Response to Referee Comments on T5 (Cost Uniqueness): Clarification of the d'Alembert Functional Equation Requirement

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Abstract

We respond to the referee's valid critique regarding the uniqueness theorem T5 for the cost functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$. The referee correctly identifies that the five conditions stated in the manuscript—Reciprocity, Convexity, Minimality, Normalization, and Reciprocal-invariance—are insufficient for uniqueness, providing an explicit one-parameter family of counterexamples. We acknowledge this error and provide the corrected theorem statement based on the d'Alembert composition law, which is the actual constraint used in our Lean 4 formalization. We prove that the referee's counterexample family violates the d'Alembert equation for all $\varepsilon > 0$.

1 Acknowledgment of the Referee's Critique

The referee has provided a rigorous and valuable critique. We fully acknowledge that the counterexample family

$$J_\varepsilon(x) = \frac{1}{2}(x + x^{-1} - 2) + \varepsilon(x + x^{-1} - 2)^2, \quad \varepsilon \geq 0 \tag{1}$$

satisfies all five conditions (1)–(5) as stated in the manuscript:

1. **Reciprocity:** $J_\varepsilon(x) = J_\varepsilon(x^{-1})$ holds because $x + x^{-1}$ is symmetric under $x \mapsto x^{-1}$.
2. **Convexity:** The referee proves $J''_\varepsilon(x) > 0$ for all $x > 0$ and $\varepsilon \geq 0$.
3. **Minimality:** $J_\varepsilon(1) = 0$ and $J_\varepsilon(x) > 0$ for $x \neq 1$ follows from $u(x) := x + x^{-1} - 2 \geq 0$ with equality iff $x = 1$.
4. **Normalization:** The referee verifies $J''_\varepsilon(1) = 1$.
5. **Reciprocal-invariance:** By construction, $J_\varepsilon(x) = g_\varepsilon(f(x))$ where $f(x) = x + x^{-1}$.

This is a valid counterexample to the theorem as incorrectly stated in the manuscript.

2 The Correct Uniqueness Theorem

The actual uniqueness of $J(x)$ comes from the **d'Alembert composition law**, which is Axiom A2 in our Lean 4 formalization. The correct theorem statement is:

Theorem 1 (T5: Cost Uniqueness—Corrected Statement)**.** *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfy:*

(A1) Normalization: $F(1) = 0$.

(A2) d'Alembert Composition Law: *For all $x, y > 0$,*

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y). \tag{2}$$

(A3) Calibration: *In log coordinates $G(t) := F(e^t)$, we have $G''(0) = 1$.*

(A4) Continuity: *F is continuous on $\mathbb{R}_{>0}$.*

Then F is uniquely determined:

$$F(x) = \frac{1}{2}(x + x^{-1}) - 1. \tag{3}$$

Remark 2. The five conditions (1)–(5) in the manuscript are *consequences* of (A1)–(A4), not equivalent to them. Specifically:

- Reciprocity follows from the d'Alembert equation by setting $y = x$.
- Strict convexity follows from the functional form once uniqueness is established.
- Reciprocal-invariance follows from reciprocity.

The d'Alembert equation (A2) is the crucial constraint that was missing from the stated theorem.

3 Why the Counterexample Fails Under d'Alembert

We now prove that J_ε violates the d'Alembert equation for all $\varepsilon > 0$.

Theorem 3. *For $\varepsilon > 0$, the function $J_\varepsilon(x) = \frac{1}{2}(x + x^{-1} - 2) + \varepsilon(x + x^{-1} - 2)^2$ does* not *satisfy the d'Alembert composition law.*

Proof. Work in log coordinates: let $G_\varepsilon(t) := J_\varepsilon(e^t)$. Then

$$G_\varepsilon(t) = \frac{1}{2}(e^t + e^{-t} - 2) + \varepsilon(e^t + e^{-t} - 2)^2 \tag{4}$$

$$= (\cosh t - 1) + 4\varepsilon(\cosh t - 1)^2. \tag{5}$$

The d'Alembert equation in log coordinates is:

$$G(t + u) + G(t - u) = 2G(t)G(u) + 2G(t) + 2G(u). \tag{6}$$

For $G_0(t) = \cosh t - 1$, this identity holds (this is well-known for cosh-shifted functions). For $\varepsilon > 0$, let us check at $t = u$. The LHS of (**??**) becomes:

$$\text{LHS} = G_\varepsilon(2t) + G_\varepsilon(0) \tag{7}$$

$$= (\cosh 2t - 1) + 4\varepsilon(\cosh 2t - 1)^2 + 0. \tag{8}$$

Using $\cosh 2t = 2\cosh^2 t - 1$:

$$\text{LHS} = 2\cosh^2 t - 2 + 4\varepsilon(2\cosh^2 t - 2)^2. \tag{9}$$

The RHS of (**??**) at $t = u$ becomes:

$$\text{RHS} = 2G_\varepsilon(t)^2 + 4G_\varepsilon(t) \tag{10}$$

$$= 2[(\cosh t - 1) + 4\varepsilon(\cosh t - 1)^2]^2 + 4[(\cosh t - 1) + 4\varepsilon(\cosh t - 1)^2]. \tag{11}$$

Let $c := \cosh t - 1 \geq 0$. Then:

$$\text{LHS} = 2(c + c^2) + 4\varepsilon \cdot 4(c + c^2)^2 \tag{12}$$
$$= 2c(1 + c) + 16\varepsilon c^2(1 + c)^2. \tag{13}$$

Meanwhile:

$$\text{RHS} = 2(c + 4\varepsilon c^2)^2 + 4(c + 4\varepsilon c^2) \tag{14}$$
$$= 2c^2(1 + 4\varepsilon c)^2 + 4c(1 + 4\varepsilon c). \tag{15}$$

For the linear terms in ε:

$$\text{LHS (linear in } \varepsilon) = 16\varepsilon c^2(1 + c)^2, \tag{16}$$
$$\text{RHS (linear in } \varepsilon) = 2c^2 \cdot 2 \cdot 4\varepsilon c + 4c \cdot 4\varepsilon c = 16\varepsilon c^3 + 16\varepsilon c^2. \tag{17}$$

For these to match:

$$16\varepsilon c^2(1 + c)^2 = 16\varepsilon c^2(1 + c), \tag{18}$$

which requires $(1 + c)^2 = (1 + c)$, i.e., $c = 0$ or $c = -1$.

Since $c = \cosh t - 1 \geq 0$ and $c > 0$ for $t \neq 0$, the equation fails for all $t \neq 0$ when $\varepsilon > 0$. □

Corollary 4. *The unique continuous solution to the d'Alembert composition law (A2) with normalization (A1) and calibration (A3) is* $J(x) = \frac{1}{2}(x + x^{-1}) - 1$.

4 Lean 4 Verification

The correct uniqueness theorem is verified in our Lean 4 repository. The key modules are:

- `IndisputableMonolith/Foundation/CostAxioms.lean`: Defines the three primitive axioms (A1), (A2), (A3) as type classes.
- `IndisputableMonolith/Cost/FunctionalEquation.lean`: Proves that the d'Alembert equation in log coordinates reduces to the ODE $H'' = H$, whose unique solution with $H(0) = 1$, $H'(0) = 0$ is cosh.
- `IndisputableMonolith/CostUniqueness.lean`: The main theorem `T5_uniqueness_complete` requiring the `CoshAddIdentity` (d'Alembert) hypothesis.

5 Required Corrections to the Manuscript

Based on the referee's critique, the following corrections must be made:

1. **Theorem T5 Statement:** Replace the five conditions (1)–(5) with the three primitive axioms (A1)–(A3) plus continuity.
2. **Remove Misleading Claims:** The current text suggests that reciprocity, convexity, and minimality are *axioms*. They should be presented as *consequences* of the d'Alembert law.
3. **Clarify the Forcing Chain:** The d'Alembert composition law is the *primitive* constraint. The derivation should read:

   $$\text{d'Alembert (A2)} + \text{Normalization (A1)} + \text{Calibration (A3)} \implies J \text{ unique}.$$

4. **Update Proof Sketch:** The proof sketch should explicitly invoke the reduction to the ODE $H'' = H$ via d'Alembert, not merely cite the conditions.

6 Physical Interpretation

The d'Alembert composition law has a clear physical interpretation: it encodes **multiplicative consistency** of costs. If we compare x to y and x to $1/y$, the total "information" should combine coherently. This is analogous to the composition of Lorentz boosts or the addition formula for hyperbolic functions.

The referee's counterexamples J_ε for $\varepsilon > 0$ have "extra curvature" that violates this multiplicative consistency. Only the cosh-based solution maintains the algebraic structure required for a coherent cost theory.

7 Conclusion

We thank the referee for this careful and constructive critique. The error in the manuscript was a *presentation* error, not a foundational one: the Lean 4 proofs use the correct d'Alembert axiom (A2), but the manuscript incorrectly stated weaker conditions. The corrected Theorem **??** is sound and machine-verified.

The key lesson: the d'Alembert functional equation

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y)$$

is a *much stronger* constraint than reciprocity, convexity, and normalization combined. It is this constraint—not the weaker conditions (1)–(5)—that forces uniqueness.