FINAL FIXES AND HONEST STATUS OF THE ZERO-FREE REGION PROOF

TECHNICAL COMPANION TO PAPER1_ZEROZETA-V19

1. Fixes implemented in this round

1.1. **Fix A: "holomorphic and nonvanishing" for harmonicity. Complaint:** The proof claimed $\log|\mathcal{J}_{\text{neut}}|$ is harmonic because $\mathcal{J}_{\text{neut}}$ is holomorphic, but holomorphic alone doesn't imply harmonicity of the log-modulus (zeros reintroduce singularities).

Fix: The theorem proof now explicitly states that $\mathcal{J}_{\text{neut}}$ is *holomorphic and nonvanishing* on D, with a sentence explaining why: the poles of $\mathcal{J}_{\text{out}}$ are exactly canceled (with multiplicity) by the zeros of B_{box}, and $\mathcal{J}_{\text{out}}$ has no zeros in D (the only zero at $s = 1$ lies outside D for large t_0).

1.2. **Fix B: Proposition restated for neutralized energy. Complaint:** The proposition's headline bound was for $E(I) = \iint |\nabla \log|\mathcal{J}_{\text{out}}||^2\sigma$, which is *infinite* if there are poles in the box.

Fix: The proposition now defines and bounds $E_{\text{neut}}(I)$, the energy of the *neutralized harmonic function* $\log|\mathcal{J}_{\text{neut}}| = 2\log|B| + \widetilde{W}$. The infinite-energy near-Blaschke singularities are explicitly excluded from the bound.

1.3. **Fix C: Constant notation. Complaint:** The proposition stated $C(\alpha', c)$ but the theorem proof treated it as $C(\alpha')$ independent of c.

Fix: The proposition now states $C(\alpha')$ throughout, with an explicit note that independence from c comes from the L-cancellation in the Poisson integral.

1.4. **Fix D: B_{box} definition and multiplicities. Complaint:** B_{box} was described as "zeros with ordinate in D" without specifying both coordinates or multiplicities.

Fix: The theorem proof now says "the zeros of ζ with ordinate in D (i.e. $|\gamma - \gamma_0| \le \alpha'' L$ and $\beta > 1/2$)" and states that each Blaschke factor cancels the corresponding pole "with multiplicity."

2. The remaining open step: the singular inner factor

2.1. **What was claimed before.** Previous versions claimed the singular inner factor S of the inner reciprocal $\mathcal{I} = B^2/\mathcal{J}_{\text{out}}$ could be controlled by the pointwise bound $-\log|S| \le W \le N\log(2+|t|)+C$.

2.2. **Why this is insufficient.** The bound $W(s) \le N\log(2 + |t|) + C$ at fixed height $\sigma > 0$ relies on a polynomial lower bound for $|\mathcal{I}(s)|$, which in turn requires a lower bound on $|\zeta(1/2 + \sigma + it)|$. For *fixed* σ, such bounds exist (Hadamard product + zero repulsion). But on the Whitney schedule $\sigma = \alpha'' L = \alpha'' c_0/\log^2\langle t_0\rangle \to 0$, the exponent $N(\sigma)$ degrades (grows as $1/\sigma$), and the resulting bound on M picks up an extra factor of $\log\langle t_0\rangle$.

Concretely: the singular measure ν_S has uniformly bounded mass $\nu_S([t_0 - 1, t_0 + 1]) \le \nu_* < \infty$ (from the bounded values of W at $\Re s = 3/2$). The Poisson integral at height $\sigma = \alpha'' L$ is $P_\sigma[\nu_{S,\text{near}}] \le \nu_*/(\pi\alpha'' L) = O(\log\langle t_0\rangle/c)$. With $c = c_0/\log$: this is $O(\log^2/c_0)$, adding one uncancelable log to M.

The energy then grows as $M^2|I| = O(\log^4/c_0^2 \cdot c_0/\log^2) = O(\log^2/c_0)$, and $\sqrt{E}\cdot L = O(\sqrt{c_0}/\log)$ while the lower bound is $O(c_0/\log^2)$. The ratio Upper/Lower $= O(\log/\sqrt{c_0}) \to \infty$.

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2.3. **What would close the proof.** The proof is complete **if** $S \equiv 1$ (equivalently $\nu_S = 0$). Under this condition:

- $M = O(\log\langle t_0 \rangle)$ with constant independent of c.
- $E_{\mathrm{neut}} = O(c_0)$ (height-independent after the $c = c_0/\log$ trick).
- Ratio Upper/Lower $= A\sqrt{c_0}/11 < 1$ for $c_0 < (11/A)^2/2$. Contradiction. ✓

Three potential routes to establishing $S \equiv 1$:

(1) Show that the boundary log-modulus limits of each factor of $\mathcal{I}$ converge in $L^1(\mathbb{R}, (1+t^2)^{-1}dt)$ (not just L^1_{loc}), and that the sum converges to 0 in that weighted sense. This would give $\int W(\sigma, \cdot)/(1+t^2)\,dt \to 0$ as $\sigma \to 0$, which is the standard criterion for $S \equiv 1$.

(2) Show that $(s-1)\zeta(s)/s$ restricted to Ω belongs to the *Cartwright class* (entire of exponential type with $\log^+$ in $L^1(\mathbb{R}, (1+t^2)^{-1}dt)$), which implies its inner factor is a pure Blaschke product with no singular part (Krein/Koosis theory).

(3) Bound the singular measure ν_S directly from the explicit formula for $\mathcal{I} = B\mathcal{O}_\zeta\zeta/\det_2$, using the Carleson energy of $\det_2$ and the boundary trace properties of ζ.

3. Current state of the proof

Component	**Status**	**Unconditional?**
Inner reciprocal $\mathcal{I}$, $\lvert\mathcal{I}\rvert \le 1$	Proved	Yes
Neutralized ratio $\mathcal{J}_{\mathrm{neut}}$	Proved	Yes
Phase-velocity lower bound	Proved	Yes
CR–Green on harmonic $\log\lvert\mathcal{J}_{\mathrm{neut}}\rvert$	Proved	Yes
Blaschke-tail energy bound ($S \equiv 1$ case)	Proved	Yes
$c = c_0/\log$ cancellation algebra	Proved	Yes
Contradiction ($S \equiv 1$ case)	Proved	Yes
Singular inner factor $S \equiv 1$	**Open**	—

The paper's Theorem 1 is proved **conditional on** $S \equiv 1$. Establishing $S \equiv 1$ would make the proof fully unconditional.