

# Referee Report on "Reciprocal Convex Costs for Ratio Matching: Axiomatic Characterization"

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The paper studies an axiomatic, optimization-based model of reference, where meaning is defined by minimizing a mismatch cost between configurations (tokens) and objects. The authors restrict their investigation to ratio-induced reference costs and fix a specific reciprocal functional  $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ . Within this case, they prove existence and stability results, explicit decision boundaries, and several composition and mediation properties.

Overall, the manuscript is mathematically clear, precise, and well structured. The results are new and interesting for further research.

In the beginning, the main assumptions are given. The scale maps, admissibility, and the choice of  $J$  are stated explicitly.

Once the admissible (ratio-induced) structure and the functional  $J$  are fixed, the further analysis becomes concrete.

The choice of the reciprocal cost  $J(x)$  leads to algebra and inequalities, in particular in the mediation results. The calculations are correct.

It is important to note that the main structure is largely imposed by definition. In particular,

1. Admissibility (Definition 6) restricts reference costs to be ratio-induced. This already excludes many possible reference mechanisms by construction.
2. The explicit choice of  $J$  in Section 2.2 already encodes reciprocity, convexity, and the appearance of geometric means.
3. Meaning is defined as cost minimization (Definition 7), rather than derived from independent principles.

As a consequence, the main theorems analyze consequences of a specified model, rather than uncovering a structure that holds independently of these choices. Proposition 2 and Appendix A apply classical results on d'Alembert's functional equation. Proposition 2 shows that all admissible costs reduce to  $\cosh(a \log x) - 1$ , the normalization  $a = 1$  is fixed without further use of the general family. Theorem 9 identifies geometric means as decision boundaries. Lemma 3 states that the chosen  $J(x)$  grows to infinity when the scale ratio tends to 0 or to  $\infty$ , so bounding the mismatch automatically bounds the ratio and makes all sublevel sets  $\{J(x) \leq M\}$  compact. Theorems 5 and 6 are well aligned with the structure of the cost  $J$ .

There are some technical points regarding the axiomatic presentation:

- Axiom (A2), follows directly from normalization  $J(1) = 0$  together with the multiplicative d'Alembert identity (A4), by setting  $x = 1$ . Therefore, (A2) is redundant and should either be removed or explicitly stated as a derived property.
- Strict convexity (A3) is stronger than strictly necessary for some early results, such as uniqueness of the zero-cost point. A weaker assumption would suffice in places.
- Axiom (A4) is the dominant structural assumption. Together with mild regularity, it essentially fixes the functional form of  $J$ . Presenting all axioms as having equal status can obscure this hierarchy.
- Theorem 6 proves optimal mediation when the geometric mean  $b_{\text{geo}} \in Y_M$ . The paper does not quantify how suboptimal the solution is when this condition does not hold.
- Theorem 7 appears to be an extension of Theorem 2 and I think it could be presented as a corollary.
- Section 8 is interpretative. It could be shortened or merged with the discussion, keeping only mathematically explicit examples.

The paper provides a clear, new and mathematically correct analysis of a chosen reference model, **and I recommend it for publication in *Axioms* after minor revision.**