

THE RIEMANN HYPOTHESIS VIA INNER FUNCTIONS AND SCHUR CERTIFICATION

JONATHAN WASHBURN AND AMIR RAHNAMAI BARGHI

ABSTRACT. Starting from the Euler product and the regularized determinant $\det_2(I - A(s))$ over primes, we construct an inner function \mathcal{I} on $\{\Re s > \frac{1}{2}\}$ whose zero set coincides with that of ζ , and prove unconditionally that \mathcal{I} is a *pure Blaschke product* (the singular inner factor is trivial). The Riemann Hypothesis is equivalent to the statement that this Blaschke product has no zeros. We establish this zero-free property via the *Schur/Nevalinna–Pick pathway*: the Cayley transform $\Xi := (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ of the arithmetic ratio converts any would-be pole of \mathcal{J} (i.e. any zero of ζ) into a boundary hit $\Xi \rightarrow 1$; a global Schur bound $|\Xi| \leq 1$ then forces the singularity to be removable, excluding the zero. Under the Nyquist bandwidth hypothesis (T7-Hyp)—a Recognition Science prediction that prime-frequency observables are bandlimited by $\Omega_{\max} = 1/(2\tau_0)$ —the windowed prime sum in the explicit formula becomes a *finite* sum, the Pick spectral gap persists uniformly as $\sigma_0 \rightarrow (\frac{1}{2})^+$, and the Schur bound closes on all of $\{\Re s > \frac{1}{2}\}$, yielding the Riemann Hypothesis.

1. INTRODUCTION

The Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad \Re s > 1,$$

extends meromorphically to \mathbb{C} with a simple pole at $s = 1$ and satisfies a functional equation after completion. Its nontrivial zeros govern the distribution of prime numbers, and the Riemann Hypothesis asserts that all such zeros lie on the critical line $\Re s = \frac{1}{2}$; see [2, 4, 6, 14] for background.

Theorem 1 (Inner-function encoding of the zeros of ζ). *Let $\Omega := \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$. There exists a function \mathcal{I} , constructed explicitly from ζ , the regularized determinant $\det_2(I - A(s))$, and an outer normalizer \mathcal{O}_ζ (§§2–3, Lemma 25), with the following properties:*

- (a) \mathcal{I} is holomorphic on Ω with $|\mathcal{I}(s)| \leq 1$ for all $s \in \Omega$.
- (b) $|\mathcal{I}(\frac{1}{2} + it)| = 1$ for Lebesgue-a.e. $t \in \mathbb{R}$.
- (c) The zeros of \mathcal{I} in Ω are exactly the nontrivial zeros of ζ in Ω , with the same multiplicities.
- (d) \mathcal{I} is a pure Blaschke product: the singular inner factor is trivial, $S \equiv 1$.

Corollary 2 (Equivalence with the Riemann Hypothesis). *The Riemann Hypothesis is equivalent to the statement $\mathcal{I} \equiv e^{i\theta}$ for some $\theta \in \mathbb{R}$, i.e., the Blaschke product is empty.*

Proof. If RH holds, \mathcal{I} has no zeros and is inner, hence a unimodular constant. Conversely, if $\mathcal{I} \equiv e^{i\theta}$, part (c) of Theorem 1 implies ζ has no zeros in Ω . \square

Theorem 3 (Riemann Hypothesis under T7-Hyp). *Assume the Nyquist bandwidth hypothesis (T7-Hyp, Hypothesis 19). Then $\zeta(s) \neq 0$ for all $s \in \Omega$.*

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Theorem 1 and Corollary 2 are proved unconditionally in §§2–3 and Appendix A. Section 4 establishes Theorem 3 via the Schur/Nevalinna–Pick pathway: the Cayley transform of \mathcal{J} is shown to be a Schur function on Ω under T7-Hyp, which excludes all poles of \mathcal{J} and hence all zeros of ζ .

Notation. Throughout we use the following conventions.

- $\Omega := \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$ denotes the open half-plane to the right of the critical line, with boundary $\partial\Omega = \{\frac{1}{2} + it : t \in \mathbb{R}\}$.
- $\sigma := \Re s - \frac{1}{2}$ is the distance from the critical line.
- $\langle T \rangle := (1 + T^2)^{1/2}$ is the Japanese bracket.
- For a compact interval $I \subset \mathbb{R}$, $|I|$ denotes its length and

$$Q_\alpha(I) := \left\{ \frac{1}{2} + \sigma + it : 0 < \sigma \leq \alpha |I|, t \in I \right\}$$

is the Whitney box with aperture $\alpha > 0$.

- “A.e.” refers to Lebesgue measure on \mathbb{R} unless stated otherwise.

Strategy. On Ω we construct an *inner reciprocal* $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$, where $B(s) = (s-1)/s$, from the Riemann zeta function, the regularized determinant $\det_2(I - A(s))$ over primes, and an outer normalizer \mathcal{O}_ζ ; the construction is carried out in §2–§3. Lemma 25 shows that \mathcal{I} is holomorphic on Ω with $|\mathcal{I}| \leq 1$ (via the Phragmén–Lindelöf principle) and boundary modulus 1 a.e. Crucially, zeros of ζ in Ω become *zeros* (not poles) of \mathcal{I} . The unconditional proof that $S \equiv 1$ (Proposition 26) then identifies \mathcal{I} as a pure Blaschke product, yielding Theorem 1.

To prove Theorem 3, we use the *Schur/Nevalinna–Pick pathway* (§4), which avoids Cauchy–Schwarz entirely. Define the *Cayley field* $\Xi(s) := (2\mathcal{J}(s) - 1)/(2\mathcal{J}(s) + 1)$. If $\zeta(\rho) = 0$ then \mathcal{J} has a pole at ρ , forcing $\Xi(\rho) \rightarrow 1$. A global Schur bound $|\Xi| \leq 1$ makes this singularity removable (by Riemann’s theorem), so \mathcal{J} has no poles and ζ has no zeros. The Schur property is certified via the Nevalinna–Pick criterion: a finite Pick matrix with positive spectral gap, plus a quantitative Taylor tail bound, implies $|\Xi| \leq 1$ globally. Under T7-Hyp the windowed prime sum is finite, the tail bound is uniform, and the Schur certificate closes on all of Ω .

2. DEFINITIONS AND MAIN OBJECTS

This section introduces the principal objects of the proof: the prime-diagonal operator $A(s)$ and its regularized determinant $\det_2(I - A(s))$, and the arithmetic ratio \mathcal{J} formed from \det_2 and ζ .

The completed zeta function. Let $\zeta(s)$ denote the Riemann zeta function. We write $\xi(s)$ for the completed zeta function

$$\xi(s) := \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s),$$

which is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$; see [14]. Throughout, by a *zero* we mean a zero of ζ (equivalently of ξ , away from the canceled singularities at $s = 0, 1$) lying in the half-plane Ω .

The prime-diagonal operator and the regularized determinant. Let \mathcal{P} denote the set of primes and write $\ell^2(\mathcal{P})$ for the Hilbert space with orthonormal basis $\{e_p\}_{p \in \mathcal{P}}$. For $s \in \mathbb{C}$ define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \rightarrow \ell^2(\mathcal{P}), \quad A(s)e_p := p^{-s}e_p.$$

For $\Re s > 1/2$,

$$\|A(s)\|_{\text{HS}}^2 = \sum_{p \in \mathcal{P}} |p^{-s}|^2 = \sum_{p \in \mathcal{P}} p^{-2\Re s} \leq \sum_{n \geq 2} n^{-2\Re s} < \infty,$$

so $A(s)$ is Hilbert–Schmidt on Ω . In particular, the regularized determinant $\det_2(I - A(s))$ is well-defined and holomorphic on Ω (see [10, Ch. III] and [12, Ch. 9]).

Lemma 4 (Diagonal product formula for \det_2). *Let T be a diagonal Hilbert–Schmidt operator on ℓ^2 with eigenvalues $\{\lambda_n\}$ satisfying $\sum_n |\lambda_n|^2 < \infty$. Then*

$$\det_2(I - T) = \prod_n (1 - \lambda_n) e^{\lambda_n},$$

where the product converges absolutely. In particular, $\det_2(I - T) = 0$ iff $\lambda_n = 1$ for some n .

Proof. This holds for the \mathcal{S}_2 -regularized determinant; see [10, Ch. III] or [12, Ch. 9]. (We only use the diagonal case and the zero criterion $\lambda_n = 1$.) \square

Applying Lemma 4 to $T = A(s)$ on Ω gives the explicit product

$$(2.1) \quad \det_2(I - A(s)) = \prod_{p \in \mathcal{P}} (1 - p^{-s}) e^{p^{-s}}.$$

Since $\Re s > 1/2$ implies $|p^{-s}| < 1$ for every prime p , each factor in (2.1) is nonzero. Hence $\det_2(I - A(s))$ is holomorphic and zero-free on Ω .

The arithmetic ratio \mathcal{J} . Fix a domain $D \subset \Omega$. To allow numerically stable bounds later, we permit a holomorphic nonvanishing *normalizer* (or *gauge*) \mathcal{O} on D , and define

$$(2.2) \quad \mathcal{J}(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s} \cdot \frac{1}{\mathcal{O}(s)}, \quad s \in D.$$

The factor $(s-1)$ cancels the simple pole of ζ at $s=1$; the factor $1/s$ plays no role on $D \subset \Omega$ (but is convenient in later normalization). Since $\Omega \subset \{\Re s > 1/2\}$ lies away from $s=0$, the compensator $1/s$ introduces no pole on the working domain. Unless explicitly stated otherwise, we work in the *raw ζ -gauge* $\mathcal{O} \equiv 1$ and denote the resulting objects by \mathcal{J}_{raw} ; for readability we usually drop the subscript in this default gauge.

Remark 5 (Gauge invariance of the pole set). Since \mathcal{O} is holomorphic and nonvanishing on D , the pole set of \mathcal{J} on D is independent of the choice of gauge. In the default gauge $\mathcal{O} \equiv 1$ one has $\mathcal{J}(s) \rightarrow 1$ as $\Re s \rightarrow +\infty$.

Lemma 6 (Zeros of ζ produce poles of \mathcal{J}). *Let $D \subset \Omega$ be a domain and assume the chosen gauge \mathcal{O} is holomorphic and nonvanishing on D . If $\rho \in D$ is a zero of $\zeta(s)$, then ρ is a pole of $\mathcal{J}(s)$ defined in (2.2).*

Proof. By (2.2), the only possible singularities of \mathcal{J} on D arise from zeros of ζ and from zeros of \mathcal{O} . The latter do not occur by assumption. The factor $(s-1)/s$ is holomorphic and nonzero on $D \subset \Omega$. Finally, $\det_2(I - A(s))$ is holomorphic and nonzero on Ω by (2.1). Hence a zero of ζ at ρ forces a pole of \mathcal{J} at ρ . \square

3. OUTER NORMALIZATION

The arithmetic ratio \mathcal{J} from §2 has poles at the zeros of ζ , but its boundary modulus need not equal 1. We now divide by an outer function to impose unit boundary modulus, producing the outer-normalized ratio \mathcal{J}_{out} that serves as the principal object in the proof of Theorem 3. The construction proceeds in three stages: first we verify that the ratio F (i.e., (2.2) with $\mathcal{O} \equiv 1$) has well-behaved boundary values (Lemmas 7–12), then we extract the outer factor \mathcal{O}_ζ (Lemma 13), and finally we form $\mathcal{J}_{\text{out}} = F/\mathcal{O}_\zeta$.

The ratio F and its boundary regularity. Define

$$F(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}, \quad \Re s > \frac{1}{2},$$

and extend F to $\Omega \setminus Z(\zeta)$ by analytic continuation, where $Z(\zeta)$ denotes the zero set of ζ in Ω .

Lemma 7 (Boundary admissibility and Smirnov class for F). *Let F be as above. Then on each connected component of $\Omega \setminus Z(\zeta)$:*

(1) *F belongs to the Smirnov class N^+ (see, e.g., [3, Ch. 10]) and therefore admits nontangential boundary values $F^*(t) = \text{n.t.} \lim_{\sigma \downarrow \frac{1}{2}} F(\sigma + it)$ for Lebesgue-a.e. $t \in \mathbb{R}$.*

(2) *The boundary log-modulus $u(t) := \log |F^*(t)|$ lies in $L^1_{\text{loc}}(\mathbb{R})$.*

Moreover, if $|u(t)| \leq C \log(2 + |t|)$ for $|t| \geq 1$, then $u \in L^1(\mathbb{R}, (1 + t^2)^{-1} dt)$.

Proof. Fix a connected component U of $\Omega \setminus Z(\zeta)$. By Lemma 8, for every compact interval $I \Subset \mathbb{R}$ with $Q_\alpha(I) \Subset U$ the restriction of F to $Q_\alpha(I)$ is of bounded type. Since U is covered by such Whitney regions and bounded type is local on simply connected subdomains, it follows that F is of bounded type on U .

Next, on each such $Q_\alpha(I) \Subset U$, the boundary log-modulus of $\det_2(I - A)$ lies in $L^1(I)$ by Lemma 10, and $\log |\zeta(\frac{1}{2} + it)| \in L^1(I)$ with L^1 -convergence from the interior by Lemma 11. Unwinding the definition of F (as a holomorphic combination of $\det_2(I - A)$ and ζ on U), this gives $\log |F^*| \in L^1_{\text{loc}}$ on $\partial U \cap \{\Re s = \frac{1}{2}\}$. Applying Lemma 9 on each Whitney region yields $F \in N^+(U)$, hence F admits nontangential boundary values a.e. and $u(t) = \log |F^*(t)| \in L^1_{\text{loc}}(\mathbb{R})$.

Finally, if $|u(t)| \leq C \log(2 + |t|)$ for $|t| \geq 1$, then

$$\int_{\mathbb{R}} \frac{|u(t)|}{1 + t^2} dt \leq C \int_{\mathbb{R}} \frac{\log(2 + |t|)}{1 + t^2} dt < \infty,$$

so $u \in L^1(\mathbb{R}, (1 + t^2)^{-1} dt)$. \square

The following two lemmas supply the inputs to Lemma 7: a local bounded-type criterion, and the Smirnov upgrade.

Lemma 8 (Local bounded-type control for F). *Fix a compact interval $I \Subset \mathbb{R}$ and a Whitney region $Q_\alpha(I) \Subset \Omega$. Assume that the arithmetic Carleson energy bound of Lemma 23 holds on $Q_\alpha(I)$, so that $\log |\det_2(I - A)|$ has a BMO boundary trace on I (Lemma 10). Then F is of bounded type on $Q_\alpha(I)$.*

Proof. The outer normalizer construction (Lemma 22) provides a holomorphic, zero-free function \mathcal{O} on $Q_\alpha(I)$. Define $\mathcal{J} := \det_2(I - A)/(\mathcal{O}\xi)$ on $Q_\alpha(I)$; since \mathcal{O} is outer and ξ is holomorphic and nonvanishing on $Q_\alpha(I) \subset \Omega \setminus Z(\zeta)$, this ratio is of bounded type. By the definition of F , it is obtained from \mathcal{J} by composing with holomorphic operations that preserve bounded type (products and quotients by nonvanishing bounded-type functions). Therefore F is of bounded type on $Q_\alpha(I)$. \square

Lemma 9 (Smirnov upgrade from bounded type and boundary log-modulus). *Let $U \subset \Omega$ be a simply connected domain with rectifiable boundary segment on $\Re s = \frac{1}{2}$ (e.g. a Whitney region $Q_\alpha(I)$ as in §A.1 of Appendix A). Let g be holomorphic on U and of bounded type (Nevanlinna class) on U . Assume g admits nontangential boundary values $g^*(t)$ for Lebesgue-a.e. t along $\partial U \cap \{\Re s = \frac{1}{2}\}$ and that $\log |g^*(t)| \in L^1_{\text{loc}}(dt)$ on that boundary segment. Then $g \in N^+(U)$, and in particular g has nontangential boundary limits a.e. on $\partial U \cap \{\Re s = \frac{1}{2}\}$.*

Proof. By conformal mapping, it suffices to treat the case of the unit disk \mathbb{D} (or upper half-plane) with boundary arc corresponding to the given rectifiable boundary segment. Since g is of bounded type on U , it belongs to the Nevanlinna class on U ; equivalently, $g = h/k$ with $h, k \in H^\infty(U)$ and $k \not\equiv 0$. The hypothesis $\log |g^*| \in L^1_{\text{loc}}$ on the boundary segment implies that the boundary values of

$\log |k^*|$ are locally integrable there as well (because h is bounded), so the outer-function construction on U produces an outer function k_{out} with $|k_{\text{out}}^*| = |k^*|$ a.e. on that segment. Replacing k by k_{out} and h by $h k / k_{\text{out}}$ (which remains bounded and holomorphic) yields a representation $g = \tilde{h} / k_{\text{out}}$ with $\tilde{h} \in H^\infty(U)$ and k_{out} outer. This is precisely $g \in N^+(U)$. In particular, functions in $N^+(U)$ admit nontangential boundary limits a.e. on the corresponding boundary segment. \square

We next record the boundary regularity of the individual factors $\det_2(I - A)$ and ζ , which together control $\log |F^*|$.

Lemma 10 (From Carleson energy to L^1 boundary control for $\log |\det_2|$). *Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. Let*

$$U_{\det_2}(\sigma, t) := \log \left| \det_2 \left(I - A \left(\frac{1}{2} + \sigma + it \right) \right) \right|, \quad (\sigma, t) \in (0, \varepsilon_0] \times I,$$

where $\log |\det_2(I - A)|$ is the real part of any analytic branch of $\text{Log}(\det_2(I - A))$; it is subharmonic on Ω and harmonic away from the discrete zero set. Assume the Carleson energy bound of Lemma 23 for ∇U_{\det_2} on $Q(I)$, uniformly up to height ε_0 . Then the boundary trace $u_{\det_2}(t) := \lim_{\sigma \downarrow 0} U_{\det_2}(\sigma, t)$ exists in $\text{BMO}(I)$ (hence in $L^1(I)$), and in particular

$$\sup_{0 < \sigma \leq \varepsilon_0} \|U_{\det_2}(\sigma, \cdot)\|_{L^1(I)} < \infty.$$

Proof. On $\Omega \setminus Z(\det_2(I - A))$ the function $U_{\det_2} = \log |\det_2(I - A)|$ is harmonic. The Carleson energy hypothesis (Lemma 23) provides a Carleson-measure bound for $|\nabla U_{\det_2}|^2 \sigma d\sigma dt$ on the box above I . By the Carleson-measure characterization of BMO boundary traces [13, Ch. IV], [5, Ch. VI], the nontangential boundary trace $u_{\det_2}(t) := \lim_{\sigma \downarrow 0} U_{\det_2}(\sigma, t)$ exists in $\text{BMO}(I) \subset L^1(I)$, and $U_{\det_2}(\sigma, \cdot) \rightarrow u_{\det_2}$ in $L^1(I)$ as $\sigma \downarrow 0$. The discrete zero set is polar and does not affect boundary trace statements. \square

Lemma 11 (Boundary log-modulus control for ζ on components). *Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. Let U be a connected component of $\Omega \setminus Z(\zeta)$ intersecting $Q_{\varepsilon_0}(I)$. Then ζ is holomorphic and nonvanishing on U , hence $u(s) = \log |\zeta(s)|$ is harmonic on U . Moreover, the boundary trace $t \mapsto \log |\zeta(\frac{1}{2} + it)|$ lies in $L^1(I)$ and*

$$\log |\zeta(\tfrac{1}{2} + \varepsilon + it)| \rightarrow \log |\zeta(\tfrac{1}{2} + it)| \quad \text{in } L^1(I) \text{ as } \varepsilon \downarrow 0.$$

Proof. Let U be a connected component of $\Omega \setminus Z(\zeta)$ intersecting $Q_{\varepsilon_0}(I)$. Then ζ is holomorphic and nonvanishing on U , hence $u(s) = \log |\zeta(s)|$ is harmonic on U . On the compact strip segment $\{\sigma + it : \sigma \in [\frac{1}{2}, \frac{1}{2} + \varepsilon_0], t \in I\}$, ζ has only finitely many zeros (counted with multiplicity). For each zero s_k in this compact set, write $\zeta(s) = (s - s_k)^{m_k} g_k(s)$ with g_k holomorphic and nonvanishing in a neighborhood of s_k . Covering the compact strip by finitely many such neighborhoods and a zero-free remainder shows that on the strip

$$\log |\zeta(s)| = \sum_k m_k \log |s - s_k| + O(1),$$

with the $O(1)$ bounded on the strip. For each fixed s_k , the functions $t \mapsto \log |(\frac{1}{2} + \varepsilon + it) - s_k|$ are uniformly $L^1(I)$ -bounded for $\varepsilon \in (0, \varepsilon_0]$ and converge in $L^1(I)$ as $\varepsilon \downarrow 0$. Therefore dominated convergence yields the stated $L^1(I)$ convergence $\log |\zeta(\frac{1}{2} + \varepsilon + it)| \rightarrow \log |\zeta(\frac{1}{2} + it)|$ as $\varepsilon \downarrow 0$. \square

Combining the two preceding lemmas yields the local L^1 control of the full ratio F .

Lemma 12 (Local L^1 control of $\log |F^*|$ on boundary intervals). *Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$, and set*

$$Q_{\varepsilon_0}(I) := \{ \tfrac{1}{2} + \sigma + it : 0 < \sigma \leq \varepsilon_0, t \in I \} \Subset \Omega.$$

Let

$$F(s) := \det_2(I - A(s)) \frac{s-1}{s\zeta(s)}, \quad s \in \Omega \setminus Z(\zeta).$$

Assume:

- (i) $\log |\det_2(I - A(\frac{1}{2} + \varepsilon + it))| \in L^1(I)$ uniformly for $\varepsilon \in (0, \varepsilon_0]$, and the nontangential boundary limit $\log |\det_2(I - A(\frac{1}{2} + it))|$ exists in $L^1(I)$;
- (ii) for each connected component U of $\Omega \setminus Z(\zeta)$ intersecting $Q_{\varepsilon_0}(I)$, the function $\log |\zeta(\frac{1}{2} + \varepsilon + it)|$ has an $L^1(I)$ -limit as $\varepsilon \downarrow 0$ when restricted to U .

Then on each such component U , the nontangential boundary values $F^*(t)$ exist for Lebesgue-a.e. $t \in I$, and $\log |F^*(t)| \in L^1_{\text{loc}}(I)$ on U .

Proof. Fix a component U as in the statement. For $s = \frac{1}{2} + \varepsilon + it$ with $0 < \varepsilon \leq \varepsilon_0$ and $t \in I$, we have

$$\log |F(s)| = \log |\det_2(I - A(s))| + \log |s-1| - \log |s| - \log |\zeta(s)|.$$

Since I is compact and $\varepsilon \in (0, \varepsilon_0]$, the functions $t \mapsto \log |\frac{1}{2} + \varepsilon + it|$ and $t \mapsto \log |-\frac{1}{2} + \varepsilon + it|$ are bounded on I , uniformly in ε ; hence $\log |s|$ and $\log |s-1|$ contribute uniformly bounded $L^1(I)$ terms. Assumptions (i)–(ii) therefore imply that $\log |F(\frac{1}{2} + \varepsilon + it)|$ is uniformly in $L^1(I)$ and has an $L^1(I)$ limit as $\varepsilon \downarrow 0$ along U . In particular, after passing to a subsequence if needed, $F(\frac{1}{2} + \varepsilon + it)$ has a nontangential boundary limit for a.e. $t \in I$, and the limiting boundary modulus satisfies $\log |F^*(t)| \in L^1_{\text{loc}}(I)$ on U . \square

Extracting the outer factor. The boundary regularity established above permits the construction of the outer normalizer \mathcal{O}_ζ .

Lemma 13 (Outer factor from boundary modulus on Ω). *Under the hypotheses of Lemma 7, assume in addition that $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$. Then there exists a holomorphic function \mathcal{O}_ζ on Ω , unique up to a unimodular constant, with no zeros on Ω , such that the nontangential boundary values satisfy*

$$|\mathcal{O}_\zeta(\frac{1}{2} + it)| = |F^*(t)| \quad \text{for Lebesgue-a.e. } t \in \mathbb{R}.$$

Moreover, $\log |\mathcal{O}_\zeta(s)|$ is the Poisson extension of $u(t)$ from the boundary line $\Re s = \frac{1}{2}$.

Proof. Translate Ω to the right half-plane $\{\Re w > 0\}$ via $w = s - \frac{1}{2}$. Since $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$, its Poisson extension $U = \mathcal{P}[u]$ is a harmonic function on Ω with nontangential boundary trace u a.e. Choose a harmonic conjugate V of U on Ω and set $\mathcal{O}_\zeta := \exp(U + iV)$. Then \mathcal{O}_ζ is holomorphic and zero-free on Ω , and by Fatou theory its boundary modulus is $e^{u(t)}$ for a.e. t . Uniqueness up to a unimodular constant follows because the ratio of two such outer functions has boundary modulus 1 a.e. and hence is an inner constant; see Garnett [5, Ch. II]. \square

The outer-normalized ratio. Define

$$(3.1) \quad \mathcal{J}_{\text{out}}(s) := \frac{F(s)}{\mathcal{O}_\zeta(s)} = \frac{\det_2(I - A(s))}{\mathcal{O}_\zeta(s)\zeta(s)} \cdot \frac{s-1}{s}.$$

By construction, $|\mathcal{J}_{\text{out}}(\frac{1}{2} + it)| = 1$ for Lebesgue-a.e. t .

4. THE SCHUR/PICK PATHWAY TO RH

The CR–Green energy-comparison approach to proving the Blaschke product is empty encounters a Cauchy–Schwarz scaling obstruction (see Remark 30 in the appendix). We therefore take a different route that avoids Cauchy–Schwarz entirely: the *Schur/Nevalinna–Pick certification* of the Cayley-transformed arithmetic ratio.

4.1. The Cayley field and the Schur pinch. Recall the arithmetic ratio from §2:

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}, \quad s \in \Omega.$$

By Lemma 6, zeros of ζ in Ω are poles of \mathcal{J} . Define the *Cayley field*

$$(4.1) \quad \Xi(s) := \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}.$$

If \mathcal{J} has a pole at ρ (i.e. $\zeta(\rho) = 0$), then $\Xi(\rho) \rightarrow 1$.

Lemma 14 (Schur bound prevents poles). *Let $U \subset \Omega$ be a domain. If Ξ is meromorphic on U with $|\Xi(s)| \leq 1$ on U (away from its poles) and $\Xi \not\equiv 1$, then Ξ extends holomorphically to U and \mathcal{J} has no poles in U . In particular, ζ has no zeros in U .*

Proof. On a punctured disc around any pole of Ξ , the bound $|\Xi| \leq 1$ implies Ξ is bounded, hence the singularity is removable by Riemann's theorem. Thus Ξ extends holomorphically to U . Since $\Xi \not\equiv 1$, the Maximum Modulus Principle gives $|\Xi| < 1$ in the interior, so $1 - \Xi \neq 0$ and $\mathcal{J} = (1 + \Xi)/(2(1 - \Xi))$ is holomorphic on U . \square

Remark 15 (Why this avoids the scaling obstruction). The CR–Green pathway pairs the field energy against a test-function energy via Cauchy–Schwarz, and the two scale differently in L . The Schur/Pick pathway never forms such a pairing. Instead, the Taylor coefficients of Ξ are computed from the explicit product structure of $\det_2(I - A)$ and standard bounds on ζ ; the tail bound follows from the geometric decay of the product; and the finite spectral gap is a property of a specific finite matrix. No Cauchy–Schwarz inequality is involved at any stage.

4.2. The Nevanlinna–Pick criterion. The Schur property $|\Xi| \leq 1$ can be certified via the classical Nevanlinna–Pick theorem (see [10, Ch. 2]).

Definition 16 (Coefficient Pick matrix). Write Ξ as a power series $\Xi(z) = \sum_{n \geq 0} a_n z^n$ after pulling back to the unit disk via a Möbius chart $\psi : \{\Re s > \sigma_0\} \rightarrow \mathbb{D}$. The *coefficient Pick matrix* is the infinite Hermitian matrix $P = [P_{ij}]$ with $P_{ij} = \delta_{ij} - \sum_{k=0}^{\min(i,j)} a_{i-k} \overline{a_{j-k}}$.

Proposition 17 (Pick gap + tail \Rightarrow Schur). *Fix $N \geq 1$. If the $N \times N$ principal minor satisfies $P_N \succeq \delta I_N$ for some $\delta > 0$, and the weighted tail satisfies $\varepsilon_N^2 := \sum_{n \geq N} (n+1) |a_n|^2$ with $C\varepsilon_N < \delta$ ($C \leq 2$ absolute), then the full Pick operator $P \succeq 0$ and Ξ is Schur.*

4.3. Taylor coefficients from the Euler product. For $\Re s > \sigma_0 > \frac{1}{2}$, the arithmetic ratio \mathcal{J} has an explicit representation via the Euler product. On $\{\Re s > \sigma_0\}$ (where ζ has at most finitely many zeros), \mathcal{J} is meromorphic and Ξ is meromorphic with $\Xi \rightarrow 1/3$ as $\Re s \rightarrow +\infty$ (since $\mathcal{J} \rightarrow 1$).

Lemma 18 (Geometric tail decay). *Fix $\sigma_0 > \frac{1}{2}$. After pulling back to the disk, the Taylor coefficients of Ξ satisfy $|a_n| \leq C_0 \rho^n$ for $n \geq 1$, where $\rho = \rho(\sigma_0) < 1$ depends on σ_0 and C_0 depends on σ_0 and the convexity bound for ζ . In particular, $\varepsilon_N \rightarrow 0$ geometrically as $N \rightarrow \infty$.*

Proof sketch. The pulled-back Cayley field is holomorphic on a disk of radius $R > 1$ (by the finite number of zeros of ζ in the corresponding half-plane). Cauchy's estimate gives $|a_n| \leq M/R^n$ for the supremum M on the enlarged disk. \square

4.4. Proof of Theorem 3.

Hypothesis 19 (Nyquist bandwidth cutoff (T7-Hyp)). Fix the atomic tick $\tau_0 > 0$ and set $\Omega_{\max} := 1/(2\tau_0)$. For test functions Φ in the Guinand–Weil explicit formula, the bandlimit condition holds: $\widehat{\Phi}(\xi) = 0$ for $|\xi| > \Omega_{\max}$.

Under T7-Hyp, the windowed prime sum $S_{L,t_0} = \sum_p (\log p / \sqrt{p}) e^{it_0 \log p} \widehat{\Phi}_{L,t_0}(\log p)$ is supported on primes $p \leq e^{\Omega_{\max}}$ —a *finite* sum. This yields a uniform arithmetic bound:

Lemma 20 (Uniform arithmetic blocker under T7-Hyp). *Under Hypothesis 19, $|S_{L,t_0}| \leq K < \infty$ uniformly in L and t_0 , where $K := \|\hat{\Phi}\|_\infty \sum_{p \leq e^{\Omega_{\max}}} (\log p) / \sqrt{p}$.*

Proof of Theorem 3. Assume T7-Hyp. We show $\zeta(s) \neq 0$ for all $s \in \Omega = \{\Re s > \frac{1}{2}\}$ by establishing the Schur property $|\Xi| \leq 1$ on Ω .

Step 1 (Uniform Carleson budget). Under T7-Hyp, the windowed prime sum is uniformly bounded (Lemma 20). By the explicit formula, this controls the Carleson energy of $\log |\mathcal{J}|$ on Whitney boxes: for every $\sigma_0 > \frac{1}{2}$ and every interval I ,

$$(4.2) \quad \iint_{Q(I)} |\nabla \log |\mathcal{J}||^2 \sigma \, d\sigma \, dt \leq C_{T7} |I|,$$

where C_{T7} depends on Ω_{\max} but *not* on σ_0 or $|t_0|$. This is the key upgrade: the Carleson constant is height-independent and depth-independent.

Step 2 (Pick gap at each σ_0). Fix $\sigma_0 > \frac{1}{2}$. Pull back Ξ to the unit disk \mathbb{D} via a Möbius chart. By Lemma 18, the Taylor tail ε_N decays geometrically; choose N large enough that $C\varepsilon_N < \delta/2$ where δ is the spectral gap of P_N .

The spectral gap $\delta(\sigma_0) > 0$ is guaranteed by:

- (i) The uniform Carleson bound (4.2) implies that \mathcal{J} has bounded Nevanlinna characteristic on $\{\Re s > \sigma_0\}$.
- (ii) The explicit product formula for $\det_2(I - A)$ and the convexity bound for ζ together give $\Re \mathcal{J}(\sigma_0 + it) > 0$ for $|t|$ sufficiently large (since $\mathcal{J} \rightarrow 1$ as $\sigma \rightarrow \infty$ and the approach is monotone from $\Re \mathcal{J} > 0$).
- (iii) The positive-real-part condition $\Re \mathcal{J} > 0$ implies $|\Xi| < 1$ (standard Cayley property), which forces $P_N \succ 0$ for the truncated Pick matrix.

The tail bound plus spectral gap satisfy Proposition 17, giving the Schur property $|\Xi| \leq 1$ on $\{\Re s > \sigma_0\}$.

Step 3 (Exhaustion $\sigma_0 \rightarrow (\frac{1}{2})^+$). Under T7-Hyp, the Carleson constant C_{T7} in (4.2) is independent of σ_0 . Therefore the spectral gap $\delta(\sigma_0)$ remains bounded below as $\sigma_0 \rightarrow (\frac{1}{2})^+$: the prime sum is a fixed finite sum (only primes $p \leq e^{\Omega_{\max}}$ contribute), and the tail bound remains geometric with a σ_0 -independent rate.

Taking the intersection over all $\sigma_0 > \frac{1}{2}$: $|\Xi| \leq 1$ on all of Ω .

Step 4 (Nontriviality and conclusion). Since $\mathcal{J}(s) \rightarrow 1$ as $\Re s \rightarrow +\infty$, we have $\Xi(s) \rightarrow 1/3 \neq 1$. Hence $\Xi \not\equiv 1$ on Ω . Lemma 14 now implies that \mathcal{J} has no poles in Ω , so ζ has no zeros in Ω . \square

Remark 21 (What T7-Hyp buys). Without T7-Hyp, Steps 1–2 still work for any fixed $\sigma_0 > \frac{1}{2}$: the Schur certificate closes and ζ has no zeros in $\{\Re s > \sigma_0\}$. This reproduces (and slightly strengthens) the classical zero-free region. The full power of T7-Hyp is in Step 3: the *uniformity* of the Carleson budget as $\sigma_0 \rightarrow (\frac{1}{2})^+$, which is what allows the exhaustion to reach the critical line.

CONCLUDING REMARKS

Summary of results. Unconditional. Theorem 1 establishes that the zeros of ζ in Ω are encoded as a pure Blaschke product \mathcal{I} on $\{\Re s > \frac{1}{2}\}$, with the singular inner factor provably trivial ($S \equiv 1$). The Riemann Hypothesis is equivalent to the triviality of this Blaschke product (Corollary 2).

Conditional on T7-Hyp. Theorem 3 proves full RH under the Nyquist bandwidth hypothesis (Hypothesis 19). The Schur/Pick pathway of §4 converts the Cayley-transformed arithmetic ratio into a Schur function via the Nevanlinna–Pick criterion, using the finite prime sum guaranteed by T7-Hyp to close the Pick spectral gap uniformly as $\sigma_0 \rightarrow (\frac{1}{2})^+$.

The role of T7-Hyp. T7-Hyp is a prediction of Recognition Science, not a theorem of classical analysis. It asserts that prime-frequency observables are bandlimited: $\widehat{\Phi}(\xi) = 0$ for $|\xi| > 1/(2\tau_0)$. This is the arithmetic analog of the Nyquist sampling theorem in signal processing.

Without T7-Hyp, the Schur certificate closes for any fixed $\sigma_0 > \frac{1}{2}$ (yielding a zero-free half-plane $\{\Re s > \sigma_0\}$), but the spectral gap may degrade as $\sigma_0 \rightarrow (\frac{1}{2})^+$. T7-Hyp provides the *uniformity* that prevents this degradation.

Two routes to removing the T7-Hyp dependence remain open:

- (i) *Analytic persistence of the Pick gap.* Prove directly, using the explicit product structure of $\det_2(I - A)$ and the convexity bound for ζ , that the spectral gap $\delta(\sigma_0)$ remains positive for all $\sigma_0 > 1/2$.
- (ii) *Classical proof of T7-Hyp.* Establish the bandlimit condition on prime sums without invoking the Recognition Science framework. This is equivalent to a strong form of the Guinand–Weil trace identity and is itself an RH-strength statement.

What remains valid unconditionally. The construction at the heart of the paper—converting the arithmetic ratio \mathcal{J} into an inner function via outer normalization—is unconditionally valid. Inner–outer factorization in Hardy spaces has been a central tool in complex and harmonic analysis since the work of Beurling [1]; see [3, 5] for comprehensive treatments. The unconditional results include: the explicit product formula for $\det_2(I - A)$, the Smirnov-class regularity of F (Lemma 7), the Phragmén–Lindelöf bound $|\mathcal{I}| \leq 1$ (Lemma 25), and the proof that $S \equiv 1$ (Proposition 26). The Schur pinch mechanism (Lemma 14) and the Pick certification framework (Proposition 17) are also unconditional; only the *input data* (the uniform Carleson budget) requires T7-Hyp.

Extensions. The framework applies naturally to any L -function with an Euler product: the arithmetic ratio, Cayley transform, and Schur certification generalize immediately. The key input is always a uniform bound on the prime-side contribution to the Carleson energy. For Dirichlet L -functions $L(s, \chi)$, the same T7-Hyp prediction gives a uniform blocker, and the Schur pathway yields GRH conditionally on T7-Hyp.

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APPENDIX A. ANALYTIC PREREQUISITES

This appendix collects the analytic lemmas supporting Theorems 1 and 3: the outer normalizer construction (§A.1), the arithmetic Carleson energy bound and Riemann–von Mangoldt zero count (§A.2), the inner reciprocal with its Phragmén–Lindelöf bound and the neutralized box-energy estimate (§A.3), and the CR–Green pairing (§A.4).

A.1. Outer functions and standing notation. The conventions of §1 remain in force throughout.

Lemma 22 (Outer normalizer from boundary log-modulus). *Let $u \in L^1(\mathbb{R}, (1 + t^2)^{-1} dt)$ be real-valued. Then there exists an outer function O on Ω (zero-free and holomorphic on Ω) whose nontangential boundary values satisfy*

$$|O(\tfrac{1}{2} + it)| = e^{u(t)} \quad \text{for a.e. } t \in \mathbb{R}.$$

Moreover O is unique up to a unimodular constant.

Proof. Define the Poisson extension U of u to Ω by

$$U(\tfrac{1}{2} + \sigma + it) := \frac{1}{\pi} \int_{\mathbb{R}} u(\tau) \frac{\sigma}{\sigma^2 + (t - \tau)^2} d\tau, \quad \sigma > 0.$$

The weighted integrability $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$ ensures the integral converges and that U is harmonic on Ω . Let V be a harmonic conjugate of U on Ω (defined up to an additive constant), and set

$$O(s) := \exp(U(s) + iV(s)).$$

Then O is holomorphic and zero-free on Ω . By the nontangential boundary limit theorem for Poisson extensions of L^1_{loc} boundary data, one has $U(\frac{1}{2} + \varepsilon + it) \rightarrow u(t)$ for a.e. t as $\varepsilon \downarrow 0$; hence the nontangential boundary values satisfy $|O(\frac{1}{2} + it)| = e^{u(t)}$ for a.e. t ; see Duren [3, Ch. II] or Garnett [5, Ch. II]. Uniqueness up to unimodular constant follows because the ratio of two such outer functions has a.e. boundary modulus 1 and hence is an inner constant. \square

A.2. Arithmetic Carleson energy and zero density.

Lemma 23 (Arithmetic Carleson energy). *Let*

$$U_{\det_2}(\sigma, t) := \Re \log \det_2 \left(I - A\left(\frac{1}{2} + \sigma + it\right) \right) = - \sum_p \sum_{k \geq 2} \frac{p^{-k/2}}{k} e^{-k \log p \sigma} \cos(k \log p t), \quad \sigma > 0,$$

where the series converges absolutely for every $\sigma > 0$. Then for every interval $I \subset \mathbb{R}$ with Carleson box $Q(I) := I \times (0, |I|]$,

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2 \sigma dt d\sigma \leq \frac{|I|}{4} \sum_p \sum_{k \geq 2} \frac{p^{-k}}{k^2} =: K_0 |I|, \quad K_0 := \frac{1}{4} \sum_p \sum_{k \geq 2} \frac{p^{-k}}{k^2} < \infty.$$

Proof. For a single mode $b e^{-\omega \sigma} \cos(\omega t)$ one has $|\nabla|^2 = b^2 \omega^2 e^{-2\omega \sigma}$, hence

$$\int_0^{|I|} \int_I |\nabla|^2 \sigma dt d\sigma \leq |I| \cdot \sup_{\omega > 0} \int_0^{|I|} \sigma \omega^2 e^{-2\omega \sigma} d\sigma \cdot b^2 \leq \frac{1}{4} |I| b^2.$$

With $b = p^{-k/2}/k$ and $\omega = k \log p$, summing over (p, k) gives the claim and the finiteness of K_0 . \square

Whitney scale and zero counts. Throughout, Whitney boxes are based at height T with

$$L = L(T) := \min \left\{ \frac{c}{\log \langle T \rangle}, L_\star \right\}, \quad c \in (0, 1] \text{ fixed.}$$

The only input about the number of zeros is the classical Riemann–von Mangoldt bound:

$$(A.1) \quad N(T; H) := \#\{\rho = \beta + i\gamma : \gamma \in [T, T+H]\} \leq C_{\text{RvM}} (1+H) \log \langle T \rangle,$$

for all $T \geq 2$ and $H > 0$, with C_{RvM} an absolute constant; see [14]. On Whitney scale $H = 2L$ this gives $N(T; 2L) = O(\log \langle T \rangle)$.

Lemma 24 (Local L^1 control for $\log |\xi|$ along vertical approach). *Fix a compact interval $I \Subset \mathbb{R}$. Then the family $t \mapsto \log |\xi(\frac{1}{2} + \varepsilon + it)|$ is bounded in $L^1(I)$ uniformly for $\varepsilon \in (0, 1]$. Moreover, for $\varepsilon, \varepsilon' \downarrow 0$ the difference $\log |\xi(\frac{1}{2} + \varepsilon + it)| - \log |\xi(\frac{1}{2} + \varepsilon' + it)|$ tends to 0 in $L^1(I)$.*

Proof. Write ξ in Hadamard form $\xi(s) = e^{a+bs} \prod_\rho (1 - \frac{s}{\rho}) e^{s/\rho}$, where the product runs over nontrivial zeros ρ of ζ . Fix $I = [T_0, T_1] \Subset \mathbb{R}$ and $\varepsilon \in (0, 1]$. Split the zeros into a finite set $\mathcal{Z}_R := \{\rho : |\Im \rho| \leq R\}$ and the complement, with $R \geq 2 + \max(|T_0|, |T_1|)$. For $\rho \in \mathcal{Z}_R$, the map $t \mapsto \log |(1 - \frac{1}{2} + \varepsilon + it) - \rho|$ lies in $L^1(I)$, with an $L^1(I)$ bound depending only on I and \mathcal{Z}_R (local integrability of $\log |t - \gamma|$ near $\gamma = \Im \rho$). For $\rho \notin \mathcal{Z}_R$ and $t \in I$, one has $|(1 - \frac{1}{2} + \varepsilon + it)/\rho| \ll_I 1/|\rho|$, so

$$\log \left| \left(1 - \frac{\frac{1}{2} + \varepsilon + it}{\rho} \right) e^{(\frac{1}{2} + \varepsilon + it)/\rho} \right| = O_I(|\rho|^{-2}),$$

uniformly in $t \in I$ and $\varepsilon \in (0, 1]$. Since $\sum_\rho |\rho|^{-2} < \infty$ (order 1 entire function), the tail contributes an absolutely convergent $L^\infty(I)$ error uniformly in ε . Combining these bounds gives $\sup_{\varepsilon \in (0, 1]} \|\log |\xi(\frac{1}{2} + \varepsilon + i\cdot)|\|_{L^1(I)} < \infty$.

For the Cauchy property, write the difference as a sum over the same factorization. The finite set \mathcal{Z}_R contributes a term that tends to 0 in $L^1(I)$ as $\varepsilon, \varepsilon' \downarrow 0$ by dominated convergence away from the finitely many points $t = \Im \rho$ and the local integrability of $\log |t - \Im \rho|$. The tail is uniformly $O_I\left(\sum_{\rho \notin \mathcal{Z}_R} |\rho|^{-2}\right)$ and hence uniformly small; letting $R \rightarrow \infty$ yields the $L^1(I)$ -Cauchy claim. \square

A.3. Inner reciprocal and energy estimates. The key device is the *inner reciprocal* $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$, which converts poles of \mathcal{J}_{out} (at ζ -zeros) into zeros, yielding an inner function whose nonnegative potential provides the energy estimates needed in the proof of Theorem 1.

Lemma 25 (Inner reciprocal and nonnegative potential). *Let \mathcal{J}_{out} be as in (3.1) and $B(s) := (s-1)/s$. Define*

$$\mathcal{I}(s) := \frac{B(s)^2}{\mathcal{J}_{\text{out}}(s)} = \frac{B(s) \mathcal{O}_\zeta(s) \zeta(s)}{\det_2(I - A(s))}.$$

Then:

- (1) \mathcal{I} is holomorphic on Ω . (The simple pole of ζ at $s=1$ is canceled by B ; zeros of ζ become zeros of \mathcal{I} ; the denominator $\det_2(I - A)$ is nonvanishing on Ω .)
- (2) $|\mathcal{I}(\frac{1}{2} + it)| = 1$ for Lebesgue-a.e. t . (On $\partial\Omega$: $|B| = 1$ and $|\mathcal{J}_{\text{out}}| = 1$ a.e.)
- (3) $|\mathcal{I}(s)| \leq 1$ for all $s \in \Omega$. (Phragmén–Lindelöf: $\log |\mathcal{I}|$ is subharmonic on Ω with boundary trace 0 a.e. and at most polynomial growth; see below.)

In particular, the function

$$W(s) := -\log |\mathcal{I}(s)| \geq 0 \quad (s \in \Omega)$$

is nonnegative, and one has the identity

$$U(s) := \log |\mathcal{J}_{\text{out}}(s)| = 2\log |B(s)| + W(s) \quad (s \in \Omega \setminus Z(\zeta)).$$

Proof. Part (1). Write $\mathcal{I} = B \mathcal{O}_\zeta \zeta / \det_2(I - A)$. The factor $B\zeta = (s-1)\zeta(s)/s$ is holomorphic on Ω (the simple pole of ζ at $s=1$ is canceled by the zero of $s-1$, and $s=0 \notin \Omega$). The remaining factors \mathcal{O}_ζ (outer, zero-free) and $1/\det_2(I - A)$ (nonvanishing by (2.1)) are holomorphic on Ω . Hence \mathcal{I} is holomorphic on Ω , with zeros exactly at the nontrivial zeros of ζ in Ω (same multiplicities).

Part (2). On $\partial\Omega$: $|B(\frac{1}{2} + it)|^2 = |(-\frac{1}{2} + it)/(\frac{1}{2} + it)|^2 = (\frac{1}{4} + t^2)/(\frac{1}{4} + t^2) = 1$, and $|\mathcal{J}_{\text{out}}(\frac{1}{2} + it)| = 1$ a.e. by construction. Hence $|\mathcal{I}(\frac{1}{2} + it)| = |B|^2/|\mathcal{J}_{\text{out}}| = 1$ a.e.

Part (3): $|\mathcal{I}| \leq 1$ via Phragmén–Lindelöf. Since \mathcal{I} is holomorphic on Ω , $u := \log |\mathcal{I}|$ is subharmonic on Ω .

Boundary trace. For $\varepsilon > 0$ set $s_\varepsilon := \frac{1}{2} + \varepsilon + it$. Each factor of $\mathcal{I} = B \mathcal{O}_\zeta \zeta / \det_2(I - A)$ has L^1_{loc} -convergent log-modulus as $\varepsilon \downarrow 0$:

- $\log |B(s_\varepsilon)| \rightarrow 0$ uniformly (B is continuous and $|B^*| = 1$);
- $\log |\mathcal{O}_\zeta(s_\varepsilon)| \rightarrow u(t)$ in L^1_{loc} (\mathcal{O}_ζ is the Poisson extension of $u := \log |F^*|$);
- $\log |\zeta(s_\varepsilon)| \rightarrow \log |\zeta^*(t)|$ in L^1_{loc} (Lemma 11 or 24);
- $\log |\det_2(s_\varepsilon)| \rightarrow \log |\det_2^*(t)|$ in L^1_{loc} (BMO boundary trace from the arithmetic Carleson energy, Lemma 23).

Since $u = \log |\det_2^*| - \log |\zeta^*|$ by construction of \mathcal{O}_ζ , the sum of boundary traces is $0 + u + \log |\zeta^*| - \log |\det_2^*| = 0$. Hence $u^*(\frac{1}{2} + it) = \log |\mathcal{I}^*(t)| = 0$ for a.e. t . No Smirnov or Hardy class membership is invoked; only the L^1_{loc} convergence of each factor's log-modulus is needed.

Growth. $|\mathcal{I}(s)| \leq C(1 + |t|)^N$ for some N and all $s = \frac{1}{2} + \sigma + it$ with $\sigma \in (0, 1]$ (this follows from the convexity bound for ζ , the absolutely convergent product for \det_2 , and the Poisson-controlled modulus of \mathcal{O}_ζ). Hence $u(s) = O(\log(2 + |s|)) = o(|s|)$ as $|s| \rightarrow \infty$ in Ω .

Conclusion. By the Phragmén–Lindelöf principle for subharmonic functions on the half-plane (e.g. [7, Ch. III] or [9, Thm. 5.3.4]): a subharmonic function on Ω with nontangential boundary trace ≤ 0 a.e. and growth $o(|s|)$ satisfies $u \leq 0$ on Ω . Hence $|\mathcal{I}| \leq 1$ and $W = -\log |\mathcal{I}| \geq 0$. \square

The inner reciprocal \mathcal{I} in hand, we turn to the energy estimates that drive the contradiction. The following proposition is the quantitative heart of the appendix.

Proposition 26 (Neutralized box-energy bound on Whitney scales). *Let $W = -\log |\mathcal{I}| \geq 0$ be the nonnegative potential from Lemma 25, and let $\widetilde{W} := -\log |B_{\text{far}} \cdot S|$ be the neutralized harmonic field obtained by factoring out the near Blaschke product (see Step 1 below). For each Whitney interval $I = [t_0 - L, t_0 + L]$ with $L = c/\log\langle t_0 \rangle$ and aperture $\alpha' > 1$, define the neutralized box energy*

$$E_{\text{neut}}(I) := \iint_{Q(\alpha' I)} |\nabla(2\log |B| + \widetilde{W})|^2 \sigma \, dt \, d\sigma.$$

(This is the energy of $\log |\mathcal{J}_{\text{neut}}|$, the harmonic function on D from the main theorem proof; it does not include the infinite-energy near-Blaschke singularities.) Then

$$(A.2) \quad E_{\text{neut}}(I) \leq C(\alpha') \log^2\langle t_0 \rangle |I|,$$

where $C(\alpha')$ depends only on the apertures (α', α'') , the RvM density constant, and the convexity exponent—not on c .

In particular, the windowed-phase product satisfies

$$(A.3) \quad \sqrt{E_{\text{neut}}(I)} \cdot L \leq \sqrt{C(\alpha')} \frac{c^{3/2}}{\sqrt{\log\langle t_0 \rangle}},$$

which tends to 0 as $c \rightarrow 0$, uniformly in t_0 .

Remark 27. The $\log^2\langle t_0 \rangle$ growth is not an obstruction to the main theorem: in the proof of Theorem 3, the Whitney parameter is chosen as $c = c_0/\log\langle \gamma_0 \rangle$ (depending on the height of the hypothetical zero), which causes $\log^2\langle t_0 \rangle \cdot |I|$ to collapse to a height-independent constant $2Cc_0$ (see ??). Replacing \log^2 with a uniform constant would allow a fixed c and simplify the argument, but is not logically required.

Proof. Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ with $L = c/\log\langle t_0 \rangle$ and $\alpha' > 1$. Choose a slightly larger aperture parameter $\alpha'' > 2\alpha'$, and let $D := Q(\alpha'' I)$ (a dilated Whitney box).

Since $U = 2\log |B| + W$ and $B = (s-1)/s$ is explicit and smooth on D (for t_0 large, the clip $L \leq L_*$ keeps D away from $s = 1$), $\nabla(2\log |B|)$ contributes $O_{\alpha'}(|I|)$ to the weighted energy. It therefore suffices to bound the W -energy:

$$E_W(I) := \iint_{Q(\alpha' I)} |\nabla W|^2 \sigma \, dt \, d\sigma.$$

Step 1 (Whitney neutralization). By Lemma 25, \mathcal{I} is an inner function on Ω with zeros exactly at the nontrivial zeros of ζ in Ω . Factor $\mathcal{I} = e^{i\theta} B_{\text{near}} B_{\text{far}} S$, where B_{near} is the finite Blaschke product over zeros $\rho = \beta + i\gamma$ of \mathcal{I} with $|\gamma - t_0| \leq \alpha'' L$, B_{far} is the Blaschke product of the remaining zeros, and S is the (possibly trivial) singular inner factor. By (A.1), B_{near} has at most $C_{\text{RvM}}(1 + 2\alpha'' L) \log\langle t_0 \rangle = O(\log\langle t_0 \rangle)$ factors. (On Whitney scale the count is $O(\log\langle t_0 \rangle)$, not $O(1)$; but see Step 2—the near-zero charges do not enter the Cauchy–Schwarz energy bound.)

Define the neutralized field

$$\widetilde{W}(s) := W(s) + \log |B_{\text{near}}(s)| = -\log |B_{\text{far}}(s)| - \log |S(s)|.$$

Every term on the right is ≥ 0 (each inner factor has modulus ≤ 1 on Ω), so $\widetilde{W} \geq 0$ on Ω . On $\partial\Omega$ ($\sigma = 0$): all inner factors have boundary modulus 1, so $\widetilde{W} = 0$. Moreover, \widetilde{W} is *harmonic* on D : the zeros in B_{far} have $|\gamma - t_0| > \alpha'' L$, hence lie outside the t -span of D , and S is zero-free.

The zeros in B_{near} lie *inside* the box D , so $\log |B_{\text{near}}|$ has logarithmic singularities there and its weighted Dirichlet energy on $Q(\alpha' I)$ is infinite. This is not a problem: the near Blaschke factors are absorbed into the neutralization step in the main theorem proof (see the neutralization step in §4),

where they cancel the poles of \mathcal{J}_{out} and produce the harmonic function $\log |\mathcal{J}_{\text{neut}}| = 2 \log |B| + \widetilde{W}$ on D . The energy estimate below bounds the harmonic field \widetilde{W} only.

Step 2 (boundary bound on ∂D). Since $\widetilde{W} \geq 0$ and $\widetilde{W} = 0$ on $\sigma = 0$, it remains to bound \widetilde{W} on the top/side edges of D .

Each far zero $\rho = \beta + i\gamma$ with $\delta := \beta - \frac{1}{2} \in (0, \frac{1}{2}]$ contributes

$$-\log |b_\rho(s)| = G_\Omega(s, \rho) = \frac{1}{2} \log \frac{(\sigma + \delta)^2 + (t - \gamma)^2}{(\sigma - \delta)^2 + (t - \gamma)^2} \leq \frac{2\sigma\delta}{(\sigma - \delta)^2 + (t - \gamma)^2} \leq \frac{\alpha' L}{(t - \gamma)^2}$$

(using $\log(1+x) \leq x$, $\sigma \leq \alpha' L$, $\delta \leq \frac{1}{2}$, and $|t - \gamma| \geq (\alpha'' - \alpha')L \gg \sigma$). Summing over all far zeros and using the zero density (A.1) (at most $C_{\text{RvM}}(1+R) \log \langle t_0 \rangle$ zeros with $|\gamma - t_0| \leq R$):

$$\sum_{\text{far } \rho} G_\Omega(s, \rho) \leq \alpha' L \int_{\alpha'' L}^{\infty} \frac{C_{\text{RvM}} \log \langle t_0 \rangle}{r^2} dr = \frac{\alpha' C_{\text{RvM}} \log \langle t_0 \rangle}{\alpha''} \ll \log \langle t_0 \rangle$$

on ∂D (with the implied constant depending only on α', α'').

Key independence of L and c . The integral $\alpha' L \cdot C_{\text{RvM}} \log \langle t_0 \rangle / (\alpha'' L) = \alpha' C_{\text{RvM}} \log \langle t_0 \rangle / \alpha''$: the L in the numerator ($\sigma \leq \alpha' L$) cancels the L in the denominator ($\int_{\alpha'' L}^{\infty} 1/r^2 dr = 1/(\alpha'' L)$). The Blaschke tail bound does not depend on L or c , and does not require short-interval zero control at scale L —only the coarse $O(\log \langle t_0 \rangle)$ count per unit ordinate interval.

Singular inner contribution and the $S \equiv 1$ condition. The singular inner factor S of \mathcal{I} contributes $-\log |S(s)| = P_\sigma[\nu_S](t)$, the Poisson integral of a positive singular measure ν_S on $\partial\Omega$. At $\Re s = \frac{3}{2}$: $P_1[\nu_S](t) \leq W(\frac{3}{2} + it) \leq C_0$ (bounded), so ν_S has uniformly bounded mass per unit interval: $\nu_S([t_0 - 1, t_0 + 1]) \leq 2\pi C_0 =: \nu_*$.

On ∂D at height $\sigma = \alpha'' L$: the near singular mass ($|\tau - t_0| \leq 1$) contributes at most $\nu_*/(\pi \alpha'' L) = \nu_* \log \langle t_0 \rangle / (\pi \alpha'' c)$. If $S \equiv 1$ (i.e. $\nu_S = 0$), this vanishes and

$$M := \sup_{\partial D} \widetilde{W} \leq \frac{\alpha' C_{\text{RvM}}}{\alpha''} \log \langle t_0 \rangle =: C_* \log \langle t_0 \rangle,$$

with C_* depending only on $(\alpha', \alpha'', C_{\text{RvM}})$ —not on c . In this case the energy bound closes unconditionally (see the remark below).

If $S \not\equiv 1$: the near singular Poisson spike contributes $O(\log \langle t_0 \rangle / c)$ to M , which with $c = c_0 / \log \langle t_0 \rangle$ becomes $O(\log^2 / c_0)$ and introduces one extra power of \log that the cancellation trick does not absorb. Proving $S \equiv 1$ for the specific inner function $\mathcal{I} = B \mathcal{O}_\zeta \zeta / \det_2$ would therefore complete the unconditional proof; this is recorded as an open step below.

Proof that $S \equiv 1$. The singular inner factor satisfies $S \equiv 1$ if and only if

$$\lim_{\sigma \rightarrow 0^+} \int_{\mathbb{R}} \frac{W(\frac{1}{2} + \sigma + it)}{1 + t^2} dt = 0$$

(see Garnett [5, Ch. II]). We prove this by showing that each factor of $\mathcal{I} = B \mathcal{O}_\zeta \zeta / \det_2$ has log-modulus converging in $L^1(\mathbb{R}, dt/(1+t^2))$ as $\sigma \rightarrow 0$, and that the boundary traces sum to 0.

Term $\log |B|$. $B = (s-1)/s$ is continuous with $|B^*| = 1$; convergence is uniform.

Term $\log |\mathcal{O}_\zeta|$. \mathcal{O}_ζ is the outer function with boundary modulus $\exp(u)$, so $\log |\mathcal{O}_\zeta(\sigma)| = P_\sigma[u] \rightarrow u$ in $L^1(dt/(1+t^2))$ by Poisson convergence.

Term $\log |\det_2|$. By explicit Fourier computation,

$$\int_{\mathbb{R}} \frac{\log |\det_2(\sigma, t)|}{1 + t^2} dt = -\pi \sum_p \sum_{k \geq 2} \frac{p^{-k(\frac{3}{2} + \sigma)}}{k},$$

which converges absolutely to $-\pi \sum_p \sum_{k \geq 2} p^{-3k/2}/k$ as $\sigma \rightarrow 0$.

Term $\log |\zeta|$ (the key term). We must show $\int \log |\zeta(\frac{1}{2} + \sigma + it)|/(1+t^2) dt \rightarrow \int \log |\zeta^*(t)|/(1+t^2) dt$ as $\sigma \rightarrow 0$.

(a) *The \log^+ part.* $\log^+ |\zeta(\frac{1}{2} + \sigma + it)| \leq A \log(2 + |t|)$ uniformly for $\sigma \in (0, 1]$ (convexity bound; Titchmarsh [14, Ch. V]). Since $A \log(2 + |t|)/(1 + t^2) \in L^1$, dominated convergence applies.

(b) *The \log^- part.* Cover \mathbb{R} by unit intervals $I_n = [n, n + 1]$. On each I_n , Jensen's inequality for the subharmonic function $\log |\zeta(\frac{1}{2} + \sigma + i \cdot)|$ on a disc of radius 2 centered at $n + \frac{1}{2} + i\sigma$ gives

$$\int_{I_n} \log^- |\zeta(\frac{1}{2} + \sigma + it)| dt \leq \pi \cdot 4 \cdot (A \log(3 + |n|) + C) + \pi \cdot 4 \cdot N_n \cdot \log 4,$$

where N_n is the number of ζ -zeros with $|\gamma - (n + \frac{1}{2})| \leq 4$ and the right side comes from the standard Jensen bound ($\int \log^- |f| \leq \text{mean of } \log^+ |f|$ on a larger circle + zero count $\cdot \log(\text{ratio})$). By (A.1): $N_n \leq C_1(1 + 4) \log \langle n \rangle = O(\log \langle n \rangle)$. Hence

$$\int_{I_n} \log^- |\zeta(\sigma, t)| dt \leq C_2 \log(2 + |n|) \quad \text{uniformly for } \sigma \in (0, 1].$$

Dividing by $1 + t^2 \geq 1 + n^2$ and summing: $\int_{\mathbb{R}} \log^- |\zeta(\sigma)|/(1 + t^2) \leq \sum_n C_2 \log(2 + |n|)/(1 + n^2) < \infty$. This bound is uniform in σ .

(c) *Convergence.* L^1_{loc} convergence $\log |\zeta(\sigma)| \rightarrow \log |\zeta^*|$ holds by Lemma 24. Combined with the σ -uniform $L^1(dt/(1 + t^2))$ bound from (a) and (b), Vitali's convergence theorem gives $\int \log |\zeta(\sigma)|/(1 + t^2) \rightarrow \int \log |\zeta^*|/(1 + t^2)$.

Assembly. By the construction of \mathcal{O}_ζ : $u = \log |\det_2^*| - \log |\zeta^*|$, so the boundary traces satisfy $0 + u + \log |\zeta^*| - \log |\det_2^*| = 0$. Hence

$$\lim_{\sigma \rightarrow 0} \int \frac{W(\sigma, t)}{1 + t^2} dt = 0 - (-u) - (-\log |\zeta^*|) + (-\log |\det_2^*|) = 0.$$

Therefore $S \equiv 1$. (This argument uses only: the convexity bound for ζ , the convergence of $\sum 1/(1 + \gamma^2)$, the outer construction of \mathcal{O}_ζ , and the explicit Fourier series for \det_2 . No zero-free hypothesis is used.)

Hence

$$M := \sup_{\partial D} \widetilde{W} \leq C_* \log \langle t_0 \rangle,$$

with C_* independent of c .

Step 3 (interior gradient estimate). Since \widetilde{W} is harmonic on D with $0 \leq \widetilde{W} \leq M$ and $\widetilde{W} = 0$ on $\sigma = 0$, the standard interior estimate (odd reflection + Cauchy) gives $\sup_{Q(\alpha'I)} |\nabla \widetilde{W}|^2 \leq C_2 M^2 / L^2$. Integrating with the weight σ :

$$\iint_{Q(\alpha'I)} |\nabla \widetilde{W}|^2 \sigma \leq C_3 M^2 |I| \leq C_3 C_*^2 \log^2 \langle t_0 \rangle |I|.$$

Step 4 (assembly). The energy of the neutralized harmonic function \widetilde{W} on $Q(\alpha'I)$ controls the smooth part of the boundary phase derivative via the CR–Green pairing (Lemma 32). The $O(\log \langle t_0 \rangle)$ zeros of \mathcal{I} inside D contribute nonnegative charges to the total windowed phase via the distributional Green identity, but they do not enter the Cauchy–Schwarz energy bound. A hypothetical zero ρ_0 with $\delta_0 \geq \varepsilon > \alpha' L$ lies outside D , so its Poisson contribution enters the smooth part and produces the lower bound in Theorem 1.

The effective energy bound is therefore

$$E_{\text{eff}}(I) := \iint_{Q(\alpha'I)} |\nabla \widetilde{W}|^2 \sigma \leq C \log^2 \langle t_0 \rangle |I|,$$

where $C = C_3 C_*^2$ depends only on (α', α'') and is independent of c . With $c = c_0 / \log \langle t_0 \rangle$ in the main theorem, $E_{\text{eff}} = 2C c_0$, independent of height. \square

A.4. CR–Green pairing lemmas. The final set of tools converts boundary phase integrals into Dirichlet-energy estimates via Green’s theorem on Whitney boxes.

Definition 28 (Admissible window class with atom avoidance). Fix an even C^∞ window ψ with $\psi \equiv 1$ on $[-1, 1]$ and $\text{supp } \psi \subset [-2, 2]$. For an interval $I = [t_0 - L, t_0 + L]$, an aperture $\alpha' > 1$, and a parameter $\varepsilon \in (0, \frac{1}{4}]$, define $\mathcal{W}_{\text{adm}}(I; \varepsilon)$ to be the set of C^∞ , nonnegative, mass-1 bumps ϕ supported in the fixed dilate $2I = [t_0 - 2L, t_0 + 2L]$ that can be written as

$$\phi(t) = \frac{1}{Z} \frac{1}{L} \psi\left(\frac{t - t_0}{L}\right) m(t), \quad Z = \int_{2I} \frac{1}{L} \psi\left(\frac{t - t_0}{L}\right) m(t) dt,$$

where $2I := [t_0 - 2L, t_0 + 2L]$ and the mask $m \in C^\infty(2I; [0, 1])$ satisfies:

- (i) *Atom avoidance.* There is a union of disjoint open subintervals $E = \bigcup_{j=1}^J J_j \subset I$ with total length $|E| \leq \varepsilon L$ such that $m \equiv 0$ on E and $m \equiv 1$ on $I \setminus E'$, where each transition layer $E' \setminus E$ has thickness $\leq \varepsilon L$.
- (ii) *Uniform smoothness.* $\|m'\|_\infty \lesssim (\varepsilon L)^{-1}$ and $\|m''\|_\infty \lesssim (\varepsilon L)^{-2}$ with implicit constants independent of I, t_0, L and of the number/placement of the holes $\{J_j\}$.

Every $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$ is supported in $2I$. This class contains the unmasked profile $\varphi_{L, t_0}(t) = Z_0^{-1} L^{-1} \psi((t - t_0)/L)$ with $Z_0 := \int_{-2}^2 \psi(x) dx$ (take $E = \emptyset$, $m \equiv 1$) and also allows dodging boundary atoms by punching out small neighborhoods while keeping total deleted length $\leq \varepsilon L$.

Lemma 29 (Poisson–energy bound for admissible tests). *Let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$ to the half-plane, and fix a cutoff to $Q(\alpha' I)$ with $\alpha' > 1$ as in the CR–Green pairing. Then*

$$(A.4) \quad \iint_{Q(\alpha' I)} |\nabla V_\phi(\sigma, t)|^2 \sigma dt d\sigma \leq \frac{1}{2} \|\phi\|_{L^2}^2 = \frac{\mathcal{A}_{\text{adm}}(\psi, \varepsilon)^2}{L},$$

where $\mathcal{A}_{\text{adm}}(\psi, \varepsilon)^2 := \|\psi m\|_{L^2}^2 / (2Z^2)$ depends only on (ψ, ε) , not on L . In particular, the energy scales as $1/L$, not L .

Proof. On the full half-plane the σ -weighted Dirichlet energy of a Poisson extension satisfies the identity

$$\iint_{\mathbb{R}_+^2} |\nabla V_\phi|^2 \sigma d\sigma dt = \frac{1}{2} \|\phi\|_{L^2(\mathbb{R})}^2,$$

which follows from Plancherel and the explicit formula $\int_0^\infty \sigma \omega^2 e^{-2\omega\sigma} d\sigma = \frac{1}{4}$ applied mode-by-mode. Since the box $Q(\alpha' I) \subset \mathbb{R}_+^2$, restriction to $Q(\alpha' I)$ can only decrease the integral. For the unmasked profile $\phi(t) = (Z_0 L)^{-1} \psi((t - t_0)/L)$:

$$\|\phi\|_{L^2}^2 = \frac{1}{Z_0^2 L^2} \int_{\mathbb{R}} |\psi((t - t_0)/L)|^2 dt = \frac{\|\psi\|_{L^2}^2}{Z_0^2 L}.$$

Hence the energy is $\|\psi\|_{L^2}^2 / (2Z_0^2 L) = O(1/L)$. □

Remark 30 (Scaling error and its impact). An earlier version of this paper claimed the energy in (A.4) was $\lesssim L$, which would make the constant C_{test} in Proposition 33 independent of L . The correct scaling $1/L$ causes C_{test} to blow up like $1/\sqrt{L}$. When this corrected bound is propagated through the CR–Green pathway (Appendix A.4) (see Remark ?? below), the upper bound on the windowed phase becomes proportional to \sqrt{L} rather than to L , and the contradiction mechanism in Step 3 fails. See §4 for details.

The next two lemmas implement the Cauchy–Riemann/Green pairing that converts the boundary phase integral into a box energy.

Lemma 31 (Cutoff pairing on boxes). *Fix parameters $\alpha' > \alpha > 1$. Let $\chi_{L,t_0} \in C_c^\infty(\mathbb{R}_+^2)$ satisfy $\chi \equiv 1$ on $Q(\alpha I)$, $\text{supp } \chi \subset Q(\alpha' I)$, $\|\nabla \chi\|_\infty \lesssim L^{-1}$ and $\|\nabla^2 \chi\|_\infty \lesssim L^{-2}$. Let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$. Then one has the Green pairing identity*

$$\int_{\mathbb{R}} u(t) \phi(t) dt = \iint_{Q(\alpha' I)} \nabla U \cdot \nabla(\chi_{L,t_0} V_\phi) dt d\sigma + \mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}},$$

with remainders satisfying

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \lesssim \left(\iint_{Q(\alpha' I)} |\nabla U|^2 \sigma \right)^{1/2} \cdot \left(\iint_{Q(\alpha' I)} (|\nabla \chi|^2 |V_\phi|^2 + |\nabla V_\phi|^2) \sigma \right)^{1/2}.$$

Proof. Let $Q := Q(\alpha' I)$. Assume U is C^2 on \overline{Q} and harmonic on Q , with boundary trace $u(t) = U(0, t)$ on the bottom edge $\{\sigma = 0\}$. Since $\chi_{L,t_0} V_\phi$ is compactly supported in \overline{Q} and smooth on Q , Green's identity gives

$$\iint_Q \nabla U \cdot \nabla(\chi V_\phi) dt d\sigma = \int_{\partial Q} (\chi V_\phi) \partial_n U ds - \iint_Q (\chi V_\phi) \Delta U dt d\sigma.$$

Since $\Delta U = 0$ on Q , only the boundary integral remains. On the bottom edge one has $\partial_n = -\partial_\sigma$, $\chi \equiv 1$, and $V_\phi(0, t) = \phi(t)$, hence that contribution equals

$$\int_I \phi(t) (-\partial_\sigma U)(0, t) dt.$$

If U is the real part of a holomorphic logarithm $U = \Re \log J$ with $|J(\frac{1}{2} + it)| = 1$ a.e., then $U(0, t) = 0$ a.e. and $-\partial_\sigma U(0, t) = \partial_t \text{Arg } J(\frac{1}{2} + it)$ in distributions by Cauchy–Riemann; in particular, this term is the tested boundary phase derivative in Lemma 32 below. The remaining boundary pieces (two vertical sides and the top edge) are, by definition, the remainders $\mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}}$.

For the remainder estimate, we apply Cauchy–Schwarz in the scale-invariant measure $\sigma dt d\sigma$ on Q :

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \lesssim \left(\iint_Q |\nabla U|^2 \sigma \right)^{1/2} \left(\iint_Q |\nabla(\chi V_\phi)|^2 \sigma \right)^{1/2}.$$

Expanding $\nabla(\chi V_\phi) = \chi \nabla V_\phi + (\nabla \chi) V_\phi$ yields

$$\iint_Q |\nabla(\chi V_\phi)|^2 \sigma \lesssim \iint_Q (|\nabla V_\phi|^2 + |\nabla \chi|^2 |V_\phi|^2) \sigma,$$

which gives the displayed estimate. \square

Lemma 32 (CR–Green pairing for boundary phase). *Let J be analytic on Ω with a.e. boundary modulus $|J(\frac{1}{2} + it)| = 1$, and write $\log J = U + iW$ on Ω , so U is harmonic with $U(\frac{1}{2} + it) = 0$ a.e. Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ and let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$. Then, with a cutoff χ_{L,t_0} as in Lemma 31,*

$$\int_{\mathbb{R}} \phi(t) (-W'(t)) dt = \iint_{Q(\alpha' I)} \nabla U \cdot \nabla(\chi_{L,t_0} V_\phi) dt d\sigma + \mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}},$$

and the remainders satisfy the same estimate as in Lemma 31. In particular, by Cauchy–Schwarz and Lemma 29,

$$\int_{\mathbb{R}} \phi(t) (-w'(t)) dt \leq \frac{C_{\text{rem}}(\alpha', \psi)}{\sqrt{L}} \left(\iint_{Q(\alpha' I)} |\nabla U|^2 \sigma \right)^{1/2},$$

where C_{rem} depends only on (α', ψ) . (The factor $1/\sqrt{L}$ arises from $\mathcal{A}_{\text{adm}} \sim 1/\sqrt{L}$ in Lemma 29; see Remark 30.)

Proof. On the bottom edge $\{\sigma = 0\}$ the outward normal is $\partial_n = -\partial_\sigma$. By Cauchy–Riemann for $\log J = U + iW$ on the boundary line $\{\Re s = \frac{1}{2}\}$ one has $\partial_n U = -\partial_\sigma U = \partial_t W$. Thus the bottom-edge term in Green’s identity is

$$-\int_{\partial Q \cap \{\sigma=0\}} \chi V_\phi \partial_n U \, dt = -\int_{\mathbb{R}} \phi(t) \partial_t W(t) \, dt = \int_{\mathbb{R}} \phi(t) (-w'(t)) \, dt,$$

which yields the stated identity after including the interior term and remainders. The final inequality is Cauchy–Schwarz together with the uniform Poisson-energy bound from Lemma 29. \square

Proposition 33 (Upper bound for admissible tests). *Let J be holomorphic on $\Omega \setminus Z(\zeta)$ with a.e. boundary modulus 1, write $\log J = U + iW$ on $\Omega \setminus Z(\zeta)$, and let $-w'$ denote the boundary phase distribution. For every interval $I = [t_0 - L, t_0 + L]$, every $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$, and every fixed cutoff to $Q(\alpha' I)$,*

$$(A.5) \quad \int_{\mathbb{R}} \phi(t) (-w')(t) \, dt \leq \frac{C_{\text{test}}(\psi, \varepsilon, \alpha')}{\sqrt{L}} \left(\iint_{Q(\alpha' I)} |\nabla U|^2 \sigma \, dt \, d\sigma \right)^{1/2}$$

with $C_{\text{test}}(\psi, \varepsilon, \alpha') := C_{\text{rem}}(\alpha', \psi) \mathcal{A}_{\text{adm}}(\psi, \varepsilon)$ independent of I, t_0, L . The factor $1/\sqrt{L}$ is forced by the correct Poisson-energy scaling (Lemma 29, Remark 30).

Proof. Apply Lemma 32 with $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$. The window-side Cauchy–Schwarz factor is $(\iint |\nabla(\chi V_\phi)|^2 \sigma)^{1/2} \lesssim \mathcal{A}_{\text{adm}}/\sqrt{L}$ by Lemma 29. \square

REFERENCES

- [1] A. Beurling, On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81** (1949), 239–255.
- [2] J. B. Conrey, The Riemann hypothesis, *Notices Amer. Math. Soc.* **50** (2003), no. 3, 341–353.
- [3] P. L. Duren, *Theory of H^p Spaces*, Academic Press, 1970.
- [4] H. M. Edwards, *Riemann’s Zeta Function*, Academic Press, 1974; reprinted by Dover, 2001.
- [5] J. B. Garnett, *Bounded Analytic Functions*, Graduate Texts in Mathematics, vol. 236, Springer, 2007.
- [6] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications, vol. 53, American Mathematical Society, 2004.
- [7] P. Koosis, *The Logarithmic Integral I*, Cambridge Studies in Advanced Mathematics, vol. 12, Cambridge University Press, 1988.
- [8] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, 2007.
- [9] T. Ransford, *Potential Theory in the Complex Plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, 1995.
- [10] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford University Press, 1985.
- [11] A. Selberg, On the zeros of Riemann’s zeta-function, *Skr. Norske Vid.-Akad. Oslo I* (1942), no. 10, 1–59.
- [12] B. Simon, *Trace Ideals and Their Applications*, 2nd ed., Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, 2005.
- [13] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.
- [14] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, 1986.

RECOGNITION PHYSICS RESEARCH INSTITUTE, AUSTIN, TX, USA
Email address: jon@recognitionphysics.org

RECOGNITION PHYSICS RESEARCH INSTITUTE, AUSTIN, TX, USA
Email address: arahnamab@gmail.com