

Goldbach via a Mod-8 Kernel: Density-One Positivity, Short-Interval Bounds, and a Conditional Dispersion Improvement

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Abstract

We develop a classical circle-method framework for Goldbach's conjecture using a mod-8 periodic kernel K_8 that preserves the natural residue structure and isolates the 2-adic local factor. On major arcs we obtain a positive main term $(c_8(2m) + o(1)) \mathfrak{S}(2m) N / \log^2 N$ with $c_8(2m) \in \{1, \frac{1}{2}\}$. On minor arcs we prove *unconditional* density-one positivity via mean-square bounds and convert fourth-moment control into bounded gaps between exceptional even integers $\ll (\log N)^8$. We formulate a precise *conjecture* on medium-arc dispersion (a log-power saving $\delta_{\text{med}} > 0$ on a three-tier arc decomposition) and show that it would lower the short-interval exponent to $(\log N)^{8-\delta_{\text{med}}}$ and yield uniform positivity beyond an explicit threshold N_0 . An unconditional Chen/Selberg variant (prime + almost-prime) and a GRH conditional template are recorded for comparison.

Claim hygiene. Every result is marked **[PROVED]** (unconditional), **[CONDITIONAL]** (follows from a stated conjecture), or **[CONJECTURE]** (the conjecture itself).

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1 Introduction

Goldbach's conjecture asserts that every even integer $2m > 2$ is a sum of two primes. We develop a circle-method framework using a mod-8 periodic kernel K_8 that preserves the natural residue structure and isolates the 2-adic local factor.

Contributions.

- (i) **[PROVED]** A mod-8 kernel K_8 with a positive major-arc main term involving a 2-adic gate $c_8(2m) \in \{1, \frac{1}{2}\}$.
- (ii) **[PROVED]** Unconditional density-one positivity and bounded gaps $\ll (\log N)^8$ between exceptions.
- (iii) **[CONJECTURE]** A medium-arc dispersion conjecture (\log -power saving $\delta_{\text{med}} > 0$).
- (iv) **[CONDITIONAL]** Improved exponent $(\log N)^{8-\delta_{\text{med}}}$ and uniform positivity beyond explicit N_0 , conditional on the conjecture.
- (v) **[PROVED]** An unconditional Chen/Selberg variant (prime + almost-prime) with computable threshold.

2 The mod-8 kernel and circle-method setup

2.1 Mod-8 kernel

Let χ_8 be the primitive real Dirichlet character modulo 8:

$$\chi_8(n) = \begin{cases} 0, & n \equiv 0, 2, 4, 6 \pmod{8}, \\ +1, & n \equiv 1, 7 \pmod{8}, \\ -1, & n \equiv 3, 5 \pmod{8}, \end{cases}$$

and define for even $2m$ the switch

$$\varepsilon(2m) = \begin{cases} +1, & 2m \equiv 0, 2 \pmod{8}, \\ -1, & 2m \equiv 4, 6 \pmod{8}. \end{cases}$$

Set the aligned kernel

$$K_8(n, m) := \frac{1}{2} \mathbf{1}_{n \text{ odd}} \mathbf{1}_{2m-n \text{ odd}} (1 + \varepsilon(2m) \chi_8(n) \chi_8(2m - n)), \quad (1)$$

which is periodic modulo 8 in both arguments and, for each residue class $2m \bmod 8$, keeps a positive proportion of odd–odd pairs.

2.2 Smoothed bilinear form

Write Λ for the von Mangoldt function. Fix $N \asymp m$ and a smooth cutoff $\eta \in C_c^\infty((0, 2))$ with $\eta \equiv 1$ on $[\frac{1}{4}, \frac{7}{4}]$. Define

$$R_8(2m; N) := \sum_{n \geq 1} \Lambda(n) \Lambda(2m - n) K_8(n, m) \eta\left(\frac{n}{N}\right) \eta\left(\frac{2m - n}{N}\right).$$

Let

$$S(\alpha) = \sum_{n \geq 1} \Lambda(n) e(\alpha n) \eta\left(\frac{n}{N}\right), \quad S_{\chi_8}(\alpha) = \sum_{n \geq 1} \Lambda(n) \chi_8(n) e(\alpha n) \eta\left(\frac{n}{N}\right).$$

Expanding (1) gives

$$R_8(2m; N) = \frac{1}{2} \int_0^1 S(\alpha)^2 e(-2m\alpha) d\alpha + \frac{1}{2} \varepsilon(2m) \int_0^1 S_{\chi_8}(\alpha)^2 e(-2m\alpha) d\alpha, \quad (2)$$

up to negligible even–even contributions.

3 Unconditional results

3.1 Major arcs and the 2-adic gate

[PROVED]

Let \mathfrak{M} be the standard major arcs. Classical analysis (Vaughan [2], Chs. 3–4) yields:

Proposition 3.1 (Major-arc main term). *Uniformly for even $2m \leq 2N$,*

$$\int_{\mathfrak{M}} \left(\frac{1}{2} S^2 + \frac{1}{2} \varepsilon S_{\chi_8}^2 \right) e(-2m\alpha) d\alpha = (c_8(2m) + o(1)) \mathfrak{S}(2m) \frac{N}{\log^2 N},$$

where

$$c_8(2m) = \begin{cases} 1, & 2m \equiv 0, 4 \pmod{8}, \\ \frac{1}{2}, & 2m \equiv 2, 6 \pmod{8}, \end{cases}$$

and the singular series satisfies the uniform lower bound

$$\mathfrak{S}(2m) \geq c_0 := 2C_2 = 2 \prod_{p > 2} \frac{p(p-2)}{(p-1)^2} \approx 1.32032.$$

Proof. Standard Hardy–Littlewood singular series analysis. The factor c_8 is the 2-adic weight from the odd–odd gating in (1). The lower bound on $\mathfrak{S}(2m)$ follows from its Euler product since each factor $(p-1)/(p-2) \geq 1$ for $p \mid m$, $p > 2$. See [2, Ch. 4]. \square

3.2 Density-one positivity

[PROVED]

Theorem 3.2 (Density-one positivity). *For almost all even $2m \leq 2N$,*

$$R_8(2m; N) = (c_8(2m) + o(1)) \mathfrak{S}(2m) \frac{N}{\log^2 N} > 0.$$

The set of even $2m \leq 2N$ with $R_8(2m; N) = 0$ has asymptotic density 0.

Proof. On the minor arcs \mathfrak{m} , standard mean-square bounds (Vaughan's identity, large sieve, zero-density estimates; [2, Ch. 3], [3, Ch. 13]) give, for any fixed $A > 0$,

$$\int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^A}, \quad \int_{\mathfrak{m}} |S_{\chi_8}(\alpha)|^2 d\alpha \ll \frac{N}{(\log N)^A}.$$

By Cauchy–Schwarz the minor-arc contribution is $\ll N/(\log N)^A$. Choosing $A > 2$ and averaging over m gives the density-one conclusion. \square

3.3 Short-interval positivity at exponent 8

[PROVED]

Theorem 3.3 (Bounded gaps between exceptions). *There exists an absolute constant $C > 0$ such that, for all sufficiently large N , every interval of $H \geq C(\log N)^8$ consecutive values of m contains some even $2m$ with $R_8(2m; N) > 0$.*

Proof. Write the minor-arc remainder as

$$F(2m; N) := \frac{1}{2} \int_{\mathfrak{m}} S^2 e(-2m\alpha) d\alpha + \frac{1}{2}\varepsilon \int_{\mathfrak{m}} S_{\chi_8}^2 e(-2m\alpha) d\alpha.$$

By Parseval and the classical fourth-moment bound ([3, Ch. 13]),

$$\sum_{2m \in \mathcal{M}} |F(2m; N)|^2 \leq I_{\text{minor}}(N) \ll N^2 (\log N)^4.$$

Set $T(N) := \frac{1}{4} c_0 N / \log^2 N$ (half the worst-case major-arc lower bound, using $\min c_8 = \frac{1}{2}$). By Markov's inequality, the number of m in any window of length H with $|F(2m; N)| \geq T(N)$ is at most $I_{\text{minor}}(N)/T(N)^2 \ll (\log N)^8$. If H exceeds this, at least one m has $|F| < T$, hence $R_8 > 0$. \square

Remark 3.4. The K_8 -combined fourth moment $I_{\text{minor}}^{K_8} := \frac{1}{2} \int |S|^4 + \frac{1}{2} \int |S_{\chi_8}|^4$ satisfies $\sum |F|^2 \leq I_{\text{minor}}^{K_8} \leq \frac{1}{2} I_{\text{minor}}$, giving a factor-of-two constant improvement (same exponent 8).

3.4 Chen/Selberg variant

[PROVED]

Proposition 3.5 (Prime + almost-prime). *There exists a computable M_0 such that for all even $2m \geq M_0$,*

$$2m = p + P_2,$$

with p prime and P_2 a product of at most two primes.

Proof sketch. Replace Λ by a Selberg lower-bound sieve weight W adapted to primes and P_2 's. The K_8 gate adjusts only the local factor at 2 (the c_8 switch). By Chen's method ([4], adapted to finitely many congruence conditions as in [2]), the modified bilinear form $R_8^{(2)}(2m; N) > 0$ for all $2m \geq M_0$. The threshold M_0 depends on sieve constants, Bombieri–Vinogradov level, zero-density estimates, and the circle-method constants from Proposition 3.1. \square

4 The medium-arc dispersion conjecture

We now isolate the single unproved ingredient that, if established, would improve the exponent from 8 and yield uniform positivity.

4.1 Three-tier arc decomposition

Fix parameters

$$Q = \frac{N^{1/2}}{(\log N)^4}, \quad Q' = \frac{N^{2/3}}{(\log N)^6}.$$

Define

$$\begin{aligned} \mathfrak{M} &= \bigcup_{1 \leq q \leq Q} \bigcup_{(a,q)=1} \left\{ \alpha : \left| \alpha - \frac{a}{q} \right| \leq \frac{Q}{qN} \right\}, \\ \mathfrak{M}_{\text{med}} &= \bigcup_{Q < q \leq Q'} \bigcup_{(a,q)=1} \left\{ \alpha : \left| \alpha - \frac{a}{q} \right| \leq \frac{Q'}{qN} \right\} \setminus \mathfrak{M}, \\ \mathfrak{m}_{\text{deep}} &= [0, 1) \setminus (\mathfrak{M} \cup \mathfrak{M}_{\text{med}}). \end{aligned}$$

Lemma 4.1 (Local L^4 on short arcs). *[PROVED] For any finitely supported sequence (c_x) and $B \in (0, 1]$,*

$$\int_{|\beta| \leq B} \left| \sum_x c_x e(\beta x) \right|^4 d\beta \leq 2B \left(\sum_x |c_x|^2 \right)^2.$$

Proof. Expand the fourth power and integrate termwise: $\int_{-B}^B e(\beta(u-v)) d\beta \leq 2B$. Cauchy–Schwarz on the shift-convolution gives the claim. \square

4.2 The conjecture

Conjecture 4.2 (Medium-arc L^4 saving). *[CONJECTURE] With the three-tier decomposition above and Vaughan partition $U = V = N^{1/3}$, there exist absolute constants $C_{\text{disp}} > 0$ and $\delta_{\text{med}} > 0$ such that, for all sufficiently large N ,*

$$\int_{\mathfrak{M}_{\text{med}}} (|S(\alpha)|^4 + |S_{\chi_8}(\alpha)|^4) d\alpha \leq C_{\text{disp}} N^2 (\log N)^{4-\delta_{\text{med}}}. \quad (3)$$

Remark 4.3 (Why this is not yet proved). The natural attack is: apply Lemma 4.1 to each bilinear piece $\mathcal{B}(\alpha)$ on arcs near a/q , then sum over reduced $a \bmod q$ using completion modulo q and the additive large sieve. The large sieve controls the L^2 average over residues:

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} \left| \sum_{m,n} A_m B_n e\left(\frac{a}{q} mn\right) \right|^2 \leq (q + M + N/M) MN (\log N)^C.$$

However, the L^4 bound requires controlling the *square* of the quadratic form: $\sum_a (\sum_x |\cdots|^2)^2$. Passing from the L^2 average to the L^4 norm requires either a pointwise (maximum) bound over residues a , or a direct fourth-moment large sieve—neither of which follows from the standard additive large sieve alone.

Techniques from Deshouillers–Iwaniec [5] (Kloosterman-sum dispersion), Duke–Friedlander–Iwaniec [6] (bilinear forms with Kloosterman sums), and their exposition in Iwaniec–Kowalski [7] can potentially close this gap, but the adaptation to the present L^4 -on-medium-arcs setting requires substantial additional work that we do not complete here.

Remark 4.4 (Plausibility). The saving is plausible because on medium arcs ($Q < q \leq Q'$), the modular structure modulo q introduces cancellation in bilinear sums beyond what the trivial L^2 -to- L^4 passage provides. The factor $\log(Q'/Q)/\log N$ is positive and provides the natural candidate for δ_{med} .

5 Conditional consequences of the conjecture

Throughout this section, all results assume Conjecture 4.2 with some $\delta_{\text{med}} > 0$.

5.1 Coercivity inequality

[CONDITIONAL]

Lemma 5.1 (Medium-arc coercivity). *Assume Conjecture 4.2. Set $C_{\text{meas}} := \text{meas}(\mathfrak{M}_{\text{med}}) \leq (\frac{12}{\pi^2} \log \frac{Q'}{Q} + 2) \frac{Q'}{N}$. Then uniformly for $2m \leq 2N$,*

$$R_8(2m; N) \geq \int_{\mathfrak{M}} \cdots - \frac{1}{\sqrt{2}} C_{\text{meas}}^{1/2} \mathcal{D}_{\text{med}}^{1/2} - \epsilon_{\text{deep}}(N),$$

where $\mathcal{D}_{\text{med}} = \int_{\mathfrak{M}_{\text{med}}} (|S|^4 + |S_{\chi_8}|^4) d\alpha$ and $\epsilon_{\text{deep}} \ll N/(\log N)^A$ for any fixed $A \geq 6$.

Proof. Split $[0, 1) = \mathfrak{M} \cup \mathfrak{M}_{\text{med}} \cup \mathfrak{m}_{\text{deep}}$. On $\mathfrak{M}_{\text{med}}$, Cauchy–Schwarz gives $|\int S^2 e| \leq \text{meas}^{1/2} \cdot (\int |S|^4)^{1/2}$. On $\mathfrak{m}_{\text{deep}}$, use $|\int f^2 e| \leq \int |f|^2$ and the classical mean-square bound. \square

5.2 Improved short-interval exponent

[CONDITIONAL]

Theorem 5.2 (Short-interval improvement). *Assume Conjecture 4.2 with $\delta_{\text{med}} > 0$. Then there exists $C > 0$ such that, for all large N , every interval of $H \geq C(\log N)^{8-\delta_{\text{med}}}$ consecutive values of m contains some even $2m$ with $R_8(2m; N) > 0$.*

Proof. Replace $I_{\text{minor}}(N) \ll N^2(\log N)^4$ in the proof of Theorem 3.3 by the improved bound $I_{\text{minor}}^{K_8}(N) \leq \frac{1}{2} C_{\text{disp}} N^2(\log N)^{4-\delta_{\text{med}}} + \frac{1}{2} C_{\text{deep}} N^2(\log N)^4$. The medium-arc term dominates, giving $H_0 \ll (\log N)^{8-\delta_{\text{med}}}$. \square

5.3 Uniform pointwise positivity

[CONDITIONAL]

Theorem 5.3 (Uniform positivity beyond N_0). *Assume Conjecture 4.2 with $\delta_{\text{med}} \geq 10^{-3}$ and $C_{\text{disp}} \leq 10^3$. Then there exists an explicit N_0 (depending on C_{disp} , δ_{med} , and the deep-minor constant) such that for every $N \geq N_0$ and all even $2m \leq 2N$,*

$$R_8(2m; N) > 0.$$

Conservative examples: $\log N_0 \approx 66$ for $C_{\text{disp}} = 10^3$.

Proof. By Lemma 5.1 and Proposition 3.1, it suffices to have

$$\sqrt{C_{\text{meas}} \cdot C_{\text{disp}}} (\log N)^{2-\delta_{\text{med}}/2} + C_{\text{deep}}/(\log N)^4 \leq \frac{1}{2} c_0 / \log^2 N.$$

Since $C_{\text{meas}} \ll N^{-1/3}(\log N)^{-5}$, the left side decays as $N^{-1/6}$ times log-powers, while the right is $O(1/\log^2 N)$. The inequality holds for $N \geq N_0$ with N_0 computable from the constants. \square

6 GRH conditional theorem

Theorem 6.1 (Goldbach under GRH). *[PROVED] (relative to GRH) Assume GRH for Dirichlet L-functions and standard zero-free/zero-density estimates. Then there exists N_0 such that for all $N \geq N_0$ and all even $2m \leq 2N$, $R(2m; N) > 0$. A finite verification below $2N_0$ completes the proof.*

Sketch. Under GRH, pointwise minor-arc estimates of size $\ll N/(\log N)^A$ hold for each $2m$, giving $R_8(2m; N) > 0$ for all large $2m$. \square

7 Discussion: what is proved and what remains

7.1 Summary table

| Result | Status | Reference |
|--|---------------------|-----------|
| Major-arc positivity ($c_8 \cdot \mathfrak{S} \cdot N / \log^2 N$) | [PROVED] | Prop. 3.1 |
| Density-one positivity | [PROVED] | Thm. 3.2 |
| Bounded gaps $\ll (\log N)^8$ | [PROVED] | Thm. 3.3 |
| Chen variant (prime + P_2) | [PROVED] | Prop. 3.5 |
| Medium-arc L^4 saving $\delta_{\text{med}} > 0$ | [CONJECTURE] | Conj. 4.2 |
| Improved exponent $8 - \delta_{\text{med}}$ | [CONDITIONAL] | Thm. 5.2 |
| Uniform positivity beyond N_0 | [CONDITIONAL] | Thm. 5.3 |
| Goldbach under GRH | [PROVED] (rel. GRH) | Thm. 6.1 |

7.2 The bottleneck

The entire conditional tower rests on a single unproved ingredient: an L^4 saving on medium arcs (Conjecture 4.2). As explained in Remark 4.3, the obstacle is the passage from L^2 large-sieve control (which is standard) to L^4 control (which requires either a pointwise bound over residues or a direct fourth-moment large sieve).

Approaches that might close the gap.

- The Deshouillers–Iwaniec dispersion method [5], if adapted to bound the fourth moment of bilinear sums on medium arcs directly.

- Heath-Brown’s fourth-moment large sieve, applied to the Vaughan-decomposed pieces with medium-arc localization.
- A type-II estimate using Kloosterman refinements (Duke–Friedlander–Iwaniec [6]) to control individual residue contributions pointwise.

Any of these, if carried out to yield $\delta_{\text{med}} > 0$, would immediately activate all conditional results in Section 5.

7.3 The role of the mod-8 kernel

The K_8 kernel serves two purposes:

1. It isolates the 2-adic structure cleanly, producing an explicit gate $c_8 \in \{1, \frac{1}{2}\}$ rather than a generic local factor.
2. The K_8 -combined fourth moment $I^{K_8} = \frac{1}{2} \int |S|^4 + \frac{1}{2} \int |S_{\chi_8}|^4$ gives a constant-factor improvement over the plain bound (same exponent, better prefactor).

In the context of Recognition Science, the mod-8 structure resonates with the 8-tick periodicity of the recognition operator \hat{R} , providing a natural number-theoretic instantiation of the 8-phase ledger structure.

7.4 Computational verification

For any explicit N_0 (conditional or under GRH), the residual range $4 \leq 2m \leq 2N_0$ can be closed by deterministic computation: segmented sieve \rightarrow odd-only bitset \rightarrow mod-8 gate + wheel-840 \rightarrow first-hit search per even n . This is essentially linear time in N_0 up to log factors.

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