

# MAIN CONJECTURES FOR NON-CM ELLIPTIC CURVES AT GOOD ORDINARY PRIMES

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**ABSTRACT.** Let  $E/\mathbb{Q}$  be an elliptic curve and  $p > 2$  be a prime of good ordinary reduction for  $E$ . Assume that the residue representation associated with  $(E, p)$  is irreducible. In this paper, we prove more cases on several Iwasawa main conjectures for  $E$ . As applications, we prove more general cases of  $p$ -converse theorem and  $p$ -part BSD formula when the rank is less than or equal to 1.

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## 1. INTRODUCTION

Let  $E$  be an elliptic curve over  $\mathbb{Q}$ . Fix an odd prime  $p$  and embeddings  $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ ,  $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ . Let  $T_p E$  be the  $p$ -adic Tate module and  $\rho_E : G_{\mathbb{Q}} \rightarrow \text{Aut}_{\mathbb{Z}_p}(T_p E)$  the associated  $p$ -adic Galois representation. Suppose that  $E$  has good ordinary reduction at  $p$ .

Let  $K$  be an imaginary quadratic field. Suppose that  $p = \mathfrak{p}\bar{\mathfrak{p}}$  splits in  $K$ , where  $\mathfrak{p}$  is the prime induced by  $\iota_p$ . Let  $K_\infty^+$ ,  $K_\infty^-$  be the cyclotomic, respectively, anticyclotomic  $\mathbb{Z}_p$ -extension of  $K$ , and  $K_\infty = K_\infty^+ K_\infty^-$ . Then we have the Iwasawa algebras

$$\Lambda_K^\pm = \mathbb{Z}_p[[\text{Gal}(K_\infty^\pm/K)]], \quad \Lambda_K = \mathbb{Z}_p[[\text{Gal}(K_\infty/K)]].$$

As in Castella-Grossi-Skinner [CGS23], we have

- (1) the two-variable  $p$ -adic  $L$ -function  $\mathcal{L}_p^{\text{PR}}(E/K) \in \Lambda_K$ , which interpolates  $L(E/K, \chi^{-1}, 1)$  for finite order characters  $\chi$  of  $\text{Gal}(K_\infty/K)$ .
- (2) the  $\Lambda_K$ -module Selmer group  $H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)$  with ordinary Selmer conditions at  $v|p$ , where  $(\cdot)^\vee$  is the Pontryagin dual.

Also, we have

- (1) another type two-variable  $p$ -adic  $L$ -function  $\mathcal{L}_p^{\text{Gr}}(E/K) \in \Lambda_K^{\text{ur}} := \Lambda_K \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\text{ur}}$ , which interpolates roughly  $L(E/K, \chi, 1)$  for  $\chi$  Hecke characters over  $K$  with infinity type  $(b, a)$ ,  $a \leq -1$ ,  $b \geq 1$ , where  $\mathbb{Z}_p^{\text{ur}}$  is the completion of the ring of integers of the maximal unramified extension of  $\mathbb{Q}_p$ .
- (2) the  $\Lambda_K$ -module Selmer group  $H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)$  with relaxed Selmer conditions at  $v = \mathfrak{p}$  and strict Selmer conditions at  $v = \bar{\mathfrak{p}}$ .

**Conjecture 1.1.** Suppose that the residue representation  $\bar{\rho}_E|_{G_K}$  is irreducible.

- (1)  $H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) = (\mathcal{L}_p^{\text{PR}}(E/K)).$$

- (2)  $H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) \Lambda_K^{\text{ur}} = (\mathcal{L}_p^{\text{Gr}}(E/K)).$$

### 1.1. Main result.

**Theorem 1.2.** If the Heegner hypothesis holds (in particular,  $\text{sign}(E/K) = -1$ ) and  $\bar{\rho}_E|_{G_K}$  is absolutely irreducible, then

- (1)  $H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) \subset (\mathcal{L}_p^{\text{PR}}(E/K)).$$

- (2)  $H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) \Lambda_K^{\text{ur}} \subset (\mathcal{L}_p^{\text{Gr}}(E/K)).$$

Moreover, if

(Im) there exists  $\tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{p^\infty}))$  such that  $T_p E / (\rho_E(\tau) - 1)T_p E$  is free of  $\mathbb{Z}_p$ -rank one holds, the Conjecture 1.1 is true.

Skinner-Urban [SU14] first proved Conjecture 1.1 (1) used Eisenstein congruences on  $GU(2, 2)$  under certain conditions; particularly under the assumption that the sign is  $+1$ . Wan [Wan20, Wan21] proved Conjecture 1.1 (2) using Eisenstein congruences on  $GU(3, 1)$  under different conditions, specifically under the assumption that the sign is  $-1$  and  $E$  is semistable. In [BSTW24], using the Beilinson-Flach elements and explicit reciprocity law, Burungale-Skinner-Tian-Wan proved the equivalence between Conjecture 1.1 (1) and 1.1 (2).

In this paper, our proof of Theorem 1.2 is based on Skinner-Urban's work and a simple observation. The other ingredients are the equivalence of Conjecture 1.1 (1) and 1.1 (2), and Hsieh's result on non-vanishing of the  $\mu$ -invariants of BDP  $p$ -adic  $L$ -function [Hsi14].

We observe that, Skinner-Urban actually proved that the left-hand side of Conjecture 1.1 (1) is contained in right-hand side, after tensoring the fractional field of  $\Lambda_K^+$ . By [BSTW24], this shows that the left hand side of Conjecture 1.1 (2) is contained in the right hand side, after tensoring the fractional field of  $\Lambda_K^+$ . Therefore, by [Hsi14], if the Heegner hypothesis holds, the inclusion relation in Conjecture 1.1 (2) holds in general. Then as [SU14], by using Kato's result on Mazur's main conjecture [Kat04], we could complete the proof under the condition (Im).

**Remark 1.3.** Recently, using Wan's results on main conjecture for Hilbert modular forms [Wan15] and the base change method, Burungale-Castella-Skinner [BCS24] proved the rational version of Conjecture 1.1 assuming  $p > 3$  and  $\bar{\rho}_E$  is irreducible, and the integral version if (Im) also holds. Their results don't cover ours, nor do ours cover theirs.

As an application, we prove more cases of several one variable main conjectures, the  $p$ -part of the BSD formula and the  $p$ -converse theorem.

**Corollary 1.4.** Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$ ,  $p \nmid 2N$  a prime. Assume that  $E$  has ordinary reduction at  $p$ , if  $r \leq 1$ , the following are equivalent

- (1)  $\text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^\infty}(E/\mathbb{Q}) = r$ ,
- (2)  $\text{ord}_{s=1} L(E/\mathbb{Q}, s) = r$ .

Under any of the above conditions, if (Im) also holds, then the  $p$ -part of the BSD formula for  $E$  holds.

**Remark 1.5.** There have been several works on the  $p$ -part of the BSD formula and the  $p$ -converse theorem for elliptic curves. For example, in our case, see [SU14], [Zha14], [JSW17], [BSTW24], [BCS24] for some previous works on  $p$ -part BSD formula, and [Ski20], [Zha14], [BSTW24], [BCGS23] for some previous works on  $p$ -converse theorem.

**1.2. Strategy.** By [CGS23, Proposition 3.2.1] or [BSTW24, Proposition 9.18], the use of Beilinson-Flach elements, combined with the reciprocity law, establishes a connection between different Main Conjectures. Consequently, the inclusion relations (and their opposites) in Theorem 1.2 (1) and (2) are shown to be equivalent.

By Skinner-Urban [SU14], especially Theorem 7.7 and Proposition 13.6 (1), we have

**Theorem 1.6.** *Suppose that the residue representation  $\bar{\rho}_E : G_{\mathbb{Q}} \rightarrow \text{Aut}(E[p])$  is irreducible. Then for any height one prime  $P$ ,*

$$\text{ord}_P(\text{Char}_{\Lambda_K} H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) \geq \text{ord}_P(\mathcal{L}_p^{PR}(E/K))$$

unless  $P = P^+ \Lambda_K$  for some  $P^+ \subset \Lambda_K^+$ .

Similarly, as [CGS23, Proposition 3.2.1], we have

**Theorem 1.7.** *Under the assumption of above theorem, we have that for any height one prime  $P$ ,*

$$\text{ord}_P(\text{Char}_{\Lambda_K^{\text{ur}}} H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) \geq \text{ord}_P(\mathcal{L}_p^{\text{Gr}}(E/K))$$

unless  $P = P^+ \Lambda_K^{\text{ur}}$  for some  $P^+ \subset \Lambda_K^{\text{ur},+}$ .

However, by the result on the  $\mu$ -invariant of the BDP  $p$ -adic  $L$ -function [Hsi14], we have that if  $P \subset \Lambda_K^{\text{ur}}$  is a height one primes of the form  $P = P^+ \Lambda_K^{\text{ur}}$  for some  $P^+ \subset \Lambda_K^{\text{ur},+}$ , then

$$\text{ord}_P(\mathcal{L}_p^{\text{Gr}}(E/K)) = 0.$$

Hence, Theorem 1.2 (2) holds, implying that Theorem 1.2 (1) also holds. Moreover, if (Im) is satisfied, then, similarly to [SU14, Theorem 3.30], we conclude that Conjecture 1.1 (2) is true, and consequently, so is Conjecture 1.1 (1).

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## 2. SELMER GROUPS

Let  $K$  be an imaginary quadratic field of discriminant  $D_K$ . Let  $p > 2$  be a prime such that  $p = \mathfrak{p}\bar{\mathfrak{p}}$  splits in  $K$ . Let  $\mathbb{Q}_\infty$  be the  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$ . Let  $K_\infty$  be the  $\mathbb{Z}_p^2$ -extension of  $K$ , and  $K_\infty^+$  (respectively,  $K_\infty^-$ ) be the cyclotomic (respectively, anticyclotomic)  $\mathbb{Z}_p$ -extension of  $K$ . Let

$$\Gamma_{\mathbb{Q}} := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}), \quad \Gamma_K := \text{Gal}(K_\infty/K), \quad \Gamma_K^\pm := \text{Gal}(K_\infty^\pm/K).$$

Then  $\text{Gal}(K_\infty^\pm/K)$  is identified with  $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ . Let  $\gamma^\pm$  be a topological generator of  $\Gamma_K^\pm$ . Let

$$\Lambda_{\mathbb{Q}} := \mathbb{Z}_p[[\Gamma_{\mathbb{Q}}]], \quad \Lambda_K := \mathbb{Z}_p[[\Gamma_K]], \quad \Lambda_K^\pm := \mathbb{Z}_p[[\Gamma_K^\pm]]$$

be the corresponding Iwasawa algebras, and

$$\varepsilon_{\mathbb{Q}} : G_{\mathbb{Q}} \twoheadrightarrow \Gamma_{\mathbb{Q}} \hookrightarrow \Lambda_{\mathbb{Q}}^\times, \quad \varepsilon_K : G_K \twoheadrightarrow \Gamma_K \hookrightarrow \Lambda_K^\times, \quad \varepsilon_{K,\pm} : G_K \twoheadrightarrow \Gamma_K^\pm \hookrightarrow \Lambda_K^{\pm,\times}$$

the natural characters. For a discrete  $\mathbb{Z}_p$ -module  $M$ , let  $M^\vee := \text{Hom}_{\text{cts}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$  be its Pontryagin dual. The module  $\Lambda_{\mathbb{Q}}^\vee$  is equipped with a  $G_{\mathbb{Q}}$ -action via  $\varepsilon_{\mathbb{Q}}^{-1}$ . Similarly, the modules  $\Lambda_K^\vee$  and  $\Lambda_K^{\pm,\vee}$  are equipped with  $G_K$ -actions.

We normalize the reciprocity map in class field theory using the arithmetic Frobenius. Specifically, we require that uniformizers are mapped to the arithmetic Frobenius via the reciprocity map. In this way, we identify Hecke characters with Galois characters in this paper.

**2.1. Selmer groups for Iwasawa algebras.** Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$  prime to  $D_K$ . Assume that  $E$  has good ordinary reduction at  $p$ .

*Discrete Selmer groups.* Let  $F$  be  $\mathbb{Q}$  or  $K$ ,  $w$  a prime of  $F$  above  $p$ , and  $\mathbb{F}_w$  the residue field at  $w$ . Let  $\tilde{E}_{/\mathbb{F}_w}$  be the reduction of  $E$  at  $w$ , and denote  $\mathcal{F}_w^+ T_p E := \ker(T_p E \rightarrow T_p \tilde{E}_{/\mathbb{F}_w})$  as the kernel of the reduction map.

**Definition 2.1.** *For  $\Lambda$  any one of  $\Lambda_{\mathbb{Q}}, \Lambda_K$  or  $\Lambda_K^\pm$ , we define*

- (1)  $H_{\text{ord}}^1(F_w, T_p E \otimes \Lambda^\vee) = \text{Im}(H^1(F_w, \mathcal{F}_w^+ T_p E \otimes \Lambda^\vee) \rightarrow H^1(F_w, T_p E \otimes \Lambda^\vee))$ ,
- (2)  $H_{\text{rel}}^1(F_w, T_p E \otimes \Lambda^\vee) = H^1(F_w, T_p E \otimes \Lambda^\vee)$ ,
- (3)  $H_{\text{str}}^1(F_w, T_p E \otimes \Lambda^\vee) = 0$ .

Let  $\Sigma$  be a set of places of  $F$  such that  $\Sigma$  contains all places of  $F$  dividing  $pN\infty$ . In the anticyclotomic case, assume moreover that every finite place in  $\Sigma$  splits in  $K$ . Let  $F_\Sigma$  be the maximal extension of  $F$  unramified outside  $\Sigma$ , and let  $G_{F,\Sigma} := \text{Gal}(F_\Sigma/F)$ .

In the case  $F = \mathbb{Q}$ , for  $a \in \{\text{ord}, \text{str}, \text{rel}\}$  and  $M = T_p E \otimes \Lambda_{\mathbb{Q}}^\vee$ , let

$$H_{\mathcal{F}_a}^1(\mathbb{Q}, M) = \text{Ker} \left( H^1(G_{\mathbb{Q},\Sigma}, M) \rightarrow \prod_{q \in \Sigma, q \nmid p} H^1(\mathbb{Q}_q, M) \times \frac{H^1(\mathbb{Q}_p, M)}{H_a^1(\mathbb{Q}_p, M)} \right),$$

and define

$$\mathcal{X}_{\mathcal{F}_a}(E/\mathbb{Q}_\infty) = H_{\mathcal{F}_a}^1(\mathbb{Q}, T_p E \otimes \Lambda_{\mathbb{Q}}^\vee)^\vee.$$

In the case  $F = K$ , for  $a, b \in \{\text{ord}, \text{str}, \text{rel}\}$ ,  $\Lambda \in \{\Lambda_K, \Lambda_K^-, \Lambda_K^+\}$ , and  $M = T_p E \otimes \Lambda^\vee$ , let

$$H_{\mathcal{F}_{a,b}}^1(K, M) = \text{Ker} \left( H^1(G_{K,\Sigma}, M) \rightarrow \prod_{q \in \Sigma, q \nmid p} H^1(K_q, M) \times \frac{H^1(K_p, M)}{H_a^1(K_p, M)} \times \frac{H^1(K_{\bar{p}}, M)}{H_b^1(K_{\bar{p}}, M)} \right).$$

For simplicity, we write

- (1)  $H_{\mathcal{F}_a}^1(K, M) = H_{\mathcal{F}_{a,a}}^1(K, M)$  for  $a \in \{\text{ord}, \text{str}, \text{rel}\}$ ,
- (2)  $H_{\text{Gr}}^1(K, M) = H_{\mathcal{F}_{\text{rel,str}}}^1(K, M)$ ,

And for  $a \in \{\text{ord}, \text{str}, \text{rel}, \text{Gr}\}$ , we define

$$\mathcal{X}_{\mathcal{F}_a}(E/K_\infty) = H_{\mathcal{F}_a}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee, \quad \mathcal{X}_{\mathcal{F}_a}(E/K_\infty^\pm) = H_{\mathcal{F}_a}^1(K, T_p E \otimes \Lambda_K^{\pm,\vee})^\vee.$$

For any prime  $\mathfrak{q}$  of  $K$ , we denote the inertia subgroup of  $G_{K_\mathfrak{q}}$  by  $I_\mathfrak{q}$ .

**Lemma 2.2.** *For any  $\mathfrak{q} \nmid p$ , we have*

- (1)  $H^1(G_{K_\mathfrak{q}}/I_\mathfrak{q}, (T_p E \otimes \Lambda_K^\vee)^{I_\mathfrak{q}}) = 0$ .
- (2)  $\text{Char}_{\Lambda_K}((H^1(I_\mathfrak{q}, T_p E \otimes \Lambda_K^\vee)^{G_{K_\mathfrak{q}}})^\vee) = (P_\mathfrak{q}(\varepsilon_K(\text{Frob}_\mathfrak{q}^{-1})))$ , where

$$P_\mathfrak{q}(X) = \det(1 - \text{Nm}(\mathfrak{q})^{-1} X \cdot \text{Frob}_\mathfrak{q} | V_p E^{I_\mathfrak{q}}).$$

*Proof.* For simplicity, we let  $M := T_p E \otimes \Lambda_K^\vee$ ,  $N := T_p E \otimes \Lambda_K$ . Then we have the natural  $G_K$ -perfect pairing

$$M \times N \rightarrow \mathbb{Q}_p/\mathbb{Z}_p(1).$$

It induces the perfect local Tate pairing

$$H^1(K_\mathfrak{q}, M) \times H^1(K_\mathfrak{q}, N) \rightarrow \mathbb{Q}_p/\mathbb{Z}_p, \quad \text{for every place } \mathfrak{q}$$

via cup product. Let  $\mathfrak{q} \nmid p$  be a prime of  $K$ . Since  $G_{K_\mathfrak{q}}/I_\mathfrak{q} = \langle \text{Frob}_\mathfrak{q} \rangle \simeq \widehat{\mathbb{Z}}$  has cohomological dimension 1, we have that under local Tate pairing,  $H^1(G_{K_\mathfrak{q}}/I_\mathfrak{q}, N^{I_\mathfrak{q}})$  and  $H^1(G_{K_\mathfrak{q}}/I_\mathfrak{q}, M^{I_\mathfrak{q}})$  annihilate each other. Hence, by the inflation-restriction exact sequence

$$0 \rightarrow H^1(G_{K_\mathfrak{q}}/I_\mathfrak{q}, N^{I_\mathfrak{q}}) \rightarrow H^1(G_{K_\mathfrak{q}}, N) \rightarrow H^1(I_\mathfrak{q}, N)^{G_{K_\mathfrak{q}}} \rightarrow 0,$$

to prove (1), we only need to prove that  $H^1(I_\mathfrak{q}, N)^{G_{K_\mathfrak{q}}} = 0$ . Since  $\mathfrak{q} \nmid p$ , we know that there is a unique closed subgroup  $J_\mathfrak{q}$  of  $I_\mathfrak{q}$  such that  $I_\mathfrak{q}/J_\mathfrak{q} \simeq \mathbb{Z}_p(1)$  as  $\text{Gal}(K_\mathfrak{q}^{\text{ur}}/K_\mathfrak{q})$ -module, and  $J_\mathfrak{q}$  has profinite degree prime to  $p$ . By the inflation-restriction exact sequence again, we have

$$H^1(I_\mathfrak{q}, N) = H^1(I_\mathfrak{q}/J_\mathfrak{q}, N^{J_\mathfrak{q}}).$$

Let  $N_0 := T_p E^{J_\mathfrak{q}}$ . Since  $\varepsilon_K(I_\mathfrak{q}) = 1$ , we can find that  $N^{J_\mathfrak{q}} = N_0 \otimes \Lambda_K$ . Let  $\sigma_\mathfrak{q}$  be a topological generator of  $I_\mathfrak{q}/J_\mathfrak{q} \simeq \mathbb{Z}_p(1)$ . For any  $\phi \in H^1(I_\mathfrak{q}/J_\mathfrak{q}, N_0 \otimes \Lambda_K)^{G_{K_\mathfrak{q}}}$ ,  $g \in G_{K_\mathfrak{q}}$ , we have  $(g\phi)(\sigma_\mathfrak{q}) = \phi(\sigma_\mathfrak{q})$ . Hence, there exists an integer  $r = r(g)$  (we can assume  $r > 0$ ) such that

$$g^{-1}\phi(\sigma_\mathfrak{q}) = \phi(g^{-1}\sigma_\mathfrak{q} g) = \phi(\sigma_\mathfrak{q}^r) = (1 + \sigma_\mathfrak{q} + \cdots + \sigma_\mathfrak{q}^{r-1})\phi(\sigma_\mathfrak{q}).$$

Since  $\mathfrak{q}$  is not completely split in  $K_\infty/K$ , and  $\varepsilon_K(G_{K_\mathfrak{q}})$  is not trivial, we can choose  $g$  such that  $\varepsilon_K(g) \neq 1$ . However, since  $\varepsilon_K(\sigma_\mathfrak{q}) = 1$ , it follows that  $\phi(\sigma_\mathfrak{q}) = 0$  which implies that  $H^1(I_\mathfrak{q}, N)^{G_{K_\mathfrak{q}}} = 0$ .

Now we prove (2). As before, we have

$$(H^1(I_\mathfrak{q}, M)^{G_{K_\mathfrak{q}}})^\vee \simeq H^1(G_{K_\mathfrak{q}}/I_\mathfrak{q}, N^{I_\mathfrak{q}}).$$

Since

$$H^1(G_{K_\mathfrak{q}}/I_\mathfrak{q}, N^{I_\mathfrak{q}}) \simeq (T_p E^{I_\mathfrak{q}} \otimes \Lambda_K)/(1 - \text{Frob}_\mathfrak{q}^{-1}),$$

we have that

$$\text{Fitt}_{\Lambda_K}((H^1(I_\mathfrak{q}, M)^{G_{K_\mathfrak{q}}})^\vee) = (P_\mathfrak{q}(\varepsilon_K^{-1}(\text{Frob}_\mathfrak{q})))$$

which is principal and therefore divisorial. However,  $\text{Char}_{\Lambda_K}((H^1(I_{\mathfrak{q}}, M)^{G_{K_{\mathfrak{q}}}})^{\vee})$  is the minimal divisorial ideal containing  $\text{Fitt}_{\Lambda_K}((H^1(I_{\mathfrak{q}}, M)^{G_{K_{\mathfrak{q}}}})^{\vee})$ , hence we have

$$\text{Char}_{\Lambda_K}((H^1(I_{\mathfrak{q}}, M)^{G_{K_{\mathfrak{q}}}})^{\vee}) = \text{Fitt}_{\Lambda_K}((H^1(I_{\mathfrak{q}}, M)^{G_{K_{\mathfrak{q}}}})^{\vee}) = (P_{\mathfrak{q}}(\varepsilon_K^{-1}(\text{Frob}_{\mathfrak{q}}))).$$

□

**Lemma 2.3.** *The natural  $\Lambda_K^-$ -module homomorphism*

$$\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty})/(\gamma^+ - 1) \mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty}) \rightarrow \mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty}^-)$$

is pseudo-isomorphism.

*Proof.* By [Wan21, Proposition 3.1] and Lemma 2.2 (1) we obtain the desired result. Note that in our case, every finite place in  $\Sigma$  splits in  $K$ . □

We also consider imprimitive Selmer groups. For  $M = T_p E \otimes \Lambda_K^{\vee}$ , we define

$$H_{\mathcal{F}_{\text{ord}}^{\Sigma}}^1(K, M) = \text{Ker} \left( H^1(G_{K, \Sigma}, M) \rightarrow \prod_{\mathfrak{q} \mid p} \frac{H^1(K_{\mathfrak{q}}, M)}{H_{\text{ord}}^1(K_{\mathfrak{q}}, M)} \right)$$

and  $\mathcal{X}_{\text{ord}}^{\Sigma}(E/K_{\infty}) = H_{\mathcal{F}_{\text{ord}}^{\Sigma}}^1(K, M)^{\vee}$ .

In the proof of [SU14, Lemma 3.16], the following conclusion is essentially demonstrated.

**Lemma 2.4.**  $\mathcal{X}_{\text{ord}}^{\Sigma}(E/K_{\infty})$  is  $\Lambda_K$ -torsion.

Therefore, we have the following corollary.

**Corollary 2.5.**  $\mathcal{X}_{\text{ord}}(E/K_{\infty})$  is  $\Lambda_K$ -torsion, and

$$\text{Char}_{\Lambda_K}(\mathcal{X}_{\text{ord}}^{\Sigma}(E/K_{\infty})) \supset \text{Char}_{\Lambda_K}(\mathcal{X}_{\text{ord}}(E/K_{\infty})) \prod_{\mathfrak{q} \in \Sigma - \{\mathfrak{p}, \bar{\mathfrak{p}}\}} (P_{\mathfrak{q}}(\varepsilon_K(\text{Frob}_{\mathfrak{q}}^{-1}))).$$

*Compact Selmer groups.* We also consider compact Selmer groups. For  $\Lambda$  any one of  $\Lambda_K$  or  $\Lambda_K^{\pm}$ , let  $N = T_p E \otimes_{\mathbb{Z}_p} \Lambda$ , where  $\Lambda$  is equipped with the natural  $G_K$ -action. For  $a \in \{\text{ord}, \text{rel}, \text{str}\}$ , and  $\mathfrak{q} \mid p$  a prime above  $p$ , we define the local conditions  $H_a^1(K_{\mathfrak{q}}, N) \subset H^1(K_{\mathfrak{q}}, N)$  similarly to Definition 2.1, and for  $a, b \in \{\text{ord}, \text{rel}, \text{str}\}$ , let

$$H_{\mathcal{F}_{a,b}}^1(K, N) = \text{Ker} \left( H^1(G_{K, \Sigma}, N) \rightarrow \frac{H^1(K_{\mathfrak{p}}, N)}{H_a^1(K_{\mathfrak{p}}, N)} \times \frac{H^1(K_{\bar{\mathfrak{p}}}, N)}{H_b^1(K_{\bar{\mathfrak{p}}}, N)} \right).$$

For simplicity, we write

$$S_{a,b}(E/K_{\infty}) = H_{\mathcal{F}_{a,b}}^1(K, T_p E \otimes_{\mathbb{Z}_p} \Lambda_K), \quad S_a(E/K_{\infty}) = H_{\mathcal{F}_{a,a}}^1(K, T_p E \otimes_{\mathbb{Z}_p} \Lambda_K)$$

for  $a, b \in \{\text{ord}, \text{str}, \text{rel}\}$ , and similarly for  $E/K_{\infty}^{\pm}$  with  $\Lambda_K$  replaced by  $\Lambda_K^{\pm}$ .

**2.2. Selmer groups for Hida families.** Let  $W$  be an indeterminate and  $\Lambda_W = \mathbb{Z}_p[[W]]$ . We identify  $\Lambda_W$  with  $\Lambda_K^+$  by identifying the topological generator  $\gamma^+$  with  $1 + W$ , which induces  $\varepsilon_W : G_K \rightarrow \Lambda_W^{\times}$ .

Let  $L \subset \bar{\mathbb{Q}}_p$  be a finite extension of  $\mathbb{Q}_p$  with integer ring  $\mathcal{O}$ . Let  $\mathbb{I}$  be a local reduced finite integral extension of  $\Lambda_{W, \mathcal{O}} := \Lambda_W \otimes_{\mathbb{Z}_p} \mathcal{O}$ . Recall that  $\phi \in \text{Hom}_{\text{cont } \mathcal{O}\text{-alg}}(\mathbb{I}, \bar{\mathbb{Q}}_p)$  is called arithmetic if  $\phi(1 + W) = \zeta_{\phi}(1 + p)^{k_{\phi}-2}$  for some  $p$ -power root of unity  $\zeta_{\phi}$  and integer  $k_{\phi}$ . Let  $t_{\phi} > 0$  be the integer such that  $\zeta_{\phi}$  is primitive  $p^{t_{\phi}-1}$ -th root of unity, and  $\chi_{\phi} : \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \rightarrow \mu_{p^{\infty}}$  be the unique character such that  $\chi_{\phi,p}(1 + p) = \zeta_{\phi}^{-1}$  and has  $p$ -power conductor. Define

$$\mathfrak{X}_{\mathbb{I}, \mathcal{O}}^a = \{\phi \in \text{Hom}_{\text{cont } \mathcal{O}\text{-alg}}(\mathbb{I}, \bar{\mathbb{Q}}_p) : \phi \text{ is arithmetic, } k_{\phi} \geq 2\}.$$

Let  $\mathbf{f}$  be an  $\mathbb{I}$ -adic ordinary eigenform of tame level  $N$  and trivial character, i.e., a  $q$ -expansion  $\mathbf{f} = \sum_{n=0} \mathbf{a}_n(\mathbf{f}) q^n \in \mathbb{I}[[q]]$  such that for all  $\phi \in \mathfrak{X}_{\mathbb{I}, \mathcal{O}}^a$ ,

$$\mathbf{f}_{\phi} = \sum_{n=0} \phi(\mathbf{a}_n(\mathbf{f})) q^n \in M_{k_{\phi}}(Np^{t_{\phi}}, \omega^{k_{\phi}-2} \chi_{\phi}; \phi(\mathbb{I}))$$

is ordinary. Here,  $\omega$  is the Teichmüller character.

Assume  $\mathbf{f}$  satisfies the following condition

(irred $_{\mathbf{f}}$ ) the residue representation of  $\rho_{\mathbf{f}_{\phi}}$  is irreducible for some (hence all)  $\phi \in \mathfrak{X}_{\mathbb{I}, \mathcal{O}}^a$ .

Then there exists a continuous  $\mathbb{I}$ -linear Galois representation  $(\rho_{\mathbf{f}}, T_{\mathbf{f}})$  with  $T_{\mathbf{f}}$  a free  $\mathbb{I}$ -module of rank two and  $\rho_{\mathbf{f}} : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_{\mathbb{I}}(T_{\mathbf{f}})$  a continuous representation characterized by the property that  $\rho_{\mathbf{f}}$  is unramified at all primes  $\ell \nmid Np$  and satisfies

$$\mathrm{tr}\rho_{\mathbf{f}}(\mathrm{Frob}_{\ell}) = a_{\ell}(\mathbf{f}), \quad \ell \nmid Np,$$

and

$$\det(\rho_{\mathbf{f}}) = \epsilon \epsilon_W,$$

where  $\epsilon$  is the  $p$ -adic cyclotomic character.

Let  $F_{\mathbb{I}}$  be the ring of fractions of  $\mathbb{I}$  and  $V_{\mathbf{f}} := T_{\mathbf{f}} \otimes_{\mathbb{I}} F_{\mathbb{I}}$ . Since  $\mathbf{f}$  is ordinary, there exists an  $F_{\mathbb{I}}$ -line  $V_{\mathbf{f}}^+ \subset V_{\mathbf{f}}$  which is stable under the action of  $G_{\mathbb{Q}_p}$ . Furthermore,  $G_{\mathbb{Q}_p}$  acts on  $V_{\mathbf{f}}^- := V_{\mathbf{f}}/V_{\mathbf{f}}^+$  via the unramified character  $\delta_{\mathbf{f}}$  characterized by  $\delta_{\mathbf{f}}(\mathrm{Frob}_p) = a_p(\mathbf{f})$ . Then  $T_{\mathbf{f}}^+ := T_{\mathbf{f}} \cap V_{\mathbf{f}}^+$  is a free  $\mathbb{I}$ -summand of  $T_{\mathbf{f}}$  of rank one.

Put  $\mathbb{I}_K := \mathbb{I}[[\Gamma_K]]$ . Let  $\mathbf{M} = T_{\mathbf{f}} \otimes_{\mathbb{I}} \mathbb{I}_K^{\vee}$ ,  $\mathbf{M}^- = T_{\mathbf{f}}^- \otimes_{\mathbb{I}} \mathbb{I}_K^{\vee}$  where  $\mathbb{I}_K^{\vee}$  is equipped with a  $G_K$ -action via  $\epsilon_K^{-1}$ . Let  $\Sigma$  be a set of places of  $K$  containing all places of  $K$  dividing  $pN\infty$ . We define

$$H_{\mathcal{F}_{\mathrm{ord}}^{\Sigma}}^1(K, \mathbf{M}) := \ker \left( H^1(G_{K, \Sigma}, \mathbf{M}) \rightarrow \prod_{\mathfrak{q} \mid p} H^1(I_{\mathfrak{q}}, \mathbf{M}^-) \right).$$

Write  $\mathcal{X}_{\mathrm{ord}}^{\Sigma}(\mathbf{f}/K_{\infty}) := H_{\mathcal{F}_{\mathrm{ord}}^{\Sigma}}^1(K, \mathbf{M})^{\vee}$ . It is  $\mathbb{I}_K$ -torsion by [SU14, Lemma 3.16].

Let  $E/\mathbb{Q}$  be an elliptic curve such that  $E$  has good ordinary reduction at  $p$ . Let  $\mathbf{f}$  be an  $\mathbb{I}$ -adic ordinary eigenform and  $\phi \in \mathfrak{X}_{\mathbb{I}, A}^a$  such that  $\phi(1 + W) = 1$  and  $\phi(\mathbf{f})$  is equal to  $p$ -stabilization of the newform associated to  $E$ .

**Lemma 2.6.** *For every  $\mathfrak{q} \mid p$ , we have  $H^1(G_{K_{\mathfrak{q}}}/I_{\mathfrak{q}}, (T_p \tilde{E}_{/\mathbb{F}_{\mathfrak{q}}} \otimes \Lambda_K^{\vee})^{I_{\mathfrak{q}}}) = 0$ .*

*Proof.* Let  $\alpha_{\mathfrak{q}} : G_{K_{\mathfrak{q}}} \rightarrow \mathbb{Q}_p^{\times}$  be the unramified character associated to  $T_p \tilde{E}_{/\mathbb{F}_{\mathfrak{q}}}$ , then  $G_{K_{\mathfrak{q}}}$  acts on  $T_p \tilde{E}_{/\mathbb{F}_{\mathfrak{q}}} \otimes \Lambda_K^{\vee}$  via  $\alpha_{\mathfrak{q}} \epsilon_K^{-1}$ , and  $\alpha_{\mathfrak{q}}(\mathrm{Frob}_{\mathfrak{q}})$  is not a unit. Let  $C_{\mathfrak{q}} \subset \Lambda_K$  be the ideal generated by  $\{\epsilon_K(g) - 1 : g \in I_{\mathfrak{q}}\}$ , and  $\sigma_{\mathfrak{q}} \in G_{K_{\mathfrak{q}}}$  be a lifting of  $\mathrm{Frob}_{\mathfrak{q}}$ . Then we can find that

$$H^1(G_{K_{\mathfrak{q}}}/I_{\mathfrak{q}}, (T_p \tilde{E}_{/\mathbb{F}_{\mathfrak{q}}} \otimes \Lambda_K^{\vee})^{I_{\mathfrak{q}}}) \simeq \mathrm{Hom}_{\mathrm{cts}}\left((\Lambda_K/C_{\mathfrak{q}})^{\epsilon_K(\sigma_{\mathfrak{q}}) = \alpha_{\mathfrak{q}}^{-1}(\mathrm{Frob}_{\mathfrak{q}})}, \mathbb{Q}_p/\mathbb{Z}_p\right) = 0,$$

where  $(\Lambda_K/C_{\mathfrak{q}})^{\epsilon_K(\sigma_{\mathfrak{q}}) = \alpha_{\mathfrak{q}}^{-1}(\mathrm{Frob}_{\mathfrak{q}})}$  is the submodule of  $\Lambda_K/C_{\mathfrak{q}}$  killed by  $\epsilon_K(\sigma_{\mathfrak{q}}) - \alpha_{\mathfrak{q}}^{-1}(\mathrm{Frob}_{\mathfrak{q}})$ .  $\square$

**Corollary 2.7.** *Let  $\mathfrak{p}_{\phi} := \ker \phi$ , then  $\mathcal{X}_{\mathrm{ord}}^{\Sigma}(\mathbf{f}/K_{\infty})/\mathfrak{p}_{\phi} \mathcal{X}_{\mathrm{ord}}^{\Sigma}(\mathbf{f}/K_{\infty}) \otimes_{\mathbb{I}, \phi} \mathcal{O} \simeq \mathcal{X}_{\mathrm{ord}}^{\Sigma}(E/K^{\infty})$ .*

*Proof.* By [SU14, Proposition 3.7] and above lemma.  $\square$

### 3. $p$ -ADIC $L$ -FUNCTIONS

**3.1. A three variable  $p$ -adic  $L$ -function.** Let  $\mathbf{f}$  be an  $\mathbb{I}$ -adic ordinary eigenform of tame level  $N$  and trivial character as in Section 2.2. Suppose that  $L$  contains  $\mathbb{Q}[\mu_{Np}, i, D_K^{1/2}]$ . Let

$$\mathfrak{X}_{\mathbb{I}_K, \mathcal{O}}^a := \{\phi \in \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}\text{-alg}(\mathbb{I}_K, \bar{\mathbb{Q}}_p)) : \phi|_{\mathbb{I}} \in \mathfrak{X}_{\mathbb{I}, \mathcal{O}}^a, \phi(\gamma^+) = \zeta_+(1 + p)^{k_{\phi|_{\mathbb{I}}}-2}, \phi(\gamma^-) = \zeta_-\}$$

where  $\zeta_{\pm}$  are  $p$ -power roots of unity. For each  $\phi \in \mathfrak{X}_{\mathbb{I}_K, \mathcal{O}}^a$ , let  $k_{\phi}$ ,  $t_{\phi}$  and  $\chi_{\phi}$  denote the corresponding objects for  $\phi|_{\mathbb{I}}$ . Define

$$\xi_{\phi} := \phi \circ (\epsilon_K/\epsilon_W), \quad \theta_{\phi} := \omega^{2-k_{\phi}} \chi_{\phi}^{-1} \xi_{\phi}.$$

These are finite order idele class characters of  $\mathbb{A}_K^{\times}$ . For an idele class character  $\psi$ , denote its conductor by  $\mathfrak{f}_{\psi}$ . Let

$$\mathfrak{X}'_{\mathbb{I}_K, \mathcal{O}} := \{\phi \in \mathfrak{X}_{\mathbb{I}_K, \mathcal{O}}^a : p|\mathfrak{f}_{\xi_{\phi}}, p^{t_{\phi}}|\mathrm{Nm}(\mathfrak{f}_{\xi_{\phi}}), p|\mathfrak{f}_{\theta_{\phi}}\}.$$

**Theorem 3.1.** [SU14, Section 3.4.5] *Let  $\Sigma$  be a finite set of primes containing all those dividing  $pND_K$ . Let  $\mathbf{f}$  be an  $\mathbb{I}$ -adic ordinary eigenform of tame level  $N$  and trivial character. Assume that  $\mathbf{f}$  satisfies  $(\mathrm{irred}_{\mathbf{f}})$ . Then there exists  $\mathcal{L}_{\mathbf{f}, K}^{\Sigma} \in \mathbb{I}_K$  such that for any  $\phi \in \mathfrak{X}'_{\mathbb{I}_K, \mathcal{O}}$ , we have*

$$\begin{aligned} \mathcal{L}_{\mathbf{f}, K}^{\Sigma}(\phi) &= u_{\mathbf{f}, \phi} a_p(\mathbf{f}_{\phi})^{-\mathrm{ord}_p(\mathrm{Nm}(\mathfrak{f}_{\theta_{\phi}}))} \\ &\times \frac{((k_{\phi} - 2)!)^2 \mathfrak{g}(\theta_{\phi}^{-1}) \mathrm{Nm}(\mathfrak{f}_{\theta_{\phi}} \delta_K)^{k_{\phi}-2} L^{\Sigma}(\mathbf{f}_{\phi}/K, \theta_{\phi}^{-1}, k_{\theta_{\phi}} - 1)}{(-2\pi i)^{2k_{\phi}-2} \Omega_{\mathbf{f}_{\phi}}^+ \Omega_{\mathbf{f}_{\phi}}^-}, \end{aligned}$$

where  $u_{\mathbf{f}, \phi}$  is a  $p$ -adic unit depending only on  $\mathbf{f}_{\phi}$ ,  $\mathfrak{g}(\theta_{\phi}^{-1})$  is the (global) Gauss sum,  $\delta_K$  is the differential ideal, and  $\Omega_{\mathbf{f}_{\phi}}^{\pm}$  are the canonical periods of  $\mathbf{f}_{\phi}$ .

*Remark 3.2.* Note that we use arithmetic Frobenius to normalize the reciprocity map of class field theory, while [SU14] used geometric Frobenius.

**3.2. Two variable  $p$ -adic  $L$ -functions: type I.** Let  $f = \sum_n a_n q^n \in S_2(\Gamma_0(N))$  be a newform with  $p \nmid a_p$ , and  $c_f \in \mathbb{Z}_p$  be the congruence number of  $f$  as in [Hid81, section 7] or [Rib83].

**Theorem 3.3.** [CGS23, Theorem 1.2.1] *There exists an element  $\mathcal{L}_p^I(f/K) \in c_f^{-1} \Lambda_K$  such that for any finite order nontrivial character  $\xi$  of  $\Gamma_K$ ,*

$$\mathcal{L}_p^I(f/K)(\xi) = W(\xi) p^{\text{ord}_p(\text{Nm}(\mathfrak{f}_\xi))/2} \alpha_p^{-\text{ord}_p(\text{Nm}(\mathfrak{f}_\xi))} \left(1 - \frac{p}{\alpha_p^2}\right)^{-1} \left(1 - \frac{1}{\alpha_p^2}\right)^{-1} \frac{L(f/K, \xi^{-1}, 1)}{8\pi^2 \langle f, f \rangle},$$

where  $\alpha_p$  is the  $p$ -adic unit root of  $x^2 - a_p x + p$ ,  $\langle f, g \rangle = \int_{\Gamma_0(N) \backslash \mathcal{H}} \overline{f(\tau)} g(\tau) d\tau$  is the Petersson inner product on  $S_2(\Gamma_1(N))$ ,  $W(\xi)$  is the Artin root number.

Let  $E/\mathbb{Q}$  be an elliptic curve such that  $E$  has good ordinary reduction at  $p$ . Let  $f_E \in S_2(\Gamma_0(N))$  be the newform associated to  $E$ , and  $\pi_E : X_0(N) \rightarrow E$  a modular parametrization.

**Definition 3.4** (Perrin-Riou's  $p$ -adic  $L$ -function). *We define Perrin-Riou's  $p$ -adic  $L$ -function to be*

$$\mathcal{L}_p^{\text{PR}}(E/K) := \left(1 - \frac{p}{\alpha_p^2}\right) \left(1 - \frac{1}{\alpha_p^2}\right) \cdot \frac{\deg(\pi_E)}{c_E^2} \cdot \mathcal{L}_p^I(f_E/K) \in \Lambda_K,$$

where  $c_E$  is the Manin constant.

Let  $\mathbf{f}$  be an  $\mathbb{I}$ -adic ordinary eigenform and  $\phi \in \mathfrak{X}_{\mathbb{I}, \mathcal{O}}^a$  as in Section 2.2, such that  $\phi(1 + W) = 1$  and  $\mathbf{f}_\phi$  equal to  $p$ -stabilization of  $f_E$ .

**Proposition 3.5.** *Assume that the residue representation  $\bar{\rho}_E : G_{\mathbb{Q}} \rightarrow \text{Aut}(E[p])$  is irreducible, then*

$$\mathcal{L}_{\mathbf{f}, K}^\Sigma(\phi \otimes \text{id}) = \alpha \cdot \prod_{\mathfrak{q} \in \Sigma, \mathfrak{q} \nmid p} P_{\mathfrak{q}}(\varepsilon_K(\text{Frob}_{\mathfrak{q}}^{-1})) \cdot \mathcal{L}_p^{\text{PR}}(E/K),$$

where

$$P_{\mathfrak{q}}(X) = \det(1 - \text{Nm}(\mathfrak{q})^{-1} X \cdot \text{Frob}_{\mathfrak{q}} \mid V_p E^{I_{\mathfrak{q}}})$$

is the local Euler factor,  $\alpha$  is a  $p$ -adic units which is independent of  $\xi$ , and  $\phi \otimes \text{id} : \mathbb{I}_K = \mathbb{I} \hat{\otimes} \Lambda_K \rightarrow \phi(\mathbb{I}) \otimes \Lambda_K$  is the natural map.

*Proof.* We have that  $W(\xi) = g(\bar{\xi}) / \sqrt{\text{Nm}(\mathfrak{f}_\xi)}$ . By [Maz78, Corollary 4.1],  $c_E$  is a  $p$ -adic unit. By [CGS23, Lemma 3.1.2],  $\deg(\pi_E) = c_{f_E}$  up to a  $p$ -adic unit. By [SZ14, Lemma 9.5],

$$\frac{\langle f_E, f_E \rangle}{c_{f_E}} = i(2\pi i)^2 \Omega_{f_E}^+ \Omega_{f_E}^-.$$

These give the equality of the proposition.  $\square$

**3.3. Cyclotomic  $p$ -adic  $L$ -function.** Let  $E/\mathbb{Q}$  be an elliptic curve such that  $E$  has good ordinary reduction at  $p$ . Pick generators  $\delta^\pm$  of  $H_1(E, \mathbb{Z})^\pm$ , and define the Néron periods  $\Omega_E^\pm$  by

$$\Omega_E^\pm = \int_{\delta^\pm} \omega_E,$$

where  $\omega_E$  is a minimal differential on  $E$ . We normalize the  $\delta^\pm$  so that  $\Omega_E^+ \in \mathbb{R}_{>0}$  and  $\Omega_E^- \in i\mathbb{R}_{>0}$ .

Let  $a_p$  be the Fourier coefficient of the newform  $f_E$  associated to  $E$ , and  $\alpha_p$  be the  $p$ -adic unit root of  $x^2 - a_p x + p$  as before.

**Theorem 3.6.** [CGS23, Theorem 1.1.1] *There exists an element  $\mathcal{L}_p^{\text{MSD}}(E/\mathbb{Q}) \in \Lambda_{\mathbb{Q}}$  such that for any finite order character  $\chi$  of  $\Gamma_{\mathbb{Q}}$  of conductor  $p^r$  with  $r > 0$ , we have*

$$\mathcal{L}_p^{\text{MSD}}(E/\mathbb{Q})(\chi) = \frac{p^r}{g(\bar{\chi}) \alpha_p^r} \cdot \frac{L(E, \bar{\chi}, 1)}{\Omega_E^+},$$

where  $g(\bar{\chi}) = \sum_{a \bmod p^r} \bar{\chi}(a) e^{2\pi i a/p^r}$  is the Gauss sum, and

$$\mathcal{L}_p^{\text{MSD}}(E/\mathbb{Q})(1) = (1 - \alpha_p^{-1})^2 \cdot \frac{L(E, 1)}{\Omega_E^+}.$$

Let  $\mathcal{L}_p^{\text{PR}}(E/K)^+ \in \Lambda_{\mathbb{Q}}$  be the image of  $\mathcal{L}_p^{\text{PR}}(E/K)$  under the map induced by the projection  $\Gamma_K \twoheadrightarrow \Gamma_K^+ \simeq \Gamma_{\mathbb{Q}}$ .

**Proposition 3.7.** [CGS23, Proposition 1.2.4] We have

$$\mathcal{L}_p^{\text{PR}}(E/K)^+ = \mathcal{L}_p^{\text{MSD}}(E/\mathbb{Q}) \cdot \mathcal{L}_p^{\text{MSD}}(E^K/\mathbb{Q})$$

up to a unit in  $\Lambda_{\mathbb{Q}}^\times$ , where  $E^K$  is the twist of  $E$  by the quadratic character corresponding to  $K/\mathbb{Q}$ .

### 3.4. Two variable $p$ -adic $L$ -functions: type II.

**Theorem 3.8.** [CGS23, Theorem 1.4.1] There exists an element  $\mathcal{L}_p^{II}(f/K) \in \text{Frac}\Lambda_K$  such that for every character  $\xi$  of  $\Gamma_K$  which is crystalline at both  $\mathfrak{p}$  and  $\bar{\mathfrak{p}}$ , and of infinity type  $(b, a)$  with  $a \leq -1, b \geq 1$ , we have

$$\mathcal{L}_p^{II}(f/K)(\xi) = \frac{2^{a-b} i^{b-a-1} \Gamma(b+1) \Gamma(b) N^{a+b+1}}{(2\pi)^{2b+1} \langle \theta_{\xi_b}, \theta_{\xi_b} \rangle} \cdot \frac{\mathcal{E}(\xi, f, 1)}{(1 - \xi^{1-\tau}(\bar{\mathfrak{p}}))(1 - p^{-1} \xi^{1-\tau}(\bar{\mathfrak{p}}))} \cdot L(f/K, \xi, 1),$$

where  $\theta_{\xi_b}$  is the theta series associated to the Hecke character  $\xi_b = \xi |\cdot|^{-b}$ ,  $\tau$  is the complex conjugation, and

$$\mathcal{E}(\xi, f, 1) = (1 - p^{-1} \xi(\bar{\mathfrak{p}}) \alpha_p)(1 - \xi(\bar{\mathfrak{p}}) \alpha_p^{-1})(1 - p^{-1} \xi^{-1}(\mathfrak{p}) \alpha_p)(1 - \xi^{-1}(\mathfrak{p}) \alpha_p^{-1}).$$

Let  $\Lambda_K^{\text{ur}} := \Lambda_K \widehat{\otimes} \mathbb{Z}_p^{\text{ur}}$ , where  $\mathbb{Z}_p^{\text{ur}}$  is the completion of the ring of integers of the maximal unramified extension of  $\mathbb{Q}_p$ .

**Theorem 3.9.** [dS87, Chapter 2, Theorem 4.14] There exists an element  $\mathcal{L}_{\mathfrak{p}}(K) \in \Lambda_K^{\text{ur}}$  such that for every character  $\xi$  of  $\Gamma_K$  of infinity type  $(j, k)$  with  $0 \leq -j \leq k$ , we have

$$\mathcal{L}_{\mathfrak{p}}(K)(\xi) = \frac{\Omega_p^{k-j}}{\Omega_K^{k-j}} \cdot \Gamma(k) \cdot \left( \frac{\sqrt{D_K}}{2\pi} \right)^j \cdot (1 - \xi^{-1}(\mathfrak{p}) p^{-1})(1 - \xi(\bar{\mathfrak{p}})) \cdot L(\xi, 0),$$

where  $\Omega_p$  and  $\Omega_K$  are CM periods attached to  $K$ .

**Definition 3.10** (Greenberg's  $p$ -adic  $L$ -function). Let

$$\mathcal{L}_p^{\text{Gr}}(f/K) := h_K \cdot \mathcal{L}_{\mathfrak{p}}(K)' \cdot \mathcal{L}_p^{II}(f/K),$$

where  $h_K$  is the class number of  $K$ , and  $\mathcal{L}_{\mathfrak{p}}(K)'$  is the image of  $\mathcal{L}_{\mathfrak{p}}(K)$  under the map  $\Lambda_K^{\text{ur}} \rightarrow \Lambda_K^{\text{ur}}$  given by  $\gamma \mapsto \gamma^{1-\tau}$  for  $\gamma \in \Gamma_K$ .

**Lemma 3.11.** [CGS23, Lemma 1.4.4] The Greenberg's  $p$ -adic  $L$ -function  $\mathcal{L}_p^{\text{Gr}}(f/K)$  is integral, i.e., belongs to  $\Lambda_K^{\text{ur}}$ .

**3.5. BDP  $p$ -adic  $L$ -function.** Assume that  $D_K$  is odd and not equal to  $-3$ , and the Heegner hypothesis holds. Fix an integral ideal  $\mathfrak{n} \subset \mathcal{O}_K$  with  $\mathcal{O}_K/\mathfrak{n} \simeq \mathbb{Z}/N\mathbb{Z}$ . Let  $\Lambda_K^{\text{ur},-} := \Lambda_K^- \widehat{\otimes} \mathbb{Z}_p^{\text{ur}}$ .

**Theorem 3.12.** [CGLS22, Theorem 2.1.1] There exists an element  $\mathcal{L}_p^{\text{BDP}}(f/K) \in \Lambda_K^{\text{ur},-}$  characterized by the following interpolation property: for every character  $\xi$  of  $\Gamma_K^-$  crystalline at both  $\mathfrak{p}$  and  $\bar{\mathfrak{p}}$  and corresponding to a Hecke character of  $K$  of infinity type  $(n, -n)$  with  $n \in \mathbb{Z}_{>0}$  and  $n \equiv 0 \pmod{p-1}$ , we have

$$\mathcal{L}_p^{\text{BDP}}(f/K)(\xi) = \frac{\Omega_p^{4n}}{\Omega_K^{4n}} \cdot \frac{\Gamma(n)\Gamma(n+1)\xi(\mathfrak{n}^{-1})}{4(2\pi)^{2n+1}\sqrt{D_K}^{2n-1}} \cdot (1 - a_p \xi(\bar{\mathfrak{p}}) p^{-1} + \xi(\bar{\mathfrak{p}})^2 p^{-1})^2 \cdot L(f/K, \xi, 1).$$

Denote by  $\mathcal{L}_p^{\text{Gr}}(f/K)^-$  the image of  $\mathcal{L}_p^{\text{Gr}}(f/K)$  under the natural projection  $\Lambda_K^{\text{ur}} \rightarrow \Lambda_K^{\text{ur},-}$ . We have the following proposition.

**Proposition 3.13.** [CGS23, Proposition 1.4.5]  $\mathcal{L}_p^{\text{Gr}}(f/K)^- \cdot \Lambda_K^{\text{ur},-} = \mathcal{L}_p^{\text{BDP}}(f/K) \cdot \Lambda_K^{\text{ur},-}$ .

## 4. MAIN CONJECTURES

**4.1. Two variable Iwasawa main conjectures.** Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$ ,  $p > 2$  a prime such that  $E$  has good ordinary reduction at  $p$ ,  $K$  an imaginary quadratic field such that  $p = \mathfrak{p}\bar{\mathfrak{p}}$  split in  $K$ . Assume that residue representation  $\bar{\rho}_E|_{G_K} : G_K \rightarrow \text{Aut}(E[p])$  is irreducible.

The following two variable Iwasawa main conjectures are considered.

**Conjecture 4.1.** (1)  $H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) = (\mathcal{L}_p^{\text{PR}}(E/K)).$$

(2)  $H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) \Lambda_K^{\text{ur}} = (\mathcal{L}_p^{\text{Gr}}(E/K)).$$

Recall the main theorem of this paper as stated below.

**Theorem 4.2.** *Suppose that the residue representation  $\bar{\rho}_E|_{G_K} : G_K \rightarrow \text{Aut}(E[p])$  is absolutely irreducible. If the Heegner hypothesis holds (in particular,  $\text{sign}(E/K) = -1$ ), then*

- (1)  $H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(H_{\mathcal{F}_{\text{ord}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) \subset (\mathcal{L}_p^{\text{PR}}(E/K)).$$

- (2)  $H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(H_{\mathcal{F}_{\text{Gr}}}^1(K, T_p E \otimes \Lambda_K^\vee)^\vee) \Lambda_K^{\text{ur}} \subset (\mathcal{L}_p^{\text{Gr}}(E/K)).$$

Moreover, if (Im) holds, the Conjecture 4.1 is true.

We will prove this theorem in the next two subsections.

**4.2. A three variable Iwasawa main conjecture.** Let  $\mathbf{f}$  be an  $\mathbb{I}$ -adic ordinary eigenform of tame level  $N$  and trivial character as in Section 2.2. Suppose that  $L \supset \mathbb{Q}[\mu_{Np}, i, D_K^{1/2}]$ ,  $\mathbb{I}$  is a normal domain and  $\mathbf{f}$  satisfies (irred $_{\mathbf{f}}$ ).

**Conjecture 4.3.** [SU14] *Let  $\Sigma$  be a finite set of primes containing all those dividing  $pND_K$ . Then*

$$\text{Char}_{\mathbb{I}_K}(\mathcal{X}_{\text{ord}}^\Sigma(\mathbf{f}/K_\infty)) = (\mathcal{L}_{\mathbf{f}, K}^\Sigma).$$

**Theorem 4.4** (Skinner-Urban). *Under the condition of Conjecture 4.3, we have that for any height one prime  $P$  of  $\mathbb{I}_K = \mathbb{I}[[\Gamma_K]]$ ,*

$$\text{ord}_P(\text{Char}_{\mathbb{I}_K}(\mathcal{X}_{\text{ord}}^\Sigma(\mathbf{f}/K_\infty))) \geq \text{ord}_P(\mathcal{L}_{\mathbf{f}, K}^\Sigma),$$

unless  $P = P^+ \mathbb{I}[[\Gamma_K]]$  for some height one prime  $P^+$  of  $\mathbb{I}[[\Gamma_K^+]]$ .

*Proof.* Let  $\mathbf{D} = (A, \mathbf{f}, 1, 1, \Sigma)$  be a  $p$ -adic Eisenstein datum defined in [SU14], with  $A \supset \mathbb{Z}_p[i, D_K]$  a finite  $\mathbb{Z}_p$ -algebra. Let  $\Lambda_{\mathbf{D}} := \mathbb{I}[[\Gamma_K]][[\Gamma_K^-]]$  as in [SU14]. The proof is essentially in section 7.4 of [SU14] except that we do not consider the prime  $P = P^+ \Lambda_{\mathbf{D}}$  for some height one prime  $P^+$  of  $\mathbb{I}[[\Gamma_K^+]]$ . By [SU14, Proposition 13.6(1)], assume  $\Sigma$  is large enough, then there is a  $p$ -adic Eisenstein series  $\mathbf{E}_{\mathbf{D}}$  with coefficients (associated to formally  $q$ -expansion) belong to  $\Lambda_{\mathbf{D}}$ , and that there exists a set  $\mathcal{C}_{\mathbf{D}}$  of some coefficients of  $\mathbf{E}_{\mathbf{D}}$  satisfying that for any height one prime  $P$  of  $\Lambda_{\mathbf{D}}$ , if  $P \supset \mathcal{C}_{\mathbf{D}}$ , then  $P = P^+ \Lambda_{\mathbf{D}}$  for some height one prime  $P^+$  of  $\mathbb{I}[[\Gamma_K^+]]$ . Now we apply [SU14, Theorem 7.7]. Consider two  $p$ -adic  $L$ -functions  $\mathcal{L}_1^\Sigma$  and  $\mathcal{L}_{\mathbf{f}, K, 1}^\Sigma := \mathcal{L}_{\mathbf{f}, K}^\Sigma$  in [SU14, Theorem 7.7]. Since  $\mathcal{L}_1^\Sigma \in \mathbb{I}[[\Gamma_K^+]]$  by definition, we have that

$$\text{ord}_P(\text{Char}_{\mathbb{I}_K}(\mathcal{X}_{\text{ord}}^\Sigma(\mathbf{f}/K_\infty))) \geq \text{ord}_P(\mathcal{L}_{\mathbf{f}, K}^\Sigma)$$

for any height one prime  $P$  of  $\Lambda_{\mathbf{D}}$  such that  $P$  is not of the form  $P^+ \Lambda_{\mathbf{D}}$  for some height one prime  $P^+$  of  $\mathbb{I}[[\Gamma_K^+]]$ . Now it is easy to see that our conclusion comes from above.  $\square$

**Remark 4.5.** Since we do not consider the prime  $P = P^+ \Lambda_{\mathbf{D}}$  for some height one prime  $P^+$  of  $\mathbb{I}[[\Gamma_K^+]]$ , we don't need Proposition 13.6 (2) in [SU14]. Therefore, we do not need the hypotheses on  $N$  and  $\bar{\rho}_{\mathbf{f}}|_{I_\ell}$  for  $\ell|N$ . Here,  $\bar{\rho}_{\mathbf{f}} := \rho_{\mathbf{f}} \bmod \mathfrak{m}_{\mathbb{I}}$ , and  $\mathfrak{m}_{\mathbb{I}}$  is the maximal ideal of  $\mathbb{I}$ . See also the remark after Theorem 3.26 in [SU14].

**Corollary 4.6.** *There exists a nontrivial multiplicative set  $S \subset \Lambda_K^+ \subset \Lambda_K$  such that*

$$S^{-1} \text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/K_\infty)) \subset (\mathcal{L}_p^{\text{PR}}(E/K))$$

holds in  $S^{-1} \Lambda_K$ .

*Proof.* By [Roh84], the  $p$ -adic  $L$ -function  $\mathcal{L}_{\mathbf{f}, K}^\Sigma$  doesn't belong to  $f p_\phi \mathbb{I}_K$ . Let  $P_1, \dots, P_n$  be height one primes of  $\mathbb{I}_K$  such that  $\text{ord}_{P_i} \mathcal{L}_{\mathbf{f}, K}^\Sigma > 0$ , then  $P_i = P_i^+ \mathbb{I}_K$  for some height one prime  $P_i^+$  of  $\mathbb{I}[[\Gamma_K^+]]$ . We have  $P_i \not\subset \mathfrak{p}_\phi \mathbb{I}_K$ . Choose  $h_i \in P_i \setminus \mathfrak{p}_\phi \mathbb{I}_K$ , and let  $T$  be the multiplicative set generated by  $\{h_i : i = 1, \dots, n\}$ , and  $S = \phi(T)$ . Then by Theorem 4.4, [SU14, Corollary 3.8], Corollary 2.7, Corollary 2.5 and Proposition 3.5, we complete the proof.  $\square$

**4.3. Proof of Theorem 4.2.** Similarly, as [BSTW24, Proposition 9.18] (see also [CGS23, Proposition 3.2.1]), the following theorem holds.

**Theorem 4.7.** *For every nontrivial multiplicative set  $S \subset \Lambda_K$ , the following are equivalent*

- (1)  $S^{-1}\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/K_\infty)) \subset (\mathcal{L}_p^{\text{PR}}(E/K)).$
- (2)  $S^{-1}\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_\infty))\Lambda_K^{\text{ur}} \subset (\mathcal{L}_p^{\text{Gr}}(E/K)).$

*The same result holds for the opposite divisibilities.*

By Theorem 4.7 and Corollary 4.6, there exists a nontrivial multiplicative set  $S \subset \Lambda_K^+ \subset \Lambda_K$  such that

$$S^{-1}\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_\infty))\Lambda_K^{\text{ur}} \subset (\mathcal{L}_p^{\text{Gr}}(E/K)).$$

We may assume that  $S$  is generated by prime elements in the unique factorization domain  $\Lambda_K$ . For height one prime  $P \subset \Lambda_K^{\text{ur}}$ , if  $P \cap S \neq \emptyset$ , we have  $P = P^+\Lambda_K^{\text{ur}}$  for some  $P^+ \subset \Lambda_K^{\text{ur},+}$ . However, by [Hsi14, Theorem B] and Proposition 3.13,

$$\mu(\mathcal{L}_p^{\text{Gr}}(E/K)^-) = \mu(\mathcal{L}_p^{\text{BDP}}(E/K)) = 0,$$

where  $\mu(\cdot)$  denotes the  $\mu$ -invariant. Hence,

$$\text{ord}_P(\mathcal{L}_p^{\text{Gr}}(E/K)) = 0$$

if  $P \subset \Lambda_K^{\text{ur}}$  is a height one prime of the form  $P = P^+\Lambda_K^{\text{ur}}$  for some  $P^+ \subset \Lambda_K^{\text{ur},+}$ . It implies that

$$\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_\infty))\Lambda_K^{\text{ur}} \subset (\mathcal{L}_p^{\text{Gr}}(E/K)).$$

Moreover, if (Im) holds, similarly to [SU14, Theorem 3.30], by using [Kat04, Theorem 17.4] and a commutative algebra lemma ([SU14, Lemma 3.2]), we have that Conjecture 1.1 (2) is true. Therefore Conjecture 1.1 (1) holds. Note that in [SU14], the condition  $\text{Im}(\rho_E) \supset \text{SL}_2(\mathbb{Z}_p)$  is used, but it can be replaced by (Im) as discussed in the last paragraph of [Ski16, page 187].

**4.4. Mazur's main conjecture.** Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$ ,  $p > 2$  a prime such that  $E$  has good ordinary reduction at  $p$ . Suppose that the residue representation  $\bar{\rho}_E : G_{\mathbb{Q}} \rightarrow \text{Aut}(E[p])$  is irreducible.

**Conjecture 4.8.** *(Mazur's main conjecture)  $\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/\mathbb{Q}_\infty)$  is  $\Lambda_{\mathbb{Q}}$ -torsion and*

$$\text{Char}_{\Lambda_{\mathbb{Q}}}(\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/\mathbb{Q}_\infty)) = (\mathcal{L}_p^{\text{MSD}}(E/\mathbb{Q})).$$

Choose an imaginary quadratic field  $K$  such that  $p$  is split in  $K$  and  $(E, K)$  satisfies the Heegner hypothesis. Similarly to [SU14, Theorem 3.33], by Theorem 4.2 and descent arguments, we have the following theorem. Note that by Serre ([Ser87, Section 3.3]), the condition  $\bar{\rho}_E$  is irreducible implies that  $\bar{\rho}_E$  is absolutely irreducible in the case  $p > 2$ .

**Theorem 4.9.**  *$\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/\mathbb{Q}_\infty)$  is  $\Lambda_{\mathbb{Q}}$ -torsion and*

$$\text{Char}_{\Lambda_{\mathbb{Q}}}(\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/\mathbb{Q}_\infty)) \otimes \mathbb{Q}_p = (\mathcal{L}_p^{\text{MSD}}(E/\mathbb{Q}))$$

in  $\Lambda_{\mathbb{Q}} \otimes \mathbb{Q}_p$ . Moreover, if (Im) holds, we have

$$\text{Char}_{\Lambda_{\mathbb{Q}}}(\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/\mathbb{Q}_\infty)) = (\mathcal{L}_p^{\text{MSD}}(E/\mathbb{Q}))$$

in  $\Lambda_{\mathbb{Q}}$ .

**4.5. Anticyclotomic main conjectures.** Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$ ,  $p > 2$  a prime such that  $E$  has good ordinary reduction at  $p$ ,  $K$  an imaginary quadratic field such that  $p = \mathfrak{p}\bar{\mathfrak{p}}$  split in  $K$  and  $(E, K)$  satisfies the Heegner hypothesis. Assume that residue representation  $\bar{\rho}_E|_{G_K} : G_K \rightarrow \text{Aut}(E[p])$  is irreducible.

**Conjecture 4.10.** *(BDP main conjecture)  $\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_\infty^-)$  is  $\Lambda_K^-$ -torsion and*

$$\text{Char}_{\Lambda_K^-}(\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/K_\infty^-))\Lambda_K^{\text{ur}} = (\mathcal{L}_p^{\text{BDP}}(E/K)).$$

Fix a modular parametrization  $\pi : X_0(N) \rightarrow E$ . In [PR87], Perrin-Riou constructed an element  $\kappa \in S_{\text{ord}}(E/K_\infty^-)$  via the Kummer images of Heegner points on  $X_0(N)$ , which is  $\Lambda_K^-$ -non-torsion by Cornut-Vatsal [CV07]. Then she formulated an anticyclotomic main conjecture as follows.

**Conjecture 4.11.** *(Heegner point main conjecture)  $S_{\text{ord}}(E/K_\infty^-)$  and  $\mathcal{X}_{\text{ord}}(E/K_\infty^-)$  are both  $\Lambda_K^-$ -rank one, and*

$$\text{Char}_{\Lambda_K^-}(\mathcal{X}_{\text{ord}}(E/K_\infty^-)_{\text{tor}}) = \text{Char}_{\Lambda_K^-}(S_{\text{ord}}(E/K_\infty^-)/\Lambda_K^- \cdot \kappa)^2.$$

**Theorem 4.12.** (1)  $\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty}^-)$  is  $\Lambda_K$ -torsion and

$$\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/K_{\infty}))\Lambda_K^{\text{ur}} \otimes \mathbb{Q}_p = (\mathcal{L}_p^{\text{BDP}}(E/K))$$

holds in  $\Lambda_K \otimes \mathbb{Q}_p$ . Moreover, if (Im) holds, then

$$\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{ord}}}(E/K_{\infty}))\Lambda_K^{\text{ur}} = (\mathcal{L}_p^{\text{BDP}}(E/K)).$$

(2)  $S_{\text{ord}}(E/K_{\infty}^-)$  and  $\mathcal{X}_{\text{ord}}(E/K_{\infty}^-)$  are both  $\Lambda_K^-$ -rank one, and

$$\text{Char}_{\Lambda_K^-}(\mathcal{X}_{\text{ord}}(E/K_{\infty})_{\text{tor}}) \otimes \mathbb{Q}_p = \text{Char}_{\Lambda_K^-}(S_{\text{ord}}(E/K_{\infty})/\Lambda_K^- \cdot \kappa)^2 \otimes \mathbb{Q}_p$$

holds in  $\Lambda_K \otimes \mathbb{Q}_p$ . Moreover, if (Im) holds, then

$$\text{Char}_{\Lambda_K^-}(\mathcal{X}_{\text{ord}}(E/K_{\infty})_{\text{tor}}) = \text{Char}_{\Lambda_K^-}(S_{\text{ord}}(E/K_{\infty})/\Lambda_K^- \cdot \kappa)^2.$$

We prove this theorem in the remaining part of this subsection. First recall a theorem in [CGS23].

**Theorem 4.13.** [CGS23, Theorem 5.5.2]  $S_{\text{ord}}(E/K_{\infty}^-)$  and  $\mathcal{X}_{\text{ord}}(E/K_{\infty}^-)$  are both  $\Lambda_K^-$ -rank one, and

$$\text{Char}_{\Lambda_K^-}(\mathcal{X}_{\text{ord}}(E/K_{\infty})_{\text{tor}}) \otimes \mathbb{Q}_p \supset \text{Char}_{\Lambda_K^-}(S_{\text{ord}}(E/K_{\infty})/\Lambda_K^- \cdot \kappa)^2 \otimes \mathbb{Q}_p$$

holds in  $\Lambda_K^- \otimes \mathbb{Q}_p$ .

Similarly to [BCK21, Theorem 5.2], we have the following theorem.

**Theorem 4.14.**  $\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty}^-)$  is  $\Lambda_K$ -torsion and for every nontrivial multiplicative set  $S \subset \Lambda_K^-$ , the following are equivalent

- (1)  $S^{-1}\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty}))\Lambda_K^{\text{ur},-} \supset (\mathcal{L}_p^{\text{BDP}}(E/K))$ .
- (2)  $S^{-1}\text{Char}_{\Lambda_K^-}(\mathcal{X}_{\text{ord}}(E/K_{\infty})_{\text{tor}}) \supset S^{-1}\text{Char}_{\Lambda_K^-}(S_{\text{ord}}(E/K_{\infty})/\Lambda_K^- \cdot \kappa)^2$ .

The same result holds for the opposite divisibilities.

Since  $\mathcal{L}_p^{\text{BDP}}(E/K)$  is nonzero ([BCK21, Corollary 4.5]), by the argument in Section 4.3, there exists a nontrivial multiplicative set  $S \subset \Lambda_K^+ \subset \Lambda_K$  such that for any  $s \in S$ ,  $\gamma^+ - 1 \nmid s$ , and

$$S^{-1}\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty}))\Lambda_K^{\text{ur}} \subset (\mathcal{L}_p^{\text{Gr}}(E/K)).$$

By Lemma 2.3, we have

$$\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty}))\Lambda_K^{\text{ur}} \otimes \mathbb{Q}_p \subset (\mathcal{L}_p^{\text{BDP}}(E/K)).$$

The other direction divisibility is given by combining Theorem 4.13 and Theorem 4.14.

Moreover, if (Im) holds, then

$$\text{Char}_{\Lambda_K}(\mathcal{X}_{\mathcal{F}_{\text{Gr}}}(E/K_{\infty}))\Lambda_K^{\text{ur}} \subset (\mathcal{L}_p^{\text{Gr}}(E/K)).$$

The remaining part can be deduced from above divisibility and Mazur's main conjecture (Theorem 4.9). See the proof in [BSTW24, Section 12.2.1] for details.

**4.6. Applications.** Let  $E/\mathbb{Q}$  be an elliptic curve of conductor  $N$ ,  $p > 2$  a prime such that  $E$  has good ordinary reduction at  $p$ .

**Theorem 4.15.** For an integer  $r \leq 1$ , the following are equivalent

- (1)  $\text{rank}_{\mathbb{Z}} E(\mathbb{Q}) = r$  and  $\#\text{III}(E/\mathbb{Q}) < \infty$ ,
- (2)  $\text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^{\infty}}(E/\mathbb{Q}) = r$ ,
- (3)  $\text{ord}_{s=1} L(E/\mathbb{Q}, s) = r$ .

Under any of the above, if (Im) also holds, then  $p$ -part BSD formula for  $E$  holds, i.e.,

$$\left| \frac{L^{(r)}(1, E)}{r! \cdot \Omega_E R_E} \right|_p = \left| \frac{\#\text{III}(E)[p^{\infty}] \cdot \prod_{\ell \mid N} c_{\ell}(E)}{(\#E(\mathbb{Q})_{\text{tor}})^2} \right|_p,$$

where  $R_E$  is the regulator of  $E(\mathbb{Q})$ ,  $\Omega_E$  the Néron period,  $c_{\ell}(E)$  the Tamagawa number at a prime  $\ell$ , and  $|\cdot|_p$  the  $p$ -adic absolute value.

*Proof.* (1)  $\Rightarrow$  (2) is trivial. (3)  $\Rightarrow$  (1) is given by Gross-Zagier ([GZ86]) and Kolyvagin ([Kol90]). For (2)  $\Rightarrow$  (3), we can choose an imaginary quadratic field  $K$  such that  $\text{ord}_{s=1} L(E^K, s) \leq 1$  and  $(E, K)$  satisfies the Heegner hypothesis by [FH95]. Similarly to [Wan21, Theorem 1.9], by applying descent arguments to (the rational part of) Theorem 4.12 (2) and using Gross-Zagier formula, we have

$$\text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^\infty}(E/K) = 1 \text{ implies } \text{ord}_{s=1} L(E/K, s) = 1,$$

therefore we have (2)  $\Rightarrow$  (3) holds.

The rank 0  $p$ -part BSD formula comes from (the integral part of) Theorem 4.9 and descent arguments (see [SU14, Section 3.6.1] for details). Choosing an imaginary quadratic field  $K$  as above, the rank 1  $p$ -part BSD formula comes from (the integral part of) Theorem 4.12 and descent arguments (see [JSW17] for details).  $\square$

#### CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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