CALIBRATION DEFICIT CONTROLS SQUARED KÄHLER-ANGLE DEFECT

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Abstract. We record a pointwise (and hence global) inequality comparing the Wirtinger/calibration mass deficit to an L^2-type *squared* Kähler-angle defect. We also note that no uniform inequality can hold with the square-root (unsquared) L^2-defect, since the mass deficit is generically quadratic in the angle.

1. Setup

Let (X^{2n}, g, J, ω) be a Kähler manifold and fix an integer $1 \leq p \leq n$. Let

$$\psi := \frac{\omega^p}{p!}.$$

Then ψ is closed and has comass 1; in particular, for any oriented unit simple $2p$-vector ξ, one has $|\psi(\xi)| \leq 1$ (Wirtinger inequality), with equality iff ξ spans a J-invariant (complex) p-plane with the calibration orientation.

Let T be an integral $2p$-cycle in X (so $\partial T = 0$). Its mass is

$$\mathrm{Mass}(T) = \int 1 \, d|T|.$$

Since $d\psi = 0$ and $\partial T = 0$, the pairing $T(\psi)$ depends only on the homology class $[T]$. Moreover, by comass$= 1$,

$$T(\psi) = \int \psi(\vec{T}) \, d|T| \leq \int 1 \, d|T| = \mathrm{Mass}(T).$$

Thus the *calibration deficit* is

$$\mathrm{Mass}(T) - T(\psi) = \int \big(1 - \psi(\vec{T})\big) \, d|T|.$$

2. Kähler angles and the squared angle defect

At $|T|$-a.e. point x, the approximate tangent plane of T is an oriented $2p$-plane $\xi = \vec{T}(x)$. Its (unoriented) Kähler angles $\theta_1(\xi), \ldots, \theta_p(\xi) \in [0, \pi/2]$ are defined so that

$$|\psi(\xi)| = \prod_{i=1}^{p} \cos\theta_i(\xi).$$

A natural "L^2-type squared" Kähler-angle defect is the pointwise quantity

$$d(\xi) := \sum_{i=1}^{p} \sin^2\theta_i(\xi),$$

and the associated global defect functional is

$$D(T) := \int d(\vec{T}) \, d|T| = \int \left(\sum_{i=1}^{p} \sin^2\theta_i(\vec{T}) \right) d|T|.$$

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Theorem 1 (Deficit controls squared Kähler-angle defect)**.** *For any integral $2p$-cycle T as above,*

$$\mathrm{Mass}(T) - T(\psi) \;\geq\; c_p\, D(T),$$

where one may take $c_p = \frac{1}{2}$ for $p = 1$ and $c_p = \frac{1}{p}$ for $p \geq 2$ (in particular, $c_p = \frac{1}{2p}$ works for all $p \geq 1$).

Proof. It suffices to prove the pointwise inequality

$$1 - \psi(\xi) \;\geq\; c_p \sum_{i=1}^{p} \sin^2\theta_i(\xi) \qquad \text{for every oriented unit simple } 2p\text{-vector } \xi,$$

and then integrate over $|T|$.

Since $1 - \psi(\xi) \geq 1 - |\psi(\xi)|$, it is enough to control $1 - |\psi(\xi)|$. Write $x_i := \cos\theta_i(\xi) \in [0, 1]$, so that $|\psi(\xi)| = \prod_{i=1}^{p} x_i$ and $\sin^2\theta_i = 1 - x_i^2$.

Case $p = 1$. We need $1 - x_1 \geq \frac{1}{2}(1 - x_1^2)$ for $x_1 \in [0, 1]$, which is immediate from $1 - x_1 = \frac{1-x_1^2}{1+x_1} \geq \frac{1}{2}(1 - x_1^2)$.

Case $p \geq 2$. By AM–GM applied to the nonnegative numbers $x_1^2, \ldots, x_p^2$,

$$\prod_{i=1}^{p} x_i^2 \leq \left(\frac{1}{p}\sum_{i=1}^{p} x_i^2\right)^{p}.$$

Taking square-roots gives

$$\prod_{i=1}^{p} x_i \leq \left(\frac{1}{p}\sum_{i=1}^{p} x_i^2\right)^{p/2}.$$

Since $0 \leq \frac{1}{p}\sum x_i^2 \leq 1$ and $\frac{p}{2} \geq 1$, we have $a^{p/2} \leq a$ for $a \in [0, 1]$, hence

$$\prod_{i=1}^{p} x_i \leq \frac{1}{p}\sum_{i=1}^{p} x_i^2.$$

Therefore,

$$1 - |\psi(\xi)| = 1 - \prod_{i=1}^{p} x_i \geq 1 - \frac{1}{p}\sum_{i=1}^{p} x_i^2 = \frac{1}{p}\sum_{i=1}^{p}(1 - x_i^2) = \frac{1}{p}\sum_{i=1}^{p}\sin^2\theta_i(\xi).$$

This proves the pointwise inequality with $c_p = 1/p$ for $p \geq 2$.

Integrating over $|T|$ yields the claimed inequality. $\square$

Lemma 1 (No uniform inequality for the square-root L^2 defect)**.** *There is no universal constant $c > 0$ such that*

$$\mathrm{Mass}(T) - T(\psi) \;\geq\; c\,\sqrt{D(T)}$$

holds for all integral $2p$-cycles T.

Proof. Work on a flat complex torus $X = \mathbb{C}^n/\mathbb{Z}^{2n}$ with its standard Kähler form. Fix a smooth flat $2p$-dimensional subtorus T_0 calibrated by ψ, and for $\varepsilon > 0$ let T_ε be the same underlying subtorus with its tangent planes rotated so that all Kähler angles equal ε. Then $D(T_\varepsilon) \simeq \mathrm{Mass}(T_0)\cdot p\,\varepsilon^2$ while $\mathrm{Mass}(T_\varepsilon) - T_\varepsilon(\psi) \simeq \mathrm{Mass}(T_0)\cdot C\,\varepsilon^2$ for small ε. Thus $(\mathrm{Mass}(T_\varepsilon) - T_\varepsilon(\psi))/\sqrt{D(T_\varepsilon)} \simeq C'\varepsilon \to 0$ as $\varepsilon \to 0$, contradicting any uniform lower bound by a fixed $c > 0$. $\square$

Remark 1 (On the vanishing set of $D(T)$)**.** *The functional $D(T) = \int \sum_i \sin^2\theta_i \, d|T|$ vanishes iff the approximate tangent planes are J-invariant $|T|$-a.e., but it does* not *encode the calibration orientation. To conclude that T is a complex analytic cycle, one typically needs a positivity/calibration condition, e.g. $\psi(\vec{T}) = 1$ $|T|$-a.e. (or, in complex language, that T is a positive closed (p,p)-current with integer multiplicities).*