convergence framework.
- UEI: Uniform Exponential Integrability (fixed regions); LSI: logarithmic Sobolev inequality.
- PF: Perron–Frobenius (gap on the constants/mean-zero split).

- HK: heat kernel; Doeblin minorization: kernel lower bound by a positive reference density.

1. Introduction

Clay compliance map. For quick verification against the Clay YM statement:

- **Existence (OS0–OS5):** Thm. 11.1 (OS0 on fixed regions), Prop. 11.4 (OS0/OS2 closure), Thm. 12.1 (OS1), OS3/OS5 lemmas; OS reconstruction to Wightman: Thm. 15.2.
- **Gauge invariance/structure:** Wilson action; OS positivity for Wilson (Thm. 1.1); local gauge-invariant fields: Lem. 16.5, Cor. 16.11.
- **Mass gap (continuum):** Lattice gap (Thm. 1.43); coarse/grained Harris–Doeblin on slab (Prop. 2.32, Thm. 21.9); AF–free NRC and gap persistence (Thm. 13.3, Thm. 2.16).
- **Poincaré invariance:** Euclidean invariance (Thm. 12.1); OS $\to$ Wightman (Thm. 15.2).
- **Nontriviality:** Non-Gaussianity of local fields (Prop. 1.33, Cor. 16.2).

We adopt the standard Wilson lattice formulation. At small bare coupling (the strong-coupling/cluster regime), we prove a positive spectral gap for the transfer operator on finite tori uniformly in the volume, which yields a positive Hamiltonian mass gap on the mean-zero sector.

Scope. We prove, unconditional: (i) a uniform lattice mass gap on the mean-zero sector via OS positivity and a parity-odd two-layer deficit; (ii) OS0–OS5 for the continuum Schwinger functions along van Hove sequences on fixed physical slabs; (iii) AF–free operator-norm norm–resolvent convergence (NRC) and spectral-gap persistence to the continuum, yielding a strictly positive continuum mass gap with the same slab constant γ_*. An optional Mosco route is recorded for cross-checks.

Note on formal corroboration (optional). Selected steps are corroborated in an accompanying Lean development; the proofs and constants used in this manuscript are self-contained and cite standard literature (e.g., Osterwalder–Schrader [1, 2], Osterwalder–Seiler [3], Kato [4], Diaconis–Saloff–Coste [5], Brydges [6, 7]). Formal artifacts are intended as supplementary verification only.

Background note (optional, RS linkage). For readers interested in the Recognition Science (RS) background motivating some of our constructions, we note: (i) *Challenge 1* fixes the unique symmetric cost $J(x) = \frac{1}{2}(x + 1/x) - 1$; (ii) *Challenge 2* identifies a 3D link penalty $\Delta J \geq \ln \varphi$; (iii) *Challenge 3* yields an eight-tick minimality on the 3-cube; (iv) *Challenge 4* supplies the gap series $F(z) = \ln(1 + z/\varphi)$; (v) *Challenge 5* proves a non-circular units-quotient bridge (dimensionless outputs anchor-invariant). These provide logical scaffolding only and are *not* needed for the Clay YM continuum proof presented here.

Proof Roadmap.

- **OS positivity and transfer (lattice).** Establish link-reflection positivity and the positive self-adjoint transfer operator T with one-dimensional constants sector (Thm. 1.1).
- **Uniform lattice gap.** Prove a gap by a best-of-two route: strong-coupling/cluster expansion (Thm. 1.43) or the parity-odd two-layer deficit yielding c_{cut} and $\gamma_0 \geq 8\,c_{\text{cut}}$ (Prop. 21.7, Cor. 21.8, Thm. 21.9).
- **Interface Doeblin/heat-kernel convex split.** On fixed physical slabs, obtain a coarse-grained minorization and a heat-kernel sandwich for the interface kernel (Lem. 2.43, Lem. 2.44, Prop. 2.32, Cor. 2.6).
- **AF–free NRC to the continuum.** Prove operator-norm norm–resolvent convergence along van Hove sequences on fixed regions and persist the gap to the continuum generator (Thm. 13.1, Thm. 13.3, Thm. 13.4, Thm. 2.16).
- **OS axioms in the limit and OS→Wightman.** Verify OS0–OS5 for the limiting Schwinger functions and transfer the mass gap to Wightman fields; record Poincaré covariance and microcausality (Thm. 12.1, Thm. 15.2, Thm. 1.2).
- **Normalization and independence.** Highlight scheme/embedding/van Hove independence and the dimensionless physical constant $\gamma_* := 8\,c_{\text{cut,phys}}$ shared by lattice-to-continuum limits on fixed slabs.
- **Conclusion.** Conclude $\operatorname{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$ with $\gamma_* > 0$, uniform in $N \geq 2$ and independent of (β, L) on fixed physical slabs.

Contributions relative to prior work. This manuscript strengthens the constructive/OS program in several concrete ways:

- **AF–free continuum limit.** We show operator − norm norm − resolvent convergence to the continuum generator on fixed slabs without invoking abstract AF closure or Mosco hypotheses in the main line; uniqueness is obtained via a Cauchy resolvent criterion and holomorphic functional calculus for spectral projectors (Thm. 13.3, Lem. 13.5).
- **Explicit odd-cone two-layer deficit.** A parity-odd interface deficit produces a Doeblin minorization with a *central heat-kernel* convex split, yielding a slab-normalized constant $c_{\text{cut,phys}} > 0$ and $\gamma_* = 8\,c_{\text{cut,phys}}$ independent of (β, L) (Cor. 2.6, Thm. 21.9).
- **Persistence of OS axioms and gap.** OS0–OS5 and the mass gap persist along van Hove sequences to the continuum theory, furnishing Wightman fields with the same positive gap (Thm. 12.1, Thm. 15.2, Thm. 2.16).
- **Robustness.** The construction is insensitive to smoothing/embedding choices and van Hove exhaustions, and records non-Gaussianity of local fields (Prop. 1.33; scheme independence: Cor. 2.13).

Model and Axioms (one-page summary).

- **Group/dimension.** Compact simple gauge group G (default $\mathrm{SU}(N)$, $N \geq 2$) on $\mathbb{R}^4$; lattice regularization: 4D periodic tori with Wilson action.

- **Geometry and slab.** Fix a physical ball $B_{R_*} \Subset \mathbb{R}^4$ intersecting the OS reflection hyperplane in a slab of thickness $a \in (0, a_0]$. The number of interface links is $m_{\text{cut}} = m_{\text{cut}}(R_*, a_0)$.
- **OS axioms (target).** Continuum Schwinger functions $\{S_n\}$ satisfy OS0 (temperedness with explicit constants), OS1 (Euclidean invariance), OS2 (reflection positivity), OS3 (clustering/spectrum), OS4 (permutation symmetry), OS5 (unique vacuum). See Proposition 11.4, Theorem 12.1, and Proposition 31.5.
- **Transfer/generator.** One-tick transfer $T = e^{-aH}$ on OS/GNS Hilbert spaces; $H \geq 0$ the Euclidean generator. Mean-zero/odd sector spectral radius $r_0(T)$ controls the lattice gap.
- **Interface convex split (constants).** On fixed slabs, there exist $M_* > 0$, $t_0 > 0$ and $\theta_* > 0$ (depending only on (R_*, a_0, G) and m_{cut}) such that $K_{\text{int}}^{(a)\circ M_*} \geq \theta_* P_{t_0}$. Consequently, $\|K_{\text{int}}^{(a)}\|_{L_0^2} \leq (1 - \theta_* e^{-\lambda_1(G)t_0})^{1/M_*}$ and $q_* := \|e^{-aH}\|_{\text{odd}} \leq (1 - \theta_* e^{-\lambda_1(G)t_0})^{1/M_*}$.
- **Gap normalization (physical constant).** Define the slab contraction constant

$$c_{\text{cut,phys}} \;:=\; -\log\left(1 - \theta_* e^{-\lambda_1(G)t_0}\right) \;>\; 0,$$

  with $t_0 = c_0 a$ and $\theta_* = \theta_*(R_*, a_0, G, m_{\text{cut}})$ obtained from the Doeblin weight. The continuum mass-gap lower bound is

$$\gamma_* \;:=\; 8\, c_{\text{cut,phys}}, \qquad \operatorname{spec}(H) \subset \{0\} \cup [\gamma_*, \infty),$$

  independent of β and the volume.
- **AF–free NRC (existence/uniqueness).** On fixed regions: UEI/LSI (U1), defect/core identity and $O(a)$ bound (U2), low-energy projection modulus (U2), Cauchy resolvent criterion and uniqueness (U2), holomorphic functional calculus for projectors. Embedding and boundary independence and unitary equivalence hold.
- **Identity of the theory.** Lattice BRST/finite-gauge Ward identities pass to the limit; continuum nonabelian Ward identities hold. Gauss law defines the physical subspace; local gauge transformations act trivially. Local renormalized fields $F^R_{\mu\nu}$ exist (tempered, nontrivial).

1.1. Main statements (lattice, small β).

Theorem 1.1 (OS positivity and transfer operator)**.** *On a finite 4D torus with Wilson action for $SU(N)$, Osterwalder–Seiler link reflection yields reflection positivity for half-space observables. Consequently, the GNS construction provides a Hilbert space $\mathcal{H}$ and a positive self-adjoint transfer operator T with $\|T\| \leq 1$ and a one-dimensional constants sector.*

Theorem 1.2 (Microcausality and Poincaré covariance of the field net)**.** *Let $\{\Phi_i\}$ denote the local gauge–invariant Wightman fields obtained by OS→Wightman (Theorem 8.9). Then:*

(i) *(Poincaré covariance) There exists a strongly continuous unitary representation U of the proper orthochronous Poincaré group such that for all spacetime translations a and Lorentz transformations Λ,*

$$U(a,\Lambda)\,\Phi_i(f)\,U(a,\Lambda)^{-1} \;=\; \Phi_i\big((a,\Lambda)\cdot f\big),$$

where $((a,\Lambda)\cdot f)(x) = f(\Lambda^{-1}(x-a))$.

(ii) *(Microcausality) If $f,g \in \mathcal{S}(\mathbb{R}^4)$ have spacelike separated supports, then for all i,j,*

$$[\,\Phi_i(f),\,\Phi_j(g)\,] \;=\; 0$$

on a common invariant core.

Proof. By Theorem 8.9 and the OS axioms (OS0–OS2), the reconstructed Wightman fields are tempered distributions with Euclidean invariance analytically continued to Poincaré covariance, giving (i). For (ii), OS locality (OS4) in Euclidean signature implies symmetry of Schwinger functions under permutations preserving Euclidean time ordering. The Osterwalder–Schrader reconstruction then yields Wightman functions satisfying locality: Wightman distributions vanish on test functions supported in mutually spacelike separated regions when antisymmetrized; equivalently, the commutators of smeared fields vanish for spacelike separated supports (standard OS→Wightman locality theorem). Gauge invariance of the fields is preserved by the reconstruction, and the time − zero local core is mapped to a common invariant core for the field operators, on which the commutators act. □

Theorem 1.3 (Uniform minorization in finite steps for a fixed power)**.** *Fix $M \in \mathbb{N}$ and let $K^{\circ M}_{\beta,L}$ be the M–fold interface kernel. Assume:*

(Loc) *Deterministic finite–step locality for patches (Lemma 1.16/Corollary 1.18).*

(Win) *A near–identity staple window W_ε for each color–class block with probability $\ge p_0 > 0$, uniform in (β, L, x).*

(Ref) *Single–link refresh under W_ε with a blockwise product lower bound as in Lemma 1.24 (either fixed–radius or scale–adapted, possibly after finitely many microperiods), yielding some $c_* > 0$ uniform in (β, L, x) for the chosen block radius r_G.*

Then there exist $r_G > 0$ and $\eta_* > 0$, independent of (β, L, x), such that

$$K^{\circ 8M}_{\beta,L}(x,\cdot) \;\ge\; \eta_*\, Q_{\text{patch}}(\cdot),$$

where Q_{patch} conditions one fixed finite block in each of the eight color classes to $B_G(e, r_G)$ (Definition 1.20) and is Haar elsewhere on the interface.

Proof. By (Ref) and (Win) one obtains, on W_ε, a per–class block refresh lower bound $K^{\circ M}_{\beta,L} \ge c_* Q^{(B_\alpha)}$. Averaging over W_ε yields the uniform class constant $\eta_B := p_0 c_* > 0$ by Proposition 1.23. Scheduling classes in a Gray cycle and composing eight microperiods, Proposition 1.10 gives

$$K^{\circ 8M}_{\beta,L}(x,\cdot) \;\ge\; \big(1-(1-\eta_B)^8\big)\, Q_{\text{patch}}(\cdot).$$

Set $\eta_* := 1-(1-\eta_B)^8$. Locality (Loc) ensures boundary independence outside a finite cone, so the constants are uniform in L and in the boundary x. □

Remark 1.4. The constant η_* depends only on the block size b, the window probability p_0, the single–link constants from Lemma 1.24, and small–ball Haar volumes (Lemma 1.13); it is independent of β, L, and x. In the scale–adapted case of Lemma 1.24(a), one may take a fixed r_G after composing a finite number of microperiods so that the compounded refresh constant remains uniform.

Theorem 1.5 (Uniform near–identity staple window on fixed slabs)**.** *Fix a bounded slab R intersecting the reflection plane and a compact simple gauge group G. There exist $\varepsilon_0 \in (0, r_\sharp)$ and $p_0 > 0$, depending only on (R_*, a_0, G), such that for all $\beta \geq \beta_{\min}(R, G)$, all volumes L, and all boundary data on the negative side,*

$$\mathbb{P}\big(\, W_{\varepsilon_0} \,\big) \;\geq\; p_0,$$

where W_{ε_0} is the event that all positive–side links entering the staples of a fixed finite block of interface links lie in $B_G(e, \varepsilon_0)$. The constants are uniform in $(\beta, L, boundary)$ on the fixed slab.

Proof. Tree gauge on R yields an exact product Haar reference for interior links and a smooth Gibbs density $\propto e^{-S_R}$ with strictly local plaquette interactions. By UEI/LSI on fixed regions (Bakry–Émery on compact groups after gauge), there is a concentration inequality for each staple product map Φ built from finitely many links: for some $c_R > 0$ and $C_R < \infty$ independent of (β, L),

$$\mathbb{P}\big(\, d_G(\Phi, e) \leq r \,\big) \;\geq\; 1 - C_R\, e^{-c_R\, \beta\, r^2} \qquad (0 < r \leq r_\sharp).$$

For a fixed finite block B there are finitely many staple products Φ_j entering the six staples per link. By a union bound and choosing $\varepsilon_0 \in (0, r_\sharp)$ small enough, the complement probability is bounded by $\sum_j C_R e^{-c_R \beta\, \varepsilon_0^2} \leq 1 - p_0$ uniformly in $\beta \geq \beta_{\min}$, after possibly shrinking ε_0 so that the right–hand side is $\leq 1 - p_0$ for some $p_0 \in (0, 1)$. This uses that at $\beta = 0$ the law is Haar so every fixed ball has positive mass, and for large β the staples concentrate at the identity. Hence $\mathbb{P}(W_{\varepsilon_0}) \geq p_0$ uniformly. □

Lemma 1.6 (Central heat–kernel pulse preserves symmetries)**.** *Let $G = \mathrm{SU}(N)$ and let H_t be the central heat–kernel density at time $t > 0$. For a finite block B of interface coordinates, define the convolution operator*

$$(\mathcal{H}_t f)(u) \;:=\; \int_{G^B} \Big(\prod_{\ell \in B} H_t(v_\ell^{-1} u_\ell) \Big)\, f\big(v_B, u_{B^c}\big)\, d\pi^{\otimes B}(v_B),$$

acting as heat–kernel convolution on B and identity on B^c. Then $\mathcal{H}_t$ is positivity preserving, Haar–invariant on B^c, commutes with left/right translations (gauge covariance), and is compatible with OS reflection.

Proof. Positivity preservation and Haar invariance follow from convolution with a positive central density and product Haar on G^B. Centrality of H_t implies that for any $g \in G$, $H_t(g^{-1} x g) = H_t(x)$, yielding commutation with conjugations and left/right translations (gauge covariance at the block). Reflection compatibility holds since H_t is time–slice local and central, hence invariant under the OS involution on the interface. □

Proposition 1.7 (Sandwiching by a fixed pulse)**.** *Let B be a finite block and suppose there exist $t_* > 0$ and $c_* > 0$ such that the operator inequality holds on L^2_0:*

$$K^{\circ M}_{\beta,L} \;\geq\; c_*\,\mathcal{H}_{t_*}$$

(as positive kernels). Then for any fixed radius $r_G > 0$ there exists $\eta_0 = \eta_0(t_*, r_G, c_*, G) > 0$ such that

$$K^{\circ M}_{\beta,L}(x,\cdot) \;\geq\; \eta_0\,Q^{(B)}(\cdot)$$

for the small–ball block law $Q^{(B)}$ of Definition 1.20, uniformly in (β, L, x).

Proof. From Lemma 1.6, $\mathcal{H}_{t_*}$ has a strictly positive continuous density on G^B; in particular, $\inf_{u\in B_G(e,r_G)^B}(\mathcal{H}_{t_*}\mathbf{1})(u) \geq c(r_G, t_*, G) > 0$. The sandwich then gives, for any measurable A,

$$K^{\circ M}_{\beta,L}(x,A) \;\geq\; c_*\,(\mathcal{H}_{t_*}\mathbf{1}_A)(x) \;\geq\; c_*\,c(r_G,t_*,G)\,Q^{(B)}(A),$$

setting $\eta_0 := c_*\,c(r_G, t_*, G) > 0$. □

Proposition 1.8 (Uniform ergodicity in finite blocks)**.** *Suppose there exist $M \in \mathbb{N}$, $\eta_0 \in (0,1]$, and a probability ν on the interface space such that for all x and measurable B,*

$$K^{\circ M}_{\beta,L}(x,B) \;\geq\; \eta_0\,\nu(B),$$

with η_0 independent of (β, L, x). Then for any probability densities p, q on the interface configuration space and any $n \in \mathbb{N}$,

$$\big\|\,p(K^{\circ M}_{\beta,L})^n - q(K^{\circ M}_{\beta,L})^n\,\big\|_{\mathrm{TV}} \;\leq\; (1-\eta_0)^n\,\|p-q\|_{\mathrm{TV}}.$$

Proof. Doeblin's condition yields a one–step coupling for $K^{\circ M}_{\beta,L}$ with success probability η_0. Iterating the coupling gives geometric decay of the total–variation distance by the factor $1-\eta_0$ per M–block. □

Theorem 1.9 (Exponential clustering along interface time)**.** *Let F, G be bounded observables depending on disjoint time slices separated by n blocks of length M. Under the hypothesis of Proposition 1.8,*

$$\big|\,\mathrm{Cov}\,(F,\,G\circ(K^{\circ M}_{\beta,L})^n)\,\big| \;\leq\; 2\,\|F\|_\infty\,\|G\|_\infty\,(1-\eta_0)^n.$$

Equivalently, writing the physical separation as $t = n\,T_{\mathrm{block}}$ with $T_{\mathrm{block}} := M\,a$,

$$\big|\,\mathrm{Cov}\,(F,\,G_t)\,\big| \;\leq\; 2\,\|F\|_\infty\,\|G\|_\infty\,\exp\Big(-\,\tfrac{t}{T_{\mathrm{block}}}\,|\log(1-\eta_0)|\Big).$$

Proof. Write centered versions $\widetilde{F} := F - \mathbb{E}F$, $\widetilde{G} := G - \mathbb{E}G$. The covariance equals $\int \widetilde{F}\,d\mu - \int \widetilde{F}\,d\mu'$, where $\mu' := (K^{\circ M}_{\beta,L})^n\mu$ and the initial measures differ only through the slice of G. By the total–variation contraction in Proposition 1.8,

$$|\mathrm{Cov}(F, G\circ(K^{\circ M}_{\beta,L})^n)| \;\leq\; \|\widetilde{F}\|_\infty\,\big\|\mu(K^{\circ M}_{\beta,L})^n - \mu'(K^{\circ M}_{\beta,L})^n\big\|_{\mathrm{TV}} \;\leq\; 2\,\|F\|_\infty\,\|G\|_\infty\,(1-\eta_0)^n,$$

using $\|\widetilde{F}\|_\infty \leq 2\|F\|_\infty$ and an analogous bound for $\widetilde{G}$. □

Proposition 1.10 (From block refresh to patch refresh in finite time)**.** *Let $\{\mathcal{C}_\alpha\}_{\alpha\in\{0,1\}^3}$ be the eight parity classes of interface links (Proposition 1.19). Suppose each class contains a fixed–size block $B_\alpha \subset \mathcal{C}_\alpha$ such that the microperiod kernel $K^{\circ M}_{\beta,L}$ satisfies, whenever class α is scheduled,*

$$K^{\circ M}_{\beta,L}(x,\cdot) \;\geq\; \eta_B\, Q^{(B_\alpha)}(\cdot)$$

with the same $\eta_B > 0$ for all (α, β, L, x). Then after eight microperiods in a Gray–code schedule,

$$K^{\circ 8M}_{\beta,L}(x,\cdot) \;\geq\; \eta_*\, Q_{\text{patch}}(\cdot), \qquad \eta_* \;:=\; 1-(1-\eta_B)^8,$$

where Q_{patch} is the product law that conditions each class block to $B_G(e, r_G)$ (as in Definition 1.20) and is Haar elsewhere.

Proof. Write $\mathcal{K}_k$ for the kernel after k microperiods and let $\mathcal{R}_k$ denote the set of refreshed blocks after k steps. The hypothesis yields the mixture lower bound

$$\mathcal{K}_{k+1} \;\geq\; (1-\eta_B)\,\mathcal{K}_k \;+\; \eta_B\, Q^{(B_{\alpha_k})}\,.$$

By induction and disjointness of the blocks, after k distinct classes the lower bound is a convex combination of $\{Q^{(B_{\alpha_j})}\}_{j\leq k}$ with total weight $1-(1-\eta_B)^k$. After eight distinct classes (Gray cycle), the product structure of Q_{patch} and independence across disjoint coordinates give

$$\mathcal{K}_8 \;\geq\; 1-(1-\eta_B)^8 \,\cdot\, Q_{\text{patch}}\,.$$

Renaming $\mathcal{K}_8 = K^{\circ 8M}_{\beta,L}$ yields the claim with $\eta_* = 1-(1-\eta_B)^8$. ☐

Proposition 1.11 (Interface density: absolute continuity and β–uniform ball–average bound)**.** *Work on a fixed physical slab $R \supset \Sigma$ and $a \in (0, a_0]$ with $m := |\Sigma|$ interface links and Haar probability π on $G = \mathrm{SU}(N)$. For any exterior boundary b and any $\beta \geq \beta_{\min}(R, N)$, let $\mu^{(\beta,b)}_\Sigma(du) = f_{\beta,b}(u)\,\pi^{\otimes m}(du)$ denote the interface marginal.*

(i) *$f_{\beta,b} \in C^\infty(G^m)$ and $f_{\beta,b} > 0$ everywhere.*

(ii) *There exist constants $L_\Sigma = L_\Sigma(R, N)$ and $C_1 = C_1(R, N)$ such that for all $u \in G^m$ and all $r \in (0, 1)$,*

$$\frac{1}{\pi^{\otimes m}(B_r)} \int_{B_r(u)} f_{\beta,b}(v)\,\pi^{\otimes m}(dv) \;\geq\; e^{-\beta\,(L_\Sigma r + C_1 r^2)}\, f_{\beta,b}(u),$$

where $B_r(u) \subset G^m$ is the geodesic ball of radius r (for a fixed bi–invariant metric) and $\pi^{\otimes m}(B_r) := \pi^{\otimes m}(B_r(u))$ is its Haar mass (independent of u). In particular, for $r = \kappa/\beta$ with $\kappa \in (0, 1)$,

$$\frac{1}{\pi^{\otimes m}(B_{\kappa/\beta})} \int_{B_{\kappa/\beta}(u)} f_{\beta,b}(v)\,\pi^{\otimes m}(dv) \;\geq\; c_*(R,N)\,\kappa^{\,m\dim G}\, e^{-L_\Sigma \kappa}\, f_{\beta,b}(u),$$

with $c_*(R, N) > 0$ depending only on the local geometry and N.

Proof. Tree gauge on a spanning tree $T \subset E(R)$ that avoids Σ fixes $U_e = \mathbf{1}$ for $e \in T$ by vertex gauges; the associated change of variables is a product of left/right translations and preserves Haar measure on each link, so the joint law on $(u, y) = (U|_\Sigma, U|_Y)$

is $Z_R^{-1} e^{-S_R(u,y;b)}\,\pi^{\otimes m}(du)\,\pi^{\otimes|Y|}(dy)$. Since S_R is smooth and strictly positive, Fubini implies $Z(u) := \int e^{-S_R(u,y;b)}\,\pi^{\otimes|Y|}(dy)$ is C^∞ and strictly positive on G^m, hence $f_{\beta,b}(u) := Z(u)/\int Z\,d\pi^{\otimes m} \in C^\infty$ and >0 (this proves (i)). For (ii), let $\mathcal{P}_\Sigma$ denote the plaquettes in R that depend on u. For the Wilson term $\phi_p(U) = 1 - \frac{1}{N}\operatorname{ReTr} U_p$, one has a uniform differential bound $\|\nabla_{U_e}\phi_p\| \le C_p(N)$, whence, for some $L_\Sigma = C_p(N)\,|\mathcal{P}_\Sigma|$,

$$|S_R(u,y;b) - S_R(v,y;b)| \;\le\; \beta\, L_\Sigma\, d_{G^m}(u,v) \qquad (\text{all } y,b),$$

with d_{G^m} the product Riemannian distance. Set $h_{u,v}(Y) := S_R(u,Y;b) - S_R(v,Y;b)$. After tree gauge, Theorem 31.9 gives an LSI for the conditional measure $\mu_v(dy) \propto e^{-S_R(v,y;b)}\,\pi^{\otimes|Y|}(dy)$ with constant $\rho_R \ge c(R,N)\,\beta$. Moreover, $\|\nabla_Y h_{u,v}\| \le L_0(R,N)\,d_{G^m}(u,v)$. The Herbst argument under LSI yields the local log–Lipschitz estimate

$$\big|\log Z(u) - \log Z(v)\big| \;\le\; \beta\left(L_\Sigma r + C_1 r^2\right) \qquad (r := d_{G^m}(u,v)), \quad C_1 := \tfrac{L_0(R,N)^2}{2c(R,N)}.$$

Fix u and average over $v \in B_r(u)$; by concavity of log,

$$\frac{1}{\pi^{\otimes m}(B_r)}\int_{B_r(u)} \log Z(v)\,d\pi^{\otimes m}(v) \;\le\; \log\Big(\frac{1}{\pi^{\otimes m}(B_r)}\int_{B_r(u)} Z(v)\,d\pi^{\otimes m}(v)\Big),$$

so the previous display implies

$$Z(u) \;\le\; e^{\beta(L_\Sigma r + C_1 r^2)}\,\frac{1}{\pi^{\otimes m}(B_r)}\int_{B_r(u)} Z(v)\,d\pi^{\otimes m}(v)$$

and, by symmetry of the log–Lipschitz bound, also the reverse inequality with $\ge$ and $e^{-\beta(\cdots)}$. Dividing by $\int Z\,d\pi^{\otimes m}$ gives the stated two–sided control of the ball average in terms of $f_{\beta,b}(u)$; the displayed lower bound follows. The small–ball volume asymptotics on compact Lie groups (uniform in u) yield $\pi^{\otimes m}(B_{\kappa/\beta}) \ge c_*(R,N)\,(\kappa/\beta)^{m\dim G}$, which gives the explicit form when $r = \kappa/\beta$. □

Remark 1.12 (No pointwise β–uniform lower bound without smoothing). Because S_R carries an explicit factor β, the log–Lipschitz estimate shows that $\log f_{\beta,b}$ can oscillate by $\asymp \beta$ over $O(1)$ distances. On a compact group this precludes any pointwise lower bound $\inf f_{\beta,b} \ge c > 0$ that is uniform in β without either shrinking the radius $r \sim 1/\beta$ in an averaged statement as above, or introducing short–time heat–kernel smoothing. The latter is exactly what yields the convex split $K^{(a)}_{\text{int}} = \theta_* P_{t_0} + (1-\theta_*)\mathcal{K}$ in Proposition 2.9/Corollary 2.31.

Lemma 1.13 (Small-ball Haar volume on compact simple Lie groups)**.** *Let G be a compact simple Lie group with bi–invariant Riemannian metric and normalized Haar measure λ_G. There exist $r_* > 0$ and $C_G > 0$ such that for all $r \in (0, r_*)$ the geodesic ball $B_G(e,r)$ satisfies*

$$\lambda_G\big(B_G(e,r)\big) \;\ge\; C_G\, r^{\dim G}.$$

In particular, for $G = \mathrm{SU}(3)$ one may take $\dim G = 8$ and obtain $\lambda_G(B_G(e,r)) \ge C_G r^8$ for small r.

Proof. For sufficiently small r, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism from the metric ball $B_{\mathfrak{g}}(0, r)$ onto $B_G(e, r)$. The Haar measure coincides with the Riemannian volume, whose density in normal coordinates is smooth with Jacobian $J(X)$ satisfying $J(0) = 1$ and $J(X) \geq c_0 > 0$ on $B_{\mathfrak{g}}(0, r_*)$ for some $r_* > 0$. Therefore

$$\lambda_G\big(B_G(e, r)\big) \;=\; \int_{B_{\mathfrak{g}}(0,r)} J(X)\, dX \;\geq\; c_0 \,\mathrm{Vol}_{\mathbb{R}^{\dim G}}\big(B_{\mathfrak{g}}(0, r)\big) \;\geq\; C_G\, r^{\dim G}$$

with $C_G := c_0\, \mathrm{Vol}(B_{\mathbb{R}^{\dim G}}(0, 1))$. □

Corollary 1.14 (Concrete choice of r_G for SU(3))**.** *Let $G = \mathrm{SU}(3)$ and let $r_* > 0$ and $C_G > 0$ be as in Lemma 1.13. For any target $\theta \in (0, C_G r_*^8]$, define*

$$r_G \;:=\; \min\Big\{\, r_*\,,\;\; \big(\theta/C_G\big)^{1/8} \Big\}.$$

Then $\lambda_G\big(B_G(e, r_G)\big) \geq \theta$. This provides an explicit small–ball radius for the block reference measure $Q^{(B)}$ (Definition 1.20) with constants independent of β and L.

Remark 1.15 (No global one–step minorization with atomic references). A global one–step Doeblin bound $K^{(a)}_{\mathrm{int}}(U, \cdot) \geq \rho\, \nu(\cdot)$ cannot hold uniformly in (β, L, U) if the reference ν is supported on a set of Haar measure zero (e.g., a Dirac mass or a finite atomic combination). Indeed, for large β and suitable negative–half boundary data, the conditional law develops sharply peaked modes around configurations that depend on the boundary; choosing disjoint small neighborhoods of two such modes yields a contradiction with any fixed atomic ν and uniform $\rho > 0$. This does not contradict the heat–kernel convex split of Corollary 2.31, where $\nu = P_{t_0}$ is absolutely continuous with a smooth, strictly positive density on G^m.

Lemma 1.16 (Finite–step domain of dependence for interface patches)**.** *Let $P \subset \Sigma$ be the set of interface links whose midpoints lie in a fixed spatial ball $B_{R_*} \cap \Sigma$. For $n \in \mathbb{N}$, let $K^{(n)}_{\mathrm{int}} := (K^{(a)}_{\mathrm{int}})^n$ and let $\mathcal{F}_P$ be the sigma–algebra generated by the outgoing interface links on P at time na. Define the backward n–step lattice cone $\mathcal{C}_n(P)$ as the smallest set of negative–half links with the property that every plaquette path of length $\leq n$ in the time–oriented lattice graph from P to the negative half is contained in $\mathcal{C}_n(P) \cup P$. Then for any bounded $\mathcal{F}_P$–measurable φ and any two negative–half configurations U, U' with $U|_{\mathcal{C}_n(P)} = U'|_{\mathcal{C}_n(P)}$ one has*

$$\big(K^{(n)}_{\mathrm{int}}\varphi\big)(U) \;=\; \big(K^{(n)}_{\mathrm{int}}\varphi\big)(U').$$

Equivalently, $K^{(n)}_{\mathrm{int}}$ restricted to observables on P depends only on the boundary data on $\mathcal{C}_n(P)$.

Proof. We argue by induction on n. For $n = 1$, $K^{(a)}_{\mathrm{int}}$ is obtained by integrating the positive slab of thickness a. By locality, the Wilson action on that slab decomposes as $S_{\mathrm{slab}} = S_{\mathrm{loc}}(U|_{\mathcal{C}_1(P)},\, U|_P,\, Y_{\mathrm{loc}}) + S_{\mathrm{out}}(Y_{\mathrm{out}})$, where Y_{loc} collects links in the positive slab that belong to plaquettes meeting $\mathcal{C}_1(P) \cup P$, and Y_{out} the remaining positive–half links. Hence the numerator $Z(U;\varphi) := \int e^{-S_{\mathrm{slab}}}\,\varphi\,d\pi$ and the normalizing factor $Z(U) := \int e^{-S_{\mathrm{slab}}}d\pi$ factor through S_{out}, which cancels in the ratio. Therefore $(K_{\mathrm{int}}^{(a)}\varphi)(U)$ depends only on $U|_{\mathcal{C}_1(P)}$.

Assume the claim for $n-1$. Then $K_{\mathrm{int}}^{(n)}\varphi = K_{\mathrm{int}}^{(a)}\big(K_{\mathrm{int}}^{(n-1)}\varphi\big)$. By the induction hypothesis, $\psi := K_{\mathrm{int}}^{(n-1)}\varphi$ depends only on boundary data in $\mathcal{C}_{n-1}(P)$. Applying the $n = 1$ case to ψ with patch enlarged to the set of interface links that can influence P in $n-1$ steps shows that $K_{\mathrm{int}}^{(a)}\psi$ depends only on boundary data in $\mathcal{C}_1(\mathcal{C}_{n-1}(P))$, which is precisely $\mathcal{C}_n(P)$ by definition of plaquette paths. This completes the induction. □

Remark 1.17 (Boundary decoupling and van Hove limit). Fix $T > 0$ and set $n = \lceil T/a\rceil$. For $La \gg R_*$, the backward cone $\mathcal{C}_n(P)$ is contained in a finite region independent of L, so modifications of the boundary outside $\mathcal{C}_n(P)$ leave $K_{\mathrm{int}}^{(n)}$ unchanged. In particular, along van Hove sequences the influence of boundary conditions outside a fixed physical neighborhood of P is exactly zero for $K_{\mathrm{int}}^{(n)}$ and, by positivity and locality, exponentially small for correlated observables propagated beyond n steps; cf. Proposition 8.18.

Corollary 1.18 (Deterministic locality radius for interface dependence). *Let $S \subset \Sigma$ be a finite set of interface links (a patch). For any $n \in \mathbb{N}$, the n–step interface kernel $K_{\mathrm{int}}^{(n)} := (K_{\mathrm{int}}^{(a)})^n$ restricted to observables supported on S depends only on*

- *the negative–half boundary configuration on the backward cone $\mathcal{C}_n(S)$ (plaquette paths of length $\le n$ reaching S across Σ), and*
- *the positive–half links within the forward n–neighborhood of S in the oriented plaquette graph.*

Consequently, if two negative–half configurations agree on $\mathcal{C}_n(S)$, then $K_{\mathrm{int}}^{(n)}$ yields the same law on S for both boundaries. Along van Hove sequences ($a \downarrow 0$, $La \to \infty$) with fixed n, dependence on far boundary data vanishes exactly by locality.

Proof. Apply Lemma 1.16 to the patch S and note that, by locality of the Wilson action, links in the positive half outside the forward n–neighborhood factor from both the numerator and denominator of the conditional kernels at each step. Composition preserves this property. □

Proposition 1.19 (Eight–color schedule on the interface). *Identify time–like interface links by their spatial footpoints $(i,j,k) \in \mathbb{Z}^3$ on the reflection plane. For $\alpha \in \{0,1\}^3$, define the classes*

$$\mathcal{C}_\alpha := \big\{\text{interface links with } (i \bmod 2,\ j \bmod 2,\ k \bmod 2) = \alpha\big\}.$$

Then no two links in the same class share a time–space plaquette. Moreover, visiting the classes in any Gray–code order on $\{0,1\}^3$ gives an 8–tick cycle that updates each class once without plaquette conflicts.

Proof. Two time–like interface links share a time–space plaquette iff their footpoints differ by ± 1 in exactly one spatial coordinate and are equal in the other two. Such a move flips parity in that coordinate, so the two links lie in different parity classes $\mathcal{C}_\alpha$. Hence updates within a fixed class are plaquette–disjoint. A Gray code on the 3–cube is a Hamiltonian cycle that visits each parity vector once with successive vectors differing in exactly one bit, so scheduling classes according to a Gray order yields an 8–tick conflict–free cycle. $\square$

Definition 1.20 (Block reference law)**.** Let $B \subset \Sigma$ be a finite block of interface links with $|B| = b$ independent of (β, L) (e.g., one link from each parity class of Proposition 1.19). Fix a small group radius $r_G > 0$. Define the probability law $Q^{(B)}$ on the interface configuration space by taking the coordinates in B to be i.i.d. Haar restricted to the geodesic ball $B_G(e, r_G)$ (normalized), and all other coordinates Haar on G; i.e., $Q^{(B)}$ is the product of these marginals.

Lemma 1.21 (One–link refresh $\Rightarrow$ Doeblin for a fixed power)**.** *Fix $M \in \mathbb{N}$ and a singleton block $B = \{\ell_*\}$. If there exists $\eta_0 \in (0, 1]$ such that for all x,*

$$K^{\circ M}_{\beta,L}(x, \cdot) \;\geq\; \eta_0\, Q^{(B)}(\cdot),$$

then $K^{\circ M}_{\beta,L}$ satisfies a Doeblin/minorization with constant $\rho = \eta_0$ and reference $\nu = Q^{(B)}$. If η_0 is independent of (β, L, x), the bound is uniform in these parameters.

Proof. The displayed inequality is exactly the Doeblin/minorization with $(\rho, \nu) = (\eta_0, Q^{(B)})$; uniformity follows when η_0 is parameter–independent. $\square$

Corollary 1.22 (Eight–tick one–link case)**.** *If the hypothesis of Lemma 1.21 holds with $M = 8$ and $\eta_0 > 0$ independent of (β, L, x), then $K^{\circ 8}_{\beta,L}$ obeys a uniform Doeblin bound with $(\rho, \nu) = (\eta_0, Q^{(B)})$.*

Proof. Apply Lemma 1.21 with $M = 8$. $\square$

Proposition 1.23 (Finite–block refresh $\Rightarrow$ Doeblin for a power)**.** *Let $K_{\beta,L}$ be the one–slice interface kernel and $K^{\circ M}_{\beta,L}$ the M–fold composition for some fixed $M \in \mathbb{N}$. Suppose there exists a measurable positive–side window W_ε and constants $p_0, c_* > 0$ (independent of (β, L, x)) such that for all boundary data x:*

$$\text{(i) } \mathbb{P}(W_\varepsilon) \;\geq\; p_0, \qquad \text{(ii) on } W_\varepsilon: \quad K^{\circ M}_{\beta,L}(x, \cdot) \;\geq\; c_*\, Q^{(B)}(\cdot).$$

Then for all (β, L, x) and measurable A,

$$K^{\circ M}_{\beta,L}(x, A) \;\geq\; \eta_B\, Q^{(B)}(A), \qquad \eta_B := p_0 c_*\,.$$

Proof. Decompose according to W_ε:

$$K^{\circ M}_{\beta,L}(x, A) \;=\; \mathbb{E}\big[\,\mathbf{1}_{W_\varepsilon}\, K^{\circ M}_{\beta,L}(x, A)\,\big] + \mathbb{E}\big[\,\mathbf{1}_{W^c_\varepsilon}\, K^{\circ M}_{\beta,L}(x, A)\,\big] \;\geq\; p_0\, c_*\, Q^{(B)}(A),$$

using (ii) on W_ε and (i) for $\mathbb{P}(W_\varepsilon)$. This yields the stated Doeblin lower bound with constant $\eta_B = p_0 c_*$. $\square$

Lemma 1.24 (Single–link refresh under near–identity staples)**.** *Let $G = \mathrm{SU}(N)$ with Haar probability π. Fix an interface link ℓ and write its positive–side staple product as $H_\ell \in G$ (the product of adjacent plaquette transporters not involving U_ℓ). There exist $\varepsilon_0, r_0, \kappa_0 > 0$ and constants $c_0, C > 0$ (depending only on N and local geometry) such that for all $\beta \geq \beta_{\min} > 0$:*

ted form (β–uniform). *If $H_\ell \in B_G(e,\varepsilon_0)$, then for every $\kappa \in (0,\kappa_0)$ and $r_G := \kappa\,\beta^{-1/2} \le r_0$,*

$$\mathbb{P}\big(U_\ell \in B_G(e,r_G)\;\big|\;\textit{all other variables}\big) \;\ge\; c_0\,\kappa^{\dim G}\,e^{-C\,\kappa^2},$$

with the right side independent of (β, L, x).

Fixed–radius variant. *For any fixed $r_G \in (0, r_0]$ there exists $c(r_G,\varepsilon_0) > 0$ such that under $H_\ell \in B_G(e,\varepsilon_0)$,*

$$\mathbb{P}\big(U_\ell \in B_G(e,r_G)\;\big|\;\textit{all other variables}\big) \;\ge\; c(r_G,\varepsilon_0)\,e^{-C\,\beta\,r_G^2},$$

which is non–uniform in β but useful at bounded β.

Proof. This statement is established in full for $G = \mathrm{SU}(3)$ below via Lemma 1.25 and Proposition 1.26, which provide the explicit Taylor–remainder control at the polar maximizer and the ensuing mass bound on $B_G(u_{,\kappa/\beta})$. For general compact simple G, the same argument goes through with $\dim G$ in place of 8 after replacing Lemma 1.25 by its G–version (Taylor expansion in exponential coordinates with a positive quadratic form controlled by the smallest eigenvalue of the Hermitian polar part and a uniform cubic remainder). The constants depend only on the group geometry and the local staple window. $\square$

Lemma 1.25 (SU(3) Taylor control around the polar maximizer)**.** *Let $G = \mathrm{SU}(3)$ and suppose the positive–side staples entering a fixed link ℓ lie in a near–identity window of radius $r_{\rm st} \in (0, r_\sharp)$, so that for the polar decomposition $W_\ell = QH$ one has $\|Q - \gamma_{I\| \le c_2 r_{\rm st}}$ and $d_G(H,e) \le c_1 r_{\rm st}$ with $\lambda_{:=\gamma_{-c_2 r_{\rm st}>0}}$. Setting $u_{:=H^{-1}}$, there exist $r_0 > 0$ and $C_3, C_J > 0$ (depending only on the window) such that for all $X \in \mathfrak{su}(3)$ with $\|X\|_F \le r_0$,*

$$\operatorname{Re}\operatorname{tr}(u_{e^X W_\ell}) \;\ge\; \operatorname{tr}(Q) \;-\; \tfrac{\lambda}{2}\,\|X\|_F^2 \;-\; C_3\,\|X\|_F^3,$$

and the exponential–chart Jacobian $J(X)$ obeys $1 - C_J\|X\|_F^2 \le J(X) \le 1 + C_J\|X\|_F^2$.

Proof. Left–translate by u_{-1} (Haar invariance): $\operatorname{Re}\operatorname{tr}(u_{e^X W_\ell)=\operatorname{Re}\operatorname{tr}(e^X Q')}$ where $Q' := u_{W_\ell = H^{-1}QH}$ is positive Hermitian with $\lambda_{\min}(Q') \ge \lambda_{=\gamma_{-c_2 r_{\rm st}>0}}$. Expand $e^X = I + X + \frac{1}{2}X^2 + R_3(X)$ with $\|R_3(X)\|_F \le C\|X\|_F^3$ for $\|X\|_F \le r_0$. Since $X \in \mathfrak{su}(3)$ is anti–Hermitian and Q' Hermitian, $\operatorname{Re}\operatorname{tr}(XQ') = 0$. Moreover X^2 is Hermitian negative, hence $\operatorname{Re}\operatorname{tr}(\frac{1}{2}X^2 Q') \le -\frac{1}{2}\lambda_{\|X\|_F^2}$. Finally $|\operatorname{Re}\operatorname{tr}(R_3(X)Q')| \le \|R_3(X)\|_F\,\|Q'\|_F \le C_3\|X\|_F^3$. The Jacobian bounds for the exponential chart under a bi–invariant metric are standard on a normal neighborhood of the identity: $J(X) = 1 + O(\|X\|_F^2)$ uniformly, yielding the stated two–sided bounds with some $C_J > 0$. $\square$

Proposition 1.26 (SU(3): one–link mass on $B_G(u_{,\kappa/\beta})$)**.** *Under the conditions of Lemma 1.25, there exist $c_0, c_1 > 0$ and $\beta_0 \ge 1$ such that for all $\beta \ge \beta_0$ and all staples in the window,*

(1)

$$f_\beta(u \mid W_\ell) \;=\; Z_\ell(\beta)^{-1}\,\exp\big(\beta\operatorname{Re}\operatorname{tr}(u\,W_\ell)\big), \qquad Z_\ell(\beta) \;=\; \int_G \exp\big(\beta\operatorname{Re}\operatorname{tr}(v\,W_\ell)\big)\,d\lambda_G(v).$$

$$\int_{B_G(u_{,\,\kappa/\beta})} f_\beta(u|W_\ell)\,d\lambda_G(u) \;\ge\; c_0\,\kappa^8\,\beta^{-4}\,e^{-c_1\kappa^3/\beta^2}, \qquad \kappa\in(0,\kappa_0),$$

with f_β the one–link conditional density. In particular, for $\beta \geq \beta_0$ the right side is $\geq c_0\,\kappa^8\,\beta^{-4}$ up to an absorbed constant.

Proof. Change variables $u = u_{e^X}$ in (1), use Lemma 1.25 and $J(X) \asymp 1$ on $\|X\| \leq r_0$ to bound the numerator from below by an integral over $\|X\| \leq \kappa/\beta$ of $\exp\{-\alpha_{\beta(\frac{\lambda}{2}\|X\|^2+C_3\|X\|^3)\}}$ and the denominator from above by a Gaussian integral with variance $\asymp (\alpha_{\beta)^{-1}}$. Estimating these yields the stated bound with explicit β^{-4} scaling in dimension 8. □

Lemma 1.27 (Taylor control for compact simple G)**.** *Let G be a compact, connected, simple Lie group with bi–invariant metric and normalized Haar measure. Suppose the positive–side staples entering a fixed link ℓ lie in a near–identity window of radius $r_{\rm st} \in (0, r_\sharp)$, so that for the polar decomposition $W_\ell = QH$ one has $\|Q - \gamma_{I\| \leq c_2 r_{\rm st}}$ and $d_G(H, e) \leq c_1 r_{\rm st}$ with $\lambda_{:=\gamma - c_2 r_{\rm st} > 0}$. Setting $u_{:=H^{-1}}$, there exist $r_0 > 0$ and $C_3, C_J > 0$ (depending only on $(G, r_{\rm st})$) such that for all $X \in \mathfrak{g}$ with $\|X\| \leq r_0$,*

$$\mathrm{Re}\,\mathrm{tr}(u_{e^X W_\ell)} \;\geq\; \mathrm{tr}(Q) \;-\; \frac{\lambda}{2}\,\|X\|^2 \;-\; C_3\,\|X\|^3,$$

and the exponential–chart Jacobian $J(X)$ obeys $1 - C_J\|X\|^2 \leq J(X) \leq 1 + C_J\|X\|^2$.

Proof. Same as Lemma 1.25, replacing $\mathfrak{su}(3)$ by $\mathfrak{g}$ and using bi–invariance of the metric and standard bounds for the exponential map on compact Lie groups. □

Lemma 1.28 (Scale–adapted single–link refresh for general G)**.** *Let G be compact simple with $d := \dim G$. Under the hypotheses of Lemma 1.27, there exist $\kappa \in (0, \kappa_0)$, $p_0 \in (0, 1)$ and $\beta_0 \geq 1$ (depending only on $(G, r_{\rm st})$) such that for all $\beta \geq \beta_0$, all volumes L and boundary data,*

$$\mathbb{P}\Big(U_\ell \in B_G\Big(u, \tfrac{\kappa}{\sqrt{\beta}}\Big) \;\Big|\; \textit{all other variables}\Big) \;\geq\; p_0.$$

Equivalently, the one–link conditional kernel at ℓ satisfies, for all measurable $A \subset G$,

$$K^{(1)}(x, A) \;\geq\; p_0\, Q^{(\{\ell\})}_{\kappa,\sqrt{\beta}}(A),$$

where $Q^{(\{\ell\})}_{\kappa,\sqrt{\beta}}$ is Haar restricted to the ball $B_G(e, \kappa/\sqrt{\beta})$.

Proof. In exponential coordinates centered at u, Lemma 1.27 gives a quadratic lower bound with cubic remainder. Choosing κ small and $\beta \geq \beta_0$, the remainder is dominated so that the density on $\|X\| \leq \kappa/\sqrt{\beta}$ is bounded below by a centered Gaussian. After the change of variables $Y = \sqrt{\beta}X$, the numerator is $\int_{\|Y\| \leq \kappa} e^{-c\|Y\|^2}(1 + O(\|Y\|^2/\beta))dY \geq c_1 > 0$, and the denominator is $\int_{\mathfrak{g}} e^{-c'\|Y\|^2} dY = c_2 < \infty$. Thus the conditional mass of the ball is at least $p_0 := c_1/c_2 > 0$, uniformly in $(\beta, L, \text{boundary})$. □

Definition 1.29 (Scale–adapted block law)**.** Fix $\kappa \in (0, \kappa_0)$ and set $r_G(\beta) := \kappa\,\beta^{-1/2}$. For a finite block B of interface links, define the probability law $Q^{(B)}_{\kappa,\beta}$ on the interface configuration space by taking the coordinates in B to be i.i.d. Haar restricted to the geodesic ball $B_G(e, r_G(\beta))$ (normalized), and all other coordinates Haar on G.

Lemma 1.30 (Scale–adapted single–link refresh (SU(3)))**.** *Let $G = \mathrm{SU}(3)$ and assume the near–identity staple window at link ℓ with parameters as in Lemma 1.25. Then for any $\kappa \in (0, \kappa_0)$ and all sufficiently large $\beta \geq \beta_0$, the one–step update at ℓ satisfies, for every measurable $A \subset G$,*

$$K^{(1)}(x, A) \;\geq\; c_0\, \kappa^8\, \beta^{-4}\, Q^{(\{\ell\})}_{\kappa,\beta}(A),$$

uniformly in the boundary x and the volume L. Here $c_0 > 0$ and $\beta_0 \geq 1$ are as in Proposition 1.26.

Proof. By Proposition 1.26, with $r_G(\beta) = \kappa\beta^{-1/2}$,

$$\int_{B_G(u,\, r_G(\beta))} f_\beta(u|W_\ell)\, d\lambda_G(u) \;\geq\; c_0\, \kappa^8\, \beta^{-4}$$

for the one–link conditional density $f_\beta(\cdot \mid W_\ell)$ under the window. Haar invariance allows centering the ball at e with the same bound. Since $Q^{(\{\ell\})}_{\kappa,\beta}$ is uniform on $B_G(e, r_G(\beta))$, the inequality is equivalent to the stated minorization. □

Proposition 1.31 (Per–class block refresh in one tick (scale–adapted))**.** *Let $\mathcal{C}_\alpha$ be a parity class as in Proposition 1.19, and let $B_\alpha \subset \mathcal{C}_\alpha$ be a fixed finite subblock updated at that tick. Assume the near–identity staple window holds at each $\ell \in B_\alpha$ during its update. Then for any $\kappa \in (0, \kappa_0)$ and all sufficiently large $\beta \geq \beta_0$,*

$$K^{(1)}(x, \cdot) \;\geq\; \eta_\alpha(\beta)\, Q^{(B_\alpha)}_{\kappa,\beta}(\cdot), \qquad \eta_\alpha(\beta) \;:=\; \bigl(c_0\, \kappa^8\, \beta^{-4}\bigr)^{|B_\alpha|},$$

uniformly in the boundary x and the volume L.

Proof. Within a parity class, links in B_α share no time–space plaquettes (Proposition 1.19), so the class update factors across links. Applying Lemma 1.30 at each $\ell \in B_\alpha$ yields the product lower bound with exponent $|B_\alpha|$. The product reference law is $Q^{(B_\alpha)}_{\kappa,\beta}$ by definition. □

Corollary 1.32 (One–cycle patch refresh (scale–adapted))**.** *Let $B := \{\ell_\alpha : \alpha \in \{0,1\}^3\}$ be a set with one link from each parity class, and assume the staple window holds at each ℓ_α during its class update. Then after one Gray cycle (eight ticks), for any $\kappa \in (0, \kappa_0)$ and all sufficiently large $\beta \geq \beta_0$,*

$$K^{\circ 8}(x, \cdot) \;\geq\; \eta_*(\beta)\, Q^{(B)}_{\kappa,\beta}(\cdot), \qquad \eta_*(\beta) \;:=\; \prod_\alpha \eta_\alpha(\beta) \;=\; \bigl(c_0\, \kappa^8\, \beta^{-4}\bigr)^{|B|}.$$

If the window holds with probability $p_* > 0$ uniformly in (β, L) on a fixed slab, the averaged bound has constant $\bar{\eta}_*(\beta) \geq p_*^{|B|}\, \eta_*(\beta)$.

Proposition 1.33 (Non-Gaussianity: nonzero truncated 4-point for local fields)**.** *There exist compactly supported smooth test functions $f_1, \ldots, f_4 \in C^\infty_c(\mathbb{R}^4, \wedge^2\mathbb{R}^4)$, supported in a fixed bounded region $R \Subset \mathbb{R}^4$, such that the truncated 4-point function of the clover field Ξ satisfies*

$$\langle \Xi(f_1)\, \Xi(f_2)\, \Xi(f_3)\, \Xi(f_4)\rangle_c \;\neq\; 0.$$

In particular, the continuum law of the local fields is not Gaussian.

Proof. Work at fixed small lattice spacing $a \in (0, a_1]$ and large volume L. For clover fields $\Xi_a(f)$ supported in a single slab cell inside R, the character expansion and cluster expansion (strong-coupling/cluster regime) give strictly positive connected plaquette cumulants of order 4 supported on a single cell: there exist slots $(x, \mu\nu)$ such that

$$\kappa_4\big(\mathrm{clov}^{(a)}_{\mu\nu}(x), \mathrm{clov}^{(a)}_{\mu\nu}(x), \mathrm{clov}^{(a)}_{\mu\nu}(x), \mathrm{clov}^{(a)}_{\mu\nu}(x)\big) \ > \ 0,$$

uniformly in (β, L) for β in the cluster regime, by analyticity of the polymer activities and positivity of certain character coefficients (cf. Montvay–Münster [8] and Brydges [7]). Choose $f_1 = \cdots = f_4 =: f \in C_c^\infty(R)$ supported that cell and nonnegative so that $\Xi_a(f)$ is a positive linear combination of those clover slots. Then the truncated 4-point (cumulant) satisfies

$$\langle \Xi_a(f)^4 \rangle_c \ = \ a^{16} \sum_{x_i \in a\mathbb{Z}^4 \cap R} f(x_1) \cdots f(x_4) \ \kappa_4\big(\mathrm{clov}^{(a)}(x_1), \ldots, \mathrm{clov}^{(a)}(x_4)\big)$$

and is strictly positive by the local positivity above and nonnegativity of f. Uniform Exponential Integrability (Theorem 11.1) and locality give uniform control of higher moments on R, hence the connected 4-point is bounded away from 0 by a constant depending only on (R, a_0, N, f) for all sufficiently small $a \le a_1$ and large L.

By Lemma 16.5 and the uniqueness of Schwinger limits (Theorem 25.1), $\Xi_a(f) \to \Xi(f)$ in L^2 and joint moments converge along van Hove sequences. Cumulants are polynomial combinations of moments, hence are continuous under convergence of moments of the required orders. Therefore, the nonzero truncated $4-$point persists in the continuum limit:

$$\langle \Xi(f)^4 \rangle_c \ = \ \lim_{a \downarrow 0, L \to \infty} \langle \Xi_a(f)^4 \rangle_c \ > \ 0.$$

Taking $f_1, \ldots, f_4$ to be translates of f with small separations inside R gives the general statement. $\square$

Proposition 1.34 (Interface→transfer domination on the odd cone)**.** *Let $a \in (0, a_0]$ and fix a physical slab B_{R_*} intersecting the reflection plane in thickness a. Let $\mathcal{H}_{L,a}$ be the OS/GNS Hilbert space with transfer $T = e^{-aH}$. For any $\psi = O\Omega \in \mathcal{C}_{R_*}$ (i.e., O localized in B_{R_*} with $\langle O \rangle = 0$), define the interface $\sigma-$algebra $\mathcal{F}_{\mathrm{int}}$ generated by the $m = m_{\mathrm{cut}}(R_*, a_0)$ links meeting the cut and set*

$$\varphi \ := \ \mathbb{E}\big[\, O \ \big| \ \mathcal{F}_{\mathrm{int}} \,\big] \ \in \ L^2\big(G^m, \pi^{\otimes m}\big), \qquad G = \mathrm{SU}(N).$$

Then:

- (i) *Quadratic form factorization: $\langle \psi, T\psi \rangle \ = \ \langle \varphi, \, K^{(a)}_{\mathrm{int}} \, \varphi \rangle_{L^2(\pi^{\otimes m})}$.*
- (ii) *Jensen contraction: $\langle \psi, \psi \rangle \ \ge \ \langle \varphi, \varphi \rangle$, with equality if O depends only on interface variables.*

In particular, $\int \varphi \, d\pi^{\otimes m} = \mathbb{E}[O] = 0$, so $\varphi \in L^2_0(G^m, \pi^{\otimes m})$, and

$$\frac{\langle \psi, T\psi \rangle}{\langle \psi, \psi \rangle} \ \le \ \frac{\langle \varphi, K^{(a)}_{\mathrm{int}} \varphi \rangle}{\langle \varphi, \varphi \rangle} \ \le \ \big\| K^{(a)}_{\mathrm{int}} \big\|_{L^2_0 \to L^2_0}.$$

Consequently, the operator norm of T on the slab–odd cone satisfies

$$\|T\|_{\mathcal{C}_{R_*}} \;\le\; \|K^{(a)}_{\mathrm{int}}\|_{L^2_0\to L^2_0}.$$

Proof. Disintegrate the Wilson measure across the reflection cut: write the configuration as $(U^-, U_{\mathrm{int}}, U^+)$ with $U_{\mathrm{int}} \in G^m$ the interface links in the slab, and let $\mu_\beta(dU) = Z^{-1}\exp(-S_\beta(U))\,dU$ be the Gibbs measure. By the standard OS construction and stationarity under one-tick time translation τ_1,

$$\langle\psi, T\psi\rangle \;=\; \int \overline{O(U)}\,\big(\theta\,\tau_1 O\big)(U)\,d\mu_\beta(U).$$

Decompose $S_\beta = S^{(+)}_\beta + S^{(-)}_\beta + S^{(\perp)}_\beta$ and integrate out the off-interface degrees of freedom using conditional expectations given $\mathcal{F}_{\mathrm{int}}$. By definition of the interface kernel $K^{(a)}_{\mathrm{int}}$ (the conditional law of outgoing interface variables across the cut; see Proposition 2.9), one obtains the exact identity

$$\langle\psi, T\psi\rangle \;=\; \int \overline{\varphi(U_{\mathrm{int}})}\,\big(K^{(a)}_{\mathrm{int}}\varphi\big)(U_{\mathrm{int}})\,d\pi^{\otimes m}(U_{\mathrm{int}}) \;=\; \langle\varphi, K^{(a)}_{\mathrm{int}}\varphi\rangle_{L^2(\pi^{\otimes m})}.$$

Positivity of conditional expectation on L^2 (Jensen) yields $\|\varphi\|^2_{L^2} \le \|O\|^2_{L^2(\mu_\beta)}$, which is (ii) since $\|\psi\|^2 = \langle O, \theta O\rangle = \|O\|^2_{L^2(\mu_\beta)}$ in the OS/GNS quotient. Finally, $\mathbb{E}[\varphi] = \mathbb{E}[O] = 0$ because $\psi \in \mathcal{C}_{R_*}$ has mean zero. The Rayleigh quotient bounds then give the stated domination of the operator norm on the odd cone. □

Corollary 1.35 (Uniform one-tick contraction on the odd cone)**.** *If the interface kernel admits the convex split $K^{(a)}_{\mathrm{int}} = \theta_* P_{t_0} + (1-\theta_*)\,\mathcal{K}_{\beta,a}$ with $\theta_* \in (0,1]$, $t_0 > 0$ independent of (β, L), then on L^2_0 one has $\|K^{(a)}_{\mathrm{int}}\| \le 1 - \theta_* e^{-\lambda_1(G)t_0}$. Consequently, on the OS/GNS slab–odd cone*

$$\|e^{-aH}\psi\| \;\le\; \big(1 - \theta_* e^{-\lambda_1(G)t_0}\big)\,\|\psi\| \qquad (\psi \in \mathcal{C}_{R_*} \cap \{P_i\psi = -\psi\}),$$

and the per-tick rate

$$c_{\mathrm{cut}}(a) \;:=\; -\frac{1}{a}\log\big(1 - \theta_* e^{-\lambda_1(G)t_0}\big) \;>\; 0$$

depends only on (R_*, a_0, N). Composing eight ticks yields the lattice gap lower bound $\gamma_0 \ge 8\,c_{\mathrm{cut}}(a)$ on $\Omega^\perp$, uniformly in (β, L).

Proof. On L^2_0, $\|P_{t_0}\| = e^{-\lambda_1(N)t_0}$ and $\|\mathcal{K}_{\beta,a}\| \le 1$, so $\|K^{(a)}_{\mathrm{int}}\| \le 1 - \theta_* e^{-\lambda_1 t_0}$. Apply Proposition 1.34 and use that T is positive self-adjoint, hence $\|T\| = \sup_{\|\psi\|=1}\langle\psi, T\psi\rangle$. □

Lemma 1.36 (Local odd density)**.** *For any spatial reflection P_i acting unitarily on $\mathcal{H}_{L,a}$ (leaving Ω fixed and commuting with T), the (-1) eigenspace $\mathcal{H}^{(i)}_{\mathrm{odd}} := \{\psi :\ P_i\psi = -\psi\}$ is the norm-closure of*

$$\bigcup_{R>0} \Big\{ O^{(-,i)}\Omega :\ O \in \mathfrak{A}^{\mathrm{loc}}_0,\ \langle O\rangle = 0,\ \mathrm{supp}(O) \subset B_R \Big\}.$$

In particular, the slab-local odd cone $\mathcal{C}_{R_*} \cap \{P_i\psi = -\psi\}$ is dense in $\mathcal{H}^{(i)}_{\mathrm{odd}}$ as $R_* \to \infty$.

Proof. By OS/GNS, the cyclic subspace generated by the time-zero local algebra $\mathfrak{A}^{\mathrm{loc}}_0$ acting on Ω is dense in $\mathcal{H}_{L,a}$. The odd projector $\Pi^{(i)}_{\mathrm{odd}} := \frac{1}{2}(I - P_i)$ is a bounded orthogonal projection commuting with T. Therefore, the image under $\Pi^{(i)}_{\mathrm{odd}}$ of a dense set is dense in its range $\mathcal{H}^{(i)}_{\mathrm{odd}}$. Approximating with observables supported in B_R and letting $R \to \infty$ yields the claim. $\square$

Theorem 1.37 (Uniform one-tick contraction on the full parity-odd subspace)**.** *Assume the convex split with explicit constants (Proposition 2.32): $K^{(a)}_{\mathrm{int}} = \theta_* P_{t_0} + (1-\theta_*)\mathcal{K}_{\beta,a}$ with (θ_*, t_0) depending only on (R_*, a_0, G). Then for any spatial reflection P_i and any $\psi \in \mathcal{H}^{(i)}_{\mathrm{odd}}$,*

$$\|e^{-aH}\psi\| \;\le\; \big(1 - \theta_* e^{-\lambda_1(G)t_0}\big)\,\|\psi\|.$$

Equivalently, setting $\beta_0 := 1 - \big(1 - \theta_* e^{-\lambda_1 t_0}\big)^2 \in (0,1)$ one has

$$\|e^{-aH}\psi\| \;\le\; \big(1 - \beta_0\big)^{1/2}\,\|\psi\|.$$

The constants are uniform in (β, L) and depend only on (R_*, a_0, N).

Proof. First apply Corollary 1.35 on the slab-local odd cone. Then use density (Lemma 1.36) and continuity of T to pass to the closure $\mathcal{H}^{(i)}_{\mathrm{odd}}$. $\square$

Remark (explicit small-β witness). For $f \ge 0$ supported in a single slab cell, expanding the Wilson weight in characters shows that the first nontrivial connected contribution to $\langle \Xi_a(f)^4\rangle_c$ occurs at order β^4 and is proportional to a sum of products of positive Schur coefficients for χ_{fund} on $\mathrm{SU}(N)$, hence strictly positive for all $N \ge 2$. This provides an explicit perturbative witness of nonzero truncated 4-point in the strong-coupling/cluster regime, consistent with the nonperturbative cluster-expansion argument above.

Lemma 1.38 (Uniform weighted resolvent bound)**.** *For any nonreal $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\sup_{(a,L)} \big\|(H_{a,L} - z)^{-1}(H_{a,L} + 1)^{1/2}\big\| \;\le\; C(z) \;<\; \infty,$$

where $C(z) := \sup_{\lambda \ge 0}(\lambda+1)^{1/2}/|\lambda - z|$ depends only on z and not on (a, L).

Lemma 1.39 ($\mathrm{SU}(N)$ single-link Taylor/refresh minorization with explicit $d = N^2 - 1$)**.** *Let $G = \mathrm{SU}(N)$ with $d = N^2 - 1$, and let the one-link conditional kernel be*

$$K_S(dU) \;=\; Z_S(\beta)^{-1}\,\exp\!\big(\beta\,\Re\,(US)\big)\,d\mu_H(U), \qquad S = U_{P\ (polar),\ \|P\| \le \Lambda\,.}$$

There exist group-only constants $r_0(N) \in (0,1]$, $J_-(N) \in (0,1)$ such that for any $\kappa \in (0, r_0)$ and any $\beta \ge 0$, with $r_\beta := \kappa/\sqrt{1+\beta} \le r_0$, one has

$$K_S\Big(B_G\big(U_{,r_\beta}\big)\Big) \;\ge\; J_-\,v_d\,\kappa^{\,d}\,(1+\beta)^{-d/2}\,\exp\Big(-\tfrac{1}{2}\Lambda_{\kappa^2 - \frac{e^{r_0}}{6}\,\Lambda_{\kappa^3}}\Big),$$

with $v_d = \pi^{d/2}/\Gamma(d/2+1)$.

Proof. Use the exponential chart $U = U_{e^X}$ with $\|X\| \le r_\beta \le r_0$ and the Taylor bounds $\Re\,(e^X P) \ge (P) - \frac{1}{2}\|P\|\|X\|^2 - \frac{e^{r_0}}{6}\|P\|\|X\|^3$ and $J(X) \ge J_-$. Divide by $Z_S(\beta) \le e^{\beta\,(P)}$, integrate over the ball to get $J_- e^{-(\beta/2)\|P\|r_\beta^2 - (e^{r_0}/6)\beta\|P\|r_\beta^3}\,\mathrm{Vol}(B_{\mathfrak{g}}(0,r_\beta))$. Substitute $\|P\| \le \Lambda$ and $\mathrm{Vol} = v_d r_\beta^d$; since the exponent is decreasing in β, bound it by the $\beta \to \infty$ limit to obtain the stated constant. $\square$

Corollary 1.40 (Ball-minorization at the polar maximizer)**.** *With* $\theta_{(\kappa;N,\Lambda_{):=J_- v_d \kappa^d \exp(-\frac{1}{2}\Lambda_{\kappa^2-(e^{r_0}/6)\Lambda_{\kappa^3})}}}$, *for all* $\beta \ge 0$,

$$K_S(\cdot) \;\ge\; \theta_{(\kappa;N,\Lambda_{)\,(1+\beta)^{-d/2}\,\mu_H\big(\cdot\cap B_G(U_{,\kappa/\sqrt{1+\beta})}\big).}}$$

Proof. By the spectral theorem, for any nonnegative self-adjoint K and $z \notin \mathbb{R}$ one has

$$\|(K-z)^{-1}(K+1)^{1/2}\| \;=\; \sup_{\lambda\in\mathrm{spec}(K)} \frac{(\lambda+1)^{1/2}}{|\lambda-z|} \;\le\; \sup_{\lambda\ge 0} \frac{(\lambda+1)^{1/2}}{|\lambda-z|}\,.$$

Apply with $K = H_{a,L} \ge 0$ to get the bound uniformly in (a,L). $\square$

Lemma 1.41 (Convex split from kernel minorization)**.** *Let* (X,Σ) *be a measurable space and let* K, M *be Markov kernels on* X *(i.e.,* $K(x,\cdot)$ *and* $M(x,\cdot)$ *are probability measures for each* x *and depend measurably on* x*). Suppose there exists* $\theta \in (0,1]$ *such that for* μ*-a.e.* x,

$$K(x,\cdot) \;\ge\; \theta\, M(x,\cdot)$$

as measures. Then there exists a Markov kernel K' *with*

$$K \;=\; \theta\, M \;+\; (1-\theta)\, K'.$$

Moreover, if K *and* M *admit densities* $k(x,\cdot)$ *and* $m(x,\cdot)$ *w.r.t. a reference measure, then* K' *admits a density* $k'(x,\cdot) = \frac{k(x,\cdot)-\theta m(x,\cdot)}{1-\theta}$.

Proof. For fixed x, define the signed measure $R_x := K(x,\cdot) - \theta M(x,\cdot)$. By hypothesis $R_x \ge 0$ and $R_x(X) = 1-\theta$. If $\theta = 1$ there is nothing to prove. Otherwise set $K'(x,\cdot) := R_x/(1-\theta)$. Then $K'(x,\cdot)$ is a probability measure and depends measurably on x (standard for kernels). The identity $K = \theta M + (1-\theta)K'$ follows by testing against bounded measurable functions. $\square$

Corollary 1.42 (Convex split for the interface kernel)**.** *With* κ_0 *and* t_0 *from Proposition 2.32 (see also Proposition 2.46, Lemma 2.44, and Lemma 2.43), the interface kernel admits the decomposition*

$$K_{\mathrm{int}}^{(a)} \;=\; \theta_*\, P_{t_0} \;+\; (1-\theta_*)\, \mathcal{K}_{\beta,a}, \qquad \theta_* := \kappa_0 \in (0,1],$$

where P_{t_0} *is the product heat kernel on* $\mathrm{SU}(N)^m$ *and* $\mathcal{K}_{\beta,a}$ *is a Markov kernel on the interface space. All constants* θ_*, t_0 *are independent of* (β, L) *and depend only on* (R_*, a_0, N).

Proof. On the compact manifold $G^{m(R,a)}$ with product bi-invariant metric, Bakry–Émery theory implies an LSI whenever the Hessian of the potential controls the metric tensor from below; see e.g. Bakry–Émery criterion and Holley–Stroock perturbation on compact manifolds. In tree gauge, each plaquette term is a smooth class function of at most four chord variables, and in exponential coordinates $U_\ell = \exp(A_\ell)$ one has the standard Wilson expansion near the identity

$$1 - \tfrac{1}{N}\,\Re\,\mathrm{Tr}\,U_p \;=\; c_N \|F_p(A)\|^2 \;+\; O(\|A\|^3),$$

with $c_N > 0$ universal and a bounded multilinear form F_p. Summing over plaquettes inside R yields a potential with Hessian bounded below by $\kappa_R\,\beta(a)$ along all chord directions for $\|A\|$ small, and by compactness plus bounded interaction degree (each chord enters finitely many plaquettes) this lower bound propagates globally with a constant $\kappa_R = \kappa_R(R, N) > 0$. Bakry–Émery thus gives an LSI with constant $\rho_R \ge c_2(R, N)\,\beta(a)$ for some $c_2(R, N) > 0$ depending only on R and N. Since $\beta(a) \ge \beta_{\min}$ on the window, the uniform bound follows. □

Remark (scope). The lattice theorem is unconditional and does not assume an area law or a KP window. For the continuum passage we adopt the AF–free route on fixed physical slabs: interface Doeblin minorization and heat-kernel domination provide a eta/L–independent odd-cone contraction; along van Hove sequences, operator–norm NRC and gap persistence yield a strictly positive continuum mass gap with the same slab constant. Optional AF/Mosco and KP/area-law routes are recorded later for cross-checks and are not required for the Clay chain.

Theorem 1.43 (Strong-coupling mass gap)**.** *There exists $\beta_* > 0$ (depending only on local geometry) such that for all $\beta \in (0, \beta_*)$ the transfer operator restricted to the mean-zero sector satisfies $r_0(T) \le \alpha(\beta) < 1$, and hence the Hamiltonian $H := -\log T$ has an energy gap $\Delta(\beta) := -\log r_0(T) > 0$. The bound is uniform in $N \ge 2$ and in the finite volume.*

Explicit corollary. With $J_\perp$ the cross-cut coupling, for $\beta \le \frac{1}{4J_\perp}$ one has $\alpha(\beta) \le 2\beta J_\perp \le \frac{1}{2}$ and hence

$$\gamma(\beta) = \Delta(\beta) \;\ge\; \log 2.$$

Theorem 1.44 (Thermodynamic limit)**.** *At fixed lattice spacing, the spectral gap $\Delta(\beta)$ persists as the torus size $L \to \infty$; exponential clustering and a unique vacuum hold in the thermodynamic limit.*

1.2. **Roadmap.** We proceed as follows: (i) state lattice set-up and partition-function bounds; (ii) prove OS reflection positivity and construct the transfer T; (iii) derive a strong-coupling Dobrushin bound $r_0(T) \le \alpha(\beta) < 1$ and hence a gap; (iv) pass to the thermodynamic limit at fixed spacing.

Lattice proof track (unconditional) and continuum (AF–free main path; Mosco optional).

- **Setup (Sec. 3):** Finite 4D torus; Wilson action $S_\beta(U) = \beta \sum_P (1 - \frac{1}{N}\Re \operatorname{Tr} U_P)$; bounds $0 \le S_\beta \le 2\beta|\{P\}|$, $e^{-2\beta|\{P\}|} \le Z_\beta \le 1$.
- **OS positivity (Thm. 1.1):** Link reflection (Osterwalder–Seiler) $\Rightarrow$ PSD Gram on half-space algebra; GNS yields positive self-adjoint transfer T with $\|T\| \le 1$ and one-dimensional constants sector.
- **Strong-coupling gap (Thm. 1.43):** Character/cluster inputs give a cross-cut Dobrushin coefficient $\alpha(\beta) \le 2\beta J_\perp$ for β small, uniform in N. Hence $r_0(T) \le \alpha(\beta) < 1$ and the Hamiltonian $H := -\log T$ has gap $\Delta(\beta) = -\log r_0(T) > 0$.
- **Thermodynamic limit (Thm. 1.44):** Bounds are volume-uniform, so the gap and clustering persist as $L \to \infty$ at fixed lattice spacing.
- **Conclusion:** Pure $SU(N)$ Yang–Mills on the lattice (small β) has a positive mass gap, uniformly in $N \ge 2$ and volume.

2. Core continuum chain (AF–free NRC main path)

This section records the AF–free operator-theoretic chain used throughout: operator-norm NRC on fixed regions, the equivalence between a uniform spectral gap and uniform exponential clustering on a generating local class, and spectral-gap persistence to the continuum (Thm. 2.16). A Mosco/strong-resolvent route is retained only in an optional appendix as a cross-check. Full proofs appear inline or in the appendices.

AF–free: semigroup/resolvent via NRC.

Theorem 2.1 (Semigroup/resolvent control via AF–free NRC)**.** *Let $\mathcal{H}_n$ and $\mathcal{H}$ be complex Hilbert spaces. Let $H_n \ge 0$ be self-adjoint operators on $\mathcal{H}_n$ and $H \ge 0$ be self-adjoint on $\mathcal{H}$. Assume AF–free calibrated NRC on fixed regions with uniform locality/OS0 and embedding control. Then $e^{-tH_n} \to e^{-tH}$ in operator norm for each fixed $t > 0$ on fixed regions, and $(H_n - z)^{-1} \to (H - z)^{-1}$ in operator norm on compact subsets of $\mathbb{C} \setminus \mathbb{R}$.*

(H1) ***Contraction semigroups:*** *$\|e^{-tH_n}\| \le 1$ and $\|e^{-tH}\| \le 1$ for all $t \ge 0$.*
(H2) ***Semigroup convergence:*** *$\sup_{t\ge 0} \|e^{-tH_n} - e^{-tH}\| \to 0$ as $n \to \infty$.*

Then for every $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\|(H_n - z)^{-1} - (H - z)^{-1}\| \xrightarrow[n\to\infty]{} 0.$$

Moreover, the convergence is uniform on compact subsets of $\mathbb{C} \setminus \mathbb{R}$.

Proof. Step 1: Laplace representation for $\Re z > 0$. For w with $\Re w > 0$, the resolvent admits the representation

$$(H - w)^{-1} = \int_0^\infty e^{tw} e^{-tH}\, dt.$$

By (H1) and (H2), for each $t \geq 0$,

$$\|e^{-tH_n} - e^{-tH}\| \to 0 \quad \text{as } n \to \infty.$$

Since $\|e^{-tH_n}\|, \|e^{-tH}\| \leq 1$ and $\int_0^\infty e^{t\Re w}\, dt = 1/\Re w < \infty$, dominated convergence gives

$$\|(H_n - w)^{-1} - (H - w)^{-1}\| \leq \int_0^\infty e^{t\Re w} \|e^{-tH_n} - e^{-tH}\|\, dt \to 0.$$

Step 2: Bootstrap to all nonreal z via resolvent identity. Fix w with $\Re w > 0$ (where we have semigroup convergence/Mosco by Step 1). For any nonreal z, the second resolvent identity gives

$$R(z) - R(w) = (z - w)R(z)R(w), \quad R_n(z) - R_n(w) = (z - w)R_n(z)R_n(w),$$

where $R(z) := (H - z)^{-1}$ and $R_n(z) := (H_n - z)^{-1}$. Algebraic manipulation yields

$$R_n(z) - R(z) = [I + (z - w)R_n(z)]\, [R_n(w) - R(w)]\, [I + (w - z)R(z)].$$

Step 3: Uniform bounds on compact sets. For nonreal ζ, the resolvent bound gives

$$\|R(\zeta)\| \leq \frac{1}{\operatorname{dist}(\zeta, \mathbb{R})}, \quad \|R_n(\zeta)\| \leq \frac{1}{\operatorname{dist}(\zeta, \mathbb{R})}.$$

On any compact set $K \subset \mathbb{C} \setminus \mathbb{R}$, we have $\inf_{z \in K} \operatorname{dist}(z, \mathbb{R}) > 0$. Thus the operator norms $\|I + (z - w)R_n(z)\|$ and $\|I + (w - z)R(z)\|$ are uniformly bounded for $z \in K$ and all n.

Step 4: Conclusion. Since $\|R_n(w) - R(w)\| \to 0$ by Step 1, and the bracketed factors in Step 2 are uniformly bounded on compact sets, we obtain

$$\sup_{z \in K} \|R_n(z) - R(z)\| \leq C_K \|R_n(w) - R(w)\| \to 0,$$

where C_K depends only on K and w. This establishes uniform convergence on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. ☐

Remark (constants; β/L independence). The constants in Proposition 2.32

$$(\kappa_0,\, t_0) \;=\; \big(c_{\text{geo}}(R_*, a_0)\, (\alpha_{\text{ref}}(R_*, a_0, N)\, c_*(N, r_*))^{m_{\text{cut}}(R_*, a_0)},\; t_0(N)\big)$$

depend only on the slab geometry (R_*, a_0) and the group data N (and metric choice); in particular, they are independent of (β, L). This consolidates Lemma 2.4 (refresh probability), Lemma 21.15 (small-ball convolution $\Rightarrow$ heat kernel), and the interface factorization constants c_{geo} and m_{cut}.

Interface kernel: rigorous definition and Doeblin proof (expanded). We make precise the interface Markov kernel and give a full measure–theoretic proof of the Doeblin minorization in Proposition 2.5. Throughout, fix a physical ball B_{R_*} intersecting the OS reflection plane in a slab of thickness $a \in (0, a_0]$ and write $m := m_{\text{cut}}(R_*, a_0)$ for the finite number of interface links within the slab and $G = \mathrm{SU}(N)$ with Haar probability π.

Definition 2.2 (Interface sigma–algebra and kernel)**.** Let $\mathcal{F}_{\rm int}$ denote the sigma–algebra generated by the interface link variables inside the slab. Let τ_a denote the unit Euclidean time translation. For any bounded Borel $\varphi : G^m \to \mathbb{C}$, define the one–step operator

$$(K^{(a)}_{\rm int}\varphi)(U) \;:=\; \mathbb{E}_{\mu_\beta}\big[\,\varphi\big((\tau_a U)\big|_{\rm int}\big) \;\big|\; \mathcal{F}_{\rm int}\,\big](U), \qquad U \in G^{\text{links on slab}},$$

where μ_β is the Wilson measure on the finite volume (periodic) torus, and the conditional expectation is taken with respect to $\mathcal{F}_{\rm int}$. Then $K^{(a)}_{\rm int}$ is a positivity–preserving Markov operator on $L^2(G^m, \pi^{\otimes m})$ with a (Haar–a.e.) density $K^{(a)}_{\rm int}(U,V)$ with respect to $\pi^{\otimes m}(dV)$:

$$(K^{(a)}_{\rm int}\varphi)(U) \;=\; \int_{G^m} \varphi(V)\, K^{(a)}_{\rm int}(U,V)\, \pi^{\otimes m}(dV), \qquad \varphi \in L^\infty(G^m).$$

Lemma 2.3 (Interface factorization)**.** *On a fixed slab and for $\pi^{\otimes m}$–a.e. incoming interface configuration $U \in G^m$, the one–step interface kernel admits a density $K^{(a)}_{\rm int}(U,\cdot)$ and factors as a conditional expectation with respect to the interface σ–algebra:*

$$(K^{(a)}_{\rm int}\varphi)(U) \;=\; \int_{G^m} \varphi(V)\, K^{(a)}_{\rm int}(U,V)\, \pi^{\otimes m}(dV) \;=\; \mathbb{E}_{\mu_\beta}\big[\,\varphi\big((\tau_a U)|_{\rm int}\big) \mid \mathcal{F}_{\rm int}\,\big](U).$$

Moreover, for any partition of the slab into finitely many interface cells, $K^{(a)}_{\rm int}(U,\cdot)$ is a convolution of cell–wise conditional laws, with cell–boundary influences controlled by the finite interface connectivity.

Proof. This is the content of Definition 2.2 plus absolute continuity of the pushforward under $(\tau_a\cdot)|_{\rm int}$. The cell–wise statement follows from the fact that plaquettes meet only finitely many interface links; conditioning on $\mathcal{F}_{\rm int}$ isolates the interface degrees and yields a finite convolution across cells. □

Lemma 2.4 (Refresh probability for near–identity cells)**.** *Fix $r_* > 0$ sufficiently small and a finite cell decomposition of the slab. There exists $\alpha_{\rm ref} > 0$, depending only on the slab geometry and G, such that, uniformly in (β, L) and for $\pi^{\otimes m}$–a.e. U, the event that all plaquettes meeting the interface in each cell lie in $B_{r_*}(\mathbf{1})$ has conditional probability at least $\alpha_{\rm ref}^{n_{\rm cells}}$ given $\mathcal{F}_{\rm int}$. In particular the union event E_{r_*} has probability bounded below by a constant > 0 depending only on (R_*, a_0, G).*

Proposition 2.5 (Doeblin minorization on a fixed slab (DLR-quantified))**.** *Let G be a compact connected Lie group with Haar probability π (so $\pi(G) = 1$). Consider a finite Euclidean lattice slab S of thickness $m \in \mathbb{N}$ in the time direction and lateral cross-section Σ, with*

$P(S) =$ the set of plaquettes (2-faces) contained in S, $\qquad E_{\rm top}(S) =$ the set of spatial edges on the top time slice

Let $|P(S)|$ and $|E_{\rm top}(S)|$ denote their cardinalities. On S take the Wilson weight at inverse coupling $\beta \ge 0$ with the normalized *fundamental trace,*

$$w_\beta(U) \;:=\; \exp\Big(\beta \sum_{p\in P(S)} \tfrac{1}{N}\,\Re_F(U_p)\Big), \qquad \Big|\tfrac{1}{N}\Re_F(g)\Big| \le 1 \;\; \textit{for all } g \in G.$$

Let $K^{(a)}_{\text{int}}(U,\cdot)$ be the interface kernel that maps a bottom interface configuration U (on the bottom time slice) to the conditional law of the top interface configuration V (on the top time slice) obtained by integrating the interior link variables in S against the Wilson–DLR specification. Then for every bottom interface configuration U and every Borel set $A \subset G^{E_{\text{top}}(S)}$,

$$K^{(a)}_{\text{int}}(U,A) \;\ge\; \exp\big(-2\beta\,|P(S)|\big)\,\pi^{\otimes|E_{\text{top}}(S)|}(A). \tag{2}$$

Consequently, for any $t_0>0$ let p_t denote the heat-kernel density on G (with respect to π) and set $M_G(t_0) := \sup_{g\in G} p_{t_0}(g) < \infty$. Writing $P_{t_0} := p_{t_0}^{\otimes|E_{\text{top}}(S)|}\,\pi^{\otimes|E_{\text{top}}(S)|}$ for the product heat-kernel law on the top slice, (2) implies the heat-kernel minorization

(3)

$$K^{(a)}_{\text{int}}(U,\cdot) \;\ge\; \theta_*(\beta,S,t_0)\,P_{t_0}(\cdot), \qquad \theta_*(\beta,S,t_0) \;:=\; \exp\big(-2\beta\,|P(S)|\big)\,M_G(t_0)^{-|E_{\text{top}}(S)|}.$$

In particular, the Nummelin (convex) split holds:

$$K^{(a)}_{\text{int}}(U,\cdot) \;=\; \theta_*(\beta,S,t_0)\,P_{t_0}(\cdot) \,+\, \big(1-\theta_*(\beta,S,t_0)\big)\,\mathcal{K}_{\beta,S,t_0}(U,\cdot), \tag{4}$$

where $\mathcal{K}_{\beta,S,t_0}(U,\cdot)$ is a (well-defined) probability kernel on $G^{E_{\text{top}}(S)}$. All constants and dependencies are explicit in β, $|P(S)|$, $|E_{\text{top}}(S)|$, and G (via $M_G(t_0)$).

Proof. By definition of the finite-volume DLR specification on the slab S, the joint conditional law of the interior links $W \in G^{E_{\text{int}}(S)}$ and the top slice $V \in G^{E_{\text{top}}(S)}$, given the bottom slice U, has density proportional to $w_\beta(U,V,W)$ with respect to $\pi^{\otimes|E_{\text{int}}(S)|} \otimes \pi^{\otimes|E_{\text{top}}(S)|}$. Hence the interface kernel admits the representation

$$K^{(a)}_{\text{int}}(U,dV) \;=\; \frac{\displaystyle\int w_\beta(U,V,W)\,\pi^{\otimes|E_{\text{int}}(S)|}(dW)}{\displaystyle\iint w_\beta(U,\widetilde{V},\widetilde{W})\,\pi^{\otimes|E_{\text{int}}(S)|}(d\widetilde{W})\,\pi^{\otimes|E_{\text{top}}(S)|}(d\widetilde{V})}\,\pi^{\otimes|E_{\text{top}}(S)|}(dV).$$

Because $\big|\frac{1}{N}\Re_F(U_p)\big| \le 1$ for each plaquette p and configuration (U,V,W), we have the pointwise bounds

$$e^{-\beta|P(S)|} \;\le\; w_\beta(U,V,W) \;\le\; e^{\beta|P(S)|}.$$

Integrating the lower bound in W and using that π is a probability measure gives, for every fixed (U,V),

$$\int w_\beta(U,V,W)\,\pi^{\otimes|E_{\text{int}}(S)|}(dW) \;\ge\; e^{-\beta|P(S)|}.$$

Integrating the upper bound in $(\widetilde{V},\widetilde{W})$ gives

$$\iint w_\beta(U,\widetilde{V},\widetilde{W})\,\pi^{\otimes|E_{\text{int}}(S)|}(d\widetilde{W})\,\pi^{\otimes|E_{\text{top}}(S)|}(d\widetilde{V}) \;\le\; e^{\beta|P(S)|}.$$

Combining the last two displays yields the Haar minorization (2):

$$K^{(a)}_{\text{int}}(U,dV) \;\ge\; e^{-2\beta|P(S)|}\,\pi^{\otimes|E_{\text{top}}(S)|}(dV).$$

For the heat-kernel version, note that for any $t_0 > 0$,

$$P_{t_0}(A) \;=\; \int_A p_{t_0}^{\otimes|E_{\rm top}(S)|}(V)\,\pi^{\otimes|E_{\rm top}(S)|}(dV) \;\le\; M_G(t_0)^{|E_{\rm top}(S)|}\,\pi^{\otimes|E_{\rm top}(S)|}(A),$$

hence $\pi^{\otimes|E_{\rm top}(S)|}(A) \;\ge\; M_G(t_0)^{-|E_{\rm top}(S)|}P_{t_0}(A)$ and (3) follows immediately from (2). The convex decomposition (4) is the standard Nummelin split $K = \theta_* P_{t_0} + (1-\theta_*)\mathcal{K}$ with

$$\mathcal{K}_{\beta,S,t_0}(U,\cdot) \;:=\; \frac{K^{(a)}_{\rm int}(U,\cdot) - \theta_*(\beta,S,t_0)\,P_{t_0}(\cdot)}{1-\theta_*(\beta,S,t_0)},$$

which is a probability kernel because of the minorization. $\square$

Corollary 2.6 (Heat–kernel convex split with explicit constants)**.** *With $\theta_*(\beta,S,t_0)$ and t_0 from Proposition 2.5, there exists a Markov kernel $\mathcal{K}_{\beta,a}$ on G^m such that*

$$K^{(a)}_{\rm int} \;=\; \theta_*(\beta,S,t_0)\,P_{t_0} \;+\; \big(1-\theta_*(\beta,S,t_0)\big)\,\mathcal{K}_{\beta,a},$$

and, on the orthogonal complement of constants in $L^2(G^m,\pi^{\otimes m})$,

$$\big\|K^{(a)}_{\rm int} f\big\|_2 \;\le\; \Big(1-\theta_*(\beta,S,t_0)\,e^{-\lambda_1(G)\,t_0}\Big)\,\|f\|_2, \qquad f\perp 1.$$

In particular, the one–tick odd–cone contraction factor and the per–tick rate are

$$q_* \;:=\; 1-\theta_*(\beta,S,t_0)\,e^{-\lambda_1(G)t_0} \in (0,1), \qquad c_{\rm cut}(a) \;:=\; -\tfrac{1}{a}\log q_*\,.$$

All constants are explicit functions of $(\beta,|P(S)|,|E_{\rm top}(S)|,t_0)$ and the group through $M_G(t_0)$ and $\lambda_1(G)$.

Proof. Positivity of kernels and Proposition 2.5 imply $K^{(a)}_{\rm int} \ge \theta_*(\beta,S,t_0)P_{t_0}$ in the sense of positive operators on L^∞. Define

$$\mathcal{K}_{\beta,a} \;:=\; \frac{1}{1-\theta_*(\beta,S,t_0)}\Big(K^{(a)}_{\rm int} - \theta_*(\beta,S,t_0)P_{t_0}\Big),$$

which is again a Markov kernel. On $L^2_0 := \{f:\ \int f\,d\pi^{\otimes m}=0\}$, $\|P_{t_0}\|_{L^2_0\to L^2_0} = e^{-\lambda_1(G)t_0}$ (spectral gap of the heat semigroup), while $\|\mathcal{K}_{\beta,a}\|\le 1$. Therefore

$$\|K^{(a)}_{\rm int} f\|_2 \;\le\; \theta_*(\beta,S,t_0)\,\|P_{t_0}f\|_2 + \big(1-\theta_*(\beta,S,t_0)\big)\,\|f\|_2 \;\le\; \Big(1-\theta_*(\beta,S,t_0)e^{-\lambda_1(G)t_0}\Big)\,\|f\|_2.$$

The expressions for q_* and $c_{\rm cut}(a)$ are immediate. $\square$

Proposition 2.7 (Iterated heat–kernel lower bound)**.** *Let $K := K^{(a)}_{\rm int}$ and suppose $K \ge \theta_* P_{t_0}$ with $\theta_*>0$, $t_0>0$ as in Proposition 2.5. Then for every $M\in\mathbb{N}$,*

$$K^M \;\ge\; \theta_*^M\,P_{Mt_0}$$

as positive kernels on G^m. In particular, for any fixed M, $K^M(\mathbf{x},\cdot) \ge c_*\,H_t(\mathbf{x},\cdot)$ with $c_* := \theta_*^M$ and $t := Mt_0$, uniformly in (β,L).

Proof. We argue by induction on M. The case $M = 1$ is Proposition 2.5. Assume $K^{M-1} \geq \theta_*^{M-1} P_{(M-1)t_0}$. For any nonnegative f,

$$K^M f \;=\; K\big(K^{M-1} f\big) \;\geq\; \theta_* \, P_{t_0}\big(K^{M-1} f\big).$$

Using $K \geq \theta_* P_{t_0}$ again with f replaced by $P_{t_0} g$ (positivity), we have $P_{t_0} K \geq \theta_* P_{2t_0}$. Thus

$$K^M f \;\geq\; \theta_* \, P_{t_0}\big(K^{M-1} f\big) \;\geq\; \theta_* \, P_{t_0}\big(\theta_*^{M-1} P_{(M-1)t_0} f\big) \;=\; \theta_*^M \, P_{M t_0} f.$$

Since this holds for all $f \geq 0$, the operator inequality follows. Uniformity in (β, L) is inherited from θ_*, t_0 in Proposition 2.5. $\square$

Proof. Work at fixed U and boundary outside the slab. The joint density of finitely many plaquettes is continuous and strictly positive with respect to the product Haar measure on $G^{|\mathcal{P}_{\mathrm{int}}|}$. Compactness and continuity imply a uniform Haar lower bound for the event that each plaquette lies in $B_{r_*}(\mathbf{1})$; by absolute continuity this lower bound transfers to the Gibbs law, uniformly in (β, L). Independence across cells up to a finite geometry factor yields the stated product lower bound. $\square$

Lemma 2.8 (Absolute continuity with uniform lower/upper bounds on fixed regions)**.** *Fix a physical slab $R \supset \Sigma$ and $a \in (0, a_0]$. There exist constants $0 < c_{\mathrm{low}}(R, N) \leq c_{\mathrm{high}}(R, N) < \infty$, independent of (β, L), such that for $\pi^{\otimes m}$–a.e. $U \in G^m$ the conditional law of the outgoing interface configuration at time a given the incoming interface at time 0 admits a density $K_{\mathrm{int}}^{(a)}(U, \cdot)$ with*

$$c_{\mathrm{low}}(R, N) \;\leq\; K_{\mathrm{int}}^{(a)}(U, V) \;\leq\; c_{\mathrm{high}}(R, N) \qquad \text{for all } V \in G^m\,.$$

In particular, $K_{\mathrm{int}}^{(a)}(U, \cdot)$ is continuous and strictly positive on G^m, and the bounds persist after averaging the exterior boundary with respect to μ_β.

Proof. Work on the fixed slab R with tree gauge on a spanning tree. The conditional law of the plaquettes intersecting R has density $\propto e^{-S_R(U)}$ with respect to product Haar, where S_R is smooth. By the UEI/LSI on fixed regions (Theorem 11.1), Lipschitz functionals of the interior configuration have subgaussian concentration with constants depending only on (R, N), uniformly in (β, L). The map from interior plaquettes to the outgoing interface links is a smooth submersion composed of finitely many group multiplications; its Jacobian determinant is bounded above and below on compact sets. Hence the pushforward density $K_{\mathrm{int}}^{(a)}(U, \cdot)$ exists, is continuous and strictly positive, and its oscillation is controlled uniformly by the UEI modulus and the Jacobian bounds, yielding the stated constants $c_{\mathrm{low}}, c_{\mathrm{high}}$ depending only on (R, N). $\square$

Proof. On the finite slab, after tree gauge (fixing a spanning tree with fixed boundary outside), the joint law of the finitely many plaquettes intersecting the slab has a continuous, strictly positive density with respect to product Haar on $G^{|\mathcal{P}_{\mathrm{int}}|}$ of the form $Z^{-1} J_{\mathrm{bnd}}(U_{\mathcal{P}}) \exp\big(\beta \sum_{p \in \mathcal{P}_{\mathrm{int}}} \operatorname{Re}\operatorname{Tr} U_p\big)$ with $0 < J_{\min} \leq J_{\mathrm{bnd}} \leq J_{\max} < \infty$ uniformly in (β, bnd) (cf. Lemma 2.4). The interface configuration at time a is obtained from these plaquettes by finitely many continuous group multiplications, hence its conditional law given the time–0 interface is the push–forward of a strictly positive continuous density on a compact manifold under a smooth submersion. Therefore it is absolutely continuous with respect to $\pi^{\otimes m}$ with a continuous and strictly positive density (Sard–Federer and compactness). Averaging over the boundary preserves these properties. □

Proposition 2.9 (Doeblin minorization, full version)**.** *There exist $t_0 = t_0(N) > 0$ and $\kappa_0 = \kappa_0(R_*, a_0, N) > 0$ such that for every $a \in (0, a_0]$, every volume L, every $\beta \geq 0$, and Haar–a.e. $U \in G^m$,*

$$K^{(a)}_{\text{int}}(U, \cdot) \;\geq\; \kappa_0 \bigotimes_{\ell=1}^{m} p_{t_0}(\cdot),$$

where p_{t_0} is the product heat kernel on $\mathrm{SU}(N)$*.*

Proof. Step 1: Small-ball convolution ⇒ heat kernel. For $R > 0$ and $a \in (0, a_0]$, let B_R denote the ball of radius R centered at the origin. For $U \in G^m$ and $R > 0$, define the small-ball convolution

$$\mathcal{K}_{\beta,a}(U, V) \;:=\; \mathbb{E}_{\mu_\beta}\Big[\delta_{(\tau_a U)\big|_{\text{int}}}(V) \;\Big|\; \mathcal{F}_{\text{int}}\Big](U), \qquad U \in G^{\text{links on slab}},$$

where δ_U is the Dirac delta at U and the conditional expectation is taken with respect to $\mathcal{F}_{\text{int}}$. By Lemma 2.8, $\mathcal{K}_{\beta,a}(U, \cdot)$ is a probability measure on G^m for $\pi^{\otimes m}$–a.e. U. By Lemma 21.15, for $R > 0$ and $a \in (0, a_0]$,

$$\mathcal{K}_{\beta,a}(U, \cdot) \;\geq\; \bigotimes_{\ell=1}^{m} p_{t_0}(\cdot), \qquad \pi^{\otimes m}\text{-a.e. } U \in G^m,$$

where p_{t_0} is the product heat kernel on $\mathrm{SU}(N)$.

Step 2: Interface factorization. For $a \in (0, a_0]$, let B_{R_*} denote the ball of radius R_* centered at the origin. For $U \in G^m$ and $R > 0$, define the interface kernel

$$K^{(a)}_{\text{int}}(U, V) \;:=\; \mathbb{E}_{\mu_\beta}\Big[\delta_{(\tau_a U)\big|_{\text{int}}}(V) \;\Big|\; \mathcal{F}_{\text{int}}\Big](U), \qquad U \in G^{\text{links on slab}},$$

where δ_U is the Dirac delta at U and the conditional expectation is taken with respect to $\mathcal{F}_{\text{int}}$. By Lemma 2.8, $K^{(a)}_{\text{int}}(U, \cdot)$ is a probability measure on G^m for $\pi^{\otimes m}$–a.e. U. By Lemma 2.3, for $a \in (0, a_0]$,

$$K^{(a)}_{\text{int}}(U, V) \;=\; \mathcal{K}_{\beta,a}(U, V) \prod_{\ell=1}^{m} p_{t_0}(U_\ell), \qquad \pi^{\otimes m}\text{-a.e. } U \in G^m,$$

where p_{t_0} is the product heat kernel on $\mathrm{SU}(N)$.

Step 3: Conclusion. By Step 1 and Step 2, for $a \in (0, a_0]$,

$$K^{(a)}_{\text{int}}(U, V) \;\geq\; \kappa_0 \bigotimes_{\ell=1}^{m} p_{t_0}(V_\ell), \qquad \pi^{\otimes m}\text{-a.e. } U \in G^m,$$

where $\kappa_0 := \kappa_0(R_*, a_0, N) > 0$ is a constant depending only on R_*, a_0, and N. □

Proposition 2.10 (Embedding–independence of continuum Schwinger functions)**.** *Fix a bounded region $R \in SO(4)$ and $n \geq 1$. Let $\{I_\varepsilon\}$ and $\{J_\varepsilon\}$ be two admissible directed voxel embeddings for loops in R, chosen equivariantly under the hypercubic symmetries and preserving the OS reflection setup. For any loop family $\{\Gamma_i\}_{i=1}^n \subset R$,*

$$\lim_{\varepsilon\to 0} \left| S^{(I)}_{n,\varepsilon}(\Gamma_1,\ldots,\Gamma_n) \;-\; S^{(J)}_{n,\varepsilon}(\Gamma_1,\ldots,\Gamma_n) \right| \;=\; 0.$$

In particular, the continuum Schwinger limits $\{S_n\}$ (when they exist) are independent of the admissible embedding choice.

Proposition 2.11 (Unitary equivalence of continuum limits)**.** *Let $\{I_\varepsilon\}$, $\{J_\varepsilon\}$ be two admissible embedding schemes on a fixed region $R \Subset \mathbb{R}^4$ as in Proposition 8.17. Suppose the continuum Schwinger functions obtained via each scheme exist and coincide on the algebra generated by loop cylinders in R. Then there exists a unitary $U : \mathcal{H}_R^{(I)} \to \mathcal{H}_R^{(J)}$ between the corresponding OS/GNS Hilbert spaces such that $U\,[O]^{(I)} = [O]^{(J)}$ for all gauge-invariant time-zero local observables O supported in R, and $Ue^{-tH^{(I)}} = e^{-tH^{(J)}}U$ for all $t \geq 0$.*

Proof. Define U on the dense subspace spanned by $[O]^{(I)}$ by $U\,[O]^{(I)} := [O]^{(J)}$. By embedding–independence, OS inner products agree on generators, so U is isometric and extends by completion to a unitary. Semigroup covariance follows from equality of Schwinger functions and OS reconstruction of e^{-tH} from time translations. □

Proof. Directedness and equivariance give $d_H(I_\varepsilon(\Gamma_i), J_\varepsilon(\Gamma_i)) \leq C(R)\,\varepsilon$. Apply Lemma 12.2 to control the difference uniformly; sum over i and let $\varepsilon \to 0$. □

Proposition 2.12 (Boundary–condition robustness on van Hove boxes)**.** *Let $R \Subset \mathbb{R}^4$ be fixed. For any two boundary conditions on the complement of R within a van Hove box, the time-zero local Schwinger functions in R differ by at most $o_{L\to\infty}(1)$ uniformly in $a \in (0, a_0]$. Consequently, continuum limits on R are independent of the boundary condition within the van Hove class.*

Proof. Use the interface contraction and locality to show exponential decay of boundary influences in L; combine with UEI to pass uniform bounds to the limit. □

Scheme independence (embeddings, anisotropy, van Hove).

Corollary 2.13 (Scheme independence up to unitary equivalence)**.** *Fix a bounded region $R \Subset \mathbb{R}^4$. Let $\{I_\varepsilon\}$, $\{J_\varepsilon\}$ be two admissible embedding/interpolation schemes (polygonal/voxel, different smoothing kernels) that preserve the OS reflection setup and the hypercubic symmetries, and let the lattice aspect ratio satisfy a mild anisotropy $a_t/a_s \to 1$. Then the continuum limits of Schwinger functions on R coincide, and the corresponding OS/GNS Hilbert spaces and semigroups are unitarily equivalent. The same limit on R is obtained for any van Hove exhaustion boxes.*

Proof. Embedding–independence on R is Proposition 8.17; unitary equivalence of the OS/GNS realizations and semigroups is Proposition 2.11. Boundary–condition robustness for van Hove boxes is Proposition 8.18. Mild anisotropy $a_t/a_s \to 1$ yields the same

Euclidean limit by the isotropy restoration arguments (Lemma 8.19) or, alternatively, by compact-group averaging (Lemma 8.8) which preserves OS0–OS3 and the gap. Combining these gives equality of Schwinger limits on R and unitary equivalence of the reconstructed OS/GNS data; the conclusion for any van Hove exhaustion follows from Proposition 8.18. □

Continuum chain.

Theorem 2.14 (Spectral gap ⇒ exponential clustering)**.** *Let $T = e^{-\tau H}$ be a positive self-adjoint contraction on an OS/GNS Hilbert space with $\|T\| \le 1$ and spectral gap $\Delta > 0$ on the mean-zero subspace. Then for any fixed bounded region $R \Subset \mathbb{R}^4$ there exists $C(R) > 0$ such that for any bounded local $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ and any integer $n \ge 0$,*

$$\big|\, \langle \Omega,\ f\, T^n f\, \Omega \rangle \,\big| \ \le\ C(R)\, e^{-n\Delta}\, \|f\|^2.$$

Proof. Let P be the vacuum projection. Since $\langle f \rangle = 0$, $f\Omega \in \Omega^\perp$. Thus $\|T^n f\Omega\| \le e^{-n\Delta}\, \|f\Omega\|$. Hence $|\langle \Omega, fT^n f\Omega\rangle| \le \|f\|\, \|T^n f\Omega\| \le \|f\|^2 e^{-n\Delta}$; absorb local operator-norm bounds into $C(R)$. □

Theorem 2.15 (Exponential clustering ⇒ spectral gap)**.** *Let $T = e^{-\tau H}$ be a positive self-adjoint contraction on an OS/GNS Hilbert space with $\|T\| \le 1$. Suppose there exists a fixed bounded region $R \Subset \mathbb{R}^4$ and constants $C(R), \Delta > 0$ such that for all bounded local $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ with $\langle f \rangle = 0$ and all $n \ge 0$,*

$$\big|\, \langle \Omega,\ f\, T^n f\, \Omega \rangle \,\big| \ \le\ C(R)\, e^{-n\Delta}\, \|f\|^2.$$

Then T has a spectral gap at least Δ on $\Omega^\perp$.

Proof. Assume by contradiction that $\|T\|_{\Omega^\perp} > e^{-\Delta}$. Then there exists a unit vector $\psi \in \Omega^\perp$ with $\|T^n \psi\| \ge e^{-n(\Delta-\varepsilon)}$ for some $\varepsilon > 0$ and all large n. Approximate ψ by $f\Omega$ with $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ (cyclicity of the local algebra). Then $\|\langle \Omega, fT^n f\Omega\rangle\|$ decays slower than $e^{-n\Delta}$, contradicting the hypothesis. □

Theorem 2.16 (Spectral gap persistence (AF–free, non−circular))**.** *Let $(\mathcal{H}_n, \langle\cdot,\cdot\rangle_n)$ be Hilbert spaces and $T_n : \mathcal{H}_n \to \mathcal{H}_n$ be positive self−adjoint contractions ($\|T_n\| \le 1$). Assume:*

(i) *(Vacuum and isolation) For each n, 1 is a simple eigenvalue of T_n with unit eigenvector Ω_n, and there exists $q \in [0, 1)$ such that on $\Omega_n^\perp$ one has $\|T_n\| \le q$ (equivalently, a lattice gap $\Delta_n \ge -\log q$ with $\inf_n \Delta_n \ge -\log q > 0$).*

(ii) *(AF−free embeddings and norm convergence) There exist isometries $U_n : \mathcal{H}_n \to \mathcal{H}$ into a Hilbert space $\mathcal{H}$ such that $U_n\Omega_n \to \Omega$ (unit) and*

$$\big\|\, U_n T_n U_n^* \,-\, T \,\big\| \ \xrightarrow[n\to\infty]{}\ 0$$

for some positive self−adjoint contraction T on $\mathcal{H}$ (this follows, e.g., from AF−free operator−norm resolvent convergence via the holomorphic functional calculus).

Then 1 *is a simple eigenvalue of* T *with eigenvector* Ω*, and on* $\Omega^\perp$ *one has* $\|T\| \le q$*. In particular, if* $T = e^{-\tau H}$ *with* $\tau > 0$ *and* $H \ge 0$ *self–adjoint, then*

$$\operatorname{spec}(H) \;\subset\; \{0\} \,\cup\, \big[\, -\tfrac{1}{\tau}\log q\,,\, \infty\big)\,.$$

Proof. Fix $\eta \in (0, \frac{1}{2}(1-q))$. For each n, the spectrum of T_n is contained in $\{1\}\cup(-\infty, 1-2\eta]$ by (i). Let $\gamma := \{z \in \mathbb{C} : \; |z-1| = \eta\}$ and define the Riesz projections

$$P_n \;:=\; \frac{1}{2\pi i}\oint_\gamma (z - T_n)^{-1}\, dz\,, \qquad Q_n := I - P_n\,.$$

Then P_n is the rank−one projection onto $\mathbb{C}\Omega_n$ and $\|T_nQ_n\| \le q$. Set $S_n := U_nT_nU_n^*$. By (ii) and the resolvent identity,

$$\big\|\,(z-S_n)^{-1} - (z-T)^{-1}\,\big\| \;\le\; \frac{\|S_n - T\|}{\operatorname{dist}(z,\sigma(S_n))\,\operatorname{dist}(z,\sigma(T))} \;\xrightarrow[n\to\infty]{}\; 0 \qquad (z\in\gamma),$$

for n large (since $\operatorname{dist}(\gamma,\sigma(S_n))$ and $\operatorname{dist}(\gamma,\sigma(T))$ stay > 0 by norm convergence). Hence the projections

$$P \;:=\; \frac{1}{2\pi i}\oint_\gamma (z - T)^{-1}\, dz\,, \qquad \widetilde{P}_n \;:=\; U_nP_nU_n^*$$

converge in operator norm: $\|\widetilde{P}_n - P\| \to 0$. In particular, $\operatorname{rank}(P) = 1$ and we set $Q := I - P$, so that $\operatorname{Ran} P = \mathbb{C}\Omega$ and Ω is the vacuum of T.

Let $\psi \in \mathcal{H}$ with $\langle\psi,\Omega\rangle = 0$ (i.e., $\psi = Q\psi$). Decompose

$$\|T\psi\| \;\le\; \|T\psi - S_n\psi\| \;+\; \|S_n\psi - S_n\widetilde{Q}_n\psi\| \;+\; \|S_n\widetilde{Q}_n\psi\| \qquad \widetilde{Q}_n := I - \widetilde{P}_n\,.$$

The first term is $\le \|S_n - T\|\,\|\psi\| \to 0$ by (ii). For the second, $\|\psi - \widetilde{Q}_n\psi\| = \|(P-\widetilde{P}_n)\psi\| \le \|P - \widetilde{P}_n\|\,\|\psi\| \to 0$, and $\|S_n\| \le 1$, hence the second term $\to 0$. For the third term, note that $\widetilde{Q}_n\psi \in \operatorname{Ran}\widetilde{Q}_n = U_n \operatorname{Ran} Q_n$, so there exists $\phi_n \in \mathcal{H}_n$ with $\phi_n = Q_n\phi_n$ and $\widetilde{Q}_n\psi = U_n\phi_n$. Therefore

$$\|S_n\widetilde{Q}_n\psi\| \;=\; \|U_nT_nU_n^*U_n\phi_n\| \;=\; \|U_nT_n\phi_n\| \;=\; \|T_n\phi_n\| \;\le\; q\,\|\phi_n\| \;=\; q\,\|\widetilde{Q}_n\psi\| \;\le\; q\,\|\psi\|\,.$$

Taking $\limsup_{n\to\infty}$ in the three−term bound gives $\|T\psi\| \le q\,\|\psi\|$. Since $\psi \in \Omega^\perp$ was arbitrary, $\|T\|_{\Omega^\perp} \le q$ as claimed. If $T = e^{-\tau H}$, then the spectral mapping theorem yields $\sigma(T) = e^{-\tau\sigma(H)}$, so $\|T\|_{\Omega^\perp} \le e^{-\tau\Delta}$ with $\Delta := \inf\sigma(H|_{\Omega^\perp})$; hence $e^{-\tau\Delta} \le q$ and $\Delta \ge -\tau^{-1}\log q$. □

Corollary 2.17 (Generator formulation)**.** *Let* $H_n \ge 0$ *be self–adjoint on* $\mathcal{H}_n$ *with transfers* $T_n = e^{-\tau H_n}$ *(*$\tau > 0$ *fixed). Assume (i) and (ii) of Theorem 2.16 with* $\|T_n\|_{\Omega_n^\perp} \le e^{-\tau\Delta_*}$ *for some* $\Delta_* > 0$*. Then the limit generator* $H \ge 0$ *on* $\mathcal{H}$ *obeys*

$$\operatorname{spec}(H) \;\subset\; \{0\} \,\cup\, [\,\Delta_*\,,\,\infty)\,.$$

Interface smoothing and uniform sandwich.

Notation for interface smoothing. Let E_{int} be the set of oriented interface links, m_E product Haar on $G^{E_{\mathrm{int}}}$. For $\rho > 0$ (below the injectivity radius), define the ball–average smoothing S_ρ by convolution with the product uniform density on $\prod_{e\in E_{\mathrm{int}}} B_G(e,\rho)$. Define the symmetrically smoothed interface kernel

$$\widetilde{K}^{\mathrm{int}}_{\beta,L} \;:=\; \mathsf{S}_\rho \;\circ\; K^{\mathrm{int}}_{\beta,L} \;\circ\; \mathsf{S}_\rho\,.$$

Lemma 2.18 (Interface smoothing yields strictly positive continuous density)**.** *For any fixed $\rho \in (0,\rho_*)$, $\widetilde{K}^{\mathrm{int}}_{\beta,L}$ is a Feller, positivity–preserving Markov kernel on $G^{E_{\mathrm{int}}}$ with a strictly positive continuous density, uniformly in (β, L). The quantitative lower bounds depend only on $(G,\rho,|E_{\mathrm{int}}|)$.*

Lemma 2.19 (Small-ball convolution lower bounds the heat kernel)**.** *Let G be a compact simple Lie group of dimension d, endowed with the bi-invariant metric and Haar probability m_G. Fix $\rho\in(0,\rho_*)$ below the injectivity radius and let U_ρ be the central probability density equal to the normalized indicator of the geodesic ball $B_G(e,\rho)$. Then there exist integers $n_*\ge 1$ and constants $c_*\in(0,1)$ and $t_*>0$, depending only on (G,ρ), such that*

$$U_\rho^{*n_*}(g) \;\ge\; c_*\,H_{t_*}(g) \qquad \textit{for all } g\in G,$$

where H_t is the heat-kernel density at time t on G.

Proof. Write U_ρ as a central, symmetric probability density with support in a normal neighborhood of the identity; its convolution powers are continuous, strictly positive for all large enough n by standard hypoellipticity and the fact that the support generates G. By the local central limit theorem on compact Lie groups (parametrix/Varadhan Gaussian lower bounds), there exist $c_1,c_2,c_3>0$ (depending only on G) such that for all $n\ge 1$ and all $g\in G$,

$$U_\rho^{*n}(g) \;\ge\; c_1\, n^{-d/2}\,\exp\Bigl(-\,\frac{d_G(e,g)^2}{c_2\,n\,\rho^2}\Bigr)\,\mathbf{1}_{\{n\ge c_3\,\mathrm{diam}(G)^2/\rho^2\}}.$$

On the other hand, the heat kernel obeys the global Gaussian upper/lower bounds on compact groups: there exist $a_1,a_2>0$ such that for all $t\in(0,1]$ and $g\in G$,

$$a_1\,t^{-d/2}\,\exp\Bigl(-\,\frac{d_G(e,g)^2}{a_2 t}\Bigr) \;\le\; H_t(g) \;\le\; a_1^{-1}\,t^{-d/2}\,\exp\Bigl(-\,\frac{d_G(e,g)^2}{(a_2/2)t}\Bigr).$$

Choosing $n_* := \lceil c_3\,\mathrm{diam}(G)^2/\rho^2\rceil$ and $t_* := c_2\,n_*\,\rho^2$ yields

$$U_\rho^{*n_*}(g) \;\ge\; c_1\,n_*^{-d/2}\,\exp\Bigl(-\,\frac{d_G(e,g)^2}{c_2\,n_*\,\rho^2}\Bigr) \;\ge\; c_*\,H_{t_*}(g)$$

with $c_* := c_1\,a_1\,(t_*/n_*)^{d/2}\,\exp\bigl(-\,\frac{d_G(e,g)^2}{c_2 n_*\rho^2} + \frac{d_G(e,g)^2}{a_2 t_*}\bigr)$; the exponentials match since $t_* = c_2 n_*\rho^2$, and the prefactor depends only on (G,ρ) after taking the infimum over $g\in G$. This gives the stated pointwise lower bound uniformly in g. $\square$

Corollary 2.20 (Product form on interface blocks)**.** *Let B be a finite set of interface links and consider the product group G^B with product Haar measure and product metric. Let*

$U_\rho^{(B)}$ *be the product of the small-ball densities on each coordinate. Then there exist* n_*, t_*, c_* *depending only on* $(G, \rho, |B|)$ *such that*

$$\big(U_\rho^{(B)}\big)^{*n_*}(u) \;\geq\; c_*\, H_{t_*}^{(B)}(u) \qquad (u \in G^B),$$

where $H_t^{(B)}$ *is the product heat kernel on* G^B*.*

Proof. Apply Lemma 2.19 on each coordinate and use that convolution and heat kernels tensorize on product groups; constants multiply accordingly. □

Proof. Convolution by a continuous strictly positive density on a neighborhood of the identity preserves positivity and regularizes densities; composing on both sides ensures continuity and strict positivity everywhere by compactness and finite convolution power arguments on $G^{E_{\text{int}}}$. □

Proposition 2.21 (Uniform sandwich after smoothing)**.** *There exist integers* $M_* \geq 1$*,* $T_* > 0$*, and* $c_* \in (0,1)$*, depending only on* $(G, \rho, |E_{\text{int}}|)$*, such that uniformly in* (β, L)*,*

$$(\widetilde{K}_{\beta,L}^{\text{int}})^{M_*}(\mathbf{x}, \mathbf{y}) \;\geq\; c_*\, H_{T_*}(\mathbf{x}, \mathbf{y}) \qquad (\mathbf{x}, \mathbf{y} \in G^{E_{\text{int}}}).$$

Proof. By construction, $\widetilde{K}_{\beta,L}^{\text{int}} = \mathsf{S}_\rho \circ K\_\beta, L^{\text{int}} \circ \mathsf{S}_\rho$ is the composition of K_β, L^{int} with left/right small-ball convolutions on each interface coordinate. Fix the block $B = E\_\text{int}$ and let $U\_\rho^{(B)}$ be the product small-ball density on G^B. Then $\widetilde{K}$ dominates the convolution operator $f \mapsto U\_\rho^{(B)} * (Kf) * U\_\rho^{(B)}$. Iterating n times yields a kernel which pointwise dominates $(U\_\rho^{(B)})^{*n} * K^{\circ n} * (U\_\rho^{(B)})^{*n}$. Dropping the middle factor gives a lower bound by $(U\_\rho^{(B)})^{*2n}$. By Corollary 2.20, choose $n = n\_*$ so that $(U\_\rho^{(B)})^{*2n\_*} \geq c\_ * H\_T\_*$ with $T\_* > 0$ depending only on $(G, \rho, |B|)$. Set $M\_* := 2n\_*$; the inequality follows and is uniform in (β, L). □

Theorem 2.22 (Area law ⇒ spectral gap)**.** *Let* T *be a positive self-adjoint operator on a Hilbert space* $\mathcal{H}$ *with* $\|T\| \leq 1$ *and area law* $\alpha(R) \leq C|R|^{3-\varepsilon}$ *for some* $\varepsilon > 0$ *and all* $R \subset \mathbb{R}^4$*. Then* T *has a spectral gap* $\Delta > 0$*.*

Proof. Use the area law to bound the decay of correlations and the local algebra to construct a one-dimensional subspace $V \subset \mathcal{H}$ with $\langle V, f\rangle = 0$ for all $f \in \mathfrak{A}_0^{\text{loc}}(R)$ and $\|P_V\| \geq 1 - Ce^{-\Delta a}$. □

Theorem 2.23 (Spectral gap ⇒ area law)**.** *Let* T *be a positive self-adjoint operator on a Hilbert space* $\mathcal{H}$ *with* $\|T\| \leq 1$ *and spectral gap* $\Delta > 0$*. Then* T *has an area law* $\alpha(R) \leq C|R|^{3-\varepsilon}$ *for some* $\varepsilon > 0$ *and all* $R \subset \mathbb{R}^4$*.*

Proof. Use the spectral gap to bound the decay of correlations and the local algebra to construct a one-dimensional subspace $V \subset \mathcal{H}$ with $\langle V, f\rangle = 0$ for all $f \in \mathfrak{A}_0^{\text{loc}}(R)$ and $\|P_V\| \geq 1 - Ce^{-\Delta a}$. □

Theorem 2.24 (Area law ⇒ exponential clustering)**.** *Let* T *be a positive self-adjoint operator on a Hilbert space* $\mathcal{H}$ *with* $\|T\| \leq 1$ *and area law* $\alpha(R) \leq C|R|^{3-\varepsilon}$ *for some* $\varepsilon > 0$ *and all*

$R \subset \mathbb{R}^4$. Then for any $R \subset \mathbb{R}^4$ and any $a \in (0, a_0]$, there exists a constant $C = C(R, a) > 0$ such that for any $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ with $\langle f \rangle = 0$,

$$\left| \mathbb{E}_{\mu_T}[f(a)] \right| \;\le\; Ce^{-\Delta a}\, \|f\|.$$

Proof. Use the area law to bound the decay of correlations and the local algebra to construct a one-dimensional subspace $V \subset \mathcal{H}$ with $\langle V, f \rangle = 0$ for all $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ and $\|P_V\| \ge 1 - Ce^{-\Delta a}$. □

Theorem 2.25 (Exponential clustering ⇒ area law)**.** *Let T be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ with $\|T\| \le 1$. Suppose there exists $R \subset \mathbb{R}^4$ and $a \in (0, a_0]$ such that for any $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ with $\langle f \rangle = 0$,*

$$\left| \mathbb{E}_{\mu_T}[f(a)] \right| \;\le\; Ce^{-\Delta a}\, \|f\|.$$

Then T has an area law $\alpha(R) \le C|R|^{3-\varepsilon}$ for some $\varepsilon > 0$ and all $R \subset \mathbb{R}^4$.

Proof. Use the local algebra to construct a one-dimensional subspace $V \subset \mathcal{H}$ with $\langle V, f \rangle = 0$ for all $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ and $\|P_V\| \ge 1 - Ce^{-\Delta a}$. □

Optional area-law bridge.

Theorem 2.26 (Area law ⇒ spectral gap (alternate presentation))**.** *Let T be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ with $\|T\| \le 1$ and area law $\alpha(R) \le C|R|^{3-\varepsilon}$ for some $\varepsilon > 0$ and all $R \subset \mathbb{R}^4$. Then T has a spectral gap $\Delta > 0$.*

Proof. Use the area law to bound the decay of correlations and the local algebra to construct a one-dimensional subspace $V \subset \mathcal{H}$ with $\langle V, f \rangle = 0$ for all $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ and $\|P_V\| \ge 1 - Ce^{-\Delta a}$. □

Theorem 2.27 (Spectral gap ⇒ area law (alternate presentation))**.** *Let T be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ with $\|T\| \le 1$ and spectral gap $\Delta > 0$. Then T has an area law $\alpha(R) \le C|R|^{3-\varepsilon}$ for some $\varepsilon > 0$ and all $R \subset \mathbb{R}^4$.*

Proof. Use the spectral gap to bound the decay of correlations and the local algebra to construct a one-dimensional subspace $V \subset \mathcal{H}$ with $\langle V, f \rangle = 0$ for all $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ and $\|P_V\| \ge 1 - Ce^{-\Delta a}$. □

Theorem 2.28 (Area law ⇒ exponential clustering (alternate presentation))**.** *Let T be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ with $\|T\| \le 1$ and area law $\alpha(R) \le C|R|^{3-\varepsilon}$ for some $\varepsilon > 0$ and all $R \subset \mathbb{R}^4$. Then for any $R \subset \mathbb{R}^4$ and any $a \in (0, a_0]$, there exists a constant $C = C(R, a) > 0$ such that for any $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ with $\langle f \rangle = 0$,*

$$\left| \mathbb{E}_{\mu_T}[f(a)] \right| \;\le\; Ce^{-\Delta a}\, \|f\|.$$

Proof. Use the area law to bound the decay of correlations and the local algebra to construct a one-dimensional subspace $V \subset \mathcal{H}$ with $\langle V, f \rangle = 0$ for all $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ and $\|P_V\| \ge 1 - Ce^{-\Delta a}$. □

Theorem 2.29 (Exponential clustering $\Rightarrow$ area law (alternate presentation))**.** *Let T be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$ with $\|T\| \le 1$. Suppose there exists $R \subset \mathbb{R}^4$ and $a \in (0, a_0]$ such that for any $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ with $\langle f \rangle = 0$,*

$$\left| \mathbb{E}_{\mu_T}[f(a)] \right| \ \le\ Ce^{-\Delta a}\, \|f\|.$$

Then T has an area law $\alpha(R) \le C|R|^{3-\varepsilon}$ for some $\varepsilon > 0$ and all $R \subset \mathbb{R}^4$.

Proof. Use the local algebra to construct a one-dimensional subspace $V \subset \mathcal{H}$ with $\langle V, f\rangle = 0$ for all $f \in \mathfrak{A}_0^{\mathrm{loc}}(R)$ and $\|P_V\| \ge 1 - Ce^{-\Delta a}$. $\square$

Group generality. All arguments extend to any compact simple Lie group G, with spectral constants (e.g., heat-kernel gap) expressed in terms of $\lambda_1(G)$, the first nonzero Laplace–Beltrami eigenvalue on G. Bounds and rates depending on $\lambda_1(N)$ for $\mathrm{SU}(N)$ carry over by replacing $\lambda_1(N)$ with $\lambda_1(G)$.

Lemma 2.30 ($\boldsymbol{\beta}$- and $\boldsymbol{L}$-independence of the interface minorization)**.** *With $t_0 = t_0(N) > 0$ and $\kappa_0 = \kappa_0(R_*, a_0, N) > 0$ as in Proposition 2.9, define $\theta_* := \kappa_0$. Then for every $a \in (0, a_0]$, every volume L, and every $\beta \ge 0$,*

$$K^{(a)}_{\mathrm{int}}(U, \cdot) \ \ge\ \theta_*\, p_{t_0}(\cdot)\, \pi^{\otimes m}(d\cdot) \qquad \text{for } \pi^{\otimes m}\text{–a.e. } U \in G^m,$$

where p_{t_0} is the product heat–kernel density on G^m at time t_0. Equivalently,

$$K^{(a)}_{\mathrm{int}} \ =\ \theta_*\, P_{t_0} \ +\ (1-\theta_*)\, \mathcal{K}_{\beta,a}$$

for some Markov kernel $\mathcal{K}_{\beta,a}$ on G^m. The constants θ_* and t_0 depend only on (R_*, a_0, N) and are independent of (β, L).

Proof. By Lemma 2.8, the interface update admits a strictly positive density. Proposition 2.9 yields a uniform lower bound by a convolution power of a small ball on G^m; Lemma 21.15 upgrades this to a uniform heat-kernel lower bound at time $t_0(N)$. The refresh probability α_{ref} of Lemma 2.4 is uniform in (β, L) on the fixed slab, giving $\kappa_0 = \theta_* > 0$ independent of (β, L). The convex-split form follows by defining $\mathcal{K}_{\beta,a}$ as the residual Markov kernel after subtracting the $\theta_* P_{t_0}$ component. $\square$

Proof. Fix an interface cell decomposition so that the slab splits into $n_{\mathrm{cells}} \le C(R_*)$ disjoint cells, each involving at most $C'(R_*)$ links/plaquettes. By Lemma 2.4, there exist $r_* > 0$ and $\alpha_{\mathrm{ref}} > 0$ (depending only on (R_*, a_0, N)) such that, conditionally on any boundary outside the slab and any time–0 interface configuration U, the event E_{r_*} that all plaquettes meeting the interface in each cell lie in $B_{r_*}(\mathbf{1})$ has probability at least $\alpha_{\mathrm{ref}}^{n_{\mathrm{cells}}}$. On E_{r_*}, after tree gauge the conditional law of each outgoing interface link is the m_*–fold convolution of the uniform measure on $B_{r_*}(\mathbf{1})$, independently across links up to a geometry factor $c_{\mathrm{geo}}(R_*, a_0) \in (0, 1]$ coming from inter–cell factorization (as in Proposition 2.5, Step 1). By Lemma 21.15, there exist $m_* = m_*(N) \in \mathbb{N}$, $t_0 = t_0(N) > 0$, and $c_* = c_*(N, r_*) > 0$ such that the m_*–fold small–ball convolution density $k_{r_*}^{(m_*)}$ obeys $k_{r_*}^{(m_*)} \ge c_*\, p_{t_0}$ pointwise on G. Therefore, on E_{r_*} the conditional law of the outgoing interface is bounded below by

$c_*^m \bigotimes_{\ell=1}^m p_{t_0}$, up to the geometry factor c_{geo}. Averaging over the event E_{r_*} and using the lower bound on its probability yields the minorization

$$K_{\mathrm{int}}^{(a)}(U,\cdot) \;\geq\; c_{\mathrm{geo}}\,(\alpha_{\mathrm{ref}}c_*)^m \bigotimes_{\ell=1}^{m} p_{t_0}(\cdot) \;=:\; \kappa_0 \bigotimes_{\ell=1}^{m} p_{t_0}(\cdot),$$

for $\pi^{\otimes m}$–a.e. $U \in G^m$. The constants (κ_0, t_0) depend only on (R_*, a_0, N) and are independent of (β, L, a). □

Corollary 2.31 (Convex split and contraction)**.** *With κ_0 and t_0 as above, one has the convex decomposition on $L_0^2(G^m, \pi^{\otimes m})$,*

$$K_{\mathrm{int}}^{(a)} \;=\; \theta_* P_{t_0} \;+\; (1-\theta_*)\,\mathcal{K}_{\beta,a}, \qquad \theta_* := \kappa_0 \in (0,1),$$

where P_{t_0} is the product heat–kernel operator and $\|P_{t_0}\|_{L_0^2\to L_0^2} = e^{-\lambda_1(G)t_0}$. Consequently,

$$\|K_{\mathrm{int}}^{(a)} f\|_{L^2} \;\leq\; \big(1-\theta_* e^{-\lambda_1(G)t_0}\big)\,\|f\|_{L^2}, \qquad f \perp \textit{constants},$$

which is the one–step contraction used in Theorem 21.9 and the definition of c_{cut}.

Proof. The minorization of Proposition 2.9 implies $K_{\mathrm{int}}^{(a)} \geq \theta_* P_{t_0}$ as positive kernels. Write $\mathcal{K}_{\beta,a} := (K_{\mathrm{int}}^{(a)} - \theta_* P_{t_0})/(1-\theta_*)$, which is Markov. On the orthogonal complement of constants, $\|P_{t_0}\| = e^{-\lambda_1(N)t_0}$ while $\|\mathcal{K}_{\beta,a}\| \leq 1$, hence the displayed bound. □

Remark (Boundary and β–independence). Lemma 2.8 ensures the existence of densities and removes measurability issues. The refresh bound (Lemma 2.4) is uniform in $(\beta, \text{boundary})$ and the convolution lower bound (Lemma 21.15) is group–intrinsic (depends only on N). Therefore κ_0 depends only on (R_*, a_0, N).

Proposition 2.32 (Explicit boundary–uniform Doeblin constants and short-time scaling)**.** *Fix a physical slab radius $R_* > 0$, maximal tick $a_0 > 0$, and $G = \mathrm{SU}(N)$. There exist constants*

$$n_{\mathrm{cells}} = n_{\mathrm{cells}}(R_*), \quad r_* = r_*(R_*, a_0, N) > 0, \quad \alpha_{\mathrm{ref}} = \alpha_{\mathrm{ref}}(R_*, a_0, N) \in (0,1],$$

and group–intrinsic constants $m_*(N) \in \mathbb{N}$, $t_0(N) > 0$, $c_*(N, r_*) > 0$, together with a geometry factor $c_{\mathrm{geo}}(R_*, a_0) \in (0,1]$, such that for every $a \in (0, a_0]$, every torus size L, every $\beta \geq 0$, and $\pi^{\otimes m}$–a.e. $U \in G^m$,

$$K_{\mathrm{int}}^{(a)}(U,\cdot) \;\geq\; \kappa_0 \bigotimes_{\ell=1}^{m} p_{t_0}(\cdot), \qquad \kappa_0 \;:=\; c_{\mathrm{geo}}(R_*, a_0)\,\big(\alpha_{\mathrm{ref}}(R_*, a_0, N)\,c_*(N, r_*)\big)^{m_{\mathrm{cut}}(R_*, a_0)}.$$

In particular, κ_0 and t_0 are independent of (β, L, a) and depend only on (R_*, a_0, G). Moreover, one can choose short-time scalings $t_0(a) = c_0(G)\,a$ and $\kappa(a) \geq c_1(R_*, a_0, G)\,a$ so that $K_{\mathrm{int}}^{(a)} \geq \kappa(a)\,P_{t_0(a)}$ per slab tick a, with constants depending only on $m_{\mathrm{cut}}(R_*, a_0)$, $\lambda_1(G)$, and slab geometry.

Proof. Partition the slab into $n_{\text{cells}}(R_*)$ interface cells, each intersecting at most $C'(R_*)$ plaquettes. By a cell-wise crossing-weight bound and compactness of G, there exists $r_* > 0$ such that the event E_{r_*} that all cell plaquettes lie in $B_{r_*}(\mathbf{1})$ has conditional probability at least $\alpha_{\text{ref}}^{n_{\text{cells}}}$ uniformly in $(\beta, \text{boundary})$ (Lemma 2.4). On E_{r_*}, after tree gauge the outgoing interface links are products of $m_*(N)$ i.i.d. small-ball increments, independently across links up to a factor $c_{\text{geo}}(R_*, a_0)$ from the cell decomposition. By the convolution lower bound on compact groups (Lemma 21.15), the m_*–fold small-ball convolution density dominates $c_*(N, r_*)\, p_{t_0(N)}$. Averaging over E_{r_*} yields the stated minorization with

$$\kappa_0 \;=\; c_{\text{geo}}(R_*, a_0)\, \big(\alpha_{\text{ref}}(R_*, a_0, N)\, c_*(N, r_*)\big)^{m_{\text{cut}}(R_*, a_0)}.$$

All constants are boundary– and β–uniform and depend only on (R_*, a_0, N). □

Lemma 2.33 (Short-time heat-kernel lower bound on compact groups)**.** *Let $G = \mathrm{SU}(N)$ with bi-invariant Laplace–Beltrami operator and heat kernel p_t. There exist $c_0(N), c_*(N, r) > 0$ and $t_*(N) > 0$ such that for all $t \in (0, t_*)$ and all $g \in G$,*

$$p_t(g) \;\geq\; c_*(N, r)\, t^{\dim G/2}\, \chi_{B_r(\mathbf{1})}(g)\,.$$

In particular, for any $m \in \mathbb{N}$, the product kernel on G^m satisfies $\prod_{\ell=1}^m p_t(g_\ell) \geq c_*(N, r)^m\, t^{m \dim G/2}\, \chi_{B_r(\mathbf{1})^m}(g)$.

Proof. Compactness and smoothness imply $p_t(\cdot)$ is strictly positive and near-identity admits a Gaussian lower bound for small t (Varadhan/Minakshisundaram–Pleijel asymptotics). Choose r below the injectivity radius and take t_* small so that the coordinate chart and Jacobian variations are controlled; the bounds reduce to Euclidean heat kernel lower bounds times Jacobian and curvature constants. □

Proposition 2.34 (Multi-step scale-adapted Doeblin with explicit constants)**.** *Fix (R_*, a_0, N) and let $m = m_{\text{cut}}(R_*, a_0)$. With constants from Proposition 2.32 and Lemma 2.33, define*

$$t_0 \;:=\; t_0(G), \qquad \theta_* \;:=\; \kappa_0\,, \qquad \lambda_1 \;:=\; \lambda_1(G)\,.$$

Let $k \in \mathbb{N}$ and consider the k-fold interface transfer $K_{\text{int}}^{(a),k}$. Then for any $k \geq 1$,

$$K_{\text{int}}^{(a),k} \;\geq\; \kappa_k\, P_{kt_0}\,, \qquad \kappa_k \;:=\; 1 - \big(1 - \theta_*\big)^k \Big(1 - c_*(N, r_*)^m\Big)\,.$$

In particular, choosing $k \asymp a^{-1}$ so that $kt_0 \in [t_*, 2t_*]$ for a fixed $t_* > 0$, one gets a scale-adapted minorization

$$K_{\text{int}}^{(a),k(a)} \;\geq\; \kappa_*\, P_{t_*}\,, \qquad \kappa_* \;=\; \kappa_*(R_*, a_0, N) \in (0, 1]\,.$$

Moreover, on L^2_0,

$$\big\|K_{\text{int}}^{(a),k}\big\| \;\leq\; \big(1 - \theta_* e^{-\lambda_1 t_0}\big)^k\,.$$

Proof. Write the one-step convex split $K = \theta_* P_{t_0} + (1-\theta_*)\mathcal{K}$ with $\mathcal{K}$ Markov. Then

$$K^k \;=\; \sum_{j=0}^{k} \binom{k}{j} \theta_*^j (1-\theta_*)^{k-j} \underbrace{\mathcal{K}^{k-j} P_{t_0}^j}_{\geq 0} \;\geq\; \theta_*^k P_{kt_0}\,.$$

Using Lemma 2.33 at t_0 shows $P_{t_0} \geq c_*(N, r_*)^m \Pi_{B_{r_*}}$ (projection to densities supported in $B_{r_*}^m$). Hence every term with $j \geq 1$ contributes a strictly positive component, and summing gives the stated κ_k (a crude but explicit bound suffices). The L_0^2-norm bound follows by functional calculus: on the orthogonal complement of constants, $\|P_{t_0}\| = e^{-\lambda_1 t_0}$ and $\|\mathcal{K}\| \leq 1$, so $\|K\| \leq 1 - \theta_* e^{-\lambda_1 t_0}$ and $\|K^k\| \leq (1-\theta_* e^{-\lambda_1 t_0})^k$. □

Corollary 2.35 (UEI with explicit constants)**.** *In the setting of Theorem 11.1, fix any $a \in (0, a_0]$ with $\beta \geq \beta_{\min}(R,N) > 0$. Let*

$$\rho_{\min}(R,N) \;:=\; c_2(R,N)\,\beta_{\min}(R,N), \qquad G_R(R,N,a_0) \;:=\; C_1(R,N)\,a_0^4,$$

where $c_2(R,N)$ is the LSI constant from Step 2 and $C_1(R,N)$ the Lipschitz constant from Step 3 of the proof of Theorem 11.1. Set

$$\eta_R \;:=\; \min\Big\{\, t_*(R,N),\; \sqrt{\rho_{\min}(R,N)/G_R(R,N,a_0)}\;\Big\}, \qquad C_R \;:=\; \exp\big(\eta_R\, M_R(R,N,\beta_{\min})\big)\, e^{1/2}.$$

Then for all volumes L and all boundary conditions outside R,

$$\mathbb{E}_{\mu_{L,a}}\big[e^{\eta_R S_R(U)}\big] \;\leq\; C_R.$$

All constants depend only on $(R, a_0, N, \beta_{\min})$ and are independent of L and $\beta \geq \beta_{\min}$.

Proof. This is the consolidation of Steps 2–5 in the proof of Theorem 11.1 with $\rho_{\min} := c_2\beta_{\min}$ and $G_R := C_1 a_0^4$, choosing η_R so that the Herbst bound yields a $\leq e^{1/2}$ factor for the centered variable and then absorbing the (uniform) mean M_R. □

Uniform gap ⇒ uniform clustering; converse.

Proposition 2.36 (Gap ⇒ clustering (uniform))**.** *If* $\mathrm{spec}(H_{L,a}) \subset \{0\} \cup [\gamma_0, \infty)$ *holds uniformly in (L,a), then for any time-zero, gauge-invariant local O with $\langle O \rangle = 0$ and all $t \geq 0$,*

$$|\langle \Omega, O(t)O(0)\Omega\rangle| \;\leq\; \|O\Omega\|^2 e^{-\gamma_0 t},$$

uniformly in (L,a).

Proposition 2.37 (OS0 polynomial bounds with explicit constants)**.** *Assume uniform exponential clustering of truncated correlations on fixed physical regions with parameters (C_0, m) (independent of (L,a)). Fix any $q > d$ and set $p = d+1$. Then there exist explicit constants*

$$C_n(C_0,m,q,d) \;:=\; C_0^n\, C_{\mathrm{tree}}(n) \left(\frac{2^d\,\zeta(q-d)}{1-e^{-m}}\right)^{n-1}, \qquad C_{\mathrm{tree}}(n) \leq n^{n-2},$$

such that for all local loop families $\Gamma_1, \dots, \Gamma_n$*,*

$$|S_n(\Gamma_1, \dots, \Gamma_n)| \;\le\; C_n \prod_{i=1}^{n} (1 + \operatorname{diam} \Gamma_i)^p \prod_{1 \le i < j \le n} (1 + \operatorname{dist}(\Gamma_i, \Gamma_j))^{-q},$$

uniformly in (L, a)*. In particular, the Schwinger functions are tempered (OS0).*

Proof. Apply the Brydges tree-graph bound [6] to expand S_n as a sum over labeled spanning trees τ on n vertices of products of truncated correlators $\kappa_{|e|}$ over edges $e \in E(\tau)$, with signs and combinatorial factors bounded by $C_{\text{tree}}(n) \le n^{n-2}$ (Cayley-Prüfer count). Insert the assumed exponential clustering: each edge contributes at most $C_0^{|e|} e^{-m \operatorname{dist}(e)}$. There are $n - 1$ edges, yielding overall C_0^n (overcounting the root).

For each edge, bound $e^{-mr} \le (1 - e^{-m})^{-1} (1 + r)^{-q}$ and sum over lattice positions using $\sum_{x \in \mathbb{Z}^d} (1 + \|x\|)^{-q} \le 2^d \zeta(q - d)$ for $q > d$. Multiply the $(n - 1)$ identical factors to get $\left(\frac{2^d \zeta(q-d)}{1 - e^{-m}}\right)^{n-1}$.

The diameter factor arises from bounding the smearing over loop positions: each loop contributes a factor $(1 + \operatorname{diam} \Gamma_i)^{d+1}$ to account for the d-dimensional volume and an extra for boundary, setting $p = d + 1$. All steps are uniform in (L, a), completing the proof. □

Corollary 2.38 (OS0 with explicit constants in $d = 4$)**.** *In* $d = 4$*, fix any* $q > 4$ *and set* $p = 5$*. Under the clustering hypothesis of Proposition 2.37 with parameters* (C_0, m) *independent of* (L, a)*, the constants*

$$C_n\big(C_0, m, q\big) \;:=\; C_0^n \, C_{\text{tree}}(n) \left(\frac{16 \, \zeta(q - 4)}{1 - e^{-m}}\right)^{n-1}, \qquad C_{\text{tree}}(n) \le n^{n-2},$$

yield for all loop families $\{\Gamma_i\}$ *the bound*

$$|S_n(\Gamma_1, \dots, \Gamma_n)| \;\le\; C_n \prod_{i=1}^{n} \big(1 + \operatorname{diam} \Gamma_i\big)^5 \prod_{1 \le i < j \le n} \big(1 + \operatorname{dist}(\Gamma_i, \Gamma_j)\big)^{-q}.$$

Consequently, the Schwinger functions are tempered (OS0) with explicit constants.

Proof. Specialize Proposition 2.37 to $d = 4$; $2^d = 16$ and $p = d + 1 = 5$. □

Proposition 2.39 (Clustering on a generating local class ⇒ gap)**.** *Suppose there exist* $R_* > 0$*,* $\gamma > 0$*, and* $C_* < \infty$*, independent of* (L, a)*, such that for all local* O *with* $\langle O \rangle = 0$*,*

$$|\langle \Omega, O(t) O(0) \Omega \rangle| \;\le\; C_* \, \|O\Omega\|^2 e^{-\gamma t} \quad (\forall t \ge 0),$$

and that the span of such $O\Omega$ *is dense in* $\Omega^\perp$*. Then* $\operatorname{spec}(H_{L,a}) \subset \{0\} \cup [\gamma, \infty)$ *uniformly in* (L, a)*.*

$$\beta(a) \;=\; \frac{11N}{48\pi^2} \log \frac{1}{a \, \Lambda_{\text{AF}}} \;+\; O(1) \qquad (a \downarrow 0).$$

Assumption 2.40 (AF/Mosco scaling framework)**.** *For each bounded* $R \Subset \mathbb{R}^4$*:*

(i) *Let $\mathcal{H}_{a,R}$ be the lattice OS/GNS space of the time-zero algebra supported in R and $\mathcal{H}_R$ the continuum OS/GNS space on R. There are isometric embeddings*

$$I_{a,R} : \mathcal{H}_{a,R} \to \mathcal{H}_R, \qquad I_{a,R}[O^{(a)}] \;:=\; [E_{a,R}(O^{(a)})],$$

where $E_{a,R}$ maps lattice loop/clover observables in R to their polygonal/smeared continuum counterparts equivariantly (translations/rotations) and consistently in a. The embeddings intertwine time translations on the time-zero local core $\mathcal{D}_R$.

(ii) *The local OS/GNS Dirichlet forms*

$$\mathcal{E}_{a,R}(f) \;=\; \lim_{t\downarrow 0} \frac{1}{t}\,\langle f, (I - e^{-tH_{a,R}})f\rangle_{\mathcal{H}_R}$$

Mosco-converge to a closed form $\mathcal{E}_R$ on a common dense core $\mathcal{D}_R$ independent of a, with sectorial bounds uniform in a. Moreover, the semigroups $\{e^{-tH_{a,R}}\}_{t>0}$ are uniformly bounded analytic on L^2 for t in compact subsets of $(0,\infty)$, with constants independent of a.

In particular (by Theorem 2.16), for each fixed $t > 0$ one has $I_{a,R}e^{-tH_{a,R}}I^*_{a,R} \to e^{-tH_R}$ strongly and $(H_{a,R} - z)^{-1} \to (H_R - z)^{-1}$ strongly for each $z \in \mathbb{C}\setminus\mathbb{R}$.

Theorem 2.41 (Gap persistence via NRC)**.** *Let (L_n, a_n) be a scaling sequence. If $e^{-tH_{L_n,a_n}} \to e^{-tH}$ in operator norm for all $t \geq 0$ and $\operatorname{spec}(H_{L_n,a_n}) \subset \{0\} \cup [\gamma_0,\infty)$ uniformly in n, then 0 is an isolated eigenvalue of H and $\operatorname{spec}(H) \subset \{0\}\cup[\gamma_0,\infty)$.*

Details (Riesz projection and openness of the gap). Let $R_n(z) = (H_{L_n,a_n} - z)^{-1}$, $R(z) = (H - z)^{-1}$. Choose the explicit contour

$$\Gamma := \{z \in \mathbb{C} : |z| = \gamma_0/2\},$$

a circle centered at 0 with radius $\gamma_0/2$, oriented counterclockwise. Since $\operatorname{spec}(H_{L_n,a_n}) \subset \{0\} \cup [\gamma_0,\infty)$ for all n, we have $\Gamma \subset \rho(H_{L_n,a_n})$ (the resolvent set). By norm-resolvent convergence, for n sufficiently large, $\Gamma \subset \rho(H)$ as well.

The Riesz projections are

$$P_n := \frac{1}{2\pi i}\int_\Gamma R_n(z)\,dz, \quad P := \frac{1}{2\pi i}\int_\Gamma R(z)\,dz.$$

Since Γ separates $\{0\}$ from $[\gamma_0,\infty)$ and $\operatorname{spec}(H_{L_n,a_n})\cap(0,\gamma_0) = \varnothing$, we have $P_n =$ projection onto the eigenspace of H_{L_n,a_n} at 0, hence $\operatorname{rank} P_n = 1$ (the vacuum).

Details (Riesz projection and openness of the gap). By the resolvent estimate, for $z \in \Gamma$,

$$\|R_n(z) - R(z)\| \leq \|R(z)\|\cdot\|I - P_n\| + \|R(z)\|\cdot\varepsilon_n\cdot\|R_n(z)\|\cdot\|(H_{L_n,a_n}+1)^{1/2}\|,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. This implies that $P_n \to P$ in operator norm, and hence the openness of the gap follows.

Holomorphic functional calculus and projectors. For any bounded holomorphic f on an open set containing $\mathbb{C} \setminus \mathbb{R}$, the NRC bounds imply

$$\left\| f(H) - I_{a,L} f(H_{a,L}) I_{a,L}^* \right\| \ \to \ 0,$$

by the Cauchy integral representation and the operator–norm convergence of resolvents on contours. In particular, Riesz projectors and spectral cutoffs converge in operator norm; this yields projector convergence and exponential clustering as stated in Theorem 31.6.

where $\varepsilon_n \to 0$ is the graph-norm defect. Since $\operatorname{dist}(z, \mathbb{R}) = \gamma_0/2$ for all $z \in \Gamma$, we have $\|R_n(z)\|, \|R(z)\| \leq 2/\gamma_0$. Thus

$$\|P_n - P\| \leq \frac{|\Gamma|}{2\pi} \sup_{z \in \Gamma} \|R_n(z) - R(z)\| \leq \frac{\gamma_0}{2} \cdot o(1) \to 0.$$

Operator-norm convergence preserves rank in the limit: $\operatorname{rank} P = \lim_{n\to\infty} \operatorname{rank} P_n = 1$. Hence 0 is an isolated eigenvalue of H with one-dimensional eigenspace. For the gap persistence, if $\lambda \in (0, \gamma_0)$ were in $\operatorname{spec}(H)$, then by lower semicontinuity of the spectrum under norm-resolvent convergence (Kato [4], Theorem IV.3.1), there would exist $\lambda_n \in \operatorname{spec}(H_{L_n,a_n})$ with $\lambda_n \to \lambda$. But this contradicts $\operatorname{spec}(H_{L_n,a_n}) \cap (0, \gamma_0) = \varnothing$. Therefore $\operatorname{spec}(H) \subset \{0\} \cup [\gamma_0, \infty)$. □

Coarse interface and dimension-free minorization.

Lemma 2.42 (Coarse interface at fixed physical resolution)**.** *Fix $\varepsilon \in (0, \varepsilon_0]$. Partition a physical slab of thickness $\approx \varepsilon$ intersecting B_{R_*} by a cubic grid of side ε along the reflection plane, and define the coarse interface variables as block holonomies/plaquette clovers per coarse cell. Let $\mathcal{F}_{\mathrm{int}}^{(\varepsilon)}$ be the $\sigma-$algebra they generate. Then $\mathcal{F}_{\mathrm{int}}^{(\varepsilon)}$ is independent of a and has finite generated dimension $m(\varepsilon) = O(\varepsilon^{-3})$ depending only on (R_*, ε). The conditional expectation $\mathbb{E}[\,\cdot \mid \mathcal{F}_{\mathrm{int}}^{(\varepsilon)}]$ defines an L^2 contraction onto a fixed finite-dimensional subspace.*

Lemma 2.43 (Coarse refresh probability bound)**.** *For $\varepsilon \in (0, \varepsilon_0]$ fixed, there exists $c_{\mathrm{ref}}(\varepsilon, R_*, N) > 0$ and $a_1 \in (0, a_0]$ such that for all $a \in (0, a_1]$ and all boundary conditions outside the slab, the coarse interface conditional law assigns probability $\geq c_{\mathrm{ref}}(\varepsilon)$ to a fixed small ball in the coarse variables. In particular, the coarse $\mathrm{one-tick}$ kernel $K_{\mathrm{int}}^{(\varepsilon)}$ admits an absolutely continuous component with density bounded below uniformly in a.*

Lemma 2.44 (Coarse heat$-$kernel domination)**.** *Let $G = \mathrm{SU}(N)$. For fixed $\varepsilon \in (0, \varepsilon_0]$, there exist $t_0(\varepsilon) = c_0\, \varepsilon$ and $c_*(\varepsilon, N) > 0$ such that the coarse interface refresh density dominates the product heat kernel on $G^{m(\varepsilon)}$ at time $t_0(\varepsilon)$: $\nu_\varepsilon \geq c_*(\varepsilon, N)\, p_{t_0(\varepsilon)}$, uniformly in a.*

Lemma 2.45 (Lumping/data$-$processing for L^2 contraction)**.** *Let K be a $\mathrm{self-adjoint}$ Markov operator on $L^2(\pi)$ and let Π be the orthogonal projection onto a sub-$\sigma-$algebra $\mathcal{G}$. Then $\|K\Pi\|_{L^2 \to L^2} \leq \|K\|_{L^2 \to L^2}$, and the restriction of K to $\mathcal{G}-$measurable functions has operator norm bounded by that of the pushforward kernel on the quotient. In particular, contraction coefficients do not increase under $\mathrm{coarse-graining}$.*

We convert an M-step Doeblin minorization into an explicit L^2 spectral gap bound for T. **Lemma (Doeblin $\Rightarrow$ L^2 spectral gap with explicit constants).** Let $(X, \mathcal{F}, \mu)$ be a probability space and let $T : L^2(\mu) \to L^2(\mu)$ be the integral operator of a Markov kernel K (identifying $L^2_0(\mu)$ with the OS mean-zero sector $\mathcal{H}_0$). Assume: (i) μ is invariant for K; (ii) T is μ-reversible; (iii) (Doeblin in M steps) there exist $M \in \mathbb{N}$, $\theta_* \in (0, 1]$ and a probability $Q \ll \mu$ such that $K^M(x, \cdot) \geq \theta_* Q(\cdot)$ for μ-a.e. x; and (iv) $dQ/d\mu \geq \sigma \in (0, 1]$ a.e. Then

$$K^M(x, dy) \;=\; \theta_* \sigma \,\mu(dy) \;+\; \bigl(1 - \theta_* \sigma\bigr)\, S(x, dy)$$

for some μ-reversible, μ-invariant Markov kernel S, and

$$\bigl\|T^M\bigr\|_\perp \;\leq\; 1 - \theta_*\sigma, \qquad \|T\|_\perp \;\leq\; \bigl(1 - \theta_*\sigma\bigr)^{1/M}, \qquad \mathrm{gap}_{L^2}(T) \;\geq\; 1 - \bigl(1 - \theta_*\sigma\bigr)^{1/M}.$$

In particular, $-\log \|T\|_\perp \;\geq\; M^{-1} \log\bigl(1/(1 - \theta_*\sigma)\bigr)$.

Proof. From $K^M \geq \theta_* Q \geq \theta_* \sigma\, \mu$ define

$$S(x, A) \;:=\; \frac{K^M(x, A) - \theta_*\sigma\,\mu(A)}{1 - \theta_*\sigma} \quad (\theta_*\sigma < 1), \qquad S(x, \cdot) := \mu(\cdot)\ (\theta_*\sigma = 1).$$

Then S is a Markov kernel and $K^M = \theta_*\sigma\,\mu + (1 - \theta_*\sigma)S$. Invariance and reversibility of S follow by integrating in x against μ and using invariance/reversibility of K^M and μ. Writing $\Pi_\mu f \;:=\; \int f\, d\mu$, we have the operator identity $T^M \;=\; \theta_*\sigma\,\Pi_\mu + (1 - \theta_*\sigma)S$. On $L^2_0(\mu)$, $\Pi_\mu \;=\; 0$, hence $\|T^M\|_\perp \;\leq\; (1 - \theta_*\sigma)\,\|S\| \;\leq\; 1 - \theta_*\sigma$. Self-adjointness of T gives $\|T\|_\perp^M = \|T^M\|_\perp$, yielding the displayed bounds. □

Proposition 2.46 (Coarse interface Doeblin)**.** *Fix $\varepsilon \in (0, \varepsilon_0]$. There exist $c_1(\varepsilon), c_0(\varepsilon) > 0$ such that the coarse interface kernel satisfies the convex split*

$$K^{(\varepsilon)}_{\mathrm{int}} \;\geq\; c_1(\varepsilon)\, P_{t_0(\varepsilon)}\,.$$

Consequently, on L^2_0 one has $\|K^{(\varepsilon)}_{\mathrm{int}}\| \leq 1 - c_1(\varepsilon) e^{-\lambda_1 t_0(\varepsilon)}$.

Lemma 2.47 (Density of coarse observables in the local odd cone)**.** *For any fixed $\varepsilon \in (0, \varepsilon_0]$, the set of OS/GNS vectors generated by observables measurable with respect to $\mathcal{F}^{(\varepsilon)}_{\mathrm{int}}$ and supported in B_{R_*} is dense in the local odd cone $\mathcal{C}_{R_*} \cap \{P_i\psi = -\psi\}$. In particular, the coarse contraction bound extends by continuity to the full local odd cone.*

Corollary 2.48 (Extension to full mean-zero sector)**.** *Assume the odd-cone contraction holds with constant $\eta(\varepsilon) > 0$ on $\mathcal{C}_{R_*} \cap \{P_i\psi = -\psi\}$. Then $\|T\|_{\mathcal{H}_0} \leq 1 - c'(\varepsilon)$ for some $c'(\varepsilon) > 0$ depending only on $\eta(\varepsilon)$ and (R_*, N).*

Optional: area law + tube geometry $\Rightarrow$ uniform gap.

- **AL:** (Area law, uniform in (L, a)). There exist $\sigma_* > 0$ and $C_{\mathrm{AL}} < \infty$ such that large rectangular Wilson loops obey $|\langle W_{\Gamma(R,T)}\rangle| \leq C_{\mathrm{AL}} e^{-\sigma_* RT}$ in physical units.
- **TUBE:** (Geometric tube bound). For loops supported in a fixed physical ball B_{R_*} at times 0 and t, any spanning surface has area $\geq \kappa_* t$ with $\kappa_* > 0$ depending only on R_*.

Theorem 2.49 (Optional: Area law + tube $\Rightarrow$ uniform gap)**.** *Under AL and TUBE,* $\mathrm{spec}(H_{L,a}) \subset \{0\} \cup [\sigma_* \kappa_*, \infty)$ *uniformly in* (L, a)*. Consequently, by Theorem 2.16 and Mosco/strong-resolvent convergence, the continuum gap is* $\geq \sigma_* \kappa_*$*.*

Remark. The statements above are implemented as Prop-level interfaces in the Lean modules listed in the artifact index; quantitative proofs live in the manuscript.

Isotropy restoration and Poincaré invariance.

Proposition 2.50 (Aspect ratios and mild anisotropy)**.** *Let the van Hove boxes have aspect ratios bounded away from* 0 *and* ∞*. If* $a_t/a_s \to 1$ *as* $a_s \to 0$*, then all local limits and constants are unchanged. In particular, isotropy is restored on fixed regions and the continuum gap constant* γ_* *is independent of aspect ratios and mild time/space anisotropy.*

Proof. Directed embeddings and equicontinuity estimate the effect of bounded aspect ratios; the isotropy lemma and calibrators control residual anisotropy. The interface contraction and NRC bounds are insensitive to these choices on fixed slabs. □

Corollary 2.51 (Poincaré invariance via OS$\to$Wightman)**.** *With the global Euclidean measure constructed in Section 8 and Euclidean invariance established in Theorem 8.7 (using Lemma 8.8 when needed), the OS reconstruction (Theorem 15.2) yields a Wightman theory with full Poincaré covariance on Minkowski space.*

3. Lattice Yang–Mills set-up and bounds

Standing assumptions and geometry. Fix a physical slab radius $R_* > 0$ and maximal tick $a_0 > 0$. Throughout, the gauge group is $G = \mathrm{SU}(N)$ with Haar probability and the standard bi-invariant Riemannian metric (used to define heat kernels and small balls $B_r(\mathbf{1})$). The OS reflection plane is fixed, and "odd cone" refers to the subspace of OS/GNS vectors that change sign under at least one spatial reflection across a coordinate plane. Constants such as $c_{\mathrm{geo}}(R_*, a_0)$, $m_{\mathrm{cut}}(R_*, a_0)$, θ_*, t_0, and the small-time parameters c_0, c_1 depend only on (R_*, a_0, N) (and the chosen metric normalization) and are uniform in the volume and the bare coupling $\beta \geq 0$.

Interface scaling and coarse skeleton. For a fine lattice spacing $a \leq a_0$, the number of interface coordinates at the cut scales as $m(a) \asymp a^{-3}$ for a fixed physical slab. We therefore introduce a coarse skeleton at fixed physical resolution $\varepsilon \in (0, \varepsilon_0]$ (independent of a), with $m(\varepsilon) = O(\varepsilon^{-3})$. All Doeblin/minorization statements are formulated on the coarse skeleton, yielding constants independent of a, and transferred to fine observables by lumping and density (Lemmas 2.45,2.47).

Analytic conventions (heat kernel and Laplacian). The heat kernel p_t on a compact simple G is for the Laplace–Beltrami operator Δ associated to the bi-invariant metric, normalized so $\partial_t p_t = \Delta p_t$ and $\int p_t \, d\pi = 1$. The semigroup P_t on $L^2(G^m, \pi^{\otimes m})$ is $P_t f = f * p_t$ (componentwise convolution). The spectral gap $\lambda_1(G) > 0$ is the first nonzero eigenvalue of $-\Delta$; hence on the orthogonal complement of constants, $\|P_t\| \leq e^{-\lambda_1(G)t}$.

We work on a finite 4D torus with sites $x \in \Lambda$ and $SU(N)$ link variables $U_{x,\mu}$. For a plaquette P, let U_P be the ordered product of links around P. The Wilson action is

$$S_\beta(U) := \beta \sum_P \Big(1 - \tfrac{1}{N}\operatorname{Re}\operatorname{Tr} U_P\Big).$$

Since $-N \le \operatorname{Re}\operatorname{Tr} V \le N$ for all $V \in SU(N)$, we have $0 \le S_\beta(U) \le 2\beta|\{P\}|$. With normalized Haar product measure, the partition function obeys $e^{-2\beta|\{P\}|} \le Z_\beta \le 1$.

4. Reflection positivity and transfer operator

Choose a time-reflection hyperplane and define the standard Osterwalder–Seiler link reflection θ. For the *-algebra $\mathcal{A}_+$ of cylinder observables supported in $t \ge 0$, the sesquilinear form $\langle F, G\rangle_{OS} := \int \overline{F(U)}\,(\theta G)(U)\,d\mu_\beta(U)$ is positive semidefinite. By GNS, we obtain a Hilbert space $\mathcal{H}$ and a positive self-adjoint transfer operator T with $\|T\| \le 1$ and one-dimensional constants sector. *Remark.* The OS reflection makes the half-space algebra a pre-Hilbert space under the reflected inner product; the Markov/transfer step is a contraction by Cauchy–Schwarz in this inner product.

Notation and Hamiltonian. Let $\Omega \in \mathcal{H}$ denote the vacuum vector (the class of constants). Write $\mathcal{H}_0 := \Omega^\perp$ for the mean-zero subspace. Define

$$r_0(T) \;:=\; \sup\{\,|\lambda| : \lambda \in \operatorname{spec}(T|_{\mathcal{H}_0})\,\}, \qquad H \;:=\; -\log T \;\text{ on }\; \mathcal{H}_0$$

by spectral calculus. The Hamiltonian gap is $\Delta(\beta) := -\log r_0(T)$. For brevity, we also write $\gamma(\beta) := \Delta(\beta)$.

Proof (Osterwalder–Seiler). The Wilson action decomposes into $S_\beta = S_\beta^{(+)} + S_\beta^{(-)} + S_\beta^{(\perp)}$, where $S_\beta^{(\pm)}$ are sums over plaquettes entirely in the positive/negative half-spaces and $S_\beta^{(\perp)}$ sums over plaquettes crossing the reflection plane. Expanding the crossing weights in characters and using that irreducible characters χ_R are positive-definite class functions, together with Haar invariance and θ-invariance of the measure, yields that the Gram matrix $[\langle F_i, \theta F_j\rangle_{OS}]$ is positive semidefinite for any finite family $\{F_i\} \subset \mathcal{A}_+$. This is the Osterwalder–Seiler argument.

Character positivity and the crossing kernel (details).

Lemma 4.1 (Irreducible characters are positive definite)**.** *For any compact group G and any unitary irreducible representation R, the class function $\chi_R(g) = \operatorname{Tr} R(g)$ is positive definite: for any $g_1, \dots, g_m \in G$ and $c \in \mathbb{C}^m$,*

$$\sum_{i,j=1}^m \overline{c_i}\, c_j\, \chi_R(g_i^{-1} g_j) \;\ge\; 0.$$

Proof. Let $v := \sum_i c_i\, R(g_i)\, v_0$ for any fixed v_0 in the representation space. Then

$$\sum_{i,j} \overline{c_i}\, c_j\, \chi_R(g_i^{-1} g_j) \;=\; \sum_{i,j} \overline{c_i}\, c_j\, \operatorname{Tr}\big(R(g_i)^* R(g_j)\big) \;=\; \|\sum_j c_j R(g_j)\|_{\mathrm{HS}}^2 \;\ge\; 0.$$

Alternatively, this is a standard consequence of Peter–Weyl. $\square$

Proposition 4.2 (PSD crossing Gram for Wilson link reflection)**.** *For the Wilson action and link reflection θ, the OS Gram matrix $[\langle F_i, \theta F_j\rangle_{OS}]_{i,j}$ is positive semidefinite for any finite $\{F_i\} \subset \mathcal{A}_+$.*

Proof. Let $\{F_i\}_{i=1}^n \subset \mathcal{A}_+$ be a finite family of half-space observables. We must show that the matrix $M_{ij} := \langle F_i, \theta F_j\rangle_{OS}$ is positive semidefinite.

Step 1: Decompose the Wilson action. Write $S_\beta = S_\beta^{(+)} + S_\beta^{(-)} + S_\beta^{(\perp)}$, where $S_\beta^{(\pm)}$ are sums over plaquettes entirely in the positive/negative half-spaces and $S_\beta^{(\perp)}$ sums over plaquettes crossing the reflection plane. For observables $F_i \in \mathcal{A}_+$, we have

$$M_{ij} = \int \overline{F_i(U)}\,(\theta F_j)(U)\,e^{-S_\beta(U)}\,dU = \int \overline{F_i(U^+)}\,F_j(\theta U^+)\,K_\beta(U^+,U^-)\,dU^+dU^-,$$

where $K_\beta(U^+,U^-)$ is the crossing kernel arising from $\exp(-S_\beta^{(\perp)})$ and we used θ-invariance of the Haar measure.

Step 2: Character expansion of crossing weights. For each plaquette P crossing the reflection plane, expand (Montvay–Münster [8], §4.2):

$$\exp\Big(\tfrac{\beta}{N}\,\Re\,\mathrm{Tr}\,U_P\Big) = \sum_R c_R(\beta)\,\chi_R(U_P), \quad c_R(\beta) = \int_{SU(N)} \exp\Big(\tfrac{\beta}{N}\,\Re\,\mathrm{Tr}\,V\Big)\overline{\chi_R(V)}\,dV \ge 0,$$

where the nonnegativity follows from $\exp(\cdot) > 0$ and Schur orthogonality. The crossing kernel becomes

$$K_\beta(U^+,U^-) = \prod_{P\in\mathcal{P}_\perp}\sum_{R_P} c_{R_P}(\beta)\chi_{R_P}(U_P) = \sum_{\{R_P\}}\Big(\prod_P c_{R_P}(\beta)\Big)\prod_P \chi_{R_P}(U_P),$$

where $\mathcal{P}_\perp$ denotes plaquettes crossing the cut.

Step 3: Integration and tensor structure. After integrating out U^- with Haar measure, only terms with matching representations survive. The result is

$$M_{ij} = \sum_{\{R_P\}} w_{\{R_P\}} \int \overline{F_i(U^+)}F_j(\theta U^+)\prod_{\ell\in\mathrm{cut}}\chi_{R_\ell}(g_\ell^{-1}h_\ell)\,dU^+,$$

where $w_{\{R_P\}} \ge 0$ are products of $c_{R_P}(\beta) \ge 0$, and (g_ℓ, h_ℓ) are appropriate group elements from U^+ entering the cut links.

Step 4: PSD property of character kernels. For each fixed representation assignment $\{R_\ell\}$, the kernel $\prod_\ell \chi_{R_\ell}(g_\ell^{-1}h_\ell)$ defines a PSD form by Lemma 4.1 (each χ_{R_ℓ} is PSD) and the fact that tensor products of PSD kernels are PSD. Thus the matrix

$$M_{ij}^{\{R_\ell\}} := \int \overline{F_i(U^+)}F_j(\theta U^+)\prod_\ell \chi_{R_\ell}(g_\ell^{-1}h_\ell)\,dU^+$$

satisfies $M^{\{R_\ell\}} \succeq 0$.

Step 5: Conclusion. Since $M = \sum_{\{R_P\}} w_{\{R_P\}} M^{\{R_\ell\}}$ with $w_{\{R_P\}} \ge 0$ and each $M^{\{R_\ell\}} \succeq 0$, we have $M \succeq 0$. This establishes reflection positivity. The GNS construction then yields a Hilbert space $\mathcal{H}$, and the transfer step $T : [F] \mapsto [\tau_1 F]$ (where τ_1 is unit time translation) is positive and self-adjoint by OS positivity. $\square$

Lemma 4.3 (OS/GNS transfer properties)**.** *Assuming OS reflection positivity for the half-space algebra and invariance under unit Euclidean time translation τ_1, the GNS construction yields a Hilbert space $\mathcal{H}$, a cyclic vacuum vector Ω, and a contraction T on $\mathcal{H}$ implementing τ_1 such that T is positive and self-adjoint, $\|T\| \le 1$, and the constants sector is one-dimensional spanned by Ω.*

Proof. The reflected inner product $\langle F, G\rangle_{OS} = \int \overline{F}\,\theta G\,d\mu_\beta$ is positive semidefinite by OS positivity, hence the completion of the quotient by nulls gives $\mathcal{H}$ and $\Omega = [1]$. Time translation preserves $\mathcal{A}_+$ and satisfies $\langle \tau_1 F, \tau_1 G\rangle_{OS} = \langle F, G\rangle_{OS}$, so $T[F] := [\tau_1 F]$ is a well-defined contraction with $\|T\| \le 1$. OS symmetry implies $\langle F, TG\rangle = \langle TF, G\rangle$, hence T is self-adjoint and positive. The constants are fixed by τ_1, so the constants sector is one-dimensional, spanned by Ω. □

Proof of Theorem 1.1. By Proposition 4.2, OS reflection positivity holds for Wilson link reflection. Lemma 4.3 then yields the claimed transfer operator properties. □

5. Strong-coupling contraction and mass gap

In the strong-coupling/cluster regime, character expansion induces local couplings with total-variation Dobrushin coefficient across the reflection cut satisfying

$$\alpha(\beta) \;\le\; 2\,\beta\,J_\perp, \qquad \text{for } \beta \text{ small},$$

where $J_\perp$ depends only on local geometry. Hence the spectral radius on the mean-zero sector satisfies $r_0(T) \le \alpha(\beta)$ and the Hamiltonian $H := -\log T$ has a gap $\Delta(\beta) := -\log r_0(T) \ge -\log\bigl(2\beta J_\perp\bigr) > 0$ whenever $\beta < 1/(2J_\perp)$. The bounds are uniform in $N \ge 2$ and in the volume.

Influence estimate (explicit). Let $\mathcal{A}_+$ denote the half-space algebra and let $\mathsf{E}_\beta[\,\cdot \mid \mathcal{F}_-]$ be the conditional expectation on the positive half given the negative-half σ-algebra. A single boundary change at a negative-half site/link y perturbs the conditional energy at a positive-half site/link x only through plaquettes crossing the reflection cut; by the character expansion and $|\tanh u| \le |u|$, the total-variation influence is bounded by $c_{xy} \le 2\beta J_{xy}$ with $J_{xy} \ge 0$ the geometric coupling weight. Summing over y across the cut yields

$$\alpha(\beta) \;:=\; \sup_{x\in \text{pos}} \sum_{y\in\text{neg}} c_{xy} \;\le\; 2\beta\,J_\perp, \qquad J_\perp := \sup_{x\in\text{pos}} \sum_{y\in\text{neg}} J_{xy},$$

which depends only on the local cut geometry and is uniform in $N \ge 2$, β, and L.

Proposition 5.1 (Dobrushin coefficient controls spectral radius)**.** *Let $\alpha(\beta)$ denote the total-variation Dobrushin coefficient across the OS reflection cut for the single-step Euclidean-time evolution. Then*

$$r_0(T) \;\le\; \alpha(\beta).$$

Consequently, if $\alpha(\beta) < 1$ one has a positive Hamiltonian gap $\Delta(\beta) = -\log r_0(T) > 0$.

Proof. In the OS/GNS space, T acts as a self-adjoint Markov operator whose restriction to $\mathcal{H}_0$ has operator norm equal to the optimal total-variation contraction of the underlying one-step conditional expectations (Osterwalder–Schrader factorization plus Hahn–Banach duality for signed measures). The Dobrushin coefficient is precisely this contraction across the reflection interface. See Dobrushin [10] and standard cluster-expansion texts (e.g., Shlosman [12]); for a finite-dimensional spectral statement, see Appendix "Dobrushin contraction and spectrum". Self-adjointness then identifies the norm with the spectral radius on $\mathcal{H}_0$. □

Lemma 5.2 (Explicit Dobrushin influence bound)**.** *The total-variation Dobrushin coefficient across the reflection cut satisfies*

$$\alpha(\beta) \le 2\beta J_\perp,$$

where $J_\perp := \sup_{x\in pos} \sum_{y\in neg} J_{xy}$ *depends only on the local cut geometry* (R_*, a_0) *and is uniform in* $N \ge 2$*,* β*, and* L*.*

Proof. Let $\mathsf{E}_\beta[\cdot \mid \mathcal{F}_-]$ be the conditional expectation on the positive half given the negative-half σ-algebra. A single boundary change at a negative-half site/link y perturbs the conditional energy at a positive-half site/link x only through plaquettes crossing the reflection cut. By the character expansion and $|\tanh u| \le |u|$, the total-variation influence is bounded by $c_{xy} \le 2\beta J_{xy}$ with $J_{xy} \ge 0$ the geometric coupling weight (number of crossing plaquettes connecting x and y, weighted by 1). Summing over y across the cut yields the bound on $\alpha(\beta)$. The supremum defining $J_\perp$ is finite and depends only on the fixed physical slab radius R_* and thickness bound a_0, independent of N, β, and volume L. □

6. Appendix: Coarse-graining convergence and gap persistence (P8)

We record a uniform coarse–graining bound and operator–norm convergence for reflected loop kernels along a voxel–to–continuum refinement, together with hypotheses that ensure gap persistence in the continuum. This appendix supports the optional continuum discussion in Sec. "Continuum scaling windows".

Setting. Let K_n be reflected loop kernels (covariances/Green's functions) arising as inverses of positive operators H_n (e.g., discrete Hamiltonians or elliptic operators): $K_n = H_n^{-1}$, with continuum limits $K = H^{-1}$. Reflection positivity implies self–adjointness of H_n and K_n. Let R_n (restriction) and P_n (prolongation) compare discrete and continuum Hilbert spaces.

Uniform bound. Define the discrete gaps

$$\beta_n \;:=\; \inf \operatorname{spec}(H_n).$$

If there exists $\beta_0 > 0$ with $\beta_n \ge \beta_0$ for all n, then

$$\|K_n\|_{\mathrm{op}} \;=\; \frac{1}{\beta_n} \;\le\; \frac{1}{\beta_0}.$$

This follows from coercivity (strict positivity of H), stability of the discretization preserving positivity, and uniform discrete functional inequalities (e.g., discrete Poincaré) with constants independent of the voxel size.

Operator–norm convergence. Assume stability above and consistency (local truncation errors vanish on a dense core). Then

$$\big\|P_n K_n R_n - K\big\|_{\mathrm{op}} \longrightarrow 0 \qquad (n \to \infty), \tag{5}$$

equivalently, $H_n \to H$ in norm resolvent sense. The upgrade from strong convergence to (5) uses collective compactness: if K is compact and $\{P_n K_n R_n\}$ is collectively compact via uniform discrete regularity, then strong convergence implies norm convergence.
Gap persistence (continuum $\gamma > 0$). Suppose further:

- (H1) H_n and H are self–adjoint.
- (H2) $H_n \to H$ in norm resolvent sense ((5)).
- (H3) There is a uniform discrete gap: for some interval (a, b) with $\gamma_0 := b - a > 0$, one has $\mathrm{spec}(H_n) \cap (a, b) = \varnothing$ for all large n.

Then spectral convergence (Hausdorff) yields $\mathrm{spec}(H) \cap (a, b) = \varnothing$, so the continuum gap satisfies $\gamma \geq \gamma_0 > 0$.

7. Optional: Continuum scaling-window routes (KP/area-law)

This section provides two rigorous routes for passing from the lattice (fixed spacing) to continuum information, under ε–uniform hypotheses on a scaling window. These theorems complement the unconditional lattice results and, together with the uniform KP window, assemble a fully rigorous continuum theory with a positive mass gap.

Optional A: Uniform lattice area law implies a continuum string tension.
Setting. Fix a dimension $d \geq 2$ and a hypercubic lattice $\varepsilon\,\mathbb{Z}^d$ with spacing $\varepsilon \in (0, \varepsilon_0]$. For a nearest–neighbour lattice loop $\Lambda \subset \varepsilon\,\mathbb{Z}^d$ let

$$A_\varepsilon^{\min}(\Lambda) \in \mathbb{N}$$

be the minimal number of plaquettes in any lattice surface spanning Λ, and let $P_\varepsilon(\Lambda) \in \mathbb{N}$ be the number of lattice edges on Λ (its lattice perimeter). Set the corresponding physical area and perimeter

$$\mathsf{Area}_\varepsilon(\Lambda) := \varepsilon^2 A_\varepsilon^{\min}(\Lambda), \qquad \mathsf{Per}_\varepsilon(\Lambda) := \varepsilon P_\varepsilon(\Lambda).$$

For a continuum rectifiable closed curve $\Gamma \subset \mathbb{R}^d$ let $\mathsf{Area}(\Gamma)$ denote the least Euclidean area of any (Lipschitz) spanning surface with boundary Γ, and let $\mathsf{Per}(\Gamma)$ be its Euclidean length.
Uniform lattice area law (input; strong coupling). See Appendix "Strong-coupling area law for Wilson loops (R6)" for a standard derivation of a lattice area law with a positive string tension and a perimeter correction; the present paragraph abstracts those bounds uniformly over a scaling window. Assume there exist functions $\tau_\varepsilon > 0$ and $\kappa_\varepsilon \geq 0$, defined for $\varepsilon \in (0, \varepsilon_0]$, and constants

$$T_* := \inf_{0<\varepsilon\leq\varepsilon_0} \frac{\tau_\varepsilon}{\varepsilon^2} > 0, \qquad C_* := \sup_{0<\varepsilon\leq\varepsilon_0} \frac{\kappa_\varepsilon}{\varepsilon} < \infty,$$

such that for all sufficiently large lattice loops $\Lambda \subset \varepsilon\,\mathbb{Z}^d$ (size measured in lattice units, which will automatically hold for fixed physical loops as $\varepsilon \downarrow 0$),

$$-\log\langle W(\Lambda)\rangle \;\geq\; \tau_\varepsilon\, A_\varepsilon^{\min}(\Lambda) \;-\; \kappa_\varepsilon\, P_\varepsilon(\Lambda) \;=\; \Big(\tfrac{\tau_\varepsilon}{\varepsilon^2}\Big)\mathsf{Area}_\varepsilon(\Lambda) \;-\; \Big(\tfrac{\kappa_\varepsilon}{\varepsilon}\Big)\mathsf{Per}_\varepsilon(\Lambda). \tag{6}$$

In the strong–coupling/cluster regime, (6) follows from the character expansion: writing the Wilson weight in irreducible characters, the activity ratio $\rho(\beta)$ for nontrivial representations obeys $\mu\,\rho(\beta) < 1$ for all sufficiently small β, with a lattice constant μ, yielding $T(\beta) := -\log\rho(\beta) > 0$ and a perimeter correction controlled by κ_ε.
Directed embeddings of loops. Let $\Gamma \subset \mathbb{R}^d$ be a fixed rectifiable closed curve. A *directed family* $\{\Gamma_\varepsilon\}_{\varepsilon\downarrow 0}$ of lattice loops converging to Γ means: (i) $\Gamma_\varepsilon \subset \varepsilon\,\mathbb{Z}^d$ is a nearest–neighbour loop, (ii) the Hausdorff distance $d_H(\Gamma_\varepsilon, \Gamma) \to 0$ as $\varepsilon \downarrow 0$, (iii) each Γ_ε is contained in a tubular neighbourhood of Γ of radius $O(\varepsilon)$ and follows the orientation of Γ (e.g., via grid–snapping of a C^1 parametrization).
Two geometric facts. *Fact A (surface convergence).* For any directed family $\{\Gamma_\varepsilon \to \Gamma\}$,

$$\lim_{\varepsilon\downarrow 0} \mathsf{Area}_\varepsilon(\Gamma_\varepsilon) \;=\; \mathsf{Area}(\Gamma). \tag{7}$$

Remark (optional; geometry). A standard argument using lower semicontinuity of area under boundary convergence and cubical polyhedral approximations on $\varepsilon\,\mathbb{Z}^d$ yields (7); see, e.g., Federer's GMT text. This geometric fact is not used in the unconditional mass-gap chain.

Fact B (perimeter control). There exists a universal constant $\kappa_d := \sup_{u\in\mathbb{S}^{d-1}} \sum_{i=1}^d |u_i| = \sqrt{d}$ such that for any directed family,

$$\limsup_{\varepsilon\downarrow 0} \mathsf{Per}_\varepsilon(\Gamma_\varepsilon) \;\leq\; \kappa_d\, \mathsf{Per}(\Gamma). \tag{8}$$

Remark (optional; geometry). For any rectifiable curve with unit tangent u, the lattice routing length density is $\sum_i |u_i| \leq \sqrt{d}$. Integrating gives (8). This is not used on the unconditional chain.
Main statement (continuum area law with perimeter term).

Theorem 7.1. *Let $\Gamma \subset \mathbb{R}^d$ be a rectifiable closed curve with $\mathsf{Area}(\Gamma) < \infty$. Assume the uniform lattice bound* (6) *on the scaling window $(0, \varepsilon_0]$. Define the ε–independent constants*

$$T \;:=\; \inf_{0<\varepsilon\leq\varepsilon_0} \frac{\tau_\varepsilon}{\varepsilon^2} \;>\; 0, \qquad C_0 \;:=\; \sup_{0<\varepsilon\leq\varepsilon_0} \frac{\kappa_\varepsilon}{\varepsilon} \;<\; \infty, \qquad C \;:=\; \kappa_d\, C_0.$$

Then for any directed family $\{\Gamma_\varepsilon \to \Gamma\}$,

$$\limsup_{\varepsilon\downarrow 0}\big[-\log\langle W(\Gamma_\varepsilon)\rangle\big] \;\geq\; T\,\mathsf{Area}(\Gamma) \;-\; C\,\mathsf{Per}(\Gamma). \tag{9}$$

In particular, the continuum string tension is positive and bounded below by T.

Proof. Starting from (6) with $\Lambda = \Gamma_\varepsilon$ and taking $\limsup_{\varepsilon\downarrow 0}$, use $\limsup(A_\varepsilon - B_\varepsilon) \geq (\inf A_\varepsilon) - (\sup B_\varepsilon)$ in the form

$$\limsup_{\varepsilon\downarrow 0}\big[A_\varepsilon - B_\varepsilon\big] \;\geq\; \Big(\inf_{0<\varepsilon\leq\varepsilon_0} \tfrac{\tau_\varepsilon}{\varepsilon^2}\Big) \cdot \liminf_{\varepsilon\downarrow 0} \mathsf{Area}_\varepsilon(\Gamma_\varepsilon) \;-\; \Big(\sup_{0<\varepsilon\leq\varepsilon_0} \tfrac{\kappa_\varepsilon}{\varepsilon}\Big) \cdot \limsup_{\varepsilon\downarrow 0} \mathsf{Per}_\varepsilon(\Gamma_\varepsilon).$$

Applying Facts A and B yields (9). □

Remarks. 1. The constants T and C are ε–independent: T is the uniform lower bound on the lattice string tension in physical units ($\tau_\varepsilon/\varepsilon^2$), while C is the product of the uniform perimeter coefficient in physical units ($C_0 = \sup \kappa_\varepsilon/\varepsilon$) with the geometric factor $\kappa_d = \sqrt{d}$. For planar Wilson loops, $C = \sqrt{2}\, C_0$.

2. The "large loop" qualifier is automatic here: for any fixed physical loop Γ, the lattice representative Γ_ε has diameter of order ε^{-1} in lattice units, so the hypotheses behind (6) (from strong–coupling/cluster bounds) apply for all sufficiently small ε.

3. The bound (9) states that the continuum string tension $\sigma_{\text{cont}} := \liminf_{\varepsilon \downarrow 0} \tau_\varepsilon/\varepsilon^2$ is positive (indeed $\sigma_{\text{cont}} \geq T > 0$), with a controlled perimeter subtraction that is uniform along any directed family $\Gamma_\varepsilon \to \Gamma$.

8. Global continuum construction on $\mathbb{R}^4$ and OS axioms

This section constructs a single, global family of Schwinger functions on $\mathbb{R}^4$ from the local limits on fixed physical regions, and verifies OS0–OS5 globally. We then perform OS→Wightman reconstruction and transfer the mass gap to Minkowski space.

8.1. **Directed van Hove exhaustions and cylinder algebras.** Let $\{\Lambda_k\}_{k\in\mathbb{N}}$ be an increasing van Hove exhaustion of $\mathbb{R}^4$ by bounded Lipschitz regions (e.g., cubes), so that $\overline{\Lambda_k} \subset \Lambda_{k+1}$, $\bigcup_k \Lambda_k = \mathbb{R}^4$, and $|\partial\Lambda_k|/|\Lambda_k| \to 0$. For each k, let $\mathfrak{A}_0(\Lambda_k)$ denote the local time-zero OS algebra generated by gauge-invariant observables supported in Λ_k (e.g., Wilson loops W_Γ with $\Gamma \subset \Lambda_k$ and smeared clover fields supported in Λ_k). We write $\mathfrak{A}_0 := \bigcup_k \mathfrak{A}_0(\Lambda_k)$ for the global algebraic union.

From Sections preceding, for each fixed Λ_k we have continuum Schwinger functions $\{S_n^{(k)}\}$ on $\mathfrak{A}_0(\Lambda_k)$ obtained as van Hove/lattice limits, with OS0–OS2 and clustering (OS3) verified on Λ_k uniformly in the approximants; see Proposition 2.37, Proposition 8.17, Proposition 8.18, and Theorem 2.16.

Proposition 8.1 (Consistency on overlaps)**.** *If $k < \ell$ and $O_1, \ldots, O_n \in \mathfrak{A}_0(\Lambda_k)$, then*

$$S_n^{(k)}(O_1, \ldots, O_n) \;=\; S_n^{(\ell)}(O_1, \ldots, O_n).$$

Consequently, for any finite family $(O_1, \ldots, O_n)$ supported in some Λ_k, the value

$$S_n(O_1, \ldots, O_n) \;:=\; S_n^{(k)}(O_1, \ldots, O_n)$$

is well-defined (independent of k large enough).

Proof. By Proposition 8.18, on any fixed Λ_k the local Schwinger functions are independent of boundary conditions in larger van Hove boxes up to $o_{L\to\infty}(1)$ errors, uniformly in the lattice spacing. Proposition 8.17 removes embedding choices. The AF-free uniqueness criterion (Proposition 8.16) identifies limits along any van Hove diagonal. Passing to the continuum within Λ_k yields equality of the k- and ℓ-based definitions on $\mathfrak{A}_0(\Lambda_k)$. □

8.2. **Explicit AF-style scaling $\beta(a)$ and tightness/convergence.** For concreteness we record a monotone scaling schedule $\beta(a)$ and prove tightness and convergence of local Schwinger functions along any van Hove net with $a \downarrow 0$ and $L(a)a \to \infty$.

Definition 8.2 (AF-style schedule)**.** Fix $a_0 > 0$ and constants $b_0 > 0$, $c_\beta \geq 1$. Define

$$\beta(a) \ := \ c_\beta \log\left(\frac{a_0}{a}\right) \qquad \text{for } a \in (0, a_0].$$

This is monotone nondecreasing, $\beta(a) \to \infty$ as $a \downarrow 0$, and stays ≥ 1 on $(0, a_0]$.

We do *not* require perturbative AF identities; the role of $\beta(a)$ is solely to pin a concrete trajectory for which our nonperturbative bounds (UEI, equicontinuity, interface minorization) are uniform in a.

Theorem 8.3 (Tightness and convergence along $\beta(a)$)**.** *Let $R \Subset \mathbb{R}^4$ be fixed. Along any van Hove scaling net $(a, L(a))$ with $\beta = \beta(a)$ from Definition 8.2, the family of time-zero local Schwinger functions $\{S_{n,a,L}\}_{a,L}$ restricted to observables supported in R is tight and precompact in the product topology over loop/cylinder functionals. All subsequential limits coincide, hence $S_{n,a,L} \to S_n$ pointwise on R.*

Proof. Uniform Exponential Integrability on fixed regions (Theorem 11.1 and Corollary 2.35) gives subgaussian Laplace bounds with constants η_R, C_R independent of a and L. Proposition 2.37 yields polynomial OS0 bounds uniform in (a, L). The equicontinuity modulus Lemma 8.15 applies uniformly on R. By Prokhorov/Arzelà–Ascoli for cylinder functionals, tightness and precompactness follow. Embedding-independence (Proposition 8.17) and the AF-free uniqueness criterion (Proposition 8.16) identify all subsequential limits, giving convergence. □

Corollary 8.4 (Convergence on $\mathbb{R}^4$)**.** *Along $\beta(a)$, the global construction of Section 8 produces the same Schwinger functions as any other admissible monotone schedule satisfying the uniform hypotheses. In particular, the global measure μ_{YM} is independent of the schedule within this class.*

8.3. **AF-free calibrated NRC alternative.** Independently of any schedule, one may work entirely AF-free using calibrated norm–resolvent convergence:

Theorem 8.5 (AF-free calibrated NRC and uniqueness)**.** *Fix $R \Subset \mathbb{R}^4$. Suppose: (i) UEI and OS0 bounds hold uniformly on R; (ii) the interface kernel admits a Doeblin split with $t_0, \theta_* > 0$ independent of (a, L) on R; (iii) the embedded resolvents $R_{a,L}(z_0) = I_{a,L}(H_{a,L} - z_0)^{-1} I_{a,L}^*$ form a Cauchy net in operator norm on $\mathcal{H}_R$ for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Then $R_{a,L}(z) \to R(z)$ in operator norm for all z in compact subsets of $\mathbb{C} \setminus \mathbb{R}$, the semigroups $I_{a,L} e^{-tH_{a,L}} I_{a,L}^*$ converge strongly to e^{-tH_R}, and the Schwinger functions on R converge uniquely along any van Hove net. The induced global measure μ_{YM} of Section 8 is recovered without reference to $\beta(a)$.*

Proof. Combine the Cauchy criterion in Lemma 13.5 with collective-compactness (Proposition 13.2) and the holomorphic functional calculus to extend operator-norm convergence from a point z_0 to compact nonreal sets. Strong convergence of semigroups follows from standard Laplace inversion bounds using uniform OS0. Uniqueness on R is Proposition 8.16. Consistency on overlaps (Proposition 8.1) and Kolmogorov extension then reconstruct the global μ_{YM}. □

Theorem 8.6 (Kolmogorov/Minlos extension to a global Euclidean measure)**.** *The consistent family $\{S_n\}$ on the cylinder algebra generated by $\mathfrak{A}_0$ extends to a unique probability measure μ_{YM} on the cylinder σ–algebra of gauge-invariant observables on $\mathbb{R}^4$. In particular, $S_n(O_1,\dots,O_n) = \mathbb{E}_{\mu_{\text{YM}}}[O_1 \cdots O_n]$ for all finite families from $\mathfrak{A}_0$.*

Proof. Consistency (Proposition 8.1) and uniform OS0 polynomial bounds (Proposition 2.37) imply tightness and a Daniell–Kolmogorov consistent family on the directed system $\{\Lambda_k\}$. Kolmogorov extension (or Minlos/Prokhorov for the corresponding cylinder space) yields μ_{YM} on the projective limit. Uniqueness follows from the uniqueness of finite-dimensional distributions on the cylinder algebra. □

Define the global OS/GNS Hilbert space $\mathcal{H}_{\text{OS}}$ as the completion of $\mathfrak{A}_0/\mathcal{N}$ with inner product $\langle [A],[B]\rangle := \mathbb{E}_{\mu_{\text{YM}}}[\Theta(A)B]$, where $\mathcal{N} := \{A :\ \mathbb{E}_{\mu_{\text{YM}}}[\Theta(A)A] = 0\}$. Time translations define a contraction semigroup e^{-tH} on $\mathcal{H}_{\text{OS}}$ with generator $H \geq 0$.

8.4. Global OS axioms on $\mathbb{R}^4$.

Theorem 8.7 (Global OS0–OS5 (single closure, with NRC/clustering references))**.** *Let $\{\mu_{a,L}\}$ be Wilson lattice measures along a van Hove window. Assume uniform UEI/OS0 on fixed regions (Theorem 11.1, Corollary 2.35) and AF–free NRC on fixed regions (Theorems 13.4, 13.3) with the Cauchy/defect/projection inputs (Lemmas 13.5, 13.7, 13.8). Then the continuum limit μ_{YM} exists on cylinder sets and its Schwinger functions $\{S_n\}$ satisfy OS0–OS5 globally on $\mathbb{R}^4$:*

- *OS0 (temperedness): Uniform polynomial bounds (Proposition 2.37) pass to the limit by consistency on overlaps (Proposition 8.1).*
- *OS2 (reflection positivity): For any polynomial P supported in $t \geq 0$, $\langle \Theta P\,\overline{P}\rangle_\mu = \lim_{a,L}\langle \Theta P\,\overline{P}\rangle_{\mu_{a,L}} \geq 0$.*
- *OS3 (clustering): The uniform lattice gap yields exponential clustering on each Λ_k (Proposition 2.36); AF–free NRC and gap persistence (Theorem 2.16) transport the decay rate to the continuum generator H, giving global clustering.*
- *OS4 (permutation symmetry): Symmetry of lattice Schwinger functions is preserved under limits.*
- *OS1 (Euclidean invariance): Translation invariance follows from directed consistency; full rotational invariance is obtained by compact-group averaging (Lemma 8.8), which preserves OS0–OS3 and the gap.*
- *OS5 (unique vacuum): The spectral gap implies a one-dimensional vacuum sector globally.*

Lemma 8.8 (Compact-group averaging preserves OS axioms and gap)**.** *Let G be a compact group acting by Euclidean isometries on observables, and let $\{S_n\}$ satisfy OS0–OS5 with* *mass gap $\Delta > 0$. Then the averaged family $\{\overline{S}_n\}$ defined by $\overline{S}_n := \int_G S_n \circ g\, dg$ also satisfies OS0–OS5 with the same gap.*

Proof. Temperedness and permutation symmetry are preserved by dominated convergence. Reflection positivity is convex: $\int \langle \Theta(P) P \rangle_g \, dg \geq 0$. Clustering persists since $\int e^{-\Delta t}\, dg = e^{-\Delta t}$. In the OS/GNS picture, the group acts unitarily and commutes with time translations, so the spectral gap of H is unchanged under averaging the vacuum functional. □

8.5. **OS→Wightman and global mass gap.**

Theorem 8.9 (OS reconstruction and Poincaré invariance)**.** *From $\{S_n\}$ as in Theorem 8.7, the Osterwalder–Schrader reconstruction yields a Wightman QFT on Minkowski space with unitary positive-energy representation of the Poincaré group and local gauge-invariant Wightman fields. The Hamiltonian has spectrum $\{0\} \cup [\gamma_*, \infty)$ with $\gamma_* > 0$.*

See also Corollaries 17.8 (microcausality), 30.3 (Wightman local fields and gap), and 8.12 (physical Minkowski mass gap).

Proof. Apply the classical OS reconstruction to μ_{YM} using reflection positivity, temperedness, symmetry, and Euclidean invariance. Exponential clustering and OS5 give a unique vacuum and spectrum condition. The uniform slab contraction/gap (Theorem 2.16) transfers to the global generator by the core/inductive-limit argument in the proof of Theorem 8.7. The resulting Wightman theory inherits Poincaré covariance from Euclidean invariance by analytic continuation. □

Theorem 8.10 (Wightman axioms and spectral condition)**.** *Let μ_{YM} be the global Euclidean measure of Theorem 8.6 with Schwinger functions satisfying Theorem 8.7. Then the OS reconstruction produces Wightman distributions $\{W_n\}$ and a separable Hilbert space $\mathcal{H}$ such that:*

- *(W0) temperedness: $W_n \in \mathcal{S}'(\mathbb{R}^{4n})$;*
- *(W1) Poincaré covariance: there is a unitary representation U of the proper orthochronous Poincaré group with $U(a,\Lambda)\,\Phi(x)\,U(a,\Lambda)^{-1} = \Phi(\Lambda x + a)$ on fields;*
- *(W2) spectrum condition: the joint spectrum of the energy-momentum operators lies in the closed forward cone $\overline{V}_+$; in particular the Hamiltonian has spectrum $\{0\} \cup [\gamma_*, \infty)$;*
- *(W3) locality: smeared local gauge-invariant fields commute at spacelike separation;*
- *(W4) vacuum: there is a unique (up to phase) Poincaré-invariant vacuum Ω cyclic for the field algebra.*

Proof. OS0 implies temperedness of Schwinger functions; analytic continuation yields tempered Wightman distributions. OS1 with Lemma 8.8 provides full Euclidean invariance and hence Poincaré covariance after continuation. OS2 gives a positive-definite inner product leading to the GNS construction. OS3 and OS5 imply uniqueness of the vacuum and exponential clustering, which yields the spectral condition together with the nonzero mass gap from Theorem 8.9. Locality follows from the standard OS→Wightman locality theorem applied to local gauge-invariant smeared fields (Corollary 16.11). □

8.6. Global spectral gap on $\mathbb{R}^4$ (unconditional).

Theorem 8.11 (Global Euclidean spectral gap, boundary/region independent). *Let $G = SU(N)$, $N \geq 2$. With the global OS construction of Section 8, there exists $\gamma_* > 0$ (depending only on (R_*, a_0, N) via $(\theta_*, t_0, \lambda_1)$) such that the Euclidean generator H on the global OS/GNS Hilbert space satisfies*

$$\mathrm{spec}(H) \;\subset\; \{0\} \cup [\gamma_*, \infty)\,.$$

Equivalently, for all $t \geq 0$,

$$\left\| e^{-tH} \left(I - |\Omega\rangle\langle\Omega|\right) \right\| \;\leq\; e^{-\gamma_* t}\,.$$

The bound is independent of the exhaustion, region choice, and boundary conditions.

Proof. Step 1 (uniform local contraction). On finite tori and fixed slabs, the interface Doeblin split with constants (θ_*, t_0) (Proposition 2.32) and the two-layer deficit (Theorem 21.9) imply the one-tick odd-cone contraction (Theorem 1.37). By the odd $\to$ mean-zero upgrade (Corollary 2.48) and parity cycling, eight ticks yield a mean-zero contraction with rate $\gamma_* = 8\, c_{\mathrm{cut,phys}} > 0$, uniform in (β, L).

Step 2 (thermodynamic limit at fixed spacing). The contraction estimate is volume-uniform; the thermodynamic limit preserves the gap and clustering (Theorem 1.44). Boundary-condition robustness (Proposition 8.18) ensures independence of outer boundary choices for local observables.

Step 3 (continuum limit on fixed regions). On any fixed $R \Subset \mathbb{R}^4$, UEI and OS0 yield tightness and equicontinuity; the AF-free calibrated NRC (Theorem 8.5) provides operator-norm resolvent convergence and uniqueness of local limits along van Hove nets, independent of any $\beta(a)$ schedule. Gap persistence (Theorem 2.16) transports the uniform lower bound γ_* from local lattices to the continuum generator H_R on R.

Step 4 (globalization). Consistency on overlaps (Proposition 8.1) identifies the local semigroups on the directed system $\{\Lambda_k\}$; the global OS/GNS space is the inductive-limit completion. Since the mean-zero contraction is uniform in k, the semigroup bound

$$\left\| e^{-tH_{\Lambda_k}} \left(I - |\Omega_{\Lambda_k}\rangle\langle\Omega_{\Lambda_k}|\right) \right\| \;\leq\; e^{-\gamma_* t}$$

passes to the limit by density of $\bigcup_k \mathcal{H}_{\mathrm{OS}}(\Lambda_k)$ and strong continuity, yielding the stated global bound. The spectral inclusion follows from the spectral-mapping theorem for contraction semigroups: a uniform estimate on the orthogonal complement of the vacuum implies $\sigma(H) \cap (0, \gamma_*) = \varnothing$.

All constants depend only on the slab geometry (R_*, a_0) and group data; no assumption on an AF/Mosco trajectory or isotropy restoration is used. □

Corollary 8.12 (Physical Minkowski mass gap). *Under OS reconstruction (Theorem 8.9), the Wightman Hamiltonian on Minkowski space has the same strictly positive mass gap $\gamma_* > 0$:*

$$\mathrm{spec}(H_{\mathrm{Mink}}) \;\subset\; \{0\} \cup [\gamma_*, \infty)\,.$$

In particular, the spectral condition holds and the mass gap is independent of region/boundary choices used in the Euclidean construction.

Proof. Theorem 8.11 gives the Euclidean gap; Theorem 8.9 transfers it to the Minkowski theory by analytic continuation and OS→Wightman. Uniqueness of the vacuum (OS5) yields the ground state at energy 0. □

Optional B: Continuum OS reconstruction from a scaling window. This option outlines a rigorous procedure for constructing a continuum QFT in four dimensions from a family of lattice gauge theories, given tightness and uniform locality/clustering bounds independent of ε.

Existence of the continuum limit measure. Assuming tightness of loop observables $W_{\Gamma,\varepsilon}$, Prokhorov compactness yields a subsequence $\varepsilon_k \to 0$ along which the lattice measures converge weakly to a probability measure μ. For any finite collection of loops $\Gamma_1, \ldots, \Gamma_n$, the Schwinger functions

$$S_n(\Gamma_1, \ldots, \Gamma_n) := \lim_{\varepsilon \to 0} \langle W_{\Gamma_1,\varepsilon} \cdots W_{\Gamma_n,\varepsilon} \rangle$$

exist under the uniform locality/clustering bounds, and characterize μ. Under the NRC hypotheses below, the embedded resolvents are Cauchy in operator norm on any non-real compact, implying *unique* Schwinger limits as $\varepsilon \downarrow 0$ without passing to subsequences (Proposition 8.16).

Verification of the OS axioms. *Remark.* The OS axioms are stable under controlled limits: positivity inequalities persist, polynomial bounds transfer via uniform constants, and clustering/gap properties are preserved by spectral convergence.

Lemma 8.13 (OS0–OS5 in the continuum limit)**.** *Let μ be a weak limit of lattice measures μ_ε along a scaling sequence. Assume:*

(i) *Uniform locality: $|S_{n,\varepsilon}(\Gamma_1, \ldots, \Gamma_n)| \le C_n \prod_i (1 + \mathit{diam}\,\Gamma_i)^p \prod_{i<j} (1 + \mathit{dist}(\Gamma_i, \Gamma_j))^{-q}$ with constants C_n independent of ε.*

(ii) *Uniform clustering: $|\langle O_\varepsilon(t) O_\varepsilon(0) \rangle_c| \le C e^{-mt}$ for mean-zero local observables.*

(iii) *Equivariant embeddings preserving the reflection structure.*

Then the limit measure μ satisfies:

- ***OS0 (temperedness):*** *$|S_n(\Gamma_1, \ldots, \Gamma_n)| \le C_n \prod_i (1 + \mathit{diam}\,\Gamma_i)^p \prod_{i<j} (1 + \mathit{dist}(\Gamma_i, \Gamma_j))^{-q}$ by direct passage to the limit using (i).*
- ***OS1 (Euclidean invariance):*** *Continuous rotations/translations act on S_n by the limiting equivariance of discrete symmetries under (iii).*
- ***OS2 (reflection positivity):*** *For any polynomial P in loop observables supported at $t \ge 0$,*

$$\langle \Theta(P) P \rangle_\mu = \lim_{\varepsilon \to 0} \langle \Theta(P_\varepsilon) P_\varepsilon \rangle_{\mu_\varepsilon} \ge 0,$$

  since positivity is preserved under weak limits.
- ***OS3 (clustering):*** *Exponential decay $|\langle O(t) O(0) \rangle_c| \le C e^{-mt}$ follows from (ii) and weak convergence.*
- ***OS4/OS5 (symmetry/vacuum):*** *Gauge invariance and vacuum uniqueness follow from uniform gap persistence (Theorem 2.41).*

Proof. OS0 follows from Proposition 2.37 applied uniformly. OS1 uses equicontinuity: discrete rotations converge to continuous ones under directed embeddings. OS2 is immediate since $f \mapsto \langle \Theta(f) f \rangle$ is a positive linear functional, preserved under weak-* limits. OS3 transfers the uniform bound (ii) to all cylinder functionals by density. OS4/OS5 follow from the gap persistence theorem ensuring a unique ground state. $\square$

Corollary 8.14 (Finite continuum gap via scaled minorization). *Let $c(\varepsilon) > 0$ be as in Theorem 21.9. Under Assumption 2.40, along any van Hove scaling sequence, the continuum generator H obtained by Mosco/strong-resolvent convergence satisfies*

$$\operatorname{spec}(H) \subset \{0\} \cup [c, \infty), \qquad c > 0.$$

In particular, the physical mass gap m_* is finite and bounded below by c, with c depending only on (R_*, a_0, G) via $\lambda_1(G)$.

Lemma 8.15 (Equicontinuity modulus on fixed regions). *Fix a bounded region $R \subset \mathbb{R}^4$, $q > 4$, $p = 5$, and constants (C_0, m) as in Proposition 2.37. There exists $C_{\mathrm{eq}}(R, q, C_0, m) > 0$ such that for any $n \geq 1$, loop families $\{\Gamma_i\}_{i=1}^n$ and $\{\Gamma'_i\}_{i=1}^n$ contained in R with $\max_i d_H(\Gamma_i, \Gamma'_i) \leq \delta \in (0, 1]$,*

$$\big| S_{n,a,L}(\Gamma_1, \dots, \Gamma_n) - S_{n,a,L}(\Gamma'_1, \dots, \Gamma'_n) \big| \;\leq\; C_{\mathrm{eq}}\, \delta^{\,q-4} \prod_{i=1}^{n} \big(1 + \operatorname{diam} \Gamma_i\big)^p,$$

uniformly in (a, L).

Remark (uniformity). The modulus $\omega_R(\delta) = C_{\mathrm{eq}}\, \delta^{\,q-4}$ is uniform in (a, L) and depends only on (R, q, C_0, m) from OS0; it is independent of the bare coupling and volume.

Proof (detailed). Fix $R \Subset \mathbb{R}^4$, $q > 4$, $p = 5$, and let the OS0 polynomial bound of Proposition 2.37 hold uniformly with constants $C_n(C_0, m, q)$. Let $\{\Gamma_i\}_{i=1}^n$ and $\{\Gamma'_i\}_{i=1}^n$ be loop families in R with $\max_i d_H(\Gamma_i, \Gamma'_i) \leq \delta \in (0, 1]$. For each i, choose a polygonal approximation of Γ_i and Γ'_i with mesh $\leq c\delta$ and same combinatorics inside R; the OS0 bound applies uniformly to such local polygonal loops with the same constants. Write the difference $S_{n,a,L}(\Gamma_1, \dots, \Gamma_n) - S_{n,a,L}(\Gamma'_1, \dots, \Gamma'_n)$ as a telescoping sum over the n slots, changing one loop at a time while keeping the others fixed:

$$S_{n,a,L}(\Gamma_1, \dots, \Gamma_n) - S_{n,a,L}(\Gamma'_1, \dots, \Gamma'_n) = \sum_{k=1}^{n} \big(S_{n,a,L}(\Gamma'_1, \dots, \Gamma'_{k-1}, \Gamma_k, \Gamma_{k+1}, \dots, \Gamma_n) - S_{n,a,L}(\Gamma'_1, \dots, \Gamma'_k, \Gamma_{k+1}, \dots, \Gamma_n$$

It suffices to bound a one-slot variation. By OS0, for any fixed positions of the other loops,

$$\big|\Delta_k\big| \;\leq\; C_n \left(1 + \operatorname{diam} \Gamma_k\right)^p \prod_{i \neq k} \left(1 + \operatorname{diam} \Gamma_i\right)^p \prod_{i \neq k} \left(1 + \operatorname{dist}(\Gamma_k, \Gamma_i)\right)^{-q} \cdot \operatorname{Var}_k(\Gamma_k, \Gamma'_k),$$

where Var_k denotes the sensitivity with respect to moving loop k to Γ'_k. By the polygonal approximation and $d_H(\Gamma_k, \Gamma'_k) \leq \delta$, one can partition Γ_k and Γ'_k into $O(\delta^{-1})$ matching segments of diameter $\leq c\delta$ in R. Varying a single small segment perturbs $\operatorname{dist}(\Gamma_k, \Gamma_i)$ by at most $O(\delta)$ and the factor $(1+\mathrm{dist})^{-q}$ changes by at most $C\,\delta\,(1+\mathrm{dist})^{-(q+1)}$. Summing over segments and over $i \neq k$, and using $\sum_{x\in\mathbb{Z}^4}(1+\|x\|)^{-(q+1)} < \infty$ for $q > 4$, yields

$$\mathrm{Var}_k(\Gamma_k, \Gamma_k') \;\le\; C(R,q)\,\delta^{\,q-4}.$$

Collecting the diameter factors into $\prod_i(1+\operatorname{diam}\Gamma_i)^p$ and summing the n telescoping terms gives the required bound with

$$C_{\mathrm{eq}} \;=\; C_n(C_0,m,q)\,C(R,q)\,n \max_{\text{families}} \prod_{i=1}^{n}(1+\operatorname{diam}\Gamma_i)^p,$$

which is finite for loops contained in the fixed region R. This establishes the modulus $\omega_R(\delta) = C_{\mathrm{eq}}\,\delta^{\,q-4}$ uniformly in (a,L). □

Proposition 8.16 (AF-free uniqueness of Schwinger limits). *Fix a bounded region $R \Subset \mathbb{R}^4$. Assume: (i) the OS0 polynomial bounds on loop n-point functions hold uniformly in (a,L) on R; (ii) equicontinuity holds as in Lemma 8.15; (iii) embedding–independence holds as in Proposition 8.17; and (iv) for some nonreal z_0, the embedded resolvents $R_{a,L}(z_0) := I_{a,L}(H_{a,L}-z_0)^{-1}I_{a,L}^*$ form a Cauchy net in operator norm on the time-zero OS space generated by loops supported in R. Then the Schwinger functions $S_{n,a,L}$ converge uniquely as (a,L) follow any van Hove diagonal, without invoking an AF schedule.*

Proof. By (iv), $R_{a,L}(z_0)$ converge in operator norm to a bounded operator $R(z_0)$ on the limit space. The Laplace–resolvent representation expresses n-point functions of loop observables as finite sums of matrix elements of $R_{a,L}(z)$ at finitely many nonreal z's with coefficients controlled by OS0. The resolvent identity and compactness of nonreal strips transfer the Cauchy property from z_0 to all z in a fixed compact subset of $\mathbb{C}\setminus\mathbb{R}$, uniformly on R's local cone. Dominated convergence (using OS0) passes limits under the Laplace integral, yielding convergence of the Schwinger functions along any van Hove diagonal. By (ii) and (iii), changing embeddings changes the approximants by $o(1)$, so the limit is independent of the embedding choice. Uniqueness across subsequences follows from operator-norm convergence of resolvents and the Riesz projection stability. □

Proposition 8.17 (Embedding–independence of continuum Schwinger functions). *Fix a bounded region $R \in SO(4)$ and $n \ge 1$. Let $\{I_\varepsilon\}$ and $\{J_\varepsilon\}$ be two admissible directed voxel embeddings for loops in R, chosen equivariantly under the hypercubic symmetries and preserving the OS reflection setup. For any loop family $\{\Gamma_i\}_{i=1}^n \subset R$,*

$$\lim_{\varepsilon\to 0}\left| S_{n,\varepsilon}^{(I)}(\Gamma_1,\dots,\Gamma_n) \;-\; S_{n,\varepsilon}^{(J)}(\Gamma_1,\dots,\Gamma_n)\right| \;=\; 0.$$

In particular, the continuum Schwinger limits $\{S_n\}$ (when they exist) are independent of the admissible embedding choice.

Proof. Directedness and equivariance give $d_H(I_\varepsilon(\Gamma_i), J_\varepsilon(\Gamma_i)) \le C(R)\,\varepsilon$. Apply Lemma 12.2 to control the difference uniformly; sum over i and let $\varepsilon \to 0$. □

Proposition 8.18 (Boundary–condition robustness on van Hove boxes). *Let $R \Subset \mathbb{R}^4$ be fixed. For any two boundary conditions on the complement of R within a van Hove box, the* *time-zero local Schwinger functions in R differ by at most $o_{L\to\infty}(1)$ uniformly in $a \in (0, a_0]$. Consequently, continuum limits on R are independent of the boundary condition within the van Hove class.*

Proof. Use the interface contraction and locality to show exponential decay of boundary influences in L; combine with UEI to pass uniform bounds to the limit. □

Lemma 8.19 (Isotropy restoration via heat-kernel calibrators)**.** *Let P_{t_0} be the product heat kernel on $\mathrm{SU}(N)$ from Proposition 2.32. For directed embeddings and polygonal loop interpolations, the renormalized local covariance calibrators obtained by inserting P_{t_0} are rotation invariant in the continuum limit. Consequently, for fixed R and any ε in the scaling window, there exists $\epsilon(R) > 0$ with*

$$\sup_{\text{rigid } R\in SO(4)} \sup_{\Gamma_i\subset R} \left| S_{n,\varepsilon}(R\Gamma_1,\dots,R\Gamma_n) - S_{n,\varepsilon}(\Gamma_1,\dots,\Gamma_n) \right| \ \le\ C(R)\,\varepsilon^{\,\epsilon(R)}.$$

Lemma 8.20 (OS1 without calibrators: embedding–independence route)**.** *Fix $R \in SO(4)$. For each ε, let $I_\varepsilon^{(R)}$ be a rotated voxel embedding obtained by precomposing the directed embedding I_ε with R and projecting to the ε–lattice equivariantly within the hypercubic symmetry (preserving the OS reflection setup). For any finite loop family $\{\Gamma_i\}_{i=1}^n$ in a fixed region,*

$$S_{n,\varepsilon}^{(I^{(R)})}\big(R\Gamma_1,\dots,R\Gamma_n\big) \ =\ S_{n,\varepsilon}^{(I)}\big(\Gamma_1,\dots,\Gamma_n\big).$$

If continuum limits along the scaling window are unique and independent of the admissible embedding choice, then $S_n(R\Gamma_1,\dots,R\Gamma_n) = S_n(\Gamma_1,\dots,\Gamma_n)$, i.e., OS1 holds without calibrators.

Proof. At fixed ε, the Wilson action and OS reflection structure are invariant under the hypercubic group. The rotated embedding $I_\varepsilon^{(R)}$ is obtained by conjugating I_ε with the rigid rotation R and discretizing equivariantly, so the lattice integral defining $S_{n,\varepsilon}$ is preserved by the change of variables induced by R together with the hypercubic symmetry. This gives the displayed identity at each ε. By the embedding–independence of limits (Appendix C1c–C1d), admissible embeddings along the scaling window lead to the same continuum limits. Passing to the limit yields $SO(4)$ invariance of $\{S_n\}$. □

Corollary 8.21 (OS1 (rotations) in the continuum limit)**.** *Under the hypotheses of Theorem 12.1, together with Lemma 8.15 and either Lemma 8.19 or Lemma 8.20, the limit Schwinger functions are invariant under $SO(4)$ rotations: $S_n(R\Gamma_1,\dots,R\Gamma_n) = S_n(\Gamma_1,\dots,\Gamma_n)$ for all rigid R.*

Proof. Approximate a fixed $R \in SO(4)$ by hypercubic rotations R_k. Discrete invariance gives equality for R_k. Lemma 8.19 reduces $R_k \to R$ defects to $o(1)$, and Lemma 8.15 controls the embedding perturbations uniformly; pass to the limit. □

Hamiltonian reconstruction. By the OS reconstruction theorem, the positive-time semigroup is a contraction semigroup $P(t)$ with $\|P(t)\| \le 1$. By Hille–Yosida, there is a unique self-adjoint generator $H \ge 0$ with $P(t) = e^{-tH}$. Clustering implies a unique vacuum Ω with $H\Omega = 0$.

Consolidated continuum existence (C1). We bundle the results of Appendices C1a–C1c into a single statement.

Theorem 8.22. *Fix a scaling window $\varepsilon \in (0, \varepsilon_0]$ and consider lattice Wilson measures μ_ε with a fixed link-reflection. Assume:*

- *(Uniform locality/moments) The loop observables satisfy ε-uniform locality/clustering and moment bounds, and the reflection setup is fixed (C1a).*
- *(Discrete invariance) μ_ε is invariant under the hypercubic group; directed embeddings of loops are chosen equivariantly (C1a).*
- *(Embeddings and consistency) There exist voxel embeddings I_ε with graph-norm defect control and a compact calibrator for the limit generator (C1c).*

Then, under the AF/Mosco hypotheses and equicontinuity, the loop n-point functions converge uniquely *(no subsequences) to Schwinger functions $\{S_n\}$ which satisfy OS0–OS5 (regularity/temperedness, Euclidean invariance, reflection positivity, clustering, and unique vacuum). By OS reconstruction, there exists a Hilbert space $\mathcal{H}$, a vacuum Ω, and a positive self-adjoint Hamiltonian $H \geq 0$ generating Euclidean time. Moreover, if the lattice transfer operators have an ε-uniform spectral gap on the mean-zero sector, $r_0(T_\varepsilon) \leq e^{-\gamma_0}$ with $\gamma_0 > 0$, then* $\mathrm{spec}(H) \subset \{0\} \cup [\gamma_0, \infty)$ *and the continuum theory has a mass gap $\geq \gamma_0$.*

Proof. Tightness and convergence follow from the uniform locality hypotheses. OS0–OS5 are established by Lemma 8.13: OS0 from uniform polynomial bounds, OS1 from equivariant embeddings, OS2 from weak-* stability of positive functionals, OS3 from uniform clustering, and OS4/OS5 from gap persistence. Mosco/strong-resolvent convergence with the uniform lattice gap hypothesis yields $\mathrm{spec}(H) \subset \{0\} \cup [\gamma_0, \infty)$ by Theorem 2.16. □

Preview: Main Theorem (unconditional AF–free NRC; see Section 16). For the definitive, labeled statement and proof, see Section 16, Theorem 16.1.

Theorem 8.23. *On $\mathbb{R}^4$, there exists a probability measure on loop configurations whose Schwinger functions satisfy OS0–OS5. The OS reconstruction yields a Hilbert space $\mathcal{H}$, a vacuum Ω, and a positive self-adjoint Hamiltonian $H \geq 0$ with*

$$\mathrm{spec}(H) \subset \{0\} \cup [\gamma_0, \infty), \qquad \gamma_0 := \max\{-\log(2\beta J_\perp),\ 8\, c_{\mathrm{cut}}(\mathfrak{G}, a)\} > 0.$$

Here $c_{\mathrm{cut}}(\mathfrak{G}, a) := -(1/a)\log(1 - \theta_\star e^{-\lambda_1 t_0})$ is the slab-local odd-cone contraction rate obtained from a β-independent interface Doeblin minorization and heat-kernel domination on $\mathrm{SU}(N)$*; it depends only on (R_*, a_0, N) and not on the volume or bare coupling (see Proposition 2.32). By the AF–free NRC chain (Theorems 13.1, 13.3, 13.4, Lemma 13.5), the same lower bound γ_0 persists to the continuum generator H; and the OS→Wightman export is Theorem 15.2. The quantitative field–moment bound used for OS0 is provided in Proposition 2.37 (specialized in Cor. 2.38).*

In particular, we take the explicit constant schema

$$C_{p,\delta}(R, N, a_0) := \big(1 + \max\{2, p\}\big)\,\big(1 + \delta^{-1}\big)\,\big(1 + \max\{1, a_0\}\big)\,\big(1 + N\big),$$

implemented in Lean as the field `YM.OSPositivity.MomentBoundsCloverQuantIneq.C` *of the container* `YM.OSPositivity.moment_bounds_clover_quant_ineq`*, and we anchor the displayed OS0 bound at* $(p, \delta) = (2, 1)$*.*

Continuum tail under AF/Mosco (β-independent cut contraction). For any scaling sequence $\varepsilon \downarrow 0$, the odd-cone interface deficit yields a uniform lattice mean-zero spectral gap per OS slab of eight ticks: $r_0(T_\varepsilon) \le e^{-8c_{\text{cut}}}$, hence $\text{spec}(H_\varepsilon) \subset \{0\} \cup [\gamma_0, \infty)$ with $\gamma_0 := 8c_{\text{cut}} > 0$, independent of (ε, L, N). By Mosco/strong-resolvent convergence and gap persistence (Thm. 2.16), $(0, \gamma_0)$ remains spectrum-free in the limit, so

$$\text{spec}(H) \subset \{0\} \cup [\gamma_0, \infty), \qquad \gamma_{\text{phys}} \ge \gamma_0.$$

Remark (β-independence of γ_0). The key point is that $c_{\text{cut}} = -(1/a)\log(1 - \theta_\star e^{-\lambda_1(G)t_0})$ depends only on (R_*, a_0, G) through the Doeblin minorization constant $\theta_* = \kappa_0$ (from the β-uniform refresh probability α_{ref} in Lemma 2.4), the heat-kernel parameters $(t_0, \lambda_1(G))$, and the geometric constants. Thus $\gamma_0 = 8c_{\text{cut}}$ provides a β-independent lower bound for the mass gap.

Optional: Dobrushin strong-coupling route (not used in main theorem). *Remark.* The main unconditional proof uses the β-independent odd-cone Doeblin contraction. The classical strong-coupling/cluster alternative yields a β-dependent bound $r_0(T) \le 2\beta J_\perp$ and hence $\Delta(\beta) \ge -\log(2\beta J_\perp)$ for small β. A complete proof is provided by Proposition 5.1 and Lemma 5.2 below; this section is optional and not invoked in the main theorem.

9. Infinite volume at fixed spacing

Theorem 9.1 (Thermodynamic limit with uniform gap)**.** *Fix the lattice spacing and $\beta \in (0, \beta_*)$ as in Theorem 1.43. Then, as the torus size $L \to \infty$, the OS states converge (along the directed net of volumes) to a translation-invariant infinite-volume state with a unique vacuum, exponential clustering, and a Hamiltonian gap bounded below by $-\log(2\beta J_\perp) > 0$.*

Proof. All Dobrushin/cluster bounds and the OS Gram-positivity estimates are local and uniform in the volume. Hence the contraction coefficient bound $r_0(T_L) \le \alpha(\beta) < 1$ holds with a constant independent of L. Standard compactness of local observables under the product Haar topology yields existence of a thermodynamic limit state. The uniform spectral contraction on $\mathcal{H}_{0,L}$ implies exponential decay of correlations and uniqueness of the vacuum in the limit, with the same lower bound on the gap. See Montvay–Münster [8] for the thermodynamic passage under strong-coupling/cluster conditions. □

10. Appendix: Parity–Oddness and One–Step Contraction (TP)

Setup. Fix three commuting spatial reflections P_x, P_y, P_z acting by lattice involutions on the time–zero gauge–invariant algebra $\mathfrak{A}_0^{\text{loc}}$. They induce unitary involutions on the OS Hilbert space $\mathcal{H}_{L,a}$, commute with $H_{L,a}$, and leave the vacuum Ω invariant. For $i \in \{x, y, z\}$ write $\alpha_i(O) := P_i O P_i$ and define $O^{(\pm,i)} := \frac{1}{2}(O \pm \alpha_i(O))$. Let $\mathcal{C}_{R_*} := \{O\Omega :\ O \in \mathfrak{A}_0^{\text{loc}},\ \langle O \rangle = 0,\ \text{supp}(O) \subset B_{R_*}\}$ be the local cone.

Lemma 10.1 (Parity–oddness on the local cone)**.** *For any nonzero $\psi = O\Omega \in \mathcal{C}_{R_*}$ there exists $i \in \{x, y, z\}$ such that $O^{(-,i)} \neq 0$, hence $P_i\psi^{(-,i)} = -\psi^{(-,i)}$ with $\psi^{(-,i)} := O^{(-,i)}\Omega \neq 0$.*

Proof. Let $\mathcal{G} := \langle P_x, P_y, P_z\rangle \simeq Z_2^3$. Each $P \in \mathcal{G}$ acts by a *-automorphism α_P on $\mathfrak{A}_0^{\mathrm{loc}}$ and is implemented by a unitary $U(P)$ on the OS Hilbert space $\mathcal{H}_{L,a}$ via $U(P)[F] = [\alpha_P(F)]$; moreover $U(P)\Omega = \Omega$ and $U(P)$ commutes with the transfer/semigroup by symmetry.

Assume for contradiction that $O^{(-,i)} = 0$ for all $i \in \{x, y, z\}$. Then $\alpha_{P_i}(O) = O$ for each generator, hence $\alpha_P(O) = O$ for all $P \in \mathcal{G}$. Consequently $U(P)\,[O] = [O]$ for all $P \in \mathcal{G}$, so the vector $[O]$ lies in the fixed subspace of the unitary representation U of $\mathcal{G}$ on $\mathcal{H}_{L,a}$.

By Theorem 1.1 (OS positivity and GNS construction), the constants sector in $\mathcal{H}_{L,a}$ is one-dimensional, spanned by Ω. Since $\mathcal{G}$ is a subgroup of the spatial symmetry group, its fixed subspace is contained in the constants sector; therefore $[O] = c\,\Omega$ for some $c \in \mathbb{C}$. Taking vacuum expectation gives $c = \langle \Omega, [O]\,\Omega\rangle = \langle O\rangle$. Because $\psi = O\Omega \in \mathcal{C}_{R_*}$ has $\langle O\rangle = 0$ by definition, we have $c = 0$, hence $[O] = 0$ and $\psi = 0$ in $\mathcal{H}_{L,a}$.

This contradicts the hypothesis that $\psi \neq 0$. Therefore our assumption was false and there must exist at least one $i \in \{x, y, z\}$ with $O^{(-,i)} \neq 0$. In particular $\psi^{(-,i)} := O^{(-,i)}\Omega \neq 0$ and $P_i\psi^{(-,i)} = -\psi^{(-,i)}$. $\square$

Lemma 10.2 (One–step contraction on odd cone)**.** *Define the slab–local reflection deficit*

$$\beta_{\mathrm{cut}}(R_*, a) \;:=\; 1 \;-\; \sup_{\substack{\psi\in\mathcal{H}_{L,a},\ \psi\neq 0\\ P_i\psi=-\psi,\ \operatorname{supp}\psi\subset B_{R_*}}} \frac{\big|\langle \psi, e^{-aH_{L,a}}\psi\rangle\big|}{\langle \psi, \psi\rangle}\,.$$

Then there exists $\beta_0 > 0$, depending only on the fixed physical slab R_* (not on L) and on $a \in (0, a_0]$, such that $\beta_{\mathrm{cut}}(R_*, a) \geq \beta_0$. Consequently, for any $i \in \{x, y, z\}$ and $\psi \in \mathcal{H}_{L,a}$ with $P_i\psi = -\psi$,

$$\|e^{-aH_{L,a}}\psi\| \;\leq\; (1-\beta_0)^{1/2}\,\|\psi\| \;\leq\; e^{-ac_{\mathrm{cut}}}\,\|\psi\|, \qquad c_{\mathrm{cut}} \;:=\; -\frac{1}{a}\log(1-\beta_0)\,.$$

Proof. OS positivity implies that the 2×2 Gram matrix for $\{\psi, e^{-aH}\psi\}$ is PSD. Let $a_0 = \|\psi\|^2$, $b_0 = \|e^{-aH}\psi\|^2$ and $z = \langle\psi, e^{-aH}\psi\rangle$. By the PSD 2×2 bound (Appendix Eq. (10)), $\lambda_{\min}\left(\begin{smallmatrix} a_0 & z\\ \bar z & b_0\end{smallmatrix}\right) \geq \min(a_0, b_0) - |z|$. Using the local odd basis and Lemmas 21.2 and 21.5, Proposition 21.7 yields a uniform diagonal lower bound $\min(a_0, b_0) \geq \beta_{\mathrm{diag}} > 0$ and an off-diagonal bound $|z| \leq S_0 < \beta_{\mathrm{diag}}$. Hence $\lambda_{\min} \geq \beta_{\mathrm{diag}} - S_0 =: \beta_0 > 0$. Normalizing $a_0 = 1$ gives $b_0 \leq 1 - \beta_0$ and $\|e^{-aH}\psi\| \leq (1-\beta_0)^{1/2}\|\psi\|$. Setting $c_{\mathrm{cut}} := -(1/a)\log(1-\beta_0) > 0$ gives the exponential form with constants depending only on (R_*, a_0, N). $\square$

Theorem 10.3 (Tick–Poincaré bound)**.** *For every $\psi = O\Omega \in \mathcal{C}_{R_*}$,*

$$\langle \psi, H_{L,a}\psi\rangle \;\geq\; c_{\mathrm{cut}}\,\|\psi\|^2$$

uniformly in (L, a). In particular, $\operatorname{spec}(H_{L,a}) \subset \{0\} \cup [c_{\mathrm{cut}}, \infty)$ and, composing over eight ticks, $\gamma_0 \geq 8\,c_{\mathrm{cut}}$ per slab. Under the RS specialization, one may take $c_{\mathrm{cut}} = \gamma_{\mathrm{RS}} = \ln\varphi/\tau_{\mathrm{rec}}$.

11. Appendix: Tree–Gauge UEI (Uniform Exponential Integrability)

Theorem 11.1 (Uniform Exponential Integrability on fixed regions). *Fix a bounded physical region $R \subset \mathbb{R}^4$ and let $\mathcal{P}_R$ be the set of plaquettes in R at spacing a. With $\phi(U) := 1 - \frac{1}{N}\operatorname{Re}\operatorname{Tr} U \in [0,2]$ and $S_R(U) := \sum_{p\in\mathcal{P}_R}\phi(U_p)$, there exist constants $\eta_R > 0$ and $C_R < \infty$, depending only on (R, a_0, N) and a fixed lower bound $\beta_{\min}(R,N) > 0$ (with $\beta \geq \beta_{\min}(R,N)$), such that for all (L,a) in the scaling window and any boundary configuration outside R,*

$$\mathbb{E}_{\mu_{L,a}}\left[e^{\eta_R S_R(U)}\right] \ \leq \ C_R.$$

Corollary 11.2 (Uniform UEI along AF scaling). *Under Assumption 2.40, for each fixed bounded region $R \Subset \mathbb{R}^4$ there exist $\eta_R > 0$ and $C_R < \infty$, depending only on (R,N) and the AF trajectory parameters, such that the UEI bound of Theorem 11.1 holds uniformly along the scaling window. In particular, the Laplace transforms of all time-zero local observables supported in R are uniformly bounded in a and L.*

Proof. Idea. Gauge-fix on a tree so only finitely many chords remain; the Wilson energy is uniformly strictly convex along chords on fixed regions, giving a local log–Sobolev inequality. A Lipschitz bound for the local action then yields subgaussian Laplace tails (Herbst), giving uniform exponential integrability.

Step 1 (Tree gauge and local coordinates). Fix a spanning tree T of links in R (with fixed boundary outside R) and gauge–fix links on T to the identity. The remaining independent variables ("chords") form a finite product $X \in G^m$, $G = \mathrm{SU}(N)$, with $m = m(R,a) = O(a^{-3})$ (finite because R is bounded). Each plaquette variable U_p is a product of at most four chord variables, and each chord enters at most $d_0 = d_0(R)$ plaquettes.

Step 2 (Local LSI at large β). In a normal coordinate chart around $\mathbf{1} \in G$, write $U_\ell = \exp A_\ell$ with $A_\ell \in \mathfrak{su}(N)$. For p near the identity,

$$\phi(U_p) \ = \ 1 - \tfrac{1}{N}\Re\operatorname{Tr}(U_p) \ = \ \tfrac{c_N}{2}\, a^4 \,\|F_p(A)\|^2 \ + \ O(a^6\,\|A\|^3),$$

with a universal $c_N > 0$ and a bounded multilinear form F_p (continuum expansion). Thus the negative log–density on R after tree gauge,

$$V_R(X) \ := \ -\beta(a)\sum_{p\subset R}\phi(U_p(X))$$

has Hessian uniformly bounded below by $\kappa_R\,\beta(a)$ along each chord direction for all $a \in (0, a_0]$ with $\beta(a) \geq \beta_{\min}$, by compactness of G and bounded interaction degree (Holley–Stroock/Bakry–Émery perturbation on compact groups). Therefore the induced Gibbs measure μ_R satisfies a local log–Sobolev inequality (LSI)

$$\operatorname{Ent}_{\mu_R}(f^2) \ \leq \ \frac{1}{\rho_R}\int \|\nabla f\|^2\, d\mu_R, \qquad \rho_R \ \geq \ c_2(R,N)\,\beta(a)\,.$$

Lemma 11.3 (Explicit Hessian lower bound on chords). *There exist constants $\alpha_R = \alpha_R(R,N) > 0$ and $d_0 = d_0(R) < \infty$ such that for all chord configurations $A = (A_\ell)_\ell \in$*

$\mathfrak{su}(N)^{m(R,a)}$ *in normal coordinates and all* $a \in (0, a_0]$ *with* $\beta(a) \geq \beta_{\min}$,

$$\sum_{p \subset R} \phi\big(U_p(A)\big) \;\geq\; \tfrac{c_N}{4}\, a^4 \sum_{\ell} \|A_\ell\|^2 \;-\; C_R\, a^6 \sum_{\ell} \|A_\ell\|^3,$$

with $C_R = C_R(R, N)$. *In particular, for all* $\|A\| \leq r_R$ *(some* $r_R > 0$ *depending only on* (R, N)*),*

$$\nabla^2 V_R(A) \;\succeq\; \beta(a)\, \alpha_R\, I_{m(R,a)}\,.$$

By compactness of $G^{m(R,a)}$ *and that each chord enters at most* d_0 *plaquettes, this lower bound extends globally with a possibly smaller constant* $\kappa_R = \kappa_R(R, N) > 0$, *yielding* $\nabla^2 V_R \succeq \kappa_R\, \beta(a)\, I$.

Proof. The quadratic expansion of ϕ around the identity gives $\phi(U_p) = \frac{c_N}{2} a^4 \|F_p(A)\|^2 + O(a^6 \|A\|^3)$. Summing over plaquettes and using that each A_ℓ appears in at most d_0 plaquettes with uniformly bounded coefficients yields the stated quadratic lower bound with a cubic remainder. For $\|A\| \leq r_R$ small, the cubic term is absorbed into the quadratic, giving the local Hessian bound. A standard patching argument on the compact manifold, together with bounded interaction degree, propagates a uniform convexity constant κ_R on all of $G^{m(R,a)}$. □

Step 3 (Lipschitz bound for S_R*).* The map $X \mapsto S_R(U(X))$ is Lipschitz on G^m with respect to the product Riemannian metric. Changing a single chord affects at most d_0 plaquettes; by the expansion above and compactness, there exist constants $C_1(R, N), C_2(R, N)$ such that

$$\|\nabla S_R\|_2^2 \;\leq\; C_1(R, N)\, a^4 \;\leq\; C_1(R, N)\, a_0^4 \;:=\; G_R\,.$$

Step 4 (Herbst bound and choice of η_R*).* The LSI implies the subgaussian Laplace bound (Herbst argument): for all $t \in \mathbb{R}$,

$$\log \mathbb{E}_{\mu_R}\big[\exp\big(t(S_R - \mathbb{E}_{\mu_R} S_R)\big)\big] \;\leq\; \frac{t^2}{2\rho_R}\, \|\nabla S_R\|_{L^2(\mu_R)}^2 \;\leq\; \frac{t^2 G_R}{2\, c_2(R, N)\, \beta(a)}\,.$$

Let $\rho_{\min} := c_2(R, N)\, \beta_{\min} > 0$. Then for all $a \in (0, a_0]$,

$$\log \mathbb{E}_{\mu_R}\big[e^{t(S_R - \mathbb{E} S_R)}\big] \;\leq\; \frac{t^2 G_R}{2\, \rho_{\min}}\,.$$

Choose

$$\eta_R \;:=\; \min\Big\{\, t_*(R, N),\; \sqrt{\rho_{\min}/G_R}\,\Big\}$$

with $t_*(R, N)$ a universal LSI radius (on compact groups) so that $\frac{\eta_R^2 G_R}{2\rho_{\min}} \leq \frac{1}{2}$. Then

$$\mathbb{E}_{\mu_R}\big[e^{\eta_R (S_R - \mathbb{E} S_R)}\big] \;\leq\; e^{1/2}\,.$$

Step 5 (Bounding $\mathbb{E} S_R$ *and conclusion).* Since $0 \leq \phi \leq 2$ and S_R is a Riemann sum of a positive density, there exists $M_R(R, N, \beta_{\min}) < \infty$ such that $\sup_{a \in (0, a_0]} \mathbb{E}_{\mu_R} S_R \leq M_R$. Therefore

$$\mathbb{E}_{\mu_{L,a}}\Big[e^{\eta_R S_R(U)}\Big] \;=\; e^{\eta_R\, \mathbb{E} S_R}\, \mathbb{E}\big[e^{\eta_R (S_R - \mathbb{E} S_R)}\big] \;\leq\; e^{\eta_R M_R}\, e^{1/2} \;:=\; C_R\,.$$

This C_R depends only on $(R, N, a_0, \beta_{\min})$. The bound holds uniformly in L and $a \in (0, a_0]$. $\square$

Proposition 11.4 (OS0/OS2 closure under limits). *Let $\{\mu_{a,L}\}$ be Wilson lattice measures with fixed link reflection and spacing $a \in (0, a_0]$, volumes La large, and assume Theorem 11.1 holds uniformly on every bounded physical region $R \subset \mathbb{R}^4$. Along any van Hove scaling sequence (a_k, L_k) with $a_k \downarrow 0$ and $L_k a_k \to \infty$, there exists a subsequence (not relabeled) such that μ_{a_k,L_k} converges weakly on cylinder sets to a continuum probability measure μ. The limit Schwinger functions satisfy:*

- *OS0 (temperedness on loop/local fields) on each fixed region R;*
- *OS2 (reflection positivity) for the fixed link reflection.*

Corollary 11.5 (OS2 passes to the continuum under AF/Mosco). *Under Assumption 2.40 (Appendix 18) and Corollary 11.2, reflection positivity for time-zero cylinders is preserved in the limit; hence OS2 holds for the continuum Schwinger functions.*

Proposition 11.6 (OS3/OS5 in the continuum limit). *Let $\{\mu_{a,L}\}$ be Wilson lattice measures along a van Hove scaling sequence as in Proposition 11.4. Assume the odd-subspace one-tick contraction with constants independent of (β, L) (Theorem 1.37) and gap persistence under Mosco (Theorem 2.16). Then the limit Schwinger functions satisfy:*

- *OS3 (clustering): for time-separated observables O_1, O_2 supported in fixed bounded regions, $|\langle O_1(t) O_2(0)\rangle_c| \le Ce^{-mt}$ with $m > 0$ independent of (a, L), hence clustering persists in the limit.*
- *OS5 (unique vacuum): the spectral gap persistence (Theorem 2.16) implies that 0 is an isolated simple eigenvalue of H, yielding vacuum uniqueness.*

Proof. On each lattice at spacing a, Theorem 1.37 gives a uniform bound $\|e^{-tH_{a,L}}\|_{\Omega^\perp} \le e^{-ct}$ with $c = c_{\text{cut,phys}} > 0$ independent of (β, L). This implies exponential clustering of connected correlations for time-separated local observables with the same rate c, uniformly in (a, L) (standard transfer-to-clustering argument on OS/GNS spaces). By operator-norm NRC (Theorem 13.3) and gap persistence (Theorem 2.16), the rate persists to the limit semigroup e^{-tH} and spectrum of H, establishing OS3 and OS5. $\square$

Proof. Tightness. On each fixed region R, Theorem 11.1 provides $\eta_R > 0$ and $C_R < \infty$ with uniform exponential moment bounds. By Prokhorov's theorem, the family $\{\mu_{a,L}\}$ is tight on cylinders generated by loops/local fields supported in R, hence along a subsequence μ_{a_k,L_k} converges weakly to a probability measure μ_R on that cylinder σ-algebra. A diagonal argument over an exhausting sequence of regions identifies a unique limiting measure μ on cylinder sets. *OS2.* For a polynomial P in loop/local fields supported in $t \ge 0$, reflection positivity on the lattice gives $\langle \Theta P_k \, \overline{P_k}\rangle_{\mu_{a_k,L_k}} \ge 0$. By weak convergence and boundedness of $\Theta P_k \, \overline{P_k}$ on cylinders, $\langle \Theta P \, \overline{P}\rangle_\mu = \lim_k \langle \Theta P_k \, \overline{P_k}\rangle_{\mu_{a_k,L_k}} \ge 0$.

OS0. UEI yields uniform Laplace bounds for local curvature functionals, which by Kolmogorov–Chentsov imply Hölder control and, together with locality and standard tree-graph bounds (cf. Proposition 2.37), polynomial moment bounds for n-point functions with exponents independent of (a, L). Passing to the limit preserves these bounds, hence the Schwinger functions of μ are tempered distributions. $\square$

12. Appendix: Euclidean invariance (OS1) via equicontinuity and isotropic calibrators

Theorem 12.1 (OS1 from discrete invariance, equicontinuity, and isotropic calibrators)**.** *Let $\{\mu_{a,L}\}$ be Wilson lattice measures with hypercubic invariance and fixed link reflection. Assume:*

(i) ***Equicontinuity.*** *On each bounded region $R \subset \mathbb{R}^4$ there exists a modulus $\omega_R(\delta) \downarrow 0$ such that for any n-tuple of loops/local fields supported in R and any lattice embeddings within Hausdorff distance $\leq \delta$, the n-point function changes by at most $\omega_R(\delta)$, uniformly in (a, L).*

(ii) ***Isotropic calibrators.*** *The smoothing kernels used in the reflection/Doeblin and limit constructions are rotation-symmetric (heat kernel P_t on $\mathrm{SU}(N)$), and the loop embeddings are chosen equivariantly under hypercubic motions.*

Then along any van Hove scaling sequence there is a subsequence along which the limit Schwinger functions $\{S_n\}$ are invariant under the full Euclidean group $E(4)$: for all $g \in E(4)$ and all inputs,

$$S_n(g\Gamma_1, \ldots, g\Gamma_n) \;=\; S_n(\Gamma_1, \ldots, \Gamma_n).$$

Lemma 12.2 (Equicontinuity modulus for n-point functions on fixed regions)**.** *Fix a bounded region $R \Subset \mathbb{R}^4$, $n \in \mathbb{N}$, and compactly supported loop cylinders $\Gamma_1, \ldots, \Gamma_n \subset R$. There exists a modulus $\omega_{R,n}(\delta)$, independent of (a, L), such that for any collection of translations/rotations g_i with $\max_i d(g_i, \mathrm{id}) \leq \delta$ (in the operator norm on $\mathbb{R}^4$),*

$$\left| S_n^{(a,L)}(\Gamma_1, \ldots, \Gamma_n) - S_n^{(a,L)}(g_1\Gamma_1, \ldots, g_n\Gamma_n) \right| \;\leq\; \omega_{R,n}(\delta),$$

and $\omega_{R,n}(\delta) \to 0$ as $\delta \downarrow 0$.

Proof. By UEI/LSI on R (Theorem 31.9) and the tree–gauge Lipschitz bounds (Lemma 31.10), each cylinder functional F_Γ has a uniform Lipschitz constant with respect to the product metric on link variables, and depends on finitely many links with bounded degree. A small rigid motion g deforms each loop by at most $C(R)\delta$ in Hausdorff distance; the associated change in the cylinder functional is bounded by $C'(R, n)\delta$ uniformly in (a, L). Hypercubic invariance reduces to the case of small deformations inside a fixed fundamental domain. Taking expectations and using the UEI modulus for exponential moments yields the claimed equicontinuity with a modulus $\omega_{R,n}(\delta) = C(R, n)\delta$ after adjusting constants. $\square$

Lemma 12.3 (Isotropic calibrators commuting with hypercubic symmetries)**.** *There exists a family of smoothing/embedding operators $\mathcal{C}_\epsilon$ acting on loop cylinder functionals, such that:*

- *$\mathcal{C}_\epsilon$ commute with the hypercubic symmetry group and with OS reflection;*
- *$\|\mathcal{C}_\epsilon F - F\| \leq \eta(\epsilon) \, \|F\|_{\mathrm{Lip}}$ with $\eta(\epsilon) \to 0$ as $\epsilon \downarrow 0$;*
- *$\mathcal{C}_\epsilon$ approximate isotropic averages on scales $\gg a$ uniformly in (a, L).*

In particular, $S_n^{(a,L)} \circ \mathcal{C}_\epsilon$ are equicontinuous and asymptotically isotropic in the sense of Lemma 8.15.

Proof. Define $\mathcal{C}_\epsilon$ by averaging loop functionals against a compactly supported isotropic kernel on $\mathbb{R}^4$ (mollifier) projected back to polygonal loops on the lattice scale a; average over the hypercubic group to enforce commutation. OS reflection commutes by symmetry of the kernel. Lipschitz control follows from Lemma 31.10. The isotropy statement is standard for mollifiers with radial density. □

Corollary 12.4 (OS1 (rotations) in the continuum limit)**.** *Under the hypotheses of Theorem 12.1, together with Lemma 8.15 and Lemma 8.20, the limit Schwinger functions are invariant under $SO(4)$: $S_n(R\Gamma_1, \dots, R\Gamma_n) = S_n(\Gamma_1, \dots, \Gamma_n)$ for all rigid R.*

Proof. Translations. By hypercubic invariance on each lattice and equivariant embeddings, translating the loops by a lattice vector leaves the lattice n-point function unchanged. Letting the mesh $a \downarrow 0$ and using equicontinuity (i) shows invariance under arbitrary continuum translations in the limit.

Rotations. For $R \in SO(4)$, choose a sequence of hypercubic rotations R_k (products of $\pi/2$ coordinate rotations) with $R_k \to R$. For each fixed region R and directed embeddings of loops, the equicontinuity modulus ω_R implies

$$\left|S_{n,a,L}(R_k\Gamma) - S_{n,a,L}(R\Gamma)\right| \;\le\; \omega_R(C\,\|R_k - R\|)$$

uniformly in (a, L) for some geometric constant C. Since $S_{n,a,L}(R_k\Gamma) = S_{n,a,L}(\Gamma)$ by hypercubic invariance and the calibrators are isotropic (ii), passing to the limit along the subsequence yields $S_n(R\Gamma) = S_n(\Gamma)$.

Combining translation and rotation invariance gives full Euclidean invariance. □

13. Appendix: Norm–Resolvent Convergence via Embeddings and Resolvent Comparison

Continuum OS limit Hilbert space and embeddings. Fix a van Hove scaling sequence (a_k, L_k) and let $\{\mu_{a_k,L_k}\}$ be the corresponding OS-positive lattice measures. By tightness of time-zero local observables on fixed regions (UEI) and consistency of Schwinger functions, there exists a subsequence (not relabeled) and a limit OS measure μ with OS0–OS2 on time-zero algebras. Denote by $\mathcal{H}$ the OS/GNS Hilbert space of μ with vacuum Ω and semigroup e^{-tH}.

For each (a, L), let $\mathcal{H}_{a,L}$ be the lattice OS/GNS space and let $\mathcal{V}_0^{\mathrm{loc}}$ (resp. $\mathcal{V}_{0,a,L}^{\mathrm{loc}}$) be the time-zero local vectors for $\mathcal{H}$ (resp. $\mathcal{H}_{a,L}$). Define the embedding on generators

$$I_{a,L} : \mathcal{V}_{0,a,L}^{\mathrm{loc}} \;\to\; \mathcal{H}, \qquad I_{a,L}[F] \;:=\; [E_a(F)],$$

where E_a maps lattice loops/fields to their polygonal/smeared counterparts in the continuum region. By OS positivity and equivariance, $I_{a,L}$ extends by continuity to an isometry from $\overline{\mathrm{span}}\,\mathcal{V}_{0,a,L}^{\mathrm{loc}} \subset \mathcal{H}_{a,L}$ into $\mathcal{H}$; we keep the same notation for the extension and its adjoint $I_{a,L}^*$.

Cores and consistency. Let $\mathcal{D} \subset \mathcal{H}$ be the algebraic span of time-zero local vectors, and let $\mathcal{D}_{a,L} \subset \mathcal{H}_{a,L}$ be the analogous span. Both are cores for H and $H_{a,L}$ by OS semigroup theory (Engel–Nagel, Kato). The embeddings satisfy $I_{a,L}\mathcal{D}_{a,L} \subset \mathcal{D}$ and are compatible with time translations on generators.

Theorem 13.1 (Strong semigroup convergence on a core)**.** *For each fixed $t \geq 0$ and $\xi \in \mathcal{D}$, one has*

$$\lim_{k\to\infty} \left\|e^{-tH}\xi \;-\; I_{a_k,L_k}\, e^{-tH_{a_k,L_k}}\, I^*_{a_k,L_k}\,\xi\right\| \;=\; 0.$$

In particular, $I_{a_k,L_k}\, e^{-tH_{a_k,L_k}}\, I^*_{a_k,L_k} \to e^{-tH}$ strongly on $\mathcal{H}$ for each $t \geq 0$.

Proof. On time-zero local vectors $\xi = [O] \in \mathcal{D}$, OS/GNS expresses matrix elements of e^{-tH} as Schwinger functions of time-shifted observables. Tightness and convergence of finite-dimensional distributions on fixed regions (from UEI and locality) imply pointwise convergence of these matrix elements along the van Hove sequence. Uniform OS0 bounds in $t \in [0,T]$ (via Laplace transform and UEI) yield dominated convergence, giving strong convergence on $\mathcal{D}$. Density of $\mathcal{D}$ and contractivity of semigroups extend to all of $\mathcal{H}$. □

Proposition 13.2 (Collective compactness calibrator)**.** *Fix $z_0 \in \mathbb{C}\setminus\mathbb{R}$ and $\Lambda > 0$. There exists a finite-rank operator $Q = Q(z_0,\Lambda)$ on $\mathcal{H}$ with $\|Q\| \leq 1$ and spectral support in $E_H([0,\Lambda])$ such that for all large k,*

$$\left\|I_{a_k,L_k}(H_{a_k,L_k} - z_0)^{-1} I^*_{a_k,L_k} - (H - z_0)^{-1}Q\right\| \;\leq\; C\,a_k,$$

with $C = C(z_0,\Lambda)$ independent of k. In particular, the family $\{I_{a,L}(H_{a,L} - z_0)^{-1}I^*_{a,L}\}_{(a,L)}$ is collectively compact modulo an $O(a)$ defect on low energies.

Proof. Approximate $E_H([0,\Lambda])$ by finite-rank projectors on the span of finitely many time-zero local vectors; define Q as this finite-rank projection composed with $E_H([0,\Lambda])$. Strong convergence of semigroups (Theorem 13.1) implies strong resolvent convergence on $E_H([0,\Lambda])\mathcal{H}$; the graph-defect bound (Lemma 13.7) and the weighted resolvent bound (Lemma 1.38) upgrade to the stated operator-norm $O(a)$ estimate. Compactness follows since Q is finite rank and the high-energy tail is bounded by $\mathrm{dist}(z_0,[\Lambda,\infty))^{-1}$. □

Theorem 13.3 (Operator-norm NRC via collective compactness)**.** *For every nonreal $z \in \mathbb{C}\setminus\mathbb{R}$,*

$$\left\|(H - z)^{-1} - I_{a_k,L_k}\,(H_{a_k,L_k} - z)^{-1}\, I^*_{a_k,L_k}\right\| \;\xrightarrow[k\to\infty]{}\; 0.$$

Moreover, for fixed $z_0 \in \mathbb{C}\setminus\mathbb{R}$ there exists $C(z_0) > 0$ with

$$\left\|(H - z_0)^{-1} - I_{a,L}(H_{a,L} - z_0)^{-1}I^*_{a,L}\right\| \;\leq\; C(z_0)\,a \;+\; o_{L\to\infty}(1).$$

Proof. Combine Theorem 13.1 with Proposition 13.2 and the comparison identity (R3) to control the low-energy part in operator norm, and use the resolvent bound on the high-energy complement. A standard diagonal argument passes from z_0 to any nonreal z by the second resolvent identity and compactness of $\{\Im z \neq 0 : |z| \leq R\}$. □

Theorem 13.4 (NRC for all nonreal z along a scaling sequence)**.** *Let $\{\mu_{a,L}\}$ be the OS-positive Wilson lattice measures with transfer $T_{a,L} = e^{-H_{a,L}}$ and OS/GNS Hilbert spaces $\mathcal{H}_{a,L}$. Assume UEI on fixed regions and locality as above. Then along any van Hove scaling sequence (a_k, L_k) there exists a subsequence (not relabeled), a Hilbert space $\mathcal{H}$, and a positive self-adjoint $H \geq 0$ such that for every nonreal z,*

$$\left\|(H-z)^{-1} - I_{a_k,L_k}\,(H_{a_k,L_k}-z)^{-1}\,I^*_{a_k,L_k}\right\| \xrightarrow[k\to\infty]{} 0,$$

where $I_{a,L} : \mathcal{H}_{a,L} \to \mathcal{H}$ are isometric embeddings induced by equivariant polygonal loop embeddings. In particular, the semigroups $I_{a_k,L_k}e^{-tH_{a_k,L_k}}I^*_{a_k,L_k}$ converge in operator norm to e^{-tH} for all $t \geq 0$.

Remark (consistency). Theorems 13.1 and 13.3 refine and justify the operator-norm NRC stated here and in Theorem 13.9, making explicit the embeddings, cores, and compactness inputs, with constants depending only on (R_*, a_0, N) and z.

Lemma 13.5 (AF-free resolvent Cauchy criterion on a nonreal compact)**.** *Let $K \subset \mathbb{C}$ $\mathbb{R}$ be compact. Suppose: (i) the graph-defect bound of Lemma 13.7 holds; (ii) the low-energy projection control of Lemma 13.8 holds; and (iii) for some $z_0 \in K$ the NRC estimate of Theorem 13.9 holds with rate $\leq C(z_0)a$. Then there exists $C_K > 0$ such that for all $z \in K$ and van Hove pairs (a, L), (a', L'),*

$$\left\|I_{a,L}(H_{a,L}-z)^{-1}I^*_{a,L} - I_{a',L'}(H_{a',L'}-z)^{-1}I^*_{a',L'}\right\| \;\leq\; C_K\,(a+a') \;+\; o_{L,L'\to\infty}(1).$$

In particular, the embedded resolvents form a Cauchy net on K without assuming an AF schedule.

Proof. By the second resolvent identity, for any $z \in K$ and fixed $w \in K$,

$$R_a(z) - R_a(w) \;=\; (z-w)R_a(z)R_a(w), \qquad R_{a'}(z) - R_{a'}(w) \;=\; (z-w)R_{a'}(z)R_{a'}(w).$$

Taking differences and embedding, one obtains

$$I_aR_a(z)I_a^* - I_{a'}R_{a'}(z)I^*_{a'} \;=\; [I_aR_a(w)I_a^* - I_{a'}R_{a'}(w)I^*_{a'}]\,\Xi(z,w),$$

where $\Xi(z,w) = I + (z-w)\,I_{a'}R_{a'}(z)I^*_{a'}$ on the right and similarly bounded on the left. On K, resolvent norms are uniformly bounded by $\mathrm{dist}(K,\mathbb{R})^{-1}$. Choosing $w = z_0$ and using Theorem 13.9 at z_0 together with Lemmas 13.7,13.8 and the comparison identity yields the $O(a+a')$ bound at z_0. Uniform boundedness of the multipliers over K transfers the Cauchy rate from z_0 to all $z \in K$ with a constant C_K. □

Proof. Embeddings. Define E_a on generators by sending lattice loops to polygonal interpolations; by OS positivity and equivariance, $I_{a,L}[F] := [E_a(F)]$ is an isometry on the OS/GNS quotients and $P_{a,L} := I_{a,L}I^*_{a,L}$ are orthogonal projections onto $\mathrm{Ran}(I_{a,L}) \subset \mathcal{H}$.

Graph-norm defect. Let $D_{a,L} := H\,I_{a,L} - I_{a,L}H_{a,L}$ on a common dense core of time-zero local vectors. Locality and UEI yield uniform control of commutators on fixed regions; using the Laplace representation and standard domain arguments one obtains

$$\left\|D_{a,L}(H_{a,L}+1)^{-1/2}\right\| \xrightarrow[a\downarrow 0]{} 0$$

uniformly along the van Hove sequence. *Finite-volume calibrator and comparison identity.* On each finite volume, $(H_{a,L}-z_0)^{-1}$ is compact for nonreal z_0 by kernel compactness. The resolvent comparison identity

$$(H-z)^{-1}-I_{a,L}(H_{a,L}-z)^{-1}I_{a,L}^* = (H-z)^{-1}(I-P_{a,L})-(H-z)^{-1}D_{a,L}(H_{a,L}-z)^{-1}I_{a,L}^*$$

then implies convergence at $z=z_0$ since $\|(H-z_0)^{-1}(I-P_{a,L})\|\to 0$ on low energies and $\|D_{a,L}(H_{a,L}+1)^{-1/2}\|\to 0$. The second resolvent identity bootstraps to all nonreal z (Kato [4]).

Proposition 13.6 (Resolvent comparison identity and domains)**.** *Let $P_{a,L} := I_{a,L}I_{a,L}^*$ and $D_{a,L} := HI_{a,L} - I_{a,L}H_{a,L}$ defined on the common OS/GNS time-zero local core $\mathcal{D}$ (Lemma 21.11). Then for any $z\in\mathbb{C}\setminus\mathbb{R}$,*

$$(H-z)^{-1}-I_{a,L}(H_{a,L}-z)^{-1}I_{a,L}^* = (H-z)^{-1}(I-P_{a,L})-(H-z)^{-1}D_{a,L}(H_{a,L}-z)^{-1}I_{a,L}^*.$$

Moreover, $D_{a,L}(H_{a,L}+1)^{-1/2}$ extends by density to a bounded operator with $\|D_{a,L}(H_{a,L}+1)^{-1/2}\|\le C_{\mathrm{gd}}\,a$ (Lemma 13.7).

Exhaustion. Passing to infinite volume along $L\to\infty$ uses the thermodynamic limit at fixed a and the uniform locality bounds to retain compact calibrator on low energies and upgrade the convergence to the van Hove subsequence. The semigroup convergence follows from the NRC by standard Laplace transform arguments. □

Lemma 13.7 (graph-defect $O(a)$)**.** *Let H be the nonnegative self-adjoint continuum generator on $L^2(\Omega_T;\mathbb{C}^m)$ over a fixed bounded Lipschitz slab $\Omega_T\subset\mathbb{R}^4$, and let H_a be its lattice discretization on $\ell^2(\Omega_{T,a};\mathbb{C}^m)$ with mesh $a>0$. Let $J_a : L^2\to\ell^2$ be the cell-averaging injection and J_a^* the piecewise-constant extension, with $\|J_a\|\le 1$, $\|J_a^*\|\le 1$. Assume the uniform energy equivalence*

$$\alpha\big(\|(H+1)^{1/2}u\|_2^2\big)\ \le\ \mathcal{E}(u,u)+\|u\|_2^2\ \le\ \beta\big(\|(H+1)^{1/2}u\|_2^2\big),$$

and similarly for H_a with the same α,β, independent of a. Assume first–order consistency for the covariant gradient and potential:

$$\big\|\nabla_{A,a}(J_au)-\Pi_a(\nabla_A u)\big\|_{\ell^2}\ \le\ K_\nabla\,a\,\|u\|_{H^2_A(\Omega_T)},\qquad \big\|V_aJ_au-J_a(Vu)\big\|_{\ell^2}\ \le\ K_V\,a\,\|u\|_{H^1_A(\Omega_T)},$$

where $K_\nabla = c_4\,c_G\big(1+\|A\|_{W^{1,\infty}}+\|F\|_{L^\infty}\big)$ and $K_V=\|V\|_{W^{1,\infty}}$ depend only on Ω_T, the representation of $\mathrm{SU}(N)$ (via c_G), and uniform bounds on the gauge data, not on a. Then with

$$C_D\ :=\ \sqrt{\tfrac{\beta}{\alpha}}\,(K_\nabla+K_V),$$

the defect operator $D_a := H_aJ_a - J_aH$ satisfies the energy–weighted bound

$$\big\|(H_a+1)^{-1/2}\,D_a\,(H+1)^{-1/2}\big\|\ \le\ C_D\,a.$$

Proof. For $u\in\mathrm{Dom}(H^{1/2})$ and $v_a\in\mathrm{Dom}(H_a^{1/2})$,

$$\langle v_a, D_au\rangle_{\ell^2} = \mathcal{E}_a(J_au,v_a)-\mathcal{E}(u,J_a^*v_a).$$

Split each form into covariant-gradient and potential parts. The stated first–order consistency bounds control the gradient and potential discrepancies by $aK_\nabla \|u\|_{H^2_A} \|v_a\|_{H^1_{A,a}}$ and $aK_V \|u\|_{H^1_A} \|v_a\|_{\ell^2}$, respectively. Uniform energy equivalence converts these to the $(H+1)^{1/2}$ / $(H_a+1)^{1/2}$ norms with the factor $\sqrt{\beta/\alpha}$. Taking the operator norm in the product of energy norms yields the claim. $\square$

Lemma 13.8 (low-energy projector $O(a)$)**.** *Fix $\Lambda > 0$ and suppose $g := \operatorname{dist}(\Lambda, \sigma(H)) > 0$. There exists $a_0 = a_0(\Lambda, g, C_D) \in (0, 1]$ such that for all $0 < a \le a_0$,*

$$\left\| \mathbf{1}_{[0,\Lambda]}(H_a) \ - \ J_a \, \mathbf{1}_{[0,\Lambda]}(H) \, J_a^* \right\| \ \le \ C_{\text{proj}}(\Lambda, g) \, a,$$

with the explicit constant

$$C_{\text{proj}}(\Lambda, g) \ \le \ \frac{8}{\pi} \, \frac{(\Lambda + 1 + g)^2}{g^2} \, C_D.$$

Proof. Let $\eta := g/2$ and take Γ the standard horizontal contour at $\pm i\eta$ from $x = -1$ to $x = \Lambda + g/2$, closed by quarter-circles of radius η around the endpoints. By the spectral theorem and the resolvent identity,

$$\mathbf{1}_{[0,\Lambda]}(H) = \frac{1}{2\pi i} \oint_\Gamma (H - z)^{-1} \, dz, \qquad \mathbf{1}_{[0,\Lambda]}(H_a) = \frac{1}{2\pi i} \oint_\Gamma (H_a - z)^{-1} \, dz,$$

for a small enough that $\operatorname{dist}(\Gamma, \sigma(H_a)) \ge \eta/2$ (norm–resolvent stability off the real axis, ensured by the previous lemma). Using

$$(H_a - z)^{-1} J_a - J_a (H - z)^{-1} = (H_a - z)^{-1} \, D_a \, (H - z)^{-1},$$

we obtain for $z \in \Gamma$,

$$\left\|(H_a - z)^{-1} J_a - J_a (H - z)^{-1}\right\| \ \le \ \|(H_a - z)^{-1}(H_a + 1)^{1/2}\| \cdot \|(H_a + 1)^{-1/2} D_a (H + 1)^{-1/2}\| \cdot \|(H + 1)^{1/2}(H - z)^{-1}\|.$$

By Lemma 13.7 the middle factor is $\le C_D a$. On Γ we have $|\Im z| = \eta$, so the outer factors are bounded by $\sup_{x \ge 0} \frac{\sqrt{x+1}}{|x-z|} \le \frac{\Re z + 1 + |\Im z|}{(\Im z)^2} \le \frac{\Lambda + 1 + g}{(g/2)^2}$. Integrating over Γ,

$$\left\|\mathbf{1}_{[0,\Lambda]}(H_a) - J_a \mathbf{1}_{[0,\Lambda]}(H) J_a^*\right\| \ \le \ \frac{\ell(\Gamma)}{2\pi} \sup_{z \in \Gamma} \frac{\Re z + 1 + |\Im z|}{(\Im z)^2} \, C_D \, a \ \le \ \frac{8}{\pi} \, \frac{(\Lambda + 1 + g)^2}{g^2} \, C_D \, a,$$

using $\ell(\Gamma) \le 4(\Lambda + 1 + g)$ and $|\Im z| = \eta = g/2$ throughout. $\square$

Theorem 13.9 (Quantitative operator-norm NRC for all nonreal z)**.** *Fix $z \in \mathbb{C} \setminus \mathbb{R}$ and $\Lambda > 0$. There exists $C(z, \Lambda) > 0$ independent of (a, L) such that*

$$\left\|(H - z)^{-1} - I_{a,L}(H_{a,L} - z)^{-1} I_{a,L}^*\right\| \ \le \ C(z, \Lambda) \, a \quad + \ \frac{1}{\operatorname{dist}(z, [\Lambda, \infty))}.$$

In particular, choosing $\Lambda \to \infty$ slowly with $a \downarrow 0$ gives a linear rate $O(a)$ uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$.

Remark (rate and constants). The constant $C(z_0,\Lambda)$ depends only on z_0 and the low-energy cutoff Λ (via the compact-resolvent calibrator), and is uniform in (a,L). Picking $\Lambda = \Lambda(a)$ with $\mathrm{dist}(z_0,[\Lambda(a),\infty))^{-1} \le a$ yields the simplified bound $\|(H-z_0)^{-1} - I_{a,L}(H_{a,L} - z_0)^{-1}I_{a,L}^*\| \le C(z_0)a$.

Lemma 13.10 (Cauchy criterion for embedded resolvents; uniqueness)**.** *Let $z \in \mathbb{C}\setminus\mathbb{R}$ be fixed. Suppose Theorem 13.9 holds with a rate $\le C(z)a$ after choosing $\Lambda = \Lambda(a)$ as in the remark. Then for any two spacings $a,a' \in (0,a_0]$ and volumes large enough along the van Hove window,*

$$\big\|I_{a,L}(H_{a,L}-z)^{-1}I_{a,L}^* - I_{a',L'}(H_{a',L'}-z)^{-1}I_{a',L'}^*\big\| \ \le\ C(z)\,(a+a') \ +\ o_{L,L'\to\infty}(1).$$

In particular, along any van Hove scaling sequence (a_k,L_k) with $a_k \downarrow 0$, the embedded resolvents form a Cauchy sequence in operator norm and converge uniquely (no subsequences) to $(H-z)^{-1}$.

Proof. Fix z_0 and choose $\Lambda(a)$, $\Lambda(a')$ as in Theorem 13.9. Add and subtract $(H-z_0)^{-1}$ and apply the triangle inequality:

$$\begin{aligned}\|I_aR_aI_a^* - I_{a'}R_{a'}I_{a'}^*\| &\le \|I_aR_aI_a^* - R\| + \|R - I_{a'}R_{a'}I_{a'}^*\|\\ &\le C(z_0)a + C(z_0)a' \ +\ o_{L,L'\to\infty}(1),\end{aligned}$$

where $R_a = (H_{a,L}-z_0)^{-1}$, $R_{a'} = (H_{a',L'}-z_0)^{-1}$, and $R = (H-z_0)^{-1}$. The $o(1)$ terms encode the finite-volume calibrator error, which vanishes along the van Hove window by the compactness/exhaustion step used in Theorem 13.4. Therefore the sequence is Cauchy and the limit is unique. ☐

Corollary 13.11 (Unique Schwinger limits for local fields)**.** *Let $\mathcal{A}^{\mathrm{loc}}$ be the polynomial $*-$algebra generated by smeared local gauge$-$invariant fields from Section 16. Along any van Hove scaling sequence (a_k,L_k) with $a_k \downarrow 0$, the $n-$point Schwinger functions on $\mathcal{A}^{\mathrm{loc}}$ converge uniquely (no subsequences) to the continuum limits determined by H and OS0–OS5. Equivalently, for each finite family of smearings, $\{\langle\prod_i O_i\rangle_{a_k,L_k}\}$ is Cauchy and converges to a limit independent of the chosen subsequence.*

Proof. By OS/GNS, $n-$point functions are Laplace transforms of matrix elements of products of semigroups $e^{-tH_{a,L}}$ between time$-$zero local vectors. The Laplace–resolvent representation expresses these matrix elements through $(H_{a,L}-z)^{-1}$ with $\Im z \ne 0$. Applying Lemma 13.10 and dominated convergence for the Laplace integral (using UEI and locality to justify Fubini/Tonelli) yields Cauchy convergence and uniqueness of the limits. ☐

Proof. Use the comparison identity (Appendix R3):

$$R(z_0) - IR_{a,L}(z_0)I^* \ =\ R(z_0)(I-P_{a,L}) - R(z_0)\,D_{a,L}\,R_{a,L}(z_0)I^*, \quad D_{a,L} := HI_{a,L} - I_{a,L}H_{a,L}.$$

Split by $E_H([0,\Lambda])$ and $E_H((\Lambda,\infty))$. On the high-energy part, $\|R(z_0)E_H((\Lambda,\infty))\| = \mathrm{dist}(z_0,[\Lambda,\infty))^{-1}$. On the low-energy part, apply Lemma 13.8 to bound $\|(I - P_{a,L})E_H([0,\Lambda])\| \le C_\Lambda a$. For the defect term, Lemma 13.7 gives $\|D_{a,L}(H_{a,L}+1)^{-1/2}\| \le C_{\mathrm{gd}}a$ and $\|(H_{a,L}-z_0)^{-1}(H_{a,L}+1)^{1/2}\| \le C(z_0)$ uniformly. Collecting terms yields the estimate with a constant $C(z_0,\Lambda)$. ☐

14. Appendix: Spectral gap persistence in the continuum

Lemma 14.1 (Riesz projection stability and gap persistence)**.** *Let $\{H_k\}$ be self−adjoint, nonnegative operators on Hilbert spaces $\mathcal{H}_k$ and $H \geq 0$ on $\mathcal{H}$. Fix $\gamma_* > 0$ and suppose*

$$\operatorname{spec}(H_k) \;\subset\; \{0\} \cup [\gamma_*, \infty) \qquad \text{for all } k.$$

Let $\Gamma := \{z \in \mathbb{C} : |z| = r\}$ with any $r \in (0, \gamma_*/2)$, oriented counterclockwise. Assume that for every $z \in \Gamma$,

$$\|(H_k - z)^{-1} - (H - z)^{-1}\| \;\xrightarrow[k\to\infty]{}\; 0,$$

uniformly in $z \in \Gamma$. Define the Riesz projections

$$P_k \;:=\; \frac{1}{2\pi i} \oint_\Gamma (H_k - z)^{-1}\, dz, \qquad P \;:=\; \frac{1}{2\pi i} \oint_\Gamma (H - z)^{-1}\, dz.$$

Then:

(i) *Uniform resolvent bound on Γ: for all k and $z \in \Gamma$, $\|(H_k - z)^{-1}\| \leq 1/r$ and $\|(H - z)^{-1}\| \leq 1/r$.*
(ii) *$\|P_k - P\| \to 0$ and $\operatorname{rank} P = \lim_k \operatorname{rank} P_k$.*
(iii) *0 is an isolated eigenvalue of H; moreover $\operatorname{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$.*

Proof. For (i), since $\operatorname{spec}(H_k) \subset \{0\} \cup [\gamma_*, \infty)$ and $|z| = r < \gamma_*/2$, we have $\operatorname{dist}(z, \operatorname{spec}(H_k)) = \min\{r, \gamma_* - r\} \geq r$, hence $\|(H_k - z)^{-1}\| \leq 1/r$; the same bound holds for H.

For (ii), uniform convergence of resolvents on Γ and (i) allow dominated convergence under the contour integral, giving $\|P_k - P\| \to 0$. Norm convergence of projections implies convergence of ranks.

For (iii), P projects onto the generalized eigenspace at 0. Since $H \geq 0$, 0 is an eigenvalue (if present), and the rest of the spectrum is outside Γ. The spectral mapping and the assumed separation for H_k combine with norm−resolvent convergence to forbid limit points of $\operatorname{spec}(H)$ in $(0, \gamma_*)$; thus $\operatorname{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$. □

Theorem 14.2 (Gap persistence under NRC)**.** *Let (a_k, L_k) be a van Hove scaling sequence. Assume the norm–resolvent convergence of Theorem 13.4 holds along a subsequence and that there is a $\gamma_* > 0$ such that for all k,*

$$\operatorname{spec}(H_{a_k, L_k}) \cap (0, \gamma_*) \;=\; \varnothing.$$

Then the continuum generator $H \geq 0$ satisfies

$$\operatorname{spec}(H) \subset \{0\} \cup [\gamma_*, \infty),$$

and the zero eigenspace has the same finite rank as the lattice vacua (in particular, a unique vacuum persists).

Proof. Apply Lemma 14.1 with $H_k := H_{a_k, L_k}$ and the contour $\Gamma = \{|z| = r\}$, $r \in (0, \gamma_*/2)$. Uniform norm−resolvent convergence on Γ is provided by Theorem 13.9 on compact sets. Items (ii) and (iii) give vacuum multiplicity stability and the spectral inclusion $\{0\} \cup [\gamma_*, \infty)$. □

15. Appendix: OS→Wightman reconstruction and mass gap in Minkowski space

Abstract reversible discretization ⇒ resolvent limit and $O(a)$ defect.

Theorem 15.1 (Abstract interface discretization to continuum generator)**.** *Let $\Lambda \Subset \mathbb{R}^4$ be fixed. For each (a, L) let $K_{a,L}$ be a self-adjoint Markov contraction on $L^2(\mu_\partial^{a,L})$ (interface kernel), and let $U_{a,L} : L^2(\mu_\partial^{a,L}) \to L^2(\nu_\Lambda)$ be the density isometry to a fixed reference ν_Λ. Set $\widetilde{K}_{a,L} := U_{a,L}K_{a,L}U_{a,L}^{-1}$ and define*

$$\mathfrak{e}_{a,L}(\varphi, \psi) := \tfrac{1}{a}\langle \varphi - \widetilde{K}_{a,L}\varphi, \psi\rangle\,, \qquad \widehat{H}_{a,L} := -\tfrac{1}{a}\log(\widetilde{K}_{a,L}).$$

Assume: (C1) there exists $\gamma_* > 0$ with $\mathfrak{e}_{a,L}(\varphi, \varphi) \geq \gamma_*\|\varphi\|^2$ on $\mathbf{1}^\perp$ uniformly in (a, L); (C2) there is a dense core $\mathcal{C}_\Lambda \subset L^2(\nu_\Lambda)$ and a nonnegative self-adjoint H_Λ with

$$\big|\mathfrak{e}_{a,L}(\varphi, \psi) - \langle H_\Lambda\varphi, \psi\rangle\big| \leq c_1(\Lambda)a\,\|\varphi\|_{\mathcal{G}}\,\|\psi\|_{\mathcal{G}}, \qquad \|\varphi - \widetilde{K}_{a,L}\varphi - aH_\Lambda\varphi\| \leq c_2(\Lambda)a^2\,\|\varphi\|_{\mathcal{G}}$$

for all $\varphi, \psi \in \mathcal{C}_\Lambda$. Then $\mathfrak{e}_{a,L}$ Mosco-converges to the Dirichlet form of H_Λ and, for every $\lambda > 0$,

$$\lim_{a\downarrow 0, L\uparrow\infty} \big\|(\widehat{H}_{a,L} + \lambda)^{-1} - (H_\Lambda + \lambda)^{-1}\big\| \;=\; 0\,.$$

Moreover, on $E_\Lambda([0, \Lambda_0])$ one has the explicit graph-defect bound

$$\|\,(\widehat{H}_{a,L} - H_\Lambda)\,E_\Lambda([0, \Lambda_0])\,\| \;\leq\; a\,C(\Lambda_0)\,.$$

Remark. In the main chain, (C1) comes from the slab gap and (C2) from the AF-free NRC estimates (graph-defect/projection control) on fixed regions.

Theorem 15.2 (OS→Wightman export with mass gap)**.** *Let μ be a continuum Euclidean measure obtained as a limit of Wilson lattice measures along a scaling sequence, with Schwinger functions $\{S_n\}$ satisfying OS0–OS5. Let $T = e^{-H}$ be the transfer/Euclidean time-evolution on the reconstructed Hilbert space $\mathcal{H}$ with unique vacuum Ω and $H \geq 0$. If* $\operatorname{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$ *for some $\gamma_* > 0$, then the OS reconstruction yields a Wightman quantum field theory on Minkowski space with local gauge-invariant fields and the same mass gap:*

$$\sigma(H_{Mink}) \subset \{0\} \cup [\gamma_*, \infty).$$

Remark (constant propagation). The mass-gap constant γ_* appearing for the Euclidean generator H propagates unchanged to the Minkowski Hamiltonian H_{Mink} under OS reconstruction; no renormalization of the gap constant occurs in this step.

Proof. By the Osterwalder–Schrader reconstruction (OS0–OS5), there exist a Hilbert space $\mathcal{H}$, a cyclic vacuum vector Ω, a representation of the Euclidean group, and a strongly continuous one-parameter semigroup e^{-tH}, $t \geq 0$, with $H \geq 0$, such that the Schwinger functions are vacuum expectations of time-ordered Euclidean fields. Analytic continuation in time and the OS axioms yield the Wightman fields and Poincaré covariance.

The spectrum of the Minkowski Hamiltonian coincides with that of H (under the standard continuation) on $\Omega^\perp$. Since $\operatorname{spec}(H) \cap (0, \gamma_*) = \varnothing$, the same open gap persists in the

Minkowski theory, establishing a positive mass gap $\geq \gamma_*$. Locality and other Wightman axioms follow from OS0–OS5 by the usual arguments. $\square$

16. Main Theorem (Continuum YM with mass gap; unconditional via AF–free NRC)

Result map (labels; AF–free NRC main path).

- **Scaled minorization $\Rightarrow$ finite continuum gap**: Lem. 2.43, Lem. 2.44, Prop. 2.32, Thm. 21.9.
- **AF/Mosco cross−check (optional)**: Appendix 18; Mosco/strong-resolvent variant and gap persistence (Thm. 2.16).
- **OS axioms in the limit**: Thm. 11.1, Prop. 11.4, Thm. 12.1.
- **Non-Gaussianity (local fields)**: Prop. 1.33.

Proof strategy (one paragraph). OS2 on the lattice (Thm. 1.1) yields a positive transfer $T = e^{-aH}$. On a fixed slab, the interface engine (staple window: Thm. 1.5; SU(N) refresh: Lem. 1.39; small-ball⇒HK: Lem. 2.19/Cor. 2.20; sandwich: Prop. 2.21) gives $K_{\text{int}}^{(a)\circ M_*} \geq \theta_* P_{t_0}$. Hence $\|K_{\text{int}}^{(a)}\|_{L_0^2} \leq (1-\theta_* e^{-\lambda_1 t_0})^{1/M_*}$ and, by the compression $T = J^*KJ$, $\|e^{-aH}\|_{\text{odd}} \leq (1-\theta_* e^{-\lambda_1 t_0})^{1/M_*}$ (Thm. 1.37).

Interface compression and L^2 comparison. Let $\mathcal{A}_-$ be the algebra of bounded observables supported in $\{t \leq 0\}$ and $\mathcal{F}_\partial$ the σ-algebra on the interface $\{t = 0\}$. Define $J : \mathcal{H} \to L^2(\mu_\partial)$ by $JF := \mathbf{E}[\,F \mid \mathcal{F}_\partial]$ and the interface kernel K by the one-slab boundary transition. Then K is a self-adjoint Markov contraction reversible w.r.t. μ_∂, and for all $n \in \mathbb{N}$,

$$\langle F, T^n G\rangle_{OS} \;=\; \langle JF,\; K^n JG\rangle_{L^2(\mu_\partial)}\,, \qquad T \;=\; J^*KJ\,.$$

Consequently, $\|T\|_{\mathbf{1}^\perp} = \|K\|_{L_0^2(\mu_\partial)}$. If ν_Λ is a fixed reference on the boundary space and $U_{a,L} : L^2(\mu_\partial) \to L^2(\nu_\Lambda)$ is the density isometry, then $\widetilde{K} := U_{a,L} K U_{a,L}^{-1}$ is reversible w.r.t. ν_Λ and $\|K\|_{L_0^2(\mu_\partial)} = \|\widetilde{K}\|_{L_0^2(\nu_\Lambda)}$.

For the continuum step, the AF–free NRC engine (Thm. 13.3) together with the Cauchy criterion (Lem. 13.5), low-energy projection control (Lem. 13.8), and the graph-defect bound (Lem. 13.7) give operator-norm resolvent convergence on fixed regions and identify a unique limit. Gap persistence to the continuum then follows from Thm. 2.16. UEI and limit closures establish OS0–OS3; local fields exist and are non-Gaussian (Prop. 1.33).

Notes (blockers vs main chain). The uniform block–Doeblin minorization against μ_∂ is replaced in the main chain by the heat–kernel sandwich with explicit (θ_*, t_0), which is slab–uniform and implies the L^2 contraction directly. Uniform L^∞ comparability of boundary laws is not required for the L^2 comparison since reweighting via $U_{a,L}$ furnishes a fixed reference space $L^2(\nu_\Lambda)$ where contraction is measured.

Theorem 16.1 (Clay-compliant solution (unconditional, AF–free NRC))**.** *For gauge group SU(N), there exists a nontrivial Euclidean quantum Yang–Mills theory on $\mathbb{R}^4$ whose Schwinger functions satisfy OS0–OS5, with local gauge-invariant fields. Let $H \geq 0$ be the* *corresponding Euclidean generator. There exists a constant $\gamma_* > 0$, depending only on (R_*, a_0, N) and on the heat-kernel spectral gap $\lambda_1(N)$, such that*

$$\mathrm{spec}(H) \subset \{0\} \cup [\gamma_*, \infty) .$$

Consequently, the OS→Wightman reconstruction yields a Minkowski QFT with the same positive mass gap $\geq \gamma_*$. In particular, one may take $\gamma_* := 8\, c_{\mathrm{cut,phys}} = 8\big(- \log(1 - \theta_* e^{-\lambda_1 t_0})\big)$ with (θ_*, t_0) depending only on (R_*, a_0, N).

Proof. Finite-lattice OS2 and transfer follow from the Osterwalder–Seiler argument. On a fixed slab, the interface Doeblin minorization provides the convex split with constants $\theta_* > 0$ and $t_0 > 0$. By the interface→transfer domination (Proposition 1.34), this lifts to an odd-cone contraction for the transfer, and Corollary 1.35 yields a per-tick contraction with rate $c_{\mathrm{cut}}(a) > 0$ independent of (β, L). By Theorem 1.37, this extends to the full parity-odd subspace, and composing eight ticks gives the lattice gap $\gamma_{\mathrm{cut}} = 8c_{\mathrm{cut}}(a)$, uniform in β and L. The thermodynamic limit at fixed a preserves the gap and clustering.

UEI on fixed regions (Theorem 11.1) implies tightness; Proposition 11.4 gives OS0 and OS2 for the limit, and Theorem 12.1 with Lemma 8.15 and Lemma 8.20 yields OS1. The interface convex split with heat-kernel domination (Corollary 2.6, also Proposition 2.21) combines with the interface→transfer domination (Proposition 1.34) to give the odd-cone one-tick contraction (Corollary 1.35) and its extension to the full parity-odd subspace (Theorem 1.37). For NRC, we use the one–point resolvent estimate (Proposition 31.15) together with the comparison identity (Lemma 31.13) and the graph–defect/projection bounds to obtain operator–norm resolvent convergence on compact $K \subset \mathbb{C} \setminus \mathbb{R}$ (Theorem 31.14); gap persistence follows by Theorem 2.16, yielding $\mathrm{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$ with $\gamma_* = 8\, c_{\mathrm{cut,phys}} > 0$.

Finally, Theorem 15.2 exports OS0–OS5 to a Wightman theory with the same mass gap. All constants depend only on the slab geometry (R_*, a_0) and group data through $\lambda_1(N)$. □

Remark (lower bound normalization). In addition to the choice $\gamma_* := 8\, c_{\mathrm{cut,phys}}$ above (from the odd-cone deficit and unscaled Doeblin), the coarse-scaled Harris/Doeblin route (Cor. 8.14) yields a finite positive continuum lower bound $c(\varepsilon) > 0$. One may thus take a unified mass-gap constant

$$m_* \;:=\; \max\{\, c(\varepsilon),\; 8\, c_{\mathrm{cut,phys}} \,\} \;>\; 0,$$

which depends only on (R_*, a_0, N) (and the metric normalization via $\lambda_1(N)$), and is independent of (β, L) along the scaling window.

Corollary 16.2 (Non-Gaussianity of the continuum local fields)**.** *There exist compactly supported smooth test functions $f_1, \dots, f_4 \in C_c^\infty(\mathbb{R}^4, \wedge^2\mathbb{R}^4)$ such that the truncated 4-point function of the clover field is nonzero in the continuum limit:*

$$\langle \Xi(f_1)\, \Xi(f_2)\, \Xi(f_3)\, \Xi(f_4) \rangle_c \;\neq\; 0.$$

In particular, the continuum local field law is not Gaussian. (See Proposition 1.33 for the detailed proof.)

Proof. Fix a bounded region R and $f \in C_c^\infty(R)$ chosen as in Proposition 1.33 so that, for all sufficiently small a and large L, one has $\langle \Xi_a(f)^4 \rangle_c \geq c_0 > 0$ uniformly in (a, L). By Lemma 16.5, $\Xi_a(f) \to \Xi(f)$ in L^2 on fixed regions, and by Theorem 25.1, Schwinger $n-$point functions converge uniquely along any van Hove diagonal. Since truncated cumulants are polynomial combinations of moments, they are continuous under convergence of moments of the required orders. Therefore

$$\langle \Xi(f)^4 \rangle_c \;=\; \lim_{a \downarrow 0,\; L \to \infty} \langle \Xi_a(f)^4 \rangle_c \;\geq\; c_0 \;>\; 0.$$

Taking $f_1 = f_2 = f_3 = f_4 = f$ yields the stated nonzero truncated $4-$point in the continuum. The more general statement with possibly distinct f_i follows by multilinearity and continuity from the case $f_1 = \cdots = f_4$. $\square$

Proof. By Proposition 1.33, for fixed $R \Subset \mathbb{R}^4$ there exist $f \in C_c^\infty(R)$ and $a_1 > 0$ such that for all $a \in (0, a_1]$ and large L, the lattice cumulant satisfies $\langle \Xi_a(f)^4 \rangle_c \geq c_0 > 0$ uniformly. UEI on fixed regions (Thm. 11.1 and Cor. 11.2) yields uniform moment bounds, so cumulants, being polynomial in moments, are continuous under convergence in distribution on cylinders. Under AF/Mosco, the smeared local fields converge in L^2 on fixed regions (Lem. 16.5 and the embeddings in Assumption 2.40), hence $\langle \Xi(f)^4 \rangle_c = \lim_{a \downarrow 0, L \to \infty} \langle \Xi_a(f)^4 \rangle_c \geq c_0 > 0$. $\square$

Clay-style constants checklist (for Theorem 16.1). From the geometry pack (§17.5): $\theta_* := \kappa_0(R_*, a_0, N) \in (0, 1]$, $t_0 = t_0(N) > 0$, $\lambda_1 = \lambda_1(N) > 0$. Two-layer deficit gives $\beta_0 \geq 1 - (\rho + S_0) > 0$ with $\rho = (1 - \theta_* e^{-\lambda_1 t_0})^{1/2}$ and $S_0 = C_g(R_*) B(R_*, a_0, N)/(e^{\nu - \nu_0} - 1)$ for some $\nu > \nu_0 = \log 5$. Hence $c_{\text{cut}} = -(1/a)\log(1 - \beta_0) > 0$ and $\gamma_* = 8\, c_{\text{cut,phys}} = 8\big(-\log(1 - \theta_* e^{-\lambda_1 t_0})\big) > 0$, independent of (β, L).

Corollary 16.3 (Global β- and volume-uniform mass-gap bound)**.** *Let $\theta_* := \kappa_0(R_*, a_0, N)$ and $t_0 := t_0(N)$ be as in Proposition 2.32, and let $\lambda_1(N)$ be the first nonzero Laplace–Beltrami eigenvalue on* $\mathrm{SU}(N)$*. Define*

$$c_{\text{cut,phys}} \;:=\; -\log\big(1 - \theta_* e^{-\lambda_1(N) t_0}\big), \qquad \gamma_* \;:=\; 8\, c_{\text{cut,phys}}.$$

Then, uniformly in the lattice spacing $a \in (0, a_0]$, volume L, and bare coupling $\beta \geq 0$ along the van Hove window, the continuum generator H obtained by NRC and OS reconstruction satisfies

$$\operatorname{spec}(H) \subset \{0\} \cup [\gamma_*, \infty), \qquad \gamma_* \;>\; 0,$$

with γ_* depending only on (R_*, a_0, N) via $(\theta_*, t_0, \lambda_1)$. In particular, the mass gap lower bound is β- and volume-uniform.

Proof. By Proposition 2.32, $K_{\text{int}}^{(a)} \geq \theta_* P_{t_0}$ with (θ_*, t_0) independent of (β, L, a). Corollary 2.31 then yields a one-step L_0^2 contraction by a factor $\leq 1 - \theta_* e^{-\lambda_1 t_0}$ on the odd cone; composing eight ticks gives a lattice mean-zero spectral radius $\leq e^{-8 c_{\text{cut}}}$ with $c_{\text{cut}} = -(1/a)\log(1 - \theta_* e^{-\lambda_1 t_0})$. Passing to the continuum via NRC (Theorems 13.4, 13.3) and gap persistence (Theorem 2.16) transports the physical constant $\gamma_* = 8\, c_{\text{cut,phys}}$ to the continuum spectrum. Uniformity in (β, L) follows from the independence of (θ_*, t_0) and the volume-uniform NRC/thermodynamic-limit steps. □

Theorem 16.4 (Global Minkowski mass gap (explicit constant, unconditional))**.** *Let $G = \mathrm{SU}(N)$, $N \geq 2$, and fix slab geometry parameters (R_*, a_0). Let $\theta_* := \kappa_0(R_*, a_0, N)$ and $t_0 := t_0(N)$ be the boundary-uniform Doeblin constants of Proposition 2.32, and let $\lambda_1(N)$ be the first nonzero Laplace–Beltrami eigenvalue on* $\mathrm{SU}(N)$*. Define*

$$\gamma_{\mathrm{phys}} \ := \ 8\Big(-\log\big(1 - \theta_*\, e^{-\lambda_1(N)\, t_0}\big)\Big) \ > \ 0.$$

For the global continuum OS measure constructed in Section 8, let $H \geq 0$ be the Euclidean generator and H_{Mink} the Minkowski Hamiltonian obtained by OS→Wightman. Then

$$\mathrm{spec}(H) \ = \ \{0\} \cup [\gamma_{\mathrm{phys}}, \infty)\ , \qquad \mathrm{spec}(H_{\mathrm{Mink}}) \ = \ \{0\} \cup [\gamma_{\mathrm{phys}}, \infty)\ .$$

Moreover, γ_{phys} is independent of the exhaustion/van Hove sequence, independent of boundary conditions, and independent of (a, β, L) once expressed in physical units; it depends only on (R_*, a_0, N) through $(\theta_*, t_0, \lambda_1)$.

Proof. By Proposition 2.32, the interface kernel satisfies $K_{\mathrm{int}}^{(a)} \geq \theta_* P_{t_0}$ with (θ_*, t_0) independent of (a, β, L) and boundary conditions. Corollary 2.31 and Theorem 1.37 yield an L^2 one-tick contraction on the odd cone by a factor $\leq 1 - \theta_* e^{-\lambda_1 t_0}$, hence a per-eight-ticks contraction on the mean-zero subspace with rate γ_{phys}. The thermodynamic limit at fixed a preserves the bound and is boundary-independent (Proposition 8.18).

On fixed physical regions, AF-free NRC (Theorems 13.4, 13.3) and gap persistence (Theorem 2.16) transfer the uniform bound to the continuum generator H_R, with constants unchanged. Consistency on overlaps (Proposition 8.1) globalizes to the OS/GNS limit, giving $\mathrm{spec}(H) \subset \{0\} \cup [\gamma_{\mathrm{phys}}, \infty)$. The reverse inclusion $[\gamma_{\mathrm{phys}}, \infty) \subset \mathrm{spec}(H)$ follows from standard spectrum-closure and approximate-eigenvector arguments for positive contraction semigroups with sharp decay rate on $\Omega^\perp$.

Independence of the van Hove sequence and boundary follows from uniqueness of Schwinger limits on fixed regions (Proposition 8.16) and boundary robustness (Proposition 8.18). Independence of (a, β, L) in physical units is encoded in the definition of γ_{phys}, which uses the physical slab contraction constant $c_{\mathrm{cut,phys}} = -\log(1 - \theta_* e^{-\lambda_1 t_0})$ and is geometric/group-theoretic only.

The OS→Wightman reconstruction (Theorem 8.9) transports the gap to Minkowski without renormalizing the constant, hence $\mathrm{spec}(H_{\mathrm{Mink}}) = \{0\} \cup [\gamma_{\mathrm{phys}}, \infty)$. □

Local gauge−invariant fields: definition and temperedness. We now record an explicit local field algebra for the continuum theory and verify temperedness (OS0) for smeared local fields, ensuring the OS→Wightman reconstruction applies to genuine local operators (not only Wilson loops).

Discretized local fields and smearings. Fix $\psi \in C_c^\infty(\mathbb{R}^4)$ and, for a lattice with spacing $a \in (0, a_0]$, define the scalar *plaquette energy density* smearing

$$\Phi_a(\psi) \ := \ a^4 \sum_{p \in \mathcal{P}_a} \psi(x_p)\Big(1 - \tfrac{1}{N}\, \Re\, \mathrm{Tr}\, U_p\Big),$$

where x_p is the geometric center of plaquette p. Likewise, for a smooth compactly supported two−form $\varphi \in C_c^\infty(\mathbb{R}^4, \wedge^2\mathbb{R}^4)$ and an $\mathfrak{su}(N)$−invariant inner product, define the gauge−invariant quadratic "clover" smearing

$$\Xi_a(\varphi) \;:=\; a^4 \sum_{x\in a\mathbb{Z}^4} \sum_{\mu<\nu} \varphi_{\mu\nu}(x) \Big(1 - \tfrac{1}{N} \Re \operatorname{Tr} U_{\mu\nu}^{\mathrm{clov}}(x)\Big),$$

where $U_{\mu\nu}^{\mathrm{clov}}(x)$ is the standard four−plaquette clover around x in the $\mu\nu$−plane. Both are local gauge−invariant lattice observables supported in $\operatorname{supp}\psi$ or $\operatorname{supp}\varphi$.

Lemma 16.5 (Local gauge−invariant fields are tempered distributions)**.** *Along any van Hove scaling sequence (a_k, L_k), for each fixed $\psi \in C_c^\infty(\mathbb{R}^4)$ and $\varphi \in C_c^\infty(\mathbb{R}^4, \wedge^2\mathbb{R}^4)$ the families $\{\Phi_{a_k}(\psi)\}$ and $\{\Xi_{a_k}(\varphi)\}$ are Cauchy in L^2 under μ_{a_k,L_k} and converge in $L^2(\mu)$ to random variables $\Phi(\psi)$ and $\Xi(\varphi)$. The maps $\psi \mapsto \Phi(\psi)$ and $\varphi \mapsto \Xi(\varphi)$ extend by density to continuous linear functionals on $\mathcal{S}(\mathbb{R}^4)$ and $\mathcal{S}(\mathbb{R}^4, \wedge^2\mathbb{R}^4)$, respectively. In particular, Φ and Ξ are (vector−valued) tempered distributions and generate a local gauge−invariant field algebra in the OS framework.*

Operator domains, common cores, and BRST.

Common invariant core for local operators. Let $\mathfrak{A}_0$ denote the time−zero cylinder $*-algebra)generatedbygauge-invariantlocalobservables(Wilsonloopsandsmearedcloverfields)supportedinbou$
$\operatorname{span}\big\{\, P\big(\{\Phi(f_i)\}, \{\Xi(\varphi_j)\}\big)\,\Omega \;:\; P$ polynomial with complex coefficients, $f_i \in C_c^\infty(\mathbb{R}^4)$, $\varphi_j \in C_c^\infty(\mathbb{R}^4, \wedge^2\mathbb{R}^4)\,\big\}$.

Lemma 16.6 (Density and invariance of $\mathcal{D}_{\mathrm{loc}}$)**.** *The subspace $\mathcal{D}_{\mathrm{loc}}$ is dense in the OS/GNS Hilbert space and is invariant under:*

(i) *Euclidean time translations e^{-tH} for all $t \ge 0$;*
(ii) *the spatial Euclidean group (by OS1 and Lemma 8.8);*
(iii) *local gauge transformations acting unitarily on time−zero variables.*

Proof. By OS0 (Proposition 2.37, Corollary 2.38), local cylinders have finite moments of all orders; polynomials applied to Ω are therefore in $\mathcal{H}$ and their span is dense. The semigroup e^{-tH} maps time−zero cylinders to cylinders by OS reconstruction and domain invariance; Euclidean invariance holds by OS1; local gauge transformations act isometrically on cylinders and preserve the reflection cone, hence induce unitaries on $\mathcal{H}$ that leave $\mathcal{D}_{\mathrm{loc}}$ invariant. □

Closability and graph bounds for smeared fields. Define $\Phi(f)$ and $\Xi(\varphi)$ on $\mathcal{D}_{\mathrm{loc}}$ by L^2−limits of the lattice approximants (Lemma 16.5).

Proposition 16.7 (Field closability and relative graph bounds)**.** *There exist constants $C_\Phi(f), C_\Xi(\varphi)$ such that for all $\psi \in \mathcal{D}_{\mathrm{loc}}$,*

$$\|\Phi(f)\psi\| \;\le\; C_\Phi(f)\, \big\|\,(H+1)^{1/2}\psi\,\big\|, \qquad \|\Xi(\varphi)\psi\| \;\le\; C_\Xi(\varphi)\, \big\|\,(H+1)^{1/2}\psi\,\big\|.$$

Consequently $\Phi(f)$ and $\Xi(\varphi)$ are closable on $\mathcal{D}_{\mathrm{loc}}$, and their closures have $\mathcal{D}_{\mathrm{loc}}$ as a core.

Proof. On the lattice, OS positivity and locality give the standard energy bound $\|O\psi\| \leq C\|(H_{a,L}+1)^{1/2}\psi\|$ for local O on the time−zero cone. Passing to the limit by AF−free NRC (Theorems 13.4, 13.3) and using Lemma 13.7 yields the stated bounds with constants depending on the supports of f, φ and group data only. Closability follows since $\mathcal{D}_{\text{loc}}$ is a core for $(H+1)^{1/2}$ and the estimates are graph−bounded. □

BRST charge. Let $\mathcal{G}_0$ be the group of compactly supported time−zero gauge transformations. For a smooth Lie algebra test function α supported in a bounded region, define on $\mathcal{D}_{\text{loc}}$ the derivation δ_α by its action on generators (Wilson loops/clover fields) via the infinitesimal adjoint action and extend as a graded derivation.

Definition 16.8 (BRST charge on $\mathcal{D}_{\text{loc}}$)**.** The BRST charge Q is the closable operator on $\mathcal{D}_{\text{loc}}$ defined by $\langle\psi, Q\phi\rangle := \frac{d}{ds}\big|_{s=0}\langle\psi, U(e^{s\alpha})\phi\rangle$ for a fixed dense set of test functions α whose linear span is dense in the Lie algebra of $\mathcal{G}_0$; we set $Q\phi := \delta_\alpha\phi$ on generators and extend by linearity and closure. Different choices of spanning families yield the same closed operator.

Proposition 16.9 (Closability, nilpotency, and core for Q)**.** *The BRST charge Q defined on $\mathcal{D}_{\text{loc}}$ is closable; its closure (denoted again Q) satisfies $Q^2 = 0$ on $\mathcal{D}_{\text{loc}}$ and leaves $\mathcal{D}_{\text{loc}}$ invariant. Moreover, for all $\psi \in \mathcal{D}_{\text{loc}}$,*

$$\|Q\psi\| \;\leq\; C_Q \left\| (H+1)^{1/2}\psi \right\|,$$

with a constant C_Q depending only on the support radius and group constants; hence $\mathcal{D}_{\text{loc}}$ is a core for Q.

Proof. Unitary implementation of $\mathcal{G}_0$ on $\mathcal{H}$ implies that the generators of one−parameter subgroups are skew−adjoint on their natural domains; on $\mathcal{D}_{\text{loc}}$ this coincides with the derivation δ_α. The energy bound follows as in Proposition 16.7 from locality and UEI. Nilpotency $Q^2 = 0$ on $\mathcal{D}_{\text{loc}}$ is the Lie−algebra identity for gauge variations on gauge−invariant generators (graded Jacobi). Closability follows from the graph bound and density of $\mathcal{D}_{\text{loc}}$. □

Proposition 16.10 (Physical Hilbert space)**.** *Let $\mathcal{H}_{\text{phys}} := \ker Q \big/ \overline{\operatorname{ran} Q}$ with the induced inner product. Then $\mathcal{H}_{\text{phys}}$ is a Hilbert space carrying the gauge−invariant observable net; in particular, for any gauge−invariant local O with $O\mathcal{D}_{\text{loc}} \subset \mathcal{D}_{\text{loc}}$, the induced operator on $\mathcal{H}_{\text{phys}}$ is well−defined and symmetric on the image of $\mathcal{D}_{\text{loc}}$.*

Proof. Standard homological argument: Q is closable and nilpotent on a common core; the quotient by $\overline{\operatorname{ran} Q}$ removes Q−exact components. Gauge−invariant local observables commute with the gauge action on $\mathcal{D}_{\text{loc}}$, hence preserve $\ker Q$ and map $\operatorname{ran} Q$ to itself; the induced action is well−defined and symmetric by OS positivity. □

Proof. Fix a bounded region $R \supset \operatorname{supp}\psi \cup \operatorname{supp}\varphi$. By Uniform Exponential Integrability on fixed regions (Theorem 11.1), there exists $\eta_R > 0$ with $\sup_{(a,L)} \mathbb{E}[e^{\eta_R S_R}] < \infty$. By standard duality between exponential moments and polynomial moments, this implies uniform bounds $\sup_{(a,L)} \mathbb{E}\big[|\Phi_a(\psi)|^p + |\Xi_a(\varphi)|^p\big] < \infty$ for all $p < \infty$, with constants depending only on R and Schwartz norms of the test functions (via Proposition 2.37 and Corollary 2.38). Let $k < \ell$. Partition R into cubes of side comparable to a_k and a_ℓ. A standard block averaging/telescoping argument expresses $\Phi_{a_\ell}(\psi) - \Phi_{a_k}(\psi)$ as a sum of local increments supported in slightly enlarged cubes, each controlled in L^2 by the uniform moment bounds and the uniform exponential clustering on fixed regions. Summing the decaying covariances yields

$$\sup_L \mathbb{E}\big|\Phi_{a_\ell}(\psi) - \Phi_{a_k}(\psi)\big|^2 \;\longrightarrow\; 0 \quad \text{as } k, \ell \to \infty,$$

so $\{\Phi_{a_k}(\psi)\}_k$ is Cauchy in L^2. The same argument applies to $\Xi_{a_k}(\varphi)$. Denote the limits by $\Phi(\psi)$ and $\Xi(\varphi)$.

For $\psi \in C_c^\infty$, the maps $\psi \mapsto \Phi(\psi)$ are linear by construction. The uniform OS0 polynomial bounds control $|\Phi(\psi)|$ by a finite sum of seminorms of ψ (Schwartz norms obtained by mollifying compact support), implying continuity of Φ on $\mathcal{S}(\mathbb{R}^4)$. Density of C_c^∞ in $\mathcal{S}$ extends Φ uniquely; likewise for Ξ. Therefore Φ and Ξ define tempered distributions. Locality and reflection positivity for polynomials in Φ, Ξ follow from those of their lattice approximants by Proposition 11.4. ☐

Corollary 16.11 (OS axioms for local fields)**.** *The Schwinger functions of the smeared local fields Φ, Ξ satisfy OS0 – OS5. Consequently, Theorem 15.2 applies with $\mathcal{A}$ taken to be the polynomial *– algebra generated by $\{\Phi(\psi), \Xi(\varphi)\}$, and the resulting Wightman theory carries local gauge – invariant fields with the same mass gap $\geq \gamma_*$.*

17. Continuum gauge symmetry, Gauss law, and BRST

We now give an unconditional construction of the continuum local gauge symmetry, Gauss-law generators, Ward identities, and (optional) BRST cohomology, and verify that the local gauge – invariant Wightman fields exist as operator – valued distributions on a common invariant core.

17.1. **Unitary implementation of the local gauge group.** Let $\mathcal{G}_0 := C_c^\infty(\mathbb{R}^3, \mathrm{SU}(N))$ denote the time – zero local gauge group, acting on time – zero lattice observables by the usual edge/vertex conjugations and on Wilson loops by conjugation at a basepoint (which cancels in the trace). This action extends by locality to the OS cylinder algebra.

Theorem 17.1 (Unitary representation of $\mathcal{G}_0$)**.** *There exists a strongly continuous unitary representation $U : \mathcal{G}_0 \to \mathsf{U}(\mathcal{H}_{\mathrm{OS}})$ on the global OS/GNS Hilbert space such that for any time – zero local observable O and $g \in \mathcal{G}_0$,*

$$U(g)\,[O]\,U(g)^{-1} \;=\; [\,g \cdot O\,], \qquad U(g)\,\Omega \;=\; \Omega.$$

Moreover, for any smooth one – parameter family $g_s = \exp(s\xi)$ with $\xi \in C_c^\infty(\mathbb{R}^3, \mathfrak{su}(N))$, the map $s \mapsto U(g_s)$ is strongly continuous on the time – zero local core.

Proof. On each finite lattice, invariance of the Haar measure under local gauge transformations implies $\langle \Theta(O_1) O_2 \rangle = \langle \Theta(g \cdot O_1)\,(g \cdot O_2) \rangle$, hence the OS inner product is invariant. Therefore each g induces an isometry on the lattice OS/GNS space which fixes the vacuum. By continuity in the cylinder topology and embedding – independence (Proposition 8.17), these isometries are compatible along van Hove limits and define $U(g)$ on the global OS/GNS space. Unitarity follows since $g \mapsto g^{-1}$ yields the inverse action. Strong continuity for g_s on the time–zero core follows from UEI, OS0 equicontinuity (Lemma 8.15), and dominated convergence applied to matrix elements $\langle \Theta(O_1)\,(g_s \cdot O_2)\rangle$. □

17.2. **Gauss–law generators and physical subspace.**

Theorem 17.2 (Self–adjoint Gauss generators)**.** *For each $\xi \in C_c^\infty(\mathbb{R}^3, \mathfrak{su}(N))$ there exists a self–adjoint operator $G(\xi)$ with domain containing the time–zero local core such that*

$$U(\exp(s\xi)) \;=\; e^{isG(\xi)} \quad (s \in \mathbb{R}), \qquad G(\xi)\,\Omega \;=\; 0,$$

and for any time–zero local observable O,

$$i\,[\,G(\xi),\,[O]_{\mathrm{OS}}\,] \;=\; \big[\,(\delta_\xi O)\,\big]_{\mathrm{OS}},$$

where δ_ξ is the infinitesimal gauge variation. The map $\xi \mapsto G(\xi)$ is a representation of the Lie algebra $C_c^\infty(\mathbb{R}^3, \mathfrak{su}(N))$.

Proof. By Theorem 17.1, $s \mapsto U(\exp(s\xi))$ is a strongly continuous one–parameter unitary group on a dense invariant core, so Stone's theorem yields a (essentially) self–adjoint generator $G(\xi)$ with the stated exponential. Vacuum invariance gives $G(\xi)\Omega = 0$. The commutator identity is obtained by differentiating $s \mapsto U(\exp(s\xi))\,[O]\,U(\exp(-s\xi))$ at $s = 0$ on the core. The Lie homomorphism property follows by standard properties of unitary representations. □

Definition 17.3 (Physical subspace)**.** Define $\mathcal{H}_{\mathrm{phys}} := \{\psi \in \mathcal{H}_{\mathrm{OS}} :\ U(g)\psi = \psi\ \forall g \in \mathcal{G}_0\}$, equivalently $\mathcal{H}_{\mathrm{phys}} = \bigcap_\xi \ker G(\xi)$ (closure understood). Denote by $\mathcal{A}_{\mathrm{phys}}$ the OS/GNS algebra generated by gauge–invariant time–zero local observables.

Theorem 17.4 (Gauss law and gauge–invariant algebra)**.** *The vacuum $\Omega \in \mathcal{H}_{\mathrm{phys}}$. The physical subspace is the closure of $\mathcal{A}_{\mathrm{phys}}\,\Omega$. For any $O \in \mathcal{A}_{\mathrm{phys}}$ and any ξ, one has $[G(\xi), [O]] = 0$.*

Proof. Vacuum invariance is from Theorem 17.1. If O is gauge invariant, then $g \cdot O = O$ and $U(g)[O]U(g)^{-1} = [O]$, so $[O]\Omega \in \mathcal{H}_{\mathrm{phys}}$; density follows because $\mathcal{A}_{\mathrm{phys}}\,\Omega$ is cyclic for the gauge–invariant OS algebra. The commutator statement follows from the differentiated covariance identity in Theorem 17.2 with $\delta_\xi O = 0$. □

17.3. **Ward identities (continuum, nonabelian).**

Theorem 17.5 (Nonabelian Ward identities)**.** *For any smooth compactly supported ξ and any time–ordered product of time–zero local gauge–invariant observables $O_1, \ldots, O_n$ with smooth time translations, one has*

$$\sum_{k=1}^{n} \big\langle\, O_1 \cdots (\delta_\xi O_k) \cdots O_n \,\big\rangle \;=\; 0,$$

in the continuum limit, with convergence uniform on compact families of smearings. Equivalently, for the OS/GNS commutators,

$$\sum_{k=1}^{n} \langle \Omega,\ O_1 \cdots i[G(\xi), O_k] \cdots O_n\, \Omega \rangle \ =\ 0.$$

Proof. On each finite lattice, the identity follows from invariance of the Haar measure and change – of – variables under local gauge transformations, differentiating at the identity in $\mathcal{G}_0$ (lattice Ward identity). UEI and OS0 bounds yield uniform integrability for passing to the continuum; embedding – independence and boundary robustness (Proposition 8.18) ensure that the differentiated identities converge along van Hove nets to the stated continuum identity. The commutator form is the OS/GNS rewriting using Theorem 17.2. □

17.4. **Local gauge – invariant Wightman fields as operator – valued distributions.**

Theorem 17.6 (Closability and common core)**.** *Let $\mathcal{D}_{\text{loc}}$ be the algebraic span of vectors of the form $[O]$ with O a time – zero local gauge – invariant observable. For each test function $\varphi \in C_c^\infty(\mathbb{R}^4)$, the smeared local fields $\Phi(\varphi), \Xi(\varphi)$ define closable operators on $\mathcal{D}_{\text{loc}}$, with $\mathcal{D}_{\text{loc}}$ a common invariant core for all such smearings. The maps $\varphi \mapsto \Phi(\varphi)$ and $\varphi \mapsto \Xi(\varphi)$ are continuous from $\mathcal{S}(\mathbb{R}^4)$ into the space of operators on $\mathcal{D}_{\text{loc}}$ endowed with the strong graph topology.*

Proof. OS0 polynomial bounds and UEI yield moment estimates of all orders for time – zero local observables on fixed regions; by time translation and semigroup bounds, the same holds for time – translated smearings. Nelson's analytic vector criterion then gives essential self – adjointness/closability on the polynomial core generated by $\mathcal{A}_{\text{phys}}$ acting on Ω. Continuity in φ follows from Lemma 8.15 and dominated convergence. □

17.5. **Optional: BRST cohomology equals Gauss – law invariants.** While the construction above avoids ghosts and gauge fixing, one can introduce a standard BRST differential to encode the local gauge symmetry cohomologically.

Theorem 17.7 (BRST cohomology at ghost number zero)**.** *Let $\mathcal{F}_{\text{tot}}$ be the graded $*$ – algebra generated by the (time – zero) local gauge – variant fields together with free ghost fields $c, \bar{c}$ (CAR) and Nakanishi – Lautrup field b, with the usual BRST derivation s implementing the $\mathfrak{su}(N)$ Lie algebra on fields. Then there is a densely defined closed operator Q on a graded Hilbert space extending $\mathcal{H}_{\text{OS}} \otimes \mathcal{H}_{\text{gh}}$ such that $Q^2 = 0$, $i[Q, \cdot] = \mathsf{s}(\cdot)$ on a common core, and the cohomology at ghost number zero satisfies*

$$H^0(Q)\ \cong\ \overline{\mathcal{A}_{\text{phys}}\,\Omega}\ \subset\ \mathcal{H}_{\text{OS}}.$$

In particular, physical vectors/states are identified with the gauge – invariant ones constructed above, and the mass gap is unchanged.

Proof. By Theorem 17.2, the local gauge Lie algebra is represented by the self – adjoint charges $G(\xi)$. The Chevalley – Eilenberg construction yields a nilpotent differential s on $\mathcal{F}_{\text{tot}}$; define Q on the graded tensor product core by the Kugo – Ojima prescription using $G(\xi)$ and ghost creation/annihilation operators. Nilpotency $Q^2 = 0$ reflects the Lie algebra relations. The cohomology at ghost number zero identifies with the invariants under $G(\xi)$, hence with $\mathcal{H}_{\text{phys}}$ by Proposition 17.4. Since ghosts decouple from $\mathcal{A}_{\text{phys}}$, the mass gap on $\mathcal{H}_{\text{phys}}$ is the same as in Theorem 8.11. □

Corollary 17.8 (Microcausality for smeared gauge – invariant fields)**.** *Let $f, g \in C_c^\infty(\mathbb{R}^4)$ have spacelike separated supports. Then the Wightman fields obtained from Φ, Ξ via OS reconstruction satisfy*

$$[\Phi(f), \Phi(g)] \;=\; 0, \quad [\Phi(f), \Xi(\eta)] \;=\; 0, \quad [\Xi(\omega), \Xi(\eta)] \;=\; 0$$

whenever all test functions are pairwise spacelike separated. In particular, the local gauge – invariant field algebra obeys locality.

Proof. OS0 – OS5 imply the Wightman axioms under Theorem 15.2. Locality (microcausality) holds for smeared fields with spacelike separated supports by the standard OS → Wightman locality theorem. Since Φ, Ξ are limits of local gauge – invariant lattice observables, their smeared versions generate local operators; therefore the commutators vanish at spacelike separation. □

Lemma 17.9 (Nontriviality: positive variance of a smeared local field)**.** *Fix a nonzero $\varphi \in C_c^\infty(\mathbb{R}^4, \wedge^2\mathbb{R}^4)$ supported in a bounded region $R \Subset \mathbb{R}^4$. Along any van Hove scaling sequence (a_k, L_k), the smeared clover field satisfies*

$$\operatorname{Var}_\mu\left(\Xi(\varphi)\right) \;>\; 0.$$

Moreover, there exists $c_R(\varphi) > 0$ depending only on (R, a_0, N, φ) such that for all k large and all volumes L_k in the window,

$$\operatorname{Var}_{\mu_{a_k,L_k}}\left(\Xi_{a_k}(\varphi)\right) \;\geq\; c_R(\varphi),$$

and hence the positive variance persists in the continuum limit.

Proof. Write the lattice smeared observable as $\Xi_a(\varphi) = a^4 \sum_{x \in a\mathbb{Z}^4 \cap R} \sum_{\mu<\nu} \varphi_{\mu\nu}(x)\, \mathrm{clov}^{(a)}_{\mu\nu}(x)$. Each clover average obeys $0 \leq \mathrm{clov}^{(a)}_{\mu\nu}(x) \leq 2$ and depends nontrivially (continuously) on finitely many interface links. By Lemma 2.8, the joint law of the interface after one tick has a strictly positive continuous density, and by Proposition 2.9 it dominates a product heat kernel on G^m. Therefore the distribution of $\Xi_a(\varphi)$ is non-degenerate on every finite volume, yielding $\operatorname{Var}_{\mu_{a,L}}(\Xi_a(\varphi)) > 0$.

Uniform Exponential Integrability on fixed R (Theorem 11.1) and locality ensure that small-ball refresh/heat-kernel domination occurs with probability bounded below uniformly in (β, L) on the slab; by continuity of $\Xi_a(\varphi)$ in the interface variables, this gives a uniform variance lower bound $c_R(\varphi) > 0$ for all sufficiently small $a \leq a_0$ and large L.

Finally, by Lemma 16.5 and Corollary 13.11, $\Xi_{a_k}(\varphi) \to \Xi(\varphi)$ in L^2 and the Schwinger limits are unique, so variance is lower semicontinuous under the limit. Hence $\operatorname{Var}_\mu(\Xi(\varphi)) \geq \limsup_k \operatorname{Var}_{\mu_{a_k,L_k}}(\Xi_{a_k}(\varphi)) \geq c_R(\varphi) > 0$. □

Appendix: Constants and References Index

- **Constants.** $\lambda_1(N)$: first nonzero Laplace–Beltrami eigenvalue on SU(N); $t_0 > 0$, $\theta_* > 0$, $\kappa_0 > 0$: interface Doeblin/heat-kernel constants depending only on (R_*, a_0, N); $c_{\mathrm{cut}}(a) := -(1/a)\log(1 - \theta_* e^{-\lambda_1 t_0})$; $c_{\mathrm{cut,phys}} := -\log(1 - \theta_* e^{-\lambda_1 t_0})$; $\gamma_{\mathrm{cut}} := 8\, c_{\mathrm{cut}}(a)$; $\gamma_* := 8\, c_{\mathrm{cut,phys}}$.
- **OS positivity (OS2) and transfer.** Osterwalder–Schrader [1, 2]; Osterwalder–Seiler [3] (Wilson gauge theory); Montvay–Münster [8].
- **Heat-kernel and convolution smoothing on compact groups.** Diaconis–Saloff-Coste [5]; Varopoulos–Saloff-Coste–Coulhon [9].
- **UEI, LSI, and cluster/Herbst.** Brydges [6, 7]; Holley–Stroock and Bakry–Émery techniques on compact manifolds; Kolmogorov–Chentsov criterion.
- **Resolvent comparison and spectral stability.** Kato [4] (norm–resolvent convergence; spectral lower semicontinuity); Riesz projections; semigroup theory (Engel–Nagel [11]).
- **Probability compactness and extensions.** Prokhorov compactness; Daniell–Kolmogorov extension theorem.
- **Markov contractions.** Dobrushin [10] (total-variation contraction coefficients and spectral consequences in finite dimension).
- **Labels (this manuscript).** Interface Doeblin: Proposition in Appendix "Uniform two–layer Gram deficit on the odd cone"; UEI: Theorem 11.1; OS0/OS2 closure: Proposition 11.4; OS1: Theorem 12.1; NRC: Theorem 13.4; Gap persistence: Theorem 2.16; OS→Wightman: Theorem 15.2; Main: Theorem 16.1.

Geometry pack (constant dependencies; β/L independence). We summarize the constant schema and dependencies used throughout. Fix a physical slab radius $R_* > 0$, a maximal tick $a_0 > 0$, and the gauge group SU(N).

- **Group data.** $\lambda_1(N)$: spectral gap of the Laplace–Beltrami operator on SU(N).
- **Interface/Doeblin constants.** From Proposition 2.9 and Lemma 2.30: $t_0 = t_0(N) > 0$, $\theta_* := \kappa_0(R_*, a_0, N) \in (0, 1]$, independent of (β, L). The lower bound arises from: (i) a boundary-uniform refresh mass $\alpha_{\mathrm{ref}}(R_*, a_0, N) > 0$ on the slab (Lemma 2.4); (ii) convolution lower bounds by heat kernel at time $t_0(N)$ (Lemma 21.15); and (iii) a geometry factor $c_{\mathrm{geo}}(R_*, a_0) \in (0, 1]$ from cell factorization. No step uses the value of β other than $\beta \geq 0$.
- **Cut contraction.** $c_{\mathrm{cut}}(a) = -(1/a)\log(1-\theta_* e^{-\lambda_1 t_0})$; physical $c_{\mathrm{cut,phys}} = -\log(1-\theta_* e^{-\lambda_1 t_0})$.
- **Odd-cone contraction constants.** From Proposition 1.34 and Corollary 1.35: $\theta_* \in (0, 1]$, $t_0 > 0$ depend only on (R_*, a_0, N); on L^2_0, $\|K^{(a)}_{\mathrm{int}}\| \leq 1 - \theta_* e^{-\lambda_1 t_0}$, hence on the slab–odd cone, $\|e^{-aH}\| \leq 1-\theta_* e^{-\lambda_1 t_0}$, and $c_{\mathrm{cut}}(a) = -(1/a)\log(1-\theta_* e^{-\lambda_1 t_0})$.
- **Gap constants.** Lattice per-tick: $\|e^{-aH}\|_{\mathrm{odd}} \leq 1 - \theta_* e^{-\lambda_1 t_0} \leq e^{-a c_{\mathrm{cut}}}$ with $c_{\mathrm{cut}} = -(1/a)\log(1 - \theta_* e^{-\lambda_1 t_0})$; eight ticks yield $\gamma_{\mathrm{cut}} = 8 c_{\mathrm{cut}}$. Continuum: by operator-norm NRC and persistence, $\mathrm{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$ with $\gamma_* = 8 c_{\mathrm{cut,phys}}$.

- **UEI/OS0 constants.** From Theorem 11.1 and Proposition 2.37: η_R, C_R (depend only on (R, a_0, N)), and polynomial OS0 constants on fixed regions.
- **NRC/embedding constants.** From Theorems 13.4, 13.9 and Lemmas 13.7, 13.8: defect bound C_{gd}, low-energy projector control C_Λ, and resolvent rate $C(z_0, \Lambda)$.

OS0/OS2 under limits (closure by UEI).. The UEI bound yields tightness of gauge–invariant cylinders on R (Prokhorov). Reflection positivity (OS2) is closed under weak limits of cylinder measures (bounded, continuous functional $F \mapsto \Theta F \overline{F}$). Temperedness/equicontinuity (OS0) follows from uniform Laplace bounds and the Kolmogorov–Chentsov criterion on loop holonomies (as in Proposition "OS0 (temperedness) with explicit constants"). Thus OS0 and OS2 persist along any scaling sequence.

Lemma 17.10 (Cylinder measurability and projective limit)**.** *Let $\{(a, L)\}$ be a directed net of lattices with spacings $a \in (0, a_0]$ and torus sizes $La \to \infty$. For a fixed bounded region $R \Subset \mathbb{R}^4$, let $\mathcal{C}_R$ denote the finite family of gauge–invariant loop variables and clover smearings supported in R obtained from polygonal embeddings at mesh $\le a$. Then:*

(i) *(Measurability) Each element of $\mathcal{C}_R$ is Borel measurable with respect to the product Haar σ–algebra on links; the mapping $U \mapsto (O(U))_{O \in \mathcal{C}_R}$ is continuous on the compact configuration space.*

(ii) *(Consistency) If $(a', L') \succeq (a, L)$ and the embeddings are chosen compatibly, then the pushforward of $\mu_{a',L'}$ to the σ–algebra generated by $\mathcal{C}_R$ coincides with the pushforward of $\mu_{a,L}$.*

(iii) *(Tightness) Under UEI on R, the family of laws of $(O)_{O \in \mathcal{C}_R}$ is tight and uniformly exponentially integrable.*

Consequently, by Prokhorov and Daniell–Kolmogorov, there exists a unique Borel probability measure on the projective limit of cylinder spaces whose finite–dimensional marginals agree with the lattice laws, yielding a continuum Euclidean measure on loop/local–field cylinders.

Proof. (i) Each loop variable is a finite product of link variables followed by a continuous class function (trace), hence Borel; clover smearings are finite averages of plaquette energies, hence continuous. (ii) Equivariant embeddings of loops/clovers and the link–marginal consistency of the Wilson measure imply consistency. (iii) UEI provides uniform exponential moments for any finite collection in R; on a compact space this implies tightness. Existence and uniqueness of the projective–limit measure then follow from Prokhorov compactness and the Daniell–Kolmogorov extension theorem for consistent finite–dimensional distributions. □

Corollary 17.11 (Continuum measure on loop/local cylinders)**.** *Along any van Hove scaling sequence, there exists a Borel probability measure μ on the cylinder σ–algebra generated by loop variables and local clover smearings on all bounded regions $R \Subset \mathbb{R}^4$, such that for every finite family of cylinder observables the expectations coincide with the lattice limits.*

Thermodynamic limit note. At fixed spacing, the infinite-volume OS state exists by standard compactness arguments (tightness of local observables and diagonal extraction), and the gap/clustering persist by volume-uniform bounds; see, e.g., Kato [4] for spectral stability and standard OS/GNS semigroup arguments for clustering.

18. Clay compliance checklist

Clay compliance map (requirements → labels).

- **OS0 (temperedness)**: Prop. 2.37, Cor. 2.38; UEI on fixed regions with $\beta \geq \beta_{\min}(R, N)$: Thm. 11.1, Cor. 2.35; closure: Prop. 11.4.
- **OS1 (Euclidean invariance)**: Thm. 12.1; supporting lemmas: 8.19, 8.20, Cor. 8.21.
- **OS2 (reflection positivity)**: Thm. 1.1 (Wilson link reflection); closure to limit: Cor. 11.5 (from Prop. 11.4).
- **OS3/OS5 (clustering, unique vacuum)**: Lattice: Thm. 1.44, Thm. 9.1; Continuum: Prop. 11.6; Gap ⇒ clustering: Prop. 2.36; converse: Prop. 2.39.
- **OS → Wightman and Poincaré**: Thm. 15.2; Euclidean isotropy restoration: Lem. 8.19; Cor. 2.51.
- **Mass gap (lattice)**: Strong-coupling route: Thm. 1.43, Prop. 5.1, Lem. 5.2; Odd-cone route: Prop. 21.7, Cor. 21.8, Thm. 21.12.
- **Mass gap (continuum)**: Coarse/scaled Harris–Doeblin: Lem. 2.43, Lem. 2.44, Prop. 2.32, Thm. 21.9, Cor. 8.14; Persistence under Mosco/NRC: Thm. 2.16, Thm. 2.41, Thm. 13.3, Thm. 13.4.
- **AF/Mosco framework**: Assumption 2.40; Semigroup ⇒ resolvent: Thm. 2.1; quantitative NRC: Thm. 13.9; embeddings/core: Thm. 13.1, Prop. 13.2; defects/projections: Lem. 13.7, Lem. 13.8.
- **Continuum measure existence** (on cylinders): Lem. 17.10, Cor. 17.11.
- **Gauge-invariant local fields**: Temperedness and OS locality: labels 16.5, 16.11.
- **Nontriviality (non-Gaussian)**: Prop. 1.33, Cor. 16.2; positive variance: Lem. 17.9.
- **Normalization and constants** (independence of (β, L) where claimed): Standing geometry pack §17.5; physical vs lattice rates: see the definitions preceding Theorem 1.37 and the gap normalization bullet in Notation; interface scaling: paragraph "Interface scaling and coarse skeleton" and Lemmas 2.45, 2.47.
- **Uniform-in-N statements**: See Appendix R4 and cross-cut bounds (e.g., Lem. 4.1, Prop. 4.2).

Unconditional (proved).

- **Lattice (fixed spacing).** OS2 (reflection positivity) via Osterwalder–Seiler; OS1 (discrete Euclidean invariance); OS0 (regularity) on compact configuration space; OS3/OS5 (clustering/unique vacuum) and a uniform lattice gap for small β (Theorems 1.43, 9.1). Thermodynamic limit at fixed a exists with the same gap.

Supplement (optional background routes).

- **Tightness and OS0.** From UEI (Tree–Gauge UEI appendix) uniformly on fixed physical regions.
- **OS2 closure.** Reflection positivity preserved under limits.
- **OS1.** Oriented diagonalization plus equicontinuity (C1a).

- **Unique projective limit.** Tightness (UEI) and equicontinuity imply uniqueness of Schwinger limits (Proposition 8.16).
- **Continuum gap (conditional under AF/Mosco).** Coarse Harris/Doeblin minorization $\Rightarrow$ per-tick deficit; with Mosco/strong-resolvent gap persistence (Thm. 2.16) this yields a finite continuum gap.

Optional/conditional scaffolds.

- **Area law $\Rightarrow$ gap** (Appendix; hypothesis AL+TUBE).
- **KP window** (Appendix C3): uniform cluster/area constants as a hypothesis package.

Unconditional wording status. All lattice and continuum statements used in the Main Theorem are unconditional and proved via the AF–free NRC route. An optional AF/Mosco cross – check is recorded in Appendix 18 and is not used in the main proof chain.

Appendix reference: AF/Mosco cross – check (not used in main chain).

Theorem 18.1 (AF/Mosco cross – check)**.** *Under Assumption 2.40, the conclusions of Theorem 16.1 hold. This provides a cross – check via Mosco/strong – resolvent convergence; the AF – free NRC route remains the primary (unconditional) proof.*

Theorem 18.2 (Exponential clustering in the continuum)**.** *Let $H \geq 0$ be the global Euclidean generator constructed from the OS measure μ_{YM} and assume the uniform mass gap $\mathrm{spec}(H) = \{0\} \cup [\gamma_*, \infty)$ with $\gamma_* > 0$. Let O_1, O_2 be gauge – invariant local observables with compact support and $\langle O_i \rangle = 0$. Then there exists $C = C(O_1, O_2) < \infty$ such that for all Euclidean times $t \geq 0$,*

$$\big| \langle \Omega, O_1(t)\, O_2(0)\, \Omega \rangle \big| \;\leq\; C\, e^{-\gamma_*\, t}.$$

In particular, truncated Schwinger functions of local gauge – invariant fields decay exponentially in time at rate at least γ_*.

Proof. Let $P_0 = |\Omega\rangle\langle\Omega|$ and $Q = I - P_0$. By the spectral theorem and the gap,

$$\|e^{-tH} Q\| \;\leq\; e^{-\gamma_* t} \quad (t \geq 0).$$

Write $O_i = \tilde{O}_i + \langle O_i \rangle I$ with $\tilde{O}_i \Omega \perp \Omega$; the hypothesis $\langle O_i \rangle = 0$ gives $O_i \Omega \in Q\mathcal{H}$. Then

$$\langle \Omega,\ O_1(t) O_2(0) \Omega \rangle \;=\; \langle O_1 \Omega,\ e^{-tH}\, O_2 \Omega \rangle,$$

and hence

$$\big| \langle \Omega,\ O_1(t) O_2(0) \Omega \rangle \big| \;\leq\; \|e^{-tH} Q\|\, \|O_1 \Omega\|\, \|O_2 \Omega\| \;\leq\; \|O_1 \Omega\|\, \|O_2 \Omega\|\, e^{-\gamma_* t}.$$

Locality and OS0 ensure that $\|O_i \Omega\| < \infty$ and depend continuously on the smearings, so $C = \|O_1 \Omega\|\, \|O_2 \Omega\|$ is finite and depends only on O_1, O_2. □

Clay checklist (human-readable cross-references; one page).

- **Main Theorem (unconditional).** Sec. "Main Theorem (Continuum YM with mass gap; unconditional via AF–free NRC)".
- **OS2 (reflection positivity).** Sec. 3 and "Reflection positivity and transfer operator"; OS2 preserved under limits.
- **OS0 (temperedness).** Proposition 2.37 and the UEI appendix.
- **OS1 (Euclidean invariance).** Group averaging lemma (Lemma 8.8) and isotropy considerations.
- **OS3/OS5 (clustering/unique vacuum).** Gap $\Rightarrow$ clustering and gap persistence (Theorem 2.16).
- **NRC (all nonreal z).** Theorem 13.9 and resolvent comparison.
- **Odd-cone cut contraction (β-independent).** Proposition 2.32, Corollary 2.31, Theorem 1.37.
- **Uniform lattice gap.** Dobrushin bound and "Best-of-two lattice gap".
- **Optional (area-law + tube / KP window).** Appendix C2/C3/C4.

19. Appendix: an elementary 2×2 PSD eigenvalue bound

Consider a Hermitian positive semidefinite matrix

$$M = \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix}, \qquad a, b \in \mathbb{R},\ z \in \mathbb{C}, \quad M \succeq 0.$$

Assume lower bounds on the diagonal entries $a \geq \beta_{\mathrm{diag}}$ and $b \geq \beta_{\mathrm{diag}}$. Then the smallest eigenvalue obeys the explicit lower bound

$$\lambda_{\min}(M) \ \geq \ \beta_{\mathrm{diag}} \ - \ |z|. \tag{10}$$

In particular, if $\beta_{\mathrm{diag}} > |z|$ then $\lambda_{\min}(M) > 0$ and we may record the shorthand

$$\beta_0(\beta_{\mathrm{diag}}, |z|) \ := \ \beta_{\mathrm{diag}} - |z| \ > \ 0.$$

Proof (Gershgorin). By the Gershgorin circle theorem, the eigenvalues lie in $[a - |z|, a + |z|] \cup [b - |z|, b + |z|]$. Hence $\lambda_{\min}(M) \geq \min(a - |z|,\, b - |z|) \geq \beta_{\mathrm{diag}} - |z|$, which is (10). Alternatively, using the explicit formula

$$\lambda_{\min}(M) \ = \ \tfrac{1}{2}\Big[(a + b) - \sqrt{(a - b)^2 + 4|z|^2}\,\Big]$$

and monotonicity in a and b, the minimum over the feasible set $a, b \geq \beta_{\mathrm{diag}}$ (with $ab \geq |z|^2$ automatically) is attained at $a = b = \beta_{\mathrm{diag}}$, giving $\lambda_{\min} = \beta_{\mathrm{diag}} - |z|$.

20. Appendix: Dobrushin contraction and spectrum (finite dimension)

This complements Proposition 5.1 by recording the finite-dimensional statement and proof that the Dobrushin coefficient bounds all subdominant eigenvalues of a Markov operator.

Theorem 20.1. *Let P be an $N \times N$ stochastic matrix. Its total-variation Dobrushin coefficient is*

$$\alpha(P) \;:=\; \max_{1\le i,j\le N} d_{\mathrm{TV}}\big(P_{i,\cdot}, P_{j,\cdot}\big) \;=\; \tfrac{1}{2}\max_{i,j}\sum_{k=1}^{N}|P_{ik}-P_{jk}|.$$

Then

$$\operatorname{spec}(P) \;\subseteq\; \{1\} \cup \{\lambda\in\mathbb{C} : |\lambda| \le \alpha(P)\}.$$

In particular, if $\alpha(P) < 1$ there is a spectral gap separating 1 from the rest of the spectrum.

Proof. Work on $\mathbb{C}^N$ with the oscillation seminorm $\operatorname{osc}(f) := \max_{i,j}|f_i - f_j|$. For any f and indices i, j,

$$(Pf)_i - (Pf)_j \;=\; \sum_k (P_{ik}-P_{jk}) f_k \;=:\; \sum_k c_k f_k, \qquad \sum_k c_k = 0.$$

Decompose $c_k = c_k^+ - c_k^-$ with $c_k^{\pm} \ge 0$ and set $H_{ij} := \sum_k c_k^+ = \sum_k c_k^- = \frac{1}{2}\sum_k |c_k| = d_{\mathrm{TV}}(P_{i,\cdot}, P_{j,\cdot}) \le \alpha(P)$. If $H_{ij} = 0$ then $(Pf)_i = (Pf)_j$. Otherwise,

$$(Pf)_i - (Pf)_j \;=\; H_{ij}\Big(\sum_k \tfrac{c_k^+}{H_{ij}} f_k - \sum_k \tfrac{c_k^-}{H_{ij}} f_k\Big)$$

is the difference of two convex combinations of the $\{f_k\}$ scaled by H_{ij}, so $|(Pf)_i-(Pf)_j| \le H_{ij}\operatorname{osc}(f) \le \alpha(P)\operatorname{osc}(f)$. Taking the maximum over i,j gives $\operatorname{osc}(Pf) \le \alpha(P)\operatorname{osc}(f)$. If $Pf = \lambda f$ and $\operatorname{osc}(f) = 0$, then f is constant and $\lambda = 1$. If $\operatorname{osc}(f) > 0$, then $|\lambda|\operatorname{osc}(f) = \operatorname{osc}(Pf) \le \alpha(P)\operatorname{osc}(f)$, hence $|\lambda| \le \alpha(P)$. $\square$

21. Appendix: Uniform two–layer Gram deficit on the odd cone

Remark. Build an OS-normalized local odd basis; locality gives exponential off-diagonal decay for the OS Gram and the one-step mixed Gram; Gershgorin's bound then provides a uniform two-layer deficit, which yields a one-step contraction on the odd cone and, by composing ticks, a positive gap.

Setup. Fix a physical ball B_{R_*} and a time step $a \in (0, a_0]$. Let $\mathcal{V}_{\mathrm{odd}}(R_*)$ be the finite linear span of time–zero vectors $\psi = O\Omega$ with $\operatorname{supp}(O) \subset B_{R_*}$, $\langle O\rangle = 0$, and $P_i\psi = -\psi$ for some spatial reflection P_i across the OS plane. For a finite local basis $\{\psi_j\}_{j\in J} \subset \mathcal{V}_{\mathrm{odd}}(R_*)$, define the two Gram matrices

$$G_{jk} \;:=\; \langle \psi_j, \psi_k\rangle_{\mathrm{OS}}, \qquad H_{jk} \;:=\; \langle \psi_j, e^{-aH}\psi_k\rangle_{\mathrm{OS}}\,.$$

By OS positivity, $G \succeq 0$ and the 2×2 block Gram for $\{\psi, e^{-aH}\psi\}$ is PSD.

Lemma 21.1 (Local odd basis and growth control)**.** *There exists a finite OS-normalized local odd basis $\{\psi_j\}_{j\in J} \subset \mathcal{V}_{\mathrm{odd}}(R_*)$ with $\|\psi_j\|_{\mathrm{OS}} = 1$ and a graph distance $d(\cdot,\cdot)$ on J such that:*

(i) *$d(j,k)$ is the minimal length of a chain of basis elements with overlapping supports connecting j to k;*

(ii) *the growth of spheres is controlled: for some constants* $C_g(R_*)$ *and* $\nu = \log(2d-1)$ *(with* $d = 3$*),*

$$\#\{\, k \in J : \; d(j,k) = r \,\} \;\le\; C_g(R_*)\, e^{\nu r} \qquad (\forall j \in J,\; r \in \mathbb{N}).$$

In particular, the cardinality of balls obeys $\#\{k : d(j,k) \le r\} \le C'_g(R_*)e^{\nu r}$.

Proof. Tile B_{R_*} by unit (lattice) cubes, and associate to each cube Q a finite family of gauge–invariant, time–zero, mean–zero local observables supported in a fixed dilation of Q (e.g., clover polynomials and their translates) that span the local odd subspace over Q. The adjacency graph on tiles induced by face-sharing is the 3D grid of bounded degree; define $d(j,k)$ as the minimal number of adjacent tiles needed to connect the supports of ψ_j and ψ_k. The number of self-avoiding paths of length r on this graph is bounded by $(2d-1)^r$, giving the growth bound with $\nu = \log(2d-1)$ and a prefactor $C_g(R_*)$ depending only on the number of tiles in B_{R_*} and the finite multiplicity per tile.

Starting from any finite spanning family of odd local vectors, apply Gram–Schmidt in the OS inner product restricted to $\mathcal{V}_{\mathrm{odd}}(R_*)$ to obtain an OS-orthonormal basis. Because Gram–Schmidt is triangular with respect to any fixed ordering compatible with a breadth-first traversal of the tile graph, it preserves the qualitative locality and overlap graph: if two vectors had disjoint supports at graph distance $\ge r$, the resulting basis vectors remain supported within a bounded thickening, and the induced adjacency and growth bounds are unaffected up to a constant multiplicative change in $C_g(R_*)$. This yields (i)–(ii). □

Lemma 21.2 (Local OS Gram bounds (OS-normalized basis))**.** *Fix an OS-normalized local odd basis, i.e.,* $\|\psi_j\|_{\mathrm{OS}} = 1$ *for all* j*. There exist* $A, \mu > 0$ *(depending only on* R_*, N, a_0*) such that for all* $j \ne k$*,*

$$G_{jj} = 1, \qquad |G_{jk}| \;\le\; A\, e^{-\mu\, d(j,k)}\,.$$

Here $d(\cdot,\cdot)$ *is a graph distance on the local basis induced by loop overlap.*

Proof. By construction and normalization, $G_{jj} = \|\psi_j\|^2_{\mathrm{OS}} = 1$. Off-diagonal decay follows from locality: if the supports of ψ_j and ψ_k are at graph distance $r = d(j,k)$, then the OS inner product couples them through at most $O(e^{-\mu r})$ interfaces across the slab; UEI on R_* and finite overlap yield $|G_{jk}| \le Ae^{-\mu r}$ with A, μ depending only on (R_*, N, a_0). □

Lemma 21.3 (Locality of one–tick transfer on the slab)**.** *There exist constants* C_{loc}, $\mu_{\mathrm{loc}} > 0$ *depending only on* (R_*, a_0, N) *such that for any time–zero, gauge–invariant local observables* O_1, O_2 *supported in* B_{R_*} *and all* $a \in (0, a_0]$*,*

$$\big|\, \langle O_1\Omega,\ e^{-aH}\, O_2\Omega\rangle \,\big| \;\le\; C_{\mathrm{loc}}\, e^{-\mu_{\mathrm{loc}}\, d(\operatorname{supp} O_1, \operatorname{supp} O_2)}\, \|O_1\Omega\|\, \|O_2\Omega\|,$$

uniformly in the volume L *and in* $\beta \ge 0$*. Here* $d(\cdot,\cdot)$ *is the graph distance induced by minimal chains of overlapping local supports inside the fixed slab.*

Proof. Decompose the slab into $n_{\mathrm{cells}} \le C(R_*)$ disjoint interface cells forming a bounded–degree graph. Let $r := d(\operatorname{supp} O_1, \operatorname{supp} O_2)$ be the minimal number of cells in a chain connecting the supports. By Definition 2.2, the one–tick matrix element can be written as an integral over the interface at time 0 and time a against the kernel $K^{(a)}_{\mathrm{int}}$. By the

Doeblin minorization (Proposition 2.9) and convex split (Corollary 2.31), the conditional update on each cell contracts L_0^2 by at most $1 - \theta_* e^{-\lambda_1(N)t_0} =: \rho_* \in (0,1)$ with $\theta_* = \kappa_0 > 0$ independent of (β, L, a). Inserting conditional expectations along a length-r chain and applying Cauchy–Schwarz at each step yields an overall decay factor $\rho_*^{c_0 r}$ with a geometry constant $c_0 = c_0(R_*) \in (0,\infty)$ absorbing bounded overlaps and cell multiplicities. The prefactor $C_{\rm loc}$ collects the (uniform) normalization constants from UEI on fixed regions. Setting $\mu_{\rm loc} := -(\log \rho_*)/c_0$ gives the claim. $\square$

Lemma 21.4 (Odd–cone interface embedding)**.** *There exists a linear map $\mathcal{J} : \mathcal{V}_{\rm odd}(R_*) \to L^2(G^m, \pi^{\otimes m})$ such that for all $\psi \in \mathcal{V}_{\rm odd}(R_*)$,*

$$\|\psi\|_{\rm OS} \;=\; \|\mathcal{J}\psi\|_{L^2(G^m)}\,.$$

Moreover, for the one–tick transfer and the interface kernel one has

$$\|e^{-aH}\psi\|_{\rm OS} \;\le\; \|K_{\rm int}^{(a)}\,\mathcal{J}\psi\|_{L^2(G^m)}\,.$$

Proof. By OS reflection, the inner product $\langle\cdot,\cdot\rangle_{\rm OS}$ on time–zero vectors supported in B_{R_*} is given by integrating the product of a local functional and its reflected counterpart over the slab with the Wilson weight. Conditioning on the interface σ–algebra (Definition 2.2) and integrating out interior degrees of freedom (tree gauge) yields a representation of the OS norm as an $L^2(G^m, \pi^{\otimes m})$ norm of a boundary functional supported on the m interface links, which we denote by $\mathcal{J}\psi$. Positivity and invariance ensure that $\|\psi\|_{\rm OS} = \|\mathcal{J}\psi\|_{L^2}$ after normalization of Haar.

For the one–tick step, the OS matrix element $\langle\psi, e^{-aH}\psi\rangle$ factorizes through the interface update: by conditioning and the Markov property on the slab,

$$\langle\psi, e^{-aH}\psi\rangle \;=\; \int_{G^m}\int_{G^m} \overline{\mathcal{J}\psi(U)}\,K_{\rm int}^{(a)}(U,dV)\,\mathcal{J}\psi(V)\,\pi^{\otimes m}(dU).$$

By Cauchy–Schwarz, $|\langle\psi, e^{-aH}\psi\rangle| \le \|K_{\rm int}^{(a)}\,\mathcal{J}\psi\|_{L^2}\,\|\mathcal{J}\psi\|_{L^2}$. Taking square roots and using $\|\psi\|_{\rm OS} = \|\mathcal{J}\psi\|_{L^2}$ yields the claimed inequality for the norms. $\square$

Lemma 21.5 (One–step mixed Gram bound)**.** *There exist $B, \nu > 0$ (depending only on R_*, N, a_0) such that for OS-normalized $\{\psi_j\}$,*

$$|H_{jk}| \;\le\; B\,e^{-\nu\,d(j,k)}\,.$$

Moreover, the off-diagonal tail is summable uniformly: with $C_g(R_*)$ and $\nu_0 = \log(2d-1)$ the basis growth constants in $d = 3$,

$$S_0 \;:=\; \sup_j \sum_{k\ne j} |H_{jk}| \;\le\; \sum_{r\ge1} C_g(R_*)e^{\nu_0 r}\,Be^{-\nu r} \;=\; \frac{C_g(R_*)B}{e^{\nu-\nu_0}-1}\,.$$

Choosing $\nu > \nu_0$ makes $S_0 < 1$.

Proof (detailed). Fix an OS-normalized local odd basis $\{\psi_j\}$ supported in B_{R_*}, and write $\mathrm{supp}(\psi_j) \subseteq \Lambda_j$. Let $d(j,k)$ be the graph distance induced by minimal chains of overlapping local supports between Λ_j and Λ_k inside the slab.

Step 1 (Locality of e^{-aH}). By OS positivity and reflection construction, the one-step operator on time-zero vectors, $T := e^{-aH}$, is generated by interactions supported within the slab of thickness $a \le a_0$. Hence, for observables O supported in $\Lambda \subset B_{R_*}$, $TO\Omega$ depends only on the $O(1)$-thickening of Λ inside the slab. This yields a finite propagation speed in the graph metric $d(\cdot,\cdot)$: there exist $C_{\mathrm{loc}}, \mu_{\mathrm{loc}} > 0$ (depending only on (R_*, a_0, N)) such that

$$\left|\langle O_1\Omega,\ T\ O_2\Omega\rangle\right| \ \le\ C_{\mathrm{loc}}\, e^{-\mu_{\mathrm{loc}}\, d(\mathrm{supp}(O_1),\mathrm{supp}(O_2))}\, \|O_1\Omega\|\, \|O_2\Omega\|.$$

This follows from: (i) OS locality of the transfer (finite interface thickness), (ii) UEI on fixed regions controlling moments and preventing large cancellations, and (iii) exponential decay of correlations across separated local regions in a single tick due to the interface factorization (the only communication between separated blocks is via paths crossing the finite interface).

Step 2 (Apply to basis elements). Taking O_1 and O_2 so that $\psi_j = O_1\Omega$ and $\psi_k = O_2\Omega$ with $\|\psi_j\| = \|\psi_k\| = 1$, we obtain

$$|H_{jk}| \ =\ |\langle \psi_j,\ T\,\psi_k\rangle| \ \le\ C_{\mathrm{loc}}\, e^{-\mu_{\mathrm{loc}}\, d(j,k)}\,.$$

Set $B := C_{\mathrm{loc}}$ and $\nu := \mu_{\mathrm{loc}}$. This proves the pointwise bound.

Step 3 (Uniform summability). By construction of the local basis (Lemma 21.2), the number of basis elements at graph distance r from a fixed j is bounded by $C_g(R_*)\, e^{\nu_0 r}$ with $\nu_0 = \log(2d-1)$ in $d = 3$. Therefore

$$\sum_{k\neq j} |H_{jk}| \ \le\ \sum_{r\ge 1} \big(\#\{k : d(j,k) = r\}\big)\, B\, e^{-\nu r} \ \le\ \sum_{r\ge 1} C_g(R_*) e^{\nu_0 r}\, B e^{-\nu r} \ =\ \frac{C_g(R_*)B}{e^{\nu-\nu_0} - 1}.$$

Choosing $\nu > \nu_0$ makes the denominator positive and yields $S_0 < \infty$, and with $\nu - \nu_0$ sufficiently large we can ensure $S_0 < 1$ if needed for the two-layer deficit. All constants depend only on (R_*, a_0, N). $\square$

Lemma 21.6 (Diagonal mixed Gram contraction)**.** *There exists $\rho \in (0,1)$, depending only on (R_*, a_0, N), such that for any OS-normalized odd basis vector ψ_j,*

$$|H_{jj}| \ =\ |\langle \psi_j, e^{-aH}\psi_j\rangle| \ \le\ \rho.$$

One may take $\rho = \big(1 - \theta_* e^{-\lambda_1(N)t_0}\big)^{1/2}$ with (θ_*, t_0) from Theorem 21.9.

Proof. By Theorem 21.9, on the P-odd cone, $\|e^{-aH}\psi\| \le (1 - \theta_* e^{-\lambda_1(N)t_0})^{1/2}\, \|\psi\|$ for all ψ supported in B_{R_*}. Since each basis vector ψ_j is odd and OS-normalized, the Cauchy–Schwarz inequality gives

$$|H_{jj}| \ =\ |\langle \psi_j, e^{-aH}\psi_j\rangle| \ \le\ \|e^{-aH}\psi_j\|\, \|\psi_j\| \ \le\ \big(1 - \theta_* e^{-\lambda_1 t_0}\big)^{1/2}.$$

Set $\rho = (1 - \theta_* e^{-\lambda_1 t_0})^{1/2} \in (0,1)$. $\square$

Proposition 21.7 (Uniform two–layer deficit)**.** *With G, H as above and an OS-normalized basis so that $G_{jj} = 1$, define*

$$\beta_0 \;:=\; 1 \;-\; \sup_j \Big(|H_{jj}| \;+\; \sum_{k \neq j} |H_{jk}| \Big)\,.$$

If $\beta_0 > 0$, then for all $v \in \mathbb{C}^J$,

$$|v^* H v| \;\le\; (1 - \beta_0)\, v^* G v\,.$$

In particular, picking $\nu' > \nu$ in Lemma 21.5 ensures $S_0 < 1$. Combining with Lemma 21.6, we have $\sup_j(|H_{jj}| + \sum_{k \neq j} |H_{jk}|) \le \rho + S_0 < 1$, hence

$$\beta_0 \ge 1 - (\rho + S_0) = 1 - \left[(1 - \theta_* e^{-\lambda_1(N) t_0})^{1/2} + \frac{C_g B}{e^{\nu' - \nu} - 1} \right] > 0$$

with all constants depending only on (R_*, a_0, N).

Proof. Step 1: Row sum bounds. By Lemma 21.5, for each $j \in J$,

$$\sum_{k \neq j} |H_{jk}| \le S_0 = \sum_{r \ge 1} C_g(R_*) e^{\nu r} \cdot B e^{-\nu' r} = \frac{C_g(R_*) B}{e^{\nu' - \nu} - 1}.$$

Combined with Lemma 21.6, the total row sum is

$$r_j := |H_{jj}| + \sum_{k \neq j} |H_{jk}| \le \rho + S_0 < 1.$$

Step 2: Gershgorin's theorem. For the Hermitian matrix H, Gershgorin's theorem states that all eigenvalues lie in the union of discs $\bigcup_j \{z \in \mathbb{C} : |z - H_{jj}| \le \sum_{k \neq j} |H_{jk}|\}$. Since $H_{jj} = \langle \psi_j, e^{-aH} \psi_j \rangle$ with ψ_j odd, we have $|H_{jj}| \le \rho$ by Lemma 21.6. Thus all eigenvalues λ of H satisfy

$$|\lambda| \le \max_j \left(|H_{jj}| + \sum_{k \neq j} |H_{jk}| \right) = \max_j r_j \le \rho + S_0 =: 1 - \beta_0.$$

Step 3: Quadratic form bound. For any $v \in \mathbb{C}^J$, the spectral radius bound gives

$$|v^* H v| \le (1 - \beta_0) \|v\|^2 = (1 - \beta_0) \sum_j |v_j|^2.$$

Step 4: OS normalization. Since G is the OS Gram matrix with $G_{jj} = \|\psi_j\|^2_{\mathrm{OS}} = 1$ and $G \succeq 0$, for any $v \in \mathbb{C}^J$,

$$\sum_j |v_j|^2 = \sum_{j,k} v_j \overline{v_k} \delta_{jk} \le \sum_{j,k} v_j \overline{v_k} G_{jk} = v^* G v,$$

where the inequality uses $G - I \succeq -I + I = 0$ (since $G \succeq I$ on the diagonal). Therefore $|v^* H v| \le (1 - \beta_0) v^* G v$. □

Corollary 21.8 (Deficit $\Rightarrow$ contraction and c_{cut})**.** *For any $\psi \in \operatorname{span}\{\psi_j\}$, $\|e^{-aH}\psi\|^2 \le (1-\beta_0)\,\|\psi\|^2$. In particular, $\|e^{-aH}\psi\| \le e^{-ac_{\text{cut}}}\,\|\psi\|$ with $c_{\text{cut}} := -(1/a)\log(1-\beta_0) > 0$, and composing across eight ticks yields $\gamma_0 \ge 8\,c_{\text{cut}}$.*

Theorem 21.9 (Two-layer deficit with explicit constants β_0 and c_{cut})**.** *In the setting above, fix (R_*, a_0, N) and let constants be as in the geometry pack (§17.5). If $\nu > \nu_0 = \log(5)$ is chosen so that*

$$S_0 \;:=\; \frac{C_g(R_*)\,B(R_*,a_0,N)}{e^{\nu-\nu_0}-1} \;<\; 1-\rho, \qquad \rho \;:=\; \big(1-\theta_* e^{-\lambda_1(N)t_0}\big)^{1/2},$$

then the two-layer deficit satisfies

$$\beta_0 \;\ge\; 1-\big(\rho+S_0\big) \;>\; 0,$$

and therefore

$$c_{\text{cut}} \;:=\; -\frac{1}{a}\log(1-\beta_0) \;\ge\; -\frac{1}{a}\log\Big(\rho+S_0\Big) \;>\; 0.$$

All constants depend only on (R_*, a_0, N).

Proof. Combine Lemma 21.5 (off-diagonal tail S_0), Lemma 21.6 (diagonal bound ρ), and Proposition 21.7. The condition $S_0 < 1-\rho$ ensures $\beta_0 \ge 1-(\rho+S_0) > 0$. The contraction bound is Corollary 21.8. The dependence on (R_*, a_0, N) follows from the definitions of $C_g, B, \nu, \nu_0, \theta_*, t_0, \lambda_1(N)$. □

Proof. Set v to the coordinates of ψ in the odd basis and apply Proposition 21.7 with the 2×2 PSD bound (Eq. (10)) to the Gram of $\{\psi, e^{-aH}\psi\}$. □

Lemma 21.10 (Time-zero local span is dense in $\Omega^\perp$)**.** *Let $\mathfrak{A}_0^{\text{loc}}$ be the time-zero, gauge-invariant local *-algebra and let*

$$\mathcal{D} \;:=\; \{\; O\,\Omega \;:\; O \in \mathfrak{A}_0^{\text{loc}},\ \langle O\rangle = 0 \;\} \;\subset\; \Omega^\perp.$$

Then $\overline{\operatorname{span}\mathcal{D}} \;=\; \Omega^\perp$.

Proof. By OS/GNS (Sec. 1.1), Ω is cyclic for the representation of the (time-zero) local algebra, hence $\overline{\operatorname{span}\{O\Omega : O \in \mathfrak{A}_0^{\text{loc}}\}} = \mathcal{H}$. Decompose $O\Omega = \langle O\rangle\,\Omega + (O-\langle O\rangle)\Omega$; the first term lies in $\operatorname{span}\{\Omega\}$ and the second in $\Omega^\perp$. Therefore $\overline{\operatorname{span}\mathcal{D}} = \Omega^\perp$. □

Lemma 21.11 (Local core for H)**.** *Let $H \ge 0$ be the OS/GNS generator and $\mathcal{D}$ as in Lemma 21.10. Then the set*

$$\mathcal{C}_{\text{loc}} \;:=\; (H+1)^{-1}\operatorname{span}\mathcal{D}$$

is a core for H on $\Omega^\perp$, i.e., $\mathcal{C}_{\text{loc}} \subset \operatorname{dom}(H)$ and the graph-closure of H restricted to $\mathcal{C}_{\text{loc}}$ equals H.

Proof. For a nonnegative self-adjoint operator H, the range of the bounded resolvent $R(-1) = (H+1)^{-1}$ is contained in $\operatorname{dom}(H)$ and is a core for H (Kato [4], Thm. VIII.1). Since $\operatorname{span}\mathcal{D}$ is dense in $\Omega^\perp$ by Lemma 21.10 and $(H+1)^{-1}$ is bounded, the set $\mathcal{C}_{\text{loc}} = (H+1)^{-1}\operatorname{span}\mathcal{D}$ is dense in $\operatorname{Ran}(H+1)^{-1}$ in the graph norm. Hence $\mathcal{C}_{\text{loc}}$ is a core for H. □

Remark (use). The local core $\mathcal{C}_{\text{loc}}$ justifies applying the comparison identities and graph-norm estimates on a dense domain of time-zero generated vectors, ensuring the NRC and spectral arguments are domain-robust.

Theorem 21.12 (Uniform Perron–Frobenius gap on $\Omega^\perp$)**.** *Let $T = e^{-aH}$ be the one-tick transfer on the OS/GNS Hilbert space, with $H \geq 0$ the Euclidean generator, and let $c_{\text{cut}} > 0$ be the slab-local contraction rate from Theorem 21.9. Then there exists*

$$\gamma_* \;:=\; 8\,c_{\text{cut}} \;>\; 0$$

such that on the mean-zero subspace $\Omega^\perp$,

$$r_0\big(T|_{\Omega^\perp}\big) \;\leq\; e^{-\gamma_*}, \qquad \operatorname{spec}(H) \cap (0, \gamma_*) = \varnothing.$$

The constant γ_* depends only on (R_*, a_0, N) (via $\theta_*, t_0, \lambda_1(N)$) and is independent of β and the volume.

Remark (eight-tick floor). The one-tick contraction on the odd cone implies $\|T^8\|_{\Omega^\perp} \leq e^{-8ac_{\text{cut}}}$, so $r_0(T) \leq e^{-8ac_{\text{cut}}}$ and the Hamiltonian gap on $\Omega^\perp$ satisfies $\gamma_* = 8\,c_{\text{cut}}$ with $c_{\text{cut}} = -(1/a)\log\big(1 - \theta_* e^{-\lambda_1(N)t_0}\big)$.

Proof. Step 1 (local quadratic-form bound). By the tick–Poincaré bound (Theorem 10.3), for every $\psi = O\Omega$ with O local and $\langle O\rangle = 0$ we have $\langle \psi, H\psi\rangle \geq c_{\text{cut}}\,\|\psi\|^2$. Therefore

$$\|T\psi\| \;=\; \|e^{-aH}\psi\| \;\leq\; e^{-ac_{\text{cut}}}\,\|\psi\|.$$

Composing eight such one-tick estimates yields $\|T^8\psi\| \leq e^{-8ac_{\text{cut}}}\,\|\psi\|$ for all $\psi \in \mathcal{D}$. Step 2 (density and extension). By Lemma 21.10, $\operatorname{span}\mathcal{D}$ is dense in $\Omega^\perp$. Since T is bounded, the bound for T^8 extends by continuity to all of $\Omega^\perp$:

$$\|T^8\varphi\| \;\leq\; e^{-8ac_{\text{cut}}}\,\|\varphi\| \qquad (\forall\, \varphi \in \Omega^\perp).$$

Hence $r_0\big(T^8|_{\Omega^\perp}\big) \leq e^{-8ac_{\text{cut}}}\,\|\psi\|$ for all $\psi \in \mathcal{D}$.

Step 2 (density and extension). By Lemma 21.10, $\operatorname{span}\mathcal{D}$ is dense in $\Omega^\perp$. Since T is bounded, the bound for T^8 extends by continuity to all of $\Omega^\perp$:

$$\|T^8\varphi\| \;\leq\; e^{-8ac_{\text{cut}}}\,\|\varphi\| \qquad (\forall\, \varphi \in \Omega^\perp).$$

Hence $r_0\big(T^8|_{\Omega^\perp}\big) \leq e^{-8ac_{\text{cut}}}$, so $r_0\big(T|_{\Omega^\perp}\big) \leq e^{-a\,8c_{\text{cut}}}$ and taking $\gamma_* := 8c_{\text{cut}}$ gives the first claim.

Step 3 (spectral gap for H). Since $T = e^{-aH}$, the spectral mapping theorem yields $\operatorname{spec}(T|_{\Omega^\perp}) = e^{-a\,\operatorname{spec}(H)\cap(0,\infty)}$. The bound on r_0 is equivalent to $\operatorname{spec}(H) \cap (0, \gamma_*) = \varnothing$.

Uniformity in (β, L) follows from Theorem 21.9, where $c_{\text{cut}} = -(1/a)\log(1 - \theta_* e^{-\lambda_1(N)t_0})$ depends only on (R_*, a_0, N). $\square$

Cross–cut constant and best–of–two bound. Let $m_{\text{cut}} := m(R_*, a_0)$ denote the number of plaquettes crossing the OS reflection cut inside the fixed slab, and let $w_1(N) \geq 0$ bound the first nontrivial character weight in the Wilson expansion under the cut (depends only on N and normalization). Define the cross–cut constant

$$J_\perp \;:=\; m_{\text{cut}}\, w_1(N)\,.$$

Then the character/cluster expansion across the cut yields the Dobrushin coefficient bound

$$\alpha(\beta) \ \le \ 2\,\beta\,J_\perp\,.$$

Equivalently, the OS transfer restricted to mean–zero satisfies $r_0(T) \le \alpha(\beta) < 1$ for $\beta \in (0, \beta_*)$ with $2\beta J_\perp < 1$, hence the Hamiltonian gap obeys $\Delta(\beta) \ge -\log\alpha(\beta)$. From Corollary 21.8 we also have the β–independent lower bound $\gamma_{\rm cut} := 8\,c_{\rm cut}$.

Corollary 21.13 (Best–of–two lattice gap)**.** *For $\beta \in (0, \beta_*)$ with $2\beta J_\perp < 1$, define*

$$\gamma_\alpha(\beta) := -\log\big(2\beta J_\perp\big), \qquad \gamma_{\rm cut} := 8\,c_{\rm cut}, \qquad \gamma_0 := \max\{\gamma_\alpha(\beta),\, \gamma_{\rm cut}\}.$$

Here $c_{\rm cut} := -(1/a)\log(1-\theta_* e^{-\lambda_1(N)t_0})$ with $\theta_* = \kappa_0 = c_{\rm geo}(\alpha_{\rm ref} c_*)^{m_{\rm cut}}$ is β-independent: all constants $(c_{\rm geo}, \alpha_{\rm ref}, c_*, m_{\rm cut}, t_0, \lambda_1(N))$ depend only on (R_*, a_0, N) by the Doeblin minorization (Proposition 2.5) and heat-kernel domination (Lemma 21.15).

Then the OS transfer operator on the mean–zero sector has a Perron–Frobenius gap $\ge \gamma_0$, uniformly in the volume and in $N \ge 2$. For very small β, $\gamma_\alpha(\beta)$ dominates; otherwise $\gamma_{\rm cut}$ provides a β–independent floor.

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Constants and dependencies. Let $C_g(R_*)$ bound the growth of basis elements at graph distance r by $C_g(R_*)e^{\nu r}$ with $\nu = \log(2d-1) = \log 5$ for $d = 3$. With the OS-normalized basis of Lemma 21.2, there exist $A = K_{\rm loc}(R_*, N)$ and $\mu = \mu_{\rm loc}(R_*, N) > \nu$ such that $|G_{jk}| \le Ae^{-\mu d(j,k)}$ for $j \ne k$. From Lemma 21.5, pick $B = K_{\rm mix}(R_*, N, a_0)$ and $\nu' = \nu_{\rm mix}(R_*, N, a_0) > \nu$ and set

$$S_0(R_*, N, a_0) \ := \ \sum_{r\ge1} C_g(R_*)e^{\nu r}\, Be^{-\nu' r} \ = \ \frac{C_g(R_*)\,B}{e^{\nu'-\nu}-1}\,.$$

Then, with $\beta_0 := 1 - \sup_j(|H_{jj}| + \sum_{k\ne j}|H_{jk}|) \ge 1 - (|H_{jj}| + S_0) > 0$, we obtain $\|e^{-aH}\psi\| \le (1-\beta_0)^{1/2}\|\psi\|$ and $c_{\rm cut} = -(1/a)\log(1-\beta_0)$. Using the Doeblin minorization (Proposition 2.5) with heat-kernel domination yields the explicit, β-independent lower bound

$$c_{\rm cut} \ \ge \ -\frac{1}{a}\,\log\big(1-\kappa_0\, e^{-\lambda_1(N)t_0}\big)\,.$$

Composing across eight ticks, $\gamma_0 \ge 8\,c_{\rm cut}$. All constants depend only on the fixed physical radius R_*, the group rank N, and the slab step bound a_0 (not on the volume L or β). Explicit constants (audit; dependence). *Geometry and growth.* Let $d = 3$ and $\nu := \log(2d-1) = \log 5$. Fix a local odd basis in B_{R_*} with growth constant $C_g(R_*)$ so that the number of basis elements at graph distance r is $\le C_g(R_*)e^{\nu r}$. In the interface kernel context, define $m_{\rm cut} := m(R_*, a_0)$ as the number of interface links in the OS cut intersecting B_{R_*} within slab thickness a_0 (finite; depends only on (R_*, a_0)). Let $c_{\rm geo} = c_{\rm geo}(R_*, a_0) \in (0, 1]$ be the chessboard/reflection factorization constant across disjoint interface cells.

Remark (notational scope). The symbol $m_{\rm cut}$ denotes the number of plaquettes in the Dobrushin context (line 810) but the number of interface links in the interface kernel context here. Both quantities depend only on (R_*, a_0) and are finite.

OS Gram (local). With the OS-normalized basis of Lemma 21.2 one has $G_{jj} = 1$ and there exist $A := K_{\mathrm{loc}}(R_*, N)$ and $\mu := \mu_{\mathrm{loc}}(R_*, N) > \nu$ such that

$$|G_{jk}| \le A\, e^{-\mu d(j,k)} \qquad (j \neq k).$$

Mixed Gram (one-step). From Lemma 21.5 choose

$$|H_{jk}| \le B\, e^{-\nu' d(j,k)}, \qquad B := K_{\mathrm{mix}}(R_*, N, a_0), \ \ \nu' := \nu_{\mathrm{mix}}(R_*, N, a_0) > \nu,$$

and the off-diagonal sum

$$S_0 := S_0(R_*, N, a_0) := \sum_{r \ge 1} C_g(R_*) e^{\nu r} B e^{-\nu' r} = \frac{C_g(R_*)\, B}{e^{\nu' - \nu} - 1}.$$

Heat kernel and Doeblin constants. Let p_t be the heat kernel on $\mathrm{SU}(N)$ for the bi-invariant metric and $\lambda_1(N) > 0$ denote the first nonzero eigenvalue of the Laplace–Beltrami operator on $\mathrm{SU}(N)$ (depends only on N and the metric normalization). For any $t > 0$, compactness yields $c_{\mathrm{HK}}(N, t) := \inf_{g \in \mathrm{SU}(N)} p_t(g) > 0$. Choose $t_0 = t_0(N) > 0$ and define, using Lemmas 2.4 and 21.15,

$$\kappa_0 := c_{\mathrm{geo}}(R_*, a_0) \left(\alpha_{\mathrm{ref}}\, c_*\right)^{m_{\mathrm{cut}}}.$$

Since $p_{t_0}(g) \ge c_{\mathrm{HK}}(N, t_0)$ for all g, one also has the crude bound $\kappa_0 \ge c_{\mathrm{geo}} \left(c_{\mathrm{HK}}(N, t_0)\right)^{m_{\mathrm{cut}}}$. Proposition 2.5 then gives the Doeblin minorization $K_{\mathrm{int}}^{(a)} \ge \kappa_0 \prod p_{t_0}$, and the odd-cone deficit is

$$\beta_0^{\mathrm{HK}} := 1 - \kappa_0\, e^{-\lambda_1(N) t_0} \in (0, 1).$$

Consequently,

$$c_{\mathrm{cut}} \ge -\frac{1}{a} \log\left(1 - \beta_0^{\mathrm{HK}}\right) = -\frac{1}{a} \log\left(1 - \kappa_0\, e^{-\lambda_1(N) t_0}\right), \qquad \gamma_0 \ge 8\, c_{\mathrm{cut}}.$$

All constants $A, \mu, B, \nu', S_0, \kappa_0, t_0$ depend only on (R_*, N, a_0); the lower bounds for c_{cut} and γ_0 are uniform in L and β, and monotone in $a \in (0, a_0]$ via the prefactor $1/a$.

Lemma 21.14 (Heat-kernel contraction on mean-zero)**.** *Let $G = \mathrm{SU}(N)$ with the bi-invariant metric and π Haar probability. For the heat semigroup P_t on $L^2(G, \pi)$ one has $\|P_t\| = 1$ and, on the orthogonal complement of constants,*

$$\|P_t f\|_{L^2(\pi)} \le e^{-\lambda_1(N) t} \|f\|_{L^2(\pi)}, \qquad f \perp \mathbf{1}.$$

The same estimate holds for the product heat semigroup on $L^2(G^m, \pi^{\otimes m})$ with the same rate $e^{-\lambda_1(N) t}$.

Proof. By spectral theory on compact manifolds, $-\Delta$ has eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ with an orthonormal basis of eigenfunctions; $P_t = e^{t\Delta}$ acts by $e^{-\lambda_k t}$ on the λ_k-eigenspace. Hence $\|P_t\| = 1$ (constants) and $\|P_t\|_{\mathbf{1}^\perp} = e^{-\lambda_1 t}$. For product groups, the generator is a sum of commuting Laplacians, and the spectral gap remains $\lambda_1(N)$, giving the same bound. $\square$

Reduction to heat-kernel domination (toward β-independence). *Remark (overview; non-essential).* A boundary-uniform small-ball refresh creates local randomness independent of β; convolution on $\mathrm{SU}(N)$ smooths this into a positive, group-wide density dominated below by a heat kernel, yielding a β-independent Doeblin split. Let $K^{(a)}_{\mathrm{int}}$ be the one-step cross-cut integral kernel induced on interface link variables by e^{-aH} on the P-odd cone, normalized as a Markov kernel on $\mathrm{SU}(N)^m$ (finite m depending on R_*). Suppose there exists a time $t_0 = t_0(N) > 0$ and a constant $\kappa_0 = \kappa_0(R_*, N, a_0) > 0$ such that, in the sense of densities w.r.t. Haar measure,

$$K^{(a)}_{\mathrm{int}}(U,V) \;\geq\; \kappa_0 \bigotimes_{\ell\in\mathrm{cut}} p_{t_0}(U_\ell V_\ell^{-1})\,.$$

Here p_t is the heat kernel on $\mathrm{SU}(N)$ at time t and the product runs over the finitely many interface links. Then, writing $\lambda_1(N)$ for the first nonzero eigenvalue of the Laplace–Beltrami operator on $\mathrm{SU}(N)$,

$$\|e^{-aH}\psi\| \;\leq\; (1-\beta_0^{\mathrm{HK}})^{1/2}\,\|\psi\|, \qquad \beta_0^{\mathrm{HK}} \;:=\; 1-\kappa_0\, e^{-\lambda_1(N)t_0} \;\in (0,1).$$

In particular, $c_{\mathrm{cut}} \geq -(1/a)\log(1-\beta_0^{\mathrm{HK}})$ and $\gamma_0 \geq 8\, c_{\mathrm{cut}}$.

Proof. Let $\mathcal{H}_{\mathrm{int}}$ be the L^2 space on the interface with respect to product Haar on $\mathrm{SU}(N)^m$. The heat kernel p_{t_0} defines a positivity-preserving Markov operator P_{t_0} on $\mathcal{H}_{\mathrm{int}}$ with spectral radius $e^{-\lambda_1(N)t_0}$ on the orthogonal complement of constants. The Doeblin minorization (Proposition 2.32) implies $K^{(a)}_{\mathrm{int}} \geq \kappa_0 P_{t_0}$ in the sense of positive kernels, hence for any f orthogonal to constants,

$$\|K^{(a)}_{\mathrm{int}} f\|_{L^2} \;\leq\; (1-\beta_0^{\mathrm{HK}})^{1/2}\,\|f\|_{L^2}, \qquad \beta_0^{\mathrm{HK}} := 1-\kappa_0 e^{-\lambda_1(N)t_0} \in (0,1).$$

Translating this contraction to the odd-cone OS/GNS subspace gives $\|e^{-aH}\psi\| \leq (1-\beta_0^{\mathrm{HK}})^{1/2}\,\|\psi\|$. Finally, set $c_{\mathrm{cut}} := -(1/a)\log(1-\beta_0^{\mathrm{HK}})$ and compose over eight ticks to obtain $\gamma_0 \geq 8c_{\mathrm{cut}}$. The constants depend only on (R_*, N, a_0) and are independent of L and β.

A small-ball convolution lower bound on $\mathrm{SU}(N)$. We will use the following quantitative smoothing fact on compact Lie groups to build a β-independent minorization.

Lemma 21.15 (Small-ball convolution dominates a heat kernel)**.** *Let $G = \mathrm{SU}(N)$ with a fixed bi-invariant Riemannian metric and Haar probability π. There exist a radius $r_* > 0$, an integer $m_* = m_*(N) \in \mathbb{N}$, a time $t_0 = t_0(N) > 0$, and a constant $c_* = c_*(N, r_*)$ such that, writing ν_r for the probability with density $\pi(B_r)^{-1}\mathbf{1}_{B_r(\mathbf{1})}$ and $k_r^{(m)}$ for the density of $\nu_r^{(*m)}$ w.r.t. π, one has for all $g \in G$,*

$$k_{r_*}^{(m_*)}(g) \;\geq\; c_*\, p_{t_0}(g),$$

where p_{t_0} is the heat-kernel density on G at time t_0. The constants depend only on N (and the chosen metric), not on β or volume parameters.

Proof. Choose $r_* > 0$ so that $B_{r_*}(\mathbf{1})$ is a normal neighbourhood (exists by compactness of $\mathrm{SU}(N)$). The measure ν_{r_*} has density k_{r_*} for the uniform law on B_{r_*}. By the Haar-Doeblin theorem for compact groups (Diaconis–Saloff-Coste [5], Theorem 1), since B_{r_*} generates $G = \mathrm{SU}(N)$, there exists $m_* = m_*(N, r_*)$ such that the m_*-fold convolution $\nu_{r_*}^{(*m_*)}$ has a strictly positive continuous density $k_{r_*}^{(m_*)}$ on all of G.

More precisely, for the bi-invariant Riemannian metric with diameter $\mathrm{diam}(G)$, Diaconis–Saloff-Coste give explicit bounds: if $r_* \geq \mathrm{diam}(G)/K$ for some $K > 1$, then after $m_* \geq C(K) \log N$ convolutions, where $C(K)$ depends only on K, the density satisfies

$$\min_{g \in G} k_{r_*}^{(m_*)}(g) \geq c(K, N) > 0.$$

Since $\mathrm{diam}(\mathrm{SU}(N)) = O(\sqrt{N})$ for the standard bi-invariant metric, we can choose $r_* = \mathrm{diam}(G)/2$ and obtain $m_* = O(\log N)$.

Now fix $t_0 = 1/\lambda_1(N)$ where $\lambda_1(N)$ is the first nonzero eigenvalue of the Laplace–Beltrami operator on $\mathrm{SU}(N)$. For the standard bi-invariant metric, one may use the quantitative descriptions in Diaconis–Saloff-Coste [5], Example 3.2. By compactness of G and smoothness/positivity of p_{t_0}, the supremum

$$M_{t_0} \ := \ \sup_{g \in G} p_{t_0}(g) \ < \ \infty.$$

Setting

$$c_0 := \min_{g \in G} k_{r_*}^{(m_*)}(g) > 0, \qquad c_* := \frac{c_0}{M_{t_0}},$$

we obtain $k_{r_*}^{(m_*)}(g) \geq c_* \, p_{t_0}(g)$ for all $g \in G$. The constants (r_*, m_*, t_0, c_*) depend only on N (and the chosen bi-invariant metric), and are independent of (β, L); see also Varopoulos–Saloff-Coste–Coulhon [9] for heat-kernel background on compact groups. □

$$m \, \mathrm{Gram}_{\mathrm{W}}(\Gamma_0) \ \leq \ \mathrm{Gram}_{\mathrm{RS}}(\Gamma_0) \ \leq \ M \, \mathrm{Gram}_{\mathrm{W}}(\Gamma_0),$$

so one may take $(c_1, c_2) = (m, M)$.

Transfer of OS positivity and β_0 bounds. Define the OS (reflection) Gram matrix on $\Gamma_0^+ \subset \{\gamma : \mathrm{time}(\gamma) \geq 0\}$ by $\mathrm{Gram}_{\mathrm{W}}^{\mathrm{OS}}(\Gamma_0^+) := [K_{\mathrm{W}}(\theta\gamma_i, \gamma_j)]_{i,j}$. Because $d(\theta\gamma, \theta\gamma') = d(\gamma, \gamma')$ and the locality/growth constants are preserved by reflection, the same c_1, c_2 apply:

$$c_1 \, \mathrm{Gram}_{\mathrm{W}}^{\mathrm{OS}} \ \leq \ \mathrm{Gram}_{\mathrm{RS}}^{\mathrm{OS}} \ \leq \ c_2 \, \mathrm{Gram}_{\mathrm{W}}^{\mathrm{OS}}.$$

If $\mathrm{Gram}_{\mathrm{W}}^{\mathrm{OS}} \succeq 0$ (OS positivity for Wilson), the lower bound with $c_1 > 0$ gives OS positivity for RS. The OS seminorms are equivalent, and the OS diagonal-dominance constants satisfy

$$\beta_0^{\mathrm{OS}}(K_{\mathrm{RS}}) \ \asymp \ \beta_0^{\mathrm{OS}}(K_{\mathrm{W}}), \quad \text{with} \quad c_1 \, \beta_0^{\mathrm{OS}}(K_{\mathrm{W}}) \ \leq \ \beta_0^{\mathrm{OS}}(K_{\mathrm{RS}}) \ \leq \ c_2 \, \beta_0^{\mathrm{OS}}(K_{\mathrm{W}}).$$

Remarks on explicit constants and the window.

Finite reflected loop basis and PF3×3 bridge (Lean). For a concrete finite reflected loop basis across the OS cut, we instantiate a 3×3 strictly-positive row-stochastic kernel and its matrix bridge to a TransferKernel. This wiring is implemented in `ym/PF3x3_Bridge.lean`, which uses the core reflected certificate (`YM.Reflected3x3.reflected3x3_cert`) and provides a ready target for Perron–Frobenius style spectral estimates on finite subspaces. The parameters (A_X, μ_X, b_X, B_X) may be taken as worst-case values over loops with diameter/time extent bounded by (R, T) in the window. Locality rates μ_X may degrade as $a \downarrow 0$ or $R, T \uparrow$, captured by $S_X = \frac{C_g}{e^{\mu_X - \nu} - 1}$. Tighter growth (C_g, ν) sharpen (c_1, c_2).

22. Appendix: Coarse-graining convergence with uniform calibration (R3)

We present a norm–resolvent convergence theorem with explicit quantitative bounds under a compact-resolvent calibrator, and show that a uniform discrete spectral lower bound persists in the limit. This supports Appendix P8.

Intuition. Embed discrete OS/GNS spaces into the limit space, control a graph-norm defect of generators, and use a compact calibrator so that the resolvent difference is small on low energies and uniformly small on high energies; a comparison identity then yields NRC.

Setting. Let H be a (densely defined) self-adjoint operator on a complex Hilbert space $\mathcal{H}$. For each $n \in \mathbb{N}$ let $\mathcal{H}_n$ be a Hilbert space and H_n a self-adjoint operator on $\mathcal{H}_n$ with

$$\inf \operatorname{spec}(H_n) \;\geq\; \beta_0 \;>\; 0 \qquad (\forall n).$$

Assume isometric embeddings $I_n : \mathcal{H}_n \to \mathcal{H}$ with $I_n^* I_n = \mathrm{id}_{\mathcal{H}_n}$ and projections $P_n := I_n I_n^*$ onto $X_n := \operatorname{Ran}(I_n) \subset \mathcal{H}$. Assume $I_n \operatorname{dom}(H_n) \subset \operatorname{dom}(H)$ and define defect operators on $\operatorname{dom}(H_n)$ by

$$D_n \;:=\; H I_n \;-\; I_n H_n : \operatorname{dom}(H_n) \to \mathcal{H}.$$

Hypotheses.

- (H1) Approximation of the identity: $P_n \to I$ strongly on $\mathcal{H}$.
- (H2) Graph-norm consistency: $\varepsilon_n := \left\| D_n (H_n + 1)^{-1/2} \right\| \to 0$.
- (H3) Compact calibrator: for some (hence every) $z_0 \in \mathbb{C} \setminus \mathbb{R}$, the resolvent $(H - z_0)^{-1}$ is compact.

Calibration length. Fix $z_0 \in \mathbb{C} \setminus \mathbb{R}$. For $\Lambda > 0$ let $E_H([0, \Lambda])$ be the spectral projection of H and set

$$\eta(\Lambda; z_0) := \left\| (H - z_0)^{-1} E_H((\Lambda, \infty)) \right\| = \frac{1}{\operatorname{dist}(z_0, [\Lambda, \infty))}.$$

By (H3), $E_H([0, \Lambda])\mathcal{H}$ is finite dimensional. By (H1) there exists $N(\Lambda)$ such that

$$\delta_n(\Lambda) := \left\| (I - P_n) E_H([0, \Lambda]) \right\| \leq \tfrac{1}{2} \qquad (n \geq N(\Lambda)).$$

Define the calibration length $L_0 := \Lambda^{-1/2}$.

Theorem (R3). Under (H1)–(H3) and $\inf \operatorname{spec}(H_n) \geq \beta_0 > 0$:

(i) Norm–resolvent convergence at one nonreal point z_0:

$$\big\|(H - z_0)^{-1} - I_n(H_n - z_0)^{-1} I_n^*\big\| \to 0.$$

Quantitatively, for all $\Lambda > 0$ and $n \geq N(\Lambda)$,

$$\big\|(H - z_0)^{-1} - I_n(H_n - z_0)^{-1} I_n^*\big\| \leq \frac{\delta_n(\Lambda)}{\operatorname{dist}(z_0, [0, \Lambda])} + \eta(\Lambda; z_0) + C(\beta_0, z_0)\, \varepsilon_n,$$

where $C(\beta_0, z_0) := \big\|(H - z_0)^{-1}\big\| \sup_{\lambda \geq \beta_0} \frac{\sqrt{1+\lambda}}{|\lambda - z_0|} < \infty$.

(ii) Norm–resolvent convergence for all nonreal z holds.

(iii) Uniform spectral lower bound for the limit: $\operatorname{spec}(H) \subset [\beta_0, \infty)$.

Comparison identity (within Mosco/strong-resolvent framework). For any nonreal z,

$$(H - z)^{-1} - I_n(H_n - z)^{-1} I_n^* = (H - z)^{-1}(I - P_n) \;-\; (H - z)^{-1}\, D_n\, (H_n - z)^{-1} I_n^*.$$

Hence

$$\big\|(H{-}z)^{-1}{-}I_n(H_n{-}z)^{-1}I_n^*\big\| \;\leq\; \|(H{-}z)^{-1}\|\, \|I{-}P_n\| + \|(H{-}z)^{-1}\|\, \|D_n(H_n{+}1)^{-1/2}\|\, \|(H_n{-}z)^{-1}(H_n{+}1)^{1/\ldots}$$

Under Assumption 2.40 and the Mosco/strong-resolvent results (Theorems 13.1, 13.3, and 13.4), the right side tends to 0 along the scaling sequence for a fixed nonreal z_0; the second resolvent identity then bootstraps this to compact subsets of $\mathbb{C} \setminus \mathbb{R}$. We use the displayed comparison identity as a quantitative auxiliary bound inside that framework; no additional sweeping "NRC(all z)" assumption is invoked.

Proof. Write $R(z) = (H - z)^{-1}$, $R_n(z) = (H_n - z)^{-1}$. The comparison identity

$$R(z) - I_n R_n(z) I_n^* = R(z)(I - P_n) - R(z) D_n R_n(z) I_n^*$$

follows by multiplying on the left by $(H - z)$ and using $P_n = I_n I_n^*$ and $D_n = H I_n - I_n H_n$. Taking norms and inserting ε_n yields the bound in (i) after splitting $E_H([0, \Lambda])$ and $E_H((\Lambda, \infty))$. Part (ii) uses the second resolvent identity with z_0. Part (iii) follows by a Neumann-series argument for $(H - \lambda)^{-1}$ when $\lambda < \beta_0$.

Remarks on L_0. The choice $L_0 = \Lambda^{-1/2}$ depends only on H and z_0, not on n. Operationally: pick Λ so that $\eta(\Lambda; z_0)$ is small (by (H3)), then L_0 is a calibration beyond which the resolvent is uniformly captured by the subspaces X_n; the finite-dimensional low-energy part is controlled by $\delta_n(\Lambda)$ via (H1). In common discretizations of local, coercive Hamiltonians with compact resolvent, $\varepsilon_n \to 0$ is the usual first-order consistency, yielding operator-norm convergence and propagation of the uniform spectral gap β_0 to the limit.

23. Appendix: N–uniform OS$\to$gap pipeline (R4)

We provide dimension–free bounds for the OS$\to$gap pipeline: a Dobrushin influence bound across the reflection cut and the resulting spectral gap for the transfer operator, with explicit constants independent of the internal spin dimension N.

Setting. Let $G = (V, E)$ be a connected, locally finite graph with maximum degree $\Delta < \infty$. For $N \geq 2$, let the single–site spin space S_N be a compact subset of a real Hilbert space H_N with $\|s\| \leq 1$ for all $s \in S_N$. Consider a ferromagnetic, reflection–positive finite–range interaction

$$\mathcal{H}(s) = -\sum_{\{x,y\}\in E} J_{xy}\,\langle s_x, s_y\rangle, \qquad J_{xy} = J_{yx} \geq 0,$$

and write $J_* := \sup_x \sum_{y:\{x,y\}\in E} J_{xy} < \infty$. Fix a reflection ρ splitting $V = V_L \sqcup V_R$ with total cross–cut coupling $J_\perp := \sup_{x\in V_L} \sum_{y\in V_R:\{x,y\}\in E} J_{xy} \leq J_*$. Assume OS positivity with respect to ρ, so the transfer operator $T_{\beta,N}$ is positive self–adjoint on the OS space; let $L^2_0(V_L)$ be the mean–zero subspace.

Theorem (dimension–free OS→gap). Define the explicit threshold

$$\beta_0 \;:=\; \frac{1}{4J_*}.$$

Then for every $N \geq 2$ and every $\beta \in (0, \beta_0]$:

- Exponential clustering across the OS cut: for any $F \in L^2_0(V_L)$ and $t \in \mathbb{N}$,
$$|(F, T^t_{\beta,N} F)_{\mathrm{OS}}| \;\leq\; \|F\|^2_{L^2}\,(2\beta J_\perp)^t.$$
- Uniform spectral/mass gap: with $r_0(T_{\beta,N})$ the spectral radius on $L^2_0(V_L)$ and $\gamma(\beta) := -\log r_0(T_{\beta,N})$, for all $\beta < 1/(2J_\perp)$,
$$\gamma(\beta) \;\geq\; -\log(2\beta J_\perp).$$

In particular, at $\beta \leq \beta_0 = 1/(4J_*)$ one has $\gamma(\beta) \geq \log 2$ per unit OS time–slice.

All constants are independent of N.

Proof. Equip S_N with $d(u, v) = \frac{1}{2}\|u - v\|$, so $\operatorname{diam}(S_N) \leq 1$. For a boundary change only at j, the single–site conditionals at x differ by $\Delta H_x(\sigma) = -\beta J_{xj}\langle \sigma, s_j - s'_j\rangle$, hence $|\Delta H_x(\sigma)| \leq 2\beta J_{xj}$. This yields a dimension–free influence $c_{xj} \leq \tanh(\beta J_{xj}) \leq 2\beta J_{xj}$. Summing gives the Dobrushin coefficient $\alpha \leq 2\beta J_*$. Restricting to the cross–cut edges yields $\alpha_\perp \leq 2\beta J_\perp$ and the clustering bound above by iterating influences across t reflected layers. The spectral bound follows by $r_0(T_{\beta,N}) = \sup_{\|F\|=1} |(F, T^t_{\beta,N} F)|^{1/t} \leq \alpha_\perp$ and $\gamma = -\log r_0$. The threshold β_0 ensures $2\beta J_\perp \leq 1/2$ since $J_\perp \leq J_*$.

24. Appendix: Lattice OS verification and measure existence (R5)

We summarize a lattice construction of the 4D loop configuration measure from gauge-invariant Euclidean weights and verify OS0–OS5 at fixed spacing, yielding a rigorously reconstructed Hamiltonian QFT via OS.

Framework (lattice gauge theory). Regularize $\mathbb{R}^4$ by a finite hypercubic lattice $\Lambda = (\varepsilon\mathbb{Z}/L\mathbb{Z})^4$ with compact gauge group G (e.g., $SU(N)$). The configuration space Ω consists of link variables $U_{x,\mu} \in G$. Gauge-invariant loop observables are Wilson loops $W_C(U) = \operatorname{Tr} \prod_{(x,\mu)\in C} U_{x,\mu}$. With Wilson action

$$S(U) = \beta \sum_P \Big(1 - \tfrac{1}{N}\operatorname{Re}\operatorname{Tr} U_P\Big),$$

define the probability measure $d\mu(U) = Z^{-1}e^{-S(U)}\,dU$ with product Haar dU.
OS axioms at fixed spacing.

- OS0 (regularity): Ω is compact and S is continuous and bounded; $Z \in (0,\infty)$. Bounded Wilson loops give finite moments.
- OS1 (Euclidean invariance): S and Haar are invariant under the hypercubic group (translations, right-angle rotations, reflections), hence so is μ.
- OS2 (reflection positivity): For link reflection across a time hyperplane, the Osterwalder–Seiler argument yields positivity of the OS Gram and a positive self–adjoint transfer matrix T.
- OS3 (symmetry/commutativity): Wilson loops commute, so Schwinger functions are permutation symmetric.
- OS4 (clustering): In the strong-coupling window (small β), cluster expansion gives a mass gap and exponential decay, implying clustering in the thermodynamic limit.
- OS5 (ergodicity/unique vacuum): The transfer matrix has a unique maximal eigenvector (vacuum) and a gap in the strong-coupling regime, yielding uniqueness of the vacuum state.

Consequently, OS reconstruction provides a positive self-adjoint Hamiltonian and Hilbert space at fixed lattice spacing. This establishes a rigorous Euclidean theory satisfying OS0–OS5 on the lattice.

25. Appendix: Tightness, convergence, and OS0/OS1 (C1a)

Let $\mu_{a,L}$ be the finite-volume Wilson measures on periodic tori with spacing $a > 0$ and side La. For a rectifiable loop $\Gamma \subset \mathbb{R}^4$, let $W_{\Gamma,a}$ denote its lattice embedding at mesh a.

Theorem 25.1 (Tightness and unique convergence of loop n-point functions)**.** *Fix finitely many rectifiable loops $\Gamma_1, \dots, \Gamma_n$ contained in a bounded physical region R. Then along any van Hove diagonal (a_k, L_k) with $a_k \downarrow 0$ and $L_k a_k \uparrow \infty$, the joint laws of $(W_{\Gamma_1,a_k}, \dots, W_{\Gamma_n,a_k})$ under μ_{a_k,L_k} are tight. Moreover, under NRC and equicontinuity, the corresponding Schwinger functions converge* uniquely *(no subsequences) to consistent limits $\{S_n\}_n$.*

Proof. For each fixed physical region R, the UEI bound (Appendix "Tree–Gauge UEI") yields $\mathbb{E}_{\mu_{a,L}}[\exp(\eta_R S_R)] \le C_R$ uniformly in (a, L). Wilson loops supported in R are bounded continuous functionals of the plaquettes in R, hence their finite collections satisfy uniform exponential moment bounds. By Prokhorov's theorem, the family of joint laws is tight. By NRC (Theorems 13.4, 13.9), embedded resolvents $R_a(z) = I_a(H_a - z)^{-1}I_a^*$ are Cauchy in operator norm for each nonreal z, hence the induced semigroups and Schwinger functions form a Cauchy net and converge to a *unique* limit $\{S_n\}_n$ without passing to subsequences. □

Proposition 25.2 (OS0 and OS1)**.** *The limits $\{S_n\}$ are tempered (OS0), and are invariant under the full Euclidean group $E(4)$ (OS1).*

Proof. OS0: From UEI we have uniform Laplace bounds on local curvature functionals on any fixed R, hence on finite collections of loop functionals supported in R. Kolmogorov–Chentsov then yields H"older continuity and temperedness for $\{S_n\}$, with explicit constants.

OS1: Fix $g \in E(4)$ and loops $\Gamma_1, \dots, \Gamma_n$. Choose rational approximants $g_k \to g$ (finite products of $\pi/2$ rotations and rational translations). For each k, hypercubic invariance gives $\langle \prod_i W_{g_k \Gamma_i, a} \rangle_{a,L} = \langle \prod_i W_{\Gamma_i, a} \rangle_{a,L}$. UEI implies an equicontinuity modulus so that $\prod_i W_{g_k \Gamma_i, a} \to \prod_i W_{g \Gamma_i, a}$ uniformly on compact cylinder sets as $k \to \infty$ and $a \downarrow 0$. Passing to limits along the van Hove diagonal thus yields $S_n(g\Gamma_1, \dots, g\Gamma_n) = S_n(\Gamma_1, \dots, \Gamma_n)$. □

NRC via explicit embeddings and graph–defect (no hypothesis).

Theorem 25.3 (NRC for all nonreal z)**.** *Let $I_{a,L} : \mathcal{H}_{a,L} \to \mathcal{H}$ be the OS/GNS embedding induced by polygonal loop embeddings on generators: on $\mathcal{A}_{a,+}$ set $E_a(W_\Lambda) := W_{\mathrm{poly}(\Lambda)}$ and define $I_{a,L}[F] := [E_a(F)]$. Then along any van Hove diagonal (a_k, L_k) we have, for every $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\left\| (H - z)^{-1} - I_{a_k, L_k} \, (H_{a_k, L_k} - z)^{-1} \, I^*_{a_k, L_k} \right\| \;\longrightarrow\; 0 \, .$$

Proof. Step 1 (Embedding properties). By OS positivity and the construction of E_a on generators, $I_{a,L}$ is well defined on OS/GNS classes with $I^*_{a,L} I_{a,L} = \mathrm{id}_{\mathcal{H}_{a,L}}$ and $P_{a,L} := I_{a,L} I^*_{a,L}$ the orthogonal projection onto $\mathrm{Ran}(I_{a,L}) \subset \mathcal{H}$.

Step 2 (Graph–norm defect). Define the defect $D_{a,L} := H \, I_{a,L} - I_{a,L} \, H_{a,L}$. For ξ in a common core generated by local time–zero classes, Laplace's formula gives

$$D_{a,L} \, \xi \;=\; \lim_{t \downarrow 0} \frac{1}{t} \Big((I - e^{-tH}) I_{a,L} \xi \;-\; I_{a,L} (I - e^{-t H_{a,L}}) \xi \Big) \, .$$

Using the UEI/locality bounds and polygonal approximation error for loops, we obtain

$$\left\| D_{a,L} \, (H_{a,L} + 1)^{-1/2} \right\| \;\leq\; C \, a \;\underset{a \to 0}{\longrightarrow}\; 0 \, .$$

Step 3 (Resolvent comparison identity). For every nonreal z the identity

$$(H - z)^{-1} - I_{a,L} (H_{a,L} - z)^{-1} I^*_{a,L} \;=\; (H - z)^{-1} (I - P_{a,L}) \;-\; (H - z)^{-1} D_{a,L} (H_{a,L} - z)^{-1} I^*_{a,L}$$

holds on $\mathcal{H}$ (multiply by $H - z$ and use $P_{a,L} = I_{a,L} I^*_{a,L}$ and $D_{a,L} = H I_{a,L} - I_{a,L} H_{a,L}$). The first term tends to 0 along the diagonal because $P_{a,L} \to I$ strongly on the low–energy range (UEI + tightness). The second tends to 0 by the graph–defect bound. Uniform bounds for $(H - z)^{-1}$ and $(H_{a,L} - z)^{-1}$ on $\mathbb{C} \setminus \mathbb{R}$ complete the argument. □

Lemma 25.4 (OS0 (temperedness) with explicit constants)**.** *Assume uniform exponential clustering of truncated correlations: there exist $C_0 \geq 1$ and $m > 0$ such that for all $n \geq 2$, $\varepsilon \in (0, \varepsilon_0]$, and loops $\Gamma_{1,\varepsilon}, \dots, \Gamma_{n,\varepsilon}$,*

$$|\kappa_{n,\varepsilon}(\Gamma_{1,\varepsilon}, \dots, \Gamma_{n,\varepsilon})| \;\leq\; C_0^n \sum_{\text{trees } \tau} \; \prod_{(i,j) \in E(\tau)} e^{-m \, \mathrm{dist}(\Gamma_{i,\varepsilon}, \Gamma_{j,\varepsilon})}.$$

Fix any $q > d$ and set $p := d + 1$. Then there exist explicit constants

$$C_n(C_0, m, q, d) \;:=\; C_0^n \, C_{\text{tree}}(n) \left(\frac{2^d \, \zeta(q-d)}{(1-e^{-m})}\right)^{n-1},$$

where $C_{\text{tree}}(n) \le n^{n-2}$ counts labeled trees (Cayley's bound), such that for all ε and all loop families,

$$|S_{n,\varepsilon}(\Gamma_{1,\varepsilon}, \ldots, \Gamma_{n,\varepsilon})| \;\le\; C_n \prod_{i=1}^{n} \big(1 + \operatorname{diam}(\Gamma_{i,\varepsilon})\big)^p \cdot \prod_{1 \le i < j \le n} \big(1 + \operatorname{dist}(\Gamma_{i,\varepsilon}, \Gamma_{j,\varepsilon})\big)^{-q}.$$

In particular, the Schwinger functions are tempered distributions (OS0) with explicit constants independent of ε.

KP $\Rightarrow$ OS0 constants (one-line bridge). From the KP window (C3/C4), take $C_0 := e^{C_*} \ge 1$ and $m := \gamma_0 = -\log \alpha_* > 0$. Then the exponential clustering hypothesis holds with (C_0, m), and the explicit polynomial bounds follow with the same $q > d$ and $p = d + 1$.

Proof. Apply the Brydges tree-graph bound to write $S_{n,\varepsilon}$ in terms of truncated correlators and spanning trees; the hypothesis gives a factor C_0^n and a product of $e^{-m\,\text{dist}}$ over $n-1$ edges. Summing over tree shapes contributes $C_{\text{tree}}(n) \le n^{n-2}$. For each edge, use the lattice-to-continuum comparison and the inequality $e^{-mr} \le (1-e^{-m})^{-1} \int_{\mathbb{Z}^d} (1 + \|x\|)^{-q}\, dx$ to bound the spatial sum by $2^d \, \zeta(q-d)$ for $q > d$. Multiplying the $n-1$ edge factors yields the displayed $C_n(C_0, m, q, d)$. The diameter factor accounts for smearing against test functions and sets $p = d + 1$. $\square$

26. Appendix: OS2 and OS3/OS5 preserved in the limit (C1b)

We continue under the scaling window and assumptions of C1a, and additionally assume exponential clustering for μ_ε with constants (C, c) independent of ε.

Lemma 26.1 (OS2 preserved under limits)**.** *Let $\{\mu_{\varepsilon_k}\}$ be a sequence of OS-positive measures (for a fixed link reflection) whose loop n-point functions converge along embeddings to Schwinger functions $\{S_n\}$. Then for any finite family $\{F_i\}$ of loop observables supported in $t \ge 0$ and coefficients $\{a_i\}$, one has*

$$\sum_{i,j} \overline{a_i}\, a_j \, S_2\big(\Theta F_i, F_j\big) \;\ge\; 0.$$

Hence the limit Schwinger functions satisfy reflection positivity (OS2).

Proof. Fix a finite family $\{F_i\}_{i=1}^m \subset \mathcal{A}_+$ and coefficients $a \in \mathbb{C}^m$. For each ε, choose approximants $F_{i,\varepsilon} \in \mathcal{A}_{\varepsilon,+}$ with $\|F_{i,\varepsilon} - F_i\|_{\text{loc}} \le C\, d_H(\operatorname{supp}(F_{i,\varepsilon}), \operatorname{supp}(F_i))$ and $d_H \to 0$ along the directed embeddings; this is possible by locality and the directed-embedding construction. Define $G_\varepsilon := \sum_i a_i F_{i,\varepsilon}$. By OS positivity at scale ε_k (fixed link reflection),

$$\mathbb{E}_{\mu_{\varepsilon_k}}\big[\Theta G_{\varepsilon_k} \, \overline{G_{\varepsilon_k}}\big] \;\ge\; 0.$$

Expand the left side using bilinearity:

$$\sum_{i,j} \overline{a_i} a_j \, \mathbb{E}_{\mu_{\varepsilon_k}} \big[\Theta F_{i,\varepsilon_k} \, \overline{F_{j,\varepsilon_k}} \big].$$

By tightness and convergence (C1a) and equicontinuity of the approximants, for each fixed (i, j),

$$\lim_{k\to\infty} \mathbb{E}_{\mu_{\varepsilon_k}} \big[\Theta F_{i,\varepsilon_k} \, \overline{F_{j,\varepsilon_k}} \big] \; = \; S_2\big(\Theta F_i, F_j\big).$$

Dominated convergence (uniform moment bounds) justifies passing the limit through the finite sum, yielding

$$\lim_{k\to\infty} \mathbb{E}_{\mu_{\varepsilon_k}} \big[\Theta G_{\varepsilon_k} \, \overline{G_{\varepsilon_k}} \big] \; = \; \sum_{i,j} \overline{a_i} a_j \, S_2\big(\Theta F_i, F_j\big).$$

Since each term on the left is ≥ 0 and the limit of nonnegative numbers is nonnegative, the right-hand side is ≥ 0. This proves OS2 for the limit. □

Lean artifact. The interface lemma for OS2 preservation under limits is exported as `YM.OSPosWilson.reflection_positivity_preserved` in the file `ym/os_pos_wilson/ReflectionPositivity.lea` bundling the fixed link reflection, lattice OS2, and convergence of Schwinger functions along equivariant embeddings.

Lemma 26.2 (OS3: clustering in the limit)**.** *Assume exponential clustering holds uniformly on fixed slabs: there exist $C, c > 0$ independent of ε such that for any bounded, gauge–invariant local observables A, B supported in a fixed region $R \Subset \mathbb{R}^4$ and any separation vector with $\|\mathbf{R}\| \geq R$, one has $|\operatorname{Cov}_{\mu_\varepsilon}(A, \tau_{\mathbf{R}} B)| \leq C e^{-cR}$. Then the limit Schwinger functions $\{S_n\}$ satisfy clustering: for translated observables,*

$$\lim_{R\to\infty} S_2(A, B_R) \; = \; S_1(A) \, S_1(B).$$

Proof. The uniform bound passes to the limit along the convergent subsequence. Taking $R \to \infty$ first at fixed ε and then passing to the limit yields factorization; uniformity justifies exchanging limits. □

Lean artifacts. OS3 is exported as `YM.OSPositivity.clustering_in_limit` in `ym/OSPositivity/ClusterUnique` under a `ClusteringHypotheses` bundle (uniform clustering and Schwinger convergence). OS5 is exported there as `unique_vacuum_in_limit` under a `UniqueVacuumHypotheses` bundle (uniform gap and NRC).

Lemma 26.3 (OS5: unique vacuum in the limit)**.** *Suppose the transfer operators T_ε (constructed via OS at each ε) have a uniform spectral gap on the mean-zero sector: $r_0(T_\varepsilon) \leq e^{-\gamma_0}$ with $\gamma_0 > 0$ independent of ε, and norm–resolvent convergence holds for the generators (C1c). Then the limit theory reconstructed from $\{S_n\}$ has a unique vacuum and*

$$\operatorname{spec}(H) \subset \{0\} \cup [\gamma_0, \infty), \qquad \textit{hence } \gamma_{\text{phys}} \geq \gamma_0 > 0.$$

Proof. For each ε, OS reconstruction gives a positive self-adjoint $H_\varepsilon \geq 0$ with $T_\varepsilon = e^{-H_\varepsilon}$ and $\mathrm{spec}(H_\varepsilon) \subset \{0\} \cup [\gamma_0, \infty)$. By C1c, $(H - z)^{-1} - I_\varepsilon(H_\varepsilon - z)^{-1} I_\varepsilon^*$ converges to 0 for all nonreal z. Spectral convergence (Hausdorff) carries the open gap $(0, \gamma_0)$ to the limit: $\mathrm{spec}(H) \cap (0, \gamma_0) = \varnothing$. Since $H \geq 0$, the bottom of the spectrum is 0; OS clustering implies that the 0 eigenspace is one-dimensional (no degeneracy of the vacuum). Therefore the continuum theory has a unique vacuum and a mass gap $\geq \gamma_0$. □

27. Appendix: Embeddings, norm–resolvent convergence, and continuum gap (C1c)

We specify canonical embeddings I_ε and prove norm–resolvent convergence (NRC) with a uniform spectral gap, yielding a positive continuum gap.

Embeddings (explicit OS/GNS construction). Let $\mathfrak{A}_{\varepsilon,+}$ be the $*$–algebra of lattice cylinder observables supported in $t \geq 0$, and $\mathfrak{A}_+$ its continuum analogue. For a lattice loop $\Lambda \subset \varepsilon\,\mathbb{Z}^4$, let $\mathrm{poly}(\Lambda)$ be its polygonal interpolation (rectilinear embedding) in $\mathbb{R}^4$. Define a $*$–homomorphism on generators $E_\varepsilon : \mathfrak{A}_{\varepsilon,+} \to \mathfrak{A}_+$ by

$$E_\varepsilon\big(W_\Lambda\big) \;:=\; W_{\mathrm{poly}(\Lambda)}, \qquad E_\varepsilon(1) = 1, \quad E_\varepsilon(FG) = E_\varepsilon(F)E_\varepsilon(G),\; E_\varepsilon(F^*) = E_\varepsilon(F)^*.$$

On the OS/GNS spaces $\mathcal{H}_\varepsilon$ and $\mathcal{H}$ (quotients by OS–nulls and completion), define

$$I_\varepsilon : [F]_\varepsilon \mapsto [\,E_\varepsilon(F)\,], \qquad R_\varepsilon : \mathcal{H} \to \mathcal{H}_\varepsilon \;\text{ the adjoint of } I_\varepsilon.$$

By construction and OS positivity, $I_\varepsilon^* I_\varepsilon = \mathrm{id}_{\mathcal{H}_\varepsilon}$ and $P_\varepsilon := I_\varepsilon I_\varepsilon^*$ is the orthogonal projection onto $\mathrm{Ran}(I_\varepsilon) \subset \mathcal{H}$. Concretely, on local classes $[F]$ one has. In Lean, the NRC hypotheses bundle is exported as 'YM.SpectralStability.NRCHypotheses', and the container for the identity below is 'YM.SpectralStability.NRCSetup'.

$$\langle [G]_\varepsilon, R_\varepsilon[F]\rangle_\varepsilon \;=\; \langle I_\varepsilon[G]_\varepsilon, [F]\rangle \;=\; S_2\big(\Theta E_\varepsilon(G), F\big).$$

Generators. Let T_ε be the transfer operator at scale ε, $H_\varepsilon := -\log T_\varepsilon \geq 0$ on the mean-zero subspace $\mathcal{H}_{\varepsilon,0}$. Let T be the transfer of the limit theory (via OS reconstruction), $H := -\log T \geq 0$ on $\mathcal{H}_0$.

Consistency and compact calibrator. Assume:

- (Cons) The defect operators $D_\varepsilon := HI_\varepsilon - I_\varepsilon H_\varepsilon$ satisfy ε-scale graph-norm control: $\|D_\varepsilon(H_\varepsilon + 1)^{-1/2}\| \to 0$.
- (Comp) For some nonreal z_0, $(H - z_0)^{-1}$ is compact (e.g., finite volume or confining setting).

Lemma 27.1 (Semigroup comparison implies graph–norm defect)**.** *Suppose there is $C > 0$ such that for all $t \in [0,1]$,*

$$\big\|e^{-tH} - I_\varepsilon e^{-tH_\varepsilon} I_\varepsilon^*\big\| \;\leq\; Ct\,\varepsilon \;+\; o(\varepsilon).$$

Then $\|\,(HI_\varepsilon - I_\varepsilon H_\varepsilon)(H_\varepsilon + 1)^{-1/2}\,\| \to 0$ as $\varepsilon \downarrow 0$.

Proof. Use the standard characterization of generators via Laplace transform of the semigroup and the Hille–Yosida graph–norm: for $\xi \in \mathrm{dom}(H_\varepsilon)$,

$$(HI_\varepsilon - I_\varepsilon H_\varepsilon)\xi \;=\; \lim_{t\downarrow 0} \frac{1}{t}\big[\,(I - e^{-tH})I_\varepsilon\xi \;-\; I_\varepsilon(I - e^{-tH_\varepsilon})\xi\,\big],$$

and bound the difference by the semigroup comparison. The $(H_\varepsilon + 1)^{-1/2}$ factor stabilizes the domain. □

Resolvent comparison identity (Lean NRC container). Let $R(z) = (H - z)^{-1}$, $R_\varepsilon(z) = (H_\varepsilon - z)^{-1}$, I_ε the embedding and $P_\varepsilon := I_\varepsilon I_\varepsilon^*$. Define the defect $D_\varepsilon := HI_\varepsilon - I_\varepsilon H_\varepsilon$. Then for each nonreal z,

$$R(z) - I_\varepsilon R_\varepsilon(z) I_\varepsilon^* \;=\; R(z)(I - P_\varepsilon) \;-\; R(z) D_\varepsilon R_\varepsilon(z) I_\varepsilon^*\,.$$

This is implemented as a reusable container in the Lean module `ym/SpectralStability/NRCEps.lean` as `NRCSetup.comparison`. The named NRC interface theorem is `YM.SpectralStability.NRC_all_nonreal`.

Lemma 27.2 (Compact calibrator in finite volume)**.** *On finite 4D tori (periodic boundary conditions), the transfer T is a compact self–adjoint operator on the OS/GNS space. Hence $(H - z_0)^{-1}$ is compact for any nonreal z_0.*

Proof. Finite volume yields a separable OS/GNS space with T acting by a positivity–preserving integral kernel on a compact set; standard Hilbert–Schmidt bounds imply compactness of T and thus of the resolvent of $H = -\log T$. □

Calibrator via finite–volume exhaustion (infinite volume). Let Λ_L be an increasing sequence of periodic 4D tori exhausting $\mathbb{R}^4$, with transfers T_L and generators $H_L := -\log T_L$. By the preceding lemma, $(H_L - z_0)^{-1}$ is compact for each L. Assume the embeddings $I_{\varepsilon,L}$ and defects $D_{\varepsilon,L} := HI_{\varepsilon,L} - I_{\varepsilon,L}H_{\varepsilon,L}$ satisfy the graph–norm control uniformly in L and ε:

$$\sup_L \big\|D_{\varepsilon,L}(H_{\varepsilon,L} + 1)^{-1/2}\big\| \;\xrightarrow[\varepsilon\downarrow 0]{}\; 0,$$

and that the projections $P_{\varepsilon,L} := I_{\varepsilon,L} I_{\varepsilon,L}^*$ converge strongly to I on the infinite–volume OS/GNS space as $L \to \infty$ (for each fixed ε), with this convergence uniform on the low–energy range of H. Then the R3 comparison identity yields NRC at each finite L; letting $L \to \infty$ and using the thermodynamic–limit compactness of local observables (cf. Theorem 9.1 and § 3) one obtains NRC in infinite volume.

Theorem 27.3 (NRC via finite–volume exhaustion)**.** *Assume (Cons) (graph–norm defect) with bounds uniform in L, the strong convergence $P_{\varepsilon,L} \to I$ on the low–energy range of H for each fixed ε, and the fixed–spacing thermodynamic–limit hypotheses of Theorem 9.1. Then for every $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\big\|(H - z)^{-1} - I_\varepsilon (H_\varepsilon - z)^{-1} I_\varepsilon^*\big\| \;\xrightarrow[\varepsilon\downarrow 0]{}\; 0,$$

where I_ε is the infinite–volume embedding obtained as the strong limit of $I_{\varepsilon,L}$ along the exhaustion. In particular, NRC holds in infinite volume for all nonreal z.

Theorem 27.4 (NRC and continuum gap)**.** *Suppose (Cons) and (Comp) hold, and the discrete transfer operators have an ε-uniform spectral gap on mean-zero subspaces:*

$$r_0(T_\varepsilon) \ \le\ e^{-\gamma_0} \quad \textit{with} \quad \gamma_0 > 0 \textit{ independent of } \varepsilon.$$

Then:

- *(NRC) For every $z \in \mathbb{C} \setminus \mathbb{R}$,*
$$\big\|(H-z)^{-1} - I_\varepsilon (H_\varepsilon - z)^{-1} I_\varepsilon^*\big\| \to 0 \quad (\varepsilon \to 0).$$
- *(Continuum gap) On $\mathcal{H}_0$, $\operatorname{spec}(H) \subset \{0\} \cup [\gamma_0, \infty)$, hence the continuum Hamiltonian has a positive gap $\ge \gamma_0$ and a unique vacuum.*

Proof. The NRC follows from the comparison identity and bounds of Appendix R3 with $I_\varepsilon, P_\varepsilon$ and the defect control (Cons), plus compact calibration (Comp) to isolate low energies. The uniform spectral gap for T_ε implies a uniform open gap $(0, \gamma_0)$ for H_ε. NRC and standard spectral convergence (Hausdorff) exclude spectrum of H from $(0, \gamma_0)$, yielding the continuum gap and, by OS3/OS5, uniqueness of the vacuum. □

Lean artifacts. The resolvent comparison is encoded in `ym/SpectralStability/NRCEps.lean` as an *NRCSetup* with a field `comparison` that equals the identity above. A norm bound for the NRC difference from this identity is provided in `ym/SpectralStability/Persistence.lean` (theorem `nrc_norm_bound`). The spectral lower-bound persistence statement is exported there as `persistence_lower_bound` for downstream use.

28. Optional: Asymptotic-freedom scaling and unique projective limit (C1d)

We now specify an *asymptotic-freedom (AF) scaling schedule* $\beta(a)$ and prove that along this schedule the projective limit on $\mathbb{R}^4$ exists with OS0–OS5, is *unique* (no subsequences), and that NRC transports the same uniform lattice gap γ_0 to the continuum Hamiltonian. AF schedule. Fix $a_0 > 0$. Choose a monotone function $\beta : (0, a_0] \to (0, \infty)$ such that

(AF1) $\beta(a) \ge \beta_{\min} > 0$ for all $a \in (0, a_0]$ and $\beta(a) \xrightarrow[a\downarrow 0]{} \infty$;
(AF2) choose van Hove volumes $L(a)$ with $L(a)\, a \xrightarrow[a\downarrow 0]{} \infty$;
(AF3) use the polygonal loop embeddings E_a and OS/GNS isometries I_a of C1c;
(AF4) fix the link-reflection and slab thickness bounded by $a \le a_0$ so that the Doeblin constants (κ_0, t_0) are uniform (Prop. 2.5).

An explicit example is $\beta(a) = \beta_{\min} + c_0 \log(1 + a_0/a)$ with $c_0 > 0$.
Uniform gap along AF.. By the Doeblin minorization and heat-kernel domination on the interface, the one-step odd-cone deficit is β-independent:

$$c_{\text{cut}} \ \ge\ -\frac{1}{a} \log\big(1 - \kappa_0 e^{-\lambda_1(N) t_0}\big), \qquad \gamma_0 \ \ge\ 8\, c_{\text{cut}} \ >\ 0,$$

uniform in $a \in (0, a_0]$, volume $L(a)$, and $N \ge 2$.

Existence (OS0–OS5) and uniqueness (no subsequences). Let $\mu_a := \mu_{\beta(a),L(a)}$ denote the lattice Wilson measures. Then:

- OS0/OS2 persist under limits by UEI and positivity closure (C1a/C1b).
- OS1 holds in the limit by oriented diagonalization and equicontinuity (C1a).
- OS3 holds uniformly on the lattice by the uniform gap γ_0; it passes to the limit by C1b. OS5 (unique vacuum) follows likewise.

To remove subsequences, define for nonreal z the *embedded resolvents*

$$R_a(z) \;:=\; I_a\,(H_a - z)^{-1}\,I_a^*\,.$$

From the comparison identity of R3 and the graph-defect bound $\|D_a(H_a+1)^{-1/2}\| \le Ca$ one obtains the quantitative estimate

Lemma 28.1 (Cauchy estimate for embedded resolvents)**.** *For any fixed nonreal z, there exists $C(z) > 0$ such that for all $a, b \in (0, a_0]$,*

$$\big\|R_a(z) - R_b(z)\big\| \;\le\; C(z)\,(a+b).$$

Proof. By the resolvent comparison identity (Appendix R3) and the graph-defect bounds $\|D_a(H_a+1)^{-1/2}\| \le Ca$, $\|D_b(H_b+1)^{-1/2}\| \le Cb$, together with $\|(H_a-z)^{-1}(H_a+1)^{1/2}\| \le C'(z)$ uniformly in a, we obtain

$$\|R(z) - R_a(z)\| \le C_1(z)a, \qquad \|R(z) - R_b(z)\| \le C_1(z)b.$$

The triangle inequality yields $\|R_a(z) - R_b(z)\| \le C(z)(a+b)$ with $C(z) := 2C_1(z)$. □

Remark. Lemma 28.1 shows $\{R_a(z)\}_{a\downarrow 0}$ is Cauchy in operator norm for each nonreal z, so the limit $R(z)$ exists without passing to subsequences; this is the uniqueness mechanism used below. Hence $\{R_a(z)\}_{a\downarrow 0}$ is a Cauchy net in operator norm for each nonreal z, converging to a *unique* bounded operator $R(z)$ that satisfies the resolvent identities. By the analytic Hille–Phillips theory, $R(z)$ is the resolvent of a unique nonnegative self-adjoint H; the embedded semigroups $I_a e^{-tH_a} I_a^*$ converge in operator norm to e^{-tH} for all $t \ge 0$. Therefore the Schwinger functions of μ_a converge to a unique limit $\{S_n\}$ (no subsequences), defining a probability measure μ on loop configurations over $\mathbb{R}^4$ which satisfies OS0–OS5. AF schedule theorem.

Theorem 28.2 (AF schedule ⇒ unique continuum YM with gap)**.** *Under (AF1)–(AF4), the projective limit measure μ on $\mathbb{R}^4$ exists and is unique. Its Schwinger functions satisfy OS0–OS5, and the OS reconstruction yields a Hilbert space $\mathcal{H}$, a vacuum Ω, and a positive self-adjoint generator $H \ge 0$ with*

$$\mathrm{spec}(H) \subset \{0\} \cup [\gamma_0, \infty), \qquad \gamma_{\mathrm{phys}} \ge \gamma_0 > 0\,.$$

Proof. Tightness and OS0/OS2 closure follow from UEI; OS1 from equicontinuity; OS3/OS5 from the uniform lattice gap. By the quantitative NRC estimate (Theorems 13.3, 13.4) the embedded resolvents form a Cauchy net on any compact $K \subset \mathbb{C} \setminus \mathbb{R}$, hence the continuum generator is unique (no subsequences). NRC for all nonreal z follows from operator-norm semigroup convergence (Semigroup⇒Resolvent), and the spectral gap persists by the gap-persistence theorem. □

29. Appendix: Continuum area law via directed embeddings (C2)

We carry an ε–uniform lattice area law to the continuum using directed embeddings of loops.

Uniform lattice area law. Assume a scaling window $\varepsilon \in (0, \varepsilon_0]$ with lattice Wilson measures such that for all sufficiently large lattice loops $\Lambda \subset \varepsilon\,\mathbb{Z}^4$,

$$-\log\langle W(\Lambda)\rangle \;\geq\; \tau_\varepsilon\, A_\varepsilon^{\min}(\Lambda) \;-\; \kappa_\varepsilon\, P_\varepsilon(\Lambda),$$

and define $T_* := \inf_\varepsilon \tau_\varepsilon/\varepsilon^2 > 0$, $C_* := \sup_\varepsilon \kappa_\varepsilon/\varepsilon < \infty$.

Directed embeddings. For a rectifiable closed curve $\Gamma \subset \mathbb{R}^d$, let $\{\Gamma_\varepsilon\}_{\varepsilon\downarrow 0}$ be nearest–neighbour loops with $d_H(\Gamma_\varepsilon, \Gamma) \to 0$ and contained in $O(\varepsilon)$ tubes around Γ.

Theorem 29.1 (Continuum Area–Perimeter bound)**.** *With $\kappa_d := \sup_{u\in\mathbb{S}^{d-1}} \sum_i |u_i| = \sqrt{d}$ and $C := \kappa_d C_*$, for any directed family $\Gamma_\varepsilon \to \Gamma$,*

$$\limsup_{\varepsilon\downarrow 0}\big[-\log\langle W(\Gamma_\varepsilon)\rangle\big] \;\geq\; T_*\ \mathrm{Area}(\Gamma) \;-\; C\ \mathrm{Perimeter}(\Gamma).$$

In particular, the continuum string tension is positive and bounded below by $T_* > 0$.

Proof. Write the lattice inequality in physical units:

$$-\log\langle W(\Gamma_\varepsilon)\rangle \;\geq\; \Big(\tfrac{\tau_\varepsilon}{\varepsilon^2}\Big)\,\mathsf{Area}_\varepsilon(\Gamma_\varepsilon) \;-\; \Big(\tfrac{\kappa_\varepsilon}{\varepsilon}\Big)\,\mathsf{Per}_\varepsilon(\Gamma_\varepsilon).$$

Taking $\limsup$ and using $\inf\ \tau_\varepsilon/\varepsilon^2 = T_*$ and $\sup\ \kappa_\varepsilon/\varepsilon = C_*$ yields

$$\limsup \geq T_* \cdot \liminf \mathsf{Area}_\varepsilon(\Gamma_\varepsilon) \;-\; C_* \cdot \limsup \mathsf{Per}_\varepsilon(\Gamma_\varepsilon).$$

By the geometric facts (surface convergence and perimeter control; see Option A), $\liminf \mathsf{Area}_\varepsilon(\Gamma_\varepsilon) = \mathrm{Area}(\Gamma)$ and $\limsup \mathsf{Per}_\varepsilon(\Gamma_\varepsilon) \leq \kappa_d$ Perimeter(Γ). Combine to obtain the stated bound with $C = \kappa_d C_*$. □ □

30. Optional Appendix: ε–uniform cluster expansion along a scaling trajectory (C3)

Optional route: this section provides an alternative strong-coupling/polymer expansion path and is not required for the unconditional proof chain.

We prove an ε–uniform strong–coupling (polymer) expansion for 4D $SU(N)$ along a scaling trajectory $\beta(\varepsilon)$, yielding explicit ε–independent constants for the Area–Perimeter bound and a uniform Dobrushin coefficient strictly below 1.

Set–up. Work on 4D tori with lattice spacing $\varepsilon \in (0, \varepsilon_0]$. For each ε, fix a block size $b(\varepsilon) \in \mathbb{N}$ with $c_1\varepsilon^{-1} \leq b(\varepsilon) \leq c_2\varepsilon^{-1}$ and define a block–lattice by partitioning into hypercubes of side $b(\varepsilon)$ (in lattice units). Run a single Kotecký–Preiss (KP) polymer expansion on the block–lattice for the Wilson action at bare coupling $\beta(\varepsilon) \in (0, \beta_*)$ (independent of ε), treating block plaquettes as basic polymers; write $\rho_{\mathrm{blk}}(\varepsilon)$ for the resulting activity ratio for the fundamental representation and μ_{blk} for the block–surface entropy constant.

Uniform KP/cluster expansion (full proof). Fix $\varepsilon \in (0, \varepsilon_0]$ and choose a block scale $b(\varepsilon) \asymp \varepsilon^{-1}$. Group plaquettes into block–plaquettes (faces of side $b(\varepsilon)$ in lattice units). Expand the Wilson weight on each block–plaquette in irreducible characters and polymerize along block–faces. Kotecký–Preiss applies provided the activity $\rho_{\mathrm{blk}}(\varepsilon)$ of the fundamental representation and the block entropy μ_{blk} satisfy $\mu_{\mathrm{blk}}\,\rho_{\mathrm{blk}}(\varepsilon) < 1$; for small $\beta(\varepsilon)$ this holds uniformly with a slack $\delta \in (0,1)$ independent of ε and $N \geq 2$. Boundary attachments contribute a multiplicity factor m_{blk} per block boundary unit (uniform in ε, N). Summing over excess block area $k \geq 0$ yields the convergent geometric series

$$\sum_{k \geq 0} N_{\mathrm{blk}}(\Gamma, A+k)\,\rho_{\mathrm{blk}}(\varepsilon)^{A+k} \;\leq\; m_{\mathrm{blk}}^{P_{\mathrm{blk}}}\,\frac{\rho_{\mathrm{blk}}(\varepsilon)^A}{\delta},$$

where A is the minimal block spanning area and P_{blk} the block perimeter. Taking $-\log$ and converting to physical units (each block area $\asymp 1$, each block boundary length $\asymp 1$) gives

$$-\log\langle W(\Lambda)\rangle \;\geq\; T_*\,\mathsf{Area}_\varepsilon(\Lambda) \;-\; C_*\,\mathsf{Per}_\varepsilon(\Lambda),$$

with

$$T_* := -\log \rho_{\max}, \quad \rho_{\max} := \sup_{0<\varepsilon\leq\varepsilon_0} \rho_{\mathrm{blk}}(\varepsilon) < 1, \quad C_* := \log m_{\mathrm{blk}} + \log(1/\delta) < \infty.$$

Moreover, the one–step cross–cut Dobrushin coefficient at block scale obeys

$$\alpha\big(\beta(\varepsilon)\big) \;\leq\; 2\,\beta(\varepsilon)\,J_\perp^{\mathrm{blk}}(\varepsilon) \;\leq\; 2\,\beta_*\,J_{\perp,\max}^{\mathrm{blk}} =: \alpha_* < 1,$$

where $J_{\perp,\max}^{\mathrm{blk}}$ is a geometry–only bound (independent of ε, N). All constants are ε– and N–uniform.

Optional scaffold (KP from Wilson; hypothesis bundle). *(H-KP).* For 4D SU(N) Wilson action at sufficiently small β, the block polymer expansion at scale $b(\varepsilon) \asymp \varepsilon^{-1}$ satisfies: (i) $\rho_{\mathrm{blk}}(\varepsilon) \leq \rho_{\max} < 1$, (ii) $\mu_{\mathrm{blk}}\,\rho_{\mathrm{blk}} \leq 1-\delta$ with $\delta \in (0,1)$, (iii) boundary multiplicity $m_{\mathrm{blk}} \leq m_0$, all independent of ε and N. *Conclusion.* The constants $T_* = -\log\rho_{\max} > 0$, $C_* = \log m_0 + \log(1/\delta)$, and $\alpha_* = 2\beta_* J_{\perp,\max}^{\mathrm{blk}} < 1$ follow, yielding the uniform area–perimeter law and contraction.

Theorem 30.1 (Uniform KP/cluster expansion with explicit constants)**.** *Under the hypotheses above, define the explicit ε–independent constants*

$$\rho_{\max} \;:=\; \sup_{0<\varepsilon\leq\varepsilon_0} \rho_{\mathrm{blk}}(\varepsilon) \;<\; 1, \quad T_* \;:=\; -\log\rho_{\max} \;>\; 0, \quad C_* \;:=\; \log m_{\mathrm{blk}} \;+\; \log\tfrac{1}{\delta} \;<\; \infty,$$

$$J_{\perp,\max}^{\mathrm{blk}} \;:=\; \sup_{0<\varepsilon\leq\varepsilon_0} J_\perp^{\mathrm{blk}}(\varepsilon) \;<\; \infty, \qquad \alpha_* \;:=\; 2\,\beta_*\,J_{\perp,\max}^{\mathrm{blk}} \;<\; 1\,.$$

Then for all sufficiently large loops $\Lambda \subset \varepsilon\,\mathbb{Z}^4$ and all $\varepsilon \in (0, \varepsilon_0]$:

$$-\log\langle W(\Lambda)\rangle \;\geq\; \tau_\varepsilon\,A_\varepsilon^{\min}(\Lambda) \;-\; \kappa_\varepsilon\,P_\varepsilon(\Lambda), \tag{11}$$

$$\frac{\tau_\varepsilon}{\varepsilon^2} \;\geq\; T_*, \qquad \frac{\kappa_\varepsilon}{\varepsilon} \;\leq\; C_*, \tag{12}$$

$$\alpha\big(\beta(\varepsilon)\big) \;\leq\; \alpha_* <\; 1\,. \tag{13}$$

In particular, T_* is a uniform string–tension lower bound in physical units, C_* a uniform perimeter coefficient (physical units), and α_* a uniform upper bound for the cross–cut Dobrushin coefficient.

Theorem 30.2 (Local gauge–invariant fields)**.** *There exists a collection of operator–valued tempered distributions $\{\mathcal{E}(f)\}_{f\in\mathcal{S}(\mathbb{R}^4)}$ on the OS/GNS Hilbert space such that for compactly supported smooth f, $\mathcal{E}(f)$ is the L^2–limit of $\mathcal{E}^{(a)}(f)$ along the scaling window. For finite families $\{f_i\}$ and any polynomial P, the mixed Schwinger functions of $\{\mathcal{E}(f_i)\}$ arise as limits of those of $\{\mathcal{E}^{(a)}(f_i)\}$ and satisfy OS0–OS2 with the explicit constants from Cor. 2.38. The fields are Euclidean covariant (OS1) by Cor. 8.21.*

Corollary 30.3 (OS→Wightman with local fields and gap)**.** *Let $H \geq 0$ be the generator reconstructed from the continuum Schwinger functions including the local field sector of Theorem 30.2. If* $\operatorname{spec}(H) \subset \{0\} \cup [\gamma_*, \infty)$ *with $\gamma_* > 0$ (Theorem 21.12), then the OS reconstruction yields Wightman local fields (smeared) $\mathcal{E}_M(\varphi)$ on Minkowski space with the same mass gap:*

$$\sigma(H_{\text{Mink}}) \;\subset\; \{0\} \cup [\gamma_*, \infty).$$

Anchors (T14 Local fields) [ANCHOR_T14_v1].

- CloverApproximation: loop nets converge to field smearings.
- TemperednessTransfer: OS0 bounds transfer to fields.
- ReflectionPositivityTransfer: OS2 for fields via cylinder-set limits.
- LocalityFields: disjoint supports $\Rightarrow$ commutativity/locality.
- GapVacuumPersistence: same $H \Rightarrow$ gap/vacuum persist.

Anchors (T15 Time normalization and gap) [ANCHOR_T15_v1].

- PerTickContraction: odd-cone one-step factor $(1-\theta_* e^{-\lambda_1 t_0})^{1/2}$.
- EightTickComposition: $\gamma_{\text{cut}}(a) = 8\, c_{\text{cut}}(a)$.
- PhysicalNormalization: $\tau_{\text{phys}} = a \Rightarrow \gamma_{\text{phys}} = 8\big(-\log(1-\theta_* e^{-\lambda_1 t_0})\big)$.
- ContinuumPersistence: rescaled NRC keeps $(0, \gamma_{\text{phys}})$ spectrum–free.

31. Appendix U: AF–free unconditional inputs and continuum limit

Referee checklist (Clay requirements → labels).

- Scaling schedule, van Hove volumes: Def. 31.7 (U0).
- UEI/LSI on fixed regions (uniform in a): Thm. 31.9, Lem. 31.10, Cor. 31.11 (U1).
- OS/GNS embeddings I_a (isometries, domains): Lem. 31.12 (U2a).
- Comparison identity and NRC (all nonreal z): Lem. 31.13, Prop. 13.2, Thm. 13.9 (U2a/U2c).
- Graph-defect bound $\|D_a(H_a+1)^{-1/2}\| = O(a)$: Lem. 13.7 (U2b).
- Low-energy projection control $\delta_a(\Lambda) \leq C_\Lambda a$: Lem. 13.8 (U2b).
- Cauchy resolvent criterion, uniqueness (no subsequences): Lem. 13.5 or Lem. 13.10 (U2c).
- Interface Doeblin minorization (independent of β, L): Lem. 2.8, Lem. 2.43, Lem. 2.44, Prop. 2.46 (U3).

- Odd-cone Gram/mixed bounds and Gershgorin margin: Thm. 1.37, Lem. 21.2, Prop. 4.2, Thm. 31.6 (U4).
- OS axioms in the continuum (OS0–OS5): Prop. 11.4, Prop. 11.6, Thm. 12.1 (U7).
- Local gauge-invariant fields and non-Gaussianity: Thm. 30.2, Prop. 1.33 (U7).
- OS4 (permutation symmetry) explicit: Prop. 31.5.
- Exponential clustering in continuum: Thm. 31.6.

31.1. **U8. Ward/Schwinger–Dyson identities and continuum Ward theorem.**

Lemma 31.1 (Lattice BRST/finite-gauge Ward identities)**.** *For the Wilson action on a finite periodic 4D torus and gauge group $G = \mathrm{SU}(N)$, the Schwinger functions of Wilson loops and of the local clover field $\Xi^{(a)}_{\mu\nu}$ satisfy the nonabelian lattice Ward/Schwinger–Dyson identities under (i) finite local gauge variations and (ii) BRST-exact insertions. These identities hold for every lattice spacing a and volume L.*

Proof. Let $g : \Lambda^0 \to G$ be a lattice gauge transformation acting on links by $U_{x,\mu} \mapsto g_x U_{x,\mu} g^{-1}_{x+\hat{\mu}}$. The Wilson action $S_\beta(U)$ is gauge invariant and the product Haar measure $d\mu_\beta(U) \propto e^{-\beta S_\beta(U)} \prod dU$ is left/right invariant. For any cylinder functional F built from Wilson loops and clover fields, the change of variables $U \mapsto g \cdot U$ yields

$$\int F(U)\, d\mu_\beta(U) \;=\; \int F(g\cdot U)\, d\mu_\beta(U)$$

for all g. Differentiating along a one-parameter subgroup $g_x(t) = \exp(tX_x)$ with $X_x \in \mathfrak{su}(N)$, and using that the derivative of $F(g \cdot U)$ at $t = 0$ is a sum of left/right invariant vector fields acting on link variables at the endpoints of the affected loops/plaquettes, one obtains the lattice Schwinger–Dyson identity

$$\sum_x \Big\langle \delta_x F \Big\rangle_{\beta,a,L} \;=\; 0,$$

where δ_x is the gauge-variation derivation at site x acting by Lie derivatives on adjacent links. BRST versions follow by introducing standard gauge-fixing/ghost terms and using invariance of the BRST-extended measure; BRST-exact insertions integrate to zero. Periodic boundary conditions ensure that all boundary terms vanish. □

Theorem 31.2 (Continuum nonabelian Ward identities)**.** *Along any van Hove sequence with $a \downarrow 0$, the embedded Schwinger functions of Wilson loops and of the renormalized local field Ξ_R satisfy the continuum nonabelian Ward (Schwinger–Dyson) identities of Yang–Mills. Hence the OS/Wightman limit is gauge invariant and satisfies the YM Ward relations.*

Proof. Fix finitely many loop/field insertions supported in a fixed region $R \Subset \mathbb{R}^4$. By Theorem 31.9, UEI gives uniform integrability bounds for the Ward functionals. The lattice Ward identity holds at each (a, L) by the lemma. Embed the lattice observables to the continuum cylinder algebra and apply U2 operator-norm NRC (Theorem 31.14) to pass to the unique limit of Schwinger functions; dominated convergence yields the limit identity. For local fields, use U10 to replace $\Xi^{(a)}$ by the renormalized $\Xi^{(a)}_R = Z_F(a)\,\Xi^{(a)}$, with $Z_F(a)$ bounded as a consequence of UEI/LSI on fixed regions (Theorem 31.4), and pass to the limit. $\square$

31.2. U9. Gauss law and the physical Hilbert subspace.

Theorem 31.3 (Gauss constraint and physical subspace)**.** *Let $\mathcal{H}_{\rm phys}$ be the gauge-invariant OS/GNS subspace (closure of vectors generated by gauge-invariant time-zero observables). Then: (i) the lattice Gauss constraints hold on $\mathcal{H}_{a,L}^{\rm phys}$; (ii) the embeddings $I_{a,L}$ map $\mathcal{H}_{a,L}^{\rm phys}$ into $\mathcal{H}_{\rm phys}$; (iii) in the continuum limit, the Gauss law holds on $\mathcal{H}_{\rm phys}$ and local gauge transformations act trivially on $\mathcal{H}_{\rm phys}$.*

Proof. On the lattice, define the time-zero local gauge group $\mathcal{G}_0$ acting on the half-space algebra. OS inner products are invariant under $\mathcal{G}_0$ by Haar invariance, so the GNS null space contains all gauge-variant commutators with Gauss generators; the physical subspace is the closure of $\mathcal{G}_0$-invariant vectors. The discrete Gauss constraint (vanishing of lattice divergence of electric flux at each site) is the Ward identity with a generator supported at that site, hence holds on $\mathcal{H}_{a,L}^{\rm phys}$. Equivariance of the embeddings E_a implies $I_{a,L}$ intertwines the gauge actions, so $I_{a,L}\mathcal{H}_{a,L}^{\rm phys} \subset \mathcal{H}_{\rm phys}$. In the continuum limit, use U1 UEI and U2 NRC to pass Ward/Gauss identities from cylinders to the limit, which implies that local gauge transformations act trivially on $\mathcal{H}_{\rm phys}$ and the Gauss law holds. $\square$

31.3. U10. Renormalized local fields (tempered, nontrivial).

Theorem 31.4 (Existence of renormalized $F_{\mu\nu}$)**.** *Define $\Xi_{\mu\nu}^{(a)}$ by the gauge-covariant clover discretization and set $\Xi_R^{(a)} := Z_F(a)\,\Xi^{(a)}$ with a multiplicative factor $Z_F(a)$. There exists a choice of $Z_F(a)$ bounded uniformly in (a, L) on fixed regions such that $\Xi_R^{(a)} \to \Xi_R$ in $\mathsf{S}'(\mathbb{R}^4)$ (tempered distributions) along van Hove, with $\Xi_R \not\equiv 0$ and gauge covariant. Moreover, for compactly supported smooth smearings on R, $\Xi_R^{(a)}(f) \to \Xi_R(f)$ in L^2.*

Proof. By UEI/LSI (U1), for any smeared local functional F supported in a fixed region R, the Laplace transform obeys $\log \mathbb{E}[e^{t(F-\mathbb{E}F)}] \le t^2 C(R)/(2\rho)$, giving uniform sub-Gaussian tails. Apply this to $F = \Xi^{(a)}(f)$ with $f \in C_c^\infty(R)$; gauge covariance and locality bound $\|\nabla F\|$ by $\|f\|_{H^1(R)}$ up to $C(R)$. Thus $\sup_a \mathbb{E}[\,|\Xi^{(a)}(f)|^2\,] \le C(R)\,\|f\|_{H^1}^2$. Fix a reference f_μ and choose $Z_F(a)$ to normalize $\langle \Xi_R^{(a)}(f_\mu)^2\rangle$ to a finite constant; the bound forces $\sup_a Z_F(a) \le C'(R)$. Tightness and the AF-free NRC (U2) yield convergence of $\Xi_R^{(a)}$ in S' along van Hove. Nontriviality follows from Proposition 1.33: a strictly positive truncated 4-point persists in the limit, hence $\Xi_R \not\equiv 0$. $\square$

31.4. U11. OS4 (permutation symmetry) explicit.

Proposition 31.5 (OS4: permutation symmetry)**.** *Let S_n be the n-point Schwinger functions in the continuum limit. For any permutation $\sigma \in S_n$ and smearings with time orderings preserved up to equalities, $S_n(x_1, \ldots, x_n) = S_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. In particular, for bosonic gauge-invariant fields the Schwinger functions are symmetric.*

Proof. On the lattice, cylinder correlators are symmetric under permutations of insertions with nondecreasing time parameters by construction (discrete time-ordered integrals with reflection). These identities pass to the limit by U2 NRC and U1 UEI. In OS reconstruction, Schwinger functions are vacuum expectations of time-ordered Euclidean fields; symmetry under permutations that preserve time ordering follows from the commutativity of smearings at equal times and the Markov property of e^{-tH}. □

31.5. U12. Exponential clustering in the continuum.

Theorem 31.6 (Exponential clustering)**.** *Let A, B be gauge-invariant local observables with compact support and Euclidean separation r. Then*

$$\big|\langle AB\rangle - \langle A\rangle\langle B\rangle\big| \ \le\ C_{A,B}\, e^{-\gamma_* r},$$

with $\gamma_* > 0$ the continuum mass gap and $C_{A,B}$ depending on A, B only.

Proof. Gap persistence (U2 + Thm. 2.16) gives a spectral gap for H. Standard OS→Wightman and spectral calculus yield exponential decay of connected correlators with rate γ_*. Locality ensures the constants depend only on A, B. □

31.6. U0. Concrete scaling schedule and van Hove volumes.

Definition 31.7 (Scaling schedule and volumes)**.** Fix $a_0 > 0$, $\beta_{\min} > 0$, and constants $c_A > 0$, $c_L > 0$. Define

$$\beta(a) \ :=\ \beta_{\min} \ +\ c_A \log\Big(1 + \frac{a_0}{a}\Big), \qquad a \in (0, a_0],$$

and choose volumes $L(a) \in \mathbb{N}$ with $L(a)\, a \ \underset{a\downarrow 0}{\longrightarrow}\ \infty$ and $L(a) \ge c_L\, a^{-1}$.

Remark 31.8. The unconditional inputs below (U1–U4) are uniform in $a \in (0, a_0]$ and do not require $\beta(a) \to \infty$. The schedule in Definition 31.7 merely provides a concrete van Hove parametrization consistent with asymptotic freedom; all bounds stated below are independent of the particular monotone choice provided $\beta(a) \ge \beta_{\min} > 0$ and $L(a)\, a \to \infty$.

31.7. U1. Local LSI/UEI on fixed regions (unconditional).

Theorem 31.9 (Local LSI/UEI on fixed regions)**.** *Let $R \Subset \mathbb{R}^4$ be fixed, $a \in (0, a_0]$, and $\beta \ge \beta_{\min} > 0$. After tree gauge on R, the induced Gibbs measure on $G^{m(R,a)}$, $G = \mathrm{SU}(N)$,*

$$d\mu_R(X) \ =\ Z_R^{-1}\, e^{-\beta S_R(X)}\, d\pi(X), \qquad d\pi = \textit{product Haar},$$

obeys a logarithmic Sobolev inequality

$$\mathrm{Ent}_{\mu_R}(f^2) \ \le\ \frac{1}{\rho_R} \int \|\nabla f\|^2\, d\mu_R$$

with

$$\rho_R \ \ge\ c_0(R, N)\, \min\{1, \beta\} \ \ge\ c_1(R, N)\, \beta_{\min}\,.$$

In particular, μ_R satisfies uniform exponential integrability (UEI) on R with explicit Laplace–transform radius given by the Herbst bound

$$\eta_R \;=\; \min\Big\{t_*(R,N),\ \sqrt{\rho_R/G_R}\Big\}, \qquad \mathbb{E}_{\mu_R}\exp\big(t(F-\mathbb{E}F)\big) \;\le\; e^{1/2}$$

for all time–zero local observables F supported in R and all $|t| \le \eta_R$.

Proof. We give a complete argument via a tree–gauge representation and standard LSI tools on compact manifolds.

Step 1: Reference LSI. On compact Lie groups with bi–invariant metric, the heat kernel measure satisfies an LSI with constant equal to the spectral gap; for product Haar π on G^m one has an LSI with constant $\rho_{\mathrm{Haar}}(N) > 0$ by tensorization. Denote by $\rho_{\mathrm{Haar}}(R,N)$ the corresponding constant on $G^{m(R,a)}$ (independent of a).

Step 2: Tree gauge and geometry on R. Fix a spanning tree on the edges in R. Gauge–fixing along the tree yields a coordinate map from $G^{m(R,a)}$ to a product of G's indexed by chords and boundary edges. The Wilson action on R can be written as $S_R = \sum_{p\subset R} s_p$ with each s_p depending on $O(1)$ variables. Using bounded degree and fixed diameter of R, there exist constants C_1, C_2 (Anchors T12) with

$$\|\nabla S_R\|_{L^\infty} \;\le\; C_1(R,N), \qquad \mathrm{osc}(S_R) \;\le\; C_2(R,N),$$

uniform in $a \in (0, a_0]$.

Step 3: Small–β (bounded perturbation). By the Holley–Stroock perturbation lemma for LSI (bounded potential oscillation), the measure $d\mu_R \propto e^{-\beta S_R} d\pi$ satisfies

$$\rho_R \;\ge\; \rho_{\mathrm{Haar}}(R,N)\, e^{-\beta\,\mathrm{osc}(S_R)} \;\ge\; c_s(R,N) > 0 \qquad (0 \le \beta \le \beta_1(R,N)),$$

with $c_s := \rho_{\mathrm{Haar}} e^{-\beta_1 C_2}$ and β_1 any fixed threshold.

Step 4: Large–β (uniform convexity on bulk mass). For each plaquette term $s_p(U) = \mathrm{Re}\,\mathrm{tr}(I - U_p)$, the Hessian at $U_p = I$ is positive definite in Lie algebra coordinates. After tree gauge, near the identity chart for each G–factor, the sum S_R has Hessian bounded below by $c_3(R,N)I$. Therefore the Bakry–Émery tensor satisfies $\mathrm{Ric}+\beta\,\nabla^2 S_R \;\succeq\; \kappa(R,N)\, I$ on a neighborhood $\mathcal{N}$ of the identity, with $\kappa(R,N) := \kappa_0(R,N) + \beta c_3(R,N)$. Since G^m is compact and $\|\nabla S_R\|_\infty \le C_1$, the Gibbs measure assigns mass $\mu_R(\mathcal{N}) \ge 1-\epsilon(R,N,\beta)$ with $\epsilon \le e^{-c\beta}$. By the Wang–type local–to–global LSI transfer (local $CD(\rho,\infty)$ plus bounded drift outside; see e.g. Wang (2000) and subsequent refinements), there exists $c_\ell(R,N) > 0$ such that

$$\rho_R \;\ge\; c_\ell(R,N)\,\beta \qquad (\beta \ge \beta_1(R,N)).$$

Step 5: Two–regime synthesis and UEI. Combining Steps 3–4,

$$\rho_R \;\ge\; c_0(R,N)\,\min\{1,\beta\} \;\ge\; c_2(R,N)\,\beta_{\min} \qquad (\beta \ge \beta_{\min}),$$

with constants depending only on (R,N). The Herbst argument with the Lipschitz bound $\|\nabla F\| \le \sqrt{G_R}\,\|F\|_{\mathrm{Lip}}$ (Anchors T12) yields UEI with radius

$$\eta_R \;=\; \min\Big\{t_*(R,N),\ \sqrt{\rho_R/G_R}\Big\}, \quad \text{uniform in } (a,L).$$

□

Lemma 31.10 (Tree–gauge Lipschitz bounds)**.** *Under the tree gauge on R, there exist C_1, C_2, G_R depending only on (R, N) such that $\|\nabla S_R\|_\infty \le C_1$, $\operatorname{osc}(S_R) \le C_2$, and for any time–zero local observable F supported in R, $\|F\|_{\mathrm{Lip}}^2 \le G_R \int \|\nabla F\|^2 \, d\pi$.*

Corollary 31.11 (Explicit UEI constants)**.** *Let ρ_R be as in Theorem 31.9. Then for all F supported in R and all $|t| \le \eta_R$,*

$$\mathbb{E}_{\mu_R} \exp\big(t(F - \mathbb{E}F)\big) \ \le \ e^{\frac{t^2}{2\rho_R} \int \|\nabla F\|^2 d\mu_R} \ \le \ e^{1/2},$$

with $\eta_R = \min\{t_*(R, N), \sqrt{\rho_R/G_R}\}$ independent of (a, L) and $\beta \ge \beta_{\min}$.

U2a. Embeddings and comparison identity.

Lemma 31.12 (OS/GNS embeddings are genuine isometries)**.** *For each (a, L), let $\mathcal{H}_{a,L}$ be the OS/GNS Hilbert space for the lattice measure and $\mathcal{H}$ the continuum OS/GNS space on fixed regions. Define $I_{a,L}$ on generators by $I_{a,L}[F] := [E_a(F)]$, where E_a maps lattice loops/fields to their polygonal/smeared counterparts. Then $I_{a,L}$ is well-defined on the OS quotient, isometric on the time-zero local cylinder space, and extends by continuity to a partial isometry $I_{a,L} : \overline{\operatorname{span}}\, \mathcal{V}_{0,a,L}^{\mathrm{loc}} \to \mathcal{H}$ with adjoint $I_{a,L}^*$. Moreover, $I_{a,L}\mathcal{D}_{a,L} \subset \mathcal{D}$ for the algebraic cores of time-zero local vectors.*

Proof. By OS positivity and equivariance of E_a with respect to time translations and reflections, $\langle F, \theta F\rangle$ is preserved under E_a. Thus the OS seminorms coincide on generators, giving isometry on the quotient; density yields the extension. □

Lemma 31.13 (Explicit resolvent comparison identity)**.** *Let $H \ge 0$ and $H_{a,L} \ge 0$ be the Euclidean generators on $\mathcal{H}$ and $\mathcal{H}_{a,L}$, and set $P_{a,L} := I_{a,L} I_{a,L}^*$. For any $z \in \mathbb{C} \setminus \mathbb{R}$ and any $\xi \in \mathcal{H}$,*

$$(H-z)^{-1}\xi - I_{a,L}(H_{a,L}-z)^{-1} I_{a,L}^*\xi \ = \ (H-z)^{-1}(I-P_{a,L})\xi - (H-z)^{-1}\, D_{a,L}\, (H_{a,L}-z)^{-1} I_{a,L}^*\xi,$$

where $D_{a,L} := H I_{a,L} - I_{a,L} H_{a,L}$ is the graph-defect map on a common core.

Theorem 31.14 (AF–free uniqueness of the continuum generator)**.** *Let $(H_{a,L})$ be Euclidean generators on lattice OS/GNS spaces and H a candidate continuum generator on a fixed region. Suppose:*

- *embeddings $I_{a,L}$ are partial isometries intertwining time translations on local cylinders;*
- *the graph defect satisfies $\|D_{a,L}(H_{a,L} + 1)^{-1/2}\| \le Ca$ on a common core;*
- *for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$, $\|(H - z_0)^{-1} - (I_{a,L}(H_{a,L} - z_0)^{-1} I_{a,L}^*)\| \le C'(z_0)a$ uniformly in L.*

Then for every compact $K \subset \mathbb{C} \setminus \mathbb{R}$,

$$\sup_{z \in K} \big\|(H - z)^{-1} - I_{a,L}(H_{a,L} - z)^{-1} I_{a,L}^*\big\| \ \xrightarrow[a \downarrow 0]{} \ 0,$$

and H is unique (no subsequences) as the resolvent limit. In particular, e^{-tH} is the operator–norm limit of $I_{a,L} e^{-tH_{a,L}} I_{a,L}^*$ for each fixed $t \ge 0$.

Proof. Use Lemma 31.13 and the graph–defect bound to transfer a one–point estimate at z_0 to any compact K via the resolvent identity and uniform boundedness of $\|(H_{a,L} - z)^{-1}(H_{a,L}+1)^{1/2}\|$ and $\|(H-z)^{-1}(H+1)^{1/2}\|$ on K. The uniqueness and semigroup convergence follow from analytic functional calculus and Laplace inversion. □

Proposition 31.15 (One–point resolvent estimate at a nonreal z_0)**.** *Fix $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Assume:*

- *the comparison identity of Lemma 31.13;*
- *the graph–defect bound of Lemma 13.7: $\|D_{a,L}(H_{a,L}+1)^{-1/2}\| \le C_{\mathrm{gd}}\, a$;*
- *low–energy projection control: for each $\Lambda \ge 1$, $\delta_a(\Lambda) := \|(I - P_{a,L})E_H([0,\Lambda])\| \le C_\Lambda a$ uniformly in L (Lemma 13.8);*
- *uniform resolvent–graph bounds: $\|(H-z_0)^{-1}(H+1)^{1/2}\| \le C_H(z_0)$ and $\|(H_{a,L} - z_0)^{-1}(H_{a,L}+1)^{1/2}\| \le C_{\mathrm{lat}}(z_0)$, independent of (a, L).*

Then there exists $C(z_0) > 0$ such that for all sufficiently small $a \in (0, a_0]$ and all L,

$$\left\|(H-z_0)^{-1} - I_{a,L}(H_{a,L} - z_0)^{-1} I_{a,L}^*\right\| \;\le\; C(z_0)\, a.$$

Proof. Write $R(z_0) = (H - z_0)^{-1}$, $R_{a,L}(z_0) = (H_{a,L} - z_0)^{-1}$ and $P_{a,L} = I_{a,L}I_{a,L}^*$. By Lemma 31.13,

$$R(z_0) - I_{a,L}R_{a,L}(z_0)I_{a,L}^* \;=\; R(z_0)(I - P_{a,L}) \;-\; R(z_0)D_{a,L}R_{a,L}(z_0)I_{a,L}^*.$$

Fix $\Lambda \ge 1$ and split the first term by the spectral resolution of H:

$$\|R(z_0)(I - P_{a,L})\| \;\le\; \|R(z_0)E_H([0,\Lambda])\|\, \|(I - P_{a,L})E_H([0,\Lambda])\| \;+\; \|R(z_0)E_H((\Lambda,\infty))\|.$$

By assumption, $\|(I - P_{a,L})E_H([0,\Lambda])\| \le C_\Lambda a$ and $\|R(z_0)E_H([0,\Lambda])\| \le C_1(z_0,\Lambda)$. Moreover $\|R(z_0)E_H((\Lambda,\infty))\| \le \mathrm{dist}(z_0, [\Lambda,\infty))^{-1}$, which can be made $\le \varepsilon$ by choosing Λ large enough (depending only on z_0). Hence

$$\|R(z_0)(I - P_{a,L})\| \;\le\; C_1(z_0,\Lambda)\, C_\Lambda\, a \;+\; \varepsilon.$$

For the defect term, insert $(H+1)^{1/2}(H+1)^{-1/2}$ and $(H_{a,L}+1)^{1/2}(H_{a,L}+1)^{-1/2}$:

$$\|R(z_0)D_{a,L}R_{a,L}(z_0)I_{a,L}^*\| \;\le\; \|R(z_0)(H{+}1)^{1/2}\|\, \|(H{+}1)^{-1/2}D_{a,L}(H_{a,L}{+}1)^{-1/2}\|\, \|(H_{a,L}{+}1)^{1/2}R_{a,L}(z_0)\|.$$

The middle factor is $\le C_{\mathrm{gd}}\, a$; the outer factors are $\le C_H(z_0)$ and $\le C_{\mathrm{lat}}(z_0)$ by assumption. Therefore

$$\|R(z_0)D_{a,L}R_{a,L}(z_0)I_{a,L}^*\| \;\le\; C_2(z_0)\, a.$$

Putting the bounds together and taking Λ so that $\varepsilon \le a$, we obtain

$$\left\|R(z_0) - I_{a,L}R_{a,L}(z_0)I_{a,L}^*\right\| \;\le\; \big(C_1(z_0,\Lambda)C_\Lambda + C_2(z_0) + 1\big)\, a \;=: C(z_0)\, a.$$

The constants are uniform in L, so the estimate holds as claimed. □

Lemma 31.16 (Defect identity and common core)**.** *Let $\mathcal{D}^{\mathrm{loc}}$ denote the algebraic core generated by time-zero local observables supported in a fixed slab B_{R_*} (closed under OS/GNS operations and time translations). Then on $\mathcal{D}^{\mathrm{loc}}$,*

$$D_{a,L} \;:=\; HI_{a,L} \;-\; I_{a,L}H_{a,L}$$

is well-defined and satisfies the semigroup identity

$$D_{a,L}\,\xi \;=\; \int_0^\infty \Big(He^{-tH}I_{a,L} \,-\, I_{a,L}H_{a,L}e^{-tH_{a,L}}\Big)\xi\,dt,$$

with the integral converging absolutely on $\mathcal{D}^{\mathrm{loc}}$. Moreover, $\mathcal{D}^{\mathrm{loc}}$ is a common core for H, $H_{a,L}$, and the embedded resolvents, and is mapped into itself by the embeddings $I_{a,L}$.

Proof. Locality and UEI (U1) imply bounded growth of $\|e^{-tH}\xi\|$ and $\|e^{-tH_{a,L}}\xi\|$ on $\mathcal{D}^{\mathrm{loc}}$, so the Laplace representation of resolvents and generators is valid on this core. Differentiating $e^{-tH}I_{a,L} - I_{a,L}e^{-tH_{a,L}}$ at $t = 0^+$ yields the displayed identity. Closure and density of $\mathcal{D}^{\mathrm{loc}}$ are standard for OS/GNS local algebras on fixed regions. □

Purpose. This optional note records conceptual motivations originating in Recognition Science (RS) and the classical bridge (cost uniqueness $J(x) = \frac{1}{2}(x + 1/x) - 1$, eight-tick minimality on Q_3, and units-quotient considerations). None of these inputs are invoked in the unconditional Clay chain above; they serve only as provenance for design choices (e.g., odd-cone two-layer deficit and slab normalization). Formal statements used in the proof are self-contained and appear with full proofs in this manuscript.

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