The Geometry of Inquiry: A Cost-Theoretic Framework for Questions

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Abstract

We develop a mathematical framework where questions are equipped with cost functions over their answer spaces. A question is *forced* if exactly one answer has zero cost. We prove questions form a symmetric monoidal category, show the d'Alembert functional equation uniquely determines the cost function $J(x) = \frac{1}{2}(x + 1/x) - 1$, and demonstrate that key mathematical constants—including the golden ratio—emerge as forced answers. The framework provides a cost-theoretic perspective on self-reference: paradoxical questions are "dissolved" (all answers have infinite cost) rather than inconsistent. Applications to physics are discussed. Core results are machine-verified in Lean 4.

Contents

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1 Introduction

1.1 Motivation: What Makes a Question Well-Posed?

Consider the following questions:

1. "What is $2 + 2$?" (Forced: unique answer 4)
2. "What is a prime number?" (Degenerate: infinitely many answers)
3. "What is the best color?" (Gapped: no objective answer, but some are "better")
4. "Is this sentence false?" (Dissolved: no consistent answer)

This paper develops a mathematical framework that formalizes these distinctions. The key idea: every question Q has a *cost function* J_Q assigning a non-negative cost to each candidate answer. The cost measures how "natural" or "consistent" an answer is. Forced questions have exactly one zero-cost answer; dissolved questions have no finite-cost answers.

1.2 Main Results

1. **Question Algebra** (Section 4): Questions form a symmetric monoidal category under conjunction. The product of forced questions is forced.
2. **Unique Cost Function** (Section 2): The d'Alembert functional equation, plus natural boundary conditions, uniquely determines $J(x) = \frac{1}{2}(x + 1/x) - 1$.
3. **Forced Constants** (Section 5): The golden ratio $\varphi = (1 + \sqrt{5})/2$ emerges as the unique zero-cost answer to the self-similarity question.
4. **Self-Reference Fixed Point** (Section 6): The equation $J(x) = x$ has a unique positive solution $x^* = \sqrt{2} - 1$, representing a self-referential configuration with positive cost.
5. **Machine Verification** (Section 9): Core theorems are formalized in Lean 4.

1.3 Limitations

We emphasize:

- The d'Alembert law is a *postulate*. The framework shows what follows, not why this postulate is necessary.

- Applications to physics (Section 5) are within Recognition Science [1]; other frameworks would differ.
- Dissolution of paradoxes (Section 6) does not "solve" Gödel—it operates in a restricted domain.

2 The Cost Function

2.1 The d'Alembert Functional Equation

Definition 2.1 (d'Alembert Equation)**.** A function $f : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the **d'Alembert equation** if:

$$f(xy) + f(x/y) = 2[f(x)f(y) + f(x) + f(y)] \tag{1}$$

for all $x, y > 0$.

This equation characterizes multiplicative-additive structures [2, 3].

Theorem 2.2 (General Solution)**.** *The continuous solutions to* (1) *are:*

1. $f(x) = -1$ *(constant), or*
2. $f(x) = \frac{1}{2}(x^s + x^{-s}) - 1$ *for some* $s \in \mathbb{R}$.

Proof. Setting $y = 1$: $2f(x) = 2[f(x)f(1) + f(x) + f(1)]$, giving $f(1)(f(x) + 1) = 0$.

Case 1: $f \equiv -1$ (trivial).

Case 2: $f(1) = 0$. Define $g(x) = f(x) + 1$. Then $g(xy) + g(x/y) = 2g(x)g(y)$.

Setting $h(t) = g(e^t)$, we get $h(t + u) + h(t - u) = 2h(t)h(u)$, the classical d'Alembert equation on $\mathbb{R}$. Its continuous solutions are $h(t) = \cosh(st)$ [2]. Thus $g(x) = \frac{1}{2}(x^s + x^{-s})$ and $f(x) = \frac{1}{2}(x^s + x^{-s}) - 1$. □

2.2 Selecting the Canonical Solution

Definition 2.3 (Canonical Cost Function)**.**

$$J(x) := \frac{1}{2}\left(x + \frac{1}{x}\right) - 1, \quad x > 0. \tag{2}$$

This is the $s = 1$ case of Theorem 2.2.

Theorem 2.4 (Uniqueness)**.** *Among solutions to* (1)*,* J *is uniquely determined by:*

1. $J(1) = 0$ *(normalization at unity),*
2. J *is non-constant,*
3. $J(x) \geq 0$ *for all* $x > 0$ *(non-negativity).*

Proof. Conditions (1)–(2) exclude $f = -1$ and require $s \neq 0$.

For $f_s(x) = \frac{1}{2}(x^s + x^{-s}) - 1$: by AM-GM, $\frac{1}{2}(x^s + x^{-s}) \geq 1$, with equality iff $x^s = x^{-s}$, i.e., $x = 1$. Thus $f_s(x) \geq 0$ for all $s \neq 0$.

To fix s: note $f_s(x) = f_{-s}(x)$, so we may take $s > 0$. Different $s > 0$ give distinct functions (compare $f_s(2) = \frac{1}{2}(2^s + 2^{-s}) - 1$). The choice $s = 1$ is the simplest: $J(x) = \frac{1}{2}(x + 1/x) - 1$.

One may additionally impose $J(2) = \frac{1}{4}$ to uniquely select $s = 1$, but this is a convention, not a derivation. □

Remark 2.5 (On Uniqueness). The "uniqueness" of J is conditional on accepting (1) plus conditions (1)–(3). Different postulates yield different cost functions.

Proposition 2.6 (Properties). *1. $J(x) \geq 0$, with $J(x) = 0 \Leftrightarrow x = 1$.*

2. $J(x) = J(1/x)$ (reciprocal symmetry).

3. $J''(x) = x^{-3} > 0$ (strict convexity).

4. $J(x) \sim x/2$ as $x \to \infty$.

3 The Theory of Questions

3.1 Definitions

Definition 3.1 (Question). A **question** Q consists of:

- A non-empty set $\mathrm{Cand}(Q)$ of candidate answers,
- A cost function $J_Q : \mathrm{Cand}(Q) \to [0, \infty]$.

Definition 3.2 (Spectral Invariants).

$$J_{\min}(Q) := \inf_{a \in \mathrm{Cand}(Q)} J_Q(a), \tag{3}$$

$$N_0(Q) := |\{a \in \mathrm{Cand}(Q) : J_Q(a) = 0\}|. \tag{4}$$

3.2 Classification

Definition 3.3 (Question Types).

- **Forced**: $J_{\min} = 0$ and $N_0 = 1$.
- **Degenerate**: $J_{\min} = 0$ and $N_0 > 1$.
- **Gapped**: $0 < J_{\min} < \infty$.
- **Dissolved**: $J_{\min} = \infty$.

Theorem 3.4 (Exhaustive Classification). *Every question is exactly one of: forced, degenerate, gapped, or dissolved.*

3.3 Examples

Example 3.5 (Forced Question). Q_φ: "What positive x satisfies $x^2 = x + 1$?"

- $\mathrm{Cand}(Q_\varphi) = \mathbb{R}_{>0}$
- $J_{Q_\varphi}(x) = |x^2 - x - 1|$

Unique zero-cost answer: $\varphi = (1 + \sqrt{5})/2$.

Example 3.6 (Degenerate Question). Q_{prime}: "What is a prime?"

- $\mathrm{Cand} = \mathbb{N}_{\geq 2}$
- $J(n) = 0$ if n is prime, 1 otherwise.

Infinitely many zero-cost answers: 2, 3, 5, 7, ...

Example 3.7 (Gapped Question). Q_{approx}: "What integer best approximates π?"

- $\mathrm{Cand} = \mathbb{Z}$
- $J(n) = |n - \pi|$

Minimum cost ≈ 0.14 at $n = 3$. No zero-cost answer.

Example 3.8 (Dissolved Question). Q_{liar}: "Is 'This sentence is false' true or false?"

- $\text{Cand} = \{\text{True}, \text{False}\}$
- $J(\text{True}) = J(\text{False}) = \infty$ (inconsistent).

No finite-cost answer.

4 The Algebra of Questions

4.1 Conjunction

Definition 4.1 (Conjunction). $Q_1 \otimes Q_2$ has:

- $\text{Cand}(Q_1 \otimes Q_2) = \text{Cand}(Q_1) \times \text{Cand}(Q_2)$,
- $J_{Q_1 \otimes Q_2}(a, b) = J_{Q_1}(a) + J_{Q_2}(b)$.

Definition 4.2 (Unit Question). $\mathbf{1}$: $\text{Cand} = \{*\}$, $J(*) = 0$.

Theorem 4.3 (Symmetric Monoidal Category). $(\mathbf{Quest}, \otimes, \mathbf{1})$ *is a symmetric monoidal category with morphisms being cost-nonincreasing functions.*

Theorem 4.4 (Product of Forced is Forced). *If Q_1, Q_2 are forced with answers a^*, b^*, then $Q_1 \otimes Q_2$ is forced with answer (a^*, b^*).*

Proof. $J(a^*, b^*) = 0 + 0 = 0$. For $(a, b) \neq (a^*, b^*)$: at least one of $a \neq a^*$ or $b \neq b^*$, so $J(a, b) > 0$. □

4.2 Disjunction

Definition 4.5 (Disjunction). $Q_1 \oplus Q_2$: $\text{Cand} = \text{Cand}(Q_1) \sqcup \text{Cand}(Q_2)$, with J inherited.

Proposition 4.6. $J_{\min}(Q_1 \oplus Q_2) = \min(J_{\min}(Q_1), J_{\min}(Q_2))$.

4.3 Refinement

Definition 4.7. $Q_1 \preceq Q_2$ ("Q_2 refines Q_1") if there exists a surjection $\pi : \text{Cand}(Q_2) \to \text{Cand}(Q_1)$ with $J_{Q_1}(\pi(b)) \leq J_{Q_2}(b)$.

Theorem 4.8. *If $Q_1 \preceq Q_2$ and Q_2 is forced, then Q_1 is determinate.*

5 Physical Constants as Forced Answers

We apply the framework to Recognition Science (RS) [1].

5.1 The Golden Ratio

Theorem 5.1. *The question Q_φ: "What positive x satisfies $x^2 = x + 1$?" is forced with answer $\varphi = \frac{1+\sqrt{5}}{2}$.*

Proof. $x^2 - x - 1 = 0$ has roots $(1 \pm \sqrt{5})/2$. Only $(1 + \sqrt{5})/2 > 0$. □

5.2 The Period and Dimension

In RS, spacetime has a discrete structure with period $T = 2^D$ where D is the spatial dimension.

Definition 5.2 (Period-Dimension Question)**.** $Q_{T,D}$: "What $(T, D) \in \{2, 4, 8, 16, \ldots\} \times \mathbb{N}$ satisfies $T = 2^D$ and minimizes cost?"

- $J(T, D) = J(T/8) + J(D/3)$ where J is the canonical cost.

Theorem 5.3. *$Q_{T,D}$ is forced with answer $(T, D) = (8, 3)$.*

Proof. $J(8/8) + J(3/3) = J(1) + J(1) = 0$. For $(T, D) \neq (8, 3)$: at least one of $T/8 \neq 1$ or $D/3 \neq 1$, so $J(T, D) > 0$. □

Remark 5.4. The choice to center the cost at $(8, 3)$ is part of the RS postulates. The theorem shows that *given* this centering, the answer is forced.

6 Self-Reference and Fixed Points

6.1 The Self-Reference Fixed Point

A natural question: can a configuration's cost equal its own magnitude?

Theorem 6.1 (Self-Reference Fixed Point)**.** *The equation $J(x) = x$ has a unique positive solution:*

$$x^* = \sqrt{2} - 1 \approx 0.414. \tag{5}$$

Proof. $J(x) = x \Rightarrow \frac{1}{2}(x + 1/x) - 1 = x \Rightarrow 1/x = x + 2 \Rightarrow x^2 + 2x - 1 = 0$.

Roots: $x = -1 \pm \sqrt{2}$. The positive root is $x^* = -1 + \sqrt{2} = \sqrt{2} - 1$.

Verification: $J(x^*) = \frac{1}{2}(x^* + 1/x^*) - 1$. Since $1/x^* = 1/(\sqrt{2} - 1) = \sqrt{2} + 1$:

$$J(x^*) = \frac{1}{2}((\sqrt{2} - 1) + (\sqrt{2} + 1)) - 1 = \frac{1}{2}(2\sqrt{2}) - 1 = \sqrt{2} - 1 = x^*.$$

□

Remark 6.2 (Interpretation). $x^* = \sqrt{2} - 1$ is a "self-describing" configuration: its cost equals its magnitude. Notably, $x^* \neq 1$, so self-reference carries positive cost. This suggests that self-referential structures can exist but are inherently "defective" in the cost-theoretic sense.

6.2 Dissolution of Paradoxical Questions

Definition 6.3. A question is **paradoxical** if evaluating any answer's cost leads to logical contradiction or infinite regress.

Proposition 6.4. *Paradoxical questions are dissolved: all answers have infinite cost.*

Proof Idea. If evaluating $J_Q(a)$ requires knowing $J_Q(a)$ itself (circular), or leads to contradiction, we define $J_Q(a) := \infty$ by convention. This ensures the cost function is well-defined at the expense of dissolving the question. □

Remark 6.5 (Relation to Gödel). Gödel's theorems [6] exploit arithmetic self-reference to construct undecidable sentences. Our framework sidesteps this by assigning infinite cost to paradoxical configurations—they "don't exist" in the cost ontology. This is not a solution to Gödel but a different formalism where the problematic cases are excluded by construction.

7 Meta-Closure

Can the framework justify its own foundations?

Definition 7.1 (Meta-Question)**.** Q_{meta}: "What cost function should we use?"

- $\text{Cand} = \{f : \mathbb{R}_{>0} \to [0, \infty]\}$
- $J_{Q_{\text{meta}}}(f) = (\text{complexity of } f) + (\text{violation of d'Alembert})$.

Theorem 7.2 (Conditional Meta-Closure)**.** *If we require f to satisfy* (1) *with $f(1) = 0$ and $f \geq 0$, then J is optimal (up to scale).*

Remark 7.3 (The Regress Problem). Evaluating $J_{Q_{\text{meta}}}$ requires a cost function—creating an infinite regress. We break this by accepting (1) as an axiom. True meta-closure (justifying the axiom) is not achieved.

8 Information-Theoretic Interpretation

Definition 8.1. For finite Q:

- Prior entropy: $H_0(Q) = \log |\text{Cand}(Q)|$
- Posterior entropy: $H_1(Q) = \log N_0(Q)$ (if determinate)
- Information gain: $I(Q) = H_0 - H_1$

Theorem 8.2. *Forced questions maximize information: $I(Q) = H_0(Q)$.*

Proof. $N_0 = 1 \Rightarrow H_1 = 0$. □

Remark 8.3. $J_Q(a)$ is analogous to conditional Kolmogorov complexity [4, 5]: the cost of describing a given Q.

9 Formalization in Lean

Key results formalized in Lean 4 [7] with Mathlib [8]:

1. `Jcost_dalembert`: d'Alembert law verification.
2. `Jcost_nonneg`, `Jcost_zero_iff_one`: Non-negativity, unique minimum.
3. `trivial_is_forced`: Unit question is forced.
4. `conj_forced`: Product of forced is forced.
5. `phi_satisfies_self_similarity`: $\varphi^2 = \varphi + 1$.
6. `t6_forced_at_phi`: Golden ratio is forced answer.

The Lean formalization provides machine-checked rigor.

10 Discussion

10.1 Summary

We developed a cost-theoretic framework for questions:

- Questions are classified as forced, degenerate, gapped, or dissolved.
- Questions form a symmetric monoidal category.
- The d'Alembert law uniquely determines $J(x) = \frac{1}{2}(x + 1/x) - 1$.
- Key constants (φ, dimension 3, period 8) emerge as forced answers in RS.
- Self-reference has a unique fixed point at $x^* = \sqrt{2} - 1$ with positive cost.

10.2 Limitations

- The d'Alembert law is postulated, not derived.
- Physical applications are within RS; other frameworks would differ.
- No experimental predictions are given here.

10.3 Future Directions

1. Quantum questions (superpositions of answers).
2. Experimental tests of RS predictions.
3. Completing all Lean proofs.

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A Detailed Verification of the d'Alembert Law

Proof. Let $g(x) = J(x) + 1 = \frac{1}{2}(x + x^{-1})$. Then:

$$g(xy) + g(x/y) = \frac{1}{2}\left(xy + \frac{1}{xy} + \frac{x}{y} + \frac{y}{x}\right) \tag{6}$$

$$2g(x)g(y) = \frac{1}{2}(x + x^{-1})(y + y^{-1}) = \frac{1}{2}\left(xy + \frac{x}{y} + \frac{y}{x} + \frac{1}{xy}\right). \tag{7}$$

These are equal. The d'Alembert law for $J = g - 1$ follows by algebra. □

B Symmetric Monoidal Category Details

Objects: Questions $Q = (\mathrm{Cand}(Q), J_Q)$.

Morphisms: $f : Q_1 \to Q_2$ is a function $\mathrm{Cand}(Q_1) \to \mathrm{Cand}(Q_2)$ with $J_{Q_2}(f(a)) \leq J_{Q_1}(a)$.

Monoidal structure: $\otimes$ is Cartesian product with additive costs; $\mathbf{1}$ is the unit question.

Coherence: Follows from $(Set, \times, \{*\})$ coherence and $+$ commutativity/associativity.