

## RESEARCH ARTICLE

# Velocity Direction Assumption as Sufficient Condition for Regularity of Weak Solutions to the Navier–Stokes Equations

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## ABSTRACT

In this paper, we show that a simple geometrical assumption on the direction of the velocity leads to the regularity of weak solutions of the 3D Navier–Stokes equations on a smooth bounded domain. We prove two versions of this result: In the first one, we use a Lipschitz-continuity assumption on the velocity direction in all sufficiently close space points. In the second version, only points whose distance is equal to a sufficiently small positive number dependent on the data are considered.

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## 1 | Introduction

Let  $\Omega$  be a smooth bounded simply connected three-dimensional domain or the whole space  $\mathbb{R}^3$ . We consider the initial value problem for the Navier–Stokes equations

$$\frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u + \nabla P = 0 \quad \text{in } \Omega \times (0, \infty), \quad (1)$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \times (0, \infty), \quad (2)$$

$$u|_{t=0} = u_0, \quad (3)$$

where  $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$  and  $P = P(x, t)$  denote the unknown velocity and pressure and  $u_0 = u_0(x)$  is the initial velocity. If  $\Omega$  is bounded the problem (1–3) is supplemented with the boundary conditions of the Navier type

$$(u \cdot v)|_{\partial\Omega \times (0, \infty)} = 0 \text{ and } (\omega \times v)|_{\partial\Omega \times (0, \infty)} = 0, \quad (4)$$

where  $\omega = \nabla \times u$  is the vorticity field and  $v = (v_1, v_2, v_3)$  denotes the exterior unit normal vector. On the right-hand side of (1) and the second equality (4), 0 denotes the unit three-dimensional vector.

For simplicity in the following text, if there is no danger of misunderstanding, we denote  $L^p = L^p(\Omega)$ ,  $(L^2)^3 = L^2$  and  $\|v\|_p = \|v\|_{L^p(\Omega)}$  and similarly for other spaces. We define  $H^1/\mathbb{R} = \{v \in H^1; \int_{\Omega} v = 0\}$ . Further, in the integrals over space, we do not use  $dx$ , and similarly, in the integrals over time, we do not write  $dt$ . Moreover, we use  $\int$  instead of  $\int_{\mathbb{R}^3}$ .  $L_\sigma^2$  is defined as a closure in  $L^2$  of  $C_{0,\sigma}^\infty$  (the space of infinitely differentiable divergence-free vector functions in  $\Omega$  with a compact support in  $\Omega$ ) and  $H_\sigma^1 = H^1 \cap L_\sigma^2$ .  $\angle(a, b)$  denotes the angle between the vectors  $a$  and  $b$ .

**Definition 1.** Let  $T > 0$  and  $u_0 \in L_\sigma^2$ . A function  $u \in L^\infty(0, T; L_\sigma^2) \cap L^2(0, T; H_\sigma^1)$  is called a weak solution of (1–4) (in the case of a bounded domain) or of (1–3) (in the case of the whole space) on  $(0, T)$  with the initial condition  $u_0$  if

$$\int_0^T \int_{\Omega} \left( -u \cdot \frac{\partial \phi}{\partial t} + (\nabla \times u) \cdot (\nabla \times \phi) + (u \cdot \nabla) u \cdot \phi \right) = \int_{\Omega} u_0 \cdot \phi(\cdot, 0),$$

for all infinitely differentiable functions  $\phi$  in  $\overline{\Omega \times (0, T)}$  such that  $\phi(\cdot, t) \in H_\sigma^1$  for all  $t \in [0, T]$  and  $\phi(\cdot, T) = 0$ . Moreover, if there exists  $c = c(\Omega)$  such that the energy inequality

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 \leq \|u_0\|_2^2 e^{2ct}$$

is satisfied for all  $t \in [0, T]$ , then  $u$  is called a Leray weak solution. A weak solution  $u$  is called a strong solution on  $[0, T]$  if  $u \in L^\infty(0, T; H^1)$ , and, respectively, on  $[0, T)$  if  $u \in L^\infty(0, T - \varepsilon; H^1)$  for every  $\varepsilon > 0$ .

It is known that for every  $T > 0$ , there exists at least one Leray weak solution of (1–4) on  $(0, T)$ . If  $u$  is a strong solution on  $[0, T)$ , then it is smooth in space-time, that is,  $u \in \overline{\Omega \times (0, T)}$ . If  $u_0 \in H_\sigma^1$ , then there exists  $T^* = T^*(\|\nabla u_0\|_2)$  such that there exists a unique strong solution on  $[0, T^*)$  with the initial condition  $u_0$ . See [1, 2] and the papers cited there for the facts presented in this paragraph.

In the literature, one can find interesting results on the regularity of Leray weak solutions of (1–3) both in the whole space and a bounded domain (supplemented with boundary conditions 4) stemming from assumptions on the vorticity direction. The following three theorems, which have been proved in [1, 3, 4] (see also [5, 6]), represent the contemporary state of the art. To ensure simplicity and consistency with the notation of the present paper, Theorems 1–3 are slight adjustments of the original results.

**Theorem 1.** Let  $T > 0$  and let  $u$  be a Leray weak solution of (1–3) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1(\mathbb{R}^3)$ . Suppose that  $c_1$ ,  $K$ , and  $\delta$  are given positive numbers and

$$\sin \angle(\omega(x, t), \omega(y, t)) \leq c_1 |x - y|^{\frac{1}{2}}, \quad (5)$$

for all  $(x, t), (y, t) \in \mathbb{R}^3 \times (0, T)$ , such that both  $|\omega(x, t)|$  and  $|\omega(y, t)|$  are greater than  $K$  and  $|x - y| \leq \delta$ . Then,  $u$  is regular on  $(0, T]$ .

The proof of Theorem 1 is relatively simple. Starting with the equation

$$\frac{\partial \omega}{\partial t} - \Delta \omega + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u,$$

one gets

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \|\nabla \omega\|_2^2 = \int_{\mathbb{R}^3} \omega_i \frac{\partial u_i}{\partial x_l} \omega_l \, dx.$$

Applying then the Biot–Savart law

$$\frac{\partial u_i}{\partial x_l} = -\frac{1}{3} \epsilon_{ilk} \omega_k - \epsilon_{ijk} \left( p.v. \frac{\delta_{jl} - \frac{3x_j x_l}{|x|^2}}{4\pi|x|^3} \right) * \omega_k,$$

where  $\epsilon_{ijk}$  is the Levi–Civita symbol, one can write

$$\omega_l(x) \frac{\partial u_i}{\partial x_l}(x) \omega_i(x) = \lim_{\varepsilon \rightarrow 0_+} \int_{B_\varepsilon^c(0)} (\omega(x) \times \omega(x-y))_j \omega_l(x) \left( \frac{\delta_{jl} - \frac{3y_j y_l}{|y|^2}}{4\pi |y|^3} \right) dy,$$

and it enables us to use the condition (5):

$$|(\omega(x) \times \omega(x-y))| \leq c_1 |y|^{1/2} |\omega(x)| |\omega(x-y)|.$$

The proof then follows by the use of Hardy–Littlewood–Sobolev inequality.

**Theorem 2.** *Let  $T > 0$  and let  $u$  be a Leray weak solution of (1–4) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1$ . Suppose that  $c_1 > 0$  and*

$$\sin \angle(\omega(x, t), \omega(y, t)) \leq c_1 |x - y|^{\frac{1}{2}}, \quad (6)$$

for all  $(x, t), (y, t) \in \Omega \times (0, T)$ . Then,  $u$  is regular on  $(0, T]$ .

The proof of Theorem 2 is based on an explicit representation of the velocity in terms of the vorticity by means of Green's matrices and is technically complicated.

**Theorem 3.** *Let  $T > 0$  and let  $u$  be a Leray weak solution of (1–3) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1$ . Then, there exists a constant  $c_1 = c_1(T, \|\nabla u_0\|_2) > 0$  such that if for all  $x, y$  with  $|x - y| < \lambda$  for some small  $\lambda$  and for almost all  $t \in (0, T)$*

$$\sin(\angle(\omega(x, t), \omega(y, t))) \leq c_1, \quad (7)$$

then  $u$  is regular on  $(0, T]$ .

Unlike Theorems 1 and 2, Theorem 3 only requires that the angle between vorticity vectors  $\omega(x, t)$  and  $\omega(y, t)$  remains small enough when  $x$  is close to  $y$ . For further results concerning regularity in terms of the vorticity direction, see [1, 3–16].

As the main goal of this paper, we will show two regularity results (Theorems 6 and 8) in which the regularity stems from the assumptions put on the direction of the velocity. Both the velocity and vorticity are fundamental quantities in fluid dynamics and it is surprising that except the paper by Vasseur, see [17], there are no results in the literature using assumptions on the velocity direction for the proof of the regularity. We will concentrate on the case of a smooth bounded domain with the boundary conditions (4), the case of the whole space having recently been discussed in [18]. The precise wording of the results from [18] is presented in the following Theorems 4 and 5.

**Theorem 4.** *Let  $T > 0$  and let  $u$  be a Leray weak solution of (1–3) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1$ . Suppose that  $c_1$ ,  $K$ , and  $\delta$  are given positive numbers and*

$$\sin(\angle(u(x, t), u(y, t))) \leq c_1 |x - y|, \quad (8)$$

for almost all  $(x, t), (y, t) \in \mathbb{R}^3 \times (0, T)$  such that both  $|u(x, t)|$  and  $|u(y, t)|$  are greater than  $K$  and  $|x - y| \leq \delta$ . Then,  $u$  is regular on  $(0, T]$ .

**Theorem 5.** *There exist a constant  $c > 0$  and a real function  $\lambda = \lambda(s_1, s_2, s_3, s_4, s_5) > 0$  defined on  $\{(s_1, s_2, s_3, s_4, s_5) \in (0, \infty)^5; s_1 \geq 1/(2cs_3)^{1/4}\}$  and decreasing in all five variables such that if we take positive numbers  $T, C, K$  and a Leray weak solution of (1–3) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1$  such that  $\|\nabla u_0\|_2 \geq 1/(2cT)^{1/4}$  and*

$$\sin(\angle(u(x, t), u(y, t))) \leq Ch, \quad (9)$$

for a fixed  $h \leq \lambda(\|\nabla u_0\|_2, \|u_0\|_2, T, C, K)$ , almost all  $(x, t), (y, t) \in \mathbb{R}^3 \times (0, T)$  such that  $|x - y| = h$  and both  $|u(x, t)|$  and  $|u(y, t)|$  are greater than  $K$ , then  $u$  is regular on  $(0, T]$ .

**Remark 1.** As is described in the proofs of Theorems 7 and 8, in the case of the boundary conditions (4), one can easily handle the boundary integrals. This is not the case for the Navier boundary conditions

$$(u \cdot v)|_{\partial\Omega \times (0, \infty)} = 0 \text{ and } (((\nabla u) + (\nabla u)^T)v)_\tau + \gamma u|_{\partial\Omega \times (0, \infty)} = 0, \quad (10)$$

where  $\gamma > 0$  denotes the friction coefficient between the fluid and the fixed wall and  $\tau$  denotes the projection onto the tangent plane or the Dirichlet boundary conditions

$$u|_{\partial\Omega \times (0, \infty)} = 0, \quad (11)$$

where we were unable to get rid of or suitably estimate the corresponding boundary integrals. It concerns, for example, the integral  $\int_{\partial\Omega} u' \cdot (\omega \times v)$  in (40), which is equal to zero in the case of the boundary conditions (4), but it is unclear how to estimate it for the boundary conditions (10). Similarly, it is not obvious, how to estimate the integral  $\int_{\partial\Omega} \left( \pi + \frac{|u|^2}{2} \right) (v \cdot \Delta u)$  in (41) in the case of the boundary conditions (10) and (11). To get regularity of the solution in this case, it would probably be necessary to start with a different formulation of the Navier–Stokes equations instead of the formulation (39) which is perfectly tailored to the boundary conditions of the Navier type. Thus, the regularity of Leray weak solutions of the systems (1), (2), (3), (10) and (1), (2), (3), (11) satisfying the conditions (38) or (47) remains an open problem. This also applies when considering the angle between the vorticity vectors.

## 2 | Two Lemmas

The proof of the following lemma follows from Theorems 4.4 and 4.8 in [19] and the Agmon–Douglis–Nirenberg theory for partial differential equations; see, for example, [20].

**Lemma 1.** *Let  $\Omega$  be a smooth bounded simply connected three-dimensional domain and let  $f \in H^q$ ,  $q \in \mathbb{N}_0$ . Then, there exists a unique strong solution  $(u, \mathcal{P}) \in H^2 \times H^1/\mathbb{R}$  of the Stokes system with the boundary conditions of the Navier type, that is,*

$$\begin{aligned} -\Delta u + \nabla \mathcal{P} &= f \quad \text{in } \Omega, \\ \nabla \cdot u &= 0 \quad \text{in } \Omega, \\ (u \cdot v)|_{\partial\Omega} &= 0, \\ ((\nabla \times u) \times v)|_{\partial\Omega} &= 0. \end{aligned}$$

Moreover,  $u \in H^{q+2}$ ,  $\mathcal{P} \in H^{q+1}$ , and there exists a constant  $c = c(q)$  independent of  $f$ ,  $u$ , and  $p$  such that

$$\|u\|_{H^{q+2}} + \|\mathcal{P}\|_{H^{q+1}} \leq c \|f\|_{H^q}.$$

It follows immediately from Lemma 1 that there exists a positive constant  $c$  such that if

$$v \in H^2, \nabla \cdot v = 0, (\nabla \times v) \times v = 0 \text{ on } \partial\Omega \text{ and } v \cdot v = 0 \text{ on } \partial\Omega,$$

then

$$\|v\|_{H^2} \leq c \|\Delta v\|_2. \quad (12)$$

We will further use the following inequalities. If  $v \in H^2$ ,  $\nabla \cdot v = 0$  and  $(\nabla \times v) \times v = 0$  on  $\partial\Omega$ , then

$$\|v\|_{H^1} \leq c \|\nabla \times v\|_2, \text{ see [2], (1.6).} \quad (13)$$

If  $v \in H^2$  and  $\nabla \cdot v = 0$ , then  $(\Delta v \cdot v)|_{\partial\Omega}$  can be understood as an element of  $H^{-1/2}(\partial\Omega) = H^{1/2}(\partial\Omega)^*$ , and if moreover,  $(\nabla \times v) \times v = 0$  on  $\partial\Omega$ , then

$$(\Delta v \cdot v)|_{\partial\Omega} = 0. \quad (14)$$

Equation (14) can be proved in the same way as Theorem 1.2 from [21]. Finally,

$$H^1(\Omega) \hookrightarrow L^4(\partial\Omega), \text{ see [22].} \quad (15)$$

We will also use the fact that if  $u$  is a Leray weak solutions of (1–4) with the initial condition  $u_0 \in H_\sigma^1$  and  $\nabla u \in L^\infty(t_1, t_2; L^2)$ , where  $0 \leq t_1 < t_2 < \infty$ , then  $u \in C^\infty(\bar{\Omega} \times (t_1, t_2))$  (see [1], the paragraph below Remark 2.8).

The following lemma serves as a key argument in the proof of Theorems 6–8.

**Lemma 2.** *Let  $k \in \{2, 3, 4\}$ . Then, there exists a real positive function  $\Phi_k$  defined on  $\{(s_1, s_2, s_3, s_4) \in (0, \infty)^4; s_2 < s_3\}$ , nondecreasing in  $s_1, s_3$  and  $s_4$  and nonincreasing in  $s_2$  such that if  $T > 0$ ,  $\varepsilon \in (0, T)$  and  $u$  is a Leray weak solution of (1–4) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1$  such that  $\nabla u \in L^\infty(0, T; L^2)$  then*

$$\|\nabla^k u\|_{L^\infty(\varepsilon, T; L^2)}^2 \leq \Phi_k(\|\nabla u\|_{L^\infty(0, T; L^2)}, \varepsilon, T, \|u_0\|_2). \quad (16)$$

*Proof.* Let  $T > 0$ ,  $\varepsilon \in (0, T)$  and let  $u$  be a Leray weak solution of (1–4) on  $(0, T)$  satisfying the assumptions from Lemma 2. In the following estimates, we will repeatedly use the inequalities (12–14). Multiplying (1) by  $-\Delta u = \nabla \times \omega$  and using (4), we get

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \|\Delta u\|_2^2 + \int (u \cdot \nabla) u \cdot \Delta u = 0.$$

By the use of (12) and (13),

$$\begin{aligned} \left| \int (u \cdot \nabla) u \cdot \Delta u \right| &\leq \|u\|_6 \|\nabla u\|_3 \|\Delta u\|_2 \leq c \|u\|_{H^1} \|\nabla u\|_2^{\frac{1}{2}} \|\nabla u\|_6^{\frac{1}{2}} \|\Delta u\|_2 \\ &\leq c \|\omega\|_2^{\frac{3}{2}} \|\Delta u\|_2^{\frac{3}{2}} \leq \frac{1}{2} \|\Delta u\|_2^2 + c \|\omega\|_2^6, \end{aligned}$$

and consequently,

$$\frac{d}{dt} \|\omega\|_2^2 + \|\Delta u\|_2^2 \leq c \|\omega\|_2^6.$$

Integrating the last inequality over  $(0, T)$ , we get

$$\|u\|_{L^2(0, T; H^2)}^2 \leq cT \|\nabla u\|_{L^\infty(0, T; L^2)}^6 + \|\nabla \times u_0\|_2^2 \leq cT \|\nabla u\|_{L^\infty(0, T; L^2)}^6 + c \|\nabla u\|_{L^\infty(0, T; L^2)}^2.$$

Defining  $\Phi$  as

$$\Phi(s_1, s_2, s_3, s_4) = cs_3s_1^6 + cs_1^2,$$

we can write

$$\|u\|_{L^2(0, T; H^2)}^2 \leq \Phi(\|\nabla u\|_{L^\infty(0, T; L^2)}, \varepsilon, T, \|u_0\|_2). \quad (17)$$

Multiplying (1) by  $u' \equiv \partial u / \partial t$ , we get

$$\frac{d}{dt} \|\omega\|_2^2 + \|u'\|_2^2 \leq c \|\omega\|_2^2 \|\Delta u\|_2^2,$$

and integrating over  $(0, T)$ , we arrive at

$$\begin{aligned} \|u'\|_{L^2(0, T; L^2)}^2 &\leq \|\nabla \times u_0\|_2^2 + \int_0^T c \|\omega\|_2^2 \|\Delta u\|_2^2 \\ &\leq c \|\nabla u\|_{L^\infty(0, T; L^2)}^2 + c \|\nabla u\|_{L^\infty(0, T; L^2)}^2 \|u\|_{L^2(0, T; H^2)}^2 \\ &\leq c \|\nabla u\|_{L^\infty(0, T; L^2)}^2 (1 + \|u\|_{L^2(0, T; H^2)}^2) \\ &\leq (1 + c \|\nabla u\|_{L^\infty(0, T; L^2)}^2) (1 + \Phi(\|\nabla u\|_{L^\infty(0, T; L^2)}, \varepsilon, T, \|u_0\|_2)). \end{aligned}$$

Defining  $\Phi_{\text{new}}$  as

$$\Phi_{\text{new}}(s_1, s_2, s_3, s_4) = (1 + cs_1^2)(1 + \Phi(s_1, s_2, s_3, s_4)),$$

we have  $\Phi_{\text{new}} \geq \Phi$  and

$$\|u'\|_{L^2(0, T; L^2)}^2 \leq \Phi_{\text{new}}(\|\nabla u\|_{L^\infty(0, T; L^2)}, \varepsilon, T, \|u_0\|_2).$$

In the following, we write, for simplicity,  $\Phi$  instead of  $\Phi_{\text{new}}$ , and so we have

$$\|u'\|_{L^2(0,T;L^2)}^2 \leq \Phi(\|\nabla u\|_{L^\infty(0,T;L^2)}, \varepsilon, T, \|u_0\|_2). \quad (18)$$

For simplicity, we will now write  $\Phi$  instead of  $\Phi(\|\nabla u\|_{L^\infty(0,T;L^2)}, \varepsilon, T, \|u_0\|_2)$ . It follows from (18) that there exists  $t_1 \in (\varepsilon/2, 5\varepsilon/8)$  such that

$$\|u'(t_1)\|_2^2 \leq \frac{8}{\varepsilon} \Phi. \quad (19)$$

Differentiating (1) along  $t$  and multiplying it by  $u'$ , we get

$$\frac{d}{dt} \|u'\|_2^2 + \|\omega'\|_2^2 \leq c \|\Delta u\|_2^2 \|u'\|_2^2.$$

It now follows from (17) and (19) that for every  $t \in (t_1, T)$ ,

$$\|u'(t)\|_2^2 \leq \|u'(t_1)\|_2^2 \exp\left(\int_{t_1}^T c \|\Delta u\|_2^2 dt\right) \leq \frac{8\Phi}{\varepsilon} \exp(c\Phi),$$

and

$$\int_{t_1}^T \|\omega'\|_2^2 \leq \|u'(t_1)\|_2^2 + c \int_{t_1}^T \|\Delta u\|_2^2 \|u'\|_2^2 \leq \frac{8\Phi}{\varepsilon} + c \frac{8\Phi}{\varepsilon} \exp(c\Phi) \Phi.$$

Defining  $\Phi_{\text{new}}$  as

$$\Phi_{\text{new}}(s_1, s_2, s_3, s_4) = \Phi(s_1, s_2, s_3, s_4) + \frac{8\Phi(s_1, s_2, s_3, s_4)}{\varepsilon} (1 + (1 + c\Phi) \exp(c\Phi)),$$

we see that  $\Phi_{\text{new}} \geq \Phi$  and  $\Phi_{\text{new}}$  is nondecreasing in  $s_1, s_3$ , and  $s_4$  and nonincreasing in  $s_2$ . Writing again as above  $\Phi$  instead of  $\Phi_{\text{new}}$ , we arrive at

$$\|u'\|_{L^\infty(t_1, T; L^2)}^2 \leq \Phi, \quad (20)$$

and

$$\|\omega'\|_{L^2(t_1, T; L^2)}^2 \leq \Phi. \quad (21)$$

Multiplying (1) by  $-\Delta u$ , we get

$$-(u', \Delta u) + \|\Delta u\|_2^2 - (u \cdot \nabla u, \Delta u) = 0,$$

and consequently,

$$\begin{aligned} \|\Delta u\|_2 &\leq \|u'\|_2 + \|u \cdot \nabla u\|_2 \leq \|u'\|_2 + \|u\|_6 \|\nabla u\|_3 \\ &\leq \|u'\|_2 + \|\nabla u\|_2^{\frac{3}{2}} \|\Delta u\|_2^{\frac{1}{2}} \leq \|u'\|_2 + \frac{1}{2} \|\Delta u\|_2 + c \|\nabla u\|_2^3. \end{aligned}$$

Thus,

$$\|\Delta u\|_2 \leq 2\|u'\|_2 + c \|\nabla u\|_2^3.$$

For simplicity, we now (and in the following) omit the consideration of  $\Phi_{\text{new}}$  and immediately get to the conclusion (by the use of (20)) that

$$\|u\|_{L^\infty(t_1, T; H^2)}^2 \leq \Phi. \quad (22)$$

Putting  $\Phi_2 = \Phi$ , Lemma 2 is proved for  $k = 2$ .

It now follows from (21) that there exists  $t_2 \in (5\varepsilon/8, 3\varepsilon/4)$  such that

$$\|\omega'(t_2)\|_2^2 \leq \frac{8\Phi}{\varepsilon}. \quad (23)$$

Differentiating (1) along  $t$  and multiplying it by  $-\Delta u'$ , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega'\|_2^2 + \|\Delta u'\|_2^2 &\leq \left| \int_{\Omega} u' \nabla u \Delta u' \right| + \left| \int_{\Omega} u \nabla u' \Delta u' \right| \\ &\leq c \|u'\|_3 \|\nabla u\|_6 \|\Delta u'\|_2 + c \|u\|_{\infty} \|\nabla u'\|_2 \|\Delta u'\|_2 \\ &\leq c \|\omega'\|_2 \|u\|_{H^2} \|\Delta u'\|_2 \leq \frac{1}{2} \|\Delta u'\|_2^2 + c \|\omega'\|_2^2 \|u\|_{H^2}^2. \end{aligned}$$

It now follows from (22) and (23) (redefining  $\Phi$  in a similar way as above) that

$$\|\omega'\|_{L^\infty(t_2, T; L^2)}^2 \leq \Phi, \quad (24)$$

and

$$\|\Delta u'\|_{L^2(t_2, T; L^2)}^2 \leq \Phi. \quad (25)$$

Differentiating (1) along  $t$  and multiplying it by  $u''$ , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega'\|_2^2 + \|u''\|_2^2 &\leq \left| \int_{\Omega} u' \nabla u u'' \right| + \left| \int_{\Omega} u \nabla u' u'' \right| \\ &\leq \|u'\|_6 \|\nabla u\|_3 \|u''\|_2 + \|u\|_{\infty} \|\nabla u'\|_2 \|u''\|_2 \\ &\leq \|\omega'\|_2 \|\Delta u\|_2 \|u''\|_2 \leq \frac{1}{2} \|u''\|_2^2 + c \|\omega'\|_2^2 \|\Delta u\|_2^2, \end{aligned}$$

and consequently, by the use of (22) and (24),

$$\|u''\|_{L^2(t_2, T; L^2)}^2 \leq \Phi. \quad (26)$$

Point out that we have stopped commenting on the redefinition of  $\Phi$  in every step, because it can always be done in a similar way as above. The “new”  $\Phi$  is always bigger than the “old” one and nondecreasing in  $s_1$ ,  $s_3$ , and  $s_4$  and nonincreasing in  $s_2$ .

We now write (1) as

$$-\Delta u + \nabla \mathcal{P} = -(u \cdot \nabla) u - u' \quad (27)$$

and estimate the right-hand side of (27):

$$\|u'\|_{H^1} \leq \|\nabla \times u'\|_2 = \|\omega'\|_2,$$

and

$$\begin{aligned} \|(u \cdot \nabla) u\|_{H^1} &\leq \|(u \cdot \nabla) u\|_2 + \|\nabla u \cdot \nabla u\|_2 + \|u \cdot \nabla^2 u\|_2 \\ &\leq \|u\|_6 \|\nabla u\|_3 + \|\nabla u\|_6^2 + \|u\|_{\infty} \|\nabla^2 u\|_2 \leq c \|\Delta u\|_2^2. \end{aligned}$$

It now follows from (22) and (24) that

$$\|u' + (u \cdot \nabla) u\|_{L^\infty(t_2, T; H^1)} \leq \Phi,$$

and using Lemma 1, we obtain

$$\|u\|_{L^\infty(t_2, T; H^3)}^2 \leq \Phi. \quad (28)$$

Putting  $\Phi_3 = \Phi$ , Lemma 2 is proved for  $k = 3$ .

Differentiating (1) twice along  $t$  and multiplying it by  $u''$ , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u''\|_2^2 + \|\omega''\|_2^2 &\leq \left| \int_{\Omega} u'' \nabla u u'' \right| + \left| \int_{\Omega} u' \nabla u' u'' \right| + \left| \int_{\Omega} u \nabla u'' u'' \right| \\ &\leq \|\nabla u\|_{\infty} \|u''\|_2^2 + \|u'\|_{\infty} \|\nabla u'\|_2 \|u''\|_2 + \|u\|_{\infty} \|\nabla u''\|_2 \|u''\|_2 \\ &\leq c \|u\|_{H^3} \|u''\|_2^2 + c \|\Delta u'\|_2 \|\omega'\|_2 \|u''\|_2 + c \|\Delta u\|_2 \|\omega'\|_2 \|u''\|_2 \\ &\leq c \|u\|_{H^3} \|u''\|_2^2 + \|\omega'\|_2^2 + c \|\Delta u'\|_2^2 \|u''\|_2^2 + \frac{1}{2} \|\omega''\|_2^2 + c \|\Delta u\|_2^2 \|u''\|_2^2, \end{aligned}$$

and

$$\frac{d}{dt} \|u''\|_2^2 + \|\omega''\|_2^2 \leq c(\|u\|_{H^3} + \|\Delta u'\|_2^2 + \|\Delta u\|_2^2) \|u''\|_2^2 + \|\omega'\|_2^2.$$

It now follows from (17), (24), (25), and (28) and by the use of the Grönwall lemma that

$$\|u''\|_{L^\infty(t_2, T; L^2)}^2 \leq \Phi, \quad (29)$$

and

$$\|\omega''\|_{L^2(t_2, T; L^2)}^2 \leq \Phi. \quad (30)$$

It now follows from (30) that there exists  $t_3 \in (3\varepsilon/4, 7\varepsilon/8)$  such that

$$\|\omega''(t_3)\|_2^2 \leq \frac{8\Phi}{\varepsilon}. \quad (31)$$

Differentiating (1) twice along  $t$  and multiplying it by  $-\Delta u''$ , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega''\|_2^2 + \|\Delta u''\|_2^2 &\leq \left| \int_\Omega u'' \nabla u \Delta u'' \right| + \left| \int_\Omega u' \nabla u' \Delta u'' \right| + \left| \int_\Omega u \nabla u'' \Delta u'' \right| \\ &\leq \|u''\|_2 \|\nabla u\|_\infty \|\Delta u''\|_2 + \|u'\|_\infty \|\nabla u'\|_2 \|\Delta u''\|_2 + \|u\|_\infty \|\nabla u''\|_2 \|\Delta u''\|_2 \\ &\leq \frac{1}{6} \|\Delta u''\|_2^2 + \|u''\|_2^2 \|u\|_{H^3}^2 + \frac{1}{6} \|\Delta u''\|_2^2 + \|\Delta u'\|_2^2 \|\omega'\|_2^2 + \frac{1}{6} \|\Delta u''\|_2^2 + \|\Delta u\|_2^2 \|\omega''\|_2^2. \end{aligned}$$

It now follows from (17), (24), (25), (26), (28), and (31) that

$$\|\omega''\|_{L^\infty(t_3, T; L^2)}^2 \leq \Phi, \quad (32)$$

and

$$\|\Delta u''\|_{L^2(t_3, T; L^2)}^2 \leq \Phi. \quad (33)$$

It further follows from (25) that there exists  $t_4 \in (7\varepsilon/8, \varepsilon)$  such that

$$\|\Delta u'(t_4)\|_2^2 \leq \frac{8\Phi}{\varepsilon}, \quad (34)$$

and integrating  $\Delta u''$  over time we get from (33) and (34) that

$$\|\Delta u'\|_{L^\infty(t_4, T; L^2)}^2 \leq \Phi. \quad (35)$$

We can now estimate by the use of (28)

$$\|\nabla^2 u \cdot \nabla u\|_2 \leq \|\nabla^2 u\|_4 \|\nabla u\|_4 \leq \|u\|_{H^3}^2 \leq \Phi,$$

and

$$\|u \cdot \nabla^3 u\|_2 \leq \|u\|_\infty \|u\|_{H^3} \leq \|u\|_{H^3}^2 \leq \Phi.$$

Using also (35), we have

$$\|u' + (u \cdot \nabla) u\|_{H^2} \leq \Phi \text{ for every } t \in (t_4, T).$$

It now follows from (27) and by the use of Lemma 1 that

$$\|u\|_{L^\infty(t_4, T; H^4)} \leq \Phi. \quad (36)$$

Putting  $\Phi_4 = \Phi$ , the proof of Lemma 2 is complete.  $\square$

### 3 | Proofs of Main Results

**Theorem 6.** Let  $T > 0$  and let  $u$  be a Leray weak solution of (1–4) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1$ . Suppose that  $c_1$ ,  $K$ , and  $\delta$  are arbitrary positive numbers and

$$\sin \angle(u(x, t), u(y, t)) \leq c_1 |x - y|, \quad (37)$$

for almost all  $(x, t), (y, t) \in \Omega \times (0, T)$ , such that both  $|u(x, t)|$  and  $|u(y, t)|$  are greater than  $K$  and  $|x - y| \leq \delta$ . Then,  $u$  is regular on  $(0, T]$ .

Compare Theorem 6 with Theorem 2. The proof of Theorem 2 is based on an explicit representation of the velocity in terms of the vorticity by means of Green's matrices and is technically complicated. It is not clear to us how to apply this procedure for the proof of Theorem 6, where the condition is imposed on the angle between velocity vectors. So in the proof of Theorem 6, we start with the equation  $\partial u / \partial t - \Delta u + \omega \times u + \nabla(\pi + |u|^2/2) = 0$ , which is, as it turns out, well tailored to the conditions (4). The conditions (4) enable us to nullify the boundary integrals and estimate conveniently the volume integrals. On the other hand, using the fact that  $\partial u_i / \partial x_j(x) = \lim_{h \rightarrow 0^+} (u(x + he_j) - u(x))/h$  has two consequences. Firstly, we are not able to prove Theorem 6 by the use of the  $1/2$ -Hölder continuity assumption and need to make a weaker Lipschitz-continuity assumption (see 37 and 6). Whether it is possible to replace the Lipschitz-continuity assumption in (37) by a stronger  $\beta$ -Hölder continuity assumption for some  $\beta \in [1/2, 1)$  remains an open problem. Secondly, we must be careful with  $u(x + he_j)$  not to fall outside  $\Omega$ —see the estimate of  $A_2$ .

For simplicity, we will prove the following simpler version of Theorem 6.

**Theorem 7.** Let  $T > 0$  and let  $u$  be a Leray weak solution of (1–4) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1$ . Suppose that  $c_1$  is an arbitrary positive number and

$$\sin \angle(u(x, t), u(y, t)) \leq c_1 |x - y|, \quad (38)$$

for almost all  $(x, t), (y, t) \in \Omega \times (0, T)$ . Then,  $u$  is regular on  $(0, T]$ .

*Proof.* We can assume without loss of generality that  $\nabla u \in L_{\text{loc}}^2((0, T); L^2)$  and (consequently)  $u \in C^\infty(\overline{\Omega} \times (0, T))$ . We will use the following formulation of the Equation (1) (writing again  $u'$  instead of  $\partial u / \partial t$ ):

$$u' - \Delta u + \omega \times u + \nabla(\pi + |u|^2/2) = 0. \quad (39)$$

Taking the scalar product of (39) with  $-\Delta u$  and integrating over  $\Omega$ , we get

$$-\int_{\Omega} u' \cdot \Delta u + \int_{\Omega} |\Delta u|^2 = \int_{\Omega} (\omega \times u) \cdot \Delta u + \int_{\Omega} \nabla(\pi + |u|^2/2) \cdot \Delta u.$$

Because

$$-\Delta u = \nabla \times \omega - \nabla(\nabla \cdot u) = \nabla \times \omega,$$

we have

$$\begin{aligned} -\int_{\Omega} u' \cdot \Delta u &= \int_{\Omega} u' \cdot (\nabla \times \omega) = \int_{\Omega} u'_i \epsilon_{ijk} \frac{\partial \omega_j}{\partial x_k} = \int_{\partial \Omega} u'_i \epsilon_{ijk} v_k \omega_j - \int_{\Omega} \frac{\partial u'_i}{\partial x_k} \epsilon_{ijk} \omega_j \\ &= \int_{\partial \Omega} u' \cdot (\omega \times v) + \int_{\Omega} \omega'_j \omega_j = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\omega|^2 = \frac{1}{2} \frac{d}{dt} \|\omega\|_2^2, \end{aligned} \quad (40)$$

and using (14), we obtain

$$\int_{\Omega} \nabla \left( \pi + \frac{|u|^2}{2} \right) \cdot \Delta u = \int_{\partial \Omega} \left( \pi + \frac{|u|^2}{2} \right) (\nu \cdot \Delta u) - \int_{\Omega} \left( \pi + \frac{|u|^2}{2} \right) \Delta(\nabla \cdot u) = 0. \quad (41)$$

Further,

$$\int_{\Omega} (\omega \times u) \cdot \Delta u = \int_{\Omega} (\omega \times u)_i \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \int_{\partial \Omega} (\omega \times u)_i v_j \frac{\partial u_i}{\partial x_j} - \int_{\Omega} \frac{\partial}{\partial x_j} (\omega \times u)_i \frac{\partial u_i}{\partial x_j} \equiv A_1 - A_2. \quad (42)$$

Estimating  $|A_1|$ , we can write

$$\begin{aligned} A_1 &= \int_{\partial\Omega} (\omega \times u)_i v_j \frac{\partial u_i}{\partial x_j} = \int_{\partial\Omega} (\omega \times u) \cdot \frac{\partial u}{\partial v} \\ &= - \int_{\partial\Omega} (\omega(x) \times u(x)) \cdot \lim_{h \rightarrow 0_+} \frac{u(x - hv) - u(x)}{h} \\ &= - \int_{\partial\Omega} \lim_{h \rightarrow 0_+} \frac{1}{h} (\omega(x) \times u(x)) \cdot u(x - hv), \end{aligned}$$

and so

$$\begin{aligned} |A_1| &\leq \int_{\partial\Omega} \lim_{h \rightarrow 0_+} \frac{1}{h} |(\omega(x) \times u(x)) \cdot u(x - hv)| \\ &= \int_{\partial\Omega} \lim_{h \rightarrow 0_+} \frac{1}{h} |\cos \angle(\omega(x) \times u(x), u(x - hv))| |\omega(x) \times u(x)| |u(x - hv)| \\ &\leq \int_{\partial\Omega} \limsup_{h \rightarrow 0_+} \frac{1}{h} \sin \angle(u(x), u(x - hv)) |\omega(x) \times u(x)| |u(x - hv)|. \end{aligned}$$

Using now gradually (12), (13), (15), and (38), we get

$$\begin{aligned} |A_1| &\leq \int_{\partial\Omega} \lim_{h \rightarrow 0_+} \frac{1}{h} c_1 h |\omega(x) \times u(x)| |u(x - hv)| \\ &= \int_{\partial\Omega} c_1 |\omega(x) \times u(x)| |u(x)| \leq c \int_{\partial\Omega} |\omega| |u|^2 \leq c \|\omega\|_{2,\partial\Omega} \|u\|_{4,\partial\Omega}^2 \\ &\leq c \|\omega\|_{H^1(\Omega)} \|u\|_{H^1(\Omega)}^2 \leq c \|\Delta u\|_2 \|\omega\|_2^2 \leq \frac{1}{4} \|\Delta u\|_2^2 + c \|\omega\|_2^4. \end{aligned} \tag{43}$$

To estimate  $|A_2|$ , let  $\Omega_k = \{x \in \Omega; \text{dist}(x, \partial\Omega) > 1/k\}$ ,  $k \in \mathbb{N}$ . Clearly,

$$A_2 = \lim_{k \rightarrow \infty} \int_{\Omega_k} \frac{\partial}{\partial x_j} (\omega \times u)_i \frac{\partial u_i}{\partial x_j},$$

and similarly as in the estimate of  $|A_1|$ ,

$$\begin{aligned} \left| \int_{\Omega_k} \frac{\partial}{\partial x_j} (\omega \times u)_i \frac{\partial u_i}{\partial x_j} \right| &= \left| \int_{\Omega_k} \left( \frac{\partial \omega}{\partial x_j} \times u \right) \cdot \frac{\partial u}{\partial x_j} \right| \leq \int_{\Omega_k} \left| \left( \frac{\partial \omega}{\partial x_j} \times u \right) \cdot \frac{\partial u}{\partial x_j} \right| \\ &\leq \int_{\Omega_k} \left| \left( \frac{\partial \omega}{\partial x_j}(x) \times u(x) \right) \cdot \lim_{h \rightarrow 0_+} \frac{u(x + he_j) - u(x)}{h} \right| \\ &= \int_{\Omega_k} \lim_{h \rightarrow 0_+} \left| \left( \frac{\partial \omega}{\partial x_j}(x) \times u(x) \right) \cdot \frac{u(x + he_j) - u(x)}{h} \right| \\ &= \int_{\Omega_k} \lim_{h \rightarrow 0_+} \frac{1}{h} \left| \left( \frac{\partial \omega}{\partial x_j}(x) \times u(x) \right) \cdot u(x + he_j) \right| \\ &\leq \int_{\Omega_k} \lim_{h \rightarrow 0_+} \frac{1}{h} c_1 h \left| \frac{\partial \omega}{\partial x_j}(x) \times u(x) \right| |u(x + he_j)| \\ &= \int_{\Omega_k} c_1 \left| \frac{\partial \omega}{\partial x_j}(x) \times u(x) \right| |u(x)| \leq c \int_{\Omega} |\nabla \omega| |u|^2 \\ &\leq c \|\nabla \omega\|_2 \|u\|_4^2 \leq c \|\Delta u\|_2 \|\omega\|_2^2 \leq \frac{1}{4} \|\Delta u\|_2^2 + c \|\omega\|_2^4. \end{aligned}$$

So

$$|A_2| \leq \frac{1}{4} \|\Delta u\|_2^2 + c \|\omega\|_2^4. \tag{44}$$

Finally,

$$\frac{d}{dt} \|\omega\|_2^2 + \|\Delta u\|_2^2 \leq c \|\omega\|_2^4,$$

and the regularity follows by the use of the Grönwall inequality.  $\square$

**Remark 2.** It is well known that if we take in the system (1–4)  $u_0 \in H_\sigma^1$ , then

$$\|\omega(t)\|_2^2 \leq \frac{\|\omega_0\|_2^2}{\sqrt{1 - 2ct\|\omega_0\|_2^4}} \quad \text{for every } t \in [0, 2t_0], \quad (45)$$

where

$$t_0 = \frac{1}{4c\|\omega_0\|^4} \text{ and } \omega_0 = \nabla \times u_0. \quad (46)$$

For the case of the homogeneous Dirichlet conditions, it is proved in, for example, [23] (having  $\nabla u$  instead of  $\omega$ ), for the case of the boundary conditions (4), it can be proved similarly using the formulation (39).

**Theorem 8.** *There exists a real function  $\lambda = \lambda(s_1, s_2, s_3) > 0$  defined on  $(0, \infty)^3$  and decreasing in all three variables such that if we take positive numbers  $T, C$  and a Leray weak solution of (1–4) on  $(0, T)$  with the initial condition  $u_0 \in H_\sigma^1$  such that*

$$\sin(\angle(u(x, t), u(y, t))) \leq Ch, \quad (47)$$

for a fixed positive  $h \leq \lambda(\|u_0\|_{H^1}, T, C)$ , almost all  $(x, t), (y, t) \in \Omega \times (0, T)$  such that  $|x - y| = h$ , then  $u$  is regular on  $(0, T]$ .

Compare Theorem 8 with Theorem 3 (from [4]), which has been proved for the whole space. Theorem 8 says, roughly speaking, that to obtain the regularity of  $u$  on  $(0, T]$ , it is sufficient to check the smallness of angles between the velocity vectors in points  $(x, t)$   $(y, t)$ , whose distance  $|x - y|$  is equal to a given (small) number. Thus, no assumption of the continuity of the angle between the velocity vectors is required. Notice that in Theorem 3, the smallness is required for all sufficiently close points. Tackling the velocity vectors in Theorem 8, we start again with the formulation  $\partial u / \partial t - \Delta u + \omega \times u + \nabla(\pi + |u|^2/2) = 0$ , which, as having been stressed before, complies perfectly with the conditions (4). Further, we rely heavily on Lemma 2, which is a version of Lemma 3.3 from [4] for the bounded domain and the conditions (4). Most importantly, it is necessary to estimate the term  $A_2$  and to prevent the points  $x + he_j$  and  $x + \xi_{ij}e_j$  to fall out of  $\Omega$ , which is done in detail in the text below (49).

*Proof of Theorem 8.* We proceed in the same way as in Theorem 6 up to (42). For the estimate of  $A_1$  and  $A_2$ , we now use the Taylor theorem, specifically in the case of  $A_1$  the equality

$$\frac{\partial u_i}{\partial v}(x) = \frac{u_i(x) - u_i(x - hv)}{h} + \frac{h}{2} \frac{\partial^2 u_i}{\partial v^2}(x - \xi_i v), \quad \xi_i = \xi_i(x) \in (0, h).$$

Supposing that  $h$  is sufficiently small (in dependence on  $\Omega$ ) and defining  $\Omega_h = \{x \in \Omega; \text{dist}(x, \partial\Omega) > h\}$ , we can estimate

$$\begin{aligned} |A_1| &= \left| \int_{\partial\Omega} (\omega \times u)_i \cdot \frac{\partial u_i}{\partial v} \right| = \int_{\partial\Omega} \left| (\omega \times u)_i \cdot \frac{u_i(x - hv)}{h} \right| + \left| \int_{\partial\Omega} (\omega \times u)_i \cdot \frac{h}{2} \frac{\partial^2 u_i}{\partial v^2}(x - \xi_i v) \right| \\ &\leq \int_{\partial\Omega} \frac{Ch}{h} |\omega(x) \times u(x)| |u(x - hv)| + \frac{h}{2} \|\nabla^2 u\|_{L^\infty(\Omega)} \|\omega\|_{L^2(\partial\Omega)} \|u\|_{L^2(\partial\Omega)} \\ &\leq cC \|\omega\|_{L^2(\partial\Omega)} \|u\|_{L^4(\partial\Omega)} \|u\|_{L^4(\partial\Omega)} \|u\|_{L^4(\partial\Omega_h)} + ch \|\nabla^2 u\|_{H^2} \|\Delta u\|_2 \|\omega\|_2 \\ &\leq cC \|\omega\|_{L^2(\partial\Omega)} \|u\|_{L^4(\partial\Omega)} \|u\|_{L^4(\partial\Omega_h)} + ch \|\nabla^2 u\|_{H^2} \|\Delta u\|_2 \|\omega\|_2 \\ &\leq cC \|\Delta u\|_2 \|\omega\|_2 \|u\|_{H^1(\Omega_h)} + ch \|\nabla^2 u\|_{H^2} \|\Delta u\|_2 \|\omega\|_2 \\ &\leq cC \|\Delta u\|_2 \|\omega\|_2^2 + ch \|\nabla^2 u\|_{H^2} \|\Delta u\|_2 \|\omega\|_2 \\ &\leq \frac{1}{4} \|\Delta u\|_2^2 + cC^2 \|\omega\|_2^4 + \|\omega\|_2^2 + ch^2 \|\nabla^2 u\|_{H^2}^2 \|\Delta u\|_2^2 \\ &\leq (1 + cC^2 \|\omega\|_2^2) \|\omega\|_2^2 + \left( \frac{1}{4} + ch^2 \|\nabla^2 u\|_{H^2}^2 \right) \|\Delta u\|_2^2. \end{aligned} \quad (48)$$

To estimate  $A_2$ , we will use the equality

$$\frac{\partial u_i}{\partial x_j}(x) = \frac{u_i(x + he_j) - u_i(x)}{h} - \frac{h}{2} \frac{\partial^2 u_i}{\partial x_j^2}(x + \xi_{ij} e_j), \quad \xi_{ij} = \xi_{ij}(x) \in (0, h). \quad (49)$$

In the following, we will tackle the problem that the points  $x + he_j$  and  $x + \xi_{ij} e_j$  may possibly lie outside  $\Omega$ . Because  $\partial\Omega \in C^\infty$ , there exists a sufficiently small  $h$  (depending on  $\Omega$ ) such that  $\Omega = \bigcup_{k=1}^n \Omega_k$ ,  $n \in \mathbb{N}$ , every  $\Omega_k$  has its own local orthogonal coordinate system with axes  $\{x_1^k, x_2^k, x_3^k\}$  and unit direction vectors  $\{e_1^k, e_2^k, e_3^k\}$  (which is obtained by the rotation of the original coordinate system with the axes  $\{x_1, x_2, x_3\}$  and unit direction vectors  $\{e_1, e_2, e_3\}$  and so has the same orientation) and  $x + \eta he_l^k \in \Omega$  for every  $x \in \Omega_k$ ,  $\eta \in [0, 1]$  and  $l = 1, 2, 3$ . More specifically, for  $k = 1, 2, \dots, n-1$ ,  $\Omega_k = \Omega \cap B(x_k, r_k)$ , where  $x_k \in \partial\Omega$  and  $B(x_k, r_k)$  denotes an open ball with the center  $x_k$  and radius  $r_k > 0$  and  $\Omega_n = \{x \in \Omega; \text{dist}(x, \partial\Omega) > \delta\}$  for some positive  $\delta$ .

We can write

$$|A_2| = \left| \int_{\Omega} \left( \frac{\partial \omega}{\partial x_j} \times u \right)_i \frac{\partial u_i}{\partial x_j} \right| \leq \sum_{k=1}^n \int_{\Omega_k} \left| \left( \frac{\partial \omega}{\partial x_j} \times u \right)_i \frac{\partial u_i}{\partial x_j} \right|. \quad (50)$$

Because

$$\left| \left( \frac{\partial \omega}{\partial x_j} \times u \right)_i \frac{\partial u_i}{\partial x_j} \right|$$

is a volume of a parallelepiped which is independent of the choice of the coordinate system, integrating over  $\Omega_k$ , we can assume that we use the above-described local coordinate system with axes  $\{x_1^k, x_2^k, x_3^k\}$  and unit direction vectors  $\{e_1^k, e_2^k, e_3^k\}$  which prevent us from “falling out” of  $\Omega$  as mentioned above. Nevertheless, for simplicity, we will still continue using the original denotation, that is,  $e_j$  instead of  $e_j^k$  and  $x_j$  instead of  $x_j^k$ . So using (49), we compute firstly

$$\begin{aligned} & \int_{\Omega_k} \left| \left( \frac{\partial \omega}{\partial x_j}(x) \times u(x) \right)_i \frac{u_i(x + he_j) - u_i(x)}{h} \right| \\ & \leq \frac{1}{h} \int_{\Omega_k} \left| \left( \frac{\partial \omega}{\partial x_j}(x) \times u(x) \right)_i u_i(x + he_j) \right| \\ & \leq \frac{Ch}{h} \int_{\Omega_k} \left| \frac{\partial \omega}{\partial x_j}(x) \times u(x) \right| \|u(x + he_j)\| \\ & \leq C \|u\|_4^2 \|\nabla \omega\|_2 \leq cC \|u\|_6^2 \|\Delta u\|_2 \\ & \leq cC \|\nabla u\|_2^2 \|\Delta u\|_2 \leq cC \|\omega\|_2^2 \|\Delta u\|_2 \\ & \leq cC^2 \|\omega\|_2^4 + \frac{1}{4} \|\Delta u\|_2^2. \end{aligned} \quad (51)$$

Secondly,

$$\begin{aligned} & \int_{\Omega_k} \frac{h}{2} \left| \left( \frac{\partial \omega}{\partial x_j} \times u \right)_i \frac{\partial^2 u_i}{\partial x_j^2}(x + \xi_{ij} e_j) \right| \leq h \|\nabla^2 u\|_\infty \|\nabla \omega\|_2 \|u\|_2 \\ & \leq ch \|\nabla^2 u\|_{H^2} \|\Delta u\|_2 \|\omega\|_2 \leq ch^2 \|\nabla^2 u\|_{H^2}^2 \|\Delta u\|_2^2 + \|\omega\|_2^2, \end{aligned} \quad (52)$$

and consequently, it follows from (49–52) that

$$|A_2| \leq \left( \frac{1}{4} + ch^2 \|\nabla^2 u\|_{H^2}^2 \right) \|\Delta u\|_2^2 + (1 + cC^2 \|\omega\|_2^2) \|\omega\|_2^2. \quad (53)$$

It finally follows from (40), (41), (42), (48), and (53) that

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \left( \frac{1}{2} - ch^2 \|\nabla^2 u\|_{H^2}^2 \right) \|\Delta u\|_2^2 \leq (1 + cC^2 \|\omega\|_2^2) \|\omega\|_2^2. \quad (54)$$

Take now  $t_0$  from (46). If  $2t_0 = 1/(2c\|\omega_0\|_2^4) > T$  than  $u$  is regular on  $(0, T]$ , and the proof is finished. Suppose that  $2t_0 \leq T$ . It follows from (45) that

$$\|\omega(t)\|_2^2 \leq \sqrt{2}\|\omega_0\|_2^2 \text{ for every } t \in [0, t_0]. \quad (55)$$

Define

$$T_1 = \sup \left\{ t \in (t_0, T); \|\omega(s)\|_2^2 \leq \sqrt{2}\|\omega_0\|_2^2 \exp(cC^2\|u_0\|_2^2 + 2T) \text{ for every } s \in (t_0, t) \right\}.$$

Evidently,  $T_1 > t_0$ . If  $T_1 = T$ , then  $u$  is regular on  $(0, T]$ , and the proof is completed. So to finish the proof, it is sufficient to show that the assumption  $T_1 < T$  leads to contradiction. Because

$$\|\omega\|_{L^\infty(0, T_1; L^2)}^2 \leq \sqrt{2}\|\omega_0\|_2^2 \exp(cC^2\|u_0\|_2^2 + 2T),$$

applying Lemma 2 to  $u$ , the interval  $(0, T_1)$ , and  $\varepsilon = t_0$ , we get that

$$\begin{aligned} \|\nabla^2 u\|_{L^\infty(t_0, T_1; H^2)}^2 &\leq \Psi_4(\|\nabla u\|_{L^\infty(0, T_1; L^2)}, t_0, T_1, \|u_0\|_2) \\ &\leq \Psi_4(c\|\nabla u_0\|_2 \exp(cC^2\|u_0\|_2^2 + T), t_0, T, \|u_0\|_2), \end{aligned} \quad (56)$$

where  $\Psi_4 = \Phi_2 + \Phi_3 + \Phi_4$  with the functions  $\Phi_2$ ,  $\Phi_3$ , and  $\Phi_4$  having been defined in Lemma 2. In the second inequality in (56), we used the fact that the functions  $\Phi_2$ ,  $\Phi_3$ , and  $\Phi_4$  are increasing in the first and third variables. As a consequence of (46), (56), and the fact that  $\Phi_2$ ,  $\Phi_3$ , and  $\Phi_4$  are increasing in the first, third, and fourth variables and decreasing in the second variable, there exists a function  $\lambda$ , which satisfies the assumptions from Theorem 8 and

$$c\|\nabla^2 u\|_{H^2}^2 \lambda^2(\|u_0\|_{H^1}, T, C) < \frac{1}{2} \text{on}(t_0, T_1).$$

Fixing now any  $h \in (0, \lambda(\|u_0\|_{H^1}, T, C)]$ , we get from (54) that

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 \leq (1 + cC^2\|\omega\|_2^2) \|\omega\|_2^2,$$

on  $(t_0, T_1)$  and subsequently by the use of the Grönwall inequality and the continuity of the function  $\|\nabla u(\cdot)\|_2$  on  $(0, T)$

$$\|\omega(t)\|_2^2 \leq \sqrt{2}\|\omega_0\|_2^2 \exp(cC^2\|u_0\|_2^2 + 2T_1) < \sqrt{2}\|\omega_0\|_2^2 \exp(cC^2\|u_0\|_2^2 + 2T),$$

for all  $t \in (t_0, T_1]$ . Using again the continuity of the function  $\|\nabla u(\cdot)\|_2$  on  $(0, T)$ , we get the contradiction with the definition of  $T_1$ . The proof is completed.

## Author Contributions

**Zhengguang Guo:** investigation, writing – original draft, writing – review and editing. **Zdenek Skalak:** investigation, writing – original draft, writing – review and editing.

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## Conflicts of Interest

The authors declare no conflicts of interest.

## Data Availability Statement

No data are used or shared in the manuscript.

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