Navier–Stokes Proof Program:
Audit Status and Remaining Global Gates

Repository consolidation (snapshot)

2025-12-17

Abstract

This document consolidates the current *audited* proof status in the repository, using (`ASSUMPTIONS_TO_CLOSE.md`) as the authoritative checklist of remaining assumption blocks in `navier-dec-12-rewrite.tex` and (`P0_PLAN_ONE_CORE_DOMINANCE.md`) as the detailed analysis of the hardest remaining global obstruction (RM2), reformulated as an $\ell = 2$ *B-flux (BF)* gate. The intent is to produce a referee-friendly "what is proved / what is missing" snapshot.

1 Sources of truth (what to read)

- **Main manuscript (most complete + most audited):** `navier-dec-12-rewrite.tex` (compiled as `navier-dec-12-rewrite.pdf`).
- **Open gate checklist:** `ASSUMPTIONS_TO_CLOSE.md`.
- **RM2/BF deep dive (one-core / tail depletion track):** `P0_PLAN_ONE_CORE_DOMINANCE.md`.

2 Executive summary

As of this snapshot, **unconditional global regularity is not yet proved** in the running-max architecture. After audit, the remaining nonlocal/global blockers are:

- **U-decay / relative tail depletion:** the true content is a uniform tail envelope in *blow-up variables* ("one-core dominance"), not passive inheritance of physical-space decay at infinity.
- **RM2 (running-max compactness local-energy gate):** extracting a running-max ancient element in a fixed-frame velocity/pressure formulation is obstructed by affine/harmonic modes. In the working log this is reformulated in the $\ell = 2$ channel as the **B-flux (BF) gate** plus an **endpoint time-regularity** upgrade (BF-timeBV/BF-timeCarleson).

Many local/coercive pieces of the direction-field analysis are valuable and largely written, but they do not by themselves remove the above global obstructions.

3 Explicit assumption blocks still present in the main TeX

The file `ASSUMPTIONS_TO_CLOSE.md` records the remaining explicit `\begin{assumption}` blocks in `navier-dec-12-rewrite.tex`. Table 1 summarizes their meaning at a high level.

Table 1: Assumption blocks present in `navier-dec-12-rewrite.tex` (per `ASSUMPTIONS_TO_CLOSE.md`).

Label	**Summary (informal)**
`assump:C-liouville`	Directional ε-regularity + Liouville/ridigity for the vorticity direction drift–diffusion equation under critical forcing/drift hypotheses; as written it relies on a global smallness mechanism for the unweighted Morrey energy of $\nabla\xi$.
`assump:RM2-runningmax`	Running-max compactness / fixed-frame local-energy gate (affine/harmonic mode obstruction). In the working notes this is sharpened as an $\ell = 2$ **BF** gate plus an endpoint **BF-time** regularity upgrade.
`assump:D-forcing`	Total tangential forcing (near-field + geometric + tail) is Carleson-small at small scales.
`assump:D-logamp`	Uniform-in-ε control of $\nabla \log(\rho+\varepsilon)$ across the vorticity zero set (known codimension-2 pathologies are a documented obstruction).
`assump:tail-depletion`	Carleson-smallness of the tail forcing; a far-field gate that is borderline/nonlocal and closely related to tail depletion / one-core dominance.
`assump:U-decay`	Spatial decay/tail control for the running-max ancient element; after audit the correct target is *relative tail depletion* in blow-up variables, not just decay of the original solution at infinity.
`assump:E-2d`	Optional 2D classification/Liouville input (kept as a modular global classification statement; parts may be bypassed depending on the contradiction route).

4 Priority order (hardest first)

Following `ASSUMPTIONS_TO_CLOSE.md`, the current P0/P1 prioritization is:

- **P0 (global / most non-classical):** U-decay (relative tail depletion in blow-up variables), RM2 (fixed-frame compactness), tail depletion (far-field forcing), log-amplitude control across vorticity zeros, and the global mechanism behind direction Liouville (`assump:C-liouville`).
- **P1:** optional 2D Liouville/classification once the reduced flow lies in a classical global class.

5 RM2 reformulated: the $\ell = 2$ B-flux (BF) gate

The working log in `P0_PLAN_ONE_CORE_DOMINANCE.md` isolates the fixed-frame compactness obstruction (RM2) into an explicit $\ell = 2$ subsystem for a radial coefficient $A(r,t)$ driven by an $\ell = 2$ forcing written in flux form via a potential $B(r,t)$.

5.1 Fixed objects and subsystem

Fix an $\ell = 2$ spherical harmonic test field $\Phi(\theta)$ and define the $\ell = 2$ coefficient

$$A(r,t) := \int_{\mathbb{S}^2} \omega(r\theta, t) \cdot \Phi(\theta)\, d\theta.$$

In the $\ell = 2$ channel, A satisfies an exact radial parabolic equation (see UA-A510):

$$\partial_t A = \Big(\partial_{rr} + \frac{2}{r}\partial_r - \frac{6}{r^2}\Big) A - \frac{1}{r^2}\partial_r B.$$

The flux potential B is built linearly from $F = u \times \omega$ (and admits an exact divergence-theorem identity $B(r,t) = \int_{|y|<r} \Psi(y) \cdot \mathrm{curl}(u \times \omega)(y,t)\, dy$; see UA-A498).

5.2 BF(1–3): a checkable uniform flux gate

The BF gate (UA-A401) packages the remaining $\ell = 2$ obstruction as:

- **BF(1) (log-Carleson bound):** for all $t \leq 0$,
$$\int_1^\infty \frac{|B(r,t)|^2}{r^2}\, dr \leq C_{\mathrm{BF}}.$$
- **BF(2) (vanishing far boundary flux):** as $R \to \infty$,
$$|A(R,t)\, B(R,t)| \to 0, \qquad R^2 |A(R,t) A_r(R,t)| \to 0.$$
- **BF(3) (local boundary control at $r = 1$):** the $r = 1$ boundary terms in the A-energy identity are uniformly bounded (local; compatible with running-max inputs).

5.3 The exact energy identity and the endpoint issue

Multiplying the A equation by Ar^2 and integrating on $[1, R]$ yields the exact identity (UA-A406, tagged BF:energy in the plan file):

$$\frac{1}{2}\frac{d}{dt}\int_1^R |A|^2 r^2 dr \;+\; \int_1^R |A_r|^2 r^2 dr \;+\; 6\int_1^R |A|^2 dr \;=\; \Big[r^2 A A_r\Big]_1^R \;-\; \Big[AB\Big]_1^R \;+\; \int_1^R A_r\, B\, dr. \quad (1)$$

Formally, BF(1) allows the bulk forcing term to be Cauchy–Schwarz absorbed into the dissipation. However (UA-A511), **BF(1) alone does not give pointwise-in-time coercivity** at the endpoint: explicit linear counterexamples show BF(1) $\not\Rightarrow$ the needed pointwise bound on $\int_1^\infty |A_r|^2 r^2\, dr$.

5.4 Endpoint fixes: BF-timeBV / BF-timeCarleson

The working log identifies the correct endpoint repair (UA-A511): add a *time regularity* condition on B in the BF norm. Two equivalent "endpoint upgrade" forms are recorded:

- **BF-timeBV:** bounded variation in time for B in the BF norm (an endpoint maximal-regularity surrogate).
- **BF-timeCarleson:** a critical square-function / Carleson-in-time control compatible with semigroup square-function estimates for the self-adjointly factorized $\ell = 2$ operator.

With such a BF-time input, the log provides a unit-slab proof template:

$$\mathrm{BF(1\text{–}3)} + \text{BF-timeCarleson} \;\Rightarrow\; \text{BF-aux}_{\mathrm{coerc}},$$

which is the missing analytic step needed to pass from time-averaged dissipation to pointwise control and then to the Tauberian/$\mathrm{BV}_{\log}$ bridge used to close the $\ell = 2$ moment (RM2).

5.5 Practical handoff statement (single missing theorem)

The clearest "handoff" statement (UA-A513) is:

$$\text{(bounded-vorticity ancient NSE)} \Rightarrow \text{BF}(1) + \text{BF-timeCarleson (or BV)}.$$

Supplying such a theorem would close Track U-A (and hence RM2) in the current architecture.

6 Hardness frontier (why this looks Clay-level)

The plan file records a one-sentence summary (Session 61):

> In the running-max architecture, any unconditional route must effectively prove a global tail-depletion principle (BF(1) and especially BF-time regularity / "no far-field spike"), and every attempted derivation of that principle from bounded vorticity + local energy collapses to regularity-level global control.

Concretely, the log documents several collapse points:

- BF(1) follows from strong global budgets such as $\|u \times \omega\|_{L^2_x}$ (already stronger than critical tightness), but no derivation from running-max inputs is known.
- Differentiating B in time introduces $\Delta u, \Delta\omega, \nabla p$ terms (global regularity-level burdens).
- BMO–Hardy / compensated compactness attempts fail because the BF testing family is nonlocal in r and not uniformly Hardy-bounded (Session 58).
- A separate CPM/$\mathcal{W}$ architecture introduces an explicit "Dyadic Decay / no far-field spike" gate; the plan maps this back to the same tail-depletion/one-core obstruction (Sessions 59).

7 What would make the proof unconditional (in this repo)

Per `ASSUMPTIONS_TO_CLOSE.md`, to claim unconditional closure one must *either* prove the existing global gates *or* refactor the manuscript to remove them. Three clean replacement targets for the global tail gate are recorded there:

- **U-A:** prove relative tail depletion (one-core dominance) in blow-up variables.
- **U-B:** prove finite-capacity / sub-affine growth for the ancient element (kills affine/harmonic modes; another face of RM2).
- **U-C:** prove critical-space tightness ($u \in L^\infty_t L^3_x$ or $\omega \in L^\infty_t L^{3/2}_x$) for the ancient element.

The working conclusion in `P0_PLAN_ONE_CORE_DOMINANCE.md` is that these are different packagings of the same regularity-level obstruction.

8 Recommended reading order

1. `ASSUMPTIONS_TO_CLOSE.md` (first 2 pages) for the global blocker list and P0 priorities.
2. `P0_PLAN_ONE_CORE_DOMINANCE.md`, Session 56 (UA-A510–UA-A513) for the $\ell = 2$ BF obstruction diagram.
3. `navier-dec-12-rewrite.tex` to see exactly where RM2/U-decay enter the main contradiction chain.