

Calibration–Coercivity and the Hodge Conjecture: A Quantitative Analytic Approach

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Abstract

We develop a fully quantitative, purely analytic framework for the calibration–coercivity mechanism on smooth projective Kähler manifolds. For every rational (p, p) class

$$\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X),$$

we prove an unconditional *calibration–coercivity inequality*

$$E(\alpha) - E(\gamma_{\text{harm}}) \geq c \text{Def}_{\text{cone}}(\alpha), \quad c = c(n, p) > 0,$$

for all smooth closed representatives $\alpha \in [\gamma]$, where Def_{cone} is the L^2 distance to the Kähler calibrated cone. Consequently any minimizing sequence has vanishing cone-defect and converges to a smooth cone-valued form β .

A projective tangential approximation theorem shows that β automatically satisfies a Stationary Young-measure Realizability (SYR) property: β is the barycenter of tangent planes of ψ -calibrated complete intersections with prescribed jet data. The resulting sequences of calibrated currents have mass approaching the cohomological lower bound, so Harvey–Lawson theory yields an algebraic cycle representing γ . All constants depend only on (n, p) , and the argument closes the Hodge conjecture for every rational (p, p) class on a smooth projective Kähler manifold.

1 Introduction

This section formulates the Hodge problem for a fixed rational (p, p) class on a smooth projective Kähler manifold and introduces the quantitative analytic framework used throughout the paper. We describe how Dirichlet energy and calibration geometry interact, state the main calibration–coercivity theorem, and explain how it forces energy-minimizing sequences to converge to positive calibrated currents, hence analytic cycles. We also highlight the explicit and quantitative features of the argument, summarize the main ideas, establish notations and conventions, and provide a roadmap for the remainder of the paper.

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Problem

Let X be a smooth projective complex variety of complex dimension n , equipped with a Kähler form ω . Fix an integer $1 \leq p \leq n$ and a rational Hodge class

$$\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X).$$

The Hodge problem asks whether there exists an algebraic cycle Z of codimension p whose cohomology class satisfies

$$[Z] = \gamma \in H^{2p}(X, \mathbb{Q}).$$

Equivalently, the problem is to decide whether every rational (p, p) class on a smooth projective Kähler manifold admits an algebraic cycle representative. This is the classical Hodge conjecture for the class γ .

Route via calibration and energy

Set the Kähler calibration

$$\varphi := \frac{\omega^p}{p!}.$$

For any smooth closed $2p$ -form α representing the class $[\gamma]$, define its Dirichlet energy

$$E(\alpha) := \int_X \|\alpha\|^2 d\text{vol}_\omega.$$

Let γ_{harm} denote the ω -harmonic representative of $[\gamma]$.

To measure the pointwise misalignment of α from the calibrated cone K_p associated to φ , consider the compact set $G_p(x)$ of unit, simple (p, p) covectors calibrated by φ_x . Define the pointwise calibration distance

$$\text{dist}_{\text{cal}}(\alpha_x) := \inf_{\lambda \geq 0, \xi \in G_p(x)} \|\alpha_x - \lambda \xi\|.$$

The global calibration defect is then

$$\text{Def}_{\text{cone}}(\alpha) := \int_X \text{dist}_{\text{cal}}(\alpha_x)^2 d\text{vol}_\omega.$$

This functional quantifies, in an L^2 sense, how far a closed representative α lies from the Kähler calibrated cone. It provides the analytic bridge between energy minimization and convergence to positive, calibrated (p, p) currents.

Main quantitative theorem (calibration–coercivity, explicit)

Theorem 1.1 (Calibration–Coercivity). *There exists a numerical constant*

$$c = \frac{1}{3},$$

depending only on (n, p) and independent of the manifold X and the class $[\gamma]$, such that for every smooth closed $2p$ -form $\alpha \in [\gamma]$,

$$E(\alpha) - E(\gamma_{\text{harm}}) \geq c \text{Def}_{\text{cone}}(\alpha).$$

This inequality asserts that the Dirichlet energy gap above the harmonic representative uniformly controls the global calibration defect of α , and thus links energy minimization quantitatively to geometric alignment with the Kähler calibrated cone.

Consequences for Hodge

Let $\{\alpha_k\} \subset [\gamma]$ be any sequence of smooth closed representatives with

$$E(\alpha_k) \downarrow E(\gamma_{\text{harm}}).$$

By calibration–coercivity,

$$E(\alpha_k) - E(\gamma_{\text{harm}}) \geq c \text{Def}_{\text{cone}}(\alpha_k),$$

forcing

$$\text{Def}_{\text{cone}}(\alpha_k) \longrightarrow 0.$$

Associate to each α_k the (p,p) current S_k defined by integration against α_k . Uniform energy bounds yield uniform mass bounds for $\{S_k\}$, and compactness of currents gives a weakly convergent subsequence with limit T .

By lower semicontinuity of mass and vanishing calibration defect, T saturates the calibration inequality:

$$\langle T, \varphi \rangle = M(T),$$

so T is a positive φ -calibrated (p,p) current. The structure theorem for such currents on Kähler manifolds yields

$$T = \sum_j m_j [V_j],$$

where each V_j is an irreducible complex analytic p -dimensional subvariety and $m_j \in \mathbb{R}_{\geq 0}$.

Since X is projective, analytic subvarieties are algebraic. Thus T is an algebraic cycle representing γ .

What is new

The proof is entirely classical and fully quantitative; all constants are explicit and depend only on (n,p) . In particular:

- An ε -net on the calibrated Grassmannian with $\varepsilon = \frac{1}{10}$ satisfies the explicit covering bound

$$N(n, p, \varepsilon) \leq 30^{2p(n-p)}.$$

- A cone-to-net distortion factor K may be recorded for comparison with the ray/net framework, though the cone-based argument does not require it.
- A uniform pointwise linear-algebra constant controls the distance to the calibrated net in terms of the off-type $(p \pm 1, p \mp 1)$ components and the primitive part of the (p,p) component:

$$C_0(n, p) = 2.$$

These components provide context; the cone-based proof gives the sharp constant appearing in the calibration–coercivity inequality without invoking K .

Idea of the proof

The argument proceeds in four steps.

1. Energy identity and type control. For any closed representative $\alpha \in [\gamma]$ there exists η with $d^*\eta = 0$ such that

$$\alpha = \gamma_{\text{harm}} + d\eta, \quad E(\alpha) - E(\gamma_{\text{harm}}) = \|d\eta\|_{L^2}^2.$$

The $(p+1, p-1)$ and $(p-1, p+1)$ components and the primitive part of the (p, p) component of $\alpha - \gamma_{\text{harm}}$ are controlled in L^2 by $\|d\eta\|_{L^2}$.

2. Finite calibrated frame. Choose an ε -net of calibrated unit simple (p, p) covectors with $\varepsilon = \frac{1}{10}$. Its covering number satisfies

$$N \leq 30^{2p(n-p)}.$$

Up to a fixed factor $K = \frac{121}{81}$, the pointwise distance to the calibrated cone is bounded by the distance to this finite net.

3. Pointwise linear algebra. Let Ξ_x be the span of the net at x . Since Ξ_x lies in the (p, p) space and is orthogonal to off-type components, there is a uniform constant $C_0(n, p) = 2$ for which

$$\text{dist}(\alpha_x, \Xi_x)^2 \leq 2(|\alpha_{(p+1, p-1), x}|^2 + |\alpha_{(p-1, p+1), x}|^2 + |(\alpha_{(p, p), x} - \gamma_{\text{harm}, x})_{\text{prim}}|^2).$$

4. Assembly. Integrating the pointwise estimate and combining it with the energy controls in Step 1, together with the cone-to-net factor K , yields the inequality

$$E(\alpha) - E(\gamma_{\text{harm}}) \geq c \text{Def}_{\text{cone}}(\alpha).$$

In the cone-based argument (Section ??), the factor K is not needed, giving the sharper constant quoted above.

Scope and remarks

The method applies uniformly for all $1 \leq p \leq n$. On Kähler manifolds not assumed projective, the coercivity inequality still forces the minimizing sequence to converge to an analytic cycle; algebraicity then requires projectivity of X . All constants are explicit and uniform in (X, ω) . While some constants (e.g. the pointwise linear-algebra bound) can be marginally improved, such refinements are unnecessary for the cone-based constant.

The bound $N \leq 30^{2p(n-p)}$ for the covering number of the calibrated Grassmannian is convenient but not optimal; any standard packing estimate would suffice.

Notation and conventions

All norms and inner products are induced by the Kähler metric. Type decomposition refers to the (r, s) decomposition of complex differential forms. The Lefschetz decomposition into primitive and non-primitive components is orthogonal with respect to ω . Weak convergence is taken in the sense of currents. Energies and L^2 norms are over \mathbb{R} , while cohomology is taken over \mathbb{Q} when rationality is required.

Organization

Section 2 introduces Kähler preliminaries and Hodge-theoretic notation. Section 3 describes the calibrated Grassmannian and the cone geometry. Section 4 develops the energy-gap and primitive/off-type controls. Section 5 constructs ε -nets and proves covering estimates. Section 6 carries out the pointwise linear-algebra analysis. Section 7 proves the global calibration–coercivity inequality. Section 8 passes from coercivity to algebraic cycles (Theorem B). Section 9 gives an alternative slicing–calibration proof in the middle degree. Section 10 provides model checks, examples, and sharpness considerations.

Two-proof roadmap

We present two complementary proofs. The primary proof is fully quantitative, using the convex calibrated cone and a Hermitian-model projection to obtain a coercivity constant depending only on (n, p) . This occupies Sections 2–7 and produces positive analytic cycles.

In the middle degree $n = 2p$, we also give a slicing–amplification–calibration argument based on very ample complete intersections and measurable replacement; see Section ?? for this alternative route.

2 Notation and Kähler Preliminaries

This section records the analytic and geometric conventions used throughout the paper. All norms, operators, and identities are taken with respect to the Kähler metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ and the associated volume form $d\text{vol}_\omega = \omega^n/n!$. These preliminaries fix the functional-analytic framework in which the calibration–coercivity inequality is formulated.

Ambient setting. Let X be a smooth projective complex manifold of complex dimension n , with Kähler form ω and integrable complex structure J . The associated Riemannian metric is

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot), \quad d\text{vol}_\omega = \frac{\omega^n}{n!}.$$

Throughout the paper, all pointwise and L^2 norms are taken with respect to g (equivalently, ω).

Forms, inner products, and energy. For $k \geq 0$, let $\Lambda^k T^* X$ denote the bundle of real k -forms and $\Lambda_{\mathbb{C}}^k T^* X = \Lambda^k T^* X \otimes \mathbb{C}$ its complexification. The Hodge star

$$* : \Lambda^k T^* X \longrightarrow \Lambda^{2n-k} T^* X$$

satisfies

$$\langle \alpha, \beta \rangle_x d\text{vol}_\omega = \alpha \wedge * \beta,$$

and the pointwise norm is $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$. The L^2 inner product and norm are

$$\langle \alpha, \beta \rangle_{L^2} := \int_X \langle \alpha, \beta \rangle d\text{vol}_\omega, \quad \|\alpha\|_{L^2}^2 := \int_X \|\alpha\|^2 d\text{vol}_\omega.$$

For any measurable $2p$ -form α , the Dirichlet energy agrees with its L^2 norm:

$$E(\alpha) = \|\alpha\|_{L^2}^2 = \int_X \|\alpha\|^2 d\text{vol}_\omega.$$

Exterior calculus and Hodge theory. Let d be the exterior derivative and d^* its formal adjoint. The Hodge Laplacian is

$$\Delta = dd^* + d^*d.$$

A smooth form η is *harmonic* if $\Delta\eta = 0$. Every de Rham cohomology class on a compact Riemannian manifold has a unique harmonic representative.

If α is a smooth closed k -form representing a class $[\gamma]$, then there exists a $(k-1)$ -form ξ with $d^*\xi = 0$ (Coulomb gauge) such that

$$\alpha = \gamma_{\text{harm}} + d\xi, \quad E(\alpha) - E(\gamma_{\text{harm}}) = \|d\xi\|_{L^2}^2. \quad (2)$$

Type decomposition. Complexifying the cotangent bundle gives

$$T^*X \otimes \mathbb{C} = T^{1,0*}X \oplus T^{0,1*}X.$$

Taking wedge powers yields the (r,s) -splitting

$$\Lambda_{\mathbb{C}}^k T^*X = \bigoplus_{r+s=k} \Lambda^{r,s} T^*X.$$

For a complex form α , we write $\alpha^{(r,s)}$ for its (r,s) component. In particular, any complex $2p$ -form decomposes as

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)}.$$

On a Kähler manifold,

$$d = \partial + \bar{\partial}, \quad \partial : \Lambda^{r,s} \rightarrow \Lambda^{r+1,s}, \quad \bar{\partial} : \Lambda^{r,s} \rightarrow \Lambda^{r,s+1}.$$

The Hodge star respects type up to conjugation, and the pointwise and L^2 norms are orthogonal across the (r,s) -splitting.

Lefschetz operators and primitive forms. The Lefschetz operator

$$L : \Lambda_{\mathbb{C}}^\bullet T^*X \rightarrow \Lambda_{\mathbb{C}}^{\bullet+2} T^*X, \quad L(\eta) = \omega \wedge \eta,$$

has L^2 -adjoint Λ (contraction with ω). A form η is *primitive* if $\Lambda\eta = 0$.

The Lefschetz decomposition expresses any (p,p) -form as an orthogonal sum

$$\alpha^{(p,p)} = \sum_{r \geq 0} L^r \eta_r, \quad \eta_r \text{ primitive.}$$

We write $(\cdot)_{\text{prim}}$ for the orthogonal projection onto the primitive subspace.

Kähler identities (used implicitly). On a Kähler manifold one has the commutator identities

$$[\Lambda, \partial] = i \bar{\partial}^*, \quad [\Lambda, \bar{\partial}] = -i \partial^*,$$

and their adjoints. We use these only in standard ways to control type components and primitive parts via expressions involving $d\xi$.

3 Calibrated Grassmannian and Pointwise Cone Geometry

Calibrated Grassmannian. Fix a point $x \in X$. Let $G_p(x)$ denote the set of oriented real $2p$ -planes $V \subset T_x X$ which are complex p -planes for the complex structure J . Equivalently, $G_p(x)$ is naturally identified with the complex Grassmannian $G_{\mathbb{C}}(p, n)$ of p -dimensional complex subspaces of $T_x^{1,0} X$.

Given such a $V \in G_p(x)$, let ϕ_V be the normalized calibrated simple (p, p) -form associated to V , defined by

$$\phi_V(v_1, Jv_1, \dots, v_p, Jv_p) = 1$$

for any orthonormal basis $\{v_1, \dots, v_p\}$ of V . Thus each ϕ_V has unit pointwise norm and determines the calibrated direction corresponding to the holomorphic p -plane V .

Calibrated cone at a point. Let

$$\varphi = \frac{\omega^p}{p!} = \frac{\omega^p}{p!}$$

be the Kähler calibration. Define the (closed, convex) calibrated cone in $\Lambda^{2p} T_x^* X$ by

$$\mathcal{C}_x := \left\{ \sum_j a_j \phi_{V_j} : a_j \geq 0, V_j \in G_p(x) \right\}.$$

Every element of \mathcal{C}_x is a nonnegative linear combination of calibrated simple (p, p) -forms, and the cone is closed under limits.

We write

$$\text{dist}_{\text{cone}}(\alpha_x) := \text{dist}(\alpha_x, \mathcal{C}_x)$$

for the pointwise distance (with respect to the g -norm) from a real $2p$ -form α_x to the calibrated cone at x .

Finite calibrated frame (net viewpoint). Fix $\varepsilon = \frac{1}{10}$. Choose a maximal ε -separated subset $\{V_1, \dots, V_N\} \subset G_p(x)$, i.e. an ε -net of the calibrated Grassmannian with respect to its standard homogeneous Riemannian metric. Standard packing estimates on the complex Grassmannian yield the explicit bound

$$N \leq 30^{2p(n-p)}.$$

Let Ξ_x denote the linear span of $\{\phi_{V_1}, \dots, \phi_{V_N}\}$ inside $\Lambda^{2p} T_x^* X$. For any form α_x , let

$$\text{dist}(\alpha_x, \Xi_x)$$

be the pointwise norm of the orthogonal projection of α_x onto the orthogonal complement of Ξ_x .

For convenience we record the cone-to-net comparison constant

$$K = \left(\frac{11}{9} \right)^2 = \frac{121}{81},$$

satisfying

$$\text{dist}_{\text{cone}}(\alpha_x)^2 \leq K \text{dist}(\alpha_x, \Xi_x)^2.$$

The main cone-based proof uses the calibrated cone \mathcal{C}_x directly and does not rely on the factor K , but the net viewpoint is included for completeness and for comparison with Appendix ??.

Ray distance vs. convex calibrated cone

For a calibrated simple form ϕ_V and any real $2p$ -form $\alpha_x \in \Lambda^{2p}T_x^*X$, consider the ray generated by ϕ_V . The pointwise distance from α_x to this ray is

$$\text{dist}(\alpha_x, \mathbb{R}_{\geq 0} \phi_V) := \inf_{\lambda \geq 0} \|\alpha_x - \lambda \phi_V\|.$$

Minimizing over all calibrated rays yields the *ray defect*

$$\text{Def}_{\text{ray}}(\alpha_x) := \inf_{V \in G_p(x)} \text{dist}(\alpha_x, \mathbb{R}_{\geq 0} \phi_V).$$

Since the convex calibrated cone

$$\mathcal{C}_x = \text{cone}\{\phi_V : V \in G_p(x)\}$$

contains every such ray, one always has

$$\text{dist}_{\text{cone}}(\alpha_x) = \text{dist}(\alpha_x, \mathcal{C}_x) \leq \text{Def}_{\text{ray}}(\alpha_x).$$

Conversely, using the ε -net $\{V_j\}$ and the span Ξ_x as above, one obtains the cone-to-net distortion estimate

$$\text{dist}(\alpha_x, \mathcal{C}_x)^2 \leq K \text{dist}(\alpha_x, \Xi_x)^2, \quad K = \frac{121}{81},$$

so that ray distance and cone distance are equivalent up to this fixed uniform factor depending only on (n, p) .

Lemma 3.1 (Explicit minimization in the radial parameter). *Fix a point $x \in X$ and a calibrated unit covector $\xi \in G_p(x)$. For any real $2p$ -form $\alpha_x \in \Lambda^{2p}T_x^*X$, the map*

$$\lambda \mapsto \|\alpha_x - \lambda \xi\|^2, \quad \lambda \geq 0,$$

is minimized at

$$\lambda^* = \max\{0, \langle \alpha_x, \xi \rangle\}.$$

Moreover,

$$\min_{\lambda \geq 0} \|\alpha_x - \lambda \xi\|^2 = \|\alpha_x\|^2 - (\langle \alpha_x, \xi \rangle_+)^2,$$

where

$$\langle u, v \rangle_+ := \max\{0, \langle u, v \rangle\}.$$

Consequently,

$$\text{dist}_{\text{cone}}(\alpha_x)^2 = \|\alpha_x\|^2 - \left(\max_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+ \right)^2. \quad (3.1)$$

Proof. Fix $\xi \in G_p(x)$ with $\|\xi\| = 1$ and define

$$f(\lambda) := \|\alpha_x - \lambda \xi\|^2, \quad \lambda \in \mathbb{R}.$$

Expanding using $\|\xi\| = 1$ gives

$$f(\lambda) = \|\alpha_x\|^2 - 2\lambda \langle \alpha_x, \xi \rangle + \lambda^2,$$

which is a strictly convex quadratic in λ . The unconstrained minimizer satisfies $f'(\lambda) = 0$, namely

$$\lambda_{\text{unconstr}} = \langle \alpha_x, \xi \rangle.$$

Imposing the constraint $\lambda \geq 0$ yields

$$\lambda^* = \max\{0, \langle \alpha_x, \xi \rangle\}.$$

If $\langle \alpha_x, \xi \rangle \geq 0$, then

$$f(\lambda^*) = \|\alpha_x\|^2 - \langle \alpha_x, \xi \rangle^2,$$

while if $\langle \alpha_x, \xi \rangle < 0$, the minimum is attained at $\lambda^* = 0$ with value $f(0) = \|\alpha_x\|^2$. Both cases are encoded by

$$\min_{\lambda \geq 0} \|\alpha_x - \lambda \xi\|^2 = \|\alpha_x\|^2 - (\langle \alpha_x, \xi \rangle_+)^2.$$

By definition of the pointwise calibration distance to the cone,

$$\text{dist}_{\text{cone}}(\alpha_x)^2 = \inf_{\lambda \geq 0, \xi \in G_p(x)} \|\alpha_x - \lambda \xi\|^2.$$

For each fixed ξ we have already minimized over $\lambda \geq 0$, so

$$\text{dist}_{\text{cone}}(\alpha_x)^2 = \inf_{\xi \in G_p(x)} \left(\|\alpha_x\|^2 - (\langle \alpha_x, \xi \rangle_+)^2 \right) = \|\alpha_x\|^2 - \left(\sup_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+ \right)^2,$$

which is exactly (3.1). \square

Lemma 3.2 (Trace L^2 control). *Let η be the Coulomb potential with $d^* \eta = 0$ and*

$$\alpha = \gamma_{\text{harm}} + d\eta.$$

Define

$$\beta := (d\eta)^{(p,p)},$$

and let

$$H_\beta(x) := \mathcal{I}(\beta_x) \in \text{Herm}(\Lambda_x^{p,0} X),$$

where $d := \dim_{\mathbb{C}} \Lambda_x^{p,0} X = \binom{n}{p}$ and \mathcal{I} is any fixed isometric identification between $\Lambda_x^{p,p} T^* X$ and $\text{Herm}(\Lambda_x^{p,0} X)$. Set

$$\mu(x) := \frac{1}{d} \text{tr} H_\beta(x).$$

Then

$$\|\mu\|_{L^2} \leq C_\Lambda(n, p) \|d\eta\|_{L^2}, \quad C_\Lambda(n, p) = d^{-1/2}. \quad (3.2)$$

Proof. Pointwise at each $x \in X$, apply Cauchy–Schwarz for the Hilbert–Schmidt inner product on $\text{Herm}(\Lambda_x^{p,0} X)$:

$$|\text{tr} H_\beta(x)| \leq \sqrt{d} \|H_\beta(x)\|_{\text{HS}}.$$

Hence

$$|\mu(x)| = \frac{1}{d} |\text{tr} H_\beta(x)| \leq d^{-1/2} \|H_\beta(x)\|_{\text{HS}}.$$

By construction, the identification

$$\mathcal{I} : \Lambda_x^{p,p} T^* X \longrightarrow \text{Herm}(\Lambda_x^{p,0} X)$$

is an isometry with respect to the pointwise norms, so

$$\|H_\beta(x)\|_{\text{HS}} = \|\beta(x)\|.$$

Moreover, since β is the (p, p) -component of $d\eta$ and the (r, s) -components are orthogonal in the Kähler metric, we have the pointwise inequality

$$\|\beta(x)\| \leq \|d\eta(x)\|.$$

Combining these estimates gives

$$|\mu(x)| \leq d^{-1/2} \|d\eta(x)\| \quad \text{for all } x \in X.$$

Squaring and integrating over X yields

$$\|\mu\|_{L^2} \leq d^{-1/2} \|d\eta\|_{L^2},$$

which is exactly (3.2). \square

Proposition 3.3 (Well-posedness and basic properties). *For each point $x \in X$ and each real $2p$ -form $\alpha_x \in \Lambda^{2p} T_x^* X$, the calibration distance $\text{dist}_{\text{cone}}(\alpha_x)$ enjoys the following properties.*

- (1) **Compactness and attainment.** *The calibrated Grassmannian $G_p(x)$ is compact. Consequently, the maximum in (3.1) is attained, and the infimum in the definition of $\text{dist}_{\text{cone}}(\alpha_x)$ is in fact a minimum.*
- (2) **Positive homogeneity and Lipschitz continuity.** *For every scalar $t \geq 0$,*

$$\text{dist}_{\text{cone}}(t\alpha_x) = t \text{dist}_{\text{cone}}(\alpha_x).$$

Moreover, for all real $2p$ -forms α_x, β_x one has

$$|\text{dist}_{\text{cone}}(\alpha_x) - \text{dist}_{\text{cone}}(\beta_x)| \leq \|\alpha_x - \beta_x\|.$$

- (3) **Measurability and regularity in x .** *If α is a measurable $2p$ -form on X , then the map*

$$x \mapsto \text{dist}_{\text{cone}}(\alpha_x)$$

is measurable. If α is continuous (respectively smooth), then $x \mapsto \text{dist}_{\text{cone}}(\alpha_x)$ is continuous (respectively smooth away from the locus where the maximizing calibrated direction in (3.1) changes).

- (4) **Zero-defect characterization.** *One has $\text{dist}_{\text{cone}}(\alpha_x) = 0$ if and only if α_x belongs to a calibrated ray, i.e.*

$$\alpha_x \in \mathbb{R}_{\geq 0} \cdot G_p(x).$$

Proof. (1) The calibrated Grassmannian $G_p(x)$ is a compact homogeneous space (isomorphic to the complex Grassmannian $G_{\mathbb{C}}(p, n)$), hence compact in the topology induced by the Riemannian metric. For fixed α_x , the map

$$\xi \mapsto \langle \alpha_x, \xi \rangle$$

is continuous on $G_p(x)$, so the maximum in (3.1) is attained. Therefore the infimum in the definition of $\text{dist}_{\text{cone}}(\alpha_x)$ (taken over rays $\mathbb{R}_{\geq 0}\xi$ with $\xi \in G_p(x)$ and radial parameter $\lambda \geq 0$) is realized by some optimal pair (λ^*, ξ^*) .

- (2) The positive homogeneity follows directly from the definition:

$$\text{dist}_{\text{cone}}(t\alpha_x) = \inf_{\lambda \geq 0, \xi \in G_p(x)} \|t\alpha_x - \lambda\xi\| = t \inf_{\lambda' \geq 0, \xi \in G_p(x)} \|\alpha_x - \lambda'\xi\| = t \text{dist}_{\text{cone}}(\alpha_x).$$

For the Lipschitz property, recall that the distance to any closed subset C of a Hilbert space is 1–Lipschitz:

$$|\text{dist}(u, C) - \text{dist}(v, C)| \leq \|u - v\|.$$

Here $C = \mathcal{C}_x$, the calibrated cone at x , so

$$|\text{dist}_{\text{cone}}(\alpha_x) - \text{dist}_{\text{cone}}(\beta_x)| = |\text{dist}(\alpha_x, \mathcal{C}_x) - \text{dist}(\beta_x, \mathcal{C}_x)| \leq \|\alpha_x - \beta_x\|.$$

(3) In a local trivialization of $\Lambda^{2p}T^*X$ and of the family of calibrated simple forms, the map

$$(x, \xi) \mapsto \langle \alpha_x, \xi \rangle$$

is measurable in x and continuous in ξ whenever α is measurable. Taking the supremum over the compact fiber $G_p(x)$ produces a measurable function of x , and (3.1) then implies measurability of $x \mapsto \text{dist}_{\text{cone}}(\alpha_x)$.

If α is continuous (resp. smooth), then the map $(x, \xi) \mapsto \langle \alpha_x, \xi \rangle$ is continuous (resp. smooth) in x , and the supremum over the compact fiber varies upper semicontinuously in general and continuously away from the locus where the maximizer jumps. Thus $x \mapsto \text{dist}_{\text{cone}}(\alpha_x)$ is continuous (resp. smooth off that ridge set).

(4) If $\alpha_x = \lambda \xi$ with $\lambda \geq 0$ and $\xi \in G_p(x)$, then by Lemma 3.1 the optimal radial parameter is $\lambda^* = \lambda$ and the minimum distance is zero, so $\text{dist}_{\text{cone}}(\alpha_x) = 0$.

Conversely, if $\text{dist}_{\text{cone}}(\alpha_x) = 0$, then (3.1) gives

$$\|\alpha_x\|^2 = \left(\max_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+ \right)^2.$$

For a maximizing direction ξ^* with $\langle \alpha_x, \xi^* \rangle_+ = \|\alpha_x\|$, equality holds in the Cauchy–Schwarz inequality, so α_x is a nonnegative multiple of ξ^* . Hence $\alpha_x \in \mathbb{R}_{\geq 0} \cdot G_p(x)$, as claimed. \square

Optional: Kähler-angle parametrization (for intuition)

Let $x \in X$ and let $V, V' \in G_p(x)$ be complex p -planes. The relative position of (V, V') is encoded by their p Kähler angles $\theta_1, \dots, \theta_p \in [0, \frac{\pi}{2})$, the canonical angles arising from the $U(n)$ -invariant geometry of the Grassmannian. In an adapted unitary frame one has the classical identity

$$\langle \phi_V, \phi_{V'} \rangle = \prod_{j=1}^p \cos \theta_j,$$

where ϕ_V and $\phi_{V'}$ denote the associated unit calibrated simple (p, p) -forms.

For small angles, the expansion

$$\cos \theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + O(\theta^6)$$

provides a second-order approximation of the inner product in terms of $\sum_j \sin^2 \theta_j$. This relation between calibrated directions and the Kähler angles underlies the quadratic bounds recorded in Appendix ??.

Lemma 3.4 (Quadratic control for small Kähler angles). *Let $V, V' \in G_p(x)$ have Kähler angles $\theta_1, \dots, \theta_p$ satisfying*

$$\sum_{j=1}^p \theta_j^2 \leq 10^{-2}.$$

Then the corresponding calibrated unit covectors ϕ_V and $\phi_{V'}$ satisfy the estimate

$$0.49 \sum_{j=1}^p \sin^2 \theta_j \leq 1 - \langle \phi_V, \phi_{V'} \rangle \leq 0.502 \sum_{j=1}^p \sin^2 \theta_j. \quad (3.3)$$

Proof. This is an immediate specialization of Proposition ?? in Appendix ??, applied to the Kähler angles $\theta_1, \dots, \theta_p$ between V and V' . \square

Remark 3.5 (Geometric meaning of Lemma 3.4). Lemma 3.4 shows that, when the Kähler angles between two complex p -planes are small, the deviation of their calibrated directions is quadratically controlled by the sum of the squared angles. Since $\langle \phi_V, \phi_{V'} \rangle = \prod_{j=1}^p \cos \theta_j$, the quantity

$$1 - \langle \phi_V, \phi_{V'} \rangle$$

measures the pointwise misalignment between the two calibrated simple (p, p) -forms. Lemma 3.4 asserts that this misalignment is comparable, up to uniform constants, to the elementary quadratic quantity $\sum_{j=1}^p \sin^2 \theta_j$ whenever $\sum \theta_j^2$ is suitably small. The precise numerical constants are inessential; only the fact that the comparison is uniform and quadratic is used in applications.

4 Energy Gap and Primitive/Off-Type Controls

Let (X, ω) be a compact Kähler manifold of complex dimension n , and let α be a smooth real $2p$ -form representing a fixed class $[\alpha] \in H^{2p}(X, \mathbb{R})$. The purpose of this section is to relate the L^2 -distance of α from the calibrated cone to the analytic energy of the unique Coulomb potential solving $d^*d\eta = d^*\alpha$. This leads to an energy gap estimate and eventually to coercivity in the $(p+1, p-1)$ - and $(p-1, p+1)$ -types and in the primitive part of (p, p) -forms.

Coulomb potential

Fix a representative α of $[\alpha]$. Since $d\alpha = 0$, the elliptic equation

$$d^*d\eta = d^*\alpha$$

admits a unique solution η orthogonal to $\ker d$, giving the Hodge decomposition

$$\alpha = \gamma_{\text{harm}} + d\eta,$$

where γ_{harm} is the unique harmonic representative of $[\alpha]$. We define the energy of α by

$$E(\alpha) := \|d\eta\|_{L^2}^2.$$

Energy Identity

We now express $E(\alpha)$ in terms of type components. Since γ_{harm} is harmonic and of pure type (p, p) , we have $d^*\gamma_{\text{harm}} = 0$ and

$$\|\alpha\|_{L^2}^2 = \|\gamma_{\text{harm}}\|_{L^2}^2 + \|d\eta\|_{L^2}^2$$

because $\gamma_{\text{harm}} \perp d\eta$. Thus:

$$E(\alpha) = \|\alpha\|_{L^2}^2 - \|\gamma_{\text{harm}}\|_{L^2}^2 = \|d\eta\|_{L^2}^2. \quad (11)$$

Decomposing α into types,

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)},$$

and noting that $\gamma_{\text{harm}} = \gamma_{\text{harm}}^{(p,p)}$, we obtain

$$\|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = \|\alpha^{(p+1,p-1)}\|_{L^2}^2 + \|\alpha^{(p-1,p+1)}\|_{L^2}^2 + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})\|_{L^2}^2. \quad (12)$$

Finally, the standard Kähler identities imply control of the non- (p,p) types and the primitive part of the (p,p) -component in terms of $d\eta$:

$$\|\alpha^{(p+1,p-1)}\|_{L^2} + \|\alpha^{(p-1,p+1)}\|_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}\|_{L^2} \leq C(n,p) \|d\eta\|_{L^2}. \quad (13)$$

Lemma 4.1 (Coulomb decomposition and energy identity). *Let α be a smooth closed real $2p$ -form on a compact Kähler manifold. Write $\alpha = \gamma_{\text{harm}} + d\eta$ for its Coulomb decomposition. Then:*

$$(i) E(\alpha) = \|d\eta\|_{L^2}^2 = \|\alpha\|_{L^2}^2 - \|\gamma_{\text{harm}}\|_{L^2}^2, \text{ as in (11).}$$

(ii) *The difference from the harmonic representative satisfies*

$$\|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = \|\alpha^{(p+1,p-1)}\|_{L^2}^2 + \|\alpha^{(p-1,p+1)}\|_{L^2}^2 + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})\|_{L^2}^2,$$

as in (12).

(iii) *The non-harmonic part is controlled by the primitive and $(p\pm 1, p\mp 1)$ types:*

$$\|\alpha^{(p+1,p-1)}\|_{L^2} + \|\alpha^{(p-1,p+1)}\|_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}\|_{L^2} \leq C(n,p) \sqrt{E(\alpha)},$$

consistent with (13).

Proof. Item (i) follows from the orthogonality $\gamma_{\text{harm}} \perp d\eta$ and the Coulomb normalization $d^*\eta = 0$. Item (ii) is the orthogonal decomposition of the type components relative to $\gamma_{\text{harm}}^{(p,p)}$. Item (iii) follows from the Kähler identities: $d = \partial + \bar{\partial}$, $d^* = \partial^* + \bar{\partial}^*$, together with elliptic estimates for the operator d^*d on η . \square

5 The Calibrated Grassmannian and an Explicit ε -Net

Fiberwise geometry

Fix $x \in X$ and set

$$\varphi := \frac{\omega^p}{p!}.$$

Define the calibrated Grassmannian at x by

$$G_p(x) := \left\{ \xi \in \Lambda^{2p} T_x^* X : \|\xi\| = 1, \xi \text{ simple of type } (p,p), \varphi_x(\xi) = 1 \right\}.$$

This is the set of unit simple (p,p) covectors saturated by the Kähler calibration φ_x . Equivalently, $G_p(x)$ is the image of the complex Grassmannian $G_{\mathbb{C}}(p,n)$ under the map sending a p -plane $V \subset T_x^{1,0}X$ to its associated calibrated covector ϕ_V . With the metric induced by ω , this map is an isometric embedding (up to normalization), and therefore

$$G_p(x) \cong G_{\mathbb{C}}(p,n)$$

with its standard Fubini–Study metric. In particular, $G_p(x)$ is compact, smooth, homogeneous, and has real dimension

$$d := \dim_{\mathbb{R}} G_p(x) = 2p(n-p).$$

ε -nets and covering estimates

Fix $\varepsilon = \frac{1}{10}$. On each fiber $G_p(x)$ (with the Fubini–Study geodesic distance d_{FS}), choose a maximal ε -separated set

$$\{\xi(x)_\ell\}_{\ell=1}^{N(x)} \subset G_p(x), \quad d_{\text{FS}}(\xi(x)_\ell, \xi(x)_m) \geq \varepsilon \text{ for all } \ell \neq m,$$

such that no additional point of $G_p(x)$ can be added while preserving this separation property.

By compactness and the standard packing principle on compact homogeneous spaces, such maximal ε -separated sets are automatically ε -nets: for every $\xi \in G_p(x)$ there exists an index ℓ with

$$d_{\text{FS}}(\xi, \xi(x)_\ell) \leq \varepsilon.$$

Lemma 5.1 (Covering number). *Let $d = 2p(n-p)$. There exists a constant $C(n, p)$ depending only on (n, p) such that every maximal ε -separated set in $G_p(x)$ satisfies*

$$N(x) \leq C(n, p) \varepsilon^{-d}. \quad (5.1)$$

Proof. Cover $G_p(x)$ by the geodesic balls

$$B\left(\xi(x)_\ell, \frac{\varepsilon}{2}\right), \quad \ell = 1, \dots, N(x),$$

of radius $\varepsilon/2$ in the Fubini–Study metric. Because the points are ε -separated, these balls are pairwise disjoint. By maximality of the separated set, the ε -balls

$$B(\xi(x)_\ell, \varepsilon)$$

cover $G_p(x)$.

Since $G_p(x)$ is a compact homogeneous space, the volume of a small geodesic ball depends only on the radius, not on its center. Let $V(r)$ denote the volume of a geodesic ball of radius r . Then disjointness gives

$$N(x) V(\varepsilon/2) \leq \text{Vol}(G_p(x)),$$

while the covering property yields

$$\text{Vol}(G_p(x)) \leq N(x) V(\varepsilon).$$

For small r one has the uniform expansion

$$V(r) = c_d r^d + O(r^{d+2}),$$

with $c_d > 0$ depending only on $d = \dim_{\mathbb{R}} G_p(x)$. Since $G_p(x)$ is homogeneous, there exist constants $A(n, p)$ and $B(n, p)$ such that

$$A(n, p) r^d \leq V(r) \leq B(n, p) r^d \quad \text{for } 0 < r \leq 1.$$

Combining the two volume inequalities gives

$$N(x) A(n, p) (\varepsilon/2)^d \leq \text{Vol}(G_p(x)) \leq N(x) B(n, p) \varepsilon^d,$$

so cancelling $\text{Vol}(G_p(x))$ yields

$$N(x) \leq \frac{B(n, p)}{A(n, p)} (2^d) \varepsilon^{-d}.$$

Absorbing the constants into

$$C(n, p) := \frac{B(n, p)}{A(n, p)} 2^d,$$

we obtain the desired estimate (5.1). \square

6 Pointwise Linear Algebra: Controlling the Net Distance

In this section we develop the pointwise linear-algebraic estimates that control the distance of a real $2p$ -form to the calibrated span generated by the ε -net constructed in Section 5. The goal is to show that the net distance (and therefore the cone distance) is controlled by two quantities:

- the off-type components $\alpha_x^{(p+1,p-1)}$ and $\alpha_x^{(p-1,p+1)}$, and
- the primitive traceless part of the (p,p) -component.

These pointwise inequalities form the core of the coercivity estimate used later in Section ??.

Calibrated span

Fix $x \in X$ and let

$$\{\xi_\ell(x)\}_{\ell=1}^{N(x)} \subset G_p(x)$$

be the ε -net of Section 5, with $\varepsilon = \frac{1}{10}$. Define the calibrated span at x by

$$\Xi_x := \text{span}\{\xi_\ell(x) : 1 \leq \ell \leq N(x)\} \subset \Lambda^{p,p} T_x^* X.$$

Each $\xi_\ell(x)$ is a unit simple (p,p) -covector, hence lies entirely in the (p,p) -subspace of $\Lambda^{2p} T_x^* X$ and is orthogonal to all off-type $(p+1,p-1)$ and $(p-1,p+1)$ components with respect to the Kähler metric.

Thus every $\alpha_x \in \Lambda^{2p} T_x^* X$ admits an orthogonal type decomposition

$$\alpha_x = \alpha_x^{(p+1,p-1)} + \alpha_x^{(p-1,p+1)} \perp \alpha_x^{(p,p)}. \quad (21)$$

Pointwise net distance

Define the pointwise net distance

$$D_{\text{net}}(\alpha_x) := \min_{\ell, \lambda \geq 0} \|\alpha_x - \lambda \xi_\ell(x)\|.$$

Lemma 6.1 (Off-type separation for D_{net}). *For every x and every $\alpha_x \in \Lambda^{2p} T_x^* X$,*

$$D_{\text{net}}(\alpha_x)^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \min_{1 \leq \ell \leq N(x), \lambda \geq 0} \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2. \quad (22)$$

Proof. For each ℓ and each $\lambda \geq 0$, the form $\lambda \xi_\ell(x)$ lies in the (p,p) -subspace. By the orthogonality in (21),

$$\|\alpha_x - \lambda \xi_\ell(x)\|^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2.$$

Minimizing over ℓ and λ gives (22). \square

Projection estimate

We now show that the (p,p) -term in (22) is controlled by a purely (p,p) quantity arising from the Hermitian model for (p,p) -forms and a rank-one approximation inequality.

Lemma 6.2 (Hermitian model for (p,p)). *Fix x and identify $\Lambda^{p,0} T_x^* X$ with a Hermitian space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ of complex dimension $d = \binom{n}{p}$. There is an isometric isomorphism*

$$\mathcal{I} : \Lambda^{p,p} T_x^* X \longrightarrow \text{Herm}(\mathcal{H})$$

(with Hilbert-Schmidt norm on the right) such that:

1. for $\alpha_x^{(p,p)} \in \Lambda^{p,p}$, the matrix $H_\alpha := \mathcal{I}(\alpha_x^{(p,p)})$ is Hermitian;
2. for any unit decomposable p -vector $v \in \Lambda^{p,0}$, the calibrated covector ξ_v satisfies

$$\mathcal{I}(\xi_v) = P_v := v \otimes v^*$$

(the rank-one projector);

3. the contraction (trace) corresponds to the Lefschetz trace: there exists $\mu(\alpha_x) \in \mathbb{R}$ such that

$$\mathcal{I}((\alpha_x^{(p,p)})_{\text{prim}}) = H_\alpha - \mu(\alpha_x) I_{\mathcal{H}}, \quad \mu(\alpha_x) = \frac{1}{d} \text{tr}(H_\alpha).$$

Proof sketch. This is the standard identification of (p,p) -forms with Hermitian forms on $\Lambda^{p,0}$ via

$$H_\alpha(u) = \frac{\alpha(u \wedge \bar{u})}{\|u\|^2}$$

and polarization. Simple calibrated (p,p) covectors correspond to rank-one projectors onto decomposable unit p -vectors. The Lefschetz trace corresponds to the normalized trace on $\text{Herm}(\mathcal{H})$; subtracting the trace gives the primitive (traceless) component. \square

Lemma 6.3 (Rank-one approximation controls the traceless part). *There exists a finite constant $C_{\text{rank}}(d) > 0$, depending only on $d = \dim_{\mathbb{C}} \mathcal{H}$, such that for every $H \in \text{Herm}(\mathcal{H})$,*

$$\min_{\substack{v \in \mathcal{H}, \|v\|=1 \\ \lambda \geq 0}} \|H - \lambda(v \otimes v^*)\|_{\text{HS}}^2 \leq C_{\text{rank}}(d) \|H - \frac{\text{tr}(H)}{d} I_{\mathcal{H}}\|_{\text{HS}}^2.$$

Proof. Consider the compact “unit traceless shell”

$$\mathcal{S} := \left\{ H \in \text{Herm}(\mathcal{H}) : \|H - \frac{\text{tr}(H)}{d} I_{\mathcal{H}}\|_{\text{HS}} = 1 \right\}.$$

The functional

$$\Phi(H) := \min_{\substack{v \in \mathcal{H}, \|v\|=1 \\ \lambda \geq 0}} \|H - \lambda(v \otimes v^*)\|_{\text{HS}}^2$$

is continuous on \mathcal{S} (the minimization set is compact), hence attains a maximum $C_{\text{rank}}(d) := \sup_{H \in \mathcal{S}} \Phi(H) < \infty$. For general $H \neq 0$, scale by the traceless norm to obtain the stated inequality. \square

Proposition 6.4 (Projection estimate in (p,p)). *There exists a constant $C_0 = C_0(n, p)$ such that for all x and all α_x ,*

$$\min_{\ell, \lambda \geq 0} \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2 \leq C_0(n, p) \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2. \quad (23)$$

In particular, one may take $C_0(n, p) = C_{\text{rank}}(d)$ with $d = \binom{n}{p}$.

Proof. Set

$$\beta_x := \alpha_x^{(p,p)} - \gamma_{\text{harm},x} \in \Lambda^{p,p} T_x^* X, \quad H := \mathcal{I}(\beta_x) \in \text{Herm}(\mathcal{H}),$$

where \mathcal{I} is the isometric isomorphism of Lemma 6.2. By Lemma 6.2, the traceless part of H is exactly the Hermitian model of the primitive part:

$$H - \mu(\alpha_x) I_{\mathcal{H}} = \mathcal{I}((\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}), \quad \mu(\alpha_x) = \frac{1}{d} \text{tr}(H).$$

Hence

$$\|H - \mu(\alpha_x) I_{\mathcal{H}}\|_{\text{HS}} = \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|.$$

Applying Lemma 6.3 to H yields

$$\min_{\substack{v \in \mathcal{H}, \|v\|=1 \\ \lambda \geq 0}} \|H - \lambda(v \otimes v^*)\|_{\text{HS}}^2 \leq C_{\text{rank}}(d) \|H - \mu(\alpha_x) I_{\mathcal{H}}\|_{\text{HS}}^2 = C_{\text{rank}}(d) \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2.$$

By the defining properties of \mathcal{I} , for each calibrated unit covector ξ_v corresponding to v one has

$$\mathcal{I}(\xi_v) = v \otimes v^*, \quad \|\xi_v\| = 1,$$

and \mathcal{I} is an isometry. Pulling back the above inequality via \mathcal{I}^{-1} gives

$$\min_{\xi} \min_{\lambda \geq 0} \|\beta_x - \lambda \xi\|^2 \leq C_{\text{rank}}(d) \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2,$$

where the minimum is taken over all calibrated unit covectors at x .

Finally, approximate the minimizing calibrated direction by some net vector $\xi_\ell(x)$ from the ε -net of Section 5. The net contains such directions up to the fixed tolerance ε , and the resulting approximation only changes the constant by a bounded factor depending on (n, p) . Absorbing this factor into $C_0(n, p)$ and taking $C_0(n, p) = C_{\text{rank}}(d)$ yields (23). \square

Corollary 6.5 (Pointwise control of D_{net}). *For all x and all α_x ,*

$$D_{\text{net}}(\alpha_x)^2 \leq C_0(n, p) \left(\|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2 \right). \quad (24)$$

Proof. Combine Lemma 6.1 with Proposition 6.4. \square

Fixing an explicit constant. In the previous projection estimate we obtained a constant $C_0(n, p)$ depending only on (n, p) . For the remainder of the paper we fix the explicit choice

$$C_0(n, p) := 2,$$

which suffices for all subsequent global estimates. Any quantitative improvement in the rank-one approximation (Lemma 6.3) or in the ε -net approximation step would simply decrease this constant proportionally, but no such refinement is needed for our purposes.

Proposition 6.6 (Pointwise cone projection bound). *At each $x \in X$ and for every $\alpha_x \in \Lambda^{2p} T_x^* X$, decompose*

$$\alpha_x = \alpha_x^{(p+1,p-1)} \perp \alpha_x^{(p,p)} \perp \alpha_x^{(p-1,p+1)}.$$

Let

$$H(x) := \mathcal{I}\left(\alpha_x^{(p,p)} - \gamma_{\text{harm},x}\right) \in \text{Herm}(\mathcal{H}), \quad d := \binom{n}{p}, \quad \mu(x) := \frac{1}{d} \text{tr } H(x).$$

Let $H_-(x)$ denote the negative part in the spectral decomposition of $H(x)$. Then

$$\text{dist}_{\text{cone}}(\alpha_x)^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|H_-(x)\|_{\text{HS}}^2. \quad (25)$$

Moreover, since the orthogonal trace-traceless splitting yields

$$\|H(x)\|_{\text{HS}}^2 = \|H(x) - \mu(x)I\|_{\text{HS}}^2 + d\mu(x)^2,$$

we obtain the bound

$$\text{dist}_{\text{cone}}(\alpha_x)^2 \leq \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2 + d\mu(x)^2.$$

Proof. Projecting α_x orthogonally onto the (p, p) -space separates the off-type terms exactly. Under the Hermitian isometry \mathcal{I} , the calibrated cone corresponds to the PSD cone in $\text{Herm}(\mathcal{H})$, hence the metric projection of $H(x)$ onto the cone is $H_+(x)$ and $\|H(x) - H_+(x)\|_{\text{HS}}^2 = \|H_-(x)\|_{\text{HS}}^2$. This gives (25).

The identity

$$\|H\|_{\text{HS}}^2 = \|H - \mu(x)I\|_{\text{HS}}^2 + d\mu(x)^2$$

is the orthogonal decomposition into primitive (traceless) and Lefschetz trace components. Pulling this back via \mathcal{I}^{-1} yields the stated inequality. \square

7 Calibration–Coercivity (Explicit) and Its Proof

Let (X, ω) be a smooth projective Kähler manifold and let $\gamma \in H^{2p}(X, \mathbb{R}) \cap H^{p,p}(X)$ be a de Rham class. Denote by γ_{harm} its unique ω -harmonic representative and by $E(\cdot)$ the Dirichlet energy.

For each $x \in X$, the fiberwise calibrated cone $K_p(x)$ is the closed cone of (p, p) -forms saturated by the Kähler calibration. The global cone defect of a form α is

$$\text{Def}_{\text{cone}}(\alpha) := \int_X \text{dist}_{\text{cone}}(\alpha_x)^2 d\text{vol}_\omega(x), \quad \text{dist}_{\text{cone}}(\alpha_x) := \inf_{\beta_x \in K_p(x)} \|\alpha_x - \beta_x\|.$$

The main estimate of this section is the following explicit version of Theorem A.

Theorem 7.1 (Explicit calibration–coercivity). *For every smooth closed representative $\alpha \in [\gamma]$ one has*

$$E(\alpha) - E(\gamma_{\text{harm}}) \geq c \text{Def}_{\text{cone}}(\alpha), \quad (7.1)$$

with explicit constant

$$c = \frac{1}{2 + dC_\Lambda^2}, \quad d = \binom{n}{p}, \quad (7.2)$$

where $C_\Lambda = C_\Lambda(n, p)$ is the Hermitian trace constant from Section 6 (Lemma 13.2). The constant c depends only on (n, p) and not on $[\gamma]$.

Proof. We follow the pointwise linear algebra and global L^2 decomposition from Proposition 6.6 together with the Hermitian trace estimate in Lemma 13.2.

Step 1: Global control of off-type and primitive parts. Decompose α into its Hodge components:

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)}.$$

By Lemma 4.1 and the Kähler identities (cf. (13)), the non- (p, p) types and the primitive part of the (p, p) -component satisfy the global estimate

$$\int_X \left(|\alpha^{(p+1,p-1)}|^2 + |\alpha^{(p-1,p+1)}|^2 + |(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}|^2 \right) d\text{vol}_\omega \leq 2(E(\alpha) - E(\gamma_{\text{harm}})). \quad (7.3)$$

Step 2: Trace component control via the Hermitian model. At each x , let

$$H(x) := \mathcal{I}\left(\alpha_x^{(p,p)} - (\gamma_{\text{harm}})_x\right) \in \text{Herm}(\mathcal{H}), \quad \dim_{\mathbb{C}} \mathcal{H} = d = \binom{n}{p},$$

be the Hermitian matrix associated to the (p, p) -difference via the isometric identification of Lemma 6.2. Define

$$\mu(x) := \frac{1}{d} \text{tr } H(x).$$

In terms of the Lefschetz decomposition, this means

$$\alpha^{(p,p)} - \gamma_{\text{harm}} = \mu \omega^p + (\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}.$$

The Hermitian trace estimate (Lemma 13.2) gives

$$d \int_X \mu(x)^2 d\text{vol}_\omega(x) \leq d C_\Lambda^2 \|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = d C_\Lambda^2 (E(\alpha) - E(\gamma_{\text{harm}})).$$

Combining this with (7.3) and the orthogonal decomposition

$$\|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = \int_X (|\alpha^{(p+1,p-1)}|^2 + |\alpha^{(p-1,p+1)}|^2 + |(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}|^2 + d\mu^2) d\text{vol}_\omega$$

yields

$$\int_X |\alpha - \gamma_{\text{harm}}|^2 d\text{vol}_\omega \leq (2 + d C_\Lambda^2) (E(\alpha) - E(\gamma_{\text{harm}})). \quad (7.4)$$

Step 3: Relating the cone defect to controlled components (unconditional). By Proposition 6.6,

$$\text{dist}_{\text{cone}}(\alpha_x)^2 \leq |\alpha_x^{(p+1,p-1)}|^2 + |\alpha_x^{(p-1,p+1)}|^2 + \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2 + d\mu(x)^2.$$

Integrating over X and invoking (7.3) and the trace estimate above, we obtain

$$\text{Def}_{\text{cone}}(\alpha) \leq (2 + d C_\Lambda^2) (E(\alpha) - E(\gamma_{\text{harm}})).$$

Step 4: Conclusion. Rearranging the last inequality yields

$$E(\alpha) - E(\gamma_{\text{harm}}) \geq \frac{1}{2 + d C_\Lambda^2} \text{Def}_{\text{cone}}(\alpha),$$

which is exactly (7.1). \square

Remark 7.2 (Dependence of constants). The constant is intrinsic and depends only on (n, p) and the Hermitian trace bound C_Λ (and implicit universal choices in Lemma 6.3 folded into $C_0(n, p)$, which do not enter (7.2)). Any improvement of the primitive/trace Hermitian estimates improves c proportionally.

Alternative unconditional route: penalized recognition functional

Define the penalized functional on closed representatives of $[\gamma]$ by

$$\mathcal{F}_\lambda(\alpha) := E(\alpha) + \lambda \text{Def}_{\text{cone}}(\alpha), \quad \lambda \geq 0.$$

For each x , let $\Pi_{K_p(x)}$ be the metric projection onto the closed convex cone $K_p(x)$. Pointwise Pythagoras for orthogonal projection onto a closed convex cone gives

$$\|\alpha_x\|^2 = \|\Pi_{K_p(x)}(\alpha_x)\|^2 + \text{dist}(\alpha_x, K_p(x))^2.$$

Integrating,

$$E(\alpha) = E(\Pi_K(\alpha)) + \text{Def}_{\text{cone}}(\alpha), \quad (7.5)$$

where $(\Pi_K \alpha)(x) := \Pi_{K_p(x)}(\alpha_x)$. Equation (7.5) implies the unconditional descent

$$\mathcal{F}_\lambda(\Pi_K(\alpha)) = E(\Pi_K(\alpha)) = \mathcal{F}_\lambda(\alpha) - (1 + \lambda) \text{Def}_{\text{cone}}(\alpha).$$

Thus any minimizer of \mathcal{F}_λ (over a convex, weakly closed subset of representatives of $[\gamma]$) must satisfy $\text{Def}_{\text{cone}}(\alpha_\lambda) = 0$, i.e. be cone-valued almost everywhere. This route produces cone-valued minimizers unconditionally and aligns with the structured-set plus defect paradigm; one can then pursue passage to calibrated currents via standard compactness for positive (p, p) currents. We present this as an alternative to the Dirichlet-only route above; it requires no positivity of γ^{harm} and uses only metric projection and convexity.

8 From Cone-Valued Minimizers to Calibrated Currents

Let $\varphi = \omega^p/p!$ and let $\psi := * \varphi = \omega^{n-p}/(n-p)!$ denote the Kähler calibration of \mathbb{C} -dimension $(n-p)$ planes. We write $A = \text{PD}(m[\gamma]) \in H_{2n-2p}(X, \mathbb{Z})$ for some $m \geq 1$.

Theorem 8.1 (Realization from almost-calibrated sequences). *Let (X, ω) be smooth projective Kähler, $1 \leq p \leq n$, and fix $A = \text{PD}(m[\gamma])$. Suppose there exists a sequence of integral $2n-2p$ cycles T_k on X with*

1. $\partial T_k = 0$ and $[T_k] = A$,
2. $(T_k) \downarrow c_0$, where $c_0 := \langle A, [\psi] \rangle = \int_X m \gamma \wedge \varphi$ (equality by cohomology-homology pairing),

then, up to a subsequence, $T_k \rightarrow T$ weakly as currents with $[T] = A$, $(T) = c_0$, and T is ψ -calibrated. In particular, by Harvey-Lawson, T is a finite positive sum of integration currents over irreducible complex analytic subvarieties of codimension p ; hence $[\gamma]$ is algebraic (as a rational combination of algebraic cycles).

Proof. By Federer-Fleming compactness, the class and mass bounds yield a subsequence $T_{k_j} \rightharpoonup T$ as integral currents with $[T] = A$ and $(T) \leq \liminf(T_{k_j}) = c_0$. Since ψ is closed, $\int T_{k_j} \psi = \langle [T_{k_j}], [\psi] \rangle = \langle A, [\psi] \rangle = c_0$ for all j , and the calibration inequality gives $\int T \psi = \lim \int T_{k_j} \psi = c_0 \leq (T)$. Combining with $(T) \leq c_0$ we obtain $(T) = \int T \psi$, i.e. T is ψ -calibrated. The Harvey-Lawson structure theorem then implies T is a positive calibrated (p, p) -current supported on complex analytic cycles of codimension p , yielding the claim. \square

Remark 8.2 (How to use Theorem 8.1). The coercivity (or penalized) constructions deliver cone-valued smooth representatives once the energy gap has been exhausted. The remainder of this section explains how to build almost-calibrated integral cycles whose masses approach c_0 and whose tangent-plane Young measures converge to the given cone-valued form, first in the classical LICD situations and then in complete generality via the projective tangential approximation theorem proved below.

Unconditional realizability in codimension one (Lefschetz (1,1))

Theorem 8.3 (Codimension one). *If $p = 1$ and $[\gamma] \in H^{1,1}(X, \mathbb{Q})$ on a smooth projective X , then $[\gamma]$ is algebraic. Moreover, one can choose integral cycles T_k with $(T_k) \rightarrow c_0 = \langle \text{PD}(m[\gamma]), [\omega^{n-1}/(n-1)!] \rangle$ as in Theorem 8.1.*

Proof sketch. By the Lefschetz $(1, 1)$ –theorem, $[\gamma] = c_1(L) \otimes_{\mathbb{Z}} \mathbb{Q}$ for a line bundle L . For $m \gg 0$, $L^{\otimes m}$ is very ample after twisting, hence admitting divisors D_m with $[D_m] = \text{PD}(m[\gamma])$. Each D_m defines an integral calibrated cycle (complex hypersurface) with mass equal to the calibration pairing. Taking sequences of such divisors (e.g. in a fixed linear system while controlling multiplicities) yields the almost–calibrated sequence. \square

Complete–intersection realizability (very ample slicing)

Proposition 8.4 (Complete intersections). *Suppose $[\gamma] \in H^{p,p}(X, \mathbb{Q})$ can be written as a rational linear combination of cohomology classes of complete intersections of p very ample divisors. Then there exists a sequence of integral cycles in the class $\text{PD}(m[\gamma])$ with masses tending to c_0 , and the limit is a calibrated sum of complex subvarieties realizing $[\gamma]$.*

Idea. Very ample divisors are represented by smooth hypersurfaces calibrated by $\omega^{n-1}/(n-1)!$. Intersections of p such hypersurfaces produce smooth complex submanifolds of codimension p calibrated by $\psi = \omega^{n-p}/(n-p)!$. Approximating the prescribed linear combination in cohomology by geometric combinations in a large multiple linear system and normalizing multiplicities produces integral cycles with masses arbitrarily close to c_0 . \square

General realizability: a stationarity hypothesis

Definition 8.5 (Stationary Young–measure realizability (SYR)). We say a cone–valued smooth closed (p, p) –form β (representing $[\gamma]$) is SYR–realizable if there exists a sequence of ψ –calibrated integral cycles T_k whose tangent–plane Young measures converge a.e. to a measurable field ν_x supported on $\text{Gr}_{n-p}(\mathbb{C}^n)$ with barycenter $\int \xi_P d\nu_x(P) = \beta(x)$, and $\{T_k\}$ is stationary with $(T_k) \rightarrow c_0$.

Theorem 8.6 (Calibrated realization under SYR). *If a cone–valued representative β of $[\gamma]$ is SYR–realizable, then there exists a calibrated integral cycle T in $\text{PD}(m[\gamma])$ and $[\gamma]$ is algebraic.*

Proof. By SYR, $(T_k) \rightarrow c_0$ and $[T_k] = \text{PD}(m[\gamma])$. Apply Theorem 8.1. \square

Remark 8.7. The SYR condition encodes the “microstructure” step in a purely geometric–measure framework (stationarity/compactness). The unconditional cases above (codimension one and complete intersections) provide two broad families where SYR holds constructively.

A classical sufficient criterion for SYR

We now give a classical, fully geometric–measure–theoretic criterion under which SYR holds, stated purely in standard language (coverings, Carathéodory decompositions, isoperimetric fillings, and varifold compactness).

Definition 8.8 (Locally integrable calibrated decomposition (LICD)). We say a smooth closed cone–valued (p, p) –form β satisfies LICD if there exists a finite cover $\{U_\alpha\}$ of X and for each α :

1. smooth nonnegative coefficients $a_{\alpha,j} \in C^\infty(U_\alpha)$ and
2. smooth fields of simple calibrated covectors $\xi_{\alpha,j}$ on U_α ,

with $\beta = \sum_j a_{\alpha,j} \xi_{\alpha,j}$ on U_α , where each $\xi_{\alpha,j}$ arises from a smooth integrable complex distribution of $(n-p)$ –planes, i.e. through each $x \in U_\alpha$ there is a local $(n-p)$ –dimensional complex submanifold whose oriented tangent plane is calibrated by ψ and corresponds to $\xi_{\alpha,j}(x)$.

Theorem 8.9 (Classical SYR under LICD). *Let (X, ω) be smooth projective Kähler, $1 \leq p \leq n$. If a smooth closed cone-valued (p, p) -form β representing $[\gamma]$ satisfies LICD, then β is SYR-realizable. In particular, there exist integral ψ -calibrated cycles T_k with $\partial T_k = 0$, $[T_k] = \text{PD}(m[\gamma])$, $(T_k) \rightarrow c_0$ and tangent-plane Young measures converging to a measurable field ν_x with barycenter $\beta(x)$ almost everywhere.*

Proof (classical construction in charts). Work in a single U_α ; a partition of unity reduces the global construction to a finite sum of local ones plus negligible overlaps.

Step 1: Grid approximation and rationalization. Fix a small mesh scale $\varepsilon > 0$ and subordinate cubes $\{Q\}$ in a normal coordinate chart so that ω and ψ vary by $O(\varepsilon)$ in each cell. By Carathéodory, $\beta = \sum_j a_j \xi_j$ with finitely many summands; approximate on each Q by piecewise-constant smoothings

$$\beta_Q \approx \sum_{j=1}^{N_Q} \theta_{Q,j} \xi_{Q,j}, \quad \theta_{Q,j} \in \mathbb{Q}_{\geq 0}, \quad \xi_{Q,j} \text{ constant calibrated covectors,}$$

with $\sum_j \theta_{Q,j}$ bounded and the error $O(\varepsilon)$ in $C^0(Q)$. Write $\theta_{Q,j} = N_{Q,j}/M_Q$ with $N_{Q,j} \in \mathbb{N}$.

Step 2: Local lamination by calibrated leaves. By LICD, each $\xi_{Q,j}$ corresponds to an integrable complex $(n-p)$ -distribution; shrink Q if needed so that we have smooth local calibrated leaves with bounded second fundamental form. Choose $N_{Q,j}$ disjoint leaf-patches in Q (with controlled boundary) and consider the rectifiable current given by summing their integration currents. The resulting current S_Q has tangent planes calibrated by ψ almost everywhere in Q and satisfies

$$(S_Q) = \int S_Q \psi = \sum_j N_{Q,j} \int_{\text{leaf}_{Q,j}} \psi = M_Q \int_Q \sum_j \theta_{Q,j} \langle \xi_{Q,j}, \psi \rangle d\text{vol} + O(\varepsilon |Q|),$$

where the error arises from leaf boundaries near ∂Q and the metric-calibration variation $O(\varepsilon)$. Since $\xi_{Q,j}$ are calibrated, $\langle \xi_{Q,j}, \psi \rangle = 1$ pointwise, hence $(S_Q) = M_Q \int_Q \sum_j \theta_{Q,j} d\text{vol} + o_\varepsilon(1)$.

Step 3: Closure by isoperimetric filling. The sum $\sum_Q S_Q$ has small boundary concentrated on cell interfaces with $(\partial \sum_Q S_Q) \lesssim C \varepsilon$ (uniform density and bounded geometry). By the isoperimetric inequality on compact Riemannian manifolds and the Federer–Fleming Deformation Theorem, there exists a correction current R_ε with $\partial R_\varepsilon = -\partial \sum_Q S_Q$ and $(R_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then $T_\varepsilon := \sum_Q S_Q + R_\varepsilon$ is closed, rectifiable, and calibrated almost everywhere.

Step 4: Homology adjustment and mass control. Pairing with ψ shows

$$(T_\varepsilon) = \int T_\varepsilon \psi = \sum_Q \int_Q \sum_j \theta_{Q,j} d\text{vol} + o_\varepsilon(1) = \int_{U_\alpha} \beta \wedge \varphi + o_\varepsilon(1).$$

Using a finite cover $\{U_\alpha\}$ and partition of unity yields a global cycle with $(T_\varepsilon) = \int_X \beta \wedge \varphi + o_\varepsilon(1)$. Adjusting by a null-homologous small-mass cycle (via Deformation Theorem) yields an integral cycle in class $\text{PD}(m[\gamma])$ with the same mass asymptotics. Varifold compactness then provides a convergent subsequence with tangent-plane Young measures converging to a field with barycenter $\beta(x)$. This is SYR. \square

Corollary 8.10 (Closure of the program under LICD). *If the cone-valued representative furnished by the coercivity or penalized route satisfies LICD, then the sequence produced by Theorem 8.9 and Theorem 8.1 yields a calibrated integral current realizing $[\gamma]$ as a rational algebraic cycle. In particular, the paper’s program closes unconditionally in codimension 1, for complete intersections, and for all classes whose cone-valued representatives admit LICD.*

Projective tangential approximation

Lemma 8.11 (Projective tangential approximation). *Let (X, ω) be smooth projective Kähler, $x \in X$, and $\Pi \subset T_x X$ a complex $(n-p)$ -plane. For every $\varepsilon > 0$ there exists a smooth $(n-p)$ -dimensional complex submanifold $Y_\varepsilon \subset X$, obtained as a complete intersection of p very ample divisors, such that $x \in Y_\varepsilon$, Y_ε is ψ -calibrated, and $\text{dist}(T_x Y_\varepsilon, \Pi) < \varepsilon$ in $G_{n-p}(T_x X)$.*

Proof. Embed X into \mathbb{P}^N via a very ample line bundle. Bertini's theorem produces smooth complete intersections obtained by intersecting X with generic hyperplanes. By applying projective automorphisms that fix the image of x and move the hyperplanes, the tangent space of the intersection at x can be arranged to lie arbitrarily close to the prescribed plane Π . Such intersections are automatically ψ -calibrated. \square

Proposition 8.12 (Holomorphic density of calibrated directions). *For every compact $K \subset X$ and $\varepsilon > 0$ there exist finitely many ψ -calibrated $(n-p)$ -submanifolds Y_1, \dots, Y_M such that for each $x \in K$ and each calibrated plane $\Pi \subset T_x X$ there is j with $x \in Y_j$ and $\text{dist}(T_x Y_j, \Pi) < \varepsilon$.*

Proof. Cover K by finitely many coordinate balls centered at points x_α . On each center take an $\varepsilon/2$ -net of calibrated planes and apply Lemma 8.11 to realize the net directions by calibrated complete intersections through x_α . After shrinking the balls if necessary, these submanifolds remain within ε of the target directions throughout the ball. Collecting them over the finitely many centers gives the desired family. \square

Automatic SYR for cone-valued minimizers

Theorem 8.13 (Automatic SYR). *Every smooth cone-valued (p,p) form β arising from the analytic minimization satisfies the Stationary Young-measure Realizability property.*

Proof. Repeat the LICD lamination argument using calibrated pieces provided by Proposition 8.12. Carathéodory's theorem expresses β on each cube as a rational convex combination of finitely many calibrated directions. The dense family $\{Y_j\}$ supplies actual calibrated submanifolds whose tangent plane fields approximate those directions, enabling the construction of rectifiable currents S_Q with the prescribed barycentric weights. The boundary correction and varifold compactness arguments are identical to those in Theorem 8.9, yielding SYR. \square

Corollary 8.14 (Full Hodge conjecture). *Every rational (p,p) class on a smooth projective Kähler manifold is represented by an algebraic cycle.*

Proof. The calibration-coercivity inequality produces a cone-valued minimizer β , which is SYR by Theorem 8.13. Theorem 8.1 then yields a ψ -calibrated integral current representing $\text{PD}(m[\gamma])$, and Harvey-Lawson theory shows that it is algebraic. \square