Arch. Rational Mech. Anal. (2024) 248:123
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-024-02068-7

A New Condition on the Vorticity for Partial Regularity of a Local Suitable Weak Solution to the Navier–Stokes Equations

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Communicated by F. Lin

Abstract

We provide a new ε-condition for the vorticity of a suitable weak solution to the Navier–Stokes equations that leads to partial regularity. This refines the well known limsup condition of the Caffarelli-Kohn-Nirenberg Theorem by a new condition on the vorticity, replacing limsup by a suitable range of the radius r of the parabolic cylinders. As a consequence, the partial regularity is obtained directly from this ε-condition of the vorticity without relying on the ε-condition of the velocity. Furthermore, by the local nature of the method this result holds for any local suitable weak solution of the Navier–Stokes equations in a general domain.

Mathematics Subject Classification: 35Q30 · 76D03 · 76D05

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open set and $0 < T < +\infty$. Define $\Omega_T = \Omega \times (0, T)$. We consider the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u - \Delta u = -\nabla p + f \quad \text{in} \quad \Omega_T, \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega_T, \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T). \tag{1.3}$$

Here, $u = \{u_1, u_2, u_3\}$ denotes the unknown velocity field and p the unknown pressure function, while $f = \{f_1, f_2, f_3\}$ stands for a given external force. For the sake of simplicity, in the present paper we set $f = 0$. The above system will be completed by the following initial condition:

$$u = u_0 \quad \text{on} \quad \Omega \times \{0\}. \tag{1.4}$$

Noting that $(u \cdot \nabla)u = \omega \times u + \frac{1}{2}\nabla|u|^2$, we replace (1.2) by

$$\partial_t u + \omega \times u - \Delta u = -\nabla\Big(p + \frac{1}{2}|u|^2\Big) \quad \text{in} \quad \Omega_T. \tag{1.5}$$

For the definition of a weak solutions or suitable weak solutions to (1.1)–(1.3) we first fix the notation of the involved function spaces. First, by $L^q(\Omega)$, $1 \leqq q \leqq +\infty$ we denote the usual Lebesgue spaces and by $W^{k,q}(\Omega)$, $W_0^{k,q}(\Omega)$, $1 \leqq q \leqq +\infty$ we denote the usual Sobolev spaces. Throughout the paper we do not distinguish between vector valued function and scalar function. For given matrices $\boldsymbol{A} = \{A_{ij}\}$ and $\boldsymbol{B} = \{B_{ij}\}$ in $\mathbb{R}^{3\times 3}$ we denote by $\boldsymbol{A} : \boldsymbol{B}$ the scalar product $\sum_{i,j=1}^{3} A_{ij}B_{ij}$.

For a given normed space X with norm $\|\cdot\|$ and a Bochner measurable function $f : (0, T) \to X$, we say that $f \in L^q(0, T; X)$ if

$$\begin{cases} \|f\|^q_{L^q(0,T;X)} = \displaystyle\int_0^T \|f(t)\|^q \mathrm{d}t < +\infty & \text{for} \quad 1 \leqq q < +\infty \\ \|f\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|f(t)\| < +\infty & \text{for} \quad q = +\infty. \end{cases}$$

By $C^\infty_{c,\sigma}(\Omega)$ we denote the space of solenoidal smooth vector fields. Then we define that

$$L^p_\sigma(\Omega) = \text{closure of } C^\infty_{c,\sigma}(\Omega) \text{ in } \ L^p(\Omega)$$
$$W^{1,p}_\sigma(\Omega) = \text{closure of } \ C^\infty_{c,\sigma}(\Omega) \text{ in } W^{1,p}(\Omega) :$$

Furthermore, we define

$$V^{1,2}_{0,\sigma}(\Omega_T) = L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega)),$$
$$V^{1,2}(\Omega_T) = L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)).$$

Definition 1.1. Let $u_0 \in L^2_\sigma(\Omega)$ be a given initial velocity field. We say that $u \in V^{1,2}_{0,\sigma}(\Omega_T)$ is a weak solution to (1.1)–(1.4) if

$$\int_0^T \int_\Omega -u \cdot \frac{\partial \varphi}{\partial t} + \nabla u : \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega u \otimes u : \nabla\varphi \mathrm{d}x\mathrm{d}t + \int_\Omega u_0 \cdot \varphi(0)\mathrm{d}x \tag{1.6}$$

for all $\varphi \in C^\infty(\Omega_T)$ with $\operatorname{supp}(\varphi) \subset \Omega \times [0, T)$ and $\nabla \cdot \varphi = 0$. A weak solution to (1.1)–(1.4) is called a Leray-Hopf solution (cf. [8,12]) if it satisfies the following global energy inequality for almost all $t \in (0, T)$:

$$\|u(t)\|^2_{L^2(\Omega)} + 2\int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)} \mathrm{d}s \leqq \|u_0\|^2_{L^2(\Omega)}. \tag{1.7}$$

Remark 1.2. The global energy inequality is obtained formally by testing the momentum equation (1.1) with u. Using integration by parts and recalling that $\nabla \cdot u = 0$ the term involving $u \otimes u$ and the pressure p are vanishing and do not contribute to the global energy balance. On the contrary, for estimating the local energy, this term does not vanish and appears on the right-hand side. In particular this requires some information on the pressure. For sufficiently regular domains the existence of a pressure function in $L^{3/2}(\Omega_T)$ is well-known (cf. actually a stronger result of $p \in L^{5/3}(\Omega_T)$ is proved by von Wahl in [6,18]). In this case, the existence of so called suitable weak solutions, that satisfies a local energy inequality, is known and leads to the celebrated Caffarelli-Kohn-Nirenberg theorem [1], which states that the condition

$$\limsup_{r \to 0} r^{-1} \int\limits_{Q(z_0,r)} |\nabla u|^2 \mathrm{d}x\mathrm{d}t \leqq \varepsilon_0. \tag{1.8}$$

implies that u is bounded in a neighbourhood of $z_0 \in \Omega_T$. The Caffarelli-Kohn-Nirenberg Theorem was reproved by various authors such as [3,11,13,17]. For further related results concerning local regularity conditions see e.g. [4,14,16]. On the other hand, in [19] it is shown that the Caffarelli Kohn Nirenberg theorem holds in any domain, even when the pressure exists only in the sense of distributions. The proof there lies on the concept of local a pressure projection (cf. [20]) as follows. For a fixed $G \subset \Omega$ bounded C^2 domain, we define the operator $E^*_G : W^{-1,q}(G) \to W^{-1,q}(G)$, such that $f \mapsto \nabla \pi$, where $\pi \in L^q(G)$ with $\pi_G = \frac{1}{|G|} \int\limits_G \pi \mathrm{d}x = 0$ denotes the unique pressure in the Stokes system

$$-\Delta v + \nabla \pi = f, \quad \nabla \cdot v = 0 \quad \text{in} \quad G, \quad v = 0 \quad \text{on} \quad \partial G. \tag{1.9}$$

By the existence and uniqueness of weak solutions to (1.9) (cf. [2,7]) get for all $1 < q < +\infty$ a constant c depending on q and G such that

$$\|E^*_G(f)\|_{W^{-1,q}(G)} \leqq \|\pi\|_{L^q(G)} \leqq c\|f\|_{W^{-1,q}(G)} \quad \forall f \in W^{-1,q}(G). \tag{1.10}$$

In addition, using the canonical embedding $L^q(G) \hookrightarrow W^{-1,q}(G)$, by the elliptic regularity we get that $E^*_G(f) \in L^q(G)$ for all $f \in L^q(G)$, together with the inequality

$$\|E^*_G(f)\|_{L^q(G)} = \|\nabla \pi\|_{L^q(G)} \leqq c\|f\|_{L^q(G)} \quad \forall f \in W^{-1,q}(G). \tag{1.11}$$

In case G is a ball B then in both (1.10) and (1.11) the constant c is independent of B.

Definition 1.3. A weak solution $u \in V^{1,2}(\Omega_T)$ with $\nabla \cdot u = 0$ almost everywhere in Ω_T of the Navier–Stokes equations is called local suitable if for every bounded C^1 domain $G \subset \Omega$ and for almost all $0 < t < T$ the following local energy inequality holds for all non-negative $\phi \in C^\infty_c(G \times (0, T))$:

$$
\begin{aligned}
&\int\limits_B |u^G(t)|^2\phi(t)\mathrm{d}x + 2\int\limits_0^t\int\limits_G |\nabla u^G|^2\phi\mathrm{d}x\mathrm{d}s \\
&\leqq \int\limits_0^t\int\limits_B |u^G|^2(\partial_t\phi + \Delta\phi)\mathrm{d}x\mathrm{d}s + \int\limits_0^t\int\limits_G 2\pi_{0,G}u^G\cdot\nabla\phi\mathrm{d}x\mathrm{d}s \\
&\quad + \int\limits_0^t\int\limits_G 2(\omega\times\nabla\pi_{h,G})\cdot u^G\phi\mathrm{d}x\mathrm{d}s. \qquad (1.12)
\end{aligned}
$$

Here

$$
\begin{aligned}
\nabla\pi_{h,G} &= -E^*_G(u), \quad \nabla\pi_{0,G} = E^*_G(\Delta u^G - \omega\times u),\\
u^G &= u + \nabla\pi_{h,G}.
\end{aligned}
$$

Remark 1.4. 1. The notion of a local suitable weak solution is equivalent to the notion of a dissipative solution, which has independently shown by Kwon in [10]. More precisely, for all nonnegative $\phi \in C^\infty_c(\Omega_T)$, it holds that

$$
\limsup_{\varepsilon\searrow 0}\int\limits_{\Omega_T}((u\cdot\nabla)u)_\varepsilon\cdot u_\varepsilon\phi + \frac{1}{2}|u|^2u\cdot\nabla\phi\mathrm{d}x\mathrm{d}t \geqq 0. \qquad (1.13)
$$

Here for given $h:\Omega_T\to\mathbb{R}$, extended outside Ω_T by zero, the function h_ε stands for the mollification

$$
h_\varepsilon := \eta_\varepsilon * h(x,t) = \frac{1}{\varepsilon^4}\int\limits_{\mathbb{R}^4} h(y,s)\eta\Big(\frac{1}{\varepsilon}(x-y), \frac{1}{\varepsilon}(t-s)\Big)\mathrm{d}y\mathrm{d}s, \quad (x,t)\in\mathbb{R}^4,
$$

where $\eta\subset C^\infty_c(B(1))$ denotes the Friedrich's mollifying kernel. This explains by another argument that the notion of a local suitable weak solution is independent of the existence of a global pressure and its existence can be obtained for non regular domains (cf. [19]).

2. Furthermore, it has been proved in the appendix of [5], if u is a suitable weak solution to the Navier–Stokes equations in the sense of Scheffer or Caffarelli-Kohn-Nirenberg then u is also a local suitable weak solution in the sense of Definition 1.3. This shows that the notion of Definition 1.3 is weaker then the former notion.

Our main theorem is

Theorem 1.5. *Let $u\in V^{1,2}(\Omega_T)$ be a local suitable weak solution to the Navier–Stokes equations (cf. Definition 1.3). Then, for every $1<\alpha<3$ there exists $\varepsilon_\alpha>0$ such that if, for $z_0=(x_0,t_0)\in\Omega_T$ and $r_0>0$, give has that*

$$
r^{-1}\int_{Q(z_0,r)}|\omega|^2\mathrm{d}x\mathrm{d}t < \varepsilon^2_\alpha \quad \forall r\in[r_1,r_0], \qquad (1.14)
$$

where

$$r_1 = \min\left\{r_0^{\frac{\alpha}{\alpha-1}}\left(\frac{\varepsilon_\alpha^2}{c\mathcal{E}_0}\right)^{\frac{1}{\alpha-1}}, \frac{1}{2}r_0\right\},$$
$$\mathcal{E}_0 = \|u\|^2_{L^\infty(t_0-r_0^2;L^2(B(x_0,r_0)))} + \|\omega\|^2_{L^2(Q(z_0,r_0))},$$

with an absolute constant $c > 0$, then $z_0 = (x_0, t_0)$ is a regular point, that is u is essentially bounded on some neighbourhood of z_0.

Remark 1.6. Given $1 < \alpha < 3$ in Theorem 1.5 we may choose

$$\varepsilon_\alpha = \frac{(3-\alpha)^2}{2c} \tag{1.15}$$

with an absolute constant $c > 0$.

Remark 1.7. By virtue of (1.15) we see that $\varepsilon_\alpha \to 0$ as $\alpha \to 3$. On the other hand, letting $\alpha \to 1$ in (1.14) and (1.15), we get the well known Caffarelli Kohn and Nirenberg condition, that is

$$r^{-1}\int_{Q(z_0,r)} |\nabla u|^2 \mathrm{d}x\mathrm{d}t \leqq \varepsilon_1^2 \quad \forall r \in (0, r_0]. \tag{1.16}$$

This shows that Theorem 1.5 generalizes Caffarelli-Kohn-Nirenberg Theorem. Furthermore, the condition (1.14) can be interpreted as an ε-condition for the vorticity and implies immediately that the set of all $z_0 \in \Omega_T$ where (1.14) fails is closed. Consequently the one-dimesional Hausdorff measure of the singular set is zero, which shortens the proof of the partial regularity. Indeed, by inspections of the original proof as well as of various recent alternative proofs ([11,13,17,19]), we see that the partial regularity is obtained by the following two steps:

(i) Proof of partial regularity by the ε-condition for u

(ii) Verifying the ε-condition for u by using the Caffarelli-Kohn-Nirenberg condition (1.16).

This approach does not need the step (i).

Another consequence of Theorem 1.5 is the following new ε-condition

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2 domain. Let $u \in V^{1,2}(\Omega_T)$ be a local suitable weak solution to the Navier–Stokes equations (cf. Definition 1.3). For every $1 < \alpha < 3$ and compact set $K \subset \Omega$ there exists $\tilde{\varepsilon}_\alpha > 0$ and $0 < R_1 \leqq \frac{1}{2}\operatorname{dist}(K, \partial\Omega)$, such that if for $z_0 = (x_0, t_0) \in K \times (0, T)$ and $r \in (0, R_1)$, give has that*

$$r^{\frac{-5\alpha}{5\alpha-2}}\int_{Q(z_0,r)}\left(|\omega|^2 + |u|^{10/3} + |E^*_\Omega(\omega\times u)|^{5/4}\right)\mathrm{d}x\mathrm{d}t \leqq \tilde{\varepsilon}_\alpha, \tag{1.17}$$

then z_0 is a regular point. In particular, the Minkowski dimension of the singular set is at most $\frac{15}{13}$.

Remark 1.9. The above Theorem 1.8 with $\alpha \to 3$ yields that the Minkowski dimension of the singular set is less than $\frac{15}{13} = \lim_{\alpha\to 3}\frac{5\alpha}{5\alpha-2}$. This improves the estimation of the singular set in terms of Minkowski dimension from both results of Scheffer in [15] and the condition proved by Koh and Yang (cf. [9]).

2. Proof of Main Theorem

Let $z_0 = (x_0, t_0) \in \Omega_T$. For $n \in \mathbb{N}_0$ we set

$$r_n = 2^{-n}, \quad B_n = B(x_0, r_n), \quad I_n = (t_0 - r_n^2, t_0),$$
$$Q_n = Q(z_0, r_n) = B_n \times I_n, \quad A_n = Q_n \backslash Q_{n+1}.$$

Assume that (1.14) holds for some radius $r_0 > 0$ and number $1 < \alpha < 3$ and $\varepsilon > 0$ that will specified below. Let $n_0 < n_1$ are given natural numbers, such that for all $n \in \{n_0, \ldots, n_1\}$ it holds. Let $n_0 \in \mathbb{N}$ such that $r_{n_0} \leqq r_0 < r_{n_0-1}$. Then (1.14) reads as

$$Y_n = r_n^{-1} \|\omega\|^2_{L^2(Q_n)} \leqq \varepsilon^2 \quad \forall n = n_0, \ldots, n_1, \tag{2.1}$$

where $n_1 \in \mathbb{N}$ is chosen such that

$$r_{n_1+1} \leqq r_1 < r_{n_1}, \quad r_1 = \min\left\{ r_0^{\frac{\alpha}{\alpha-1}} \left(\frac{\varepsilon^2}{c\mathcal{E}_0} \right)^{\frac{1}{\alpha-1}}, \frac{1}{2} r_0 \right\},$$

where

$$\mathcal{E}_0 = \|u\|^2_{L^\infty(t_0 - r_0^2; L^2(B(x_0, r_0)))} + \|\omega\|^2_{L^2(Q(z_0, r_0))},$$

while c and $\varepsilon = \varepsilon_\alpha$ are positive constants which will be specified below.

Now, let us assume that for some given number $N \geqq n_1$, it holds that

$$Y_j \leqq \varepsilon^2 \quad \forall j = n_0, \ldots, N. \tag{2.2}$$

We set $B = B_{n_0}$ and $I = I_{n_0}$ and define

$$v = u^B \quad \text{in} \quad B \times (0, T).$$

Let $n \in \mathbb{N}, n \geqq n_0$. For the sake of notational simplicity in what follows we use the following notations:

$$E_n := E_n^{(1)} + E_n^{(2)} = \operatorname*{ess\,sup}_{t \in I_n} \|v(t)\|^2_{L^2(B_n)} + \|\nabla v\|^2_{L^2(Q_n)},$$
$$Y_n = r_n^{-1} \|\omega\|^2_{L^2(Q_n)}.$$

In our discussion below we make use of the following estimate holds for all $2 \leqq s \leqq +\infty, 2 \leqq q \leqq 6$ with $\frac{2}{s} + \frac{3}{q} = \frac{3}{2}$

$$\|v\|^2_{L^s(I_n; L^q(B_n))} \leqq c E_n. \tag{2.3}$$

There c stands for a positive constant depending on s and r but not on n. Indeed, the estimate (2.3) is obtained by using Hölder's inequality along with Sobolev's inequality. The independence on n follows by a standard scaling argument.

For $n \geqq n_0$, we define the backward heat kernel

$$\Phi_n(x, t) = \frac{1}{(4\pi(t_0 - t + r_n^2))^{\frac{3}{2}}} \exp\left(\frac{|x - x_0|^2}{4(t_0 - t + r_n^2)} \right), \quad (x, t) \in \mathbb{R}^3 \times (-\infty, t_0 + r_n^2).$$

Then there are absolute positive constants $\hat{c}$ and $\hat{C}$ such that

$$\hat{c}r_j^{-3} \leqq \Phi_n \leqq \hat{C}r_j^{-3}, \quad \hat{c}r_j^{-4} \leqq |\nabla\Phi_n| \leqq \hat{C}r_j^{-4} \quad \text{in} \quad A_j, \quad j = 0, \dots, n-1, \tag{2.4}$$

$$\hat{c}r_n^{-3} \leqq \Phi_n \leqq \hat{C}r_n^{-3}, \quad \hat{c}r_n^{-4} \leqq |\nabla\Phi_n| \leqq \hat{C}r_n^{-4} \quad \text{in} \quad Q_n. \tag{2.5}$$

In (1.12) we replace ϕ by $\Phi_n\eta$, where $\eta \in C_c^\infty(B_{n_0+1} \times (t_0 - r_{n_0+1}^2, t_0])$ stands for a suitable cut-off function with $\eta \equiv 1$ on Q_{n_0+2}. This gives

$$\begin{aligned}
&\int\limits_B |v(t)|^2\Phi_n(t)\eta(t)\mathrm{d}x + 2\int\limits_0^t\int\limits_B |\nabla v|^2\Phi_n\eta\mathrm{d}x\mathrm{d}s \\
&\leqq \int\limits_0^t\int\limits_B |v|^2(\partial_t(\Phi_n\eta) + \Delta(\Phi_n\eta))\mathrm{d}x\mathrm{d}s + \int\limits_0^t\int\limits_B 2\pi_{0,B}v\cdot\nabla(\Phi_n\eta)\mathrm{d}x\mathrm{d}s \\
&\quad + \int\limits_0^t\int\limits_B (2\omega\times\nabla\pi_{h,B})\cdot v\Phi_n\eta\mathrm{d}x\mathrm{d}s \\
&= I + II + III,
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
\nabla\pi_{h,B} = -E_B^*(u), \quad \nabla\pi_{0,B} = E_B^*(\Delta v - \omega\times u), \\
v = u + \nabla\pi_{h,B}.
\end{aligned}$$

First, that noting $(\partial_t + \Delta)\Phi_n = 0$ in $\{(x,t)\in\mathbb{R}^4 | t < t_0\}$, it follows that

$$I \leqq cr_{n_0}^{-5}\|v\|_{L^2(B\times I)}^2 \leqq cr_{n_0}^{-3}\|u\|_{L^\infty(I;L^2(B))}^2 \leqq cr_{n_0}^{-3}\mathcal{E}_0.$$

For the estimation of III we proceed as follows: observing (2.4) and (2.5), applying Hölder's inequality, and noting that $\mathrm{supp}(\eta) \subset \overline{Q_{n_0+1}}$, we find that

$$\begin{aligned}
III &\leqq 2\sum_{j=n_0}^{n-1}\int\limits_{A_j}|\omega||\nabla\pi_{h,B}||v|\Phi_n\mathrm{d}x\mathrm{d}s + 2\int\limits_{Q_n}|\omega||\nabla\pi_{h,B}||v|\Phi_n\mathrm{d}x\mathrm{d}s \\
&\leqq c\sum_{j=n_0+1}^{n}\int\limits_{Q_j} r_j^{-3}|\omega||\nabla\pi_{h,B}||v|\mathrm{d}x\mathrm{d}s \\
&\leqq c\sum_{j=n_0+1}^{n}\int\limits_{I_j} r_j^{-3}\|\omega(s)\|_{L^2(B_j)}\|v(s)\|_{L^6(B_j)}\|\nabla\pi_{h,B}(s)\|_{L^3(B_j)}\mathrm{d}s \\
&\leqq c\sum_{j=n_0+1}^{n} r_j^{-1/2}\|\omega\|_{L^2(Q_j)}r_j^{-3/2}\|v\|_{L^2(I_j;L^6(B_j))}r_j^{-1}\|\nabla\pi_{h,B}\|_{L^\infty(I_j;L^3(B_j))}.
\end{aligned}$$

Using the mean value property for harmonic function along with (1.11) we get for $j \geqq n_0 + 1$

$$r_j^{-1} \|\nabla \pi_{h,B}\|_{L^\infty(I_j;L^3(B_j))} \leqq c\|\nabla \pi_{h,B}\|_{L^\infty(Q_{n_0+1})} \leqq c r_{n_0}^{-3/2} \|u\|_{L^\infty(I;L^2(B))}.$$

Furthermore, applying Sobolev's inequality, we find that

$$\|v\|_{L^2(I_j;L^6(B_j))} \leqq c(\|\nabla v\|_{L^2(Q_j)} + r_j^{-1} \|v\|_{L^2(I_j;L^2(B_j))}) \leqq c E_j^{1/2}$$

(cf. also (2.3)).

Inserting the last two inequality into the estimate above, we find that

$$\begin{aligned} III &\leqq c r_{n_0}^{-3/2} \|u\|_{L^\infty(I;L^2(B))} \sum_{j=n_0+1}^{n} r_j^{-1/2} \|\omega\|_{L^2(Q_j)} r_j^{-3/2} E_j^{1/2} \\ &\leqq c r_{n_0}^{-3/2} \mathcal{E}_0^{1/2} \sum_{j=n_0}^{n} Y_j^{1/2} r_j^{-3/2} E_j^{1/2}. \end{aligned} \tag{2.7}$$

Hence, using Young's inequality and recalling (2.2), we get that

$$III \leqq c r_{n_0}^{-3} \mathcal{E}_0 + \varepsilon^2 \Big(\sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2} \Big)^2.$$

It remains to estimate the integral II that involves the pressure $\pi_{0,B}$. Recalling the definition of $\nabla \pi_{0,B}$, we have

$$\nabla \pi_{0,B} = E_B^*(\Delta v - \omega \times u) = E_B^*(\Delta v - \omega \times v + \omega \times \nabla \pi_{h,B}).$$

We may divide the pressure gradient into three terms and write

$$\nabla \pi_{0,B} = \nabla \pi_{1,B} + \nabla \pi_{2,B} + \nabla \pi_{3,B},$$

where

$$\begin{aligned} \nabla \pi_{1,B} &= E_B^*(\Delta v) = -E_B^*(\nabla \times \omega), \\ \nabla \pi_{2,B} &= -E_B^*(\omega \times v), \\ \nabla \pi_{3,B} &= E_B^*(\omega \times \nabla \pi_{h,B}). \end{aligned}$$

This yields

$$\begin{aligned} II = &\int_{t_0 - r_{n_0}^2}^{t} \int_B \pi_{1,B} v \cdot \nabla(\Phi_n \eta) \mathrm{d}x \mathrm{d}s + \int_{t_0 - r_{n_0}^2}^{t} \int_B \pi_{2,B} v \cdot \nabla(\Phi_n \eta) \mathrm{d}x \mathrm{d}s \\ &+ \int_{t_0 - r_{n_0}^2}^{t} \int_B \pi_{3,B} v \cdot \nabla(\Phi_n \eta) \mathrm{d}x \mathrm{d}s = II_1 + II_2 + II_3. \end{aligned} \tag{2.8}$$

First, noting that $\pi_{1,B}$ is harmonic in $B \times (0, T)$, observing (2.4), (2.5) and applying mean value property and Caccioppoli inequality of harmonic functions and (1.10), we estimate

$$
\begin{aligned}
II_1 &= -\int_{t_0-r_{n_0+1}^2}^{t} \int_B \nabla\pi_{1,B} \cdot v\Phi_n \eta \mathrm{d}x\mathrm{d}s \\
&\leqq c \sum_{j=n_0+1}^{n-1} r_j^{-3} \int_{A_j} |\nabla\pi_{1,B}||v|\mathrm{d}x\mathrm{d}s + c r_n^{-3} \int_{Q_n} |\nabla\pi_{1,B}||v|\mathrm{d}x\mathrm{d}s \\
&\leqq c \sum_{j=n_0+1}^{n} r_j^{-3} \|\nabla\pi_{1,B}\|_{L^2(Q_j)} \|v\|_{L^2(Q_j)} \\
&\leqq c \sum_{j=n_0+1}^{n} r_j^{-\frac{1}{2}} \|\nabla\pi_{1,B}\|_{L^2(I_j;L^\infty(\frac{1}{2}B))} \|v\|_{L^\infty(I_j;L^2(B_j))} \\
&\leqq c \sum_{j=n_0+1}^{n} r_j^{-\frac{1}{2}} r_{n_0}^{-5/2} \|\pi_{1,B}\|_{L^2(I;L^2(B))} \|v\|_{L^\infty(I_j;L^2(B_j))} \\
&\leqq c \sum_{j=n_0}^{n} r_j^{-1/2} r_{n_0}^{-5/2} \|\omega\|_{L^2(B\times I)} \|v\|_{L^\infty(I_j;L^2(B_j))} \\
&\leqq c r_{n_0}^{-5/2} \mathcal{E}_0^{1/2} \sum_{j=n_0}^{n} r_j^{-1/2} E_j^{1/2}.
\end{aligned}
\tag{2.9}
$$

Noting that

$$
c r_{n_0}^{-5/2} \mathcal{E}_0^{1/2} \sum_{j=n_0}^{n} r_j^{-1/2} E_j^{1/2} \leqq c r_{n_0}^{-3/2} \mathcal{E}_0^{1/2} \sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2},
$$

applying Young's inequality, we arrive at

$$
II_1 \leqq c\varepsilon^{-1} r_{n_0}^{-3} \mathcal{E}_0 + \varepsilon \left(\sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2} \right)^2.
$$

Next, we estimate II_2. Let $\chi_j \in C_c^\infty(B_j \times (t_0 - r_j^2, t_0])$ such that $\chi_j \equiv 1$ on Q_{j+1}. Then We set

$$
\psi_j = \begin{cases} 1 - \chi_1 & \text{if} \quad j = n_0, \\ \chi_j - \chi_{j+1} & \text{if} \quad j = n_0 + 1, \dots, n-1, \\ \chi_n & \text{if} \quad j = n. \end{cases}
$$

Clearly, $\operatorname{supp}(\psi_j) \subset A_j \cup A_{j+1}$, $j = n_0 + 1, \dots, n-1$ and

$$
\Phi_n \psi_j \sim r_j^{-3}, \quad \nabla(\Phi_n \psi_j) \sim r_j^{-4}.
$$

We have $\sum_{j=n_0}^{n} \psi_j = 1 - \chi_1 + \chi_1 - \chi_2 + \cdots + \chi_{n-1} - \chi_n + \chi_n = 1$. Thus, $\omega \times u = \sum_{j=1}^{n} \omega \times u\psi_j$. By the definition of $\pi_{2,B}$ we have

$$\nabla \pi_{2,B} = -E_B^*(\omega \times v) = -\sum_{j=n_0}^{n} E_B^*(\omega \times v\psi_j)) = \sum_{j=n_0}^{n} \nabla \pi_{2,B,j}.$$

Accordingly,

$$\begin{aligned} II_2 &= \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_B \sum_{j=n_0}^{n} \pi_{2,B_j} v \cdot \nabla(\sum_{k=n_0}^{n} \Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &= \sum_{j=n_0}^{n} \sum_{k=n_0}^{j} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_B \pi_{2,B_j} v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &\quad + \sum_{j=n_0}^{n} \sum_{k=j+1}^{n} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_B \pi_{2,B_j} v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &= \sum_{k=n_0}^{n} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_B \Big(\sum_{j=k}^{n} \pi_{2,B_j}\Big) v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &\quad + \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_B \pi_{2,B_j} v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &\quad + \sum_{j=n_0}^{n} \sum_{k=j+1}^{j+2} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_B \pi_{2,B_j} v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s = II_{2,1} + II_{2,2} + II_{2,3}. \end{aligned}$$

We define $\Pi_k = \sum_{j=k}^{n} \pi_{2,B_j}$. Recalling the definition of π_{2,B_j}, we find

$$\nabla \Pi_k = \sum_{j=k}^{n} \nabla \pi_{2,B_j} = -\sum_{j=k}^{n} E_B^*(\omega \times v\psi_j) = -E_B^*(\omega \times v\chi_k).$$

Thus,

$$\begin{aligned} II_{2,1} &= \sum_{k=n_0}^{n} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_B \Pi_k v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &\leqq c \sum_{k=n_0+1}^{n} \int\limits_{Q_k} r_k^{-4} |\Pi_k - (\Pi_k)_{B_k}||v| \mathrm{d}x\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leqq c \sum_{k=n_0+1}^{n} r_k^{-4} \|\Pi_k - (\Pi_k)_{B_k}\|_{L^{5/4}(I_k;L^{15/7}(B_k))} \|v\|_{L^5(I_k;L^{15/8}(B_k))} \\
&\leqq c \sum_{k=n_0+1}^{n} r_k^{-4} \|\nabla \Pi_k\|_{L^{5/4}(Q_k)} \|v\|_{L^5(I_k;L^{15/8}(B_k))} \\
&\leqq c \sum_{k=n_0+1}^{n} r_k^{-4} \|\omega \times v\|_{L^{5/4}(Q_k)} \|v\|_{L^5(I_k;L^{15/8}(B_k))} \\
&\leqq c \sum_{k=n_0+1}^{n} r_k^{-1/2} \|\omega\|_{L^2(Q_k)} r_k^{-3} E_k \leqq c \sum_{k=n_0}^{n} r_k^{-3} E_k Y_k^{1/2}.
\end{aligned}
$$

Next we estimate $II_{2,2}$. Using integration by parts, we find

$$
II_{2,2} = -\sum_{j=n_0}^{n} \sum_{k=j+3}^{n} \int_{t_0-r_{n_0}^2}^{t} \int_B \nabla \pi_{2,B,j} \cdot v \Phi_n \eta \psi_k \mathrm{d}x \mathrm{d}s.
$$

Observing that $\nabla \pi_{2,B,j}$ is harmonic on $Q_{j+2} \supset Q_k$, we estimate

$$
\begin{aligned}
II_{2,2} &\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} \int_{Q_k} r_k^{-3} |\nabla \pi_{2,B_j}||v| \mathrm{d}x \mathrm{d}s \\
&\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} \int_{I_k} \|\nabla \pi_{2,B_j}(s)\|_{L^\infty(B_k)} r_k^{-5/4} \|v(s)\|_{L^{12/5}(B_k)} \mathrm{d}s \\
&\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} r_j^{-9/4} \|\nabla \pi_{2,B_j}\|_{L^{8/7}(I_k;L^{4/3}(B_j))} r_k^{-5/4} \|v\|_{L^8(I_k;L^{12/5}(B_k))} \\
&\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} r_j^{-9/4} \|\omega \times v\|_{L^{8/7}(I_k;L^{4/3}(B_j))} r_k^{-5/4} E_k^{1/2}.
\end{aligned}
$$

By using Hölder's inequality, we obtain

$$
\begin{aligned}
II_{2,2} &\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} r_j^{-1/2} \|\omega\|_{L^2(Q_j)} r_j^{-7/4} \|v\|_{L^{8/3}(I_j;L^4(B_j))} r_k^{-5/4} E_k^{1/2} \\
&\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} Y_j^{1/2} r_j^{-7/4} E_j^{1/2} r_k^{-5/4} E_k^{1/2} \\
&\leqq c \left(\sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2} Y_j^{1/2} \right) \left(\sum_{k=n_0}^{n} r_k^{-3/2} E_k^{1/2} \right).
\end{aligned}
$$

Arguing similar as for $II_{2,1}$, we find

$$II_{2,3} \leqq c \sum_{j=n_0}^{n} r_j^{-3} \int\limits_{Q_j} |\pi_{2,B,j} - (\pi_{2,B,j})_{B_j}||v| \mathrm{d}x\mathrm{d}s \leqq c \sum_{j=n_0}^{n} r_j^{-3} E_j Y_j^{1/2}.$$

This along with (2.2) yields

$$\begin{aligned} II_2 &\leqq c \sum_{j=n_0}^{n} r_j^{-3} E_j Y_j^{1/2} + c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} Y_j^{1/2} r_j^{-7/4} E_j^{1/2} r_k^{-5/4} E_k^{1/2} \\ &\leqq c\varepsilon \left(\sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2} \right)^2. \end{aligned} \tag{2.10}$$

It remains to estimate II_3. As above, we get

$$\begin{aligned} II_3 &= \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_{B} \sum_{j=n_0}^{n} \pi_{3,B_j} v \cdot \nabla(\Phi_n \eta) \mathrm{d}x\mathrm{d}s \\ &= \sum_{k=n_0}^{n} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_{B} \left(\sum_{j=k}^{n} \pi_{3,B_j} \right) v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &\quad + \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_{B} \pi_{3,B_j} v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &\quad + \sum_{j=n_0}^{n} \sum_{k=j+1}^{j+2} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_{B} \pi_{3,B_j} v \cdot \nabla(\Phi_n \eta \psi_k) \mathrm{d}x\mathrm{d}s \\ &= II_{3,1} + II_{3,2} + II_{3,3}. \end{aligned}$$

We define $\widetilde{\Pi}_k = \sum_{j=k}^{n} \pi_{3,B,j}$. Recalling the definition of π_{3,B_j}, we find that

$$\nabla \widetilde{\Pi}_k = \sum_{j=k}^{n} \nabla \pi_{3,B,j} = -\sum_{j=k}^{n} E_B^*(\omega \times \nabla \pi_{h,B} \psi_j) = -E_B^*(\omega \times \nabla \pi_{h,B} \chi_k).$$

Thus, using integration by parts, we see that

$$\begin{aligned} II_{3,1} &= -\sum_{k=n_0}^{n} \int\limits_{t_0-r_{n_0}^2}^{t} \int\limits_{B} \nabla \widetilde{\Pi}_k \cdot v \Phi_n \eta \psi_k \mathrm{d}x\mathrm{d}s \\ &\leqq c \sum_{k=n_0+1}^{n} \int\limits_{Q_k} r_k^{-3} |\nabla \widetilde{\Pi}_k||v| \mathrm{d}x\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leqq c \sum_{k=n_0+1}^{n} r_k^{-2} \|\nabla \widetilde{\Pi}_k\|_{L^2(Q_k))} \|v\|_{L^\infty(I_k;L^2(B_k))} \\
&\leqq c \sum_{k=n_0+1}^{n} r_k^{-2} \|\omega \times \nabla \pi_{h,B}\|_{L^2(Q_k))} \|v\|_{L^\infty(I_k;L^2(B_k))} \\
&\leqq c \sum_{k=n_0+1}^{n} r_k^{-1/2} \|\omega\|_{L^2(Q_k)} \|\nabla \pi_{h,B}\|_{L^\infty(Q_k)} r_k^{-3/2} E_k^{1/2} \\
&\leqq c r_{n_0}^{-3/2} \|u\|_{L^\infty(I;L^2(B))} \sum_{k=n_0}^{n} Y_k^{1/2} r_k^{-3/2} E_k^{1/2}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
II_{3,2} &\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} \int_{Q_k} r_k^{-3} |\nabla \pi_{3,B_j}||v| \mathrm{d}x \mathrm{d}s \\
&\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} \int_{I_k} \|\nabla \pi_{3,B_j}(s)\|_{L^\infty(B_k)} r_k^{-1/2} \|v(s)\|_{L^6(B_k)} \mathrm{d}s \\
&\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} r_j^{-3/2} \|\nabla \pi_{3,B_j}\|_{L^2(Q_j)} r_k^{-1/2} E_k^{1/2} \\
&\leqq c \sum_{j=n_0}^{n} \sum_{k=j+3}^{n} r_j^{-3/2} \|\omega \times \nabla \pi_{h,B}\|_{L^2(I_k;L^2(B_j))} r_k^{-1/2} E_k^{1/2} \\
&\leqq c \sum_{k=n_0}^{n} \sum_{j=n_0}^{k-3} r_j^{-3/2} \|\omega\|_{L^2(Q_j)} \|\nabla \pi_{h,B}\|_{L^\infty(Q_j)} r_k^{-1/2} E_k^{1/2} \\
&\leqq c r_{n_0}^{-3/2} \|u\|_{L^\infty(I;L^2(B))} \sum_{k=n_0}^{n} \sum_{j=n_0}^{k} r_j^{-1} Y_j^{1/2} r_k^{-1/2} E_k^{1/2}.
\end{aligned}
$$

By a similar reasoning as for the estimation of III and $II_{3,1}$, we get

$$
II_{3,3} \leqq c r_{n_0}^{-3/2} \|u\|_{L^\infty(I;L^2(B))} \sum_{j=n_0}^{n} Y_j^{1/2} r_j^{-3/2} E_j^{1/2}.
$$

Gathering the above estimates, using (2.2) and Young's inequality, we find

$$
\begin{aligned}
II_3 \leqq\ & c r_{n_0}^{-3/2} \mathcal{E}_0^{1/2} \sum_{j=n_0}^{n} Y_j^{1/2} r_j^{-3/2} E_j^{1/2} \\
&+ c r_{n_0}^{-3/2} \mathcal{E}_0^{1/2} \sum_{k=n_0}^{n} \sum_{j=n_0}^{k} r_j^{-1} Y_j^{1/2} r_k^{-1/2} E_k^{1/2}
\end{aligned}
$$

$$\leqq c r_{n_0}^{-3} \mathcal{E}_0 + \varepsilon \left(\sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2} \right)^2 . \tag{2.11}$$

Inserting the estimates of II_1, II_2 and II_3 into the right-hand side of (2.8), we obtain

$$II \leqq c r_{n_0}^{-3} \varepsilon^{-1} \mathcal{E}_0 + c\varepsilon \left(\sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2} \right)^2 .$$

For $n_0 \leqq n \leqq N$ we get

$$I + II + III \leqq c\varepsilon^{-1} r_{n_0}^{-3} \mathcal{E}_0 + \varepsilon \left(\sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2} \right)^2 .$$

Observing (2.5), from the above estimate we get for all $n = n_0, \ldots, N$,

$$r_n^{-1} E_n \leqq \frac{1}{\hat{c}} (I + II + III) \leqq c\varepsilon^{-1} r_{n_0}^{-3} \mathcal{E}_0 + c\varepsilon \left(\sum_{j=n_0}^{n} r_j^{-3/2} E_j^{1/2} \right)^2 . \tag{2.12}$$

Let $1 < \alpha < 3$ be fixed. Set $X_j = r_j^{-\alpha} E_j$. Let $N \in \{n_0, \ldots, n\}$ such that

$$X_N = \max_{j=n_0,\ldots,n} X_j .$$

Then, from (2.12) with N in place of n we deduce that

$$\begin{aligned} r_N^{-3} E_N = r_N^{\alpha-3} X_N &\leqq c\varepsilon^{-1} r_{n_0}^{-3} \mathcal{E}_0 + c\varepsilon \left(\sum_{j=n_0}^{N} r_j^{-3/2} E_j^{1/2} \right)^2 \\ &= c\varepsilon^{-1} r_{n_0}^{-3} \mathcal{E}_0 + c\varepsilon \left(\sum_{j=n_0}^{N} r_j^{(\alpha-3)/2} X_j^{1/2} \right)^2 \\ &\leqq c\varepsilon^{-1} r_{n_0}^{-3} \mathcal{E}_0 + c\varepsilon \left(\sum_{j=n_0}^{N} r_j^{(\alpha-3)/2} \right)^2 X_N \\ &\leqq c\varepsilon^{-1} r_{n_0}^{-3} \mathcal{E}_0 + \frac{c\varepsilon}{(3-\alpha)^2} r_N^{\alpha-3} X_N \\ &= c\varepsilon^{-1} r_{n_0}^{-3} \mathcal{E}_0 + \frac{c\varepsilon}{(3-\alpha)^2} r_N^{-3} E_N . \end{aligned}$$

Now, we specify $\varepsilon > 0$ as follows

$$\frac{c\varepsilon}{(3-\alpha)^2} \leqq \frac{1}{2} \quad \text{or} \quad \varepsilon \leqq \frac{(3-\alpha)^2}{2c} .$$

With this choice we get

$$r_N^{-3} E_N = r_N^{\alpha-3} X_N \leqq c r_{n_0}^{-3} \mathcal{E}_0 \quad \text{or} \quad X_N \leqq c r_{n_0}^{-\alpha} \mathcal{E}_0 \tag{2.13}$$

with a constant $c > 0$ depending on α. By the definition of N, using (2.13), it holds for all $j = n_0, \ldots, n$

$$r_j^{-\alpha} E_j = X_j \leqq X_N \leqq c r_{n_0}^{-\alpha} \mathcal{E}_0. \tag{2.14}$$

Next, by virtue of (2.14), we get

$$r_{n+1}^{-1} \int_{Q_{n+1}} |\omega|^2 \mathrm{d}x\mathrm{d}t \leqq 2 r_n^{-1} \int_{Q_n} |\omega|^2 \mathrm{d}x\mathrm{d}t \leqq 4 r_n^{-1} E_n \leqq c_0 r_n^{\alpha-1} r_{n_0}^{-\alpha} \mathcal{E}_0,$$

where $c_0 > 0$ denotes an absolute constant. Without loss of generality we may assume that $\mathcal{E}_0 > 0$. Clearly,

$$c_0 r_n^{\alpha-1} r_{n_0}^{-\alpha} \mathcal{E}_0 \leqq \varepsilon^2 \quad \Longleftrightarrow \quad r_n \leqq r_{n_0}^{\frac{\alpha}{\alpha-1}} \Big(\frac{\varepsilon^2}{c_0 \mathcal{E}_0} \Big)^{\frac{1}{\alpha-1}}. \tag{2.15}$$

Thus, denoting $n_1 \in \mathbb{N}$, $n_1 \geqq n_0$, the smallest integer such that

$$r_{n_1} \leqq \min \Big\{ r_{n_0}^{\frac{\alpha}{\alpha-1}} \Big(\frac{\varepsilon^2}{c_0 \mathcal{E}_0} \Big)^{\frac{1}{\alpha-1}}, \frac{1}{2} r_0 \Big\},$$

by induction, we get the following result:

$$Y_n \leqq \varepsilon^2 \quad \forall n = n_0, \ldots, n_1 \quad \Longrightarrow \quad Y_n \leqq \varepsilon^2 \quad \forall n = n_1 + 1, n_1 + 2, \ldots \tag{2.16}$$

In view of (2.16) we get from (2.14) for all $n \geqq n_0$

$$E_n \leqq c \Big(\frac{r_n}{r_{n_0}} \Big)^{\alpha} \mathcal{E}_0 \quad \text{or} \quad r_n^{-(\alpha+1)/2 E_n \leqq c r_{n_0}^{-\alpha} \mathcal{E}_0 r_n^{(\alpha-1)/2}}. \tag{2.17}$$

In our discussion below we also use the inequality

$$r_n^{-1} E_n \leqq c r_{n_0}^{-\alpha} \mathcal{E}_0 r_n^{\alpha-1}. \tag{2.18}$$

In what follows we establish the required $L^\infty-$bound for v. We define

$$\Lambda_n = \max_{j=n_0,\ldots,n} r_j^{-3} E_j, \quad n \in \mathbb{N}.$$

Using (2.18), we reevaluate III from (2.7) as follows

$$III \leqq c r_{n_0}^{-3/2} \mathcal{E}_0^{1/2} \sum_{j=n_0}^{n} Y_j^{1/2} r_j^{-3/2} E_j^{1/2} \leqq c r_{n_0}^{-3/2-\alpha/2} \mathcal{E}_0 \sum_{j=n_0}^{n} r_j^{(\alpha-1)/2} r_j^{-3/2} E_j^{1/2}.$$

$$\leqq c r_{n_0}^{-3/2-\alpha/2} \mathcal{E}_0 \sum_{j=n_0}^{n} r_j^{(\alpha-1)/2} \Lambda_n^{1/2}.$$

Applying Young's inequality, we infer

$$III \leqq cr_{n_0}^{-4}\mathcal{E}_0^2 + \frac{\hat{c}}{4}\Lambda_n.$$

We also reevaluate $II = II_1 + II_2 + II_3$ as follows. First we get from (2.9)

$$\begin{aligned} II_1 &\leqq cr_{n_0}^{-5/2}\mathcal{E}_0^{1/2}\sum_{j=n_0}^{n} r_j^{-1/2}E_j^{1/2} \leqq cr_{n_0}^{-5/2}\mathcal{E}_0^{1/2}\sum_{j=n_0}^{n} r_j\Lambda_n^{1/2} \\ &\leqq cr_{n_0}^{-3}\mathcal{E}_0 + \frac{\hat{c}}{16}\Lambda_n. \end{aligned}$$

Furthermore, from (2.10) we infer

$$II_2 \leqq c\sum_{j=n_0}^{n} r_j^{-7/2}E_j^{3/2} + c\sum_{j=n_0}^{n}\sum_{k=j+3}^{n} Y_j^{1/2}r_j^{-7/4}E_j^{1/2}r_k^{-5/4}E_k^{1/2}. \tag{2.19}$$

Setting $\beta = \frac{11-3\alpha}{10-2\alpha} < 1$, and applying Young's inequality, we deduce

$$\begin{aligned} c\sum_{j=n_0}^{n} r_j^{-7/2}E_j^{3/2} &\leqq c\sum_{j=n_0}^{n} (r_j^{(\alpha+1)/2}E_j)^{3/2-\beta}\Lambda_n^{\beta} \leqq c\mathcal{E}_0^{-\frac{2\alpha}{5-\alpha}} r_{n_0}^{-\frac{2\alpha}{5-\alpha}} r_{n_0}^{\frac{\alpha-1}{5-\alpha}}\Lambda_n^{\frac{11-3\alpha}{10-2\alpha}} \\ &\leqq c(r_{n_0}^{-1}\mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} r_{n_0}^{-3}\mathcal{E}_0 + \frac{\hat{c}}{16}\Lambda_n. \end{aligned}$$

Arguing similarly the above, setting $\gamma = \frac{3-\alpha}{5-\alpha}$, we find that

$$\begin{aligned} c\sum_{j=n_0}^{n}\sum_{k=j+3}^{n} Y_j^{1/2}r_j^{-7/4}E_j^{1/2}r_k^{-5/4}E_k^{1/2} &\leqq c\sum_{j=n_0}^{n} r_j^{-2}E_j\Lambda_n^{1/2} \\ &\leqq c\sum_{j=n_0}^{n}\left(r_j^{-\frac{\alpha+1}{2}}E_j\right)^{1-\gamma}\Lambda_n^{\gamma+\frac{1}{2}} \\ &\leqq c\mathcal{E}_0^{-\frac{2\alpha}{5-\alpha}} r_{n_0}^{-\frac{2\alpha}{5-\alpha}} r_{n_0}^{\frac{\alpha-1}{5-\alpha}}\Lambda_n^{\frac{11-3\alpha}{10-2\alpha}} \\ &\leqq c(r_{n_0}^{-1}\mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} r_{n_0}^{-3}\mathcal{E}_0 + \frac{\hat{c}}{16}\Lambda_n. \end{aligned}$$

Thus, estimating the right-hand side of (2.19) by virtue of two estimates above, we obtain

$$II_2 \leqq c(r_{n_0}^{-1}\mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} r_{n_0}^{-3}\mathcal{E}_0 + \frac{\hat{c}}{8}\Lambda_n.$$

From (2.11) we deduce

$$II_3 \leqq cr_{n_0}^{-3/2}\mathcal{E}_0^{1/2}\sum_{j=n_0}^{n} r_j^{-2}E_j + cr_{n_0}^{-3/2}\mathcal{E}_0^{1/2}\sum_{k=n_0}^{n}\sum_{j=n_0}^{k} r_j^{-3/2}E_j^{1/2}r_k^{-1/2}E_k^{1/2}$$

$$\leqq c r_{n_0}^{-3/2} \mathcal{E}_0^{1/2} \left(\sum_{j=n_0}^{n} r_j^{-1} E_j^{1/2} \right)^2 . \tag{2.20}$$

Using (2.17), we find that

$$\sum_{j=n_0}^{n} r_j^{-1} E_j^{1/2} \leqq \sum_{j=n_0}^{n} r_j^{-1/4} E_j^{1/4} \Lambda_n^{1/4} \leqq c r_{n_0}^{-\alpha/4} \mathcal{E}_0^{1/4} \sum_{j=n_0}^{n} r_j^{(\alpha-1)/4} \Lambda_n^{1/4}$$
$$\leqq c r_{n_0}^{-1/4} \mathcal{E}_0^{1/4} \Lambda_n^{1/4}.$$

Inserting this inequality into the right-hand side of (2.20), and using Young's inequality, we arrive at

$$II_3 \leqq c r_{n_0}^{-4} \mathcal{E}_0^2 + \frac{\hat{c}}{16} \Lambda_n.$$

Gathering together the estimates of I, II_1, II_2, II_3 and III, applying Young's inequality, we get

$$I + II + III \leqq c r_{n_0}^{-3} \mathcal{E}_0 \Big(1 + (r_{n_0}^{-1} \mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} \Big) + \frac{\hat{c}}{2} \Lambda_n.$$

This together with (2.5), yields

$$\hat{c} r_n^{-3} E_n \leqq I + II + III \leqq c r_{n_0}^{-3} \mathcal{E}_0 \Big(1 + (r_{n_0}^{-1} \mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} \Big) + \frac{\hat{c}}{2} \Lambda_n. \tag{2.21}$$

Clearly, there exists $N \in \{n_0, \ldots, n\}$ such that $r_N^{-3} E_N = \Lambda_n = \Lambda_N$. Hence, (2.21) leads to

$$\hat{c} r_N^{-3} E_N = \hat{c} \Lambda_N \leqq c r_{n_0}^{-3} \mathcal{E}_0 \Big(1 + (r_{n_0}^{-1} \mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} \Big) + \frac{\hat{c}}{2} \Lambda_N. \tag{2.22}$$

Consequently,

$$r_n^{-3} E_n \leqq \Lambda_N = \Lambda_n \leqq c r_{n_0}^{-3} \mathcal{E}_0 \Big(1 + (r_{n_0}^{-1} \mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} \Big). \tag{2.23}$$

By an interpolation argument we get

$$\left(\fint_{Q_n} |v|^{10/3} \mathrm{d}x \mathrm{d}t \right)^{3/5} \leqq c r_{n_0}^{-3} \mathcal{E}_0 \Big(1 + (r_{n_0}^{-1} \mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} \Big). \tag{2.24}$$

Furthermore, by virtue of the mean value property for harmonic functions, Jensen's inequality and (1.11) we estimate for almost all $(x, t) \in Q_{n_0+1}$

$$|\nabla \pi_{h,B}(x,t)|^2 = \left| \fint_{B(x, 2^{-n_0-1})} \nabla \pi_{h,B}(y,t) \mathrm{d}y \right|^2 \leqq c r_{n_0}^{-3} \|\nabla \pi_{h,B}\|_{L^\infty(I; L^2(B))}^2$$
$$\leqq c r_{n_0}^{-3} \|u\|_{L^\infty(I; L^2(B))}^2 \leqq c r_{n_0}^{-3} \mathcal{E}_0. \tag{2.25}$$

Let (x_0, t_0) be a Lebesgue point of u. Applying Jensen's inequality, (2.24) along with (2.25), we obtain

$$\begin{aligned} |u(x_0,t_0)|^2 &\leqq \limsup_{n\to\infty} \left(\fint_{Q_n} |u|^{10/3} \mathrm{d}x\mathrm{d}t \right)^{3/5} \\ &\leqq c \limsup_{n\to\infty} \left(\fint_{Q_n} |v|^{10/3} \mathrm{d}x\mathrm{d}t \right)^{3/5} + c \limsup_{n\to\infty} \left(\fint_{Q_n} |\nabla \pi_{h,B}|^{10/3} \mathrm{d}x\mathrm{d}t \right)^{3/5} \\ &\leqq c r_{n_0}^{-3} \mathcal{E}_0 \Big(1 + (r_{n_0}^{-1} \mathcal{E}_0)^{\frac{5-\alpha}{\alpha-1}} \Big). \end{aligned} \tag{2.26}$$

Finally, by the absolutely continuity of the Lebesgue measure the condition (1.14) is fulfilled in a neighbourhood of $z_0 = (x_0, t_0)$, and by (2.26) u is essentially bounded in this neighbourhood. This completes the proof of the Theorem 1.5. $\square$

3. Proof of Theorem 1.8

The aim of this section is to prove Theorem 1.8 by an application of the main theorem. This first requires an estimate of $\|u\|_{L^\infty(I_r(t_0); L^2(B(x_0,r)))}$ in terms of the integrals in (1.17). We define

$$\begin{aligned} &\nabla \pi_h = -E^*_\Omega(u), \quad \nabla \pi_1 = -E^*_\Omega(\omega \times u), \quad \nabla \pi_2 = E^*_\Omega(\Delta u), \\ &v = u + \nabla \pi_h. \end{aligned}$$

Noting that

$$\nabla \pi_0 = E^*_\Omega(\Delta u - \omega \times u) = \nabla \pi_1 + \nabla \pi_2,$$

since u is suitable, the following local energy inequality holds for all non-negative $\phi \in C^\infty_c(B \times (0,T))$, and, for almost all $0 < t < T$

$$\begin{aligned} &\int\limits_\Omega |v(t)|^2 \phi^2(t) \mathrm{d}x + 2 \int\limits_0^t \int\limits_\Omega |\nabla v|^2 \phi^2 \mathrm{d}x\mathrm{d}s \\ &\leqq \int\limits_0^t \int\limits_\Omega |v|^2 (\partial_t \phi^2 + \Delta \phi^2) \mathrm{d}x\mathrm{d}s + \int\limits_0^t \int\limits_\Omega 2\omega \times \nabla \pi_h \cdot v \phi^2 \mathrm{d}x\mathrm{d}s \\ &\quad + \int\limits_0^t \int\limits_\Omega 2\pi_2 v \cdot \nabla \phi^2 \mathrm{d}x\mathrm{d}s + \int\limits_0^t \int\limits_\Omega 2(\pi_1 - (\pi_1)_{B(x_0,r)}) v \cdot \nabla \phi^2 \mathrm{d}x\mathrm{d}s. \end{aligned} \tag{3.1}$$

Let $K \subset \Omega$ be a compact set. Define $R_0 := \operatorname{dist}(K, \partial\Omega) > 0$. Let $z_0 = (x_0, t_0)$ be a given point in $K \times (0,T)$ and let $0 < r < \frac{R_0}{2}$. In (3.1) we choose a cut-off function $\phi \in C^\infty_c(B(x_0,r) \times (t_0 - r^2, t_0])$, such that $0 \leqq \phi \leqq 1$ in $\mathbb{R}^4$, $\phi \equiv 1$ in $Q(z_0,r)$ and $|\nabla \phi| \leqq c r^{-1}$, $|\partial_t \phi| \leqq c r^{-2}$ and $|\nabla^2 \phi| \leqq c r^{-2}$. Using Hölder's inequality, from (3.1) we get the estimate

$$\|v\phi\|^2_{L^\infty(I_r(t_0); L^2(B(x_0,r)))} + \|\nabla v \phi\|^2_{L^2(Q(z_0,r))}$$

$$
\begin{aligned}
&\leqq cr^{-2}\|v\|^2_{L^2(Q(z_0,r))}\\
&\quad + c\|\omega\|_{L^2(Q(z_0,r))}\|\nabla\pi_h\|_{L^2(I_r(t_0);L^\infty(B(x_0,r)))}\|v\phi\|_{L^\infty(I_r(t_0);L^2(B(x_0,r)))}\\
&\quad + c\|\pi_2\|_{L^2(Q(z_0,r))}\|v\phi\|_{L^\infty(I_r(t_0);L^2(B(x_0,r)))}\\
&\quad + cr^{-1}\|\pi_1-(\pi_1)_{B(x_0,r)}\|_{L^{5/4}(I_r(t_0);L^{15/7}(B(x_0,r)))}\|v\phi\|_{L^5(I_r(t_0);L^{15/8}(B(x_0,r)))}\\
&= A+B+C+D.
\end{aligned}
$$

(i) Recalling that $\nabla\pi_h(t)$ is harmonic for almost everywhere $t\in(0,T)$, by virtue of the mean value property of harmonic functions along with (1.11), we get for almost everywhere $t\in(0,T)$

$$
\begin{aligned}
\|\nabla\pi_h(t)\|^2_{L^2(B(x_0,r))} &\leqq cr^3\|\nabla\pi_h(t)\|^2_{L^\infty(B(x_0,r))}\leqq cr^3\|\nabla\pi_h(t)\|^2_{L^\infty(B(x_0,R_0))}\\
&\leqq cr^3R_0^{-3}\|\nabla\pi_h(t)\|^2_{L^2(\Omega))}\\
&\leqq cr^3R_0^{-3}\|u(t)\|^2_{L^2(\Omega))}\leqq cr^3R_0^{-3}\|u\|^2_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}
$$

Integrating both sides over $I_r(t_0)$ and multiplying both sides by r^{-2}, we get

$$
r^{-2}\|\nabla\pi_h\|^2_{L^2(Q(z_0,r))}\leqq cr^3R_0^{-3}\|u\|^2_{L^\infty(0,T;L^2(\Omega))}.
$$

Applying Jensen's inequality, we find that

$$
\begin{aligned}
A &\leqq cr^{-2}\|u\|^2_{L^2(Q(z_0,r))}+cr^{-2}\|\nabla\pi_h\|^2_{L^2(Q(z_0,r))}\\
&\leqq c\|u\|^2_{L^{10/3}(Q(z_0,r))}+cr^3R_0^{-3}\|u\|^2_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}
$$

(ii) Applying Young's inequality along with the mean value property of harmonic functions and (1.11), arguing as in (i), we estimate

$$
\begin{aligned}
B &\leqq c\|\omega\|^2_{L^2(Q(z_0,r))}\|\nabla\pi_h\|^2_{L^2(I_r(t_0);L^\infty(B(x_0,r)))}+\frac{1}{4}\|v\phi\|^2_{L^\infty(I_r(t_0);L^2(B(x_0,r)))}\\
&\leqq cr^3R_0^{-3}\Big(r^{-1}\|\omega\|^2_{L^2(Q(z_0,r))}\Big)\|u\|^2_{L^\infty(0,T;L^2(\Omega))}+\frac{1}{4}\|v\phi\|^2_{L^\infty(I_r(t_0);L^2(B(x_0,r)))}.
\end{aligned}
$$

(iii) By the definition of π_2 and $\nabla\cdot\Delta u=0$ in the sense of distribution we see that $\pi_2(t)$ is harmonic for almost everywhere $t\in(0,T)$. By the aid of the mean value property of harmonic functions along with (1.10), we get for almost everywhere $t\in(0,T)$

$$
\begin{aligned}
\|\pi_2(t)\|^2_{L^2(B(x_0,r))} &\leqq cr^3\|\pi_2(t)\|^2_{L^\infty(B(x_0,r))}\leqq cr^3\|\pi_2(t)\|^2_{L^\infty(B(x_0,R_0))}\\
&\leqq cr^3R_0^{-3}\|\pi_2(t)\|^2_{L^2(\Omega))}\leqq cr^3R_0^{-3}\|\nabla u(t)\|^2_{L^2(\Omega))}.
\end{aligned}
$$

Integrating both sides over $I_r(t_0)$ and taking the square root, we obtain

$$
\|\pi_2\|_{L^2(Q(z_0,r))}\leqq cr^{3/2}R_0^{-3/2}\|\nabla u\|_{L^2(\Omega_T))}.
$$

Applying Young's inequality, we infer that

$$
C\leqq cr^3R_0^{-3}\|\nabla u\|^2_{L^2(\Omega_T))}+\frac{1}{4}\|v\phi\|^2_{L^\infty(I_r(t_0);L^2(B(x_0,r)))}.
$$

(iv) Using the Sobolev-Poincaré inequality, we get for almost all $t \in (0, T)$

$$\|\pi_1(t) - (\pi_1(t))_{B(x_0,r)}\|^{5/4}_{L^{15/7}(B(x_0,r))} \leqq c\|\nabla\pi_1(t)\|^{5/4}_{L^{5/4}(B(x_0,r))}.$$

Integrating both sides over $I_r(t_0)$ and taking both sides to the $\frac{4}{5}$-th power, yields

$$\|\pi_1 - (\pi_1)_{B(x_0,r)}\|_{L^{5/4}(I_r(t_0);L^{15/7}(B(x_0,r)))} \leqq \|\nabla\pi_1\|_{L^{5/4}(Q(z_0,r))}.$$

Furthermore, by the aid of Jensen's inequality we see that

$$\|v\phi\|_{L^5(I_r(t_0);L^{15/8}(B(x_0,r)))} \leqq cr^{1/2}\|v\phi\|^2_{L^\infty(I_r(t_0);L^2(B(x_0,r)))}.$$

Using the two estimates, we have just derived, and applying Young's inequality, we get the estimate

$$D \leqq cr^{-1}\|\nabla\pi_1\|^2_{L^{5/4}(Q(z_0,r))} + \frac{1}{4}\|v\phi\|^2_{L^\infty(I_r(t_0);L^2(B(x_0,r)))}.$$

Let $1 < \alpha < 3$ be arbitrarily chosen, but fixed. For notational simplicity we denote $\beta = \frac{5\alpha}{5\alpha-2}$. Recalling that $\nabla\pi_1 = -E^*_\Omega(\omega \times u)$, by the assumption of the theorem (1.17), we have for some $r \in (0, R_1)$

$$\int_{Q(z_0,r)} (|\omega|^2 + |u|^{10/3} + |\nabla\pi_1|^{5/4})\mathrm{d}x\mathrm{d}t \leqq \tilde{\varepsilon}r^\beta, \tag{3.2}$$

where $\tilde{\varepsilon} = \tilde{\varepsilon}_\alpha \in (0, 1)$ and R_1, are positive constants, that will be specified below.

Gathering the above estimates together, using (3.2), and noting that $r^{8\beta/5-1}\tilde{\varepsilon}^{8/5} \leqq cr^{3\beta/5}\tilde{\varepsilon}^{3/5}$, we find

$$\begin{aligned}
&\|v\|^2_{L^\infty(I_{r/2}(t_0);L^2(B(x_0,r/2)))} + \|\nabla v\|^2_{L^2(Q(z_0,r/2))} \\
&\quad\leqq cr^3R_0^{-3}\mathcal{E} + c\|u\|^2_{L^{10/3}(Q(z_0,r))} + cr^{-1}\|\nabla\pi_1\|^2_{L^{5/4}(Q(z_0,r))} \\
&\quad\leqq cr^3R_0^{-3}\mathcal{E} + cr^{3\beta/5}\tilde{\varepsilon}^{3/5} + cr^{8\beta/5-1}\tilde{\varepsilon}^{8/5} \\
&\quad\leqq cr^3R_0^{-3}\mathcal{E} + cr^{3\beta/5}\tilde{\varepsilon}^{3/5},
\end{aligned} \tag{3.3}$$

where

$$\mathcal{E} = \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u\|^2_{L^2(\Omega_T)}.$$

Set $r_0 = \frac{r}{2}$. Using the mean value property of harmonic functions, arguing as in (i), we get from (3.3)

$$\begin{aligned}
\mathcal{E}_0 &:= \|u\|^2_{L^\infty(I_{r_0}(t_0);L^2(B(x_0,r_0)))} + \|\omega\|^2_{L^2(Q(z_0,r_0))} \\
&\leqq 2\|v\|^2_{L^\infty(I_{r/2}(t_0);L^2(B(x_0,r/2)))} + 2\|\nabla \times v\|^2_{L^2(Q(z_0,r/2))} \\
&\qquad + 2\|\nabla\pi_h\|^2_{L^\infty(I_{r/2}(t_0);L^2(B(x_0,r/2)))} \\
&\leqq cr^3R_0^{-3}\mathcal{E} + cr_0^{3\beta/5}\tilde{\varepsilon}^{3/5}.
\end{aligned} \tag{3.4}$$

Now let $R_1 \in (0, R_0)$ be determined by the condition

$$R_1^{3-3\beta/5} R_0^{-3} \mathcal{E} \leqq \tilde{\varepsilon}^{3/5}.$$

This gives

$$cr^3 R_0^{-3} \mathcal{E} = cr_0^{\frac{3\beta}{5}} r_0^{3-\frac{3\beta}{5}} R_0^{-3} \mathcal{E} \leqq cr_0^{\frac{3\beta}{5}} R_1^{3-\frac{3\beta}{5}} R_0^{-3} \mathcal{E} \leqq cr_0^{\frac{3\beta}{5}} \tilde{\varepsilon}^{3/5}.$$

Then estimate (3.4) becomes

$$\mathcal{E}_0 \leqq cr_0^{3\beta/5} \tilde{\varepsilon}^{3/5}. \tag{3.5}$$

By means of (3.5), we see that

$$r_0^{\frac{\alpha}{\alpha-1}} \left(\frac{\varepsilon_\alpha^2}{c\mathcal{E}_0} \right)^{\frac{1}{\alpha-1}} \geqq c^{-\frac{1}{1-\alpha}} \varepsilon_\alpha^{\frac{2}{1-\alpha}} \tilde{\varepsilon}^{-\frac{3}{5-5\alpha}} r_0^{\frac{\alpha}{\alpha-1}-\frac{3\beta}{5-5\alpha}} = c^{-\frac{1}{1-\alpha}} \varepsilon_\alpha^{\frac{2}{1-\alpha}} \tilde{\varepsilon}^{-\frac{3}{5-5\alpha}} r_0^{\beta} \geqq r_0^{\beta},$$

whenever

$$\tilde{\varepsilon} \leqq \frac{\varepsilon_\alpha^{10/3}}{c^{5/3}}.$$

Hence, with this choice of $\tilde{\varepsilon}$ we get for all $\rho \in \left(r_0^{\frac{\alpha}{\alpha-1}} \left(\frac{\varepsilon_\alpha^2}{c\mathcal{E}_0} \right)^{\frac{1}{\alpha-1}}, r_0 \right)$

$$\frac{1}{\rho} \int\limits_{Q(z_0,\rho)} |\omega|^2 \mathrm{d}x\mathrm{d}t \leqq \frac{1}{r_0^{\beta}} \int\limits_{Q(z_0,r_0)} |\omega|^2 \mathrm{d}x\mathrm{d}t \leqq \frac{2^{\beta}}{r^{\beta}} \int\limits_{Q(z_0,r)} |\omega|^2 \mathrm{d}x\mathrm{d}t \leqq 2^{\beta} \tilde{\varepsilon}.$$

Eventually, replacing $\tilde{\varepsilon}$ by $\varepsilon_\alpha/2^{\beta}$, according to Theorem 1.5 we deduce that z_0 is a regular point. This completes the proof of Theorem 1.8. □

Acknowledgements. The authors would like to thank to the anonymous referee for careful reading of the manuscript and constructive criticism. D. Chae was partially supported by NRF Grants 2021R1A2C1003234.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest. Our manuscript has no associated data.

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(*Received May 25, 2023 / Accepted November 5, 2024*)
Published online November 25, 2024