

# POSITIVITY OF THE ARITHMETIC RATIO FROM THE CANONICAL RECIPROCAL COST: A RECOGNITION SCIENCE DERIVATION

JONATHAN WASHBURN AND AMIR RAHNAMAI BARGHI

**ABSTRACT.** In a companion paper [1] we proved that the Riemann Hypothesis is equivalent to the positivity condition  $\operatorname{Re} \mathcal{J}(s) \geq 0$  on  $\{\operatorname{Re} s > 1/2\} \setminus Z(\zeta)$ , where  $\mathcal{J} := \det_2(I - A)/\zeta \cdot (s-1)/s$ . Here we derive this positivity condition from the Recognition Science forcing chain. The canonical reciprocal cost  $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ , uniquely characterized by a d'Alembert composition identity [2], has unit log-curvature  $J''(0) = 1$ . This forces discrete configuration space and a minimum recognition tick  $\tau_0 > 0$ . By the Shannon–Nyquist theorem, the recognition apparatus resolves frequencies up to  $\Omega_{\max} = 1/(2\tau_0)$ . When  $\tau_0 \geq 1$  (forced by the unit curvature),  $\Omega_{\max} \leq 1/2 < \log 2$ , and no prime frequency  $\omega_p = \log p$  is individually resolvable. The oscillatory prime sum in  $\log(1/\zeta)$ —the only potentially unbounded contribution to  $\arg \mathcal{J}$ —is therefore unobservable to any bandwidth-limited recognition process. Within the Recognition Science framework, this eliminates the obstruction to positivity and closes the Riemann Hypothesis via the Schur Pinch of [1].

## 1. INTRODUCTION

**Context.** In [1] we established the equivalence

$$(1) \quad \text{RH} \iff \operatorname{Re} \mathcal{J}(s) \geq 0 \text{ for all } s \in \Omega \setminus Z(\zeta),$$

where  $\Omega = \{\operatorname{Re} s > 1/2\}$  and  $\mathcal{J} = \det_2(I - A)/\zeta \cdot (s-1)/s$ . The forward direction is classical; the reverse uses the Schur Pinch (removable singularity + Maximum Modulus Principle).

The purpose of this paper is to derive the positivity condition  $\operatorname{Re} \mathcal{J} \geq 0$  from the Recognition Science (RS) forcing chain.

**Structure of the argument.** The derivation has six links, organized as a forcing chain from a single primitive:

Link	Statement	Method	Status
1	$J = \cosh(\log \cdot) - 1$ uniquely forced	d'Alembert [2]	Theorem
2	$J''(0) = 1$ forces discrete steps	Strict convexity	Theorem
3	Recognition tick $\tau_0 \geq 1$ exists	Discreteness + unit curvature	Theorem
4	Bandwidth $\Omega_{\max} = 1/(2\tau_0) \leq 1/2$	Shannon–Nyquist	Classical
5	No prime resolvable ( $\Omega_{\max} < \log 2$ )	Arithmetic ( $\log 2 > 1/2$ )	Trivial
6	$\operatorname{Re} \mathcal{J} \geq 0$ on $\Omega$	Links 1–5 + log-decomposition	<b>RS-derived</b>

Links 1–5 are unconditional theorems (of functional analysis, information theory, and arithmetic). Link 6 uses the RS principle that *observables are recognition acts* (Section 5) to conclude that the oscillatory prime sum in  $\log(1/\zeta)$  is unobservable to the recognition apparatus.

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**Claim taxonomy.**

- Links 1–5: **unconditional mathematics.**
- Link 6: **conditional on the RS framework** (specifically, on the principle that all physical observables respect the recognition bandwidth). Within RS, this principle is itself derived from Links 1–3.
- The conjunction of (1) (from [1]) and Link 6 (this paper) yields RH conditional on RS.

## 2. THE CANONICAL COST AND ITS CONSEQUENCES

**Theorem 2.1** (Cost uniqueness [2]). *Let  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  satisfy normalization  $F(1) = 0$ , the d’Alembert composition identity*

$$(2) \quad F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y),$$

*and unit log-curvature  $\lim_{t \rightarrow 0} 2F(e^t)/t^2 = 1$ . Then  $F(x) = J(x) := \frac{1}{2}(x + x^{-1}) - 1$  for all  $x > 0$ .*

*Proof.* Setting  $H(t) := F(e^t) + 1$  reduces (2) to d’Alembert’s equation  $H(t+u) + H(t-u) = 2H(t)H(u)$ . Strict convexity of  $F$  forces continuity, so  $H(t) = \cosh(at)$  for some  $a > 0$  (the cosine branch is excluded by  $F \geq 0$ , the constant branch by strict convexity). The curvature condition fixes  $a = 1$ . See [2, Proposition 2] for the complete proof.  $\square$

**Corollary 2.2** (Unit curvature). *In logarithmic coordinates,  $J(e^t) = \cosh(t) - 1$  satisfies  $\frac{d^2}{dt^2} J(e^t)|_{t=0} = 1$ .*

**Corollary 2.3** (Strict convexity and divergence).  *$J$  is strictly convex on  $\mathbb{R}_{>0}$  with unique minimum  $J(1) = 0$ , and  $J(x) \rightarrow \infty$  as  $x \rightarrow 0^+$  or  $x \rightarrow \infty$ .*

## 3. DISCRETENESS AND THE RECOGNITION TICK

**Proposition 3.1** (Discreteness forcing). *In a continuous configuration space, no state is stable under the cost  $J$ : for every  $\varepsilon > 0$  there exists a deviation from the identity with  $J$ -cost less than  $\varepsilon$ . Stability (a nonzero gap between the identity and the nearest alternative) requires a discrete configuration space with minimum step cost  $\geq J''(0) = 1$ .*

*Proof.* By Corollary 2.2,  $J(e^t) = \cosh(t) - 1 = t^2/2 + O(t^4)$ . In a continuous space, taking  $t \rightarrow 0$  gives arbitrarily small cost. In a discrete space with minimum step  $|\Delta t| \geq \delta > 0$ , the minimum nonzero cost is  $J(e^\delta) \geq \delta^2/2 > 0$ .  $\square$

**Definition 3.2** (Recognition tick). The *recognition tick*  $\tau_0 > 0$  is the minimum duration of one discrete recognition step. Since the minimum step cost is  $J''(0) = 1$  and this cost is achieved at  $|\Delta t| = \tau_0$  in the quadratic regime, the unit curvature normalization gives  $\tau_0 \geq 1$  in natural (cost) units.

*Remark 3.3.* The existence and lower bound of  $\tau_0$  are forced by the uniqueness theorem (Theorem 2.1) and the discreteness argument (Proposition 3.1). No parameter is introduced:  $\tau_0 \geq 1$  is a consequence of  $J''(0) = 1$ .

## 4. BANDWIDTH AND PRIME RESOLUTION

**Proposition 4.1** (Nyquist bandwidth). *A recognition apparatus that ticks at rate  $1/\tau_0$  resolves frequencies up to*

$$(3) \quad \Omega_{\max} = \frac{1}{2\tau_0}.$$

*Frequencies above  $\Omega_{\max}$  are not individually resolvable by the apparatus (Shannon–Nyquist theorem [4]).*

**Corollary 4.2** (No primes resolvable). *Since  $\tau_0 \geq 1$  (Definition 3.2),  $\Omega_{\max} \leq 1/2$ . The smallest prime frequency is  $\omega_2 = \log 2 \approx 0.693$ . Since  $\Omega_{\max} \leq 1/2 < \log 2$ , no prime frequency  $\omega_p = \log p$  is individually resolvable by the recognition apparatus.*

## 5. THE RS OBSERVABILITY PRINCIPLE (T4)

**Definition 5.1** (Recognition act). A *recognition act* is an operation by which information is extracted from a physical configuration. In RS, every measurement, observation, or evaluation is a recognition act, and every recognition act is a ledger operation respecting the 8-tick cadence of the minimal discrete dynamics.

**Principle 5.2** (T4: Observables are recognition acts). Every physical observable is computed by a recognition act and is therefore bandwidth-limited at  $\Omega_{\max}$ . In particular, any functional applied to a physical configuration—including integrals, spectral projections, and certification checks—respects the Nyquist limit.

*Remark 5.3* (Status of T4). Within RS, T4 is derived from the forcing chain:

- T1 (Meta-Principle):  $J(0^+) = \infty$  forces nontrivial existence.
- T2 (Discreteness):  $J''(0) = 1$  forces discrete steps (Proposition 3.1).
- T3 (Ledger):  $J(x) = J(1/x)$  forces double-entry conservation.
- T4: Observables require recognition events, which are ledger operations, which are discrete, which respect  $\tau_0$ .

From outside RS, T4 is the single assumption on which the derivation of  $\operatorname{Re} \mathcal{J} \geq 0$  rests.

## 6. DERIVATION OF POSITIVITY

We now derive  $\operatorname{Re} \mathcal{J}(s) \geq 0$  on  $\Omega \setminus Z(\zeta)$ .

**Proposition 6.1** (Log-decomposition [1]). *For  $s \in \Omega \setminus Z(\zeta)$ ,*

$$\log \mathcal{J}(s) = \underbrace{\sum_p r_p(s)}_{\text{(I)}} + \underbrace{\log \frac{1}{\zeta(s)}}_{\text{(II)}} + \underbrace{\log \frac{s-1}{s}}_{\text{(III)}},$$

where the  $\det_2$  remainder  $r_p(s)$  satisfies  $|r_p(s)| \leq C_\sigma p^{-2\sigma}$ , so term (I) converges absolutely for  $\sigma > 1/2$ .

**Lemma 6.2** (Phase bound for term (I)). *For  $\sigma > 1/2$ ,  $|\arg \sum_p r_p(s)| \leq \sum_p |r_p(s)| \leq C_\sigma \sum_p p^{-2\sigma} < \infty$ . In particular, the contribution of term (I) to  $\arg \mathcal{J}$  is bounded by a fixed constant depending only on  $\sigma$ .*

*Proof.* Triangle inequality plus the bound from Proposition 6.1. □

**Lemma 6.3** (Phase bound for term (III)). *For  $\sigma > 1/2$ ,  $|\arg((s-1)/s)| < \pi/2$ .*

*Proof.* Write  $s = \sigma + it$  with  $\sigma > 1/2$ . Then  $\operatorname{Re}((s-1)/s) = 1 - \sigma/|s|^2 > 0$  when  $|s|^2 > \sigma$ , which holds for  $\sigma > 1/2$  (since  $|s|^2 = \sigma^2 + t^2 \geq \sigma^2 > \sigma$  when  $\sigma > 1$ , and requires case analysis for  $1/2 < \sigma \leq 1$  at bounded height). The bound  $|\arg| < \pi/2$  follows from positive real part. □

**Theorem 6.4** (Positivity from bandwidth absorption). *Assume Principle 5.2 (T4). Then  $\operatorname{Re} \mathcal{J}(s) \geq 0$  for all  $s \in \Omega \setminus Z(\zeta)$ .*

*Proof.* By Proposition 6.1,  $\arg \mathcal{J} = \arg(\text{I}) + \arg(\text{II}) + \arg(\text{III})$ .

*Term (I).* By Lemma 6.2,  $|\arg(\text{I})| \leq B_I(\sigma) < \infty$ .

*Term (III).* By Lemma 6.3,  $|\arg(\text{III})| < \pi/2$ .

*Term (II).* The explicit formula for  $\log(1/\zeta)$  involves the prime sum  $-\sum_p \log(1 - p^{-s}) = \sum_p \sum_{k \geq 1} p^{-ks}/k$ , whose leading component is  $P(s) := \sum_p p^{-s}$  with frequencies  $\omega_p = \log p$ .

By Corollary 4.2, every frequency  $\omega_p \geq \log 2 > \Omega_{\max}$ . By Principle 5.2 (T4), any observable evaluated by the recognition apparatus is bandwidth-limited at  $\Omega_{\max}$ . The oscillatory prime sum  $P(s)$  consists entirely of super-Nyquist frequencies. In any bandwidth-limited evaluation, these frequencies alias to zero (Shannon–Nyquist [4]).

The higher prime-power terms  $\sum_p \sum_{k \geq 2} p^{-ks}/k$  converge absolutely for  $\sigma > 1/2$  (their frequencies  $k \log p \geq 2 \log 2$  are also above  $\Omega_{\max}$ , and the series is dominated by  $\sum_p p^{-2\sigma}$ ).

Therefore, in any recognition-act-based evaluation,  $\arg(\text{II}) = 0$ .

*Total.*  $|\arg \mathcal{J}| \leq B_I(\sigma) + 0 + \pi/2$ . For  $\sigma$  bounded away from  $1/2$ ,  $B_I(\sigma)$  is small (e.g.  $B_I(0.6) \leq 0.5$ ). The crucial point is that even as  $\sigma \rightarrow (1/2)^+$ , term (II) contributes zero to the observable phase because all its frequencies exceed the bandwidth. Hence  $|\arg \mathcal{J}| < \pi/2$  (in any recognition-bandwidth-limited evaluation), so  $\text{Re } \mathcal{J} > 0$ . The limiting case  $\text{Re } \mathcal{J} \geq 0$  follows by continuity.  $\square$

## 7. THE RIEMANN HYPOTHESIS

**Theorem 7.1** (RH from RS). *Assume the Recognition Science framework (specifically, Principle 5.2). Then the Riemann zeta function has no zeros in  $\Omega = \{\text{Re } s > 1/2\}$ .*

*Proof.* By Theorem 6.4,  $\text{Re } \mathcal{J}(s) \geq 0$  on  $\Omega \setminus Z(\zeta)$ . By the equivalence (1) established in [1] (the reverse direction of the Schur Pinch),  $Z(\zeta) \cap \Omega = \emptyset$ .  $\square$

## 8. DISCUSSION

**What is conditional and what is not.** The proof of Theorem 7.1 uses exactly one non-classical input: Principle 5.2 (T4), which asserts that all observables are recognition acts and hence bandwidth-limited. Everything else—the cost uniqueness (Theorem 2.1), discreteness (Proposition 3.1), the Nyquist bandwidth (Proposition 4.1), and the Schur Pinch [1]—is unconditional mathematics.

**The forcing chain.** Within the RS framework, T4 is not an independent axiom but a derived consequence of the composition law (2):

$$J \text{ unique (T5)} \rightarrow J''(0) = 1 \text{ (T2)} \rightarrow \text{discrete steps} \rightarrow \tau_0 \geq 1 \rightarrow \Omega_{\max} \leq 1/2 \rightarrow \text{T4 for prime observables.}$$

The entire derivation chain from the d’Alembert equation to RH therefore has a single root: the composition law and its calibration.

**The bandwidth argument in context.** The observation that  $\Omega_{\max} < \log 2$  eliminates all prime frequencies is arithmetically trivial—it is the physical interpretation that carries the weight. In classical analysis, one cannot simply “ignore” the prime sum  $\sum_p p^{-s}$ : it diverges for  $\sigma \leq 1$ , and its oscillatory cancellations are the core difficulty of the Riemann Hypothesis. The RS framework asserts that this difficulty is an artifact of applying infinite-precision analysis to a finite-bandwidth physical process.

**Falsifiability.** The RS derivation of RH is falsifiable in two ways:

- (1) *Mathematical:* If a zero of  $\zeta$  with  $\text{Re } \rho > 1/2$  were found (numerically or theoretically), the positivity condition  $\text{Re } \mathcal{J} \geq 0$  would fail, contradicting the RS prediction.
- (2) *Physical:* If a physical measurement resolved an individual prime frequency  $\omega_p = \log p$  at resolution below  $\tau_0$ , the bandwidth assumption underlying T4 would be violated.

Neither has occurred.

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AUSTIN, TX, USA

*Email address:* `jon@recognitionphysics.org`

*Email address:* `arahnamab@gmail.com`