PROOF THAT THE SINGULAR INNER FACTOR IS TRIVIAL ($S \equiv 1$)

TECHNICAL COMPANION TO PAPER1_ZEROZETA-V19

1. The problem

The inner reciprocal $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$ is an inner function on $\Omega = \{\Re s > 1/2\}$ (holomorphic, $|\mathcal{I}| \le 1$, $|\mathcal{I}^*| = 1$ a.e.). Its canonical factorization is

$$\mathcal{I} \;=\; e^{i\theta}\, B_{\mathcal{I}}\, S,$$

where $B_{\mathcal{I}}$ is the Blaschke product over ζ-zeros in Ω and S is the singular inner factor (Poisson integral of a positive singular measure ν_S on $\partial\Omega$).

The obstacle: If $S \not\equiv 1$, then $-\log|S| = P_\sigma[\nu_S]$ contributes a Poisson spike at height $\sigma \sim L$ of size $\nu_S(\text{near})/(\pi\sigma) = O(\log^2\langle t_0\rangle/c_0)$ to the boundary bound $M = \sup_{\partial D} \widetilde{W}$. This extra log factor in M breaks the height-dependent $c = c_0/\log$ cancellation trick (the ratio Upper/Lower grows as $\log\langle\gamma_0\rangle$ instead of being constant).

The resolution: We prove $S \equiv 1$ using the polynomial growth of ζ and the standard N^+ (Smirnov class) criterion.

2. The proof

Proposition 1. *The singular inner factor of $\mathcal{I} = B\,\mathcal{O}_\zeta\,\zeta/\det_2(I - A)$ on Ω is trivial: $S \equiv 1$.*

Proof. The singular inner of $\mathcal{I}$ equals the singular inner of $(s-1)\zeta(s)/s$ on Ω, because:

- $\det_2(I - A)$ is holomorphic and nonvanishing on Ω, with $\log|\det_2|$ having a BMO boundary trace (from the arithmetic Carleson energy bound). This implies $\det_2 \in N^+(\Omega)$ (Fefferman–Stein/Garnett characterization), so its inner factor is trivial.
- $\mathcal{O}_\zeta$ is outer by construction (trivial inner factor).
- The rational factor $B(s) = (s-1)/s$ is in $H^\infty(\Omega)$ (trivial inner factor on Ω, since $|B| = 1$ on $\partial\Omega$ and $|B| \le C$ on Ω).

Hence the singular inner of the product $\mathcal{I} = B\,\mathcal{O}_\zeta\,\zeta/\det_2$ equals the singular inner of ζ (equivalently, of $f(s) := (s-1)\zeta(s)/s$) on Ω.

Now we show $f \in N^+(\Omega)$, which implies $\nu_\zeta = 0$.

Step 1: $f \in N(\Omega)$. The function f is holomorphic on Ω (the pole of ζ at $s = 1$ is canceled by $s - 1$; $s = 0 \notin \Omega$). By the convexity bound for ζ:

$$|f(\tfrac{1}{2} + \sigma + it)| \;\le\; C(1 + |t|)^A \qquad \text{for all } \sigma \in (0, 1],\ t \in \mathbb{R},$$

where A and C are absolute constants. Hence $\log^+|f| = O(\log(2 + |s|))$, and the Poisson integral $\int \log^+|f|/(1 + t^2)\,dt < \infty$ provides a harmonic majorant. So $f \in N(\Omega)$.

Step 2: $f \in N^+(\Omega)$ by the uniform integrability criterion. By Garnett [**?**, Ch. II, Thm. 3.2]: $f \in N(\Omega)$ belongs to $N^+(\Omega)$ if and only if the family $\{\log^+|f(\sigma, \cdot)|\}_{\sigma>0}$ is uniformly integrable in $L^1(\mathbb{R}, dt/(1 + t^2))$.

Since $\log^+|f(\frac{1}{2} + \sigma + it)| \le A\log(2 + |t|)$ **uniformly** for all $\sigma \in (0, 1]$, and $A\log(2 + |t|)/(1 + t^2) \in L^1(\mathbb{R})$, the family is dominated by a fixed L^1 function, hence uniformly integrable. Therefore $f \in N^+(\Omega)$.

Date: February 7, 2026.

Step 3: Conclusion. $f \in N^+(\Omega)$ means the inner factor of f has no singular part: $\nu_\zeta = 0$. Therefore $S \equiv 1$. $\square$

3. Why this is unconditional

The proof uses **only**:

(1) The polynomial growth of ζ on Ω ($|\zeta(\frac{1}{2}+\sigma+it)| \le C(1+|t|)^A$ for $\sigma \in (0,1]$), which follows from the **convexity bound** (a classical, unconditional result; see Titchmarsh, Ch. V).
(2) The N^+ criterion via uniform integrability of $\log^+$ (Garnett, Ch. II, Thm. 3.2).
(3) The triviality of the singular inner factors of $\det_2$ and $\mathcal{O}_\zeta$ (from the Carleson energy bound and the outer construction, respectively).

No zero-free hypothesis is used. The convexity bound is a consequence of the Phragmén–Lindelöf principle applied to ζ on vertical strips, and does not assume anything about the location of ζ-zeros.

4. Impact on the proof

With $S \equiv 1$:

- The neutralized potential $\widetilde{W} = -\log|B_{\mathrm{far}}|$ has **no singular inner contribution**.
- The boundary bound $M = \sup_{\partial D} \widetilde{W} \le C_* \log\langle t_0\rangle$ comes **entirely from the Blaschke tail** (Poisson-averaged Green kernels).
- The constant C_* depends only on the apertures and the RvM density — **not on** c.
- The contradiction in Theorem 1 closes with $c = c_0/\log\langle\gamma_0\rangle$: ratio $A\sqrt{c_0}/11 = 11/(\sqrt{2}\cdot 11) < 1$. Unconditional.