Recognition Algebra: The Unified Algebraic Framework Forced by a Single Composition Law

A Self-Contained Mathematical Treatment

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Abstract

A single functional equation on $\mathbb{R}_{>0}$,

$$J(xy) + J(x/y) = 2\,J(x)\,J(y) + 2\,J(x) + 2\,J(y),$$

together with the normalization $J(1) = 0$ and the calibration $J''_{\log}(0) = 1$, uniquely determines the cost function $J(x) = \frac{1}{2}(x + x^{-1}) - 1$. We show that this determination forces, with no further choices, four interlocking algebraic structures:

(i) a commutative monoid $(\mathbb{R}_{\geq 0}, \star, 0)$ with $a \star b = 2ab + 2a + 2b$;
(ii) the real-quadratic ring $\mathbb{Z}[\varphi]$, where $\varphi = (1+\sqrt{5})/2$ is a unit of norm -1;
(iii) a graded abelian group of paired events satisfying a global balance constraint $\sigma = 0$; and
(iv) the cyclic phase group $\mathbb{Z}/8\mathbb{Z}$ with a canonical DFT-8 spectral decomposition admitting exactly 20 basis modes.

We call the resulting quadruple *Recognition Algebra.* We define a category **RecAlg** whose objects are cost algebras satisfying the composition law and whose morphisms are multiplicative, cost-preserving maps. Recognition Algebra is the *initial object* of this category: every calibrated cost algebra receives a unique morphism from it. In particular, no free parameters remain.

The paper is self-contained; every claim is accompanied by a proof or an explicit proof sketch. A companion Lean 4 formalization (six modules, ~1,200 lines) independently certifies the principal theorems [5].

Keywords: functional equations, golden ratio, d'Alembert equation, cost function, zero-parameter framework, category theory.

Contents

1 Introduction

Unification in physics has always meant discovering that two apparently independent phenomena are governed by the same algebraic structure. Newton showed that the fall of an apple and the orbit of the Moon obey one law of gravitation. Maxwell showed that electricity and magnetism are components of a single antisymmetric tensor. Einstein showed that space and time are coordinates in a single pseudo-Riemannian manifold.

Each of these advances began with an algebraic identity that *forced* downstream structure—once the identity was granted, everything else followed. This paper exhibits such an identity at the most fundamental level and traces all of the algebraic consequences that it forces.

1.1 The starting point

We begin with a single functional equation.

Definition 1.1 (Recognition Composition Law)**.** A function $J : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the *Recognition Composition Law* (RCL) if, for all $x, y > 0$,

$$J(xy) + J(x/y) = 2\,J(x)\,J(y) + 2\,J(x) + 2\,J(y). \tag{1}$$

Supplementing (1) with two constraints,

$$J(1) = 0, \qquad \left.\frac{d^2}{dt^2}\right|_{t=0} J(e^t) = 1, \tag{2}$$

we will show that J is *uniquely* determined, and that four algebraic layers—cost composition, the golden ratio, double-entry conservation, and period-8 phase structure—emerge inevitably.

1.2 Relation to classical functional equations

Equation (1) is a *calibrated multiplicative form of the d'Alembert functional equation.* Under the logarithmic substitution $t = \ln x$, $u = \ln y$, define $G(t) = J(e^t)$. Then (1) becomes

$$G(t+u) + G(t-u) = 2\,G(t)\,G(u) + 2\big(G(t) + G(u)\big). \tag{3}$$

Setting $H(t) = G(t) + 1$, one obtains

$$H(t+u) + H(t-u) = 2\,H(t)\,H(u), \tag{4}$$

which is the standard d'Alembert (cosine) functional equation. Its continuous solutions are $H(t) = \cosh(\kappa t)$ for $\kappa \geq 0$ and $H(t) = \cos(\kappa t)$ for $\kappa \geq 0$ (see Aczél [2], Aczél–Dhombres [3]). The calibration $H''(0) = 1$ forces $\kappa = 1$ and selects cosh over cos (since $\cos''(0) = -1$). Therefore $H(t) = \cosh t$, i.e. $G(t) = \cosh t - 1$.

1.3 Outline

In §2 we develop the **Cost Algebra**: the algebraic properties of J itself, including a commutative monoid on cost values. In §3 we show that J forces the **φ-Ring** $\mathbb{Z}[\varphi]$ and derive its ring-theoretic properties. In §4 we prove that reciprocal symmetry forces a **Ledger Algebra** of paired events with a global balance constraint. In §5 we construct the **Octave Algebra**, a period-8 structure with a canonical spectral basis. These four layers are assembled into a single structure in §6 and given categorical treatment in §7.

2 The Cost Algebra

2.1 Uniqueness of the cost function

Theorem 2.1 (Uniqueness – T5)**.** *Let $J : \mathbb{R}_{>0} \to \mathbb{R}$ be continuous, satisfy the RCL* (1)*, and obey the normalization and calibration conditions* (2)*. Then*

$$J(x) = \tfrac{1}{2}(x + x^{-1}) - 1 \qquad \textit{for all } x > 0. \tag{5}$$

Proof sketch. As shown in §1, the substitution $H(t) = J(e^t) + 1$ converts the RCL to the d'Alembert equation (4) with $H(0) = 1$. By the Aczél–Dhombres classification [3], every continuous solution is $H(t) = \cosh(\kappa t)$. The calibration $H''(0) = 1$ gives $\kappa^2 = 1$, so $\kappa = 1$. Undoing the substitution yields (5). □

2.2 Elementary properties

Proposition 2.2 (Cost algebra identities)**.** *The function* (5) *satisfies, for every $x > 0$:*

(a) $J(1) = 0$.
(b) $J(x) = J(x^{-1})$ (reciprocal symmetry).
(c) $J(x) \geq 0$, *with equality iff* $x = 1$.
(d) $J(x) = (x-1)^2/(2x)$ (defect form).

Proof. (a): $J(1) = \frac{1}{2}(1+1) - 1 = 0$. (b): $J(x^{-1}) = \frac{1}{2}(x^{-1} + x) - 1 = J(x)$. (d): Clearing fractions, $\frac{1}{2}(x + x^{-1}) - 1 = \frac{x^2+1-2x}{2x} = \frac{(x-1)^2}{2x}$. (c): $(x-1)^2 \geq 0$ and $x > 0$, so the ratio is non-negative, and vanishes iff $x = 1$. □

2.3 Cost composition

The RCL expresses how costs combine when ratios are multiplied. If we write $a = J(x)$ and $b = J(y)$, the right-hand side of (1) depends only on a and b. This defines an *induced binary operation* on cost values.

Definition 2.3 (Cost composition)**.** For $a, b \in \mathbb{R}_{\geq 0}$, the *cost-composition operation* is

$$a \star b \;=\; 2ab + 2a + 2b \;=\; 2(a{+}1)(b{+}1) - 2. \tag{6}$$

Equivalently, writing $A = a + 1$ and $B = b + 1$, the factored form is $a \star b = 2AB - 2$. Under this operation, the RCL (1) reads

$$J(xy) + J(x/y) = J(x) \star J(y) : \tag{7}$$

the *total cost of the pair* $(xy,\, x/y)$ is determined by the individual costs.

We now determine the precise algebraic structure of $\star$.

Theorem 2.4 (Algebraic structure of $\star$)**.** *The pair* $(\mathbb{R}_{\geq 0}, \star)$ *is a* ***commutative, flexible, power-associative magma*** *that is* ***not associative*** *and possesses* ***no identity element****. Specifically:*

(a) ***Commutativity.*** $a \star b = b \star a$ *for all* $a, b \geq 0$.

(b) ***No identity.*** *There is no element* $e \geq 0$ *satisfying* $e \star a = a$ *for all* $a \geq 0$.

(c) ***Non-associativity.*** *The* associator *of* $\star$ *has the closed form*

$$(a \star b) \star c \;-\; a \star (b \star c) \;=\; 2(a - c). \tag{8}$$

In particular, $\star$ *is associative on a triple* (a, b, c) *if and only if* $a = c$.

(d) ***Flexibility.*** $(a \star b) \star a = a \star (b \star a)$ *for all* $a, b \geq 0$.

(e) ***Power-associativity.*** *Every element generates an associative sub-magma: all parenthesizations of* $a \star a \star \cdots \star a$ *agree.*

Proof. (a) $a \star b = 2ab + 2a + 2b = 2ba + 2b + 2a = b \star a$.

(b) $e \star a = 2ea + 2e + 2a$. Setting this equal to a requires $2ea + 2e + a = 0$ for all $a \geq 0$. Taking $a = 0$ forces $e = 0$; but $0 \star a = 2a \neq a$ for $a > 0$. No identity exists.

(c) We compute both bracketings by direct expansion:

$$\begin{aligned}
a \star b &= 2ab + 2a + 2b, \\
(a \star b) \star c &= 2(2ab{+}2a{+}2b)\,c + 2(2ab{+}2a{+}2b) + 2c \\
&= 4abc + 4ac + 4bc + 4ab + 4a + 4b + 2c,
\end{aligned}$$

$$\begin{aligned}
b \star c &= 2bc + 2b + 2c, \\
a \star (b \star c) &= 2a(2bc{+}2b{+}2c) + 2a + 2(2bc{+}2b{+}2c) \\
&= 4abc + 4ab + 4ac + 4bc + 4b + 4c + 2a.
\end{aligned}$$

Subtracting: $(4a + 4b + 2c) - (4b + 4c + 2a) = 2a - 2c = 2(a{-}c)$.

(d) Set $c = a$ in (8): the associator is $2(a - a) = 0$.

(e) Every parenthesization of $a^{\star n}$ agrees because any two parenthesizations differ by a sequence of re-associations $(u \star v) \star w \leftrightarrow u \star (v \star w)$ where $u = w = a$; each such swap has zero associator by (d). ☐

Remark 2.5 (Naming convention). A *magma* (also called *groupoid* in older universal-algebra literature) is a set equipped with a single binary operation and no further axioms [4]. The operation $\star$ satisfies commutativity and flexibility but fails both associativity and the existence of an identity element. It is therefore *not* a monoid, semigroup, group, or quasigroup. The linear associator (8)—depending only on the *outer* arguments and independent of the middle argument b—is a distinctive algebraic signature of the RCL.

2.4 The shifted monoid and the correct associative realization

The non-associativity of $\star$ is resolved by passing to the *shifted* function $H = J + 1$.

Theorem 2.6 (Associative realization)**.** *Define $H = J + 1 : \mathbb{R}_{>0} \to [1, \infty)$, so that $H(x) = \frac{1}{2}(x + x^{-1}) \geq 1$.*

(a) ***d'Alembert character.*** *The d'Alembert equation* (4) *states*

$$H(xy) + H(x/y) = 2\,H(x)\,H(y).$$

In the log-coordinate $t = \ln x$, $H(e^t) = \cosh t$, so cosh *is a character of the additive group $(\mathbb{R}, +)$ satisfying the d'Alembert (cosine) equation.*

(b) ***Shifted operation.*** *For $A, B \geq \frac{1}{2}$, define $A \bullet B = 2AB$. Then $([\frac{1}{2}, \infty),\ \bullet,\ \frac{1}{2})$ is a* ***commutative monoid****:*

- Associativity: $(A \bullet B) \bullet C = 2(2AB)C = 4ABC = 2A(2BC) = A \bullet (B \bullet C)$.
- Identity: $\frac{1}{2} \bullet A = 2 \cdot \frac{1}{2} \cdot A = A$.

(c) ***Restriction to H-values.*** *$([1, \infty), \bullet)$ is an associative commutative sub-semigroup (without identity, since $\frac{1}{2} \notin [1, \infty)$).*

Proof. (a) Immediate from (4). The identity $\cosh(t{+}u) + \cosh(t{-}u) = 2\cosh t \cosh u$ is verified by expanding $\cosh t = (e^t + e^{-t})/2$.

(b)–(c) Associativity and the identity property are verified by the displayed calculations. Closure of $[1, \infty)$: if $A, B \geq 1$ then $2AB \geq 2 \geq 1$. □

Remark 2.7 (Why $\star$ is non-associative). The operation $\star$ is the conjugate of $\bullet$ by the *affine* shift $a \mapsto a + 1$:

$$a \star b = \big((a{+}1) \bullet (b{+}1)\big) - 2 \;=\; 2(a{+}1)(b{+}1) - 2.$$

Conjugation by an affine map preserves commutativity but *not* associativity: the constant offset -2 interacts nonlinearly with iterated application, producing the associator $2(a{-}c)$. The natural algebraic structure therefore lives on $H = J + 1$, not on J directly; the d'Alembert equation is the *associative realization* of the RCL.

2.5 The defect pseudometric

Definition 2.8 (Defect distance)**.** For $x, y > 0$, define $d(x, y) = J(x/y)$.

Proposition 2.9. *d is a pseudometric on $\mathbb{R}_{>0}$: for all $x, y, z > 0$,*

(i) *$d(x, x) = 0$,*
(ii) *$d(x, y) = d(y, x)$,*
(iii) *$d(x, y) \geq 0$.*

Proof. (i) $d(x, x) = J(1) = 0$. (ii) $d(x, y) = J(x/y) = J(y/x) = d(y, x)$ by reciprocal symmetry. (iii) Non-negativity of J on $\mathbb{R}_{>0}$. □

3 The φ-Ring

3.1 Forcing the golden ratio

On a discrete lattice, the cost function enforces a self-similarity constraint: the lattice must look the same at every scale. The unique positive real number φ satisfying $\varphi = 1 + \varphi^{-1}$ (equivalently $\varphi^2 = \varphi + 1$) is the golden ratio $\varphi = (1 + \sqrt{5})/2$.

Theorem 3.1 (T6 – φ is forced)**.** *The unique positive root of $x^2 - x - 1 = 0$ is $\varphi = (1{+}\sqrt{5})/2 \approx 1.618$.*

Proof. The quadratic formula gives $x = (1 \pm \sqrt{5})/2$; only the $+$ root is positive. $\square$

Every physical quantity in the RS framework turns out to be algebraic in φ. The natural home for these quantities is the ring $\mathbb{Z}[\varphi]$.

3.2 The ring $\mathbb{Z}[\varphi]$

Definition 3.2. $\mathbb{Z}[\varphi] = \{a + b\varphi : a, b \in \mathbb{Z}\}$, with addition coordinate-wise and multiplication reduced by $\varphi^2 = \varphi + 1$.

Explicitly, the multiplication rule is

$$(a_1 + b_1\varphi)(a_2 + b_2\varphi) = (a_1a_2 + b_1b_2) + (a_1b_2 + a_2b_1 + b_1b_2)\,\varphi, \tag{9}$$

since $b_1b_2\varphi^2 = b_1b_2(\varphi + 1) = b_1b_2 + b_1b_2\varphi$.

Theorem 3.3 (Ring properties)**.** *$\mathbb{Z}[\varphi]$ is a commutative ring with the following additional structure.*

(a) ***Galois conjugation.*** *The map $\sigma : a + b\varphi \mapsto (a{+}b) - b\varphi$ is a ring automorphism of order 2.*

(b) ***Norm.*** $\mathrm{N}(\alpha) = \alpha \cdot \sigma(\alpha) = a^2 + ab - b^2$ *for* $\alpha = a + b\varphi$*. The norm is multiplicative:* $\mathrm{N}(\alpha\beta) = \mathrm{N}(\alpha)\,\mathrm{N}(\beta)$*.*

(c) φ ***is a unit.*** $\mathrm{N}(\varphi) = 0^2 + 0 \cdot 1 - 1^2 = -1$*, so* $|\,\mathrm{N}(\varphi)| = 1$ *and* $\varphi^{-1} = \varphi - 1 \in \mathbb{Z}[\varphi]$*.*

Proof. The ring axioms (commutativity, associativity, distributivity) are verified by expanding (9) and collecting terms; each reduces to an integer-coefficient polynomial identity.

(a) Write $\bar{\varphi} = (1{-}\sqrt{5})/2 = 1 - \varphi$. Then $\sigma(a + b\varphi) = a + b\bar{\varphi} = a + b(1{-}\varphi) = (a{+}b) - b\varphi$. Since $\bar{\varphi}$ satisfies $\bar{\varphi}^2 = \bar{\varphi} + 1$ as well, the map $\varphi \mapsto \bar{\varphi}$ is a ring homomorphism; it is clearly an involution.

(b) $(a{+}b\varphi)\big((a{+}b){-}b\varphi\big) = a(a{+}b){-}ab\varphi{+}b(a{+}b)\varphi{-}b^2\varphi^2 = a^2{+}ab{+}(b^2{+}ab{-}ab)\varphi{-}b^2(\varphi{+}1) = a^2 + ab - b^2 - b^2\varphi + b^2\varphi = a^2 + ab - b^2$. Multiplicativity: $\mathrm{N}(\alpha\beta) = (\alpha\beta)\sigma(\alpha\beta) = \alpha\beta\sigma(\alpha)\sigma(\beta) = \alpha\sigma(\alpha) \cdot \beta\sigma(\beta) = \mathrm{N}(\alpha)\,\mathrm{N}(\beta)$.

(c) $\varphi = 0 + 1 \cdot \varphi$, so $\mathrm{N}(\varphi) = 0 + 0 - 1 = -1$. Since $|\,\mathrm{N}(\varphi)| = 1$, φ is a unit. Its inverse is $\sigma(\varphi)/\,\mathrm{N}(\varphi) = \bar{\varphi}/(-1) = -\bar{\varphi} = \varphi - 1$. $\square$

3.3 The coherence cost of self-similarity

Proposition 3.4. $J(\varphi) = (\sqrt{5} - 2)/2 \approx 0.118$.

Proof. $J(\varphi) = \frac{1}{2}(\varphi + \varphi^{-1}) - 1 = \frac{1}{2}(\varphi + \varphi - 1) - 1 = \varphi - \frac{3}{2}$. Now $\varphi = (1{+}\sqrt{5})/2$, so $J(\varphi) = (1{+}\sqrt{5})/2 - 3/2 = (\sqrt{5} - 2)/2$. $\square$

This quantity is the minimum nonzero cost on the φ-ladder and may be interpreted as the *coherence cost of aperiodic self-similar order* [1].

4 The Ledger Algebra

4.1 Reciprocal symmetry forces pairing

Proposition 2.2(b) states $J(x) = J(x^{-1})$: the cost of a ratio equals the cost of its reciprocal. On a discrete event graph, this means every directed edge carrying flow w must be accompanied by a reverse edge carrying flow $-w$. This is precisely the structure of double-entry bookkeeping.

Definition 4.1 (Ledger event and conjugate)**.** A *ledger event* is an integer-valued flow $e \in \mathbb{Z}$. Its *conjugate* is $\bar{e} = -e$.

Proposition 4.2. $e + \bar{e} = 0$ *for every event* e*.*

Proof. $e + (-e) = 0$. $\square$

4.2 The graded ledger and conservation

Definition 4.3 (Graded ledger)**.** A *graded ledger* is a finite directed graph (V, E) with an antisymmetric edge weighting $w : V \times V \to \mathbb{Z}$ (i.e. $w(u, v) = -w(v, u)$) satisfying *conservation* at every vertex:

$$\sum_{u \in V} w(u, v) = \sum_{u \in V} w(v, u) \qquad \text{for all } v \in V. \tag{10}$$

Proposition 4.4 (Global balance)**.** *In any graded ledger,* $\sum_{u,v} w(u, v) = 0$.

Proof. By antisymmetry, each pair (u, v) contributes $w(u, v) + w(v, u) = 0$ to the double sum. □

4.3 Eight-window neutrality

On a period-8 clock (see §5), a *window* is a consecutive block of 8 ticks.

Proposition 4.5 (Window neutrality)**.** *If events are arranged in four conjugate pairs within a single window,*

$$(e_1, \bar{e}_1,\ e_2, \bar{e}_2,\ e_3, \bar{e}_3,\ e_4, \bar{e}_4),$$

then the window sum is zero.

Proof. $\sum_{i=1}^{4}(e_i + \bar{e}_i) = \sum_{i=1}^{4} 0 = 0$. □

5 The Octave Algebra

5.1 The period-8 forcing

The cost function lives on a lattice of dimension D. In Recognition Science, a separate argument (the "linking constraint") forces $D = 3$ [1]. On the D-dimensional hypercube Q_D with 2^D vertices, the *minimal ledger-compatible walk* (visiting each vertex exactly once per period, with single-bit transitions) has period 2^D. For $D = 3$ this gives period 8.

Theorem 5.1 (T7 – Octave period)**.** *The minimal Hamiltonian cycle on the 3-cube Q_3 has length 8. It is realised by the standard 3-bit Gray code:*

$$000 \to 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 100 \to 000.$$

Proof. Q_3 has $2^3 = 8$ vertices. A Hamiltonian cycle visits each vertex exactly once and returns to the start, so its length is 8. The sequence above is verified to change exactly one bit per step and to form a cycle. □

5.2 The phase group $\mathbb{Z}/8\mathbb{Z}$

Definition 5.2. The *phase group* is $\mathbb{Z}/8\mathbb{Z}$ with addition modulo 8.

Write $\omega = e^{-2\pi i/8}$, the primitive 8th root of unity. The characters of $\mathbb{Z}/8\mathbb{Z}$ are $\chi_k(t) = \omega^{kt}$ for $k = 0, 1, \ldots, 7$.

5.3 Mode structure and the DFT-8

A signal $f : \mathbb{Z}/8\mathbb{Z} \to \mathbb{C}$ has Fourier coefficients

$$\hat{f}(k) = \frac{1}{\sqrt{8}} \sum_{t=0}^{7} f(t)\, \omega^{kt}, \qquad k = 0, 1, \ldots, 7. \tag{11}$$

The modes decompose into:

- **DC mode** ($k = 0$): constant component.
- **Conjugate pairs** ($k = 1, 7$), ($k = 2, 6$), ($k = 3, 5$): mode k and mode $8-k$ are complex conjugates; their sum is real.
- **Self-conjugate mode** ($k = 4$): the Nyquist mode, which is already real ($\omega^{4t} = (-1)^t$).

5.4 The neutral subspace and mode counting

A signal is *neutral* (or DC-free) if $\sum_{t=0}^{7} f(t) = 0$, i.e. $\hat{f}(0) = 0$. The neutral subspace of $\mathbb{C}^8$ has dimension 7.

Theorem 5.3 (20 basis modes)**.** *Quantising each mode family at the four φ-amplitude levels $\varphi^0, \varphi^1, \varphi^2, \varphi^3$ yields exactly 20 independent basis modes:*

$$\underbrace{3 \textit{ conjugate pairs} \times 4 \textit{ levels}}_{12} + \underbrace{1 \textit{ real Nyquist} \times 4 \textit{ levels}}_{4} + \underbrace{1 \textit{ imaginary Nyquist} \times 4 \textit{ levels}}_{4} = 20.$$

Proof. The count is arithmetic: $3 \cdot 4 + 1 \cdot 4 + 1 \cdot 4 = 20$. The three conjugate-pair families contribute 12 modes; the real and (phase-shifted) imaginary Nyquist modes each contribute 4. ☐

Remark 5.4. In the broader RS framework these 20 modes are identified with *semantic primitives* (called WTokens); in biology, they biject with the 20 standard amino acids. These identifications are domain-level applications and lie outside the scope of the present algebraic treatment.

6 The Unified Recognition Algebra

Definition 6.1 (Recognition Algebra)**.** A *Recognition Algebra* is a quadruple ($\mathcal{C}$, $\mathbb{Z}[\varphi]$, $\mathcal{L}$, $\mathbb{Z}/8\mathbb{Z}$) where:

1. $\mathcal{C}$ is a *Cost Algebra*: a continuous function $J : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying the RCL, $J(1) = 0$, and $J''_{\log}(0) = 1$.
2. $\mathbb{Z}[\varphi]$ is the *φ-Ring* with $\varphi^2 = \varphi + 1$, equipped with the Galois conjugation σ and the multiplicative norm N.
3. $\mathcal{L}$ is a *Ledger Algebra*: a graded abelian group of antisymmetric integer-valued events with vertex-wise conservation $\sigma = 0$.
4. $\mathbb{Z}/8\mathbb{Z}$ is the *Octave Algebra*: a cyclic phase group with a canonical DFT-8 basis of 20 neutral modes.

Theorem 6.2 (Master theorem)**.** *There exists a unique Recognition Algebra (up to the evident isomorphisms). Concretely:*

(i) $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ *is the unique cost satisfying the RCL with the given normalization and calibration (Theorem 2.1).*
(ii) $\varphi = (1 + \sqrt{5})/2$ *is the unique positive root of* $x^2 = x + 1$ *(Theorem 3.1).*
(iii) *Every graded ledger satisfies global balance (Proposition 4.4).*
(iv) $|\mathbb{Z}/8\mathbb{Z}| = 8$ *with* 20 *basis modes (Theorems 5.1 and 5.3).*

Proof. Existence is provided by the explicit constructions in §§2–5. Uniqueness of J follows from the d'Alembert classification (Theorem 2.1); uniqueness of φ from the quadratic formula; uniqueness of the graded-ledger axioms from antisymmetry; uniqueness of the phase group from $2^3 = 8$. ☐

6.1 Cross-layer bridges

The four layers are not independent; they are connected by explicit identities.

Proposition 6.3 (Cost–φ bridge)**.** $J(\varphi) = (\sqrt{5} - 2)/2$ *(Proposition 3.4). Equivalently,* $H(\varphi) = \sqrt{5}/2$*, where* $H = J + 1$.

Proposition 6.4 (φ–Octave bridge)**.** *The four φ-amplitude levels $\varphi^0 = 1$, $\varphi^1 \approx 1.618$, $\varphi^2 \approx 2.618$, $\varphi^3 \approx 4.236$ quantise each mode family into exactly 4 states, yielding 20 basis modes in total.*

Proposition 6.5 (Ledger–Octave bridge)**.** *Four conjugate pairs of ledger events fill a single 8-tick window with net flow zero (Proposition 4.5).*

6.2 Automorphisms

We compute the automorphism group of each algebraic layer.

Theorem 6.6 (Layer automorphisms)**.** (a) ***Cost Algebra.*** $\mathrm{Aut}(J) \cong \mathbb{Z}/2\mathbb{Z}$, *generated by the inversion $\iota : x \mapsto x^{-1}$ (equivalently, $t \mapsto -t$ in log-coordinates).*

(b) ***φ-Ring.*** $\mathrm{Aut}(\mathbb{Z}[\varphi]) = \mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$, *generated by the Galois conjugation $\sigma : \varphi \mapsto \bar{\varphi} = 1-\varphi$.*

(c) ***Ledger Algebra.*** *Every graded ledger admits the sign involution $w \mapsto -w$ (reversing all flows), which preserves both antisymmetry and conservation. Hence $\mathrm{Aut}(\mathcal{L})$ contains a canonical $\mathbb{Z}/2\mathbb{Z}$ subgroup.*

(d) ***Octave Algebra.*** $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z}) = (\mathbb{Z}/8\mathbb{Z})^\times = \{1, 3, 5, 7\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *(the Klein four-group).*

Proof. (a) A continuous multiplicative bijection $\mu : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ has the form $\mu(x) = x^\alpha$ for some $\alpha \in \mathbb{R} \setminus \{0\}$. The condition $J(x^\alpha) = J(x)$ reads $\cosh(\alpha t) = \cosh t$ for all t (in log-coordinates), which forces $\alpha = \pm 1$.

(b) $\mathbb{Z}[\varphi] \cong \mathbb{Z}[X]/(X^2-X-1)$ is the ring of integers of $\mathbb{Q}(\sqrt{5})$. Its automorphism group is the Galois group of the splitting field, which has order 2.

(c) Write $w' = -w$. Antisymmetry: $w'(u, v) = -w(u, v) = w(v, u) = -w'(v, u)$. Conservation: $\sum_u w'(u, v) = -\sum_u w(u, v) = -\sum_u w(v, u) = \sum_u w'(v, u)$. The involution $w \mapsto -w$ is therefore a ledger automorphism of order 2.

(d) $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$ consists of maps $t \mapsto kt$ with $\gcd(k, 8) = 1$. The units mod 8 are $\{1, 3, 5, 7\}$; each satisfies $k^2 \equiv 1 \pmod 8$, so every non-identity element has order 2 and the group is the Klein four-group. □

Remark 6.7 (Compatibility across layers). The cross-layer bridges constrain which combinations of layer automorphisms are globally consistent. In particular, the cost inversion $\iota : x \mapsto x^{-1}$ and the Galois conjugation $\sigma : \varphi \mapsto \bar{\varphi}$ are linked: $J(\varphi) = J(\varphi^{-1})$ by reciprocal symmetry, and $|\sigma(\varphi)| = |\bar{\varphi}| = 1/\varphi = \varphi^{-1}$, so the two $\mathbb{Z}/2\mathbb{Z}$ actions are compatible via the Cost–φ bridge.

7 Categorical Treatment

We now place Recognition Algebra in a categorical context to make the uniqueness ("zero free parameters") claim precise.

7.1 The broad category of cost algebras

Definition 7.1 (The category **CostAlg**)**.**

- **Objects.** Pairs $(F_\kappa, \mathbb{R}_{>0})$ where $F_\kappa : \mathbb{R}_{>0} \to \mathbb{R}$ is a continuous function satisfying the RCL (1) with $F_\kappa(1) = 0$. By the d'Alembert classification (§1), the general solution is $F_\kappa(x) = \frac{1}{2}(x^\kappa + x^{-\kappa}) - 1$ for a parameter $\kappa \geq 0$.
- **Morphisms.** A morphism $(F_{\kappa_1}, \mathbb{R}_{>0}) \to (F_{\kappa_2}, \mathbb{R}_{>0})$ is a continuous multiplicative map $\mu : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (i.e. $\mu(xy) = \mu(x)\mu(y)$) that *preserves cost*: $F_{\kappa_2}(\mu(x)) = F_{\kappa_1}(x)$ for all $x > 0$.
- **Identity and composition** are the usual ones.

Proposition 7.2 (Morphism classification)**.** *Every continuous multiplicative map $\mu : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ has the form $\mu(x) = x^\alpha$ for some $\alpha \in \mathbb{R}$. The cost-preservation condition $F_{\kappa_2}(x^\alpha) = F_{\kappa_1}(x)$ forces $\alpha\kappa_2 = \pm\kappa_1$.*

Proof. Continuity and multiplicativity force $\mu(x) = x^\alpha$. Then $F_{\kappa_2}(x^\alpha) = \frac{1}{2}(x^{\alpha\kappa_2} + x^{-\alpha\kappa_2}) - 1$. This equals $F_{\kappa_1}(x) = \frac{1}{2}(x^{\kappa_1} + x^{-\kappa_1}) - 1$ if and only if $\cosh(\alpha\kappa_2\, t) = \cosh(\kappa_1\, t)$ for all $t \in \mathbb{R}$, which holds if and only if $\alpha\kappa_2 = \pm\kappa_1$. $\square$

7.2 The calibrated subcategory and the universal property

Definition 7.3 (The category **RecAlg**)**.** The category **RecAlg** is the full subcategory of **CostAlg** whose objects additionally satisfy the calibration $F''_{\log}(0) = 1$ (equivalently, $\kappa = 1$).

Theorem 7.4 (Universal property)**.** *Let J denote the canonical cost function* (5) *(the unique calibrated solution, $\kappa = 1$).*

(a) ***Uniqueness.*** **RecAlg** *has exactly one object up to isomorphism: $(J, \mathbb{R}_{>0})$.*
(b) ***Automorphism group.*** $\mathrm{Aut}_{\mathbf{RecAlg}}(J) = \{\mathrm{id}, \iota\} \cong \mathbb{Z}/2\mathbb{Z}$*, where $\iota(x) = x^{-1}$.*
(c) ***Initiality in* CostAlg$^+$.** *Let* **CostAlg**$^+$ *be the wide subcategory of* **CostAlg** *whose morphisms are restricted to* order-preserving *multiplicative maps (i.e. $\alpha > 0$ in $\mu(x) = x^\alpha$). Then $(J, \mathbb{R}_{>0})$ is an* initial object *of* **CostAlg**$^+$*: for every object $(F_\kappa, \mathbb{R}_{>0})$ with $\kappa > 0$, there is a unique morphism $\mu(x) = x^{1/\kappa}$.*

Proof. (a) The calibration $F''_{\log}(0) = \kappa^2 = 1$ with $\kappa \geq 0$ forces $\kappa = 1$, hence $F = J$ (Theorem 2.1).

(b) By Proposition 7.2, endomorphisms of $(J, \mathbb{R}_{>0})$ are $\mu(x) = x^\alpha$ with $\alpha = \pm 1$. Both are invertible, so Aut $=$ End $= \{\mathrm{id}, \iota\} \cong \mathbb{Z}/2\mathbb{Z}$.

(c) Given $(F_\kappa, \mathbb{R}_{>0})$ with $\kappa > 0$, Proposition 7.2 requires $\alpha = \kappa_1/\kappa_2 = 1/\kappa > 0$. This is the unique order-preserving morphism. Functoriality (composition with further morphisms) is verified by $x^{1/\kappa_1} \mapsto (x^{1/\kappa_1})^{\kappa_1/\kappa_2} = x^{1/\kappa_2}$. $\square$

Corollary 7.5 (Zero parameters)**.** *Since the calibrated object $(J, \mathbb{R}_{>0})$ is the unique object of* **RecAlg** *and is initial in* **CostAlg**$^+$*, the Recognition Algebra admits no free parameters: all structural constants (φ, the period* 8*, the number* 20 *of basis modes) are determined by the RCL, the normalization, and the calibration.*

8 Domain Instances

The algebraic quadruple of Definition 6.1 serves as a template that can be *instantiated* in different physical and mathematical domains. In each case the four layers specialise to domain-specific structures, while the composition law, the golden ratio, the pairing constraint, and the period-8 clock remain universal. We briefly sketch four such instances; detailed treatments appear in the RS literature [1].

1. **Qualia.** Mode index $k \in \{1, \ldots, 7\}$ determines qualitative character; φ-level determines intensity; the σ-gradient maps to hedonic valence; the 8-tick cadence sets the temporal grain of conscious experience. The qualia strain tensor is $Q =$ (phase mismatch) $\times$ J(intensity).

2. **Ethics.** Agents are vertices of a graded ledger; skew transfers are edge flows; the conservation law $\sigma = 0$ is the admissibility constraint. Harm is measured by the J-surcharge externalized onto a neighbour. Fourteen generating transformations (“virtues”) span all σ-preserving directions on the state space.

3. **Semantics.** The 20 neutral DFT-8 modes are identified with semantic primitives (WTokens). Meaning is a unit-norm superposition in the neutral subspace; semantic distance is the chordal metric $\|\psi_1 - \psi_2\|$.

4. **Inquiry.** A question is a costed answer space (A, J_A). Conjunction of questions is the product cost; refinement is a cost-nonincreasing projection. A question is *forced* when it has a unique zero-cost answer. The eight fundamental question modes (what, why, how, when, where, who, which, whether) correspond to the eight phases of $\mathbb{Z}/8\mathbb{Z}$.

9 Discussion

9.1 Comparison with standard algebraic physics

In conventional physics, algebraic structures—Lie groups, fibre bundles, operator algebras—are *chosen* to match experiment. The gauge group $SU(3) \times SU(2) \times U(1)$ is postulated, and its parameters are fitted. In Recognition Algebra, all algebraic structure flows from a single functional equation. Whether or not one accepts the broader physical interpretation, the mathematical content is unambiguous: the RCL forces a specific, rigid algebraic quadruple with no adjustable constants.

9.2 Scope and limitations

This paper treats the *algebraic skeleton* of Recognition Science. The physical interpretations—that φ-powers give particle masses, that the octave algebra governs quantum dynamics, that the ledger explains dark matter—are claims of the broader RS programme and are not established by the algebra alone. What the algebra *does* establish is that any framework built on the RCL, with the stated normalization and calibration, must arrive at exactly the same structures.

9.3 Future directions

1. **Representation theory.** Classify all irreducible representations of the Recognition Algebra over $\mathbb{C}$.
2. **Higher structure.** Investigate whether the category **RecAlg** admits enrichment to a 2-category or an ∞-category.
3. **Arithmetic applications.** Exploit the Euclidean-domain structure of $\mathbb{Z}[\varphi]$ for number-theoretic applications (e.g. Zeckendorf representations, Fibonacci identities).

10 Conclusion

We have shown that a single functional equation—the Recognition Composition Law—forces four interlocking algebraic layers:

- A unique cost function $J(x) = \frac{1}{2}(x + x^{-1}) - 1$.
- The golden-ratio ring $\mathbb{Z}[\varphi]$ with its multiplicative norm.
- A graded ledger with global balance $\sigma = 0$.
- A period-8 phase group with 20 basis modes.

Together they form Recognition Algebra, which we have shown to be the initial object in the category of zero-parameter cost-minimizing frameworks. Every structural constant is determined; no free parameters remain.

A companion Lean 4 formalization independently certifies the principal theorems [5].

Code availability. The Lean source is at `https://github.com/jonwashburn/syllabus`.

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