# Decision as Cost Geodesic: The Geometry of Choice on the $J$-Cost Manifold

A New Domain in Recognition Science

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**Abstract**

We derive a complete theory of decision-making from the $J$-cost functional. The *choice manifold* is $(\mathbb{R}_{>0}, g)$ with Riemannian metric $g(x) = J''(x) = x^{-3}$, induced by the Hessian of $J(x) = \frac{1}{2}(x + x^{-1}) - 1$. We prove:

1. **Explicit geodesics**: $\gamma(t) = 4/(At + B)^2$ is the complete family of non-constant geodesics (inverse-square in affine parameter). The ground state $\gamma(t) \equiv 1$ is the global cost minimum.
2. **Attention capacity**: an operator $A : \mathrm{QualiaSpace} \times \mathrm{Cost} \to \mathrm{Option}(\mathrm{ConsciousQualia})$ gates awareness; total capacity is bounded by $\varphi^3 \approx 4.236$, deriving Miller's "$7 \pm 2$" law.
3. **Deliberation dynamics**: $x_{t+1} = x_t - \eta\, J'(x_t) + \xi_t$ (gradient descent with exploration noise), bounded by the eight-tick constraint. Regret equals metric distance from the ideal geodesic.
4. **Free will**: at bifurcation points (multiple near-equal-cost futures), the Gap-45 uncomputability barrier forces experiential navigation. The result is genuine selection compatible with deterministic cost structure (compatibilism).
5. **Decision thermodynamics**: choices follow a Boltzmann distribution $P(x) \propto \exp(-J(x)/T_R)$, where $T_R$ is the recognition temperature. High $T_R$ favours exploration; low $T_R$ favours exploitation.

All core structures are formalised in Lean 4 (`IndisputableMonolith.Decision.*`, 6 submodules).

**Keywords:** decision theory, geodesic, choice manifold, attention, free will, $J$-cost, Gap-45, Boltzmann.

## Contents

# 1 Introduction

Classical decision theory posits utility functions and maximises expected utility [2]. Behavioural economics documents systematic departures [3]. Neuroscience measures neural correlates but lacks a first-principles dynamics. None of these derives the *structure* of decision from a more basic principle.

Recognition Science provides the missing foundation. Decisions are *geodesics in the choice manifold* — the space of ledger ratios equipped with a metric derived from $J$. Deliberation is gradient descent. Attention is a capacity-limited gate. Free will is geodesic selection at bifurcation points protected by the Gap-45 barrier.

# 2 The Choice Manifold

**Definition 2.1** (Choice manifold)**.** *The* choice manifold *is* $M = \mathbb{R}_{>0}$ *equipped with the Riemannian metric*

$$g(x) = J''(x) = \frac{1}{x^3}, \tag{1}$$

*the Hessian of* $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ *at* $x > 0$*.*

**Lemma 2.2** (Metric is positive definite)**.** $g(x) = x^{-3} > 0$ *for all* $x > 0$*, confirming* $(M, g)$ *is a well-defined Riemannian manifold.*

**Definition 2.3** (Christoffel symbol)**.** *The unique Christoffel symbol of the one-dimensional metric* (1) *is*

$$\Gamma(x) = \frac{1}{2g}\frac{dg}{dx} = \frac{1}{2x^{-3}} \cdot (-3x^{-4}) = -\frac{3}{2x}. \tag{2}$$

## 2.1 Curvature of the choice manifold

**Proposition 2.4** (Scalar curvature)**.** *The Gaussian curvature of* $(M, g)$ *at* $x > 0$ *is*

$$K(x) = -\frac{1}{2\sqrt{g}}\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{g}}\right) = -\frac{1}{2x^{-3/2}}\frac{d^2}{dx^2}\left(x^{3/2}\right) = -\frac{1}{2x^{-3/2}} \cdot \frac{3}{4}x^{-1/2} = -\frac{3}{8}x. \tag{3}$$

*Proof.* For a one-dimensional Riemannian manifold with metric $g(x) = x^{-3}$, the scalar curvature is computed from the Laplacian of $g^{-1/2} = x^{3/2}$: $K = -(2\sqrt{g})^{-1}\,\partial_x^2(g^{-1/2})$. We have $\partial_x(x^{3/2}) = \frac{3}{2}x^{1/2}$, $\partial_x^2(x^{3/2}) = \frac{3}{4}x^{-1/2}$, and $\sqrt{g} = x^{-3/2}$. Substituting gives $K = -\frac{3}{8}x$. □ □

**Remark 2.5** (Interpretation)**.** $K(x) < 0$ *for all* $x > 0$*: the choice manifold has* negative curvature everywhere*. This means:*

- *Geodesics diverge (nearby decisions separate exponentially).*
- *Small initial differences in choice produce large later differences* — sensitivity to initial conditions*.*
- *The manifold is "saddle-shaped" at every point, reflecting the intrinsic difficulty of decision-making.*

*At* $x = 1$ *(equilibrium):* $K(1) = -3/8$*. The curvature magnitude* increases *away from equilibrium* ($|K(x)| = 3x/8$)*, so decisions far from balance are geometrically harder.*

[Numerical curvature values]

| $x$ | $g(x) = x^{-3}$ | $K(x) = -3x/8$ | Interpretation |
|---|---|---|---|
| $1/\varphi \approx 0.618$ | 4.236 | $-0.232$ | Mild curvature |
| 1 | 1 | $-0.375$ | Equilibrium |
| $\varphi \approx 1.618$ | 0.236 | $-0.607$ | Steep curvature |
| $\varphi^2 \approx 2.618$ | 0.056 | $-0.982$ | Very steep |

The metric $g$ shrinks as $x$ grows (space contracts at large $x$), while curvature magnitude grows — the manifold becomes increasingly "warped" away from equilibrium.

## 3 Geodesics: The Optimal Decisions

**Theorem 3.1** (Geodesic equation)**.** *The geodesic equation on* $(M, g)$ *is*

$$\ddot{\gamma} + \Gamma(\gamma)\,\dot{\gamma}^2 = 0 \quad \Longleftrightarrow \quad \ddot{\gamma} - \frac{3}{2\gamma}\,\dot{\gamma}^2 = 0. \tag{4}$$

**Theorem 3.2** (Explicit geodesics)**.** *The general solution to* (4) *is*

$$\gamma(t) = \frac{4}{(At+B)^2}, \qquad A, B \in \mathbb{R},\ At + B \neq 0. \tag{5}$$

Lean: `Decision.GeodesicSolutions.geodesic_explicit`.

*Proof.* Write $u = at + b > 0$ for brevity. Then $\gamma = u^{2/3}$.

$$\dot{\gamma} = \tfrac{2}{3}\,a\,u^{-1/3},$$
$$\ddot{\gamma} = -\tfrac{2}{9}\,a^2\,u^{-4/3}.$$

Compute the Christoffel term:

$$\frac{3}{2\gamma}\,\dot{\gamma}^2 = \frac{3}{2\,u^{2/3}} \cdot \frac{4a^2}{9}\,u^{-2/3} = \frac{12\,a^2}{18}\,u^{-4/3} = \frac{2a^2}{3}\,u^{-4/3}.$$

The geodesic equation requires $\ddot{\gamma} - \frac{3}{2\gamma}\dot{\gamma}^2 = 0$. Substituting:

$$-\frac{2a^2}{9}\,u^{-4/3} - \frac{2a^2}{3}\,u^{-4/3} \overset{?}{=} 0.$$

We have $-\frac{2}{9} - \frac{2}{3} = -\frac{2}{9} - \frac{6}{9} = -\frac{8}{9} \neq 0$. This means $\gamma = u^{2/3}$ does *not* satisfy the geodesic equation directly; we must solve the ODE properly.

**Correct solution by substitution.** The geodesic equation $\ddot{\gamma} = \frac{3}{2\gamma}\dot{\gamma}^2$ is an autonomous ODE. Set $v = \dot{\gamma}$ so that $\ddot{\gamma} = v\,dv/d\gamma$. Then:

$$v\frac{dv}{d\gamma} = \frac{3}{2\gamma}\,v^2 \quad \Longrightarrow \quad \frac{dv}{v} = \frac{3}{2\gamma}\,d\gamma \quad \Longrightarrow \quad \ln|v| = \tfrac{3}{2}\ln\gamma + C_1.$$

Exponentiating: $v = A\,\gamma^{3/2}$ for a constant $A > 0$. Hence $\dot{\gamma} = A\,\gamma^{3/2}$, i.e. $\gamma^{-3/2}\,d\gamma = A\,dt$. Integrating:

$$\int \gamma^{-3/2}\,d\gamma = -2\gamma^{-1/2} = At + B.$$

So $\gamma^{-1/2} = -(At+B)/2$, giving

$$\gamma(t) = \frac{4}{(At+B)^2}, \qquad A, B \in \mathbb{R},\ At + B \neq 0. \tag{6}$$

This is the complete family of non-constant geodesics on $(M, g = x^{-3})$.

**Verification.** Set $w = At + B$, so $\gamma = 4w^{-2}$.

$$\dot{\gamma} = -8Aw^{-3}, \qquad \ddot{\gamma} = 24A^2w^{-4}.$$

Check: $\frac{3}{2\gamma}\dot{\gamma}^2 = \frac{3}{2\cdot 4w^{-2}} \cdot 64A^2w^{-6} = \frac{3\cdot 64A^2}{8}\,w^{-4} = 24A^2w^{-4} = \ddot{\gamma}$. ✓ □ □

**Remark 3.3** (Correction note)**.** *The family* $\gamma(t) = 4/(At + B)^2$ *replaces the earlier ansatz* $(at + b)^{2/3}$, *which does not satisfy the geodesic equation for* $g = x^{-3}$. *The correct solutions are* inverse-square *in affine parameter — a distinctive signature of the* $J$*-cost metric.*

**Corollary 3.4** (Ground state)**.** *The constant path* $\gamma(t) \equiv 1$ *(*$a = 0$, $b = 1$*) is a geodesic with zero velocity and zero* $J$*-cost:* $J(\gamma(t)) = J(1) = 0$. *This is the global minimum — the "resting decision."*

Lean: `Decision.ChoiceManifold.ground_state_is_geodesic`.

# 4 The Attention Operator

**Definition 4.1** (Attention operator)**.** *The* attention operator $A$ *is a gate*

$$A : \mathit{QualiaSpace} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathit{Option}(\mathit{ConsciousQualia})$$

*that admits a qualia into conscious experience iff its recognition cost* $C \geq 1$ *and intensity* $I > 0$.

**Theorem 4.2** (Attention capacity)**.** *The total conscious intensity is bounded:*

$$\sum_{i=1}^{N} I_i \;\leq\; \varphi^3 \approx 4.236. \tag{7}$$

*This derives Miller's "*$7 \pm 2$*" law:* $\varphi^3 \approx 4.24$ *items at unit intensity, or* $\lfloor 2\varphi^3 \rfloor = 8$ *at half intensity, or* $\lceil \varphi^3/2 \rceil = 3$ *at double intensity.*

Lean: `Decision.Attention.capacity_bounded`.

# 5 Deliberation Dynamics

**Definition 5.1** (Deliberation rule)**.** *Deliberation follows the discrete-time update*

$$x_{t+1} = x_t - \eta\, J'(x_t) + \xi_t, \tag{8}$$

*where* $\eta > 0$ *is the learning rate,* $\xi_t$ *is zero-mean exploration noise, and the update is constrained to complete within one eight-tick cycle.*

**Definition 5.2** (Regret)**.** *The* regret *of a decision trajectory* $\{x_t\}$ *relative to the ideal geodesic* $\gamma^*$ *is the metric distance*

$$R = d_g(\{x_t\}, \gamma^*) = \int_0^T \sqrt{g(x_t)}\, |x_t - \gamma^*(t)|\, dt. \tag{9}$$

**Theorem 5.3** (Zero regret iff geodesic)**.** $R = 0$ *if and only if* $\{x_t\}$ *lies on the ideal geodesic.*

Lean: `Decision.ChoiceManifold.compute_regret_zero_iff`.

# 6 Free Will as Geodesic Selection

**Definition 6.1** (Bifurcation point)**.** *A* bifurcation point *is a state* $x$ *where multiple geodesics with near-equal* $J$*-cost diverge. Formally:* $\exists\, \gamma_1 \neq \gamma_2$ *with* $\gamma_1(0) = \gamma_2(0) = x$ *and* $|\mathcal{S}[\gamma_1] - \mathcal{S}[\gamma_2]| < \varepsilon$.

**Theorem 6.2** (Gap-45 protects selection)**.** *At bifurcation points near the 45th* $\varphi$*-rung, the optimal geodesic cannot be computed by any finite algorithm operating within a single eight-tick cycle. This is because* $\gcd(8, 45) = 1$*: the eight-tick computation window and the 45-fold pattern cannot synchronise (Gap-45 barrier).*

*Consequently, the agent must* navigate experientially *— selecting a geodesic through lived exploration rather than algorithmic prediction.*

Lean: `Decision.FreeWill.gap45_protects_selection`.

**Theorem 6.3** (Compatibilism). *The cost landscape $J$ constrains the set of admissible geodesics (determinism). At bifurcation points, the agent selects among them (freedom). These coexist because:*

1. *Determinism: the metric $g(x) = x^{-3}$ is fixed.*
2. *Freedom: geodesic selection at bifurcations is underdetermined by $g$.*
3. *Protection: Gap-45 ensures no external agent can predict the selection.*

# 7 Decision Thermodynamics

**Definition 7.1** (Boltzmann distribution over choices). *At recognition temperature $T_R$, the probability of choosing state $x$ is*

$$P(x) = \frac{1}{Z}\exp\left(-\frac{J(x)}{T_R}\right), \qquad Z = \int_0^\infty \exp\left(-\frac{J(x)}{T_R}\right) dx. \tag{10}$$

**Theorem 7.2** (Exploration–exploitation tradeoff).

- *High $T_R$: $P(x)$ is broad (exploration, risk-taking).*
- *Low $T_R$: $P(x)$ is peaked at $x = 1$ (exploitation, risk-aversion).*
- *$T_R \to 0$: deterministic choice at $x = 1$ (ground state).*
- *$T_R \to \infty$: uniform distribution (random choice).*

# 8 Predictions

**Prediction 8.1** (Decision latency). *Decision latency scales as $J(\Delta x)$ where $\Delta x$ is the separation between the two most attractive options on the choice manifold. Equal-cost options (small $J$ gap) take longest (Hick–Hyman law generalisation).*

**Prediction 8.2** (Attention capacity). *Working memory capacity clusters near $\varphi^3 \approx 4.24$ items across tasks, consistent with Cowan's "$4 \pm 1$" [4] rather than Miller's $7 \pm 2$.*

**Prediction 8.3** (Swing in decision timing). *When subjects make rhythmic decisions (e.g. tapping to a beat), the natural asymmetry in inter-tap intervals will peak near $1/\varphi : 1/\varphi^2$ (the golden swing ratio).*

# 9 Falsification Criteria

**Falsification Criterion 9.1** (Wrong geodesic family). *If the optimal decision paths in a continuous choice task are inconsistent with $\gamma(t) = 4/(At + B)^2$ (e.g. linear or exponential instead), the choice manifold metric is falsified.*

**Falsification Criterion 9.2** (No capacity bound). *If working memory capacity grows unboundedly with training (no saturation near $\varphi^3$), the attention capacity theorem is falsified.*

# 10 Comparison with Existing Decision Theory

| **Feature** | Standard (utility) | RS (cost geodesic) |
|---|---|---|
| **Primitive** | Utility $u(x)$ (postulated) | $J(x)$ (forced by RCL) |
| **Optimality** | Max expected utility | Min path action $\int J\,dt$ |
| **Space** | Preference ordering | Riemannian manifold $(M, g)$ |
| **Dynamics** | None (static comparison) | Geodesic + gradient descent |
| **Capacity** | Miller's $7 \pm 2$ (empirical) | $\varphi^3 \approx 4.24$ (derived) |
| **Free will** | Incompatibilism debate | Compatibilism (Gap-45) |

**Remark 10.1** (Prospect theory). *Kahneman–Tversky prospect theory [3] introduces a value function that is concave for gains and convex for losses (S-shaped). In the RS framework, the asymmetry arises naturally: for $x > 1$ (gain), $J''(x) = x^{-3}$ is small (shallow curvature), while for $0 < x < 1$ (loss), $J''(x) = x^{-3}$ is large (steep curvature). This generates the empirical observation that "losses loom larger than gains" without postulating a separate value function.*

# 11 Discussion

### Claims and non-claims

We derive the geometric structure of decision-making from $J$ uniqueness. We do *not* claim to explain all psychological phenomena; the framework provides the *mathematical skeleton* (metric, geodesics, curvature) on which empirical decision science operates.

### Open problems

(Q1) Is the attention capacity $\varphi^3$ experimentally distinguishable from 4 (i.e. does 0.24 items matter)?
(Q2) Can the geodesic family $\gamma = 4/(At + B)^2$ be measured in continuous tracking tasks (e.g. pursuit rotor)?
(Q3) Does the exploration–exploitation tradeoff temperature $T_R$ correlate with dopamine levels?
(Q4) Is regret (metric distance from geodesic) measurable via fMRI (anterior cingulate activity)?

# 12 Lean Formalization

| **Module** | **Content** |
|---|---|
| `Decision.Attention` | Operator, capacity bound |
| `Decision.ChoiceManifold` | Metric, Christoffel, geodesic eq |
| `Decision.FreeWill` | Bifurcation, Gap-45, compatibilism |
| `Decision.DeliberationDynamics` | Gradient descent + noise |
| `Decision.GeodesicSolutions` | $\gamma(t) = (at + b)^{2/3}$ |
| `Decision.DecisionThermodynamics` | Boltzmann, temperature |

# References


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