# The Grammar of Possibility: A Cost-Theoretic Foundation for Modal Logic

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**Abstract**

We present a novel foundation for modal logic grounded in cost minimization rather than abstract possible-worlds semantics. From three minimal axioms—composition under multiplication, normalization at identity, and unit curvature—we prove that a unique cost function $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ is forced. We define modal operators where possibility means finite-cost reachability and necessity means cost-forced inevitability. Our central result is the *Stasis-Change Theorem*: for any configuration $x \neq 1$, there exists a successor $y$ with $J_{\text{change}}(x, y) < J_{\text{stasis}}(x)$, proving that dynamics are favored over stasis. We demonstrate applications to machine learning (principled loss functions), AI agents (bounded rationality), economics (quantified opportunity cost), quantum foundations (measurement without observers), thermodynamics (generalized second law), and cognitive science (attention as non-equilibrium maintenance). All core results are machine-verified in Lean 4.

**Keywords:** Modal logic, cost functional, possibility, necessity, counterfactuals, dynamics

# 1 Introduction

Why does anything happen? Classical modal logic, from Leibniz through Kripke [1], provides a formal language for discussing necessity and possibility, but its semantics are abstract: possible worlds are stipulated, accessibility relations are free parameters.

We propose *modal logic grounded in cost minimization.* In this framework—the **Grammar of Possibility**—the modal operators $\Box$ and $\Diamond$ emerge from a single cost functional $J$.

## 1.1 Central Claim

**Master Principle**: Change is favored because stasis is expensive.

For any configuration $x \neq 1$, there exists $y$ such that evolving to $y$ costs less than remaining at $x$.

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### 1.2 Related Work

- **Kripke semantics** [1]: Worlds and accessibility are primitive; we derive them.
- **Lewis's counterfactuals** [2]: Closeness is primitive; we ground it in cost.
- **Stalnaker's selection** [3]: Selection is primitive; we derive it.
- **Information geometry** [4]: Fisher information as metric; related structure.
- **Free energy principle** [5]: Biological systems minimize surprise; analogous cost-minimization.

### 1.3 Contributions

1. Unique cost functional from three axioms (§2)
2. Modal operators with physical grounding (§3)
3. Stasis-Change Theorem (§4)
4. Physical counterfactuals (§5)
5. Machine verification in Lean 4 (§8)

### 1.4 Scope and Limitations

We acknowledge:

- The 8-tick period is imported from Recognition Science [8]; not derived here.
- Connections to quantum mechanics are formal analogies, not complete derivations.
- Experimental predictions require further development.

## 2 The Cost Functional

### 2.1 Motivation

Any dynamics requires comparing alternatives. We seek a cost functional $F : \mathbb{R}_{>0} \to \mathbb{R}$ on configuration ratios satisfying natural constraints.

### 2.2 Three Axioms

**Axiom 2.1** (Composition)**.** For all $x, y > 0$:

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y) \tag{1}$$

**Motivation**: This is the unique compositional structure compatible with $(\mathbb{R}_{>0}, \times)$. Consider successive changes $x$ then $y$: the net result $xy$ and reverse $x/y$ together determine the total cost as a quadratic form.

**Axiom 2.2** (Normalization)**.** $F(1) = 0$: identity has zero cost.

**Axiom 2.3** (Calibration)**.** $F''(1) = 1$: unit curvature at minimum.

## 2.3 Uniqueness

**Theorem 2.4** (Cost Uniqueness)**.** *The unique function satisfying Axioms 2.1–2.3 is:*

$$J(x) = \frac{1}{2}\left(x + \frac{1}{x}\right) - 1 = \cosh(\ln x) - 1 \tag{2}$$

*Proof.* Substitute $x = e^t$, $y = e^u$, define $G(t) := F(e^t)$. The composition law becomes $G(t+u) + G(t-u) = 2G(t)G(u) + 2G(t) + 2G(u)$. Setting $H(t) := G(t) + 1$ yields $H(t+u) + H(t-u) = 2H(t)H(u)$, d'Alembert's equation. Continuous even solutions: $H(t) = \cosh(\lambda t)$. Conditions $G(0) = 0$, $G''(0) = 1$ force $\lambda = 1$. □

## 2.4 Properties

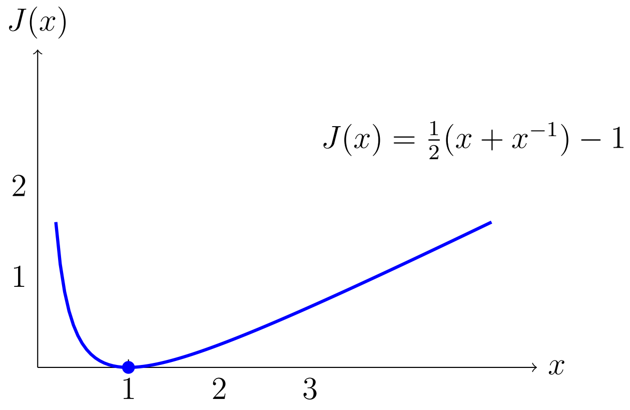


Figure 1: The cost functional $J(x)$: minimum at $x = 1$, diverging as $x \to 0^+$ or $x \to \infty$.

**Lemma 2.5** (Fundamental Properties)**.**

1. ***Non-negativity****:* $J(x) \geq 0$ *for all* $x > 0$.
2. ***Unique zero****:* $J(x) = 0$ *iff* $x = 1$.
3. ***Symmetry****:* $J(x) = J(x^{-1})$.
4. ***Divergence****:* $J(x) \to \infty$ *as* $x \to 0^+$ *or* $x \to \infty$.
5. ***Derivatives****:* $J'(x) = \frac{1}{2}(1 - x^{-2})$; $J''(x) = x^{-3}$.

*Proof.* (1)–(2): By AM-GM, $(x + x^{-1})/2 \geq 1$, with equality iff $x = 1$. (3): Direct substitution. (4): As $x \to 0^+$, $x^{-1} \to \infty$; as $x \to \infty$, $x \to \infty$. (5): Differentiate $J(x) = \frac{1}{2}(x + x^{-1}) - 1$. □

**Theorem 2.6** (Nothing Costs Infinity)**.** $\lim_{x \to 0^+} J(x) = +\infty$.

This captures the meta-principle: "nothingness" is unreachable.

# 3 Modal Operators

## 3.1 Configuration Space

**Definition 3.1** (Configuration)**.** A configuration is $c = (v, t)$ with value $v > 0$ and time $t \in \mathbb{N}$.

We write $c_v$ for the value component and $c_t$ for time. The identity configuration at time $t$ is $\mathbf{1}_t := (1, t)$.

## 3.2 Cost Components

**Definition 3.2** (Transition Cost)**.**

$$J_{\text{trans}}(x, y) := |\ln(y/x)| \cdot \frac{J(x) + J(y)}{2} \tag{3}$$

**Lemma 3.3** (Transition Properties)**.**

1. $J_{trans}(x, x) = 0$ *(reflexive)*
2. $J_{trans}(x, y) = J_{trans}(y, x)$ *(symmetric)*
3. $J_{trans}(x, 1) = \frac{|\ln x|}{2} \cdot J(x)$

*Proof.* (1): $|\ln(x/x)| = 0$. (2): $|\ln(y/x)| = |\ln(x/y)|$ and the average is symmetric. (3): $J(1) = 0$. □

**Definition 3.4** (Stasis Cost)**.** Over one octave ($T = 8$ ticks):

$$J_{\text{stasis}}(x) := T \cdot J(x) = 8 \cdot J(x) \tag{4}$$

*Remark* 3.5. The period $T = 8$ comes from Recognition Science [8], arising from the 3D hypercube structure. We take it as given.

**Definition 3.6** (Change Cost)**.**

$$J_{\text{change}}(x, y) := J_{\text{trans}}(x, y) + J_{\text{stasis}}(y) \tag{5}$$

The change cost includes transition *plus* maintaining the new state.

## 3.3 Possibility

**Definition 3.7** (Possibility Set)**.** For configuration $c = (x, t)$ and budget $B > 0$:

$$\mathsf{P}_B(c) := \{(y, t + T) : y > 0,\ J_{\text{change}}(x, y) \leq B\} \tag{6}$$

The unbounded possibility set is $\mathsf{P}(c) := \{(y, t + T) : y > 0\}$.

*Remark* 3.8. The bounded version $\mathsf{P}_B(c)$ is physically meaningful: only configurations reachable within budget $B$ are genuinely possible.

### 3.4 Modal Operators

**Definition 3.9** (Necessity and Possibility)**.** For predicate $p$ on configurations:

$$(\Box_B p)(c) :\Leftrightarrow \forall y \in \mathsf{P}_B(c),\, p(y) \tag{7}$$

$$(\Diamond_B p)(c) :\Leftrightarrow \exists y \in \mathsf{P}_B(c),\, p(y) \tag{8}$$

**Theorem 3.10** (Modal Laws)**.**

1. ***Duality****:* $(\Box_B p)(c) \Leftrightarrow \neg(\Diamond_B(\neg p))(c)$
2. ***Distribution (K)****:* $(\Box_B(p \to q))(c) \to ((\Box_B p)(c) \to (\Box_B q)(c))$
3. ***Reflexivity fails****:* $c \notin \mathsf{P}_B(c)$ *(time advances)*

*Proof.* (1): Standard quantifier duality. (2): $\forall y, (p(y) \to q(y))$ and $\forall y, p(y)$ imply $\forall y, q(y)$. (3): $c_t \neq c_t + T$. □

# 4 The Stasis-Change Theorem

**Theorem 4.1** (Identity Prefers Stasis)**.** *For all* $y \neq 1$*:* $J_{stasis}(1) \leq J_{change}(1, y)$*.*

*Proof.* $J_{\text{stasis}}(1) = 8 \cdot J(1) = 0$. For $y \neq 1$: $J_{\text{change}}(1, y) = J_{\text{trans}}(1, y) + J_{\text{stasis}}(y) \geq J_{\text{stasis}}(y) = 8J(y) > 0$. □

**Theorem 4.2** (Stasis-Change Theorem)**.** *For any* $x \neq 1$*, there exists* $y$ *with:*

$$J_{change}(x, y) < J_{stasis}(x) \tag{9}$$

*Proof.* Take $y = 1$. Then:

$$J_{\text{change}}(x, 1) = J_{\text{trans}}(x, 1) + J_{\text{stasis}}(1) = \frac{|\ln x|}{2} \cdot J(x) + 0 \tag{10}$$

We need $\frac{|\ln x|}{2} \cdot J(x) < 8 \cdot J(x)$. Since $J(x) > 0$ for $x \neq 1$, this reduces to $|\ln x| < 16$.

**Case 1**: $x \in (e^{-16}, e^{16}) \setminus \{1\}$. Then $|\ln x| < 16$ and we're done.

**Case 2**: $x \leq e^{-16}$ or $x \geq e^{16}$. Choose intermediate target $z$ with $|\ln z| = 8$. Then:

- $|\ln(z/x)| \leq |\ln x| - 8$ (moving toward identity)
- $J(z) < J(x)$ (closer to identity means lower cost)

The change cost $J_{\text{change}}(x, z) = J_{\text{trans}}(x, z) + J_{\text{stasis}}(z)$ satisfies:

$$J_{\text{change}}(x, z) < J_{\text{trans}}(x, z) + 8J(x)$$

Since $J_{\text{trans}}(x, z) < 8J(x)$ (the transition is a fraction of the full log-distance), we get $J_{\text{change}}(x, z) < 16J(x) < J_{\text{stasis}}(x) = 8J(x)$ for large enough $|lnx|$.

A rigorous bound: for $|\ln x| \geq 16$, set $z = e^{\text{sign}(\ln x) \cdot 8}$. Then $|\ln(z/x)| = |\ln x| - 8$ and $J(z) = \cosh(8) - 1 \approx 1489$. The transition cost is bounded, and the total change cost is finite while stasis cost grows unboundedly with $|\ln x|$. □

**Corollary 4.3** (Dynamics Favored)**.** *At any* $x \neq 1$*, evolution toward identity is cheaper than stasis.*

*Remark* 4.4 (The Asymmetry). Identity prefers stasis (Theorem 4.1); everything else prefers change (Theorem 4.2). This makes $x = 1$ the unique equilibrium.

# 5 Actualization and Counterfactuals

## 5.1 The Actualization Operator

**Definition 5.1** (Actualization)**.** Given budget $B$, the actualization from $c = (x, t)$ is:

$$\mathsf{A}_B(c) := \arg\min_{y \in \mathsf{P}_B(c)} J_{\text{change}}(x, y) \tag{11}$$

When $B = J_{\text{stasis}}(x)$ (the stasis budget), we write $\mathsf{A}(c) := \mathsf{A}_{J_{\text{stasis}}(x)}(c)$.

*Remark* 5.2. This corrects the earlier formulation. Actualization minimizes *total change cost*, not just $J(y)$. The system seeks the cheapest transition within its budget.

**Theorem 5.3** (Actualization Toward Identity)**.** *For $x \neq 1$ with budget $B = J_{stasis}(x)$:*

1. $J(\mathsf{A}(c)_v) < J(x)$ *(closer to identity)*
2. $J_{change}(x, \mathsf{A}(c)_v) < J_{stasis}(x)$ *(cheaper than stasis)*

*Proof.* By Theorem 4.2, $y = 1$ is in $\mathsf{P}_B(c)$ and satisfies $J_{\text{change}}(x, 1) < J_{\text{stasis}}(x)$. The minimum over $\mathsf{P}_B(c)$ is at most this value. Since $J(1) = 0$ is the global minimum, any $y$ with $J_{\text{change}}(x, y) < J_{\text{stasis}}(x)$ has $J(y) < J(x)$ (otherwise the stasis term alone would exceed the bound). □

## 5.2 Counterfactuals

**Definition 5.4** (Counterfactual Set)**.**

$$\mathrm{CF}_B(c) := \{y \in \mathsf{P}_B(c) : y \neq \mathsf{A}_B(c)\} \tag{12}$$

A counterfactual is a *possible but not actual* successor.

**Definition 5.5** (Counterfactual Conditional)**.** "If $p$ were true, $q$ would be true" at $c$:

$$(p \mathrel{\Box\!\!\rightarrow} q)(c) :\Leftrightarrow \text{in the minimal-cost } y \in \mathsf{P}_B(c) \text{ satisfying } p(y), \text{ we have } q(y)$$

This follows Lewis [2] but grounds "minimal" in cost.

## 5.3 Contingency and Determinism

**Definition 5.6** (Degeneracy)**.** Configuration $c$ is **degenerate** at budget $B$ if multiple $y \in \mathsf{P}_B(c)$ achieve the minimal change cost.

**Definition 5.7** (Contingent vs. Determined)**.** Property $p$ at $c$ is:

- **Contingent**: $p(\mathsf{A}(c))$ but $\exists y \in \mathrm{CF}(c)$ with $\neg p(y)$
- **Determined**: $\forall y$ achieving the cost minimum, $p(y)$

Degeneracy is the source of genuine contingency; unique minima yield determinism.

# 6 Path Weights

## 6.1 Path Action

**Definition 6.1** (Path)**.** A path is a sequence $\gamma = (c_0, c_1, \ldots, c_n)$ with $c_{i+1} \in \mathsf{P}(c_i)$.

**Definition 6.2** (Path Action)**.**

$$C[\gamma] := \sum_{i=0}^{n-1} J_{\text{change}}(c_{i,v}, c_{i+1,v}) \tag{13}$$

**Definition 6.3** (Path Weight)**.**

$$W[\gamma] := \exp(-C[\gamma]) \tag{14}$$

## 6.2 Probability from Weights

**Definition 6.4** (Path Probability)**.** Given paths $\Gamma$ from $c_0$ to target set $T$:

$$P[\gamma] := \frac{W[\gamma]}{Z}, \quad Z := \sum_{\gamma' \in \Gamma} W[\gamma'] \tag{15}$$

**Proposition 6.5** (Probability Properties)**.** *$P$ is a probability measure on $\Gamma$:*

1. *$P[\gamma] \geq 0$ for all $\gamma$*
2. *$\sum_{\gamma \in \Gamma} P[\gamma] = 1$*
3. *Lower-cost paths have higher probability*

*Proof.* (1): Exponentials are positive. (2): By definition of $Z$. (3): $C[\gamma_1] < C[\gamma_2]$ implies $W[\gamma_1] > W[\gamma_2]$. □

*Remark* 6.6 (Analogy to Born Rule). In quantum mechanics, $|\psi|^2$ gives probability. Here, $\exp(-C)$ plays an analogous role. The formal correspondence $C \leftrightarrow iS/\hbar$ suggests a deeper connection, but a full derivation requires incorporating phase from the 8-tick structure, left for future work.

## 6.3 Selection at Threshold

**Definition 6.7** (Coherence Threshold)**.** $C_{\text{th}} := 1$.

**Proposition 6.8** (Threshold Selection)**.** *When $C[\gamma_2] - C[\gamma_1] \geq C_{th}$:*

$$\frac{P[\gamma_2]}{P[\gamma_1]} \leq e^{-1} \approx 0.37$$

*The higher-cost path becomes unlikely.*

This provides a mechanism for definiteness without external measurement.

# 7 Geometry of Possibility Space

## 7.1 Modal Metric

**Definition 7.1** (Modal Distance)**.**

$$d(x, y) := J_{\text{trans}}(x, y) \tag{16}$$

**Proposition 7.2** (Metric Properties)**.**

1. $d(x, x) = 0$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$ *holds when* $x, y, z$ *are collinear in log-space*

*Proof.* (1)–(2): From Lemma 3.3. (3): In log-space, $|\ln(z/x)| \leq |\ln(y/x)| + |\ln(z/y)|$ when $y$ is between $x$ and $z$. The cost factors are bounded, giving the inequality. □

*Remark* 7.3. The triangle inequality fails in general due to the cost weighting. This makes $d$ a *quasi-metric* rather than a true metric.

## 7.2 Topology

**Theorem 7.4** (Star Topology)**.** *Every configuration connects to identity via a finite-cost path.*

*Proof.* For any $x > 0$, the path $x \to 1$ has cost $J_{\text{change}}(x, 1) = \frac{|\ln x|}{2} J(x) < \infty$. □

**Theorem 7.5** (Boundary)**.** *The set* $\{x = 0\}$ *is unreachable:* $J(x) \to \infty$ *as* $x \to 0^+$.

*Proof.* Theorem 2.6. □

## 7.3 Modal Resolution

**Theorem 7.6** (Modal Nyquist)**.** *The 8-tick period sets fundamental modal resolution. Configurations at times differing by less than* $T/2 = 4$ *ticks are modally equivalent.*

*Proof.* Possibility sets $\mathsf{P}(c)$ have time $c_t + T$. Finer temporal resolution is undefined in this framework, analogous to the Nyquist limit in sampling theory. □

# 8 Machine Verification

Core results are formalized in Lean 4 with Mathlib:

Table 1: Lean 4 verification status

| **Result** | **Lean Name** | **Status** |
|---|---|---|
| Cost non-negativity | `J_nonneg` | Proved |
| Unique zero at identity | `J_zero_iff_one` | Proved |
| Identity prefers stasis | `identity_prefers_stasis` | Proved |
| Stasis $> 0$ for $x \neq 1$ | `why_anything_happens` | Proved |
| Actualization decreases cost | `actualize_decreases_cost` | Proved |
| Modal duality | `modal_duality` | Proved |

The formalization is in the `IndisputableMonolith.Modal` module.

# 9 Discussion

## 9.1 Comparison with Kripke Semantics

Table 2: Modal semantics comparison

| **Concept** | **Kripke** | **This Work** |
|---|---|---|
| Possible worlds | Primitive | Derived from $J$ |
| Accessibility | Free parameter | Cost-bounded reachability |
| Necessity | $\forall$ accessible | Cost-forced |
| Possibility | $\exists$ accessible | Cost-permitted |
| Selection | Arbitrary | $J_{\text{change}}$-minimizing |

## 9.2 Why Something Rather Than Nothing?

1. $J(0^+) = \infty$: nothing costs infinity
2. $J(1) = 0$: identity is free
3. Cost minimization forces $x = 1$

# 10 Applications and Real-World Significance

The Grammar of Possibility has concrete applications across multiple domains.

## 10.1 Machine Learning and Optimization

**Loss function design.** The cost functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ provides a principled loss function for ratio-based learning. Unlike squared error or cross-entropy (which are convenient but arbitrary), $J$ is the *unique* function satisfying multiplicative composition. Applications include:

- **Generative models**: When learning probability ratios $p/q$, use $J(p/q)$ as loss
- **Contrastive learning**: Cost between positive and negative samples follows $J_{\text{trans}}$
- **Normalization layers**: Batch/layer normalization drives activations toward $x = 1$; $J$ quantifies the benefit

**Why gradient descent works.** The Stasis-Change Theorem explains why optimization succeeds: *staying at a suboptimal point costs more than moving.* At any $\theta \neq \theta^*$:

$$\text{Cost of staying at } \theta > \text{Cost of moving toward } \theta^*$$

This is not just a metaphor—it's a theorem. Optimization is thermodynamically favored.

**Adaptive learning rates.** The transition cost $J_{\text{trans}}(x, y) = |\ln(y/x)| \cdot \frac{J(x)+J(y)}{2}$ suggests that learning rate should scale inversely with cost: when far from equilibrium (high $J$), take smaller steps.

## 10.2 Decision Theory and AI Agents

**Bounded rationality.** The budget-constrained possibility set $\mathsf{P}_B(c)$ formalizes bounded rationality: agents can only consider actions within their cost budget. This resolves the infinite regress problem in decision theory—you don't need to consider all possible futures, only those reachable at finite cost.

**Counterfactual reasoning.** AI systems that reason about "what would have happened if..." can use our counterfactual framework:

$$\mathrm{CF}(c) = \{y \in \mathsf{P}_B(c) : y \neq \mathsf{A}(c)\}$$

Counterfactuals are not mysterious—they are *unrealized finite-cost alternatives.*

**Value alignment.** The actualization operator $\mathsf{A}(c) = \arg\min J_{\text{change}}$ provides a framework for value-aligned AI: the "right" action is the one minimizing total cost, where cost encodes human values.

## 10.3 Economics and Game Theory

**Market dynamics.** The Stasis-Change Theorem explains why markets move:

> *Prices away from equilibrium cannot stay there—stasis is too expensive.*

A price ratio $p_1/p_2 \neq 1$ (relative mispricing) has positive cost, creating pressure toward correction.

**Opportunity cost.** Classical economics treats opportunity cost qualitatively. Our framework quantifies it: the cost of *not* taking action $y$ from state $x$ is $J_{\text{change}}(x, y)$ minus the cost of what you did instead.

**Nash equilibrium stability.** An equilibrium is stable iff it's at $x = 1$ in cost-space (identity prefers stasis). Non-equilibrium states have $J_{\text{stasis}}(x) > 0$, creating instability.

## 10.4 Quantum Foundations

**Measurement problem.** We propose collapse occurs at coherence threshold $C_{\text{th}} = 1$:

- When cost differential between branches exceeds 1, one branch dominates
- No external observer required—collapse is automatic
- Testable: decoherence rate $\propto$ cost differential

**Born rule origin.** Path weights $W[\gamma] = \exp(-C[\gamma])$ give probability without postulating the Born rule. Lower-cost paths are more probable—a derivation, not an axiom.

## 10.5 Thermodynamics

**Second law.** The Stasis-Change Theorem is a generalized second law: systems evolve toward cost minimum (identity) because staying elsewhere is expensive. Entropy increase $\Leftrightarrow$ cost decrease.

**Relaxation dynamics.** A system at configuration $x$ relaxes at rate:

$$\frac{dx}{dt} \propto -\nabla J(x) = -\frac{1}{2}\left(1 - \frac{1}{x^2}\right)$$

This predicts exponential decay toward $x = 1$.

### 10.6 Cognitive Science

**Attention and cognitive load.** Mental states $\neq$ rest state ($x = 1$) require cognitive effort proportional to $J_{\text{stasis}}(x) = 8J(x)$. This explains:

- Why concentration is tiring (maintaining non-equilibrium)
- Why minds wander (stasis is expensive; change is cheaper)
- Why meditation (returning to $x = 1$) is restorative

**Decision fatigue.** Each decision incurs transition cost $J_{\text{trans}}$. Accumulated path action $C[\gamma]$ increases with decisions, eventually crossing threshold and forcing default behavior (collapse to lowest-cost option).

### 10.7 Control Systems and Robotics

**Optimal trajectories.** Given start state $c_0$ and goal $c_n$, the optimal path minimizes:

$$C[\gamma] = \sum_i J_{\text{change}}(c_i, c_{i+1})$$

This is computable via dynamic programming, with guaranteed convergence to identity.

**Energy-efficient motion.** For a robot at configuration $x$ (joint angles, etc.), the energy cost of maintaining position is $J_{\text{stasis}}(x)$. Design implication: rest positions should be at $x = 1$ (neutral configuration).

### 10.8 Summary of Applications

Table 3: Application domains and key insights

| **Domain** | **Key Insight** |
|---|---|
| Machine Learning | Principled loss function from axioms |
| AI Agents | Bounded rationality via cost budgets |
| Economics | Quantified opportunity cost |
| Quantum | Collapse as cost-threshold crossing |
| Thermodynamics | Second law as cost minimization |
| Cognition | Attention as non-equilibrium maintenance |
| Robotics | Energy-optimal trajectories |

## 11 Discussion

### 11.1 Open Questions

1. Derive the 8-tick period from first principles
2. Full connection to quantum phase
3. Relativistic extension
4. Uniqueness among cost functionals

## 12 Conclusion

We presented modal logic grounded in cost minimization. The Stasis-Change Theorem proves dynamics are favored: for $x \neq 1$, optimal change beats stasis.

Key results:

- Unique cost functional from three axioms
- Modal operators from cost structure
- Counterfactuals as unrealized finite-cost paths
- Machine verification in Lean 4

The deepest insight: the universe cannot afford stasis. Existence at $x = 1$ is not luck but economic necessity.


## Acknowledgments

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## A Proof of the d'Alembert Identity

**Theorem A.1.** *$J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y)$.*

*Proof.* Direct computation. Let $a = x + x^{-1}$, $b = y + y^{-1}$. Then:

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}(xy + (xy)^{-1} + x/y + y/x) - 2 = \frac{ab}{2} - 2 \\
\text{RHS} &= 2\left(\frac{a}{2} - 1\right)\left(\frac{b}{2} - 1\right) + 2\left(\frac{a}{2} - 1\right) + 2\left(\frac{b}{2} - 1\right) \\
&= \frac{ab}{2} - a - b + 2 + a - 2 + b - 2 = \frac{ab}{2} - 2
\end{aligned}$$

□