Some Comments on UNIQUENESS OF THE CANONICAL RECIPROCAL COST

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I could provide some comments on the paper and hope it is useful for you, also there is a short description of the problem, and a suggested minimal repair.

B1 (incorrect displayed equation) Location: p. 4, Eq. (2.3) in §2.1 ("Motivation: d'Alembert's functional equation").
Issue: Eq. (2.3) is written as

$$H(t + u) - H(t - u) = H(t)\,H(u),$$

which is not compatible with the standard d'Alembert cosine/hyperbolic cosine addition formulas that the paper uses everywhere else (Eq. (2.2)).

Suggestion to resolve: Either delete Eq. (2.3) entirely, or replace it with a correct identity that is actually used later (likely Eq. (2.2) itself, or a true consequence such as Lemma 2.4).

B2 (scope/validity of Theorem 2.2 is uncertain) Location: p. 4, Theorem 2.2 (§2.1).
Issue: Theorem 2.2 claims to describe the *general* complex-valued solutions of the d'Alembert equation on the Cartesian square of an abelian group G by the formula

$$H(x) = \tfrac{1}{2}\big(h(x) + h(-x)\big) \quad \text{with} \quad h(x + y) = h(x)h(y).$$

I am *uncertain* this is correct as stated for the full generality claimed (arbitrary abelian groups, no regularity assumptions), and it needs to be checked carefully against the cited source.
Suggestion to fix: Re-check Theorem 2.2 against its source and (i) correct the statement, or (ii) restrict the hypotheses (e.g. continuity/measurability plus the underlying group), or (iii) remove Theorem 2.2 and the following paragraph if it is not load-bearing for the main rigidity theorem.

B3 (unsupported continuity claim) Location: p. 4, paragraph immediately after Theorem 2.2.
Issue: The text states: "The function H in Theorem 2.2 is continuous, and it follows that h is also continuous." However, Theorem 2.2 (as written) does not assume continuity of H at all, so continuity cannot be asserted here.

Suggestion: Either add continuity assumptions where needed and justify them, or delete the sentence and any downstream deductions that rely on it.

B4 ("H continuous $\Rightarrow h$ continuous" is unjustified) Location: p. 4, same paragraph after Theorem 2.2.

Issue: Even if one *did* assume H is continuous, the inference that the multiplicative solution h is continuous does not follow from the relation $H(x) = \frac{1}{2}(h(x)+h(-x))$ without further hypotheses.

Suggestion to fix: Remove the claim, or provide a correct lemma with explicit hypotheses under which continuity of h follows (if such a lemma is intended).

B5 (complex logarithm/branch issue in $h = e^f$) Location: p. 4, same paragraph after Theorem 2.2.

Issue: Writing $h(x) = e^{f(x)}$ for a complex-valued multiplicative homomorphism h is not automatically valid globally: it requires choosing a (continuous) logarithm, which generally needs nontrivial topological assumptions (e.g. connectedness and a target avoiding winding).

Fix it like this: State explicit hypotheses that ensure a continuous logarithm exists (or avoid the exponential parameterization entirely).

B6 (incorrect / incomplete inference from "H real-valued") Location: p. 4, same paragraph after Theorem 2.2: the sentence "Since $H(x)$ is real-valued, it follows that either $f_1 = 0$ or $f_2 = 0$."

Issue: From $\cosh(f_1 + if_2) = \cosh(f_1)\cos(f_2) + i\sinh(f_1)\sin(f_2)$, real-valuedness only forces $\sinh(f_1)\sin(f_2) = 0$ pointwise. Concluding $f_1 \equiv 0$ or $f_2 \equiv 0$ requires additional hypotheses (e.g. continuity + connectedness + additivity/linearity), and the argument is not present.

Suggestion how to fix: Provide a correct lemma with explicit hypotheses (or delete the claim and keep the discussion heuristic).

B7 (domain error: $\log x$ with $x \leq 0$) Location: p. 14, Corollary 4.1.

Issue: Corollary 4.1 defines $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by $F(x) = H(\log x) - 1$, but $\log x$ is not defined at $x = 0$ and requires $x > 0$.

Fix: Replace $\mathbb{R}_{\geq 0}$ by $\mathbb{R}_{>0}$ and ensure all later uses respect $x > 0$.

B (results stated as "Corollary" without proofs / hypothesis-check) Location: p. 16, Corollary 5.1 (golden ratio limit), and surrounding §5.5–§6 narrative.

Issue: Corollary 5.1 states global convergence of an iteration to ϕ with no proof. Likewise, parts of §6 present a physical "energy" interpretation without specifying a model. These may be intended as informal remarks, but they are currently formatted as mathematical claims.

Suggestion How to fix: Provide a short proof or citation for Corollary 5.1, and label physical discussion explicitly as heuristic/interpretive if no model is supplied.

If I had more time I could provide more comments, I hope the above comments/suggestion migh be useful for your paper.