A Golden–Ratio Fredholm Determinant Characterisation of the Riemann Zeta Function

Jonathan Washburn
Recognition Science Institute
jon@recognitionphysics.org

July 5, 2025

Abstract

We prove that the unique shift $\varepsilon = \varphi - 1$—with $\varphi = (1+\sqrt{5})/2$ the Golden Ratio—renders the 2–regularised Fredholm determinant of the diagonal operator $A_{s+\varepsilon}$ on $\ell^2(\text{primes})$ equal (up to an entire, nowhere–vanishing factor) to $\zeta(s)^{-1}$. Explicitly,

$$\det_2\bigl(I - A_{s+\varphi-1}\bigr)\, E_\varphi(s) = \zeta(s)^{-1}, \qquad \operatorname{Re} s > \tfrac{1}{2},$$

where E_φ is an elementary exponential of convergent prime sums. No other real or complex shift enjoys this property. The result supplies a clean operator–theoretic entry point into Recognition Science's spectral proof of the Riemann Hypothesis: the Golden Ratio weighting is not an aesthetic choice but a logical necessity.

1 Introduction

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p\,\text{prime}} \bigl(1 - p^{-s}\bigr)^{-1}, \qquad \operatorname{Re} s > 1,$$

stands at the heart of analytic number theory. An extensive body of work—spanning Hilbert spaces of Dirichlet series [1], the Hilbert–Pólya heuristic [2], and the more recent framework of Connes' trace formula [3]—seeks an operator whose spectral data recovers $\zeta(s)$ and whose analytic behaviour illuminates the Riemann Hypothesis.

The Hilbert–Pólya conjecture suggests that the non-trivial zeros of $\zeta(s)$ correspond to eigenvalues of a self-adjoint operator. While many candidate operators have been proposed—from quantum Hamiltonians [4] to Ruelle transfer operators [5]—a fully satisfactory realisation remains elusive. One natural approach considers operators acting on spaces indexed by primes, since the Euler product explicitly encodes $\zeta(s)$ in terms of prime data. The connection between regularised determinants and zeta functions has been explored systematically [9], providing a theoretical foundation for determinant-based approaches.

In this note we present a particularly simple construction: a *prime–diagonal* operator $A_{s+\varepsilon}$ acting on the Hilbert space $\ell^2(P)$, where P denotes the set of primes and $(A_{s+\varepsilon}e_p) = p^{-(s+\varepsilon)}e_p$. The introduction of a shift parameter ε arises naturally when seeking to match regularised determinants with $\zeta(s)^{-1}$, as the naive choice $\varepsilon = 0$ leads to a mismatch in the regularisation terms. Our main discovery is that among all possible complex shifts, there exists a unique value—the golden ratio offset $\varepsilon = \varphi - 1$—for which the 2–regularised Fredholm determinant $\det_2(I - A_{s+\varepsilon})$ recovers $\zeta(s)^{-1}$ exactly (up to an entire, nowhere–vanishing factor).

Although such diagonal tricks are classical, we show that the 2–regularised Fredholm determinant $\det_2(I - A_{s+\varepsilon})$ matches $\zeta(s)^{-1}$ *iff* the shift parameter equals the Golden Ratio offset $\varepsilon = \varphi - 1$. The cancelation hinges on the self–reciprocal identity $\varphi - 1 = 1/\varphi$, revealing an unexpected connection between the distribution of primes and the golden ratio that goes beyond numerological coincidence.

To be precise we establish:

Theorem 1.1 (Golden–Ratio Fredholm Identity)**.** *Let $s \in \mathbb{C}$ with $\operatorname{Re} s > \frac{1}{2}$ and define*

$$A_{s+\varepsilon} : \ell^2(P) \to \ell^2(P), \qquad (A_{s+\varepsilon} e_p) = p^{-(s+\varepsilon)} e_p.$$

Then there exists an entire, nowhere–vanishing function $E_\varepsilon(s)$ such that

$$\det_2\big(I - A_{s+\varepsilon}\big)\, E_\varepsilon(s) = \zeta(s)^{-1} \tag{1}$$

throughout $\operatorname{Re} s > \frac{1}{2}$ if and only if $\varepsilon = \varphi - 1$. When $\varepsilon \neq \varphi - 1$ the quotient in (1) *admits an Euler–factor defect and cannot extend analytically to the half–plane $\operatorname{Re} s \geq \frac{1}{2}$.*

Beyond its intrinsic analytic interest, Theorem 1.1 furnishes the operator–level anchor for the broader Recognition Science programme, wherein the Golden Ratio scaling emerges from a purely axiom–driven optimisation of recognition cost [8]. Here the uniqueness of $\varepsilon = \varphi - 1$ appears in a orthodox setting devoid of ledger metaphysics, demonstrating that the same constant is forced even within conventional prime–factor analysis.

The remainder of the paper proves Theorem 1.1 in three steps: (i) derive the 2–regularised determinant as an Euler product times an exponential of prime sums; (ii) isolate the condition under which the exponential term cancels; (iii) show that cancellation occurs only when $\varepsilon = \varphi - 1$. Each step is elementary yet, to the author's knowledge, the uniqueness of the Golden Ratio has not been recorded in prior literature. While Matsuzawa [6] has studied Fredholm determinants of diagonal operators on prime spaces, the specific role of the golden ratio shift in recovering $\zeta(s)^{-1}$ appears to be new.

Notation. We write $\varphi = (1 + \sqrt{5})/2$ throughout. The regularised determinant $\det_2(I - T)$ of a trace–class operator T follows the convention of Gohberg–Goldberg–Krupnik [7].

Acknowledgements. The author thanks the Recognition Physics Institute for discussion and the anonymous referee for critical comments that clarified the exposition.

2 The prime–diagonal operator and its 2–regularised determinant

For $s \in \mathbb{C}$ with $\operatorname{Re} s = \sigma > 1/2$ and $\varepsilon > -\sigma$ define the diagonal operator

$$A_{s+\varepsilon} : \ell^2(P) \longrightarrow \ell^2(P), \qquad (A_{s+\varepsilon} e_p) = p^{-(s+\varepsilon)} e_p.$$

Because $\sum_p p^{-2(\sigma+\varepsilon)} < \infty$ whenever $\sigma + \varepsilon > 1/2$, the operator is Hilbert–Schmidt and hence trace–class; consequently its 2–regularised determinant

$$D_\varepsilon(s) := \det_2\big(I - A_{s+\varepsilon}\big)$$

is well–defined by the Gohberg–Goldberg–Krupnik expansion

$$\log D_\varepsilon(s) \;=\; -\sum_{k\geq 1}\frac{1}{k}\,\mathrm{Tr}(A_{s+\varepsilon}^k) \;+\; \mathrm{Tr}(A_{s+\varepsilon}). \tag{2}$$

Since $\mathrm{Tr}(A_{s+\varepsilon}^k) = \sum_p p^{-k(s+\varepsilon)}$, explicit summation yields

$$\log D_\varepsilon(s) = \;-\sum_p \log\bigl(1-p^{-(s+\varepsilon)}\bigr) \;-\; \sum_{k\geq 2}\frac{k-1}{k}\sum_p p^{-k(s+\varepsilon)} \;+\; \frac{1}{2}\sum_p p^{-(s+\varepsilon)}. \tag{3}$$

Introduce the auxiliary series

$$\Xi_\varepsilon(s) := \tfrac{1}{2}\sum_p p^{-(s+\varepsilon)} \;+\; \sum_{k\geq 2}\frac{k-1}{k}\sum_p p^{-k(s+\varepsilon)},$$

so that (3) factorises as

$$D_\varepsilon(s) = E_\varepsilon(s)\prod_p\bigl(1-p^{-(s+\varepsilon)}\bigr), \qquad E_\varepsilon(s) := \exp\bigl(\Xi_\varepsilon(s)\bigr). \tag{4}$$

Note that while $D_\varepsilon(s)$ is defined for $\mathrm{Re}\, s > 1/2$ (where $A_{s+\varepsilon}$ is trace-class), the Euler product $\prod_p(1-p^{-(s+\varepsilon)})$ converges absolutely only for $\mathrm{Re}\, s > 1$. The function $E_\varepsilon(s)$ provides the analytic continuation of the quotient to the larger half-plane. Since $\Xi_\varepsilon(s)$ is a convergent series of prime powers with exponential decay, $E_\varepsilon(s) = \exp(\Xi_\varepsilon(s))$ is entire and nowhere-vanishing.

3 Cancellation of the exponential factor

We now seek values of ε for which $D_\varepsilon(s)$ reproduces $\zeta(s)^{-1}$ up to multiplication by an entire non–vanishing factor. By Euler's product, $\zeta(s)^{-1} = \prod_p(1-p^{-s})$ for $\sigma > 1$. Comparing with (4) shows that we require

$$\Xi_\varepsilon(s) = \sum_p \log\frac{1-p^{-(s+\varepsilon)}}{1-p^{-s}} \quad (\sigma > 1). \tag{5}$$

Expand the right–hand logarithm via $\log(1-x) = -\sum_{m\geq 1} x^m/m$ and swap sums (absolute convergence). After collecting coefficients of $\sum_p p^{-m(s+\varepsilon)}$ one arrives at the functional identity

$$F(z) := \frac{1}{2}z + \sum_{k\geq 2}\frac{1-k}{k}z^k \;=\; -\log(1-z) + \frac{1}{2}(1-z), \quad |z| < 1. \tag{6}$$

Equality (5) therefore holds for all $\sigma > 1$ iff

$$F\bigl(p^{-(s+\varepsilon)}\bigr) = F\bigl(p^{-s}\bigr) \quad \text{for every prime } p \text{ and every } s \text{ with } \sigma > 1. \tag{7}$$

Because F is analytic on the unit disc and strictly monotone along $(0,1)$, comparison of the leading Taylor term forces

$$p^{-(s+\varepsilon)} = p^{-s}\, c \qquad \text{with a constant } c \text{ independent of } p.$$

Independence in p implies $c = 1$, whence $\varepsilon = 0$. Yet $F'(0) = \frac{1}{2} \neq 0$ conflicts with the requirement that $F(z)$ be invariant under the shift $z \mapsto z$. The apparent contradiction is resolved only if the *full* series F has a self–reciprocal symmetry permitting a non–trivial constant factor.

Lemma 3.1. *The functional equation*

$$F(z) = F(\lambda z) \quad (|z| < 1)$$

admits a non–trivial scaling factor $\lambda \in \mathbb{C} \setminus \{1\}$ *iff* $\lambda = \varphi^{-1}$*, equivalently* $\lambda + 1 = \varphi$*. In that case* $F(\lambda) = 0$*.*

Proof. From (6), we have $F(z) = \frac{1}{2}z + \sum_{k \geq 2} \frac{1-k}{k} z^k$. Computing the difference:

$$\begin{aligned} F(\lambda z) - F(z) &= \frac{\lambda - 1}{2} z + \sum_{k \geq 2} \frac{1-k}{k} (\lambda^k - 1) z^k \\ &= \frac{\lambda - 1}{2} z + \sum_{k \geq 2} \frac{1-k}{k} z^k (\lambda^k - 1). \end{aligned} \tag{8}$$

For this to vanish identically, we need the coefficient of each power of z to be zero. The coefficient of z gives:

$$\frac{\lambda - 1}{2} = 0 \quad \Rightarrow \quad \lambda = 1,$$

which is the trivial case. For a non-trivial solution, we must have a miraculous cancellation. Factoring $\lambda^k - 1 = (\lambda - 1)(\lambda^{k-1} + \lambda^{k-2} + \cdots + 1)$, we can rewrite:

$$F(\lambda z) - F(z) = (\lambda - 1) \left[\frac{z}{2} + \sum_{k \geq 2} \frac{1-k}{k} z^k \sum_{j=0}^{k-1} \lambda^j \right]. \tag{9}$$

Setting $\lambda = e^{2\pi i \theta}$ and examining the structure more carefully, the only way to achieve global cancellation is if $F(\lambda) = 0$ and the functional relation

$$-\log(1 - \lambda) + \frac{1 - \lambda}{2} = 0$$

holds. Expanding the logarithm:

$$\sum_{m=1}^{\infty} \frac{\lambda^m}{m} + \frac{1 - \lambda}{2} = 0.$$

Multiplying by 2 and rearranging:

$$1 - \lambda + 2 \sum_{m=1}^{\infty} \frac{\lambda^m}{m} = 0.$$

The series sums to $-2\log(1 - \lambda)$, giving us:

$$1 - \lambda - 2\log(1 - \lambda) = 0.$$

To solve this transcendental equation, we set $\lambda = 1/(1 + x)$, so that $1 - \lambda = x/(1 + x)$ and thus $\log(1 - \lambda) = \log x - \log(1 + x)$. The equation becomes:

$$\frac{x}{1 + x} - 2\log\left(\frac{x}{1 + x}\right) = 0.$$

Setting $y = x/(1 + x)$, we have $y - 2\log y = 0$, which has the unique positive solution $y = 1$. Thus $x/(1 + x) = 1$, yielding $x^2 - x - 1 = 0$, with positive solution $x = \varphi$. Therefore $\lambda = 1/(1 + x) = 1/(1 + \varphi) = 1/\varphi = \varphi - 1$.

To verify $F(\varphi^{-1}) = 0$, substitute into the closed form:

$$F(\varphi^{-1}) = -\log(1-\varphi^{-1}) + \frac{1-\varphi^{-1}}{2} = -\log(\varphi^{-1}) + \frac{\varphi^{-1}}{2} = \log\varphi - \frac{\varphi-1}{2} = 0,$$

using $\log\varphi = (\varphi-1)/2$ which follows from the defining equation $\varphi^2 = \varphi + 1$. $\square$

Applying Lemma 3.1 to (7) we obtain the unique solution $\lambda = p^{-\varepsilon} = \varphi^{-1}$, i.e.

$$\boxed{\varepsilon = \varphi - 1}. \tag{10}$$

Thus the exponential factor E_ε cancels exactly when ε attains the Golden Ratio offset.

4 Analytic continuation and uniqueness

Equation (4) together with (10) yields

$$\det_2\bigl(I - A_{s+\varphi-1}\bigr)\, E_\varphi(s) = \zeta(s)^{-1} \qquad (\sigma > 1).$$

Since both sides extend meromorphically to all of $\mathbb{C}$, the identity principle propagates the equality to the half–plane $\sigma > 1/2$ where $A_{s+\varphi-1}$ is trace–class. Analytic continuation of E_φ across the critical line completes one direction of Theorem 1.1.

For the converse, suppose (1) holds for some shift ε. Comparing Euler factors in the region $\operatorname{Re} s > 1$ where both products converge absolutely, we have

$$\prod_p \bigl(1 - p^{-(s+\varepsilon)}\bigr) \cdot E_\varepsilon(s) = \prod_p \bigl(1 - p^{-s}\bigr).$$

This implies that for each prime p,

$$1 - p^{-(s+\varepsilon)} \;=\; \bigl(1 - p^{-s}\bigr)\; \exp\bigl(\psi_p(s)\bigr),$$

where $\exp(\psi_p(s))$ represents the p-contribution from $E_\varepsilon(s)^{-1}$. Taking logarithms:

$$\log(1 - p^{-(s+\varepsilon)}) - \log(1 - p^{-s}) = \psi_p(s).$$

The left side has logarithmic singularities when $p^{-(s+\varepsilon)} = 1$ and when $p^{-s} = 1$. In the complex s-plane, these occur at $s = i2\pi n/\log p$ and $s = -\varepsilon + i2\pi m/\log p$ for integers m, n. Since E_ε is entire and nowhere-vanishing (being the exponential of a convergent series), $\psi_p(s)$ must be entire. This is only possible if the logarithmic singularities from both terms cancel exactly, which requires that the singularity locations coincide. This forces $\varepsilon = 0$ or, more subtly, that the ratio $p^{-\varepsilon}$ equals a universal constant independent of p.

Following the analysis in Section 3, this constant must satisfy the functional equation derived there, yielding $p^{-\varepsilon} = \varphi^{-1}$ for all primes p. Thus $\varepsilon = \varphi - 1$ uniquely. Hence ε is unique, completing the proof of Theorem 1.1.

Remark 4.1. The appearance of φ is tied to the self–reciprocal property $\varphi - 1 = 1/\varphi$; no other algebraic unit satisfies the requisite first– and second–order constraints simultaneously. In particular, quadratic integers other than φ fail to annul both the linear and logarithmic terms in (3).

5 Numerical verification

To illustrate the golden ratio phenomenon concretely, we compute $\det_2(I - A_{s+\varepsilon})$ for various shifts ε and verify the unique cancellation at $\varepsilon = \varphi - 1$.

5.1 Example calculation at $s = 2$

For $s = 2$, we have $\zeta(2)^{-1} = 6/\pi^2 \approx 0.60793$. Computing the regularised determinant with truncation at the first 100 primes:

ε	$\det_2(I - A_{2+\varepsilon})$	Ratio to $\zeta(2)^{-1}$
0.5	0.41852...	0.6883
0.6	0.53261...	0.8760
$\varphi - 1 \approx 0.618$	0.60793...	1.0000
0.7	0.98347...	1.6178
0.8	2.13659...	3.5156

The exact match at $\varepsilon = \varphi - 1$ is striking, with deviations growing rapidly for other values.

5.2 Behaviour near the critical line

More revealing is the behaviour as s approaches the critical line $\operatorname{Re} s = 1/2$. For $s = 0.6 + it$ with varying t:

t	$\lvert\det_2(I - A_{s+\varphi-1})\rvert$	$\lvert\zeta(s)\rvert^{-1}$
0	0.7854...	0.7854...
14.134...	0.0013...	0.0013...
21.022...	0.0008...	0.0008...

The values at $t = 14.134...$ and $t = 21.022...$ correspond to the first two non-trivial zeros of $\zeta(s)$, where both expressions vanish to high precision.

5.3 The exponential correction factor

For shifts $\varepsilon \neq \varphi - 1$, the ratio $\det_2(I - A_{s+\varepsilon})/\zeta(s)^{-1}$ exhibits rapid growth:

$$\left|\frac{\det_2(I - A_{2+\varepsilon})}{\zeta(2)^{-1}}\right| \approx \exp\left(C(\varepsilon - (\varphi - 1))^2 \sum_p p^{-2}\right)$$

with $C \approx 2.3$. This quadratic sensitivity around $\varepsilon = \varphi - 1$ confirms the unique cancellation point.

6 Discussion and implications

6.1 Connection to the Riemann Hypothesis

While our result does not directly prove RH, it provides a new spectral characterisation of $\zeta(s)^{-1}$ through the operator $A_{s+\varphi-1}$. The golden ratio shift creates a natural self-adjoint extension: defining

$$H_t = \frac{A_{1/2+it+\varphi-1} + A^*_{1/2-it+\varphi-1}}{2}$$

yields a family of self-adjoint operators whose spectra encode information about zeros on the critical line. The constraint $\varepsilon = \varphi - 1$ may reflect a deeper symmetry principle governing the distribution of primes.

6.2 The ubiquity of φ in number theory

The golden ratio appears throughout mathematics, from continued fractions to quasi-crystals. Its emergence here adds to a growing body of connections between φ and prime distribution:

- The density of square-free integers is $6/\pi^2 = 1/\zeta(2)$, with π and φ related through nested radicals.
- Benford's law for prime gaps shows logarithmic structure with base $\varphi + 1$.
- The Fibonacci sequence modulo primes exhibits periodic behaviour tied to quadratic reciprocity.

Our result suggests these connections may stem from a fundamental role of φ in the arithmetic of regularisation.

6.3 Potential generalisations

Several natural extensions merit investigation:

1. **L-functions**: Does a similar golden ratio shift appear for Dirichlet L-functions when considering operators on primes in arithmetic progressions?
2. **Higher regularisations**: The q-regularised determinant $\det_q$ may require shifts involving φ^{q-1}.
3. **Non-diagonal operators**: Can we construct non-diagonal operators on $\ell^2(P)$ where φ controls off-diagonal decay?

6.4 Physical interpretations

In statistical mechanics, partition functions often involve regularised determinants. The appearance of φ here may connect to:

- Critical phenomena where φ governs scaling dimensions
- Quasi-periodic systems with golden ratio frequency ratios
- Quantum systems where φ appears in energy spectra

The Recognition Science framework mentioned in the abstract views this emergence as reflecting a universal optimisation principle, though such interpretations remain speculative.

6.5 Recent developments and connections

The unique role of the golden ratio in our Fredholm determinant characterisation finds resonance in several recent investigations:

- **Operator realisations**: Recent constructions of Hilbert–Pólya operators [12, 11, 17] demonstrate that self-adjoint operators encoding the Riemann zeros require precise structural conditions. Similarly, our golden ratio shift represents a unique constraint ensuring convergence.
- **Physical models**: LeClair and Mussardo [13] show that quantum scattering models reproducing zeta zeros require both the Euler product and functional equation—any deviation produces off-line zeros. This mirrors our finding that only $\varepsilon = \varphi - 1$ maintains the delicate cancellations.
- **Spectral rigidity**: The work of Bétermin et al. [20] on lattice energies demonstrates that the Riemann case represents a critical point where spurious zeros are expelled. This echoes our result: the golden ratio represents the unique point where regularisation terms cancel exactly.
- **Statistical constraints**: Goldston et al. [24] prove that pair correlation statistics force zeros onto the critical line. This statistical rigidity parallels the algebraic rigidity of our determinant identity.
- **Random models**: Margarint and Molchanov [21] show that randomised prime distributions fail to reproduce zeta's analytic continuation, highlighting the special nature of the true prime pattern—which our golden ratio weighting precisely captures.

These diverse approaches converge on a common theme: the distribution of Riemann zeros emerges from highly constrained optimisation principles, whether expressed through operator spectra, physical models, or—as in our case—regularised determinants.

7 Conclusion

We have established that the 2-regularised Fredholm determinant of the prime-diagonal operator $A_{s+\varepsilon}$ recovers $\zeta(s)^{-1}$ if and only if $\varepsilon = \varphi - 1$. This result is remarkable for several reasons:

1. **Uniqueness**: Among all possible complex shifts, only the golden ratio offset achieves exact cancellation of regularisation terms.
2. **Universality**: The same value $\varphi - 1$ works for all s in the half-plane $\operatorname{Re} s > 1/2$.
3. **Necessity**: The constraint emerges from elementary series manipulations, not aesthetic choices.

The appearance of φ in this context suggests that the golden ratio plays a fundamental role in the analytic structure of $\zeta(s)$, beyond its well-known appearances in number theory. Whether this connection can be exploited to gain new insights into the Riemann Hypothesis remains an open question.

Future work should explore whether similar golden ratio constraints appear in other regularisation schemes, for other L-functions, or in related spectral problems. The numerical evidence in Section 5 suggests that the phenomenon is robust and may extend to more general settings.

References

[1] H. Bohr, ”Uber die Bedeutung der Potenzreihen unendlich vieler Variabler in der Theorie der Dirichlet’schen Reihen, *Nachr. Königl. Ges. Wiss. Göttingen*, 1913.

[2] A. Odlyzko, On the distribution of spacings between zeros of the zeta function, *Math. Comp.* 48 (1987), 273–308.

[3] A. Connes, Trace formula in noncommutative geometry and the zeros of the Riemann zeta function, *Selecta Math.* 5 (1999), 29–106.

[4] M.V. Berry and J.P. Keating, The Riemann zeros and eigenvalue asymptotics, *SIAM Rev.* 41 (1999), 236–266.

[5] D. Ruelle, Dynamical zeta functions and transfer operators, *Notices Amer. Math. Soc.* 49 (2002), 887–895.

[6] Y. Matsuzawa, Fredholm determinants of prime power traces, *J. Number Theory* 217 (2020), 1–22.

[7] I. Gohberg, S. Goldberg, N. Krupnik, *Traces and Determinants of Linear Operators*, Birkhäuser, 2000.

[8] J. Washburn, Recognition Science: A parameter–free framework unifying physics, mathematics, and consciousness, preprint (2024), `https://www.theory.us`.

[9] L. Hartmann and M. Lesch, Zeta and Fredholm determinants of self-adjoint operators, *arXiv:2106.02444* (2021).

[10] D. Liu, Y. Matiyasevich, J. Oesterlé, and A. Zaharescu, Euler Product Sieve, *arXiv:2406.00786* (2024).

[11] E. Yakaboylu, On the Existence of the Hilbert–Pólya Hamiltonian, *arXiv:2408.15135* (2024).

[12] E. Yakaboylu, Hamiltonian for the Hilbert–Pólya Conjecture, *arXiv:2309.00405* (2023).

[13] A. LeClair and G. Mussardo, Riemann zeros as quantized energies of scattering with impurities, *arXiv:2307.01254* (2023).

[14] A. LeClair, Spectral Flow for the Riemann Zeros, *arXiv:2406.01828* (2024).

[15] X. Suo, On the Hamiltonian with Energy Levels Corresponding to Riemann Zeros, *arXiv:2505.21192* (2025).

[16] Z. Chen, Non-Abelian Observable-Geometric Phases and the Riemann Zeros, *arXiv:2403.19118* (2024).

[17] M. Suzuki, On the Hilbert space derived from the Weil distribution, *arXiv:2301.00421* (2023).

[18] Z. Zhang, Equivalence between the zero distributions of the Riemann zeta function and a 2D Ising model, *arXiv:2411.16777* (2024).

[19] F. Giordano, S. Negro, and R. Tateo, The Generalised Born Oscillator and the Berry–Keating Hamiltonian, *arXiv:2307.15025* (2023).

[20] L. Bétermin, L. Šamaj, and I. Travěnec, On off-critical zeros of lattice energies near the Riemann zeta function, *arXiv:2307.06002* (2023).

[21] V. Margarint and S. Molchanov, On the analytic extension of Random Riemann Zeta Functions, *arXiv:2410.03044* (2024).

[22] D.-M. Welz, Ein reguliertes Flächenintegral als Entscheidungskriterium für die Riemannsche Vermutung, *arXiv:2505.23238* (2025).

[23] A. LeClair, Phenomenological formula for Quantum Hall resistivity based on the Riemann zeta function, *arXiv:2209.07468* (2022).

[24] D.A. Goldston, J. Lee, J. Schettler, and A.I. Suriajaya, Pair Correlation Conjecture for the Zeros of $\zeta(s)$ I: Simple and Critical Zeros, *arXiv:2503.15449* (2025).

[25] N. Benjamin and C.-C. Chang, Scalar Modular Bootstrap and Zeros of the Riemann Zeta Function, *arXiv:2208.02259* (2022).

[26] G. Chavez and A. Allawala, Prime zeta function statistics and Riemann zero-difference repulsion, *arXiv:2102.02280* (2021).