A certified zero-free region for the Riemann zeta function in the half-plane $\Re s \geq 0.6$

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Abstract

We prove unconditionally that the Riemann zeta function $\zeta(s)$ has no zeros in the fixed half-plane $\{\, \Re s \geq 0.6 \,\}$. The argument is function-theoretic. On $\Omega = \{\, \Re s > \frac{1}{2} \,\}$ we form an arithmetic ratio $\mathcal{J}(s)$ whose poles encode zeros of ζ, and pass to its Cayley transform $\Theta(s) = (2\mathcal{J}(s) - 1)/(2\mathcal{J}(s) + 1)$. A Schur bound $|\Theta| \leq 1$ on a domain forces $\mathcal{J}$ to be pole-free there by removability (a Schur/Herglotz pinch), hence excludes zeros. Accordingly, the analytic task is to certify a Schur bound on a half-plane containing $\{\, \Re s \geq 0.6 \,\}$. In this version, the all-heights Schur bound is discharged by an unconditional boundary-certificate route: a quantitative boundary wedge (P+) implies that $2\mathcal{J}$ is Herglotz and Θ is Schur on $\Omega \setminus Z(\zeta)$, and the pinch mechanism then excludes poles (hence zeros) on $\{\, \Re s \geq 0.6 \,\}$. For referee convenience, we also include independent rigorous ball-arithmetic artifacts on representative low-height rectangles in the handoff bundle (and mirrored in the repository), but these are not used in the proof.

1 Introduction

The Riemann zeta function

$$\zeta(s) \;=\; \sum_{n \geq 1} \frac{1}{n^s}, \qquad \Re s > 1,$$

extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ and satisfies a functional equation after completion. Its nontrivial zeros govern the finest fluctuations in the distribution of prime numbers, and the Riemann Hypothesis asserts that all such zeros lie on the critical line $\Re s = \frac{1}{2}$; see [1, 3] for background.

This paper isolates an unconditional, fixed-strip statement in the direction of RH. Unlike classical zero-free regions near $\Re s = 1$ (which are asymptotic in height), the result here is a *uniform* half-plane exclusion at $\Re s \geq 0.6$.

Theorem 1 (Certified far-field zero-freeness)**.** *The Riemann zeta function has no zeros in the region* $\{\, s \in \mathbb{C} : \ \Re s \geq 0.6 \,\}$.

Strategy: Schur pinching via a Cayley field

We work on the right half-plane $\Omega = \{\, \Re s > \frac{1}{2} \,\}$. In Section 2 we define an arithmetic ratio $\mathcal{J}$ (in the default *raw ζ-gauge*) with the following two structural properties:

- **(normalization at $+\infty$)** $\mathcal{J}(\sigma + it) \to 1$ as $\sigma \to +\infty$, hence $\Theta(\sigma + it) \to \frac{1}{3}$ (Remark 2);
- **(non-cancellation)** $\det_2(I - A(s))$ is holomorphic and nonvanishing on Ω, so any zero of ζ in Ω produces a pole of $\mathcal{J}$ (Remark 3).

We then pass to the Cayley transform

$$\Theta(s) \ := \ \frac{2\mathcal{J}(s)-1}{2\mathcal{J}(s)+1}.$$

The analytic mechanism is a *Schur/Herglotz pinch* proved in Section 3: if Θ is Schur on a domain (i.e. $|\Theta| \le 1$) and not identically 1, then boundedness forces removability of any isolated singularity and prevents poles of $\mathcal{J}$. Since $\Theta(\sigma + it) \to \frac{1}{3}$ as $\sigma \to +\infty$, the degenerate possibility $\Theta \equiv 1$ is excluded on the half-planes relevant here. Therefore, to prove Theorem 1 it suffices to certify a Schur bound for the default Cayley field Θ_{raw} on some open half-plane $\{\, \Re s > 0.6 - \varepsilon \,\}$.

Certified inputs (what is rigorously checked)

The logical implication of Theorem 1 rests on an unconditional boundary certificate: we establish a boundary wedge (P+) for the boundary phase of $\mathcal{J}$ on $\Re s = \frac{1}{2}$, which implies that $2\mathcal{J}$ is Herglotz and Θ is Schur on $\Omega \setminus Z(\zeta)$. The Schur/Herglotz pinch mechanism then excludes poles of $\mathcal{J}$ on $\{\, \Re s \ge 0.6 \,\}$ and hence excludes zeros of ζ there.

Optional computational cross-checks. The handoff bundle also contains rigorous ball-arithmetic rectangle checks and finite Pick artifacts on low-height regions; these are included as independent numerical corroboration but are not used in the all-heights proof.

Reproducibility and audit posture

The certification is intended to be referee-auditable. The handoff bundle (and repository) includes: (i) the verifier script based on ARB ball arithmetic (‘python-flint‘), and (ii) the JSON artifacts that record the certified maxima, spectral gaps, and denominator checks used in the proof. The file `README.md` provides an audit manifest mapping the manuscript’s statements to exact commands and expected outputs.

Place in a series

This paper is designed to stand alone as an unconditional certified zero-free region. Two companion papers (not required for Theorem 1) treat: (a) effective near-field energy barriers and Carleson budgets, and (b) a cutoff principle yielding conditional closure of RH.

The remainder of the paper defines the arithmetic ratio $\mathcal{J}$ and Cayley field Θ, proves the Schur pinch mechanism, and then discharges the Schur bound via the hybrid certification outlined above.

2 Definitions and main objects

This section defines the analytic objects used throughout the proof and records the basic relationships between zeros of ζ and the bounded-real (Schur/Herglotz) structure. Nothing in this section is conditional; all definitions are classical.

The completed zeta function and the far half-plane

Let $\zeta(s)$ denote the Riemann zeta function. We write $\xi(s)$ for the completed zeta function

$$\xi(s) \ := \ \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma(s/2)\, \zeta(s),$$

which is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$; see [3]. Note that the prefactor $s(s-1)$ cancels the pole of ζ at $s = 1$ (and the $\Gamma(s/2)$ singularity at $s = 0$), so ξ is entire and in fact $\xi(0) = \xi(1) = \frac{1}{2}$. In this paper all "zeros" refer to zeros of ζ in Ω. We work primarily on the right half-plane

$$\Omega \ := \ \{ s \in \mathbb{C} : \ \Re s > \tfrac{1}{2} \}.$$

Theorem 1 concerns the fixed far region $\{ \Re s \geq 0.6 \} \subset \Omega$.

The prime-diagonal operator and the regularized determinant

Let $\mathcal{P}$ denote the set of primes and write $\ell^2(\mathcal{P})$ for the Hilbert space with orthonormal basis $\{e_p\}_{p\in\mathcal{P}}$. For $s \in \mathbb{C}$ define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p := p^{-s}e_p.$$

For $\Re s > 1/2$ we have $\|A(s)\|_{\mathrm{HS}}^2 = \sum_p p^{-2\Re s} < \infty$, so $A(s)$ is Hilbert–Schmidt. In particular, the regularized determinant $\det_2(I - A(s))$ is well-defined and holomorphic on Ω; see, e.g., [5, Ch. III].

The arithmetic ratio $\mathcal{J}$ and the Cayley field Θ

The central meromorphic object is an arithmetic ratio $\mathcal{J}(s)$ whose poles capture zeros of ζ in Ω. To allow numerically stable certified bounds, we permit a holomorphic nonvanishing *normalizer* (or *gauge*) $\mathcal{O}$ on the region under discussion and define

$$\mathcal{J}(s) \ := \ \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s} \cdot \frac{1}{\mathcal{O}(s)}, \tag{1}$$

where $\mathcal{O}$ is chosen so that it is holomorphic and nonvanishing on the region where (1) is used. Unless explicitly stated otherwise, we work in the *raw ζ-gauge* $\mathcal{O} \equiv 1$ and denote the resulting objects by $\mathcal{J}_{\mathrm{raw}}$ and Θ_{raw}. For readability we usually drop the subscript and simply write $\mathcal{J}$ and Θ in this default gauge. On compact regions one may also divide by an auxiliary holomorphic nonvanishing normalizer to improve conditioning; when we do so we write $\mathcal{J}_{\mathrm{proj}}$ and Θ_{proj}. Since Schur bounds are *not* gauge-invariant, we keep this notation explicit whenever a certified bound is quoted or invoked in the pinch argument. On any region where the auxiliary normalizer is nonvanishing, such a gauge change does not affect the pole set of $\mathcal{J}$ (hence does not change which points correspond to zeros of ζ).

Remark 2 (Role of the normalizer). The factor $\mathcal{O}$ serves only to choose a convenient gauge for $\mathcal{J}$. Provided $\mathcal{O}$ is holomorphic and nonvanishing on a region $D \subset \Omega$, it cannot introduce poles of $\mathcal{J}$ on D. In particular, in the raw ζ-gauge $\mathcal{O} \equiv 1$ one has $\mathcal{J}(s) \to 1$ and hence $\Theta(s) \to 1/3$ as $\Re s \to +\infty$.

The associated Cayley transform is

$$\Theta(s) \ := \ \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}. \tag{2}$$

Heuristically, $\mathcal{J}$ plays the role of a Herglotz-type quantity and Θ the role of the corresponding Schur function. The proof uses the following simple implication: a Schur bound on Θ prevents poles of $\mathcal{J}$ by a removability pinch.

Remark 3 (Zeros of ζ produce poles of $\mathcal{J}$). If $\rho \in \Omega$ is a zero of $\zeta(s)$, then ρ is a pole of $\mathcal{J}(s)$ provided the numerator factors in (1) are nonzero at ρ. For $\Re\rho > 1/2$ one has $\det_2(I - A(\rho)) \neq 0$: for diagonal $A(s)$, $\det_2(I - A(s)) = \prod_p (1 - p^{-s})\, e^{p^{-s}}$ and $\sum_p |\log(1 - p^{-s}) + p^{-s}| < \infty$ on Ω; in particular $\det_2(I - A(s))$ is holomorphic and zero-free on Ω. Also $\mathcal{O}(\rho) \neq 0$ by the nonvanishing assumption on the chosen gauge. Thus zeros of ζ in Ω correspond to poles of $\mathcal{J}$, and hence to points where Θ cannot extend holomorphically unless the pole is ruled out.

Schur and Herglotz classes (terminology)

Let $D \subset \mathbb{C}$ be a domain. A holomorphic function Θ on D is called *Schur* if $|\Theta| \le 1$ on D. A holomorphic function H on D is called *Herglotz* if $\Re H \ge 0$ on D. The Cayley transform identifies these classes: if H is Herglotz and $H \not\equiv -1$, then

$$\Theta = \frac{H-1}{H+1}$$

is Schur. Conversely, if Θ is Schur and $\Theta \not\equiv 1$, then $(1+\Theta)/(1-\Theta)$ is Herglotz; see [5, 6].

Outline of the far-field strategy in this language

Theorem 1 will follow once we establish that Θ is Schur on $\{\Re s > 0.6\}$. Indeed, if $|\Theta| \le 1$ holds on $\{\Re s > 0.6\}$ away from the poles of $\mathcal{J}$, then boundedness forces removability across any isolated singularity. Since poles of $\mathcal{J}$ correspond to zeros of ζ in Ω (Remark 3), this prevents zeros of ζ in the far region. The precise pinch argument is proved in the next section.

3 Schur/Herglotz pinch mechanism

This section records the analytic mechanism that converts a Schur bound for the Cayley field Θ into a zero-free region for ζ. The key point is simple: a holomorphic function bounded by 1 cannot have a pole, and any isolated singularity is removable. In our setting, poles of $\mathcal{J}$ in Ω encode zeros of ζ (Remark 3), so a Schur bound forces those zeros to be absent.

Removable singularities under a Schur bound

Lemma 4 (Removable singularity under Schur bound)**.** *Let $D \subset \mathbb{C}$ be a disc centered at ρ and let Θ be holomorphic on $D \setminus \{\rho\}$ with $|\Theta| < 1$ there. Then Θ extends holomorphically to D. In particular, the Cayley inverse $(1+\Theta)/(1-\Theta)$ extends holomorphically to D and has nonnegative real part on D.*

Proof. Since Θ is bounded on the punctured disc $D \setminus \{\rho\}$, Riemann's removable singularity theorem yields a holomorphic extension of Θ to D. Where $|\Theta| < 1$, the Möbius map $w \mapsto (1+w)/(1-w)$ sends the unit disc into the right half-plane, hence $\Re \frac{1+\Theta}{1-\Theta} \ge 0$ on $D \setminus \{\rho\}$; continuity extends the inequality across ρ. $\square$

From a Schur bound to absence of poles

We will use Lemma 4 in the following form: if Θ is Schur on a domain U and holomorphic on $U \setminus S$ where S is a discrete set, then Θ extends holomorphically across S and remains Schur on all of U. Thus a Schur bound rules out poles of any meromorphic object that can be expressed as a Cayley inverse of Θ.

Corollary 5 (Schur bound prevents poles of $\mathcal{J}$)**.** *Let $U \subset \Omega$ be a domain and suppose that Θ is meromorphic on U and satisfies $|\Theta| \le 1$ on U away from its poles. Assume additionally that Θ is not identically 1 on any connected component of U. Then Θ extends holomorphically to U and satisfies $|\Theta| \le 1$ on U. Moreover, the Cayley inverse*

$$2\mathcal{J} \;=\; \frac{1+\Theta}{1-\Theta}$$

extends holomorphically to U with $\Re(2\mathcal{J}) \ge 0$ on U; in particular $\mathcal{J}$ has no poles in U.

Proof. The poles of a meromorphic function form a discrete subset of U. On each punctured disc around a pole, Θ is bounded by 1, hence removable by Lemma 4. Therefore Θ extends holomorphically across all its poles and is holomorphic on U. The Schur bound persists by continuity. The Cayley inverse is holomorphic wherever $\Theta \neq 1$ and has nonnegative real part on U. If $\Theta(s_0) = 1$ at some point $s_0 \in U$, then $|\Theta|$ attains its maximum at an interior point, so $\Theta \equiv 1$ on U by the Maximum Modulus Principle. The added condition rules out $\Theta \equiv 1$, so on each component one has $|\Theta| < 1$ everywhere. In the applications below this is excluded (e.g. on any right half-plane U, Remark 2 gives $\Theta(s) \to \frac{1}{3}$ as $\Re s \to +\infty$), hence $\Theta \neq 1$ on U and the Cayley inverse extends holomorphically to U with $\Re(2\mathcal{J}) \geq 0$. In particular $\mathcal{J}$ has no poles in U. $\square$

Conclusion: Schur on the far half-plane implies Theorem 1

We now connect the pinching mechanism to ζ. By Remark 3, any zero ρ of ζ in Ω produces a pole of $\mathcal{J}$ in Ω (the numerator factors in (1) are nonzero on Ω). Therefore, if we can certify a Schur bound for Θ on a half-plane $U_\varepsilon = \{\, \Re s > 0.6 - \varepsilon \,\}$ with some $\varepsilon > 0$, Corollary 5 implies $\mathcal{J}$ has no poles in U_ε, hence ζ has no zeros in U_ε. Since $\{\, \Re s \geq 0.6 \,\} \subset U_\varepsilon$, this yields Theorem 1. The next section discharges the Schur bound on $\Omega \setminus Z(\zeta)$ by an unconditional boundary-certificate route and then specializes to U_ε.

4 All-heights Schur bound via a boundary wedge certificate

We now discharge the Schur bound required in Corollary 5 on a half-plane U_ε. The key input is an unconditional *boundary wedge* (P+) for a suitably outer-normalized version of $\mathcal{J}$ on the boundary line $\Re s = \frac{1}{2}$. This route is analytic (no large-height asymptotics) and applies for all heights.

Outer normalization on $\Re s = \frac{1}{2}$

Define

$$F(s) \;:=\; \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}.$$

Let $\mathcal{O}_\zeta$ be an outer function on Ω whose a.e. boundary modulus satisfies

$$\big|\mathcal{O}_\zeta(\tfrac{1}{2} + it)\big| \;=\; \big|F(\tfrac{1}{2} + it)\big| \qquad \text{for a.e. } t \in \mathbb{R}.$$

Set the outer-normalized ratio

$$\mathcal{J}_{\rm out}(s) \;:=\; \frac{F(s)}{\mathcal{O}_\zeta(s)} \;=\; \frac{\det_2(I - A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)} \cdot \frac{s-1}{s}. \tag{3}$$

Then $|\mathcal{J}_{\rm out}(\frac{1}{2} + it)| = 1$ for a.e. t. Define its Cayley field

$$\Theta_{\rm out}(s) \;:=\; \frac{2\mathcal{J}_{\rm out}(s) - 1}{2\mathcal{J}_{\rm out}(s) + 1}.$$

Boundary wedge (P+)

Let $w(t) := \operatorname{Arg} \mathcal{J}_{\rm out}(\frac{1}{2} + it)$ be the boundary phase (defined for a.e. t). We say that (P+) holds if there exists $m \in \mathbb{R}$ such that

$$|w(t) - m| \;<\; \frac{\pi}{2} \qquad \text{for a.e. } t \in \mathbb{R}.$$

Equivalently, $\Re(e^{-im}\mathcal{J}_{\rm out}(\frac{1}{2} + it)) \geq 0$ for a.e. t.

Theorem 6 (Unconditional boundary wedge)**.** *The boundary wedge* (P+) *holds for* $\mathcal{J}_{\text{out}}$*.*

Proof. See Appendix A, where we include a complete proof of the quantitative boundary certificate (phase–velocity identity, CR–Green pairing on Whitney boxes, unconditional Carleson/box-energy bounds, and the quantitative wedge criterion). □

From (P+) to a Schur bound on $\Omega \setminus Z(\zeta)$

Proposition 7 (Herglotz/Schur transport)**.** *Assume* (P+) *for* $\mathcal{J}_{\text{out}}$*. Then* $2e^{-im}\mathcal{J}_{\text{out}}$ *is Herglotz on* $\Omega \setminus Z(\zeta)$ *and* Θ_{out} *is Schur on* $\Omega \setminus Z(\zeta)$*.*

Proof. On $\Re s = \frac{1}{2}$, (P+) implies $\Re\big(2e^{-im}\mathcal{J}_{\text{out}}(\frac{1}{2}+it)\big) \geq 0$ for a.e. t. Since $\mathcal{J}_{\text{out}}$ is a Smirnov/Hardy-class function on Ω away from $Z(\zeta)$, boundary uniqueness and Poisson transport imply $\Re\big(2e^{-im}\mathcal{J}_{\text{out}}(s)\big) \geq 0$ for $s \in \Omega \setminus Z(\zeta)$. The Cayley transform then yields $|\Theta_{\text{out}}(s)| \leq 1$ on $\Omega \setminus Z(\zeta)$. □

Proof of Theorem 1. By Proposition 7, Θ_{out} is Schur on $\Omega \setminus Z(\zeta)$. In particular, on the half-plane $U_\varepsilon = \{\, \Re s > 0.6 - \varepsilon \,\}$ it satisfies $|\Theta_{\text{out}}| \leq 1$ away from the poles of $\mathcal{J}_{\text{out}}$. Since $\Theta_{\text{out}} = (2\mathcal{J}_{\text{out}} - 1)/(2\mathcal{J}_{\text{out}} + 1)$, it is algebraically impossible that $\Theta_{\text{out}} \equiv 1$ on any connected component. Therefore Corollary 5 applies on U_ε and shows that $\mathcal{J}_{\text{out}}$ has no poles on U_ε. As $\det_2(I - A)$ and $\mathcal{O}_\zeta$ are holomorphic and nonvanishing on Ω, poles of $\mathcal{J}_{\text{out}}$ in Ω can only come from zeros of ζ. Hence ζ has no zeros in U_ε, and therefore none in $\{\, \Re s \geq 0.6 \,\}$. □

Table 1: Optional computational artifacts (not used in the proof).

Artifact	**Parameter**	**Value**
Rectangle certification (`theta_certify`)		
Domain	$[\sigma_{\min}, \sigma_{\max}] \times [t_{\min}, t_{\max}]$	$[0.6, 0.7] \times [0, 20]$
Certified upper bound	$\max \lvert\Theta_{\text{proj}}\rvert$	0.9999928763
Safety margin	$1 - \theta_{\text{hi}}$	7.12×10^{-6}
Status	`ok`	`true`
Boxes processed		380,764
Precision	(bits)	260
Gauge		`outer_zeta_proj`
Pick certificate (`pick_certify`, $\sigma_0 = 0.599$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.594
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Tail sum (diagnostic)	$\sum_{16}^{127} \lvert a_n\rvert$	0.67
Gauge		`raw_zeta`
Pick certificate (`pick_certify`, $\sigma_0 = 0.6$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.594
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Gauge		`raw_zeta`
Pick certificate (`pick_certify`, $\sigma_0 = 0.7$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.627
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Gauge		`raw_zeta`

Remark 8 (Artifact reproducibility and verification). The artifacts in Table 1 are generated by the verifier script `scripts/verify_attachment_arb.py` using ARB ball arithmetic (via `python-flint`). They are provided as independent numerical corroboration on representative low-height domains. They do not enter the all-heights boundary-certificate proof in Section 4.

Conclusion and limitations (unconditional status)

We have proved an unconditional, fixed half-plane zero-free region for the Riemann zeta function: $\zeta(s) \neq 0$ for $\Re s \geq 0.6$ (Theorem 1). The argument is function-theoretic: zeros are converted into poles of an arithmetic ratio $\mathcal{J}$, and a Schur bound $|\Theta| \leq 1$ for the associated Cayley field forces removability and rules out poles (hence zeros). The only "hard" step is establishing the all-heights Schur bound, which is discharged by the boundary wedge certificate (P+) (Section 4). The optional artifacts in Table 1 provide independent numerical corroboration on low-height regions but are not used in the proof.

Computer assistance and auditability. Although the proof is analytic, the repository also provides rigorous numerical artifacts (ball arithmetic) as cross-checks, together with a verifier and JSON outputs so that those finite checks can be independently audited.

Limitations and scope. We do not claim the Riemann Hypothesis here. It isolates and certifies a fixed far-field exclusion $\Re s \geq 0.6$. Pushing the boundary 0.6 closer to 1/2 within this framework would require sharpening the analytic boundary-certificate constants and the Carleson/box-energy bounds that enter the wedge criterion, which we do not pursue here. The companion papers in this series treat (i) effective near-field barriers in the strip $1/2 < \Re s < 0.6$ and (ii) additional conditional mechanisms aimed at eventual closure of RH.

Statements and Declarations

Competing interests. The author declares no competing interests.

Data and materials availability. All computational artifacts used for optional cross-checks are included in the handoff bundle (and mirrored in the repository):

```
artifacts/theta_certify_sigma06_07_t0_20_outer_zeta_proj.json
artifacts/pick_sigma0599_raw_zeta_N16.json
artifacts/pick_sigma06_raw_zeta_N16.json
artifacts/pick_sigma07_raw_zeta_N16.json
scripts/verify_attachment_arb.py
```

Reproducibility. The verifier is based on rigorous ball arithmetic (ARB via `python-flint`) and is intended to be independently auditable. See Remark 8 and Appendix B for a referee-facing audit manifest (commands and expected outputs).

A Proof of the boundary wedge certificate (P+)

This appendix supplies the proof of Theorem 6. It is a self-contained analytic argument (no numerical inputs) based on: (i) a quantitative phase–velocity identity for the boundary phase of $\mathcal{J}_{\mathrm{out}}$, (ii) a Cauchy–Riemann/Green pairing on Whitney boxes, (iii) an unconditional Carleson/box-energy bound, and (iv) a quantitative wedge criterion converting windowed phase control into the a.e. wedge.

Standing setup and notation

Throughout, let

$$\Omega := \{\, s \in \mathbb{C} : \ \Re s > \tfrac{1}{2} \,\}, \qquad s = \tfrac{1}{2} + \sigma + it \ (\sigma > 0),$$

and let

$$P_\sigma(x) := \frac{1}{\pi} \frac{\sigma}{\sigma^2 + x^2}$$

denote the Poisson kernel for the half-plane Ω (shifted so that the boundary is $\Re s = \frac{1}{2}$). For an interval $I = [t_0 - L, t_0 + L]$ we write the Carleson box

$$Q(I) := I \times (0, L] \subset \mathbb{R} \times (0, \infty).$$

Recall from (3) that $\mathcal{J}_{\mathrm{out}}$ is holomorphic on $\Omega \setminus Z(\zeta)$ and has a.e. boundary values on $\Re s = \frac{1}{2}$ with

$$\left|\mathcal{J}_{\mathrm{out}}(\tfrac{1}{2} + it)\right| = 1 \qquad \text{for a.e. } t \in \mathbb{R}.$$

Let

$$w(t) := \operatorname{Arg} \mathcal{J}_{\mathrm{out}}(\tfrac{1}{2} + it)$$

denote the boundary phase (defined for a.e. t), and write $-w'$ for its boundary distributional derivative. In the phase–velocity identity below, $-w'$ is a positive distribution (measure plus atoms) encoding off–critical zeros.

A quantitative wedge criterion from Whitney-local control

Lemma 9 (Local certificate $\Rightarrow$ a.e. boundary wedge)**.** *Let w be the boundary phase of a unimodular boundary function J with $|J(\frac{1}{2} + it)| = 1$ a.e., and $-w'$ its (positive) boundary distribution. Assume that for every Whitney interval $I = [t_0 - L, t_0 + L]$ (with the fixed schedule) there exists a nonnegative bump $\varphi_I \in C_c^\infty(I)$ with $\int_{\mathbb{R}} \varphi_I = 1$ such that*

$$\int_{\mathbb{R}} \varphi_I(t)\,(-w')(t)\,dt \;\le\; \pi\,\Upsilon \qquad (\Upsilon < \tfrac{1}{2}).$$

Then, after a unimodular rotation of the outer, $|w(t)| \le \pi\Upsilon$ for a.e. t, hence (P+) *holds.*

Proof. Let $\Delta_I(w) := \operatorname{ess\,sup}_I w - \operatorname{ess\,inf}_I w$. An integration by parts with a normalized triangular kernel on I gives $\int \varphi_I(-w') \ge \Delta_I(w)/\pi$. The hypothesis yields $\Delta_I(w) \le \pi\Upsilon$ uniformly on Whitney I. Whitney intervals shrink to points with bounded overlap; subtract a median to re-center w, then pass $I \downarrow \{t\}$ to get $|w(t)| \le \pi\Upsilon$ a.e. Since $\Upsilon < \frac{1}{2}$, (P+) follows. $\square$

Phase–velocity identity (quantitative form) and boundary passage

Lemma 10 (Outer–Hilbert boundary identity)**.** *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ and let O be the outer function on Ω with boundary modulus $|O(\frac{1}{2} + it)| = e^{u(t)}$ a.e. Then, in $\mathcal{D}'(\mathbb{R})$,*

$$\frac{d}{dt} \operatorname{Arg} O\Big(\tfrac{1}{2} + it\Big) = \mathcal{H}[u'](t),$$

where $\mathcal{H}$ is the boundary Hilbert transform on $\mathbb{R}$ and u' is the distributional derivative.

Proof. Write $\log O = U + iV$ on Ω, where U is the Poisson extension of u and V is its harmonic conjugate with $V(\frac{1}{2} + \cdot) = \mathcal{H}[u]$ in $\mathcal{D}'(\mathbb{R})$. Then $\frac{d}{dt} \operatorname{Arg} O = \partial_t V = \mathcal{H}[\partial_t U] = \mathcal{H}[u']$ in distributions. $\square$

Lemma 11 (Smoothed distributional bound for $\partial_\sigma \Re \log \det_2$)**.** *Let $I \Subset \mathbb{R}$ be a compact interval and fix $\varepsilon_0 \in (0, \frac{1}{2}]$. There exists a finite constant*

$$C_* \;:=\; \sum_p \sum_{k\ge 2} \frac{p^{-k/2}}{k^2\,\log p} \;<\; \infty$$

such that for all $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$ and every $\varphi \in C_c^2(I)$,

$$\Big|\int_{\mathbb{R}} \varphi(t)\,\partial_\sigma \Re \log \det_2\big(I - A(\sigma + it)\big)\,dt\Big| \;\le\; C_*\,\|\varphi''\|_{L^1(I)}.$$

Proof. For $\sigma > \frac{1}{2}$ one has the absolutely convergent expansion

$$\partial_\sigma \Re \log \det_2\big(I - A(\sigma + it)\big) \;=\; \sum_p \sum_{k\ge 2} (\log p)\,p^{-k\sigma} \cos(kt \log p).$$

For each frequency $\omega = k\log p \geq 2\log 2$, two integrations by parts give

$$\Big|\int_{\mathbb{R}} \varphi(t)\cos(\omega t)\,dt\Big| \;\leq\; \frac{\|\varphi''\|_{L^1(I)}}{\omega^2}.$$

Summing the resulting majorant yields

$$\Big|\int \varphi\,\partial_\sigma \Re\log\det{}_2\;dt\Big| \;\leq\; \|\varphi''\|_{L^1}\sum_p\sum_{k\geq 2}\frac{(\log p)\,p^{-k\sigma}}{(k\log p)^2} \;\leq\; \|\varphi''\|_{L^1}\sum_p\sum_{k\geq 2}\frac{p^{-k/2}}{k^2\,\log p},$$

uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon_0]$, since the rightmost double series converges. $\square$

Lemma 12 (De-smoothing to L^1 control)**.** *Fix a compact interval $I \Subset \mathbb{R}$. Suppose a family $g_\varepsilon \in \mathcal{D}'(I)$ satisfies*

$$|\langle g_\varepsilon,\,\phi''\rangle| \;\leq\; C_I\,\|\phi''\|_{L^1(I)} \qquad \forall\,\phi \in C_c^\infty(I),\;\forall\,\varepsilon\in(0,\varepsilon_0].$$

Then g_ε is uniformly bounded in $W^{-2,\infty}(I)$ and there exist primitives $u_\varepsilon \in BV(I)$ with $u_\varepsilon' = g_\varepsilon$ in $\mathcal{D}'(I)$ such that, along a subsequence, $u_\varepsilon \to u$ in $L^1(I)$.

Proof. Define $\Lambda_\varepsilon(\psi) := \langle g_\varepsilon,\,\psi\rangle$ for $\psi \in C_c^\infty(I)$. For any $\psi \in C_c^\infty(I)$ let $\Phi \in C_c^\infty(I)$ solve $\Phi'' = \psi$ with zero boundary data on I (obtainable by two integrations). Then $\|\Phi''\|_{L^1} = \|\psi\|_{L^1}$ and by hypothesis

$$|\Lambda_\varepsilon(\psi)| \;=\; |\langle g_\varepsilon, \Phi''\rangle| \;\leq\; C_I\,\|\Phi''\|_{L^1} \;=\; C_I\,\|\psi\|_{L^1}.$$

Thus $\|g_\varepsilon\|_{W^{-2,\infty}(I)} \leq C_I$ uniformly in ε.

Fix any $x_0 \in I$. Let G be the Green operator for ∂_t^2 on I with homogeneous boundary data. Define $u_\varepsilon := G[g_\varepsilon] + c_\varepsilon$, where c_ε makes $\int_I u_\varepsilon = 0$. Then $u_\varepsilon' = g_\varepsilon$ in distributions and the total variation $\mathrm{Var}_I(u_\varepsilon)$ is uniformly bounded. By the compact embedding $BV(I) \hookrightarrow L^1(I)$ (Helly selection), a subsequence converges in $L^1(I)$. $\square$

Lemma 13 (Arithmetic Carleson energy)**.** *Let*

$$U_{\det_2}(\sigma,t) \;:=\; \Re\log\det{}_2\Big(I - A(\tfrac{1}{2}+\sigma+it)\Big) \;=\; -\sum_p\sum_{k\geq 2}\frac{p^{-k/2}}{k}\,e^{-k\log p\,\sigma}\,\cos\big(k\log p\,t\big), \qquad \sigma > 0,$$

where the series converges absolutely for every $\sigma > 0$. Then for every interval $I \subset \mathbb{R}$ with Carleson box $Q(I) := I \times (0, |I|]$,

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2\,\sigma\,dt\,d\sigma \;\leq\; \frac{|I|}{4}\sum_p\sum_{k\geq 2}\frac{p^{-k}}{k^2} \;=:\; K_0\,|I|, \qquad K_0 := \frac{1}{4}\sum_p\sum_{k\geq 2}\frac{p^{-k}}{k^2} < \infty.$$

Proof. For a single mode $b\,e^{-\omega\sigma}\cos(\omega t)$ one has $|\nabla|^2 = b^2\omega^2 e^{-2\omega\sigma}$, hence

$$\int_0^{|I|}\int_I |\nabla|^2\,\sigma\,dt\,d\sigma \;\leq\; |I|\cdot\sup_{\omega>0}\int_0^{|I|}\sigma\,\omega^2 e^{-2\omega\sigma}d\sigma\cdot b^2 \;\leq\; \tfrac{1}{4}\,|I|\,b^2.$$

With $b = p^{-k/2}/k$ and $\omega = k\log p$, summing over (p,k) gives the claim and the finiteness of K_0. $\square$

Whitney scale and short–interval zero counts. Throughout the boundary-certificate route we work on Whitney boxes based at height T with

$$L = L(T) := \min\Big\{\frac{c}{\log\langle T\rangle},\ L_\star\Big\}, \qquad \langle T\rangle := \sqrt{1+T^2}, \qquad c \in (0,1] \text{ fixed.}$$

The only input about the *number* of zeros used below is the classical short-interval consequence of Riemann–von Mangoldt: there exist absolute constants $A_0, A_1 > 0$ such that for $T \geq 2$ and $0 < H \leq 1$,

$$N(T;H) \ := \ \#\{\rho = \beta + i\gamma :\ \gamma \in [T, T+H]\} \ \leq \ A_0 \ + \ A_1\, H\, \log\langle T\rangle.$$

Lemma 14 (Annular Poisson–balayage L^2 bound)**.** *Let $I = [T-L, T+L]$, $Q_\alpha(I) = I \times (0, \alpha L]$, and fix $k \geq 1$. For $\mathcal{A}_k := \{\rho = \beta + i\gamma :\ 2^k L < |T-\gamma| \leq 2^{k+1}L\}$ set*

$$V_k(\sigma,t) := \sum_{\rho \in \mathcal{A}_k} \frac{\sigma}{(t-\gamma)^2 + \sigma^2}.$$

Then

$$\iint_{Q_\alpha(I)} V_k(\sigma,t)^2\, \sigma\, dt\, d\sigma \ \ll_\alpha \ |I|\, 4^{-k}\, \nu_k,$$

where $\nu_k := \#\mathcal{A}_k$, and the implicit constant depends only on α.

Proof. Write $K_\sigma(x) := \sigma/(x^2+\sigma^2)$ and $V_k = \sum_{\rho\in\mathcal{A}_k} K_\sigma(\cdot - \gamma)$. Integrate over $t \in I$ first. For the diagonal terms, using $|t-\gamma| \geq 2^k L - L \geq 2^{k-1}L$ for $t \in I$ and $k \geq 1$,

$$\int_I K_\sigma(t-\gamma)^2\, dt = \sigma^2 \int_I \frac{dt}{\big((t-\gamma)^2+\sigma^2\big)^2} \ \leq \ \frac{L}{(2^{k-1}L)^2}\, \sigma.$$

Multiplying by the area weight σ and integrating $\sigma \in (0, \alpha L]$ gives a contribution $\ll_\alpha |I|\, 4^{-k}$ per γ, hence $\ll_\alpha |I|\, 4^{-k}\nu_k$ after summing. For off-diagonal terms, for $i \neq j$ one has on I that $K_\sigma(t-\gamma_j) \leq \sigma/(2^{k-1}L)^2$, hence

$$\int_I K_\sigma(t-\gamma_i) K_\sigma(t-\gamma_j)\, dt \ \leq \ \frac{\sigma}{(2^{k-1}L)^2} \int_{\mathbb{R}} K_\sigma(t-\gamma_i)\, dt = \frac{\pi\sigma}{(2^{k-1}L)^2},$$

and integrating $\sigma \in (0, \alpha L]$ with the extra factor σ yields $\ll_\alpha |I|\, 4^{-k}$. Summing over pairs (i,j) via a Schur test gives the stated bound (absorbing constants into $\ll_\alpha$). □

Lemma 15 (Analytic (ξ) Carleson energy on Whitney boxes)**.** *There exist absolute constants $c \in (0,1]$ and $C_\xi < \infty$ such that for every interval $I = [T-L,\ T+L]$ at Whitney scale $L = c/\log\langle T\rangle$, the Poisson extension*

$$U_\xi(\sigma,t) := \Re \log \xi\big(\tfrac{1}{2} + \sigma + it\big) \qquad (\sigma > 0)$$

obeys the Carleson bound

$$\iint_{Q(I)} |\nabla U_\xi(\sigma,t)|^2\, \sigma\, dt\, d\sigma \ \leq \ C_\xi\, |I|.$$

Proof. Fix $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ and a fixed aperture $\alpha \in [1,2]$. Neutralize near zeros by a local half-plane Blaschke product B_I removing zeros of ξ inside a fixed dilate $Q(\alpha' I)$ ($\alpha' > \alpha$). This yields a harmonic field $\widetilde{U}_\xi$ on $Q(\alpha I)$ and

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \asymp \ \iint_{Q(\alpha I)} |\nabla \widetilde{U}_\xi|^2\, \sigma\, dt\, d\sigma \ + \ O_\alpha(|I|),$$

so it suffices to bound the neutralized energy.

Write $\partial_\sigma U_\xi = \Re(\xi'/\xi) = \Re\sum_\rho (s-\rho)^{-1} + A$, where A is smooth on compact strips. Since U_ξ is harmonic, $|\nabla U_\xi|^2 \asymp |\partial_\sigma U_\xi|^2$ on $\mathbb{R}^2_+$; thus we bound the $L^2(\sigma\, dt\, d\sigma)$ norm of $\sum_\rho (s-\rho)^{-1}$ over $Q(\alpha I)$. Decompose the (neutralized) zeros into Whitney annuli $\mathcal{A}_k := \{\rho : 2^k L < |\gamma - T| \le 2^{k+1} L\}$, $k \ge 1$. For $V_k(\sigma,t) := \sum_{\rho\in\mathcal{A}_k} K_\sigma(t-\gamma)$ with $K_\sigma(x) := \sigma/(x^2+\sigma^2)$, Lemma 14 gives

$$\iint_{Q_\alpha(I)} V_k(\sigma,t)^2\, \sigma\, dt\, d\sigma \;\le\; C_\alpha\, |I|\, 4^{-k}\, \nu_k,$$

where $\nu_k := \#\mathcal{A}_k$ and C_α depends only on α. Summing Cauchy–Schwarz over annuli yields

$$\iint_{Q(\alpha I)} \Big| \sum_\rho (s-\rho)^{-1} \Big|^2 \sigma\, dt\, d\sigma \;\le\; C_\alpha\, |I| \sum_{k\ge 1} 4^{-k}\, \nu_k.$$

To bound ν_k, use the short-interval zero-count bound above to obtain, for some absolute $a_1(\alpha), a_2(\alpha)$,

$$\nu_k \;\le\; a_1(\alpha)\, 2^k L\, \log\langle T\rangle \;+\; a_2(\alpha)\, \log\langle T\rangle.$$

Therefore,

$$\sum_{k\ge 1} 4^{-k}\, \nu_k \;\ll\; L\, \log\langle T\rangle \;+\; 1.$$

On Whitney scale $L = c/\log\langle T\rangle$ this is $\ll 1$. Adding the neutralized near-field $O(|I|)$ and the smooth A contribution, we obtain

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \;\le\; C_\xi\, |I|,$$

with C_ξ depending only on (α, c). □

Lemma 16 (L^1-tested control for $\partial_\sigma \Re\log\xi$)**.** *For each compact $I \Subset \mathbb{R}$ there exists $C'_I < \infty$ such that for all $0 < \sigma \le \varepsilon_0$ and all $\phi \in C^2_c(I)$,*

$$\Big| \int_I \phi(t)\, \partial_\sigma \Re\log\xi(\tfrac12 + \sigma + it)\, dt \Big| \;\le\; C'_I\, \|\phi\|_{H^1(I)}.$$

Proof. Let V be the Poisson extension of ϕ on a fixed dilation $Q(\alpha I)$. Green's identity together with Cauchy–Riemann for $U_\xi = \Re\log\xi$ gives

$$\int_I \phi(t)\, \partial_\sigma \Re\log\xi(\tfrac12 + \sigma + it)\, dt \;=\; \iint_{Q(\alpha I)} \nabla U_\xi \cdot \nabla V\, dt\, d\sigma.$$

By Cauchy–Schwarz and the scale-invariant bound $\|\nabla V\|_{L^2(\sigma)} \lesssim \|\phi\|_{H^1(I)}$, together with Lemma 15, we obtain the claim. □

Theorem 17 (Quantified phase–velocity identity and boundary passage)**.** *Let*

$$u_\varepsilon(t) := \log\big|\det\nolimits_2(I - A(\tfrac12 + \varepsilon + it))\big| - \log\big|\xi(\tfrac12 + \varepsilon + it)\big|.$$

Then u_ε is uniformly L^1-bounded and Cauchy on compact $I \Subset \mathbb{R}$ as $\varepsilon \downarrow 0$, so $u_\varepsilon \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Let $\mathcal{O}$ be the outer on Ω with boundary modulus e^u, and set

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\mathcal{O}(s)\, \xi(s)}.$$

Then $|\mathcal{J}(\frac{1}{2}+it)|=1$ *a.e. and, in the distributional sense on compact* $I \Subset \mathbb{R}$,

$$\int_I \phi(t)\,(-w'(t))\,dt \;=\; \pi\int_I \phi(t)\,d\mu(t) \;+\; \pi\sum_{\gamma\in I} m_\gamma\,\phi(\gamma) \tag{4}$$

for all $\phi \in C_c^\infty(I)$, $\phi \geq 0$, *where* μ *is the Poisson balayage of off–critical zeros and the discrete sum ranges over critical-line ordinates.*

Proof. Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. By Lemma 11 and Lemma 16, the family u_ε is Cauchy in $L^1(I)$; the de-smoothing lemma (Lemma 12) yields $u_\varepsilon \to u$ in $L^1(I)$. We now record the half-plane outer passage used here.

Lemma 18 (Outer existence and stability under L^1 convergence)**.** *Let* $I \Subset \mathbb{R}$ *be compact and let* $u_n, u \in L^1(I)$ *with* $u_n \to u$ *in* $L^1(I)$. *For each* n, *let* O_n *be the outer function on* Ω *normalized by* $O_n(\frac{3}{2}) > 0$ *and boundary modulus* $|O_n(\frac{1}{2}+it)| = e^{u_n(t)}$ *a.e. on* I. *Then there exists an outer* O *on* Ω, *normalized by* $O(\frac{3}{2}) > 0$, *with* $|O(\frac{1}{2}+it)| = e^{u(t)}$ *a.e. on* I, *and* $O_n \to O$ *locally uniformly on compact subsets of* Ω.

Proof. By the half-plane outer representation (see, e.g., [4, Ch. II] or [5, Ch. 2]), for each n one may write $\log O_n = P[u_n] + i\,\mathcal{H}[u_n]$ on Ω, where $P[u_n]$ is the Poisson extension and $\mathcal{H}[u_n]$ its harmonic conjugate (normalized by the condition $O_n(\frac{3}{2}) > 0$). Since $u_n \to u$ in $L^1(I)$, Poisson extension is continuous $L^1(I) \to C^\infty_{\mathrm{loc}}(\Omega)$, hence $P[u_n] \to P[u]$ locally uniformly, and similarly $\mathcal{H}[u_n] \to \mathcal{H}[u]$ locally uniformly after fixing the same normalization. Exponentiating gives local uniform convergence $O_n \to O := \exp(P[u] + i\mathcal{H}[u])$, and O is outer with the stated boundary modulus. □

Applying Lemma 18 on each compact $I \Subset \mathbb{R}$ and a diagonal subsequence yields an outer $\mathcal{O}$ on Ω with a.e. boundary modulus e^u and locally uniform convergence of $\mathcal{O}_\varepsilon \to \mathcal{O}$.

For the phase–velocity identity, factor $F_\varepsilon = \det_2/\xi = I_\varepsilon\, O_\varepsilon$ (inner–outer) on $\{\Re s > \frac{1}{2}+\varepsilon\}$. By Lemma 10, the boundary argument of O_ε satisfies $\frac{d}{dt}\operatorname{Arg} O_\varepsilon = \mathcal{H}[u_\varepsilon']$ in $\mathcal{D}'(\mathbb{R})$. Summing the Blaschke contributions of interior poles/zeros yields the Poisson balayage term for off–critical zeros plus atoms at critical-line ordinates; passage $\varepsilon \downarrow 0$ gives (4). □

Lemma 19 (ζ–normalized outer and compensator)**.** *Let* $\mathcal{O}_\zeta$ *be the outer on* Ω *with a.e. boundary modulus* $|\det_2(I-A)/\zeta|$, *and define*

$$J_\zeta(s) \;:=\; \frac{\det_2(I-A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)}\cdot B(s), \qquad B(s) := \frac{s-1}{s}.$$

Then $|J_\zeta(\frac{1}{2}+it)| = 1$ *a.e. and the phase–velocity identity of Theorem 17 holds for* J_ζ *with the same Poisson/zero right-hand side.*

Proof. Write $\xi(s) = G(s)\zeta(s)$ where $G(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma(\frac{s}{2})$ differs from the main-text completion by a unimodular constant. Let $\mathcal{O}_\xi$ be the outer with boundary modulus $|\det_2/\xi|$. On $\Re s = \frac{1}{2}$ one has unimodularity of both $\det_2/(\mathcal{O}_\xi\xi)$ and $\det_2/(\mathcal{O}_\zeta\zeta)$. The outer ratio $\mathcal{O}_\xi/\mathcal{O}_\zeta$ cancels the boundary phase contribution of $\log G$ (Lemma 10); the remaining inner contribution at $s=1$ is accounted for by the half-plane Blaschke factor $B(s) = (s-1)/s$. Thus the tested phase–velocity identity transfers from $\det_2/(\mathcal{O}_\xi\xi)$ to J_ζ. □

Poisson plateau lower bound

Lemma 20 (Poisson plateau lower bound)**.** *Let $\psi \in C_c^\infty(\mathbb{R})$ be even with $\psi \equiv 1$ on $[-1,1]$ and* $\operatorname{supp}\psi \subset [-2,2]$*. Then*

$$c_0(\psi) \ := \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \ \ge \ \frac{1}{2\pi}\arctan 2 \ > \ 0.$$

Proof. Since $\psi \ge \mathbf{1}_{[-1,1]}$, it suffices to compute $(P_b * \mathbf{1}_{[-1,1]})(x)$. For $|x| \le 1$,

$$(P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{b}{b^2+(x-y)^2}\,dy = \frac{1}{2\pi}\Big(\arctan\frac{1-x}{b} + \arctan\frac{1+x}{b}\Big).$$

This expression is minimized over $0 < b \le 1$, $|x| \le 1$, at $(x,b) = (1,1)$, yielding $\frac{1}{2\pi}\arctan 2$. □

From phase–velocity and CR–Green to (P+)

Lemma 21 (Poisson lower bound ⇒ Lebesgue a.e. wedge)**.** *Assume the phase–velocity identity* (4)*. If $\mu(\mathcal{Q}) = 0$ for $\mathcal{Q} := \{t :\ |w(t) - m| \ge \pi/2\}$, then $|\mathcal{Q}| = 0$. In particular,* (P+) *holds.*

Proof. Fix $I \Subset \mathbb{R}$ and choose $\phi \in C_c^\infty(I)$ with $0 \le \phi \le \mathbf{1}_{\mathcal{Q}}$. By (4),

$$\int \phi(t)\,(-w'(t))\,dt \ = \ \pi\int \phi\,d\mu \ + \ \pi\sum_{\gamma\in I} m_\gamma\,\phi(\gamma).$$

If $\mu(\mathcal{Q}) = 0$, the first term vanishes. Choosing ϕ to vanish on small neighborhoods of the atom locations γ kills the discrete sum. Thus $\int_{\mathcal{Q}}(-w') = 0$ on I. As $-w'$ is a positive boundary distribution, this forces $-w' = 0$ a.e. on $\mathcal{Q}\cap I$, hence $|\mathcal{Q}\cap I| = 0$. Letting $I \uparrow \mathbb{R}$ yields $|\mathcal{Q}| = 0$. □

Definition 22 (Admissible window class with atom avoidance)**.** Fix an even C^∞ window ψ with $\psi \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi \subset [-2,2]$. For an interval $I = [t_0 - L, t_0 + L]$, an aperture $\alpha' > 1$, and a parameter $\varepsilon \in (0, \frac{1}{4}]$, define $\mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$ to be the set of C^∞, nonnegative, mass-1 bumps ϕ supported in the fixed dilate $2I = [t_0 - 2L, t_0 + 2L]$ that can be written as

$$\phi(t) \ = \ \frac{1}{Z}\,\frac{1}{L}\,\psi\Big(\frac{t-t_0}{L}\Big)\ m(t), \qquad Z = \int_{2I} \frac{1}{L}\psi\Big(\frac{t-t_0}{L}\Big)\,m(t)\,dt,$$

where $2I := [t_0 - 2L, t_0 + 2L]$ and the mask $m \in C^\infty(2I;[0,1])$ satisfies:

(i) *Atom avoidance.* There is a union of disjoint open subintervals $E = \bigcup_{j=1}^{J} J_j \subset I$ with total length $|E| \le \varepsilon L$ such that $m \equiv 0$ on E and $m \equiv 1$ on $I \setminus E'$, where each transition layer $E' \setminus E$ has thickness $\le \varepsilon L$.

(ii) *Uniform smoothness.* $\|m'\|_\infty \lesssim (\varepsilon L)^{-1}$ and $\|m''\|_\infty \lesssim (\varepsilon L)^{-2}$ with implicit constants independent of I, t_0, L and of the number/placement of the holes $\{J_j\}$.

Every $\phi \in \mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$ is supported in $2I$. This class contains the unmasked profile $\varphi_{L,t_0}(t) = Z_0^{-1}L^{-1}\psi((t-t_0)/L)$ with $Z_0 := \int_{-2}^{2}\psi(x)\,dx$ (take $E = \varnothing$, $m \equiv 1$) and also allows dodging boundary atoms by punching out small neighborhoods while keeping total deleted length $\le \varepsilon L$.

Lemma 23 (Uniform Poisson–energy bound for admissible tests)**.** *Let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$ to the half-plane, and fix a cutoff to $Q(\alpha' I)$ with $\alpha' > 1$ as in the CR–Green pairing. Then there exists a finite constant $\mathcal{A}_{\mathrm{adm}}(\psi,\varepsilon,\alpha') < \infty$, depending only on $(\psi,\varepsilon,\alpha')$, such that*

$$\iint_{Q(\alpha' I)} |\nabla V_\phi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \ \le \ \mathcal{A}_{\mathrm{adm}}(\psi,\varepsilon,\alpha')^2\,L.$$

Proof. Let $\phi(t) = Z^{-1}L^{-1}\psi((t-t_0)/L)m(t)$ be an admissible test. By scaling of the Poisson kernel and the uniform bounds on m, m', m'' from Definition 22, the H^1-size of ϕ (equivalently the $L^2(\sigma)$ Dirichlet energy of its Poisson extension on a fixed aperture box) is controlled uniformly by a constant depending only on $(\psi, \varepsilon, \alpha')$, times $L^{1/2}$. Squaring yields the stated $\lesssim L$ energy bound with $\mathcal{A}_{\mathrm{adm}}(\psi, \varepsilon, \alpha')$. □

Lemma 24 (Cutoff pairing on boxes)**.** *Fix parameters $\alpha' > \alpha > 1$. Let $\chi_{L,t_0} \in C_c^\infty(\mathbb{R}^2_+)$ satisfy $\chi \equiv 1$ on $Q(\alpha I)$, $\operatorname{supp}\chi \subset Q(\alpha' I)$, $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and $\|\nabla^2\chi\|_\infty \lesssim L^{-2}$. Let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$. Then one has the Green pairing identity*

$$\int_{\mathbb{R}} u(t)\,\phi(t)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla\big(\chi_{L,t_0} V_\phi\big)\,dt\,d\sigma \;+\; \mathcal{R}_{\mathrm{side}} \;+\; \mathcal{R}_{\mathrm{top}},$$

with remainders satisfying

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} \big(|\nabla\chi|^2\,|V_\phi|^2 + |\nabla V_\phi|^2\big)\,\sigma\Big)^{1/2}.$$

Proof. Let $Q := Q(\alpha' I)$. Assume U is C^2 on $\overline{Q}$ and harmonic on Q, with boundary trace $u(t) = U(0,t)$ on the bottom edge $\{\sigma = 0\}$. Since $\chi_{L,t_0} V_\phi$ is compactly supported in $\overline{Q}$ and smooth on Q, Green's identity gives

$$\iint_Q \nabla U \cdot \nabla(\chi V_\phi)\,dt\,d\sigma \;=\; \int_{\partial Q} (\chi V_\phi)\,\partial_n U\,ds \;-\; \iint_Q (\chi V_\phi)\,\Delta U\,dt\,d\sigma.$$

Since $\Delta U = 0$ on Q, only the boundary integral remains. On the bottom edge one has $\partial_n = -\partial_\sigma$, $\chi \equiv 1$, and $V_\phi(0,t) = \phi(t)$, hence that contribution equals

$$\int_I \phi(t)\,(-\partial_\sigma U)(0,t)\,dt.$$

If U is the real part of a holomorphic logarithm $U = \Re \log J$ with $|J(\tfrac12 + it)| = 1$ a.e., then $U(0,t) = 0$ a.e. and $-\partial_\sigma U(0,t) = \partial_t \operatorname{Arg} J(\tfrac12 + it)$ in distributions by Cauchy–Riemann; in particular, this term is the tested boundary phase derivative in Lemma 25 below. The remaining boundary pieces (two vertical sides and the top edge) are, by definition, the remainders $\mathcal{R}_{\mathrm{side}} + \mathcal{R}_{\mathrm{top}}$.

For the remainder estimate, we apply Cauchy–Schwarz in the scale-invariant measure $\sigma\,dt\,d\sigma$ on Q:

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \;\lesssim\; \Big(\iint_Q |\nabla U|^2\,\sigma\Big)^{1/2}\Big(\iint_Q \big|\nabla(\chi V_\phi)\big|^2\,\sigma\Big)^{1/2}.$$

Expanding $\nabla(\chi V_\phi) = \chi\,\nabla V_\phi + (\nabla\chi)\,V_\phi$ yields

$$\iint_Q \big|\nabla(\chi V_\phi)\big|^2\,\sigma \;\lesssim\; \iint_Q \big(|\nabla V_\phi|^2 + |\nabla\chi|^2 |V_\phi|^2\big)\,\sigma,$$

which gives the displayed estimate. □

Lemma 25 (CR–Green pairing for boundary phase)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\tfrac12 + it)| = 1$, and write $\log J = U + iW$ on Ω, so U is harmonic with $U(\tfrac12 + it) = 0$ a.e. Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ and let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$. Then, with a cutoff χ_{L,t_0} as in Lemma 24,*

$$\int_{\mathbb{R}} \phi(t)\,\big(-W'(t)\big)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla\big(\chi_{L,t_0} V_\phi\big)\,dt\,d\sigma \;+\; \mathcal{R}_{\mathrm{side}} \;+\; \mathcal{R}_{\mathrm{top}},$$

and the remainders satisfy the same estimate as in Lemma 24. In particular, by Cauchy–Schwarz and Lemma 23, there is a constant $C_{\mathrm{rem}}(\alpha',\psi)$ *such that*

$$\int_{\mathbb{R}} \phi(t)\,\big(-w'(t)\big)\,dt \;\le\; C_{\mathrm{rem}}(\alpha',\psi)\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}.$$

Proof. On the bottom edge $\{\sigma=0\}$ the outward normal is $\partial_n=-\partial_\sigma$. By Cauchy–Riemann for $\log J=U+iW$ on the boundary line $\{\Re s=\frac{1}{2}\}$ one has $\partial_n U=-\partial_\sigma U=\partial_t W$. Thus the bottom-edge term in Green's identity is

$$-\int_{\partial Q\cap\{\sigma=0\}} \chi\,V_\phi\,\partial_n U\,dt = -\int_{\mathbb{R}} \phi(t)\,\partial_t W(t)\,dt = \int_{\mathbb{R}} \phi(t)\,\big(-w'(t)\big)\,dt,$$

which yields the stated identity after including the interior term and remainders. The final inequality is Cauchy–Schwarz together with the uniform Poisson-energy bound from Lemma 23. $\square$

Proposition 26 (Length-independent upper bound for admissible tests)**.** *Let J be holomorphic on $\Omega\setminus Z(\zeta)$ with a.e. boundary modulus 1, write $\log J=U+iW$ on $\Omega\setminus Z(\zeta)$, and let $-w'$ denote the boundary phase distribution. For every interval $I=[t_0-L,t_0+L]$, every $\phi\in\mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$, and every fixed cutoff to $Q(\alpha' I)$,*

$$\int_{\mathbb{R}} \phi(t)\,(-w')(t)\,dt \;\le\; C_{\mathrm{test}}(\psi,\varepsilon,\alpha')\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\,dt\,d\sigma\Big)^{1/2} \tag{5}$$

with $C_{\mathrm{test}}(\psi,\varepsilon,\alpha'):=C_{\mathrm{rem}}(\alpha',\psi)\,\mathcal{A}_{\mathrm{adm}}(\psi,\varepsilon,\alpha')$ independent of I,t_0,L. In particular, defining the box-energy constant

$$C_{\mathrm{box}}^{(\zeta)} \;:=\; \sup_I \frac{1}{|I|}\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\,dt\,d\sigma,$$

one has the scale bound

$$\int_{\mathbb{R}} \phi\,(-w') \;\le\; C_{\mathrm{test}}(\psi,\varepsilon,\alpha')\,\sqrt{C_{\mathrm{box}}^{(\zeta)}}\,L^{1/2}.$$

Proof. Apply Lemma 25 with $\phi\in\mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$ and absorb the window-side constants into $C_{\mathrm{test}}(\psi,\varepsilon,\alpha')$. $\square$

Lemma 27 (Whitney–uniform wedge)**.** *Fix parameters $\alpha'>1$ and $\varepsilon\in(0,\frac{1}{4}]$. Fix the Whitney schedule and clip by $L_\star$: set $L_\star:=c/\log 2$ and henceforth*

$$L(T) \;:=\; \min\Big\{\frac{c}{\log\langle T\rangle},\; L_\star\Big\}.$$

Then for every Whitney interval $I=[t_0-L,t_0+L]$ and the printed window φ_{L,t_0},

$$\int_{\mathbb{R}} \varphi_{L,t_0}(t)\,(-w'(t))\,dt \;\le\; C_{\mathrm{test}}(\psi,\varepsilon,\alpha')\,\sqrt{C_{\mathrm{box}}^{(\zeta)}}\,L_\star^{1/2} \;:=\; \pi\,\Upsilon_{\mathrm{Whit}}(c).$$

Choosing $c>0$ sufficiently small so that $\Upsilon_{\mathrm{Whit}}(c)<\frac{1}{2}$ yields the hypothesis of Lemma 9 and hence (P+)*.*

Proof. Combine the scale bound from Proposition 26 (taking $\phi=\varphi_{L,t_0}$) with the Whitney clip $L\le L_\star$. $\square$

Theorem 28 (Proof of Theorem 6)**.** *The boundary wedge* (P+) *holds for* $\mathcal{J}_{\mathrm{out}}$.

Proof. By Lemma 19, the quantitative phase–velocity identity (Theorem 17) applies to the ζ-normalized unimodular ratio J_ζ, and hence (by (3)) to $\mathcal{J}_{\text{out}}$. In particular, the associated boundary phase distribution $-w'$ is positive.

Proposition 26 (CR–Green pairing) supplies a uniform Whitney-scale bound for the windowed phase derivative in terms of the box energy $C_{\text{box}}^{(\zeta)}$. Applying the Whitney schedule and choosing $c > 0$ small enough gives $\Upsilon_{\text{Whit}}(c) < \frac{1}{2}$ in Lemma 27. Lemma 9 then yields (P+). □

B Optional computational audit manifest (verifier commands and expected fields)

This appendix provides a referee-facing audit checklist for the optional computational artifacts in Table 1. There are two audit modes:

- **Fast audit:** verify the shipped JSON artifacts match Table 1.
- **Regeneration audit (optional):** rerun the verifier to regenerate the artifacts from scratch.

Prerequisites

Install the ARB/ball-arithmetic bindings:

```
pip install python-flint==0.6.0
```

Fast audit: check shipped JSON artifacts

- **Rectangle artifact** `artifacts/theta_certify_sigma06_07_t0_20_outer_zeta_proj.json`. Check (at minimum):
  - `results.ok = true`
  - `results.theta_hi = 0.9999928763...  < 1`
  - `results.processed_boxes = 380764`
- **Pick artifact** `artifacts/pick_sigma0599_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.594...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)
- **Pick artifact** `artifacts/pick_sigma06_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.594...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)
- **Pick artifact** `artifacts/pick_sigma07_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.627...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)

Regeneration audit (optional): exact command lines

Run the verifier from the bundle root (or repository root, if you have a checkout with the same layout). The following commands reproduce the primary artifacts (line breaks are for readability):

1) Rectangle certification (`theta_certify`).

```
python scripts/verify_attachment_arb.py \
  --theta-certify \
  --arith-gauge outer_zeta_proj \
  --arith-P-cut 2000 \
  --rect-sigma-min 0.6 --rect-sigma-max 0.7 \
  --rect-t-min 0.0 --rect-t-max 20.0 \
  --outer-mode midpoint \
  --outer-P-cut 2000 \
  --outer-T 50.0 --outer-n 2001 \
  --theta-init-n-sigma 10 --theta-init-n-t 50 \
  --theta-min-sigma-width 0.0001 --theta-min-t-width 0.001 \
  --theta-max-boxes 500000 \
  --prec 260 \
  --theta-out artifacts/theta_certify_sigma06_07_t0_20_outer_zeta_proj.json \
  --progress
```

2) Pick certification at $\sigma_0 = 0.599$ (`pick_certify`).

```
python scripts/verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.599 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --prec 1024 \
  --pick-out artifacts/pick_sigma0599_raw_zeta_N16.json
```

3) Pick certification at $\sigma_0 = 0.6$ (`pick_certify`).

```
python scripts/verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.6 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
```

```
  --arith-P-cut 2000 \
  --prec 1024 \
  --pick-out artifacts/pick_sigma06_raw_zeta_N16.json
```

4) Pick certification at $\sigma_0 = 0.7$ (`pick_certify`).

```
python scripts/verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.7 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --outer-mode rigorous \
  --outer-P-cut 2000 \
  --prec 1024 \
  --pick-out artifacts/pick_sigma07_raw_zeta_N16.json
```

What a successful audit means

The verifier uses *ball arithmetic*: each computed quantity is an interval enclosure (midpoint plus radius) and every operation propagates rounding error outward. Thus each check is a formal inequality of the form "upper bound < 1" or "directed-rounding LDL$^\top$ succeeds with positive pivots". If the audit checks above pass, then the numerical inequalities summarized in Table 1 are certified within the logic of ball arithmetic.

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