P0-A0 Noble Gas Closure Theorem (Mathematical Derivation + Validation Tables)

Recognition Science Derivation Campaign

2026-01-17

Abstract

This document rewrites the P0-A0 "noble gas closure" result as a complete mathematical derivation. We give explicit (piecewise) definitions of the period boundary maps prev and next, then derive the closure facts used by the repository:

- Noble gases have zero distance to the next closure: $\mathrm{dist}(Z) = 0$.
- Noble gases satisfy the complete-shell identity: $\mathrm{valence}(Z) = \mathrm{periodLen}(Z)$.
- Period lengths are recovered as differences of successive closures, yielding $[2, 8, 8, 18, 18, 32]$.

We also reproduce the preregistered validator tables stored in `artifacts/chem_noble_gas_closure.json` (PASS 4/4 tests).

1 Claim (P0-A0)

In the repository's chemistry scaffold, **noble-gas closure** means: the atomic numbers at which a period ends are exactly the closure endpoints, and at such endpoints the "distance to closure" is zero. All downstream quantities (valence position, period length, etc.) are defined deterministically from the closure endpoints, without fitting to chemical datasets.

2 Definitions (explicit piecewise form)

2.1 Closure endpoints

Define the first-six noble-gas closure endpoints

$$\mathcal{N}_6 := \{2, 10, 18, 36, 54, 86\},$$

and an extended list including the period-7 endpoint (Oganesson)

$$\mathcal{N}_7 := \{2, 10, 18, 36, 54, 86, 118\}.$$

The predicate `isNobleGas` in the Lean scaffold corresponds to membership in $\mathcal{N}_6$.

2.2 Previous/next closure maps

For $Z \in \mathbb{N}$, define the previous-closure map $\mathrm{prev}(Z)$ and next-closure map $\mathrm{next}(Z)$ by

$$\mathrm{prev}(Z) = \begin{cases} 0 & Z \le 2, \\ 2 & 2 < Z \le 10, \\ 10 & 10 < Z \le 18, \\ 18 & 18 < Z \le 36, \\ 36 & 36 < Z \le 54, \\ 54 & 54 < Z \le 86, \\ 86 & 86 < Z, \end{cases} \qquad \mathrm{next}(Z) = \begin{cases} 2 & Z \le 2, \\ 10 & 2 < Z \le 10, \\ 18 & 10 < Z \le 18, \\ 36 & 18 < Z \le 36, \\ 54 & 36 < Z \le 54, \\ 86 & 54 < Z \le 86, \\ 118 & 86 < Z. \end{cases}$$

These are exactly the closure maps implemented by `prevClosure` and `nextClosure` in the Lean file.

2.3 Derived quantities

Definition 1 (Distance to next closure)**.** *Define the (nonnegative) distance to the next closure by*

$$\mathrm{dist}(Z) := \mathrm{next}(Z) - Z.$$

Definition 2 (Valence electrons and period length)**.** *Define the valence position and period length as*

$$\mathrm{valence}(Z) := Z - \mathrm{prev}(Z), \qquad \mathrm{periodLen}(Z) := \mathrm{next}(Z) - \mathrm{prev}(Z).$$

Definition 3 (Noble gas predicate)**.** *Define*

$$\mathrm{isNobleGas}(Z) \iff Z \in \mathcal{N}_6.$$

3 Derivations

3.1 Noble gases have zero distance to closure

Theorem 1. *If* $\mathrm{isNobleGas}(Z)$ *then* $\mathrm{dist}(Z) = 0$.

Proof. Since $Z \in \mathcal{N}_6 = \{2, 10, 18, 36, 54, 86\}$, we do cases.

- If $Z = 2$, then by definition $\mathrm{next}(2) = 2$, hence $\mathrm{dist}(2) = \mathrm{next}(2) - 2 = 0$.
- If $Z = 10$, then $2 < 10 \le 10$, so $\mathrm{next}(10) = 10$, hence $\mathrm{dist}(10) = 0$.
- If $Z = 18$, then $10 < 18 \le 18$, so $\mathrm{next}(18) = 18$, hence $\mathrm{dist}(18) = 0$.
- If $Z = 36$, then $18 < 36 \le 36$, so $\mathrm{next}(36) = 36$, hence $\mathrm{dist}(36) = 0$.
- If $Z = 54$, then $36 < 54 \le 54$, so $\mathrm{next}(54) = 54$, hence $\mathrm{dist}(54) = 0$.
- If $Z = 86$, then $54 < 86 \le 86$, so $\mathrm{next}(86) = 86$, hence $\mathrm{dist}(86) = 0$.

This covers all cases. □

3.2 Noble gases satisfy the complete-shell identity

Theorem 2. *If* isNobleGas(Z) *then* valence(Z) = periodLen(Z).

Proof. Fix $Z \in \mathcal{N}_6$. By the previous theorem, next(Z) = Z at each closure endpoint. Therefore

$$\text{periodLen}(Z) = \text{next}(Z) - \text{prev}(Z) = Z - \text{prev}(Z) = \text{valence}(Z).$$

□

3.3 Uniqueness: only closures have zero distance (within the range)

Proposition 1. *If* $1 \le Z \le 118$ *and* dist(Z) = 0, *then* $Z \in \mathcal{N}_7$.

Proof. dist(Z) = 0 implies next(Z) − Z = 0, hence next(Z) = Z. But by the piecewise definition, next(Z) $\in \{2, 10, 18, 36, 54, 86, 118\}$ for all Z. Therefore Z must be one of these values, i.e. $Z \in \mathcal{N}_7$. □

3.4 Period lengths from noble-gas gaps

Let the ordered closure list be $(n_0, n_1, n_2, n_3, n_4, n_5) = (2, 10, 18, 36, 54, 86)$. Define gap lengths

$$\Delta_0 := n_0, \qquad \Delta_k := n_k - n_{k-1} \quad (k = 1, \ldots, 5).$$

Then

$$[\Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5] = [2, 8, 8, 18, 18, 32].$$

This identity is immediate by arithmetic: $10 - 2 = 8$, $18 - 10 = 8$, $36 - 18 = 18$, $54 - 36 = 18$, $86 - 54 = 32$.

3.5 Shell capacities sum to the noble-gas closure sequence

Let the (period) capacity sequence be

$$(c_0, c_1, c_2, c_3, c_4, c_5) := (2, 8, 8, 18, 18, 32).$$

Define cumulative closures

$$s_0 := c_0, \qquad s_k := s_{k-1} + c_k \quad (k = 1, \ldots, 5).$$

Then the cumulative closure list is exactly the noble-gas list:

$$(s_0, s_1, s_2, s_3, s_4, s_5) = (2, 10, 18, 36, 54, 86).$$

Indeed:

$$\begin{aligned} s_0 &= 2, \\ s_1 &= 2 + 8 = 10, \\ s_2 &= 10 + 8 = 18, \\ s_3 &= 18 + 18 = 36, \\ s_4 &= 36 + 18 = 54, \\ s_5 &= 54 + 32 = 86. \end{aligned}$$

This is the mathematical content of the Lean theorem `shell_sum_to_noble`.

4 Validation (prereg script + artifact)

The preregistered validator scripts/analysis/chem_noble_gas_closure.py writes artifacts/chem_noble_gas_closure.json. The committed artifact reports **PASS (4/4 tests)**.

4.1 Test 1: $\mathrm{dist}(Z) = 0$ at noble gases

Z	Element	$\mathrm{dist}(Z)$	Pass
2	He	0	true
10	Ne	0	true
18	Ar	0	true
36	Kr	0	true
54	Xe	0	true
86	Rn	0	true

4.2 Test 2: complete-shell identity at noble gases

Z	Element	$\mathrm{valence}(Z)$	$\mathrm{periodLen}(Z)$	Pass
2	He	2	2	true
10	Ne	8	8	true
18	Ar	8	8	true
36	Kr	18	18	true
54	Xe	18	18	true
86	Rn	32	32	true

4.3 Test 3: uniqueness (no false positives up to $Z = 118$)

The validator reports no non-noble $Z \in \{1, \ldots, 118\}$ with $\mathrm{dist}(Z) = 0$.

4.4 Test 4: period-length sequence

The validator computes the noble-gas gaps $[2, 8, 8, 18, 18, 32]$ and confirms the match.

5 Repo cross-references

Lean module:

- `IndisputableMonolith/Chemistry/PeriodicTable.lean`

Prereg, script, artifact:

- `docs/prereg/NobleGasClosure.md`
- `scripts/analysis/chem_noble_gas_closure.py`
- `artifacts/chem_noble_gas_closure.json`