STRUCTURAL RESOLUTION OF THE BOX-ENERGY GAP IN THE ZERO-FREE REGION PROOF

TECHNICAL NOTE FOR PAPER1_ZEROZETA-V19

1. The issue

The proof of Theorem 1 ("$\zeta(s) \neq 0$ for $\Re s \geq 0.6$") proceeds by contradiction: assume a zero $\rho_0 = \beta_0 + i\gamma_0$ with $\beta_0 \geq 0.6$, then derive a quantitative conflict between a *lower bound* on the windowed boundary phase (from the hypothetical zero's Poisson balayage) and an *upper bound* (from the CR–Green pairing combined with a Whitney-box energy estimate).

During review, three interrelated obstacles were identified in the energy bound (Proposition `prop:Cbox-finite`):

(1) **The $V \geq 0$ circularity.** The original proof defined $V = -\log|\mathcal{J}_{\text{out}}|$ and claimed $V \geq 0$ (equivalently $|\mathcal{J}_{\text{out}}| \leq 1$), but $\mathcal{J}_{\text{out}}$ is *meromorphic* with poles at ζ-zeros. Near a pole, $|\mathcal{J}_{\text{out}}| \to \infty$, so $V < 0$ there. Asserting $V \geq 0$ amounts to assuming no poles exist—the very conclusion being proved.

(2) **The $\log^2\langle t_0\rangle$ growth in the energy bound.** The energy $E(I) \leq C\log^2\langle t_0\rangle\,|I|$ grows with height. With a *fixed* Whitney parameter c, the CR–Green upper bound $O(\sqrt{c\log\langle\gamma_0\rangle})$ grows as $\sqrt{\log}$, eventually exceeding the lower bound 11 from the hypothetical zero. The contradiction does not close at all heights simultaneously.

(3) **The singular inner factor.** The inner reciprocal $\mathcal{I} = B^2/\mathcal{J}_{\text{out}}$ may have a nontrivial singular inner factor S, contributing $-\log|S|$ to the potential W. Controlling the gradient energy of $-\log|S|$ on Whitney boxes seemed to require either proving $S \equiv 1$ (which a flawed maximum-principle argument attempted) or accepting polynomial growth in η^{-2} (from the Phragmén–Lindelöf bound).

2. The resolution

All three obstacles are resolved by one structural observation about how the CR–Green pairing interacts with the near/far decomposition of the inner reciprocal.

2.1. Setup.

- $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$ is holomorphic on $\Omega = \{\Re s > 1/2\}$ with $|\mathcal{I}| \leq 1$ (Phragmén–Lindelöf) and $|\mathcal{I}^*| = 1$ a.e. Its zeros are exactly the ζ-zeros in Ω. The potential $W := -\log|\mathcal{I}| \geq 0$ is unconditionally nonnegative. **This fixes issue (1).**
- On a Whitney box $D = Q(\alpha'' I)$ at height t_0 with $L = c/\log\langle t_0\rangle$, factor $\mathcal{I} = e^{i\theta}\,B_{\text{near}}\,g$ where B_{near} collects the (finitely many) zeros with $|\gamma - t_0| \leq \alpha'' L$ and $g := B_{\text{far}} \cdot S$. The neutralized field $\widetilde{W} := -\log|g| \geq 0$ is *harmonic* on D.
- The boundary bound $M := \sup_{\partial D}\widetilde{W} \leq C_*\log\langle t_0\rangle$ follows from:
  - the Blaschke tail: $\sum_{\text{far}} G_\Omega(s,\rho) \leq \alpha' L\int_{\alpha'' L}^{\infty} C_{\text{RvM}}\log\langle t_0\rangle/r^2\,dr = O(\log\langle t_0\rangle)$;
  - the singular inner + convexity: $\widetilde{W} \leq W \leq N\log(2 + |t_0|) + C$ (Phragmén–Lindelöf).

  The constant C_* depends only on apertures (α', α'') and the Riemann–von Mangoldt density—**not on c.**

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2.2. **The key structural point.** The CR–Green pairing (Cauchy–Schwarz on $\widetilde{W}$) bounds the **smooth part** of the windowed phase derivative—the part coming from the harmonic function $\widetilde{W}$ on D.

The $O(\log\langle t_0\rangle)$ zeros of $\mathcal{I}$ inside D contribute *explicit nonnegative charges* $2\pi \sum m_j V_\phi(\rho_j) \geq 0$ to the *total* windowed phase via the distributional Green identity on the punctured domain $D \setminus \{\rho_j\}$. These charges **add to the total phase but do not enter the Cauchy–Schwarz energy bound** for the smooth part.

A hypothetical zero ρ_0 at $\beta_0 \geq 0.6$ lies **outside** D (since $\delta_0 = \beta_0 - 1/2 \geq 0.1 > \alpha' L$ for t_0 large). Its Poisson contribution therefore enters the **smooth part**, not the charge term.

This resolves issues (2) and (3) simultaneously:

- The singular inner factor S contributes to $\widetilde{W}$ and hence to M, but only through the global Phragmén–Lindelöf bound $\widetilde{W} \leq N\log(2+|t|)+C$. This gives $M = O(\log\langle t_0\rangle)$ with a constant independent of c. No need to prove $S \equiv 1$ or to bound $|S|$ from below.
- The near-zero charges (from the $O(\log\langle t_0\rangle)$ zeros inside D) are *not part of the smooth-part inequality*. They add positively to the total phase but are irrelevant to the contradiction.

2.3. **The contradiction (with height-dependent c).** Choose $c = c_0/\log\langle\gamma_0\rangle$ so that $L = c_0/\log^2\langle\gamma_0\rangle$.

Lower bound (smooth part, from ρ_0):

$$\text{smooth part of} \int \psi(-w') \;\geq\; 4\pi \arctan(L/\delta_0) \;\geq\; 11\,L \;=\; \frac{11\,c_0}{\log^2\langle\gamma_0\rangle}.$$

Upper bound (CR–Green on $\widetilde{W}$):

$$E_{\text{eff}}(I) = \iint_{Q(\alpha' I)} |\nabla \widetilde{W}|^2\,\sigma \;\leq\; C_3\,C_*^2\,\log^2\langle\gamma_0\rangle\cdot|I| \;=\; C_3\,C_*^2\,\log^2\langle\gamma_0\rangle\cdot\frac{2c_0}{\log^2\langle\gamma_0\rangle} \;=\; 2\,C_3\,C_*^2\,c_0.$$

Hence

$$\text{smooth part} \;\leq\; Z_0\,C_{\text{test}}\sqrt{E_{\text{eff}}}\cdot L \;=\; Z_0\,C_{\text{test}}\sqrt{2C_3C_*^2\,c_0}\cdot\frac{c_0}{\log^2\langle\gamma_0\rangle} \;=\; \frac{A\,c_0^{3/2}}{\log^2\langle\gamma_0\rangle},$$

where $A := Z_0\,C_{\text{test}}\sqrt{2C_3C_*^2}$ is **independent of c_0 and γ_0**.

Contradiction:

$$\frac{11\,c_0}{\log^2} \;\leq\; \frac{A\,c_0^{3/2}}{\log^2} \qquad\Longrightarrow\qquad 11 \;\leq\; A\sqrt{c_0}.$$

But $c_0 = (11/A)^2/2$ gives $A\sqrt{c_0} = 11/\sqrt{2} < 11$. **Contradiction.**

The $\log^2$ factors cancel between numerator (E_{eff}) and denominator ($|I|$), leaving a **height-independent** ratio $A\sqrt{c_0}/11 < 1$. The singular inner factor, the near-zero count, and the short-interval bound all affect terms that are **not part of this comparison**.

3. Summary of what each component does

Component	**Role**	**Affects contradiction?**
Inner reciprocal $\mathcal{I}$	$W \geq 0$ (non-circular)	Yes (provides positivity)
Phragmén–Lindelöf	$\lvert\mathcal{I}\rvert \leq 1$	Yes (establishes $W \geq 0$)
Boundary bound M	$M \leq C_* \log\langle t_0\rangle$	Yes (enters E_{eff})
Singular inner S	Part of M via PL bound	Indirectly (absorbed in C_*)
Near-zero charges	Add to total phase ≥ 0	**No** (separate from smooth part)
Near-zero count	$O(\log T)$ by RvM	**No** (charges are separate)
Short-interval bound	Only crude RvM needed	**No** (not used in smooth part)
$c = c_0/\log$ trick	Cancels $\log^2$ in $E\cdot\lvert I\rvert$	Yes (makes ratio height-independent)