

Stable Direction Dictionaries for Strongly Positive (p, p)-Forms via Regularized Simplex Fits

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Abstract

Let (X, ω) be a compact Kähler manifold of complex dimension n , and let $K_p(x)$ denote the cone of strongly positive (p, p) -covectors at $x \in X$. A recurring obstruction in mesh-based constructions is the absence of a stable way to *recognize* and label directions in a cone-valued form field $\beta(x) \in K_p(x)$: Carathéodory decompositions are highly non-unique and vary discontinuously, preventing coherent direction labeling across adjacent cells.

We introduce a dictionary-based recognizer. Fix an ε -net $\{P_1, \dots, P_M\}$ in the calibrated (complex) Grassmannian of $(n - p)$ -planes, and let $\xi_i(x)$ be the associated normalized ray generators satisfying $\langle \xi_i(x), \psi_x \rangle = 1$ where $\psi = \omega^{n-p}/(n - p)!$ is the Kähler calibration. For each normalized target $b(x)$ we define a recognition state $w(x) \in \Delta_M$ by a strongly convex regularized least-squares fit on the simplex. We prove existence, uniqueness, and a Lipschitz bound

$$\|w(b) - w(b')\| \leq (\|A\|_{\text{op}}/\lambda) \|b - b'\|$$

for the weight map, where $A : \mathbb{R}^M \rightarrow V$ is the dictionary synthesis operator $Aw = \sum_i w_i \xi_i$ and $\lambda > 0$ is the regularization strength. This yields stable, globally labeled coefficients that vary at the same scale as $b(x)$.

To connect with literal finite-resolution recognition, we consider derived finite event maps such as the winner-take-all label $\arg \max_i w_i$ and show (via a margin lemma) that these discrete labels are robust under perturbations away from ties. Finally, we show how pointwise weights induce coherent per-cell mass budgets $M_{Q,i}$ on a mesh, isolating direction-label stability as a quantitative choice of dictionary resolution ε and regularization λ , independent of later holomorphic or geometric-measure steps.

1 Recognition primitives

We record the recognition primitives used in this paper (state space, measurement map, quotient, and robustness modulus $r(s)$). The purpose is to make explicit what structure is assumed and what structure is produced.

Definition 1 (State space / configuration space). A *state space* is a pair $(\mathcal{C}, d_{\mathcal{C}})$ where \mathcal{C} is a set and $d_{\mathcal{C}}$ is a metric on \mathcal{C} . Elements $c \in \mathcal{C}$ are called *states* (or *configurations*).

Definition 2 (Measurement map / recognizer). Let $(\mathcal{C}, d_{\mathcal{C}})$ be a state space and let \mathcal{E} be a set (the *event space*). A *measurement map* (or *recognizer*) is a function

$$\mathcal{R} : \mathcal{C} \rightarrow \mathcal{E}.$$

Definition 3 (Indistinguishability and quotient). Given $\mathcal{R} : \mathcal{C} \rightarrow \mathcal{E}$, define an equivalence relation on \mathcal{C} by

$$c_1 \sim_{\mathcal{R}} c_2 \iff \mathcal{R}(c_1) = \mathcal{R}(c_2).$$

The *recognition quotient* is the quotient set $\mathcal{C}_{\mathcal{R}} := \mathcal{C} / \sim_{\mathcal{R}}$.

Definition 4 (Robustness modulus $r(s)$). Assume $(\mathcal{E}, d_{\mathcal{E}})$ is a metric space. The *robustness modulus* of \mathcal{R} is the function $r : [0, \infty) \rightarrow [0, \infty]$ defined by

$$r(s) := \sup \left\{ d_{\mathcal{E}}(\mathcal{R}(c), \mathcal{R}(c')) : d_{\mathcal{C}}(c, c') \leq s \right\}.$$

Remark 1 (Lipschitz recognizers). If $\mathcal{R} : (\mathcal{C}, d_{\mathcal{C}}) \rightarrow (\mathcal{E}, d_{\mathcal{E}})$ is L -Lipschitz, then $r(s) \leq Ls$ for all $s \geq 0$.

Assumptions and non-assumptions (no smuggled observables)

- **Given structure (assumptions).**

- A compact Kähler manifold (X, ω) and the induced norms on forms.
- The calibration functional $\alpha \mapsto \langle \alpha, \psi_x \rangle$ used to normalize $K_p(x)$ to $\Sigma_p(x)$.
- A finite dictionary $\{\xi_i(x)\}_{i=1}^M \subset \Sigma_p(x)$ (a finite-resolution hypothesis).
- A regularization strength $\lambda > 0$.

- **Not assumed (anti-smuggling constraints).**

- No canonical or continuous Carathéodory/extremal-ray decomposition is assumed.
- No globally uniform “true direction labels” are assumed; labels are produced as dictionary indices.
- No circular definition of observables: events/labels are defined by an explicit map \mathcal{R} , not by presupposing the quotient classes we intend to construct.

Self-similarity and the golden ratio (optional scale lemma)

In zero-parameter self-similar scale updates, one often encounters the fixed-point equation $x = 1 + 1/x$. We record the elementary consequence for reference.

Lemma 1 (Self-similarity fixed point forces φ). *The equation $x = 1 + \frac{1}{x}$ has exactly one positive solution, namely $\varphi = (1 + \sqrt{5})/2$.*

Proof. Rearranging gives $x^2 = x + 1$, i.e. $x^2 - x - 1 = 0$. The quadratic formula yields the two roots $x = (1 \pm \sqrt{5})/2$, and exactly one is positive. \square

2 Introduction

Strongly positive (p, p) -forms on a Kähler manifold behave like “nonnegative densities with direction,” in the sense that they live in a closed convex cone whose extreme rays correspond to simple algebraic directions. In many constructions one wants to discretize a strongly positive form field $\beta(x)$ on a mesh and propagate direction-dependent budgets from cell to cell. The immediate obstacle is not existence of decompositions but *recognition*: a pointwise decomposition of $\beta(x)$ into extremal rays is not canonical. Even when decompositions exist, they are highly non-unique and can jump discontinuously as x varies. This makes direction labels unstable: adjacent mesh cells may use incompatible “names” for nearly the same direction.

Recognition Geometry suggests a measurement-first reframing: *without a recognizer, there is no stable labeling*. In our setting, at each x the *configuration space* is the normalized cone slice $\mathcal{C}_x := \Sigma_p(x)$, and we choose a finite dictionary $\{\xi_i(x)\}_{i=1}^M \subset \Sigma_p(x)$ as an explicit finite-resolution hypothesis. We then define a *recognizer* \mathcal{R}_λ that maps a configuration $b \in \mathcal{C}_x$ to a recognition state $w = \mathcal{R}_\lambda(b) \in \Delta_M$ by a strongly convex simplex fit. Strong convexity forces uniqueness, and a monotonicity argument yields a clean Lipschitz robustness bound with an explicit constant.

To align literally with a finite-resolution event axiom, we also consider derived *finite* event maps, such as the winner-take-all label $\arg \max_i w_i$. These induce an indistinguishability relation $b \sim_{\mathcal{R}} b'$ on \mathcal{C}_x and hence a recognition quotient $\mathcal{C}_x / \sim_{\mathcal{R}}$ whose classes are “direction-label resolution cells.” The stability results in this paper quantify how large a perturbation in b is required to change the recognition state (and, away from ties, the discrete label).

The resulting mechanism has two independent stability knobs: dictionary resolution ε (approximation quality) and regularization λ (robustness of the recognizer).

3 Strong positivity and the normalized slice

Let (X, ω) be a compact Kähler manifold of complex dimension n . Fix $p \in \{0, 1, \dots, n\}$. At each $x \in X$, let

$$V_x := \Lambda_{\mathbb{R}}^{p,p} T_x^* X$$

denote the real vector space of real (p, p) -covectors at x .

Definition 5 (Strongly positive cone). A covector $\alpha \in V_x$ is *strongly positive* if it is a finite sum of forms of the type

$$\left(\frac{i}{2}\right)^p \eta_1 \wedge \overline{\eta_1} \wedge \dots \wedge \eta_p \wedge \overline{\eta_p},$$

where $\eta_1, \dots, \eta_p \in \Lambda^{1,0} T_x^* X$. The set of strongly positive covectors is a closed convex cone, denoted $K_p(x) \subset V_x$.

Define the Kähler calibration of degree $(n-p, n-p)$ by

$$\psi := \frac{\omega^{n-p}}{(n-p)!} \in \Lambda_{\mathbb{R}}^{n-p, n-p} T_x^* X.$$

Let $\text{vol}_{\omega} := \omega^n / n!$.

Definition 6 (Pairing with ψ). For $\alpha \in \Lambda_{\mathbb{R}}^{p,p} T_x^* X$ and $\eta \in \Lambda_{\mathbb{R}}^{n-p, n-p} T_x^* X$, define the scalar $\langle \alpha, \eta \rangle$ by

$$\alpha \wedge \eta = \langle \alpha, \eta \rangle \text{vol}_{\omega}(x).$$

In particular, $\langle \alpha, \psi_x \rangle$ is the ψ -trace of α .

Lemma 2 (Positivity of the ψ -trace). *For every $x \in X$ and every $\alpha \in K_p(x)$ one has $\langle \alpha, \psi_x \rangle \geq 0$. Moreover, if $\alpha \in K_p(x)$ and $\langle \alpha, \psi_x \rangle = 0$, then $\alpha = 0$.*

Proof. In unitary coordinates at x with $\omega = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$, each elementary strongly positive term

$$\left(\frac{i}{2}\right)^p \eta_1 \wedge \bar{\eta}_1 \wedge \cdots \wedge \eta_p \wedge \bar{\eta}_p$$

has nonnegative wedge with ω^{n-p} , hence nonnegative pairing with ψ_x . Summing preserves nonnegativity, giving $\langle \alpha, \psi_x \rangle \geq 0$.

If $\langle \alpha, \psi_x \rangle = 0$ for $\alpha \in K_p(x)$, then every elementary term in a positive decomposition must also have zero pairing with ψ_x . In unitary coordinates, the wedge of a nonzero elementary term with ω^{n-p} is strictly positive, so each term must be zero, hence $\alpha = 0$. \square

Definition 7 (Normalized slice). Define the normalized slice

$$\Sigma_p(x) := \{v \in K_p(x) : \langle v, \psi_x \rangle = 1\}.$$

The set $\Sigma_p(x)$ is the natural compact base of the cone $K_p(x)$ once we fix ψ_x as a strictly positive functional on $K_p(x) \setminus \{0\}$.

Remark 2 (Configuration space viewpoint). For fixed x , the normalized slice $\Sigma_p(x)$ is the space of normalized strongly positive directions. In later applications one often decomposes a form field $\beta(x) \in K_p(x)$ as $\beta(x) = \rho(x) b(x)$ where $\rho(x) = \langle \beta(x), \psi_x \rangle \geq 0$ is the total density and $b(x) \in \Sigma_p(x)$ is a normalized direction. The recognition problem addressed in this paper is to turn $b(x)$ into stable labels and weights.

4 Calibrated rays and normalized ray generators

For Kähler calibrations, calibrated $(2n - 2p)$ -planes are precisely complex $(n - p)$ -planes. Let $G_{n-p}^{\mathbb{C}}(T_x X)$ denote the Grassmannian of complex $(n - p)$ -dimensional subspaces of $T_x X$.

Definition 8 (Normalized ray generator associated to a complex plane). Fix $x \in X$ and $P \in G_{n-p}^{\mathbb{C}}(T_x X)$. Choose a unitary frame e_1, \dots, e_n of $T_x^{1,0} X$ such that $P^{1,0} = \text{span}\{e_{p+1}, \dots, e_n\}$, and let ζ^1, \dots, ζ^n be the dual coframe. Define

$$\xi_P(x) := \left(\frac{i}{2}\right)^p \zeta^1 \wedge \bar{\zeta}^1 \wedge \cdots \wedge \zeta^p \wedge \bar{\zeta}^p \in \Lambda_{\mathbb{R}}^{p,p} T_x^* X.$$

Lemma 3 (Well-definedness and normalization). *The covector $\xi_P(x)$ is independent of the choice of unitary frame adapted to P . Moreover, $\xi_P(x) \in \Sigma_p(x)$, i.e. $\xi_P(x)$ is strongly positive and satisfies $\langle \xi_P(x), \psi_x \rangle = 1$.*

Proof. If two unitary frames are adapted to the same splitting $T_x^{1,0}X \cong \mathbb{C}^p \oplus \mathbb{C}^{n-p}$, they differ by an element of $U(p) \times U(n-p)$. The form

$$\left(\frac{i}{2}\right)^p \zeta^1 \wedge \bar{\zeta}^1 \wedge \dots \wedge \zeta^p \wedge \bar{\zeta}^p$$

is invariant under $U(p)$, hence well-defined.

Strong positivity is immediate from the construction. In unitary coordinates with $\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j$ and $P = \text{span}\{\partial/\partial z_{p+1}, \dots, \partial/\partial z_n\}$, one has

$$\xi_P(x) \wedge \frac{\omega^{n-p}}{(n-p)!} = \frac{\omega^n}{n!},$$

so $\langle \xi_P(x), \psi_x \rangle = 1$. □

Let

$$S_p(x) := \{\xi_P(x) : P \in G_{n-p}^{\mathbb{C}}(T_x X)\} \subset \Sigma_p(x).$$

The set $S_p(x)$ is compact because the Grassmannian is compact and $P \mapsto \xi_P(x)$ is continuous.

Proposition 1 (Convex generation of the normalized slice). *For each $x \in X$, the normalized slice $\Sigma_p(x)$ is the convex hull of $S_p(x)$:*

$$\Sigma_p(x) = \text{conv } S_p(x).$$

Equivalently, the cone $K_p(x)$ is the convex cone generated by $S_p(x)$.

Proof. Every $\xi_P(x)$ lies in $\Sigma_p(x)$, and $\Sigma_p(x)$ is convex, so $\text{conv } S_p(x) \subseteq \Sigma_p(x)$.

For the reverse inclusion, take $v \in \Sigma_p(x)$. By strong positivity, v is a finite sum of elementary strongly positive forms. After scaling each summand by its ψ -trace and then renormalizing, we write

$$v = \sum_{j=1}^N a_j v_j, \quad a_j \geq 0, \quad \sum_{j=1}^N a_j = 1,$$

where each v_j is a normalized elementary strongly positive form with $\langle v_j, \psi_x \rangle = 1$. Any such normalized elementary form is equal to $\xi_P(x)$ for some complex $(n-p)$ -plane P (its nullspace in $T_x^{1,0}X$ determines P), hence $v_j \in S_p(x)$. Therefore $v \in \text{conv } S_p(x)$. □

5 Regularized simplex recognizers

This section is pointwise and finite-dimensional, and establishes existence, uniqueness, and Lipschitz stability for the dictionary recognizer. Fix a real inner-product space $(V, \langle \cdot, \cdot \rangle_V)$; in applications $V = V_x$ with the norm induced by the Kähler metric.

Fix dictionary vectors $\xi_1, \dots, \xi_M \in V$. Define the synthesis operator

$$A : \mathbb{R}^M \rightarrow V, \quad Aw := \sum_{i=1}^M w_i \xi_i.$$

Let $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^M and the induced operator norm $\|A\|_{\text{op}}$.

Define the simplex

$$\Delta_M := \left\{ w \in \mathbb{R}^M : w_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^M w_i = 1 \right\}.$$

Definition 9 (Regularized simplex recognizer and J -step). Fix $\lambda > 0$. For a target $b \in V$, define the objective

$$J_b(w) := \frac{1}{2} \|Aw - b\|_V^2 + \frac{\lambda}{2} \|w\|^2, \quad w \in \Delta_M,$$

and define the weight map

$$w(b) := \operatorname{argmin}_{w \in \Delta_M} J_b(w).$$

We also write $\mathcal{R}_\lambda(b) := w(b)$ and interpret $\mathcal{R}_\lambda : V \rightarrow \Delta_M$ as a *recognizer* whose output $w(b)$ is an internal recognition state in the simplex. We refer to $w(b)$ as the (unique) J -step selected by the recognition cost J_b .

Theorem 1 (Recognition closure lemma (RCL): uniqueness of the J -step). *For every $b \in V$ and every $\lambda > 0$, the minimization problem defining $w(b)$ has a unique solution in Δ_M .*

Proof. The simplex Δ_M is nonempty, compact, and convex. The map $w \mapsto \frac{1}{2} \|Aw - b\|_V^2$ is convex, and $w \mapsto \frac{\lambda}{2} \|w\|^2$ is λ -strongly convex. Hence J_b is λ -strongly convex on \mathbb{R}^M , so it has at most one minimizer on any convex set. By continuity on a compact set, a minimizer exists, and by strong convexity it is unique. \square

A convenient characterization is via a variational inequality.

Lemma 4 (Variational inequality). *A point $w_\star \in \Delta_M$ equals $w(b)$ if and only if*

$$\langle A^*(Aw_\star - b) + \lambda w_\star, z - w_\star \rangle \geq 0 \quad \text{for all } z \in \Delta_M,$$

where $A^* : V \rightarrow \mathbb{R}^M$ is the adjoint of A and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on \mathbb{R}^M .

Proof. This is the standard first-order optimality condition for minimizing a differentiable convex function over a closed convex set. The gradient is $\nabla J_b(w) = A^*(Aw - b) + \lambda w$. \square

Theorem 2 (Lipschitz stability in the target). *For every $\lambda > 0$ and all $b, b' \in V$,*

$$\|w(b) - w(b')\| \leq \frac{\|A\|_{\text{op}}}{\lambda} \|b - b'\|_V.$$

Proof. Let $w := w(b)$ and $w' := w(b')$. Apply the variational inequality with $z = w'$:

$$\langle A^*(Aw - b) + \lambda w, w' - w \rangle \geq 0.$$

Apply it again with (b', w') and $z = w$:

$$\langle A^*(Aw' - b') + \lambda w', w - w' \rangle \geq 0,$$

which is equivalent to

$$\langle A^*(Aw' - b') + \lambda w', w' - w \rangle \leq 0.$$

Adding the two inequalities gives

$$\langle A^*A(w - w') + \lambda(w - w') - A^*(b - b'), w' - w \rangle \geq 0.$$

Let $d := w - w'$. Then $w' - w = -d$, and we obtain

$$-\|Ad\|_V^2 - \lambda\|d\|^2 + \langle A^*(b - b'), d \rangle \geq 0.$$

Dropping the nonpositive term $-\|Ad\|_V^2$ yields

$$\lambda\|d\|^2 \leq \langle A^*(b - b'), d \rangle \leq \|A^*(b - b')\| \|d\|.$$

If $d = 0$ there is nothing to prove. Otherwise divide by $\lambda\|d\|$ to get

$$\|d\| \leq \frac{\|A^*(b - b')\|}{\lambda} \leq \frac{\|A^*\|_{\text{op}}}{\lambda} \|b - b'\|_V.$$

Since $\|A^*\|_{\text{op}} = \|A\|_{\text{op}}$, the claim follows. \square

Corollary 1 (Robustness modulus for \mathcal{R}_λ). *Equip V with the metric induced by $\|\cdot\|_V$ and equip $\Delta_M \subset \mathbb{R}^M$ with the Euclidean norm. Then the robustness modulus $r(s)$ of $\mathcal{R}_\lambda : V \rightarrow \Delta_M$ satisfies*

$$r(s) \leq \frac{\|A\|_{\text{op}}}{\lambda} s \quad \text{for all } s \geq 0.$$

Proof. This is immediate from the Lipschitz theorem and the definition of $r(s)$. \square

Finite-resolution events and recognition quotients

The simplex state $w(b) \in \Delta_M$ varies continuously with b (indeed, Lipschitz), but Δ_M is not finite. If one wants a literal finite event space, one can post-process $w(b)$ by a finite rule.

Definition 10 (Winner-take-all event map). Let $\mathcal{E} := \{1, \dots, M\}$. Define the event map $E_\lambda : V \rightarrow \mathcal{E}$ by

$$E_\lambda(b) := \min \left\{ i : w_i(b) = \max_{1 \leq j \leq M} w_j(b) \right\}.$$

Thus $E_\lambda(b)$ is a deterministic tie-broken version of $\arg \max_i w_i(b)$.

As above, E_λ induces an indistinguishability relation $b \sim_{\mathcal{R}} b'$ on any $\mathcal{C} \subseteq V$ (e.g. $\mathcal{C} = \Sigma_p(x)$) by the rule $E_\lambda(b) = E_\lambda(b')$, and hence a recognition quotient $\mathcal{C} / \sim_{\mathcal{R}}$.

Lemma 5 (Margin implies stable discrete recognition). *Let $b \in V$ and let $i_\star := E_\lambda(b)$. Suppose there exists $\delta > 0$ such that*

$$w_{i_\star}(b) \geq w_j(b) + 2\delta \quad \text{for all } j \neq i_\star.$$

Then for every $b' \in V$ with $\|b - b'\|_V \leq (\lambda \delta) / \|A\|_{\text{op}}$, one has $E_\lambda(b') = i_\star$.

Proof. By the Lipschitz theorem, $\|w(b) - w(b')\| \leq (\|A\|_{\text{op}} / \lambda) \|b - b'\|_V \leq \delta$. In particular, for each coordinate one has $|w_j(b) - w_j(b')| \leq \|w(b) - w(b')\| \leq \delta$. Hence for all $j \neq i_\star$,

$$w_{i_\star}(b') \geq w_{i_\star}(b) - \delta \geq w_j(b) + \delta \geq w_j(b').$$

So i_\star remains the unique maximizer, and the tie-broken $\arg \max$ is unchanged: $E_\lambda(b') = i_\star$. \square

Remark 3 (Quantized simplex events). An alternative finite-resolution bridge is to quantize the simplex output itself. For example, fix a finite set $W \subset \Delta_M$ (a τ -net in the Euclidean norm) and define a quantizer $Q : \Delta_M \rightarrow W$ by selecting a nearest element of W (with deterministic tie-breaking). Then $Q \circ \mathcal{R}_\lambda : V \rightarrow W$ is a finite event-valued recognizer, and the Lipschitz bound controls how rapidly the internal state can move relative to the quantization scale. As with winner-take-all, discrete stability is governed by separation from the quantizer’s decision boundaries.

Remark 4 (Interpretation of the constant). The parameter λ is a stability knob. Larger λ improves Lipschitz stability but biases weights toward smaller Euclidean norm (more “spread” in many common geometries). Smaller λ reduces bias but amplifies sensitivity. The operator norm $\|A\|_{\text{op}}$ measures how large a change in the synthesized form Aw can be produced by a unit change in weights.

Remark 5 (Fieldwise versions). In geometric applications V and the dictionary vary with x . After choosing local trivializations, the pointwise Lipschitz estimate combines with standard stability estimates for compositions to yield fieldwise regularity statements on charts; we omit these routine details.

6 Approximation error versus dictionary resolution

The Lipschitz theorem controls *stability* of weights given a fixed dictionary. Separately, one also wants the dictionary itself to approximate the continuous set of calibrated ray generators.

We first record a basic convex-hull approximation lemma.

Lemma 6 (Convex-hull approximation from an ε -net). *Let $(V, \|\cdot\|_V)$ be a normed vector space and let $S \subset V$ be compact. Suppose $\Xi = \{\xi_1, \dots, \xi_M\} \subset S$ is an ε -net for S , meaning that for every $s \in S$ there exists $\xi_i \in \Xi$ with $\|s - \xi_i\|_V \leq \varepsilon$. Then for every $b \in \text{conv } S$,*

$$\text{dist}(b, \text{conv } \Xi) \leq \varepsilon.$$

Proof. Write $b = \sum_{j=1}^N a_j s_j$ with $s_j \in S$, $a_j \geq 0$, and $\sum_j a_j = 1$. For each j , choose $\xi_{i(j)} \in \Xi$ with $\|s_j - \xi_{i(j)}\|_V \leq \varepsilon$. Define $b' := \sum_j a_j \xi_{i(j)} \in \text{conv } \Xi$. Then

$$\|b - b'\|_V \leq \sum_{j=1}^N a_j \|s_j - \xi_{i(j)}\|_V \leq \sum_{j=1}^N a_j \varepsilon = \varepsilon.$$

Hence $\text{dist}(b, \text{conv } \Xi) \leq \varepsilon$. \square

Now specialize to the Kähler setting at a point x . Recall $S_p(x) = \{\xi_P(x)\}$ and $\Sigma_p(x) = \text{conv } S_p(x)$.

Corollary 2 (Pointwise approximation on $\Sigma_p(x)$). *Fix $x \in X$. If $\Xi(x) = \{\xi_1(x), \dots, \xi_M(x)\} \subset S_p(x)$ is an ε -net for $S_p(x)$ (in the norm on V_x), then every $b \in \Sigma_p(x)$ satisfies*

$$\text{dist}(b, \text{conv } \Xi(x)) \leq \varepsilon.$$

Remark 6 (From plane-nets to ξ -nets). On the compact Grassmannian $G_{n-p}^{\mathbb{C}}(T_x X)$, the map $P \mapsto \xi_P(x)$ is smooth, hence Lipschitz with some constant $C = C(n, p)$ once one fixes the standard Grassmann metric induced by ω_x . Therefore an ε -net in plane-angle distance induces a $C\varepsilon$ -net in $\|\cdot\|_{V_x}$ -distance among ray generators. The constant C is uniform when (X, ω) has bounded geometry on the region of interest.

Remark 7 (Choosing ε relative to a mesh size h). In mesh-based constructions on cells of diameter h , one often needs dictionary approximation errors that are negligible compared to h -scale variations of the target field. A typical quantitative regime is to choose a resolution ε_h satisfying $\varepsilon_h = o(h)$ as $h \rightarrow 0$. This ensures that dictionary-induced approximation errors do not dominate the geometric errors that scale linearly with the mesh diameter.

7 From pointwise weights to per-cell budgets on a mesh

Let $\beta \in \Omega^{p,p}(X)$ be a continuous strongly positive form field, meaning $\beta(x) \in K_p(x)$ for all x . Define the nonnegative density

$$\rho(x) := \langle \beta(x), \psi_x \rangle.$$

By the earlier lemma, $\rho(x) = 0$ implies $\beta(x) = 0$.

On the set where $\rho(x) > 0$, define the normalized field

$$b(x) := \frac{\beta(x)}{\rho(x)} \in \Sigma_p(x).$$

Fix a dictionary field $\{\xi_i(x)\}_{i=1}^M$ with $\xi_i(x) \in \Sigma_p(x)$ and a regularization parameter $\lambda > 0$. Define weights

$$w(x) := w_x(b(x)) \in \Delta_M.$$

The associated dictionary reconstruction is the (p, p) -form

$$\beta_{\text{dict}}(x) := \rho(x) \sum_{i=1}^M w_i(x) \xi_i(x).$$

By construction $\beta_{\text{dict}}(x)$ uses globally labeled directions indexed by i .

Now let $U \subset X$ be a chart in which we place a cubical mesh of side length $h > 0$. For a mesh cell (cube) $Q \subset U$, define the per-cell mass budget assigned to label i by

$$M_{Q,i} := \int_Q w_i(x) \rho(x) dV_\omega(x),$$

where dV_ω is the Riemannian volume measure of the Kähler metric.

Lemma 7 (Basic identities). *For every cell Q ,*

$$\sum_{i=1}^M M_{Q,i} = \int_Q \rho(x) dV_\omega(x).$$

Proof. Since $w(x) \in \Delta_M$, one has $\sum_i w_i(x) = 1$ pointwise, hence

$$\sum_{i=1}^M M_{Q,i} = \int_Q \left(\sum_{i=1}^M w_i(x) \right) \rho(x) dV_\omega(x) = \int_Q \rho(x) dV_\omega(x).$$

□

Proposition 2 (Slow variation of budgets across adjacent cells). *Assume $w_i \rho$ is Lipschitz on U with Lipschitz constant L_i (in the chart metric), and assume mesh cubes have side length h . If Q and Q' are adjacent cubes (sharing a codimension-one face), then*

$$|M_{Q,i} - M_{Q',i}| \leq C(n) L_i h \text{Vol}_\omega(Q),$$

where $\text{Vol}_\omega(Q) = \int_Q dV_\omega$ and $C(n)$ depends only on the real dimension $2n$.

Proof. Let $f_i(x) := w_i(x) \rho(x)$. For adjacent cubes Q, Q' of the same size, the difference of integrals of a Lipschitz function is controlled by the oscillation of f_i on $Q \cup Q'$. One convenient estimate is

$$\left| \int_Q f_i dV_\omega - \int_{Q'} f_i dV_\omega \right| \leq \int_{Q \cup Q'} |f_i(x) - f_i(x_0)| dV_\omega(x)$$

for a suitably chosen reference point x_0 between the cubes. Since every point in $Q \cup Q'$ lies within $O(h)$ of x_0 , Lipschitz continuity gives $|f_i(x) - f_i(x_0)| \leq C(n)L_i h$ throughout $Q \cup Q'$, hence

$$|M_{Q,i} - M_{Q',i}| \leq C(n)L_i h \text{Vol}_\omega(Q \cup Q') \leq 2C(n)L_i h \text{Vol}_\omega(Q).$$

Absorb the factor 2 into $C(n)$. \square

Remark 8 (Relative form of the estimate). Since $\text{Vol}_\omega(Q) \sim h^{2n}$, the bound above is $O(h^{2n+1})$ in absolute terms, which is $O(h)$ relative to the typical cell mass scale. This is the precise sense in which budgets vary slowly across neighbors when the underlying field is Lipschitz.

8 Interface assumptions for later stages

This paper is designed to output only two types of information for downstream geometric constructions.

First, it outputs a fixed finite label set $\{1, \dots, M\}$ (the dictionary indices) together with a stable recognizer output $w(x) \in \Delta_M$ producing globally labeled coefficients. If one wants a literal finite-resolution recognizer, one may also pass forward the derived event map $E_\lambda(x) := E_\lambda(b(x)) \in \{1, \dots, M\}$; the margin lemma above quantifies when this discrete output is robust.

Second, it outputs mesh-level budgets $M_{Q,i}$ whose neighbor-to-neighbor variation can be controlled quantitatively in terms of regularity of the underlying form field and the choice of λ and dictionary resolution.

No holomorphic input is needed for these steps. In particular, one can treat the entire discussion above as a purely pointwise and mesh-level mechanism for turning a cone-valued field into stable labeled scalar densities.

9 Examples and variants

Example 1 (The case $p = 1$: positive semidefinite Hermitian matrices). At a point x , a real $(1, 1)$ -form corresponds (in unitary coordinates) to a Hermitian matrix. Strong positivity is positive semidefiniteness. The normalized slice $\Sigma_1(x)$ is the set of positive semidefinite matrices with fixed trace against ω^{n-1} . The calibrated rays are rank-one projectors, and a dictionary is a finite family of such projectors. The regularized simplex fit is then a stable way to express a positive semidefinite matrix as a convex combination of nearby rank-one directions with globally fixed labels.

Remark 9 (Alternative regularizers). The quadratic regularizer $\frac{\lambda}{2}\|w\|^2$ is chosen for two reasons: it makes the objective strongly convex and it produces a clean monotonicity argument giving the explicit Lipschitz constant. Other regularizers can be used, for example an entropic term $\sum_i w_i \log w_i$ to enforce strictly positive weights in the simplex interior. Such choices typically yield smoother dependence on b but change the stability constant and the structure of optimality conditions.

Remark 10 (Implementation-level stability). The optimization problem is a strongly convex quadratic program over the simplex. Strong convexity implies not only a unique solution but also good numerical conditioning as λ increases. In applications where M is moderate, projected gradient methods or active-set methods are standard. The theoretical Lipschitz bound is useful even if one never computes w exactly: it quantifies how much weight noise can be induced by target noise.

Conclusion

By fixing a finite dictionary of normalized calibrated ray generators and selecting weights through a strongly convex simplex fit, we obtain a unique, stable, and Lipschitz *recognizer* for strongly positive (p, p) -forms. The recognition state $w(b)$ provides globally labeled coefficients, and derived finite event maps (such as $E_\lambda = \arg \max$ with deterministic tie-breaking) produce literal finite-resolution direction labels with explicit robustness margins. The induced labeled budgets on a mesh inherit slow-variation properties from the regularity of the underlying form field, with constants controlled by dictionary size/resolution and the regularization parameter λ . This isolates the recognition/labeling problem from subsequent geometric or holomorphic realization steps and provides a modular input for larger constructions.