Closing Lemmas 2.6–2.9 (Dec 11 version): suggested completion

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To: Milan Zlatanović
Cc: Prof. Elshad Allahyarov
Project: NS Overleaf (link)

Dear Milan,

Thank you for clarifying. You are right: my earlier response was tied to the older (Dec 8) "single Lemma 2.6" iteration. I have now reviewed your Dec 11 restructuring into Lemmas 2.6–2.9 (singular point; normalization; rescaled domain; ancient limit) in `new-version-12-11.tex`. The structure is good and much easier to check.

What is still needed to fully close the chain:

1. **Lemma 2.6 (existence of a CKN singular point):** the logic is correct (if no singular point exists at T^*, ε-regularity gives local boundedness near $t = T^*$, hence continuation past T^*, contradiction). This is the right "anchor" for the blow-up.

2. **Lemma 2.7 (blow-up normalization):** as currently written it chooses x_k from the *global* vorticity supremum at times $t_k \uparrow T^*$. This leaves an implicit issue: x_k could in principle drift (the noncompactness worry you flagged in the Dec 8 comments).

   The clean fix is to *anchor the blow-up near a fixed singular point x^* from Lemma 2.6.* Concretely, replace the choice of x_k by either:

   - *(Local vorticity choice)* choose $x_k \in B_1(x^*)$ with $|\omega(x_k, t_k)| = \|\omega(\cdot, t_k)\|_{L^\infty(B_1(x^*))}$, and note this local supremum must diverge as $t_k \uparrow T^*$ if (x^*, T^*) is singular; or
   - *(CKN-normalization choice, recommended)* choose scales $r_k \downarrow 0$ such that the CKN functional at (x^*, T^*) satisfies

     $$r_k^{-2} \iint_{Q_{r_k}(x^*,T^*)} \left(|u|^3 + |p|^{3/2}\right) dx\, dt \geq \varepsilon_{\mathrm{CKN}},$$

     and define the blow-up by $\lambda_k := r_k$ and center $x_k := x^*$ (so no drift can occur).

   The second option makes the nontriviality in Lemma 2.9 essentially automatic by semicontinuity.

3. **Lemma 2.8 (domain exhaustion):** looks correct.

4. **Lemma 2.9 (ancient limit):** this is the main place where the proof needs to be written out. There are two distinct sub-steps:

   (a) *Compactness / passage to the limit.* State explicitly the uniform estimates on each fixed cylinder $Q_R = B_R \times (-R^2, 0)$ inherited from the local energy inequality under scaling, e.g.

   $$\sup_{s\in(-R^2,0)} \int_{B_R} |u^{(k)}(s)|^2 + \int_{Q_R} |\nabla u^{(k)}|^2 \leq C(R), \qquad \|p^{(k)}\|_{L^{3/2}(Q_R)} \leq C(R),$$

plus a standard bound on $\partial_s u^{(k)}$ in a negative Sobolev space (so Aubin–Lions applies). This yields a subsequence converging strongly in L^p_{loc} (for $p < 3$) and is enough to pass the nonlinear term and the local energy inequality, giving a suitable weak limit (u^∞, p^∞) on $\mathbb{R}^3 \times (-\infty, 0)$.

(b) *Nontriviality of the limit (your item (iii)).* This is easiest if the blow-up is anchored by a *scale-invariant lower bound* (CKN functional) rather than a pointwise vorticity normalization. With the CKN-normalization choice in Lemma 2.7, we get on Q_1:

$$\iint_{Q_1} \left(|u^{(k)}|^3 + |p^{(k)}|^{3/2}\right) dx\, dt \geq \varepsilon_{\mathrm{CKN}}.$$

By strong/weak convergence and lower semicontinuity, the same lower bound holds for the limit, hence $u^\infty \not\equiv 0$, which implies your desired statement $\int_{Q_r} |u^\infty|^3 \geq c > 0$ for some r, c.

One stylistic point: I recommend removing any claim that the ancient limit is locally L^∞ in space; from local energy bounds one naturally inherits $L^\infty_t L^2_x \cap L^2_t H^1_x$ and the scale-invariant $L^3/L^{3/2}$ controls, which are what the later "geometric depletion" steps use.

Thanks again for doing the hard work of restructuring this. If you agree, I suggest implementing the CKN-normalized blow-up (anchored at x^*) as the main route, and keeping the vorticity normalization as an optional remark (it is conceptually nice, but technically less robust for proving nontriviality of the ancient limit).

Sincerely,
Jonathan Washburn