Version-3 Comment-1 on D3

List of comments!

NEW — Leverage the published Axioms paper (axioms-4140269). Now that "Reciprocal Convex Costs for Ratio Matching: Axiomatic Characterization" (Washburn & Rahnamai Barghi, *Axioms* 2026, 15(2), 90; doi: `10.3390/axioms15020090`) is accepted, we should import its main result into this paper and cite it. This strengthens the submission by anchoring our cost functional in a *published, peer-reviewed theorem* rather than re-deriving it. Concrete insertions are described below in teal.

1. In the Introduction, we still state:

   > "Assume $\mathcal{C}_R$ is manifold-like ..."

   but never define "manifold-like" anywhere as a standalone term. We have replaced it conceptually with "admits an effective manifold model $\mathcal{M}$", but we did not propagate that terminology back into Theorem 1.2.

   Fix: Replace the "manifold-like" phrasing in Theorem 1.2 with something that directly references Definition 2.12. For example:

   > "Assume $(\mathcal{C}, \mathcal{E}, R)$ admits an effective manifold model $\mathcal{M}$ in the sense of Definition 2.12..."

   That makes paper consistent with the two-scale story.

2. We reused the symbol ω in two slightly different ways (as κ/Ω in Method 1 and as a $(4-D)$-dependent ratio in the Binet method). They are the same dimensionless ratio in the linearized regime, thus lets say:

   > "The ω in Method 2 coincides with the ratio κ/Ω in Method 1."

3. In Step (3) of Theorem 4.3, we write

   $$(\partial Q) \cdot B = Q \cdot (\partial B). \tag{1}$$

   This is not the correct boundary/intersection compatibility identity. The correct schematic identity (up to sign conventions) is

   $$\partial(Q \pitchfork B) = (\partial Q) \pitchfork B \ \pm\ Q \pitchfork (\partial B), \tag{2}$$

   i.e. there is an extra $\partial(Q \pitchfork B)$ term that is dropped.

   How to fix? The fix is to argue at the homology level:
   - In an oriented D-manifold, the intersection number $Z \cdot B$ depends only on the homology class $[Z] \in H_{p+1}(\mathcal{C}_R)$.
   - Since $H_{p+1}(\mathcal{C}_R) = 0$, we have $[Z] = 0$, hence $Z \cdot B = 0$.

- Therefore $W \cdot B = W' \cdot B$.

This avoids all chain-level sign/boundary complications and is standard.

Another one: In Proposition 3.3, we say "$p = 0$ gives $D = 1$." But Theorem 4.3 assumes $0 < p < D$, so we cannot cite it to justify $p = 0$.

Fix?

- easiest: we redefine $A_A = \{3, 5, 7, \dots\}$ by requiring $p \geq 1$ (still non-singleton, still intersects to $\{3\}$), and drop the $D = 1$ talk entirely; or
- add a short separate remark if we really want to discuss 0-dimensional "linking" (but I think it adds confusion as it doesn't help the selection argument).

New issue introduced while trying to justify p-flexibility: Remark 3.4 is conceptually broken. We call these "codimension-2 defects," but we compute the codimension:

$$D - p = D - \frac{D-1}{2} = \frac{D+1}{2},$$

which equals 2 only when $D = 3$. So as written, the remark effectively says:

> "These are codimension-2 ... and they are codimension-2 only in $D = 3$. "

Fix? Rename and reframe. The statement is:

- Same-dimension linking requires $p = \frac{D-1}{2}$, which is codimension $p+1$, not "codimension 2" in general.
- Codimension-2 defects are a different physically motivated class; if we want codim-2 specifically, that specialization directly forces $D = 3$.

4. In a statement "Prior Approaches. . . ":

   > "Freedman's exotic $\mathbb{R}^4$ theorem shows ... $\mathbb{R}^4$ admits uncountably many distinct smooth structures. . . "

   This is not safe as written. At minimum it is misattributed / oversimplified.

   Fix? We should rewrite cautiously or remove unless we can cite a precise correct attribution and statement.

5. **"Knot theory is nontrivial only in dimensions $D = 3, 4$"**
   This is false and worse, the very next clause says "surfaces link in $D = 5$," which contradicts the "only $3, 4$" part.

   Fix? we can rewrite as:

   > "Classical knot theory of embeddings $S^1 \hookrightarrow \mathbb{R}^3$ is special; in higher dimensions the behavior changes dramatically; higher-dimensional knot theory (e.g. codimension-2 sphere knots) exists."

   But we should *not* claim it is "only in $3, 4$."

6. The following statement is vague and likely wrong or at least under-specified: "chiral anomalies vanishing only in specific dimensions (e.g., $D = 2, 6, 10 \dots$). " This reads like half-memory. We need to either cut it or replace with it precise statement.

7. Appendix Green-kernel sign convention is inconsistent with the main text. In Appendix A, we write:

   > "Choosing $C < 0$ for an attractive potential . . . "

But in the main text (Proposition 4.2) we take $V_2(r) = k \ln r$ with $k > 0$ as attractive (and correctly compute $F = -k/r$). For $V(r) = C \ln r$, attraction means

$$F = -V' = -\frac{C}{r}$$

inward, so we need $C > 0$, not $C < 0$.

Fix: We need to change that line in the appendix to something like:

> "Choose the constant so that $F = -\nabla V$ is inward (attractive)..."

or explicitly: "choose $C > 0$."

8. In Section 5, we state as if $\mathcal{M}$'s rotation group is literally $SO(D)$. However, I think the global isometry group of a generic manifold is not $SO(D)$. What we want is local frame rotations, i.e. the structure group of the oriented orthonormal frame bundle is $SO(D)$ (assuming a Riemannian metric).

   Fix? We can say:

   > "the local orthonormal frame rotation group is $SO(D)$, which is non-abelian iff $D \geq 3$. "

   That removes the "global isometry group" objection cleanly.
9. Make sure all the equations have the equation numbers.
10. At the end of every proof there is a little box, one need to remove it.
11. Each equation must end either with comma or period.
12. Author names should be in alphabetical order with last name.

13. **Import the published cost-kernel theorem from [Axioms paper].**
    The paper "Reciprocal Convex Costs for Ratio Matching" (Washburn & Rahnamai Barghi, *Axioms* 2026, **15**(2), 90) is now published. It proves:

    > *Under inversion symmetry, strict convexity, coercivity, normalization at 1, and a multiplicative d'Alembert identity, the unique admissible mismatch penalty is*
    >
    > $$J(x) \;=\; \cosh(a \log x) - 1 \;=\; \tfrac{1}{2}(x^a + x^{-a}) - 1, \qquad x > 0,$$
    >
    > *for some $a > 0$; moreover a can be absorbed into the scale maps, giving the canonical choice $a = 1$.*

    Where to add in D3 paper:

    - **Introduction (1 paragraph):** Add a bridge sentence such as:

      > *"This paper builds on the axiomatic characterization of the ratio-induced cost functional established in [**?**]. There it was shown that the assumptions of inversion symmetry, strict convexity, coercivity, and a multiplicative d'Alembert compatibility identity uniquely force $J(x) = \frac{1}{2}(x^a + x^{-a}) - 1$. We take this result as given and focus on the downstream topological and dimensional consequences."*

    - **Preliminaries / Section 2:** State the result as an imported Proposition (or Assumption), e.g.:

      > ***Proposition 2.X** (Washburn–Rahnamai Barghi [**?**]). Let $J : (0, \infty) \to [0, \infty)$ satisfy (i) $J(x) = J(1/x)$, (ii) strict convexity, (iii) $J(1) = 0$, (iv) coercivity,*

and (v) the multiplicative d'Alembert identity $(1+J(xy))+(1+J(x/y)) = 2(1+J(x))(1+J(y))$. Then there exists $a>0$ such that $J(x) = \cosh(a\log x)-1$. The parameter a is absorbed by rescaling $\iota_S, \iota_O \mapsto \iota_S^a, \iota_O^a$, yielding $J(x) = \frac{1}{2}(x+x^{-1})-1$ without loss.

- **Notation alignment:** Reuse the Axioms-paper notation ι_S, ι_O, J exactly, so the two publications form a visible chain.
- **Scope sentence (Discussion / Section 1):**

  *"The novelty of the present work lies in the geometric and topological consequences of the cost kernel — specifically the forcing of $D=3$ spatial dimensions via linking constraints — rather than in the derivation of J itself, which is established in [**?**]."*

14. **Add the argmin / geometric-mean boundary result from the Axioms paper.** The Axioms paper also proves that for a finite dictionary $\{o_1,\ldots,o_n\}$ with ordered scales $y_1 < \cdots < y_n$, the decision boundary between preferring o_i and o_{i+1} is the *geometric mean* $\sqrt{y_i\, y_{i+1}}$. If the D3 paper uses any discrete selection or "best-matching" argument on scale sets, we can directly cite this as an already-published lemma instead of reproving it inline. Suggested insertion in the Preliminaries:

    Corollary 2.Y *(Geometric-mean boundaries; [**?**], Proposition 4.X). For the canonical cost $J(x) = \frac{1}{2}(x+x^{-1})-1$ and an ordered dictionary $y_1 < y_2 < \cdots < y_n$ in $\mathbb{R}_{>0}$, the argmin mapping satisfies $\mathrm{Mean}(s) = \{o_i\}$ for $\sqrt{y_{i-1}y_i} < \iota_S(s) < \sqrt{y_i\, y_{i+1}}$. In particular, decision boundaries are geometric means.*

15. **Add the compositionality / product-model result from the Axioms paper.** The Axioms paper establishes exact compositionality: for product models $S = S_1 \times S_2$, $O = O_1 \times O_2$ with product scales, the meaning set factors as $\mathrm{Mean}(s_1,s_2) = \mathrm{Mean}(s_1) \times \mathrm{Mean}(s_2)$. If the D3 paper invokes any product-structure or factorization argument (e.g. for composite configurations), this can be cited directly. Suggested sentence:

    *"By the compositionality theorem of [**?**], the argmin rule factors exactly over independent components, so the analysis extends to product models without additional assumptions."*

16. **Add the BibTeX entry.** Insert in the bibliography:

```
@article{WashburnRahnamaiBarghi2026,
author = {Washburn, Jonathan and Rahnamai Barghi, Amir},
title = {Reciprocal Convex Costs for Ratio Matching:
Axiomatic Characterization},
journal = {Axioms},
year = {2026},
volume = {15},
number = {2},
pages = {90},
doi = {10.3390/axioms15020090}
}
```