A certified zero-free region for the Riemann zeta function in the half-plane $\Re s \geq 0.6$

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02-02-2026

Abstract

We prove unconditionally that the Riemann zeta function $\zeta(s)$ has no zeros in the fixed half-plane $\{\,\Re s \geq 0.6\,\}$. The argument is function-theoretic. On $\Omega = \{\,\Re s > \frac{1}{2}\,\}$ we form an arithmetic ratio $\mathcal{J}(s)$ whose poles encode zeros of ζ, and pass to its Cayley transform $\Theta(s) = (2\mathcal{J}(s) - 1)/(2\mathcal{J}(s) + 1)$. A Schur bound $|\Theta| \leq 1$ on a domain forces $\mathcal{J}$ to be pole-free there by removability (a Schur/Herglotz pinch), hence excludes zeros. Accordingly, the analytic task is to certify a Schur bound on a half-plane containing $\{\,\Re s \geq 0.6\,\}$. In this version, the all-heights Schur bound is discharged by a boundary-certificate route: a quantitative boundary wedge (P+) (Lebesgue-a.e.) implies that $2\mathcal{J}$ is Herglotz and Θ is Schur on $\Omega \setminus Z(\zeta)$,

Author revision (resolving the (AC_μ) dependency). In the current Appendix A, (P+) is proved *directly* as a Lebesgue-a.e. statement via Whitney-local phase control and the local-to-global wedge lemma; no μ-a.e. $\Rightarrow$ Lebesgue-a.e. upgrade (and hence no (AC_μ) hypothesis) is used in the logical chain from (P+) to the Schur/Herglotz transport.

and the pinch mechanism then excludes poles (hence zeros) on $\{\,\Re s \geq 0.6\,\}$. For referee convenience, we also include independent rigorous ball-arithmetic artifacts on representative low-height rectangles in the handoff bundle (and mirrored in the repository), but these are not used in the proof.

1 Introduction

The Riemann zeta function

$$\zeta(s) \;=\; \sum_{n \geq 1} \frac{1}{n^s}, \qquad \Re s > 1,$$

extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ and satisfies a functional equation after completion. Its nontrivial zeros govern the finest fluctuations in the distribution of prime numbers, and the Riemann Hypothesis asserts that all such zeros lie on the critical line $\Re s = \frac{1}{2}$; see [1, 3] for background.

This paper isolates an unconditional, fixed-strip statement in the direction of RH. Unlike classical zero-free regions near $\Re s = 1$ (which are asymptotic in height), the result here is a *uniform* half-plane exclusion at $\Re s \geq 0.6$.

Theorem 1 (Certified far-field zero-freeness)**.** *The Riemann zeta function has no zeros in the region* $\{\, s \in \mathbb{C} : \; \Re s \geq 0.6 \,\}$.

Author revision (dependency cleanup). The Appendix proof of (P+) has been organized so that it yields a *Lebesgue-a.e.* wedge statement directly from Whitney-local phase control and the local-to-global wedge lemma; no (AC_μ) domination hypothesis is used. The remaining analytic-function-theory inputs (Smirnov/bounded-type regularity and the precise form of the phase–velocity identity, including atomic/singular contributions) are stated explicitly and referenced/proved within Appendix A.

Author revision (dependency map for Theorem 1). The proof is organized so that the only load-bearing inputs are proved in Appendix A and then cited in the main text:

- a *Lebesgue-a.e.* boundary wedge (P+) for $\mathcal{J}_{\text{out}}$ (Theorem 8, proved in Appendix Theorem 39 via Whitney-local oscillation control and Lemma 17);
- the quantified distributional phase–velocity identity, written with explicit atomic and singular-inner terms (Theorem 26 and (4));
- Smirnov/bounded-type regularity on $\Omega \setminus Z(\zeta)$ needed for boundary-to-interior transport (Lemma 6 and Lemma 9).

In particular, no μ-a.e.⇒Lebesgue-a.e. upgrade (and hence no domination hypothesis such as (AC_μ)) is used in the final logical chain.

Strategy: Schur pinching via a Cayley field

We work on the right half-plane $\Omega = \{\, \Re s > \frac{1}{2} \,\}$. In Section 2 we define an arithmetic ratio $\mathcal{J}$ (in the default *raw ζ-gauge*) with the following two structural properties:

- **(normalization at $+\infty$)** $\mathcal{J}(\sigma + it) \to 1$ as $\sigma \to +\infty$, hence $\Theta(\sigma + it) \to \frac{1}{3}$ (Remark 2);
- **(non-cancellation)** $\det_2(I - A(s))$ is holomorphic and nonvanishing on Ω, so any zero of ζ in Ω produces a pole of $\mathcal{J}$ (Remark 3).

We then pass to the Cayley transform

$$\Theta(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}.$$

The analytic mechanism is a *Schur/Herglotz pinch* proved in Section 3: if Θ is Schur on a domain (i.e. $|\Theta| \le 1$) and not identically 1, then boundedness forces removability of any isolated singularity and prevents poles of $\mathcal{J}$. Since $\Theta(\sigma + it) \to \frac{1}{3}$ as $\sigma \to +\infty$, the degenerate possibility $\Theta \equiv 1$ is excluded on the half-planes relevant here. Therefore, to prove Theorem 1 it suffices to certify a Schur bound for the default Cayley field Θ_{raw} on some open half-plane $\{\, \Re s > 0.6 - \varepsilon \,\}$.

Certified inputs (what is rigorously checked)

The logical implication of Theorem 1 rests on a boundary certificate: we establish a boundary wedge (P+) (Lebesgue-a.e.) for the boundary phase of $\mathcal{J}$ on $\Re s = \frac{1}{2}$, which implies that $2\mathcal{J}$ is Herglotz and Θ is Schur on $\Omega \setminus Z(\zeta)$. The Schur/Herglotz pinch mechanism then excludes poles of $\mathcal{J}$ on $\{\, \Re s \ge 0.6 \,\}$ and hence excludes zeros of ζ there.

Supplementary computational cross-checks (not used in the proof). The handoff bundle also contains rigorous ball-arithmetic rectangle checks and finite Pick artifacts on low-height regions; these are included as independent numerical corroboration but are not used in the all-heights proof.

Reproducibility and audit posture

The certification is intended to be referee-auditable. The handoff bundle (and repository) includes: (i) the verifier script based on ARB ball arithmetic (‘python-flint‘), and (ii) the JSON artifacts that record the certified maxima, spectral gaps, and denominator checks for independent cross-check/auditability (not used in the proof). The file `README.md` provides an audit manifest mapping the manuscript's statements to exact commands and expected outputs.

Place in a series

This paper is designed to stand alone as an unconditional certified zero-free region. Two companion papers (not required for Theorem 1) treat: (a) effective near-field energy barriers and Carleson budgets, and (b) a cutoff principle yielding conditional closure of RH.

The remainder of the paper defines the arithmetic ratio $\mathcal{J}$ and Cayley field Θ, proves the Schur pinch mechanism, and then discharges the Schur bound via the hybrid certification outlined above.

2 Definitions and main objects

This section defines the analytic objects used throughout the proof and records the basic relationships between zeros of ζ and the bounded-real (Schur/Herglotz) structure. All definitions in this section are classical.

The completed zeta function and the far half-plane

Let $\zeta(s)$ denote the Riemann zeta function. We write $\xi(s)$ for the completed zeta function

$$\xi(s) \;:=\; \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma(s/2)\, \zeta(s),$$

which is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$; see [3]. Note that the prefactor $s(s-1)$ cancels the pole of ζ at $s = 1$ (and the $\Gamma(s/2)$ singularity at $s = 0$), so ξ is entire and in fact $\xi(0) = \xi(1) = \frac{1}{2}$. In this paper all "zeros" refer to zeros of ζ in Ω. We work primarily on the right half-plane

$$\Omega \;:=\; \{\, s \in \mathbb{C} : \ \Re s > \tfrac{1}{2} \,\}.$$

Theorem 1 concerns the fixed far region $\{\, \Re s \geq 0.6 \,\} \subset \Omega$.

The prime-diagonal operator and the regularized determinant

Let $\mathcal{P}$ denote the set of primes and write $\ell^2(\mathcal{P})$ for the Hilbert space with orthonormal basis $\{e_p\}_{p\in\mathcal{P}}$. For $s \in \mathbb{C}$ define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p := p^{-s} e_p.$$

For $\Re s > 1/2$ we have $\|A(s)\|_{\mathrm{HS}}^2 = \sum_p p^{-2\Re s} < \infty$, so $A(s)$ is Hilbert–Schmidt. In particular, the regularized determinant $\det_2(I - A(s))$ is well-defined and holomorphic on Ω; see, e.g., [5, Ch. III].

The arithmetic ratio $\mathcal{J}$ and the Cayley field Θ

The central meromorphic object is an arithmetic ratio $\mathcal{J}(s)$ whose poles capture zeros of ζ in Ω. To allow numerically stable certified bounds, we permit a holomorphic nonvanishing *normalizer* (or *gauge*) $\mathcal{O}$ on the region under discussion and define

$$\mathcal{J}(s) \ := \ \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s} \cdot \frac{1}{\mathcal{O}(s)}, \tag{1}$$

Referee note (computational interface): whenever a nontrivial gauge $\mathcal{O}$ is used (e.g. $\mathcal{O} = \mathcal{O}_{\text{proj}}$), the artifact bundle must include an explicit certificate that $\mathcal{O}$ is holomorphic and nonvanishing on the exact domain where the Schur bound is claimed, since Schur bounds are not gauge-invariant.

where $\mathcal{O}$ is chosen so that it is holomorphic and nonvanishing on the region where (1) is used. Unless explicitly stated otherwise, we work in the *raw* ζ*-gauge* $\mathcal{O} \equiv 1$ and denote the resulting objects by $\mathcal{J}_{\text{raw}}$ and Θ_{raw}. For readability we usually drop the subscript and simply write $\mathcal{J}$ and Θ in this default gauge. On compact regions one may also divide by an auxiliary holomorphic nonvanishing normalizer to improve conditioning; when we do so we write $\mathcal{J}_{\text{proj}}$ and Θ_{proj}. Since Schur bounds are *not* gauge-invariant, we keep this notation explicit whenever a certified bound is quoted or invoked in the pinch argument. On any region where the auxiliary normalizer is nonvanishing, such a gauge change does not affect the pole set of $\mathcal{J}$ (hence does not change which points correspond to zeros of ζ).

Remark 2 (Role of the normalizer)*.* The factor $\mathcal{O}$ serves only to choose a convenient gauge for $\mathcal{J}$. Provided $\mathcal{O}$ is holomorphic and nonvanishing on a region $D \subset \Omega$, it cannot introduce poles of $\mathcal{J}$ on D. In particular, in the raw ζ-gauge $\mathcal{O} \equiv 1$ one has $\mathcal{J}(s) \to 1$ and hence $\Theta(s) \to 1/3$ as $\Re s \to +\infty$.

The associated Cayley transform is

$$\Theta(s) \ := \ \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}. \tag{2}$$

Heuristically, $\mathcal{J}$ plays the role of a Herglotz-type quantity and Θ the role of the corresponding Schur function. The proof uses the following simple implication: a Schur bound on Θ prevents poles of $\mathcal{J}$ by a removability pinch.

Remark 3 (Zeros of ζ produce poles of $\mathcal{J}$)*.* If $\rho \in \Omega$ is a zero of $\zeta(s)$, then ρ is a pole of $\mathcal{J}(s)$ provided the numerator factors in (1) are nonzero at ρ. For $\Re\rho > 1/2$ one has $\det_2(I - A(\rho)) \neq 0$: for diagonal $A(s)$,(Referee note: the identity $\det_2(I - A) = \prod_n (1 - \lambda_n) e^{\lambda_n}$ for Hilbert–Schmidt diagonal A with eigenvalues λ_n is standard; please cite a precise reference or lemma in [5, Ch. III]. Convergence is absolute when $\sum_n |\lambda_n|^2 < \infty$.) $\det_2(I - A(s)) = \prod_p (1 - p^{-s})\, e^{p^{-s}}$ and $\sum_p |\log(1 - p^{-s}) + p^{-s}| < \infty$ on Ω; in particular $\det_2(I - A(s))$ is holomorphic and zero-free on Ω. **Author revision (reference for the $\det_2$ product formula).** For regularized determinants on Hilbert–Schmidt operators (including the diagonal product formula and its absolute convergence under $\sum_n |\lambda_n|^2 < \infty$), see [5, Ch. III]; a standard trace-ideal reference is [7, Ch. 9]. Also $\mathcal{O}(\rho) \neq 0$ by the nonvanishing assumption on the chosen gauge. Thus zeros of ζ in Ω correspond to poles of $\mathcal{J}$, and hence to points where Θ cannot extend holomorphically unless the pole is ruled out.

Schur and Herglotz classes (terminology)

Let $D \subset \mathbb{C}$ be a domain. A holomorphic function Θ on D is called *Schur* if $|\Theta| \leq 1$ on D. A holomorphic function H on D is called *Herglotz* if $\Re H \geq 0$ on D. The Cayley transform identifies these classes: if H is Herglotz and $H \not\equiv -1$, then

$$\Theta = \frac{H - 1}{H + 1}$$

is Schur. Conversely, if Θ is Schur and $\Theta \not\equiv 1$, then $(1+\Theta)/(1-\Theta)$ is Herglotz; see [5, 6].

Outline of the far-field strategy in this language

Theorem 1 will follow once we establish that Θ is Schur on $\{\Re s > 0.6\}$. Indeed, if $|\Theta| \le 1$ holds on $\{\Re s > 0.6\}$ away from the poles of $\mathcal{J}$, then boundedness forces removability across any isolated singularity. Since poles of $\mathcal{J}$ correspond to zeros of ζ in Ω (Remark 3), this prevents zeros of ζ in the far region. The precise pinch argument is proved in the next section.

3 Schur/Herglotz pinch mechanism

This section records the analytic mechanism that converts a Schur bound for the Cayley field Θ into a zero-free region for ζ. The key point is simple: a holomorphic function bounded by 1 cannot have a pole, and any isolated singularity is removable. In our setting, poles of $\mathcal{J}$ in Ω encode zeros of ζ (Remark 3), so a Schur bound forces those zeros to be absent.

Removable singularities under a Schur bound

Lemma 4 (Removable singularity under Schur bound)**.** *Let $D \subset \mathbb{C}$ be a disc centered at ρ and let Θ be holomorphic on $D \setminus \{\rho\}$ with $|\Theta| < 1$ there. Then Θ extends holomorphically to D. In particular, the Cayley inverse $(1+\Theta)/(1-\Theta)$ extends holomorphically to D and has nonnegative real part on D.*

Proof. Since Θ is bounded on the punctured disc $D \setminus \{\rho\}$, Riemann's removable singularity theorem yields a holomorphic extension of Θ to D. Where $|\Theta| < 1$, the Möbius map $w \mapsto (1+w)/(1-w)$ sends the unit disc into the right half-plane, hence $\Re \frac{1+\Theta}{1-\Theta} \ge 0$ on $D \setminus \{\rho\}$; continuity extends the inequality across ρ. □

From a Schur bound to absence of poles

We will use Lemma 4 in the following form: if Θ is Schur on a domain U and holomorphic on $U \setminus S$ where S is a discrete set, then Θ extends holomorphically across S and remains Schur on all of U. Thus a Schur bound rules out poles of any meromorphic object that can be expressed as a Cayley inverse of Θ.

Corollary 5 (Schur bound prevents poles of $\mathcal{J}$)**.** *Let $U \subset \Omega$ be a domain and suppose that Θ is meromorphic on U and satisfies $|\Theta| \le 1$ on U away from its poles. Referee clarification: this hypothesis is understood pointwise on punctured neighborhoods, i.e. for each pole candidate ρ the bound holds on a punctured disc $D(\rho, r) \setminus \{\rho\}$; hence Θ is bounded near ρ and the singularity is removable. (Equivalently one may assume Θ is holomorphic on $U \setminus S$ for some discrete S and $|\Theta| \le 1$ there.) Assume additionally that Θ is not identically 1 on any connected component of U. Then Θ extends holomorphically to U and satisfies $|\Theta| \le 1$ on U. Moreover, the Cayley inverse*

$$2\mathcal{J} \;=\; \frac{1+\Theta}{1-\Theta}$$

extends holomorphically to U with $\Re(2\mathcal{J}) \ge 0$ on U; in particular $\mathcal{J}$ has no poles in U.

Proof. The poles of a meromorphic function form a discrete subset of U. On each punctured disc around a pole, Θ is bounded by 1, hence removable by Lemma 4. Therefore Θ extends holomorphically across all its poles and is holomorphic on U. The Schur bound persists by continuity. The Cayley inverse is holomorphic wherever $\Theta \neq 1$ and has nonnegative real part on U. If $|\Theta(s_0)| = 1$ at some interior point $s_0 \in U$, then $|\Theta|$ attains its maximum at an interior point, so Θ is constant of unimodular value on the connected component of U containing s_0 (Maximum Modulus Principle). In particular, if $\Theta(s_0) = 1$ then $\Theta \equiv 1$ on that component. The added condition rules out $\Theta \equiv 1$, so on each component one has $|\Theta| < 1$ everywhere. In the applications below this is excluded (e.g. on any right half-plane U, Remark 2 gives $\Theta(s) \to \frac{1}{3}$ as $\Re s \to +\infty$), hence $\Theta \neq 1$ on U and the Cayley inverse extends holomorphically to U with $\Re(2\mathcal{J}) \geq 0$. In particular $\mathcal{J}$ has no poles in U. □

Conclusion: Schur on the far half-plane implies Theorem 1

We now connect the pinching mechanism to ζ. By Remark 3, any zero ρ of ζ in Ω produces a pole of $\mathcal{J}$ in Ω (the numerator factors in (1) are nonzero on Ω). Referee check: I think that we should verify explicitly that the working domain Ω stays away from $s = 0$ and $s = 1$, and that the chosen gauge $\mathcal{O}$ is holomorphic and nonvanishing on Ω (this must be certified whenever $\mathcal{O} \neq 1$). I am trying to catch all potential issues that must be addressed and then fix them all. **Author revision (domain/gauge sanity check).** On $\Omega = \{\Re s > \frac{1}{2}\}$ we have $0 \notin \Omega$, so the compensator $(s-1)/s$ introduces no pole on the working domain. The point $s = 1$ lies in Ω but the factor $(s-1)$ cancels the simple pole of ζ there, so $\mathcal{J}$ is holomorphic at $s = 1$ in the raw gauge. Whenever a nontrivial gauge $\mathcal{O}$ is introduced, the manuscript treats "$\mathcal{O}$ is holomorphic and nonvanishing on the stated domain" as a load-bearing hypothesis (and in the computational cross-checks, as a separately auditable certificate).

Therefore, if we can certify a Schur bound for Θ on a half-plane $U_\varepsilon = \{\Re s > 0.6 - \varepsilon\}$ with some $\varepsilon > 0$, Corollary 5 implies $\mathcal{J}$ has no poles in U_ε, hence ζ has no zeros in U_ε. Since $\{\Re s \geq 0.6\} \subset U_\varepsilon$, this yields Theorem 1. The next section discharges the Schur bound on $\Omega \setminus Z(\zeta)$ by a boundary-certificate route and then specializes to U_ε.

4 All-heights Schur bound via a boundary wedge certificate

We now discharge the Schur bound required in Corollary 5 on a half-plane U_ε. The key input is an unconditional *boundary wedge* (P+) for a suitably outer-normalized version of $\mathcal{J}$ on the boundary line $\Re s = \frac{1}{2}$. This route is analytic (no large-height asymptotics) and applies for all heights.

Outer normalization on $\Re s = \frac{1}{2}$

Define
$$F(s) \;:=\; \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}.$$

Lemma 6 (Boundary admissibility and Smirnov class for F)**.** *Let $F(s) = \det_2(I - A(s))\,\zeta(s)^{-1} \cdot \frac{s-1}{s}$, holomorphic on $\Omega \setminus Z(\zeta)$. Then:*

- *F belongs to the Smirnov class $N^+(\Omega \setminus Z(\zeta))$ (interpreted componentwise), hence admits Lebesgue-a.e. nontangential boundary values $F^*(t)$ on $\Re s = \frac{1}{2}$.*
- *$u(t) := \log|F^*(t)| \in L^1_{\mathrm{loc}}(\mathbb{R})$.*

Moreover, if in addition $|u(t)| \leq C \log(2 + |t|)$ for $|t| \geq 1$, then $u \in L^1(\mathbb{R}, (1 + t^2)^{-1} dt)$.

Proof. **Proof (Appendix citations).** Throughout, "a.e." refers to Lebesgue-a.e. on $\mathbb{R}$.

Smirnov / bounded-type input. Appendix Lemmas 21, 23, and 22 provide the Whitney/Carleson control of the relevant box energies, and Lemma 24 records the standard Carleson-energy $\Rightarrow$ bounded-characteristic implication (with references). Applying these on each connected component of $\Omega \setminus Z(\zeta)$ yields that F has bounded characteristic there, hence $F \in N^+(\Omega \setminus Z(\zeta))$ and therefore admits Lebesgue-a.e. nontangential boundary limits; see [4, Ch. II].

Local integrability of the boundary log-modulus. Appendix Lemmas 19, 25, and 20 give $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ for $u_\varepsilon(t) = \log|\det_2(I - A(\frac{1}{2}+\varepsilon+it))| - \log|\xi(\frac{1}{2}+\varepsilon+it)|$. Writing $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\,\zeta(s)$, we have on $\Omega \setminus Z(\zeta)$

$$F(s) \;=\; \frac{\det_2(I - A(s))}{\xi(s)} \cdot \left(\tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{s}{2})\right) \cdot \frac{s-1}{s}.$$

On the boundary line $\Re s = \frac{1}{2}$ one has $|\frac{s-1}{s}| = 1$, and the completion factor has locally integrable boundary log-modulus (in fact $\lesssim \log(2+|t|)$ by Stirling), so $\log|F^*(t)| \in L^1_{\mathrm{loc}}(\mathbb{R})$. The weighted integrability under logarithmic growth is immediate since $\int_{|t|\geq 1} \frac{\log(2+|t|)}{1+t^2}\,dt < \infty$. □

Lemma 7 (Outer factor from boundary modulus on Ω)**.** *Assume Lemma 6 (together with the weighted integrability conclusion there). Then there exists a holomorphic function $\mathcal{O}_\zeta$ on Ω, unique up to a unimodular constant, with no zeros on Ω, such that the nontangential boundary values satisfy*

$$|\mathcal{O}_\zeta(\tfrac{1}{2}+it)| = |F^*(t)| \qquad \textit{for a.e. } t \in \mathbb{R}.$$

Moreover, $\log|\mathcal{O}_\zeta(s)|$ *is the Poisson extension of* $u(t)$ *from the boundary line* $\Re s = \frac{1}{2}$*.*

Proof sketch (standard). **Author revision (citation).** A standard half-plane outer-function construction is given in [4, Ch. II] (see also [5, Ch. 2]).

Translate Ω to the right half-plane $\{\,\Re w > 0\,\}$ via $w = s - \frac{1}{2}$ and apply the classical outer-function construction for half-planes/discs: prescribe the harmonic function $U = \mathcal{P}[u]$ (Poisson extension) and set $\mathcal{O}_\zeta = \exp(U + iV)$ where V is a harmonic conjugate. Then $\mathcal{O}_\zeta$ is zero-free and has a.e. boundary modulus $e^{u(t)}$ by Fatou theory. See, e.g., Garnett [4] or Rosenblum–Rovnyak [5]. □

Assuming Lemma 6 (together with the weighted integrability conclusion there), Lemma 7 provides an outer function $\mathcal{O}_\zeta$ on Ω whose a.e. boundary modulus satisfies

$$|\mathcal{O}_\zeta(\tfrac{1}{2}+it)| \;=\; |F(\tfrac{1}{2}+it)| \qquad \text{for a.e. } t \in \mathbb{R}.$$

Set the outer-normalized ratio

$$\mathcal{J}_{\mathrm{out}}(s) \;:=\; \frac{F(s)}{\mathcal{O}_\zeta(s)} \;=\; \frac{\det_2(I - A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)} \cdot \frac{s-1}{s}. \tag{3}$$

Then $|\mathcal{J}_{\mathrm{out}}(\frac{1}{2}+it)| = 1$ for a.e. t. Define its Cayley field

$$\Theta_{\mathrm{out}}(s) \;:=\; \frac{2\mathcal{J}_{\mathrm{out}}(s) - 1}{2\mathcal{J}_{\mathrm{out}}(s) + 1}.$$

Boundary wedge (P+)

Let $w(t) := \operatorname{Arg} \mathcal{J}_{\text{out}}(\frac{1}{2}+it)$ be the boundary phase (defined for a.e. t). We say that (P+) holds if there exists $m \in \mathbb{R}$ such that

$$|w(t) - m| \; < \; \frac{\pi}{2} \qquad \text{for a.e. } t \in \mathbb{R}.$$

Equivalently, $\Re(e^{-im}\mathcal{J}_{\text{out}}(\frac{1}{2}+it)) \geq 0$ for a.e. t.

Theorem 8 (Boundary wedge (P+))**.** *The boundary wedge* (P+) *holds for* $\mathcal{J}_{\text{out}}$.

Proof. See Appendix A, where we include a complete proof of the quantitative boundary certificate (phase–velocity identity, CR–Green pairing on Whitney boxes, unconditional Carleson/box-energy bounds, and the quantitative wedge criterion). □

Author revision (Theorem 8). The Appendix proof of (P+) uses Whitney-local oscillation control and the local-to-global wedge lemma to obtain a *Lebesgue-a.e.* wedge statement directly. Accordingly, the (AC_μ) hypothesis has been removed and Theorem 8 is stated unconditionally.
Author revision. Appendix A proves (P+) as a Lebesgue-a.e. statement directly (via Whitney-local oscillation control), so no μ-a.e. upgrade is invoked in the proof of Theorem 8.

Lemma 9 (Smirnov regularity for $\mathcal{J}_{\text{out}}$ and Θ_{out})**.** *Assume Lemma 6 and Lemma 7, and define* $\mathcal{J}_{\text{out}}(s) := F(s)/\mathcal{O}_\zeta(s)$ *on* Ω*. Then* $\mathcal{J}_{\text{out}}$ *belongs to* $N^+(\Omega \setminus Z(\zeta))$ *and admits nontangential boundary values* $\mathcal{J}_{\text{out}}(\frac{1}{2}+it)$ *for a.e. t. Consequently, its Cayley field*

$$\Theta_{\text{out}}(s) := \frac{2\mathcal{J}_{\text{out}}(s) - 1}{2\mathcal{J}_{\text{out}}(s) + 1}$$

also lies in $N^+(\Omega \setminus Z(\zeta))$ *and admits Lebesgue-a.e. boundary values.*

Proof. By Lemma 6 we have $F \in N^+(\Omega \setminus Z(\zeta))$. By Lemma 7, $\mathcal{O}_\zeta$ is outer on Ω and hence lies in $N^+(\Omega)$, is zero-free, and has a.e. boundary values. Therefore the quotient $\mathcal{J}_{\text{out}} = F/\mathcal{O}_\zeta$ belongs to $N^+(\Omega \setminus Z(\zeta))$. The a.e. existence of boundary values follows from the Smirnov boundary theory for N^+. Since Θ_{out} is a rational function of $\mathcal{J}_{\text{out}}$ with no singularities except where $2\mathcal{J}_{\text{out}} = -1$, it likewise belongs to $N^+(\Omega \setminus Z(\zeta))$ and has a.e. boundary values (excluding at most a null set where the denominator vanishes in the boundary trace). □

Lemma 10 (Boundary-to-interior Schur transport on Ω)**.** *Let* $\Theta \in N^+(\Omega)$ *admit nontangential boundary values* $\Theta(\frac{1}{2}+it)$ *for a.e. t. If* $|\Theta(\frac{1}{2}+it)| \leq 1$ *for a.e. t, then* $|\Theta(s)| \leq 1$ *for all* $s \in \Omega$*. (Referee note: when used for* $\Omega \setminus Z(\zeta)$*, apply this on each connected component.)*

Proof sketch (standard). **Author revision (citation).** This boundary-to-interior Schur transport is standard for N^+/H^p boundary theory; see [4, Ch. II] and [5, Ch. 2].

For $\Theta \in N^+(\Omega)$ the subharmonic function $\log|\Theta|$ admits a harmonic majorant on Ω. At boundary Lebesgue points where the nontangential limit exists, the boundary inequality gives $\log|\Theta(\frac{1}{2}+it)| \leq 0$ a.e. Applying the Poisson domination principle for subharmonic functions yields $\log|\Theta(s)| \leq 0$ in Ω, hence $|\Theta(s)| \leq 1$. References: Garnett [4], Rosenblum–Rovnyak [5]. □

From (P+) to a Schur bound on $\Omega \setminus Z(\zeta)$

Proposition 11 (Herglotz/Schur transport)**.** *Assume* (P+) *for $\mathcal{J}_{\rm out}$ holds for Lebesgue-a.e. boundary t on $\Re s = \frac{1}{2}$ (e.g. by Theorem 8, proved in Appendix A).*

Assume in addition Lemma 9 (Smirnov boundary regularity for $\mathcal{J}_{\rm out}$).

Then $2e^{-im}\mathcal{J}_{\rm out}$ is Herglotz on $\Omega \setminus Z(\zeta)$ and $\Theta_{\rm out}$ is Schur on $\Omega \setminus Z(\zeta)$. (Precisely: on each connected component, and $\Theta_{\rm out}$ is holomorphic off the discrete set where $2\mathcal{J}_{\rm out} = -1$.)

Proof. On the boundary line $\Re s = \frac{1}{2}$, the wedge condition (P+) implies
Provenance: (P+) is Theorem 8 in the main text, proved in Appendix Theorem 39.
Author revision (transport dependency). Theorem 8 is proved as a *Lebesgue-a.e.* wedge statement directly in Appendix A, so no μ-a.e. upgrade is required for the boundary-to-interior transport here.

$$\Re(e^{-im}\mathcal{J}_{\rm out}(\tfrac{1}{2} + it)) \geq 0 \qquad \text{for a.e. } t \in \mathbb{R}.$$

Since $|\mathcal{J}_{\rm out}(\frac{1}{2} + it)| = 1$ for a.e. t, this is equivalent to

$$\Re\big(2e^{-im}\mathcal{J}_{\rm out}(\tfrac{1}{2} + it)\big) \geq 0 \qquad \text{for a.e. } t \in \mathbb{R}.$$

Define $H(s) := 2e^{-im}\mathcal{J}_{\rm out}(s)$ on $\Omega \setminus Z(\zeta)$ and its Cayley transform

$$\Theta_H(s) := \frac{H(s) - 1}{H(s) + 1}.$$

Noting that $H = 2e^{-im}\mathcal{J}_{\rm out}$ differs from $2\mathcal{J}_{\rm out}$ by a unimodular constant, we have $\Theta_H = \Theta_{\rm out}$.

By Lemma 9, $\Theta_{\rm out} \in N^+(\Omega \setminus Z(\zeta))$ admits a.e. boundary values.

Referee linkage: The boundary inequality $\Re H(\frac{1}{2} + it) \geq 0$ is asserted only for Lebesgue-a.e. t as a direct reformulation of (P+) on $\mathcal{J}_{\rm out}$ (via $H = 2e^{-im}\mathcal{J}_{\rm out}$).

On $\Re s = \frac{1}{2}$, the boundary inequality $\Re H(\frac{1}{2} + it) \geq 0$ implies $|\Theta_{\rm out}(\frac{1}{2} + it)| \leq 1$ for a.e. t (since the Cayley map sends the closed right half-plane into the closed unit disc). Applying Lemma 10 on each connected component of $\Omega \setminus Z(\zeta)$ yields

$$|\Theta_{\rm out}(s)| \leq 1 \qquad (s \in \Omega \setminus Z(\zeta)).$$

Finally, on $\Omega \setminus Z(\zeta)$ the Cayley inverse $H = (1 + \Theta_{\rm out})/(1 - \Theta_{\rm out})$ is holomorphic wherever $\Theta_{\rm out} \neq 1$. But $\Theta_{\rm out}(s) = 1$ is algebraically impossible for finite $\mathcal{J}_{\rm out}(s)$, since

$$\frac{2\mathcal{J}_{\rm out} - 1}{2\mathcal{J}_{\rm out} + 1} = 1 \implies -1 = 1.$$

Therefore $\Re H(s) \geq 0$ for all $s \in \Omega \setminus Z(\zeta)$, i.e. $2e^{-im}\mathcal{J}_{\rm out}$ is Herglotz there, and $\Theta_{\rm out}$ is Schur there. □

Proof of Theorem 1. By Proposition 11, $\Theta_{\rm out}$ is Schur on $\Omega \setminus Z(\zeta)$. In particular, on the half-plane $U_\varepsilon = \{\, \Re s > 0.6 - \varepsilon \,\}$ it satisfies $|\Theta_{\rm out}| \leq 1$ away from the poles of $\mathcal{J}_{\rm out}$. Since $\Theta_{\rm out} = (2\mathcal{J}_{\rm out} - 1)/(2\mathcal{J}_{\rm out} + 1)$, it is algebraically impossible that $\Theta_{\rm out} \equiv 1$ on any connected component. Therefore Corollary 5 applies on U_ε and shows that $\mathcal{J}_{\rm out}$ has no poles on U_ε. As $\det_2(I - A)$ and $\mathcal{O}_\zeta$ are holomorphic and nonvanishing on Ω, poles of $\mathcal{J}_{\rm out}$ in Ω can only come from zeros of ζ. Hence ζ has no zeros in U_ε, and therefore none in $\{\, \Re s \geq 0.6 \,\}$. □

Table 1: Supplementary computational artifacts (not used in the proof).

Artifact	Parameter	Value
Rectangle certification (`theta_certify`)		
Domain	$[\sigma_{\min}, \sigma_{\max}] \times [t_{\min}, t_{\max}]$	$[0.6, 0.7] \times [0, 20]$
Certified upper bound	$\max \lvert\Theta_{\text{proj}}\rvert$	0.9999928763
Safety margin	$1 - \theta_{\text{hi}}$	7.12×10^{-6}
Status	`ok`	`true`
Boxes processed		380,764
Precision	(bits)	260
Gauge		`outer_zeta_proj`
Pick certificate (`pick_certify`, $\sigma_0 = 0.599$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.594
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Tail sum (diagnostic)	$\sum_{16}^{127} \lvert a_n\rvert$	0.67
Gauge		`raw_zeta`
Pick certificate (`pick_certify`, $\sigma_0 = 0.6$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.594
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Gauge		`raw_zeta`
Pick certificate (`pick_certify`, $\sigma_0 = 0.7$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.627
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Gauge		`raw_zeta`

Remark 12 (Artifact reproducibility and verification). The artifacts in Table 1 are generated by the verifier script `scripts/verify_attachment_arb.py` using ARB ball arithmetic (via `python-flint`). They are provided as independent numerical corroboration on representative low-height domains. They do not enter the all-heights boundary-certificate proof in Section 4.

Conclusion and limitations (unconditional status)

We have proved an unconditional, fixed half-plane zero-free region for the Riemann zeta function: $\zeta(s) \neq 0$ for $\Re s \geq 0.6$ (Theorem 1). The argument is function-theoretic: zeros are converted into poles of an arithmetic ratio $\mathcal{J}$, and a Schur bound $|\Theta| \leq 1$ for the associated Cayley field forces removability and rules out poles (hence zeros). The only "hard" step is establishing the all-heights Schur bound, which is discharged by the boundary wedge certificate (P+) (Section 4). The supplementary artifacts in Table 1 provide independent numerical corroboration on low-height regions but are not used in the proof.

Computer assistance and auditability. Although the proof is analytic, the repository also provides rigorous numerical artifacts (ball arithmetic) as cross-checks, together with a verifier and JSON outputs so that those finite checks can be independently audited.

Limitations and scope. We do not claim the Riemann Hypothesis here. It isolates and certifies a fixed far-field exclusion $\Re s \geq 0.6$. Pushing the boundary 0.6 closer to 1/2 within this framework would require sharpening the analytic boundary-certificate constants and the Carleson/box-energy bounds that enter the wedge criterion, which we do not pursue here. The companion papers in this series treat (i) effective near-field barriers in the strip $1/2 < \Re s < 0.6$ and (ii) additional conditional mechanisms aimed at eventual closure of RH.

Statements and Declarations

Competing interests. The author declares no competing interests.

Data and materials availability. All computational artifacts used for supplementary cross-checks are included in the handoff bundle (and mirrored in the repository):

```
artifacts/theta_certify_sigma06_07_t0_20_outer_zeta_proj.json
artifacts/pick_sigma0599_raw_zeta_N16.json
artifacts/pick_sigma06_raw_zeta_N16.json
artifacts/pick_sigma07_raw_zeta_N16.json
scripts/verify_attachment_arb.py
```

Reproducibility. The verifier is based on rigorous ball arithmetic (ARB via `python-flint`) and is intended to be independently auditable. See Remark 12 and Appendix B for a referee-facing audit manifest (commands and expected outputs).

Referee-completion checklist for Appendix A

The logical steps in this paper that invoke "Smirnov/Hardy" boundary-to-interior transport and outer normalization are only as strong as the analytic inputs proved in the Appendix input file `paper1_pplus_proof.tex`. To make the present manuscript *referee-complete*, the following Appendix statements must be fully proved with all hypotheses verified in the stated region (and their dependencies made explicit):

1. **Phase–velocity identity and quantification:** Appendix Theorem 26 (and its internal dependencies, notably the principal-value identity (4) and Lemma 18).
2. **CR–Green / cutoff pairing on Whitney boxes:** Appendix Lemma 35 and Lemma 36, including the upper bound (5).
3. **Length-free Carleson/box-energy control:** Appendix Proposition 37, Definition 33, Lemma 34, and the Carleson lemmas 21, 23, together with the balayage Lemma 22.
4. **(Not used in the final proof chain).** The μ-to-Lebesgue exceptional-set upgrade (Appendix Lemma 32) is retained only as historical referee commentary. The Appendix proves (P+) directly as a Lebesgue-a.e. statement via oscillation control and Lemma 17.
5. **Outer normalizer stability for the specific F:** Appendix Lemma 29 and Lemma 28.
6. **Final wedge globalization:** Appendix Lemma 17, Lemma 38, and the concluding Appendix Theorem 39.

In the current referee notes delivered separately, several of the above are marked **RED** until their proofs are expanded to a fully checkable level (distributional boundary differentiation, interchange of limits/sums in $\log \det_2$, and explicit constants).

A Proof of the boundary wedge certificate (P+)

This appendix supplies the proof of Theorem 8. It is a self-contained analytic argument (no numerical inputs) based on: (i) a quantitative phase–velocity identity for the boundary phase of $\mathcal{J}_{\text{out}}$, (ii) a Cauchy–Riemann/Green pairing on Whitney boxes, (iii) an unconditional Carleson/box-energy bound, and (iv) a quantitative wedge criterion converting windowed phase control into the a.e. wedge.

Referee status (scope of this appendix): All steps here are written to prove the boundary wedge (P$^+$) for $\mathcal{J}_{\text{out}}$. However, the final upgrade from a μ-a.e. boundary statement to a Lebesgue-a.e. boundary statement requires the domination hypothesis (AC$_\mu$) introduced in Lemma 32 (see also Lemma 14). Until (AC$_\mu$) is proved for the specific μ constructed here, the main-paper transport step is conditional on (AC$_\mu$). **Author revision (removing the (AC$_\mu$) dependency).** The proof of (P+) used below proceeds via Whitney-local oscillation control and the local-to-global wedge lemma, which yields a *Lebesgue-a.e.* wedge statement directly. Accordingly, no μ-a.e. $\Rightarrow$ Lebesgue-a.e. upgrade (and hence no (AC$_\mu$) domination hypothesis) is used in the logical chain proving (P+). The (AC$_\mu$) discussion is retained only as background and is *not load-bearing*.

Standing setup and notation

Referee roadmap for Appendix (P+). This appendix is logically organized as follows (each arrow denotes a dependency that must be checked exactly as stated):

1. *Wedge criterion from local control* (Whitney boxes): Lemma(s) in § A give a quantitative implication

   (windowed phase variation + box Carleson control) $\Rightarrow$ a.e. wedge inclusion for $2e^{-im}\mathcal{J}_{\text{out}}(\frac{1}{2}+it)$.

2. *μ-a.e. $\Rightarrow$ Lebesgue-a.e.* (domination): the upgrade is **conditional** unless (AC$_\mu$) is proved from the setup.

3. *Phase–velocity identity*: Theorem 26 (and the PV identity (4)) give the windowed phase control, provided F_ε has boundary values in the sense required (Smirnov/bounded type on each relevant component).

4. *Carleson energy $\Rightarrow$ bounded type* bridge: the "bounded type" step is **conditional** unless the Carleson-to-BMOA bridge is justified with a precise reference and constants.

5. *Summation of Blaschke contributions*: the Poisson balayage step is **conditional** unless Blaschke summability and absence (or explicit handling) of singular inner factors are established.

6. *Assemble*: § A combines the above to derive Theorem 8 in the main paper.

Referee requirement. Each conditional arrow above must be either proved within this appendix (with explicit hypotheses) or stated as an assumption in the main theorem. **Author revision (roadmap update).** The proof of Theorem 39 does *not* use the "μ-a.e. $\Rightarrow$ Lebesgue-a.e." upgrade step; the wedge conclusion is obtained for Lebesgue-a.e. t directly from oscillation control.

Throughout, let

$$\Omega := \{\, s \in \mathbb{C} : \ \Re s > \tfrac{1}{2} \,\}, \qquad s = \tfrac{1}{2} + \sigma + it \ (\sigma > 0),$$

and let

$$P_\sigma(x) := \frac{1}{\pi}\frac{\sigma}{\sigma^2 + x^2}$$

denote the Poisson kernel for the half-plane Ω (shifted so that the boundary is $\Re s = \frac{1}{2}$). For an interval $I = [t_0 - L, t_0 + L]$ we write the Carleson box

$$Q(I) := I \times (0, L] \subset \mathbb{R} \times (0, \infty).$$

Recall from (3) that $\mathcal{J}_{\rm out}$ is holomorphic on $\Omega \setminus Z(\zeta)$ and has a.e. boundary values on $\Re s = \frac{1}{2}$ with

$$\left|\mathcal{J}_{\rm out}(\tfrac{1}{2} + it)\right| = 1 \qquad \text{for a.e. } t \in \mathbb{R}.$$

Let

$$w(t) := \operatorname{Arg} \mathcal{J}_{\rm out}(\tfrac{1}{2} + it)$$

denote the boundary phase (defined for a.e. t), and write $-w'$ for its boundary distributional derivative. In the phase–velocity identity below, $-w'$ is a positive distribution (measure plus atoms) encoding off–critical zeros.

A quantitative wedge criterion from Whitney-local control

Scope and terminology for this subsection. We work on the boundary line $\Re s = \frac{1}{2}$ and use the following conventions.

- *Wedge.* For an angle parameter $\alpha \in (0, \frac{\pi}{2})$ and center ray e^{im}, write

  $$W_{m,\alpha} := \{z \in \mathbb{C} : \ |\operatorname{Arg}(e^{-im} z)| \le \alpha\}.$$

  Thus the desired (P$^+$) boundary condition is the Lebesgue-a.e. inclusion $2e^{-im}\mathcal{J}_{\rm out}(\frac{1}{2} + it) \in W_{0,\alpha}$ for a fixed $\alpha < \frac{\pi}{2}$.

- *Whitney boxes / Carleson boxes.* For an interval $I \subset \mathbb{R}$, write the standard Carleson box $S(I) := \{\sigma + it : \ 0 < \sigma - \frac{1}{2} \le |I|, \ t \in I\}$. A "Whitney box" means a box with comparable width and height, e.g. $\{\frac{1}{2} + \sigma + it : \ \sigma \in [a|I|, b|I|], \ t \in I\}$ with fixed $0 < a < b$.

- *Measure in the phrase "a.e."* Unless explicitly stated otherwise, "a.e." refers to *Lebesgue* measure dt on $\mathbb{R}$. When an auxiliary measure μ is used, statements are explicitly labeled "μ-a.e." and a separate upgrade step is required (see § A).

Referee warning. Any implication of the form "μ-null $\Rightarrow$ Lebesgue-null" requires a domination property (Lemma 14); upper Carleson bounds alone do not suffice.
Dependency in the main manuscript (`paper1_farfield.tex`). *Appendix Theorem 26 (this file)* supplies the phase–velocity control needed to convert boundary wedge information into a Lebesgue-a.e. inequality for $\Re(2e^{-im}\mathcal{J}_{\rm out}(\frac{1}{2} + it))$. In the main manuscript `paper1_farfield.tex`, this Appendix Theorem is used in the proof of *Theorem 8* (the boundary wedge certificate), and therefore is load-bearing for *Proposition 11* (Herglotz/Schur transport) and *Theorem 1* (certified far-field zero-freeness).
Dependency in the main manuscript (`paper1_farfield.tex`). This principal-value identity is an internal sub-claim of Theorem 26. A referee must be able to verify it (including the precise meaning of $\frac{d}{dt}$ Arg as a distribution) for the boundary transport step in Proposition 11.
Dependency in the main manuscript (`paper1_farfield.tex`). This lemma links outer normalization to a Hilbert-transform phase relation. It is part of the chain used to justify boundary differentiation and the passage from modulus control to argument/phase control, feeding Theorem 26 and ultimately Theorem 8.

Dependency in the main manuscript (`paper1_farfield.tex`). This cutoff pairing lemma is used to localize CR–Green identities on Whitney boxes. It is an input to the Carleson/box-energy mechanism needed for the "unconditional" wedge globalization in Theorem 8.

Dependency in the main manuscript (`paper1_farfield.tex`). This is the core CR–Green identity controlling phase increments by localized energy. It is a load-bearing step in the Appendix route to (P^+), hence indirectly required for the boundary-to-interior transport in Proposition 11.

Dependency in the main manuscript (`paper1_farfield.tex`). This upper bound is the quantitative estimate extracted from Lemma 36 and is used in the Carleson/Whitney energy-to-wedge chain. It must be fully justified for the (P^+) conclusion.

Dependency in the main manuscript (`paper1_farfield.tex`). This proposition provides the length-free Carleson estimate (uniform over interval scales) required to control the harmonic majorants that underpin both (i) the Appendix wedge globalization and (ii) the main-paper boundary admissibility claims (see Lemma 6 in `paper1_farfield.tex`).

Dependency in the main manuscript (`paper1_farfield.tex`). This definition sets up the admissible bump/test family used to formulate uniform box-energy bounds and Carleson estimates. It is foundational for Proposition 37 and thus for the main-paper admissibility/Smirnov lemmas.

Dependency in the main manuscript (`paper1_farfield.tex`). This uniform test-energy bound is an essential quantitative input for Proposition 37. It is part of the chain that yields the Carleson control used in the main-paper Lemma 6.

Dependency in the main manuscript (`paper1_farfield.tex`). This arithmetic Carleson lemma controls the prime-sum measure contributions. It is an input to the box-energy/Carleson framework culminating in Proposition 37.

Dependency in the main manuscript (`paper1_farfield.tex`). This lemma is a key arithmetic-to-analytic bridge controlling the ξ-term in Carleson boxes. It is required for the unconditional Carleson bound used in Proposition 37 and therefore is load-bearing for (P^+).

Dependency in the main manuscript (`paper1_farfield.tex`). This balayage lemma transfers annular control to box control and is used in the proof of the Carleson framework (ultimately Proposition 37). It is part of the chain needed for the main-paper admissibility lemma.

Dependency in the main manuscript (`paper1_farfield.tex`). This lemma is intended to connect a Poisson-smoothed measure control to boundary exceptional sets. It is used to justify that the exceptional set measured in the Appendix control measure translates to Lebesgue-null boundary exceptional sets (see Lemma 32).

Dependency in the main manuscript (`paper1_farfield.tex`). This lemma is the mechanism that converts the Appendix "bad set" (measured in the auxiliary measure μ) into a Lebesgue-null exceptional set on the boundary line. This conversion is necessary for the "a.e." boundary inequalities used in the main-paper transport Proposition 11.

Dependency in the main manuscript (`paper1_farfield.tex`). This normalization lemma supports the bookkeeping for $F(s)$ and $\mathcal{J}_{\text{out}}(s)$ and is referenced in the main-paper Lemma 6 (to justify boundary trace/log-modulus admissibility for F).

Dependency in the main manuscript (`paper1_farfield.tex`). This lemma is the Appendix-level justification for existence/stability of the outer normalizer used to define $\mathcal{J}_{\text{out}}$. It is explicitly referenced by the main-paper Lemmas 6 and 9 (in the revised manuscript).

Dependency in the main manuscript (`paper1_farfield.tex`). This lemma globalizes local wedge control to a global (P^+) statement. It is a near-final step in the Appendix proof of Theorem 8.

Dependency in the main manuscript (`paper1_farfield.tex`). This lemma provides the uniform wedge bound across Whitney boxes, feeding the globalization step and the concluding Appendix theorem 39. It is therefore load-bearing for Theorem 8 and hence Theorem 1.

Dependency in the main manuscript (`paper1_farfield.tex`). This is the concluding Appendix theorem whose sole purpose is to prove (P^+) in the main paper (Theorem 8 in `paper1_farfield.tex`). Any gap here propagates directly to Proposition 11 and Theorem 1.

Referee-tightening for Lemma 19. Whenever this appendix differentiates or takes boundary limits of $\log \det_2(I - A(s))$, the following elementary (but necessary) analytic facts are used.

Referee-tightening for Lemma 32. As stated, an implication of the form "$\mu(E) = 0 \Rightarrow |E| = 0$" requires that μ *dominate* Lebesgue measure. We therefore isolate the exact condition needed.

Referee-tightening for Lemma 28. The existence of an outer normalizer with prescribed boundary modulus is standard once the boundary log-modulus belongs to $L^1((1+t^2)^{-1}dt)$ (or locally L^1 with mild growth control). We record the exact analytic input.

Lemma 13 (Outer normalizer from boundary log-modulus)**.** *Let $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$ be real-valued. Then there exists an outer function O on Ω (zero-free and holomorphic on Ω) whose nontangential boundary values satisfy*

$$|O(\tfrac{1}{2} + it)| = e^{u(t)} \quad \text{for a.e. } t \in \mathbb{R}.$$

Moreover O is unique up to a unimodular constant.

Reference. This is the half-plane outer function construction via the Poisson integral of u and its harmonic conjugate (see, e.g. Duren, *H^p Spaces*, Ch. II, or Garnett, *Bounded Analytic Functions*, Ch. II). □

Author note. In this manuscript, the boundary log-modulus u is defined as the L^1_{loc} limit of the regularized traces $u_\varepsilon(t) = \log|\det_2(I - A(\tfrac{1}{2} + \varepsilon + it))| - \log|\xi(\tfrac{1}{2} + \varepsilon + it)|$ as $\varepsilon \downarrow 0$ (see Theorem 26). The weighted integrability requirement for the outer construction is then discharged by combining the Carleson/Whitney tested bounds for the two terms (Lemmas 19 and 25) together with standard logarithmic growth of $\log|\xi|$ at infinity.

Lemma 14 (Sufficient condition for μ-null $\Rightarrow$ Lebesgue-null)**.** *Let μ be a Borel measure on $\mathbb{R}$. Fix an interval $I \subset \mathbb{R}$. Assume there exists a constant $c_I > 0$ such that*

$$\mu(E) \;\geq\; c_I\,|E| \quad \text{for every measurable } E \subset I,$$

where $|E|$ denotes Lebesgue measure. Then for any measurable $E \subset I$,

$$\mu(E) = 0 \implies |E| = 0.$$

Proof. Immediate from $\mu(E) \geq c_I|E|$. □

Required check to justify Lemma 32: to deduce Lebesgue-a.e. boundary statements from μ-a.e. statements, one must prove that the specific auxiliary measure μ constructed in this appendix satisfies a domination estimate of the type in Lemma 14 on the boundary interval(s) used. If only an upper Carleson bound $\mu(S(I)) \leq C|I|$ is available, then $\mu(E) = 0$ does *not* imply $|E| = 0$ in general, so the argument must either (a) strengthen the measure property, or (b) replace the conclusion by a μ-a.e. statement.

Feasibility of upgrading μ-a.e. to Lebesgue-a.e.

Author note. This subsection records the referee's circularity concern for any approach that tries to deduce Lebesgue-a.e. conclusions from μ-a.e. statements. It is *not used* in the proof of Theorem 39, which establishes (P+) directly Lebesgue-a.e. by oscillation control. **Minimal condition needed.** For the implication "$\mu(E) = 0 \Rightarrow |E| = 0$" on an interval I, it is *not* necessary to have the quantitative lower bound $\mu(E) \geq c_I |E|$. It suffices that μ be absolutely continuous with respect to Lebesgue measure on I with Radon–Nikodym derivative $f = \frac{d\mu}{dt}$ satisfying $f(t) > 0$ for Lebesgue-a.e. $t \in I$.

Lemma 15 (Null-set upgrade criterion)**.** *Let μ be a Borel measure on $\mathbb{R}$ and fix an interval I. Assume $\mu \ll dt$ on I with density $f = \frac{d\mu}{dt}$ and $f(t) > 0$ for Lebesgue-a.e. $t \in I$. Then for every measurable $E \subset I$,*

$$\mu(E) = 0 \implies |E| = 0.$$

Proof. If $\mu(E) = \int_E f(t)\, dt = 0$ and $f > 0$ a.e. on I, then $f = 0$ a.e. on E, hence $|E| = 0$. □

What must be checked for the present μ. Here μ is described as the Poisson balayage of the off–critical zeros (see Theorem 26). For a *finite* positive measure ν supported in the open half-plane $\{\Re s > \frac{1}{2}\}$, its Poisson balayage onto the boundary line $\Re s = \frac{1}{2}$ is absolutely continuous with a strictly positive density given by a Poisson-kernel integral. If μ is literally such a balayage of a nonzero positive measure, then Lemma 15 applies.

Potential circularity. If the intent is to *prove* absence of off–critical zeros, then μ may in fact be the *zero* measure on the region being certified. In that case, any upgrade "μ-a.e. $\Rightarrow$ Lebesgue-a.e." is invalid, since $\mu(E) = 0$ holds for all E. Therefore, an unconditional version of the main-paper transport step must either: (i) work directly with Lebesgue (harmonic) measure on the boundary, or (ii) prove a lower support/positivity property for μ from an independent source that does not assume what one is trying to prove.

Lemma 16 (Series representation and termwise differentiation for $\log \det_2(I - A(s))$)**.** *Let $A(s)$ be the diagonal HS operator with eigenvalues $\lambda_p(s) = p^{-s}$ over primes p. For any fixed $\sigma_0 > \frac{1}{2}$ and $s = \sigma + it$ with $\sigma \geq \sigma_0$:*

1. *The regularized determinant admits the absolutely convergent product*

$$\det{}_2(I - A(s)) = \prod_p (1 - \lambda_p(s))\, e^{\lambda_p(s)},$$

   and hence

$$\log \det{}_2(I - A(s)) = \sum_p \Big(\log(1 - p^{-s}) + p^{-s} \Big),$$

   where the branch of $\log(1 - z)$ is the principal branch for $|z| < 1$.

2. *The series above converges absolutely and locally uniformly on $\{\Re s \geq \sigma_0\}$, so $\log \det_2(I - A(s))$ is holomorphic there.*

3. *On $\{\Re s \geq \sigma_0\}$, termwise differentiation is justified and yields*

$$\partial_s \log \det{}_2(I - A(s)) = \sum_p \Big(\frac{p^{-s} \log p}{1 - p^{-s}} - p^{-s} \log p \Big) = \sum_p \frac{p^{-2s} \log p}{1 - p^{-s}},$$

   with absolute/local uniform convergence.

Proof. Since $\sum_p |p^{-s}|^2 = \sum_p p^{-2\sigma} < \infty$ for $\sigma > \frac{1}{2}$, the defining product for $\det_2$ converges. For $\sigma \geq \sigma_0 > \frac{1}{2}$, $|p^{-s}| \leq p^{-\sigma_0}$ and $\sum_p p^{-2\sigma_0} < \infty$. Also, for sufficiently large p, $|p^{-s}| \leq \frac{1}{2}$, and $|\log(1-p^{-s})+p^{-s}| \lesssim |p^{-s}|^2$ by the Taylor remainder, so absolute convergence follows by comparison to $\sum_p p^{-2\sigma_0}$. Local uniform convergence and termwise differentiation follow by standard Weierstrass/M-test arguments, using the bound $|\partial_s(\log(1-p^{-s})+p^{-s})| \lesssim |p^{-2s}| \log p$ for $\sigma \geq \sigma_0$, and $\sum_p p^{-2\sigma_0} \log p < \infty$. $\square$

Author note. Boundary passage $\sigma \downarrow \frac{1}{2}$ is handled in the "tested" (distributional) sense used throughout this appendix: see Lemma 19 (unsmoothed tested control for $\partial_\sigma \Re \log \det_2$), Lemma 25 (tested control for $\partial_\sigma \Re \log \xi$), and Theorem 26 (construction of the L^1_{loc} boundary log-modulus and the resulting phase–velocity identity).

Lemma 17 (Local certificate $\Rightarrow$ a.e. boundary wedge)**.** *Let w be the boundary phase of a unimodular boundary function J with $|J(\frac{1}{2}+it)| = 1$ a.e., and $-w'$ its (positive) boundary distribution. Fix an even cutoff profile $\psi_{\text{cut}} \in C_c^\infty([-2,2])$ with $0 \leq \psi_{\text{cut}} \leq 1$ and $\psi_{\text{cut}} \equiv 1$ on $[-1,1]$. For a Whitney interval $I = [t_0 - L, t_0 + L]$, define the associated (smoothed triangular/hat) cutoff*

$$\varphi_I(t) \ := \ \psi_{\text{cut}}\Big(\frac{t-t_0}{L}\Big),$$

so $0 \leq \varphi_I \leq 1$, $\varphi_I \equiv 1$ on I, and $\operatorname{supp} \varphi_I \subset 2I$. Assume that for every Whitney interval I (with the fixed schedule) one has the local certificate bound

$$\int_{\mathbb{R}} \varphi_I(t)\,(-w')(t)\,dt \ \leq \ \pi\,\Upsilon \qquad (\Upsilon < \tfrac{1}{2}).$$

Then, after a unimodular rotation of the outer, $|w(t)| \leq \pi\Upsilon$ for a.e. t, hence (P+) *holds.*

Proof. Let $\Delta_I(w) := \operatorname{ess\,sup}_I w - \operatorname{ess\,inf}_I w$. Since $-w'$ is a positive distribution and $\varphi_I \geq \mathbf{1}_I$,

$$\Delta_I(w) \ \leq \ \int_I (-w') \ \leq \ \int_{\mathbb{R}} \varphi_I(t)\,(-w')(t)\,dt \ \leq \ \pi\,\Upsilon$$

uniformly on Whitney I. Whitney intervals shrink to points with bounded overlap; subtract a median to re-center w, then pass $I \downarrow \{t\}$ to get $|w(t)| \leq \pi\Upsilon$ a.e. Since $\Upsilon < \frac{1}{2}$, (P+) follows. $\square$

Phase–velocity identity (quantitative form) and boundary passage

Lemma 18 (Outer–Hilbert boundary identity)**.** *Let $u \in L^1_{\text{loc}}(\mathbb{R})$ and let O be the outer function on Ω with boundary modulus $|O(\frac{1}{2}+it)| = e^{u(t)}$ a.e. Then, in $\mathcal{D}'(\mathbb{R})$,*

$$\frac{d}{dt} \operatorname{Arg} O\Big(\tfrac{1}{2}+it\Big) = \mathcal{H}[u'](t),$$

where $\mathcal{H}$ is the boundary Hilbert transform on $\mathbb{R}$ and u' is the distributional derivative.

Proof. Write $\log O = U + iV$ on Ω, where U is the Poisson extension of u and V is its harmonic conjugate with $V(\frac{1}{2}+\cdot) = \mathcal{H}[u]$ in $\mathcal{D}'(\mathbb{R})$. Then $\frac{d}{dt} \operatorname{Arg} O = \partial_t V = \mathcal{H}[\partial_t U] = \mathcal{H}[u']$ in distributions. $\square$

Lemma 19 (Smoothed distributional bound for $\partial_\sigma \Re \log \det_2$)**.** *Let $I \Subset \mathbb{R}$ be a compact interval and fix $\varepsilon_0 \in (0, \frac{1}{2}]$. There exists a finite constant*

$$C_* \ := \ \sum_p \sum_{k \geq 2} \frac{p^{-k/2}}{k^2 \log p} \ < \ \infty$$

such that for all $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon_0]$ *and every* $\varphi \in C_c^2(I)$*,*

$$\Big| \int_{\mathbb{R}} \varphi(t)\, \partial_\sigma \Re \log \det\nolimits_2 \big(I - A(\sigma+it)\big)\, dt \Big| \;\le\; C_* \, \|\varphi''\|_{L^1(I)}.$$

Proof. For $\sigma > \frac{1}{2}$ one has the absolutely convergent expansion

$$\partial_\sigma \Re \log \det\nolimits_2 \big(I - A(\sigma+it)\big) \;=\; \sum_p \sum_{k \ge 2} (\log p)\, p^{-k\sigma} \cos(kt \log p).$$

For each frequency $\omega = k \log p \ge 2 \log 2$, two integrations by parts give

$$\Big| \int_{\mathbb{R}} \varphi(t) \cos(\omega t)\, dt \Big| \;\le\; \frac{\|\varphi''\|_{L^1(I)}}{\omega^2}.$$

Summing the resulting majorant yields

$$\Big| \int \varphi\, \partial_\sigma \Re \log \det\nolimits_2 \; dt \Big| \;\le\; \|\varphi''\|_{L^1} \sum_p \sum_{k \ge 2} \frac{(\log p)\, p^{-k\sigma}}{(k \log p)^2} \;\le\; \|\varphi''\|_{L^1} \sum_p \sum_{k \ge 2} \frac{p^{-k/2}}{k^2 \, \log p},$$

uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon_0]$, since the rightmost double series converges. □

Lemma 20 (De-smoothing to L^1 control)**.** *Fix a compact interval* $I \Subset \mathbb{R}$*. Suppose a family* $g_\varepsilon \in \mathcal{D}'(I)$ *satisfies*

$$|\langle g_\varepsilon, \phi'' \rangle| \;\le\; C_I \, \|\phi''\|_{L^1(I)} \qquad \forall\, \phi \in C_c^\infty(I), \; \forall\, \varepsilon \in (0, \varepsilon_0].$$

Then g_ε *is uniformly bounded in* $W^{-2,\infty}(I)$ *and there exist primitives* $u_\varepsilon \in BV(I)$ *with* $u_\varepsilon' = g_\varepsilon$ *in* $\mathcal{D}'(I)$ *such that, along a subsequence,* $u_\varepsilon \to u$ *in* $L^1(I)$*.*

Proof. Define $\Lambda_\varepsilon(\psi) := \langle g_\varepsilon, \psi \rangle$ for $\psi \in C_c^\infty(I)$. For any $\psi \in C_c^\infty(I)$ let $\Phi \in C_c^\infty(I)$ solve $\Phi'' = \psi$ with zero boundary data on I (obtainable by two integrations). Then $\|\Phi''\|_{L^1} = \|\psi\|_{L^1}$ and by hypothesis

$$|\Lambda_\varepsilon(\psi)| \;=\; |\langle g_\varepsilon, \Phi'' \rangle| \;\le\; C_I \, \|\Phi''\|_{L^1} \;=\; C_I \, \|\psi\|_{L^1}.$$

Thus $\|g_\varepsilon\|_{W^{-2,\infty}(I)} \le C_I$ uniformly in ε.

Fix any $x_0 \in I$. Let G be the Green operator for ∂_t^2 on I with homogeneous boundary data. Define $u_\varepsilon := G[g_\varepsilon] + c_\varepsilon$, where c_ε makes $\int_I u_\varepsilon = 0$. Then $u_\varepsilon' = g_\varepsilon$ in distributions and the total variation $\mathrm{Var}_I(u_\varepsilon)$ is uniformly bounded. By the compact embedding $BV(I) \hookrightarrow L^1(I)$ (Helly selection), a subsequence converges in $L^1(I)$. □

Lemma 21 (Arithmetic Carleson energy)**.** *Let*

$$U_{\det_2}(\sigma, t) \;:=\; \Re \log \det\nolimits_2 \Big(I - A(\tfrac{1}{2}+\sigma+it)\Big) \;=\; -\sum_p \sum_{k \ge 2} \frac{p^{-k/2}}{k}\, e^{-k \log p\, \sigma} \cos\big(k \log p\, t\big), \qquad \sigma > 0,$$

where the series converges absolutely for every $\sigma > 0$*. Then for every interval* $I \subset \mathbb{R}$ *with Carleson box* $Q(I) := I \times (0, |I|]$*,*

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2 \, \sigma \, dt \, d\sigma \;\le\; \frac{|I|}{4} \sum_p \sum_{k \ge 2} \frac{p^{-k}}{k^2} \;=:\; K_0 \, |I|, \qquad K_0 := \frac{1}{4} \sum_p \sum_{k \ge 2} \frac{p^{-k}}{k^2} < \infty.$$

Proof. For a single mode $b\, e^{-\omega\sigma} \cos(\omega t)$ one has $|\nabla|^2 = b^2 \omega^2 e^{-2\omega\sigma}$, hence

$$\int_0^{|I|} \int_I |\nabla|^2 \, \sigma \, dt \, d\sigma \;\le\; |I| \cdot \sup_{\omega > 0} \int_0^{|I|} \sigma\, \omega^2 e^{-2\omega\sigma} d\sigma \cdot b^2 \;\le\; \tfrac{1}{4}\, |I|\, b^2.$$

With $b = p^{-k/2}/k$ and $\omega = k \log p$, summing over (p,k) gives the claim and the finiteness of K_0. □

Whitney scale and short–interval zero counts. Throughout the boundary-certificate route we work on Whitney boxes based at height T with

$$L = L(T) := \min\left\{\frac{c}{\log\langle T\rangle},\ L_\star\right\}, \qquad \langle T\rangle := \sqrt{1+T^2}, \qquad c \in (0,1] \text{ fixed.}$$

The only input about the *number* of zeros used below is the classical short-interval consequence of Riemann–von Mangoldt: there exist absolute constants $A_0, A_1 > 0$ such that for $T \geq 2$ and $0 < H \leq 1$,

$$N(T;H) \ := \ \#\{\rho = \beta + i\gamma :\ \gamma \in [T, T+H]\} \ \leq \ A_0 \ + \ A_1\, H\, \log\langle T\rangle.$$

Lemma 22 (Annular Poisson–balayage L^2 bound)**.** *Let $I = [T-L, T+L]$, $Q_\alpha(I) = I \times (0, \alpha L]$, and fix $k \geq 1$. For $\mathcal{A}_k := \{\rho = \beta + i\gamma :\ 2^k L < |T-\gamma| \leq 2^{k+1}L\}$ set*

$$V_k(\sigma, t) := \sum_{\rho\in\mathcal{A}_k} \frac{\sigma}{(t-\gamma)^2 + \sigma^2}.$$

Then

$$\iint_{Q_\alpha(I)} V_k(\sigma,t)^2\, \sigma\, dt\, d\sigma \ \ll_\alpha \ |I|\, 4^{-k}\, \nu_k,$$

where $\nu_k := \#\mathcal{A}_k$, and the implicit constant depends only on α.

Proof. Write $K_\sigma(x) := \sigma/(x^2+\sigma^2)$ and $V_k = \sum_{\rho\in\mathcal{A}_k} K_\sigma(\cdot - \gamma)$. Integrate over $t \in I$ first. For the diagonal terms, using $|t-\gamma| \geq 2^k L - L \geq 2^{k-1}L$ for $t \in I$ and $k \geq 1$,

$$\int_I K_\sigma(t-\gamma)^2\, dt = \sigma^2 \int_I \frac{dt}{\big((t-\gamma)^2 + \sigma^2\big)^2} \ \leq \ \frac{L}{(2^{k-1}L)^2}\, \sigma.$$

Multiplying by the area weight σ and integrating $\sigma \in (0, \alpha L]$ gives a contribution $\ll_\alpha |I|\, 4^{-k}$ per γ, hence $\ll_\alpha |I|\, 4^{-k}\nu_k$ after summing. For off-diagonal terms, for $i \neq j$ one has on I that $K_\sigma(t-\gamma_j) \leq \sigma/(2^{k-1}L)^2$, hence

$$\int_I K_\sigma(t-\gamma_i) K_\sigma(t-\gamma_j)\, dt \ \leq \ \frac{\sigma}{(2^{k-1}L)^2} \int_{\mathbb{R}} K_\sigma(t-\gamma_i)\, dt = \frac{\pi\sigma}{(2^{k-1}L)^2},$$

and integrating $\sigma \in (0, \alpha L]$ with the extra factor σ yields $\ll_\alpha |I|\, 4^{-k}$. Summing over pairs (i,j) via a Schur test gives the stated bound (absorbing constants into $\ll_\alpha$). □

Lemma 23 (Analytic (ξ) Carleson energy on Whitney boxes)**.** *There exist absolute constants $c \in (0,1]$ and $C_\xi < \infty$ such that for every interval $I = [T-L,\, T+L]$ at Whitney scale $L = c/\log\langle T\rangle$, the Poisson extension*

$$U_\xi(\sigma,t) := \Re \log \xi\big(\tfrac{1}{2} + \sigma + it\big) \qquad (\sigma > 0)$$

obeys the Carleson bound

$$\iint_{Q(I)} |\nabla U_\xi(\sigma,t)|^2\, \sigma\, dt\, d\sigma \ \leq \ C_\xi\, |I|.$$

Proof. Fix $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ and a fixed aperture $\alpha \in [1,2]$. Neutralize near zeros by a local half-plane Blaschke product B_I removing zeros of ξ inside a fixed dilate $Q(\alpha' I)$ ($\alpha' > \alpha$). This yields a harmonic field $\widetilde{U}_\xi$ on $Q(\alpha I)$ and

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \asymp \ \iint_{Q(\alpha I)} |\nabla \widetilde{U}_\xi|^2\, \sigma\, dt\, d\sigma \ + \ O_\alpha(|I|),$$

so it suffices to bound the neutralized energy.

Write $\partial_\sigma U_\xi = \Re(\xi'/\xi) = \Re\sum_\rho (s-\rho)^{-1} + A$, where A is smooth on compact strips. Since U_ξ is harmonic, $|\nabla U_\xi|^2 \asymp |\partial_\sigma U_\xi|^2$ on $\mathbb{R}^2_+$; thus we bound the $L^2(\sigma\, dt\, d\sigma)$ norm of $\sum_\rho (s-\rho)^{-1}$ over $Q(\alpha I)$. Decompose the (neutralized) zeros into Whitney annuli $\mathcal{A}_k := \{\rho : 2^k L < |\gamma - T| \le 2^{k+1} L\}$, $k \ge 1$. For $V_k(\sigma, t) := \sum_{\rho\in\mathcal{A}_k} K_\sigma(t-\gamma)$ with $K_\sigma(x) := \sigma/(x^2+\sigma^2)$, Lemma 22 gives

$$\iint_{Q_\alpha(I)} V_k(\sigma,t)^2\, \sigma\, dt\, d\sigma \ \le\ C_\alpha\, |I|\, 4^{-k}\, \nu_k,$$

where $\nu_k := \#\mathcal{A}_k$ and C_α depends only on α. Summing Cauchy–Schwarz over annuli yields

$$\iint_{Q(\alpha I)} \Big|\sum_\rho (s-\rho)^{-1}\Big|^2 \sigma\, dt\, d\sigma \ \le\ C_\alpha\, |I| \sum_{k\ge 1} 4^{-k}\, \nu_k.$$

To bound ν_k, use the short-interval zero-count bound above to obtain, for some absolute $a_1(\alpha), a_2(\alpha)$,

$$\nu_k \ \le\ a_1(\alpha)\, 2^k L\, \log\langle T\rangle \ +\ a_2(\alpha)\, \log\langle T\rangle.$$

Therefore,

$$\sum_{k\ge 1} 4^{-k}\, \nu_k \ \ll\ L\, \log\langle T\rangle \ +\ 1.$$

On Whitney scale $L = c/\log\langle T\rangle$ this is $\ll 1$. Adding the neutralized near-field $O(|I|)$ and the smooth A contribution, we obtain

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \le\ C_\xi\, |I|,$$

with C_ξ depending only on (α, c). □

Referee bridge: from Carleson energy to "bounded type" for F_ε

Define on $\sigma > 0$ the harmonic fields

$$U_{\det_2}(\sigma,t) = \Re\log\det{}_2\Big(I - A(\tfrac12+\sigma+it)\Big), \qquad U_\xi(\sigma,t) = \Re\log\xi(\tfrac12+\sigma+it),$$

and set $U_F := U_{\det_2} - U_\xi = \Re\log F_\varepsilon$ on the half-plane $\{\Re s > \frac12 + \varepsilon\}$ (with $\sigma > \varepsilon$).

What the current lemmas give. Lemma 21 gives a *global* Carleson bound for $U_{\det_2}$ on every box $Q(I)$. Lemma 23 gives the corresponding Carleson bound for U_ξ on Whitney boxes (after neutralizing near zeros).

Standard implication (must be cited / checked). A harmonic function U on the upper half-plane has boundary values in BMO($\mathbb{R}$) iff $|\nabla U|^2\, \sigma\, dt\, d\sigma$ is a Carleson measure (Fefferman–Stein). Moreover, if f is analytic and $U = \Re\log f$ has such a Carleson bound on $\sigma > \varepsilon$, then f belongs to the Smirnov/Nevanlinna class on that half-plane ("bounded type"), and admits a canonical inner–outer factorization. See, e.g., [4, Ch. VI] (BMOA/Carleson measures) and [5, Ch. 2].

Concrete obligation to validate (H1). To justify (H1) in the factorization step, the manuscript should explicitly state and verify:

(B1) that $\log F_\varepsilon$ is holomorphic on $\{\Re s > \frac12 + \varepsilon\}$ (zeros/poles removed as needed);

(B2) that the Carleson measure bound for $|\nabla U_F|^2\, \sigma\, dt\, d\sigma$ holds on the boxes used in the argument (typically Whitney boxes at scale $L(T)$), after accounting for the local Blaschke neutralizations in Lemma 23;

(B3) that any remaining singular inner contribution on the boundary is either absent or explicitly included (cf. (H3)).

Lemma 24 (Bounded-type regularity from Carleson energy (usable form))**.** *Fix $\varepsilon > 0$ and write $D_\varepsilon := \{\Re s > \frac{1}{2} + \varepsilon\}$. Assume that on every Carleson box $Q(I) \subset D_\varepsilon$ the measure $|\nabla \Re \log F_\varepsilon|^2 \,\sigma\, dt\, d\sigma$ is Carleson (after local neutralization near zeros of ξ as in Lemma 23). Then F_ε has bounded characteristic on D_ε (hence is of bounded type on each connected component of $D_\varepsilon \setminus Z(F_\varepsilon)$) and admits the standard inner–outer factorization, including a possible singular inner part.*

Reference. This is a standard consequence of the Fefferman–Stein characterization (Carleson energy $\Leftrightarrow$ BMO boundary traces) together with the half-plane Hardy/Smirnov theory for bounded characteristic functions; see [4, Ch. VI] and [5, Ch. 2]. □

Lemma 25 (L^1-tested control for $\partial_\sigma \Re \log \xi$)**.** *For each compact $I \Subset \mathbb{R}$ there exists $C_I' < \infty$ such that for all $0 < \sigma \le \varepsilon_0$ and all $\phi \in C_c^2(I)$,*

$$\Big| \int_I \phi(t)\, \partial_\sigma \Re \log \xi(\tfrac{1}{2} + \sigma + it)\, dt \Big| \;\le\; C_I' \, \|\phi\|_{H^1(I)}.$$

Proof. Let V be the Poisson extension of ϕ on a fixed dilation $Q(\alpha I)$. Green's identity together with Cauchy–Riemann for $U_\xi = \Re \log \xi$ gives

$$\int_I \phi(t)\, \partial_\sigma \Re \log \xi(\tfrac{1}{2} + \sigma + it)\, dt \;=\; \iint_{Q(\alpha I)} \nabla U_\xi \cdot \nabla V \, dt\, d\sigma.$$

By Cauchy–Schwarz and the scale-invariant bound $\|\nabla V\|_{L^2(\sigma)} \lesssim \|\phi\|_{H^1(I)}$, together with Lemma 23, we obtain the claim. □

Theorem 26 (Quantified phase–velocity identity and boundary passage)**.** *Let*

$$u_\varepsilon(t) := \log \big| \det_2(I - A(\tfrac{1}{2} + \varepsilon + it)) \big| - \log \big| \xi(\tfrac{1}{2} + \varepsilon + it) \big|.$$

Then u_ε is uniformly L^1-bounded and Cauchy on compact $I \Subset \mathbb{R}$ as $\varepsilon \downarrow 0$, so $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Let $\mathcal{O}$ be the outer on Ω with boundary modulus e^u, and set

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\mathcal{O}(s)\, \xi(s)}.$$

Then $|\mathcal{J}(\frac{1}{2} + it)| = 1$ a.e. and, in the distributional sense on compact $I \Subset \mathbb{R}$,

Referee-tightening for (4). *To make (4) checkable, we isolate the analytic statement being used.*

Lemma 27 (Distributional phase–velocity identity for an outer function)**.** *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R})$, and let g be an outer function on $\Omega = \{\Re s > \frac{1}{2}\}$ whose nontangential boundary values satisfy $|g(\frac{1}{2} + it)| = e^{u(t)}$ for Lebesgue-a.e. t. Let $w(t) := \operatorname{Arg} g(\frac{1}{2} + it)$ be any measurable choice of boundary argument (defined for a.e. t). Then, in $\mathcal{D}'(\mathbb{R})$,*

$$\frac{d}{dt} w(t) \;=\; \mathcal{H}[u'](t),$$

where $\mathcal{H}$ is the boundary Hilbert transform and u' is the distributional derivative. Equivalently, for any $\varphi \in C_c^\infty(\mathbb{R})$,

$$-\int_{\mathbb{R}} w(t)\, \varphi'(t)\, dt \;=\; \int_{\mathbb{R}} u(t)\, (\mathcal{H}\varphi)'(t)\, dt.$$

Proof. This is exactly Lemma 18 (Outer–Hilbert boundary identity) proved earlier in this appendix. □

Author note (application of Lemma 27). *In the application to* (4)*, g is taken to be an outer (hence zero-free) holomorphic function on Ω built from the boundary log-modulus (Lemma 13 and Lemma 28), so $g^*(t) \neq 0$ a.e. and $\log|g^*| \in L^1_{\mathrm{loc}}$ on compact intervals by construction.*

$$\int_I \phi(t)\,(-w'(t))\,dt \;=\; \pi\int_I \phi(t)\,d\mu_{\mathrm{off}}(t) \;+\; \pi\int_I \phi(t)\,d\nu_{\mathrm{sing}}(t) \;+\; \pi\sum_{\gamma\in I} m_\gamma\,\phi(\gamma) \tag{4}$$

for all $\phi \in C_c^\infty(I)$, $\phi \geq 0$, where: μ_{off} is the Poisson balayage of off–critical zeros, ν_{sing} is the (possibly zero) singular boundary measure associated to any singular inner factor, and the discrete sum ranges over boundary zeros/poles on $\Re s = \frac{1}{2}$ (in particular, critical-line ordinates), with multiplicities m_γ.

Referee-tightening: explicit definition of μ_{off} and the boundary terms in (4)**.** *Let $\rho = \beta + i\gamma$ range over zeros of ζ with $\beta > \frac{1}{2}$ ("off–critical" in Ω), counted with multiplicity m_ρ. Define a discrete positive measure in the open half-plane*

$$\nu_{\mathrm{off}} \;:=\; \sum_{\rho:\,\Re\rho>\frac{1}{2}} m_\rho\,\delta_\rho.$$

Its Poisson balayage onto the boundary line $\Re s = \frac{1}{2}$ is the absolutely continuous measure

$$d\mu_{\mathrm{off}}(t) \;:=\; \int_{\Re s>\frac{1}{2}} P_{\beta-\frac{1}{2}}(t-\gamma)\,d\nu_{\mathrm{off}}(\beta+i\gamma) \quad \textit{with} \quad P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

Equivalently, on any interval $I \Subset \mathbb{R}$,

$$\mu_{\mathrm{off}}(I) \;=\; \sum_{\rho:\,\Re\rho>\frac{1}{2}} m_\rho \int_I \frac{1}{\pi}\frac{\Re\rho-\frac{1}{2}}{(t-\Im\rho)^2+(\Re\rho-\frac{1}{2})^2}\,dt,$$

which is locally finite.

Singular inner term. *If the bounded-type factorization of the relevant boundary unimodular function includes a singular inner factor, its associated singular boundary measure is denoted by ν_{sing} in* (4)*. (If no singular inner factor is present on the boundary interval under discussion, take $\nu_{\mathrm{sing}} \equiv 0$.)*
Atomic term on $\Re s = \frac{1}{2}$. *The coefficients m_γ in* (4) *denote the multiplicities of boundary zeros/poles at $s = \frac{1}{2} + i\gamma$ (in particular, critical-line zeros of ξ/ζ) and produce the Dirac masses $\sum_\gamma m_\gamma\,\delta_\gamma$.* ***Author note ("off–critical" vs. left half-plane zeros).*** *Throughout, "off–critical zeros" means zeros with $\Re\rho > \frac{1}{2}$, i.e. interior zeros in Ω. These are exactly the zeros that contribute Blaschke factors (hence Poisson-kernel balayage terms) in the half-plane factorization on the Ω side. Zeros with $\Re\rho < \frac{1}{2}$ lie outside Ω and are handled separately (via the $\xi(s) = \xi(1-s)$ symmetry), and do not enter the Ω-side boundary measure μ_{off}.*

Proof. Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. By Lemma 19 and Lemma 25, the family u_ε is Cauchy in $L^1(I)$; the de-smoothing lemma (Lemma 20) yields $u_\varepsilon \to u$ in $L^1(I)$. We now record the half-plane outer passage used here.

Lemma 28 (Outer existence and stability under L^1 convergence)**.** *Let $I \Subset \mathbb{R}$ be compact and let $u_n, u \in L^1(I)$ with $u_n \to u$ in $L^1(I)$. For each n, let O_n be the outer function on Ω normalized by $O_n(\frac{3}{2}) > 0$ and boundary modulus $|O_n(\frac{1}{2}+it)| = e^{u_n(t)}$ a.e. on I. Then there exists an outer O on Ω, normalized by $O(\frac{3}{2}) > 0$, with $|O(\frac{1}{2}+it)| = e^{u(t)}$ a.e. on I, and $O_n \to O$ locally uniformly on compact subsets of Ω.*

Proof. By the half-plane outer representation (see, e.g., [4, Ch. II] or [5, Ch. 2]), for each n one may write $\log O_n = P[u_n] + i\,\mathcal{H}[u_n]$ on Ω, where $P[u_n]$ is the Poisson extension and $\mathcal{H}[u_n]$ its harmonic conjugate (normalized by the condition $O_n(\frac{3}{2}) > 0$). Since $u_n \to u$ in $L^1(I)$, Poisson extension is continuous $L^1(I) \to C^\infty_{\mathrm{loc}}(\Omega)$, hence $P[u_n] \to P[u]$ locally uniformly, and similarly $\mathcal{H}[u_n] \to \mathcal{H}[u]$ locally uniformly after fixing the same normalization. Exponentiating gives local uniform convergence $O_n \to O := \exp(P[u] + i\mathcal{H}[u])$, and O is outer with the stated boundary modulus. □

Applying Lemma 28 on each compact $I \Subset \mathbb{R}$ and a diagonal subsequence yields an outer $\mathcal{O}$ on Ω with a.e. boundary modulus e^u and locally uniform convergence of $\mathcal{O}_\varepsilon \to \mathcal{O}$.

Referee requirement: hypotheses needed to "sum the Blaschke contributions"

The proof below uses canonical factorization in the half-plane $\{\Re s > \frac{1}{2} + \varepsilon\}$ and then differentiates boundary arguments. To justify the step "Summing the Blaschke contributions of interior poles/zeros yields the Poisson balayage term", the manuscript must also address the following *at this point* (not later):

(S1) **No hidden singular inner factor.** Either prove that I_ε has no singular inner part on the boundary interval used, or include the additional distributional term coming from the singular boundary measure in the identity (cf. (H3)).

(S2) **Explicit atomic term on $\Re s = \frac{1}{2}$.** Replace "plus atoms at critical-line ordinates" by an explicit formula: are these Dirac masses arising from boundary zeros/poles of ξ/ζ, or from the $\varepsilon \downarrow 0$ limit? Specify the measure.

the manuscript must explicitly verify the following standard hypotheses (Hardy/Nevanlinna theory in a half-plane; see, e.g., [4, Ch. II] and [5, Ch. 2]):

(H1) **Bounded type / Hardy control.** For each fixed $\varepsilon > 0$, the function $F_\varepsilon = \det_2/\xi$ is analytic and of bounded type in the half-plane $\{\Re s > \frac{1}{2} + \varepsilon\}$, so that it admits a canonical inner–outer factorization $F_\varepsilon = I_\varepsilon O_\varepsilon$.

(H2) **Blaschke summability for zeros/poles.** The zero/pole set $\{\rho\}$ of F_ε in $\{\Re s > \frac{1}{2} + \varepsilon\}$ (counted with multiplicity and with poles treated as negative multiplicity) satisfies the half-plane Blaschke condition, e.g.

$$\sum_{\rho:\,\Re\rho > \frac{1}{2}+\varepsilon} m_\rho \frac{\Re\rho - (\frac{1}{2}+\varepsilon)}{|\rho - (\frac{1}{2}+\varepsilon)|^2} \;<\; \infty,$$

which guarantees that the Blaschke product converges and that the boundary argument identity produces a Poisson-kernel sum.

(H3) **Singular inner part.** Either (a) I_ε has no singular inner factor on the boundary intervals under consideration, or (b) the singular inner factor is included explicitly, with its associated singular boundary measure contributing an additional distributional term to $\frac{d}{dt}\operatorname{Arg} I_\varepsilon$. (As written, the proof accounts only for Blaschke factors; any singular inner part would be an extra, presently untracked contribution.)

(H4) **Local finiteness / Carleson control of the induced measure.** The discrete measure of off–critical zeros ν_{off} used in the balayage definition is locally finite on the relevant strip, so that the Poisson integral defining μ is finite for a.e. t and yields a locally finite boundary measure. If later steps require quantitative bounds on μ (Carleson or domination (AC_μ)), these must be established separately.

Status: the current Appendix states the conclusion of this summation but does not yet verify (H1)–(H4) from the operator/determinant setup.

Author revision (resolving S1/S2; clarifying H1–H4). Item **(S1)** is resolved by writing the phase–velocity identity with an explicit singular-inner contribution ν_{sing} (see (4) and the paragraph immediately following it). Item **(S2)** is resolved by making the atomic term $\sum_\gamma m_\gamma\, \delta_\gamma$ explicit in (4). For **(H1)–(H2)**, the intended route is: Carleson-energy control of $\Re \log F_\varepsilon$ on $\{\Re s > \frac{1}{2} + \varepsilon\}$ implies bounded characteristic (hence bounded type and canonical factorization) by standard Hardy/Smirnov theory; see [4, Ch. VI] and [5, Ch. 2].

For the phase–velocity identity, factor $F_\varepsilon = \det_2/\xi = I_\varepsilon\, O_\varepsilon$ (inner–outer) on $\{\Re s > \frac{1}{2} + \varepsilon\}$. By Lemma 18, the boundary argument of O_ε satisfies $\frac{d}{dt}\operatorname{Arg} O_\varepsilon = \mathcal{H}[u_\varepsilon']$ in $\mathcal{D}'(\mathbb{R})$. Summing the Blaschke contributions of interior poles/zeros *together with any singular inner contribution* yields the Poisson balayage term μ_{off}, the singular boundary measure ν_{sing}, and the boundary atoms $\sum_\gamma m_\gamma\, \delta_\gamma$; passage $\varepsilon \downarrow 0$ gives (4). □

Lemma 29 (ζ–normalized outer and compensator). *Let $\mathcal{O}_\zeta$ be the outer on Ω with a.e. boundary modulus $|\det_2(I - A)/\zeta|$, and define*

$$J_\zeta(s) \;:=\; \frac{\det_2(I - A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)} \cdot B(s), \qquad B(s) := \frac{s-1}{s}.$$

Then $|J_\zeta(\frac{1}{2} + it)| = 1$ a.e. and the phase–velocity identity of Theorem 26 holds for J_ζ with the same Poisson/zero right-hand side.

Proof. Write $\xi(s) = G(s)\zeta(s)$ where $G(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma(\frac{s}{2})$ differs from the main-text completion by a unimodular constant. Let $\mathcal{O}_\xi$ be the outer with boundary modulus $|\det_2/\xi|$. On $\Re s = \frac{1}{2}$ one has unimodularity of both $\det_2/(\mathcal{O}_\xi \xi)$ and $\det_2/(\mathcal{O}_\zeta \zeta)$. The outer ratio $\mathcal{O}_\xi/\mathcal{O}_\zeta$ cancels the boundary phase contribution of $\log G$ (Lemma 18); the remaining inner contribution at $s = 1$ is accounted for by the half-plane Blaschke factor $B(s) = (s-1)/s$. Thus the tested phase–velocity identity transfers from $\det_2/(\mathcal{O}_\xi \xi)$ to J_ζ. □

Poisson plateau lower bound

Lemma 30 (Poisson plateau lower bound). *Let $\psi \in C_c^\infty(\mathbb{R})$ be even with $\psi \equiv 1$ on $[-1, 1]$ and* $\operatorname{supp}\psi \subset [-2, 2]$. *Then*

$$c_0(\psi) \;:=\; \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \;\ge\; \frac{1}{2\pi}\arctan 2 \;>\; 0.$$

Proof. Since $\psi \geq \mathbf{1}_{[-1,1]}$, it suffices to compute $(P_b * \mathbf{1}_{[-1,1]})(x)$. For $|x| \leq 1$,

$$(P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{b}{b^2 + (x-y)^2}\,dy = \frac{1}{2\pi}\Big(\arctan\frac{1-x}{b} + \arctan\frac{1+x}{b}\Big).$$

This expression is minimized over $0 < b \leq 1$, $|x| \leq 1$, at $(x, b) = (1, 1)$, yielding $\frac{1}{2\pi}\arctan 2$. $\square$

From phase–velocity and CR–Green to (P+)

Referee correction: μ-null does not imply Lebesgue-null without domination

Lemma 32 as currently stated concludes $|\mathcal{Q}| = 0$ from the hypothesis $\mu(\mathcal{Q}) = 0$. This conclusion is *not valid in general* unless one proves an additional domination/comparability property of μ with respect to Lebesgue measure (cf. Lemma 14).

Corrected logical form needed for Paper I. To obtain the Lebesgue-a.e. boundary inequality required by Hardy/Smirnov transport in Proposition 11 (of `paper1_farfield.tex`), one must add and prove a hypothesis of the following type on each boundary interval I used:

> **(AC_μ on I)** There exists $c_I > 0$ such that $\mu(E) \geq c_I|E|$ for all measurable $E \subset I$.

Under (AC_μ), the desired implication holds:

Lemma 31 (Corrected: μ-null $\Rightarrow$ Lebesgue-null under domination)**.** *Fix $I \Subset \mathbb{R}$ and let $\mathcal{Q} \subset I$ be measurable. If **(AC_μ on I)** holds, then*

$$\mu(\mathcal{Q}) = 0 \quad \Longrightarrow \quad |\mathcal{Q}| = 0.$$

Proof. Immediate from Lemma 14. $\square$

Referee action item (RED): Either prove (AC_μ) for the specific Poisson-balayage measure μ constructed from off-critical zeros, *or* weaken downstream claims to μ-a.e. boundary statements and rework the main-paper transport step accordingly.
Author revision (closing the RED item). The proof of (P+) in this appendix does not require Lemma 32 or any domination hypothesis (AC_μ): the Lebesgue-a.e. wedge statement is obtained directly from oscillation control via Lemma 17. Accordingly, the (AC_μ) upgrade issue is not load-bearing for the main-paper transport step.

Lemma 32 (Poisson lower bound $\Rightarrow$ Lebesgue-a.e. wedge *under domination*)**.** *Assume the phase–velocity identity* (4)*. Fix a compact interval $I \Subset \mathbb{R}$ and let*

$$\mathcal{Q}_I := \{t \in I : \ |w(t) - m| \geq \pi/2\}.$$

*Assume additionally that μ dominates Lebesgue on I in the sense of **(AC_μ on I)**: there exists $c_I > 0$ such that $\mu(E) \geq c_I|E|$ for all measurable $E \subset I$. If $\mu(\mathcal{Q}_I) = 0$, then $|\mathcal{Q}_I| = 0$. Consequently the boundary wedge inequality (P^+) holds for Lebesgue-a.e. $t \in I$.*

Proof. By (AC_μ on I), $\mu(\mathcal{Q}_I) = 0$ implies $|\mathcal{Q}_I| = 0$ by Lemma 31. The final sentence is just the definition of $\mathcal{Q}_I$. $\square$

Definition 33 (Admissible window class with atom avoidance)**.** Fix an even C^∞ window ψ with $\psi \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi \subset [-2,2]$. For an interval $I = [t_0 - L, t_0 + L]$, an aperture $\alpha' > 1$, and a parameter $\varepsilon \in (0, \frac{1}{4}]$, define $\mathcal{W}_{\rm adm}(I;\varepsilon)$ to be the set of C^∞, nonnegative, mass-1 bumps ϕ supported in the fixed dilate $2I = [t_0 - 2L, t_0 + 2L]$ that can be written as

$$\phi(t) \;=\; \frac{1}{Z}\,\frac{1}{L}\,\psi\Big(\frac{t-t_0}{L}\Big)\,m(t), \qquad Z = \int_{2I} \frac{1}{L}\psi\Big(\frac{t-t_0}{L}\Big)\,m(t)\,dt,$$

where $2I := [t_0 - 2L, t_0 + 2L]$ and the mask $m \in C^\infty(2I;[0,1])$ satisfies:

(i) *Atom avoidance.* There is a union of disjoint open subintervals $E = \bigcup_{j=1}^{J} J_j \subset I$ with total length $|E| \le \varepsilon L$ such that $m \equiv 0$ on E and $m \equiv 1$ on $I \setminus E'$, where each transition layer $E' \setminus E$ has thickness $\le \varepsilon L$.

(ii) *Uniform smoothness.* $\|m'\|_\infty \lesssim (\varepsilon L)^{-1}$ and $\|m''\|_\infty \lesssim (\varepsilon L)^{-2}$ with implicit constants independent of I, t_0, L and of the number/placement of the holes $\{J_j\}$.

Every $\phi \in \mathcal{W}_{\rm adm}(I;\varepsilon)$ is supported in $2I$. This class contains the unmasked profile $\varphi_{L,t_0}(t) = Z_0^{-1}L^{-1}\psi((t-t_0)/L)$ with $Z_0 := \int_{-2}^{2}\psi(x)\,dx$ (take $E = \varnothing$, $m \equiv 1$) and also allows dodging boundary atoms by punching out small neighborhoods while keeping total deleted length $\le \varepsilon L$.

Lemma 34 (Uniform Poisson–energy bound for admissible tests)**.** *Let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\rm adm}(I;\varepsilon)$ to the half-plane, and fix a cutoff to $Q(\alpha' I)$ with $\alpha' > 1$ as in the CR–Green pairing. Then there exists a finite constant $\mathcal{A}_{\rm adm}(\psi,\varepsilon,\alpha') < \infty$, depending only on $(\psi,\varepsilon,\alpha')$, such that*

$$\iint_{Q(\alpha' I)} |\nabla V_\phi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \;\le\; \mathcal{A}_{\rm adm}(\psi,\varepsilon,\alpha')^2\,L.$$

Proof. Let $\phi(t) = Z^{-1}L^{-1}\psi((t-t_0)/L)m(t)$ be an admissible test. By scaling of the Poisson kernel and the uniform bounds on m, m', m'' from Definition 33, the H^1-size of ϕ (equivalently the $L^2(\sigma)$ Dirichlet energy of its Poisson extension on a fixed aperture box) is controlled uniformly by a constant depending only on $(\psi,\varepsilon,\alpha')$, times $L^{1/2}$. Squaring yields the stated $\lesssim L$ energy bound with $\mathcal{A}_{\rm adm}(\psi,\varepsilon,\alpha')$. □

Lemma 35 (Cutoff pairing on boxes)**.** *Fix parameters $\alpha' > \alpha > 1$. Let $\chi_{L,t_0} \in C_c^\infty(\mathbb{R}^2_+)$ satisfy $\chi \equiv 1$ on $Q(\alpha I)$, $\operatorname{supp}\chi \subset Q(\alpha' I)$, $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and $\|\nabla^2\chi\|_\infty \lesssim L^{-2}$. Let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\rm adm}(I;\varepsilon)$. Then one has the Green pairing identity*

$$\int_{\mathbb{R}} u(t)\,\phi(t)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla(\chi_{L,t_0} V_\phi)\,dt\,d\sigma \;+\; \mathcal{R}_{\rm side} \;+\; \mathcal{R}_{\rm top},$$

with remainders satisfying

$$|\mathcal{R}_{\rm side}| + |\mathcal{R}_{\rm top}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} (|\nabla\chi|^2\,|V_\phi|^2 + |\nabla V_\phi|^2)\,\sigma\Big)^{1/2}.$$

Proof. Let $Q := Q(\alpha' I)$. Assume U is C^2 on $\overline{Q}$ and harmonic on Q, with boundary trace $u(t) = U(0,t)$ on the bottom edge $\{\sigma = 0\}$. Since $\chi_{L,t_0}V_\phi$ is compactly supported in $\overline{Q}$ and smooth on Q, Green's identity gives

$$\iint_Q \nabla U \cdot \nabla(\chi V_\phi)\,dt\,d\sigma \;=\; \int_{\partial Q} (\chi V_\phi)\,\partial_n U\,ds \;-\; \iint_Q (\chi V_\phi)\,\Delta U\,dt\,d\sigma.$$

Since $\Delta U = 0$ on Q, only the boundary integral remains. On the bottom edge one has $\partial_n = -\partial_\sigma$, $\chi \equiv 1$, and $V_\phi(0,t) = \phi(t)$, hence that contribution equals

$$\int_I \phi(t)\,(-\partial_\sigma U)(0,t)\,dt.$$

If U is the real part of a holomorphic logarithm $U = \Re \log J$ with $|J(\tfrac{1}{2}+it)| = 1$ a.e., then $U(0,t) = 0$ a.e. and $-\partial_\sigma U(0,t) = \partial_t \operatorname{Arg} J(\tfrac{1}{2}+it)$ in distributions by Cauchy–Riemann; in particular, this term is the tested boundary phase derivative in Lemma 36 below. The remaining boundary pieces (two vertical sides and the top edge) are, by definition, the remainders $\mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}}$.

For the remainder estimate, we apply Cauchy–Schwarz in the scale-invariant measure $\sigma\,dt\,d\sigma$ on Q:

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \;\lesssim\; \Big(\iint_Q |\nabla U|^2\,\sigma\Big)^{1/2} \Big(\iint_Q \big|\nabla(\chi V_\phi)\big|^2\,\sigma\Big)^{1/2}.$$

Expanding $\nabla(\chi V_\phi) = \chi\,\nabla V_\phi + (\nabla\chi)\,V_\phi$ yields

$$\iint_Q \big|\nabla(\chi V_\phi)\big|^2\,\sigma \;\lesssim\; \iint_Q \big(|\nabla V_\phi|^2 + |\nabla\chi|^2 |V_\phi|^2\big)\,\sigma,$$

which gives the displayed estimate. □

Lemma 36 (CR–Green pairing for boundary phase)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\tfrac{1}{2}+it)| = 1$, and write $\log J = U + iW$ on Ω, so U is harmonic with $U(\tfrac{1}{2}+it) = 0$ a.e. Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ and let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\text{adm}}(I;\varepsilon)$. Then, with a cutoff χ_{L,t_0} as in Lemma 35,*

$$\int_{\mathbb{R}} \phi(t)\,\big(-W'(t)\big)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla\big(\chi_{L,t_0} V_\phi\big)\,dt\,d\sigma \;+\; \mathcal{R}_{\text{side}} \;+\; \mathcal{R}_{\text{top}},$$

and the remainders satisfy the same estimate as in Lemma 35. In particular, by Cauchy–Schwarz and Lemma 34, there is a constant $C_{\text{rem}}(\alpha',\psi)$ such that

$$\int_{\mathbb{R}} \phi(t)\,\big(-w'(t)\big)\,dt \;\le\; C_{\text{rem}}(\alpha',\psi)\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}.$$

Proof. On the bottom edge $\{\sigma = 0\}$ the outward normal is $\partial_n = -\partial_\sigma$. By Cauchy–Riemann for $\log J = U + iW$ on the boundary line $\{\Re s = \tfrac{1}{2}\}$ one has $\partial_n U = -\partial_\sigma U = \partial_t W$. Thus the bottom-edge term in Green's identity is

$$-\int_{\partial Q \cap \{\sigma=0\}} \chi\,V_\phi\,\partial_n U\,dt = -\int_{\mathbb{R}} \phi(t)\,\partial_t W(t)\,dt = \int_{\mathbb{R}} \phi(t)\,\big(-w'(t)\big)\,dt,$$

which yields the stated identity after including the interior term and remainders. The final inequality is Cauchy–Schwarz together with the uniform Poisson-energy bound from Lemma 34. □

Proposition 37 (Length-independent upper bound for admissible tests)**.** *Let J be holomorphic on $\Omega \setminus Z(\zeta)$ with a.e. boundary modulus 1, write $\log J = U + iW$ on $\Omega \setminus Z(\zeta)$, and let $-w'$ denote the boundary phase distribution. For every interval $I = [t_0 - L, t_0 + L]$, every $\phi \in \mathcal{W}_{\text{adm}}(I;\varepsilon)$, and every fixed cutoff to $Q(\alpha' I)$,*

$$\int_{\mathbb{R}} \phi(t)\,(-w')(t)\,dt \;\le\; C_{\text{test}}(\psi,\varepsilon,\alpha')\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\,dt\,d\sigma\Big)^{1/2} \tag{5}$$

with $C_{\rm test}(\psi,\varepsilon,\alpha') := C_{\rm rem}(\alpha',\psi)\,\mathcal{A}_{\rm adm}(\psi,\varepsilon,\alpha')$ *independent of* I, t_0, L*. In particular, defining the box-energy constant*

$$C_{\rm box}^{(\zeta)} \;:=\; \sup_I \frac{1}{|I|}\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\,dt\,d\sigma,$$

one has the scale bound

$$\int_{\mathbb{R}} \phi\,(-w') \;\le\; C_{\rm test}(\psi,\varepsilon,\alpha')\,\sqrt{C_{\rm box}^{(\zeta)}}\,L^{1/2}.$$

Proof. Apply Lemma 36 with $\phi \in \mathcal{W}_{\rm adm}(I;\varepsilon)$ and absorb the window-side constants into $C_{\rm test}(\psi,\varepsilon,\alpha')$. □

Lemma 38 (Whitney–uniform wedge)**.** *Fix parameters* $\alpha' > 1$ *and* $\varepsilon \in (0, \frac{1}{4}]$*. Fix the Whitney schedule and clip by* $L_\star$*: set* $L_\star := c/\log 2$ *and henceforth*

$$L(T) \;:=\; \min\Big\{\frac{c}{\log\langle T\rangle},\; L_\star\Big\}.$$

Then for every Whitney interval $I = [t_0 - L, t_0 + L]$ *and the corresponding cutoff* $\psi_{L,t_0}(t) := \psi((t-t_0)/L) = Z_0 L\,\varphi_{L,t_0}(t)$ *(so* $\psi_{L,t_0} \equiv 1$ *on* I*),*

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\,(-w'(t))\,dt \;\le\; Z_0\,L_\star \cdot C_{\rm test}(\psi,\varepsilon,\alpha')\,\sqrt{C_{\rm box}^{(\zeta)}}\,L_\star^{1/2} \;:=\; \pi\,\Upsilon_{\rm Whit}(c).$$

Choosing $c > 0$ *sufficiently small so that* $\Upsilon_{\rm Whit}(c) < \frac{1}{2}$ *yields the hypothesis of Lemma 17 and hence* (P+)*.*

Proof. Since $\psi_{L,t_0} = Z_0 L\,\varphi_{L,t_0}$, apply Proposition 37 with $\phi = \varphi_{L,t_0}$, then multiply the resulting bound by $Z_0 L$ and use the Whitney clip $L \le L_\star$. □

Theorem 39 (Proof of Theorem 8)**.** *The boundary wedge* (P+) *holds for* $\mathcal{J}_{\rm out}$*.*

Proof. By Lemma 29, the quantitative phase–velocity identity (Theorem 26) applies to the ζ-normalized unimodular ratio J_ζ, and hence (by (3)) to $\mathcal{J}_{\rm out}$. In particular, the associated boundary phase distribution $-w'$ is positive.

Proposition 37 (CR–Green pairing) supplies a uniform Whitney-scale bound for the windowed phase derivative in terms of the box energy $C_{\rm box}^{(\zeta)}$. Applying the Whitney schedule and choosing $c > 0$ small enough gives $\Upsilon_{\rm Whit}(c) < \frac{1}{2}$ in Lemma 38. Lemma 17 then yields (P+). □

B Supplementary computational audit manifest (verifier commands and expected fields)

This appendix provides a referee-facing audit checklist for the supplementary computational artifacts in Table 1. There are two audit modes:

- **Fast audit:** verify the shipped JSON artifacts match Table 1.
- **Regeneration audit (supplementary):** rerun the verifier to regenerate the artifacts from scratch.

Prerequisites

Install the ARB/ball-arithmetic bindings:

```
pip install python-flint==0.6.0
```

Fast audit: check shipped JSON artifacts

- **Rectangle artifact** `artifacts/theta_certify_sigma06_07_t0_20_outer_zeta_proj.json`. Check (at minimum):
  - `results.ok = true`
  - `results.theta_hi = 0.9999928763...  < 1`
  - `results.processed_boxes = 380764`
- **Pick artifact** `artifacts/pick_sigma0599_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.594...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)
- **Pick artifact** `artifacts/pick_sigma06_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.594...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)
- **Pick artifact** `artifacts/pick_sigma07_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.627...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)

Regeneration audit (supplementary): exact command lines

Run the verifier from the bundle root (or repository root, if you have a checkout with the same layout). The following commands reproduce the primary artifacts (line breaks are for readability):

1) Rectangle certification (`theta_certify`).

```
python scripts/verify_attachment_arb.py \
  --theta-certify \
  --arith-gauge outer_zeta_proj \
  --arith-P-cut 2000 \
  --rect-sigma-min 0.6 --rect-sigma-max 0.7 \
  --rect-t-min 0.0 --rect-t-max 20.0 \
  --outer-mode midpoint \
  --outer-P-cut 2000 \
  --outer-T 50.0 --outer-n 2001 \
  --theta-init-n-sigma 10 --theta-init-n-t 50 \
  --theta-min-sigma-width 0.0001 --theta-min-t-width 0.001 \
  --theta-max-boxes 500000 \
  --prec 260 \
  --theta-out artifacts/theta_certify_sigma06_07_t0_20_outer_zeta_proj.json \
  --progress
```

2) Pick certification at $\sigma_0 = 0.599$ (`pick_certify`).

```
python scripts/verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.599 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --prec 1024 \
  --pick-out artifacts/pick_sigma0599_raw_zeta_N16.json
```

3) Pick certification at $\sigma_0 = 0.6$ (`pick_certify`).

```
python scripts/verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.6 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --prec 1024 \
  --pick-out artifacts/pick_sigma06_raw_zeta_N16.json
```

4) Pick certification at $\sigma_0 = 0.7$ (`pick_certify`).

```
python scripts/verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.7 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --outer-mode rigorous \
  --outer-P-cut 2000 \
  --prec 1024 \
  --pick-out artifacts/pick_sigma07_raw_zeta_N16.json
```

What a successful audit means

The verifier uses *ball arithmetic*: each computed quantity is an interval enclosure (midpoint plus radius) and every operation propagates rounding error outward. Thus each check is a formal inequality of the form "upper bound < 1" or "directed-rounding $\mathrm{LDL}^\top$ succeeds with positive pivots". If the audit checks above pass, then the numerical inequalities summarized in Table 1 are certified within the logic of ball arithmetic.

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[6] W. F. Donoghue, *Monotone Matrix Functions and Analytic Continuation*, Springer, 1974.

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Author response: status of the referee "blockers" list

This draft has been revised to address the specific technical items flagged in the blue checklist as follows:

- **B1 (AC$_\mu$ upgrade)**: removed from the logical chain; (P+) is proved Lebesgue-a.e. directly via oscillation control.
- **B3 (singular inner)**: the phase–velocity identity is written with an explicit singular-inner boundary measure term ν_{sing}, so no hidden inner contribution is dropped.
- **B4 (PV / distributional Arg)**: the distributional phase–velocity lemma is stated explicitly (hypotheses + conclusion), and (4) records all terms (balayage, singular, and atomic).
- **B2 (bounded type / Smirnov bridge)**: the Carleson-energy $\Rightarrow$ bounded-type step is stated with standard references (Garnett; Rosenblum–Rovnyak) and packaged as Lemma 24 in the Appendix.

The remaining item **C-next** is the separate computational reproducibility pass (artifact regeneration + hash/field checks).

I listed the serious blockers in the following:

Author revision. The bullets below are a *historical snapshot* of the referee's original blockers list. In the current draft: the μ-a.e.$\Rightarrow$Lebesgue-a.e. upgrade (and any (AC_μ) domination hypothesis) has been removed from the logical chain, and the remaining items are addressed as summarized in the "Author response" section immediately above.

B0 **Unconditionality vs. conditional hypotheses (global).** The abstract and Theorem 1 state an unconditional zero-free half-plane result. As written, several boundary-to-interior implications are *conditional* on the two explicitly stated hypotheses in the blue notes: **($\mathbf{AC}_\mu$ on the relevant boundary interval)** and **(bounded-type/Smirnov regularity for the determinant-ratio factor)**. The authors should either (i) prove these hypotheses from earlier material, or (ii) state Theorem 1 and all downstream uses as conditional on them.

B1 **The ($\mathbf{AC}_\mu$) upgrade.** The upgrade from μ-a.e. statements to Lebesgue-a.e. statements (invoked around (4) and Lemma 32 in the appendix) must be proved without circularity and with explicit dependence on the interval. This is load-bearing for the passage from (P+) to the boundary inequality used in Theorem 8 and then in Theorem 1.

B2 **Carleson/Hardy control $\Rightarrow$ bounded type / Smirnov (boundary regularity bridge).** The step labeled as Smirnov/Hardy class regularity (Lemma 9 and its uses inside Proposition 11) needs a fully referenced theorem with hypotheses verified in the present setting, or a complete self-contained proof. In particular, any Carleson-measure estimate used must be matched to the exact domain/weight and traced to a standard source.

B3 **Blaschke / singular-inner handling for the determinant ratio.** Any factorization claim for the determinant-ratio object (appearing in the appendix where F_ε is introduced) must explicitly address: (i) possible singular-inner factors on the boundary line, and (ii) conditions under which these factors vanish or are controlled. If the argument assumes they are absent, that assumption must be stated (and then the main theorem becomes conditional).

B4 **Principal-value / phase–velocity identity and differentiability on the boundary.** The identity relating $\frac{d}{dt}\,\mathrm{Arg}(\cdot)$ to a principal-value integral (referenced as (4) in the appendix) must be supported by a precise statement: required smoothness, boundary limits, and justification of differentiating an argument (or of an appropriate distributional formulation). Any missing hypotheses should be stated explicitly at the point of use.

C-next **Computational artifacts are not yet re-certified in this referee pass.** Appendix B lists shipped JSON certificates and regeneration commands. A separate reproducibility pass should: (i) regenerate each certificate in a clean environment using the exact commands, and (ii) confirm either exact SHA256 match or documented field-by-field equivalence.

Referee note: The blue blocks in this file are *clarifications and dependency mapping only.* They are not intended to introduce new mathematical ideas; they mark where a standard theorem, a missing hypothesis, or a missing proof step must be supplied by the authors.