

The Iwasawa Main Conjecture for universal families of modular motives

Olivier Fouquet & Xin Wan

Abstract

Let p be an odd prime. We prove the cyclotomic Iwasawa Main Conjecture of K.Kato for the motive attached to an eigencuspform $f \in S_k(\Gamma_0(N))$ with arbitrary reduction type at p under mild assumptions on the residual Galois representation $\bar{\rho}_f$. Under the same hypotheses, we also prove the generalized Iwasawa Main Conjecture for p -adic families of modular forms. The Iwasawa Main Conjecture for f is deduced by a limit argument involving fundamental lines from a universal Iwasawa Main Conjecture over the universal deformation space of $\bar{\rho}_f$, which itself follows from the cyclotomic Iwasawa Main Conjecture for crystalline eigencuspforms and hence from results on the Iwasawa-Greenberg Main Conjecture for Rankin-Selberg products. The main novel ingredients in our proof are as follows: a new way to study the arithmetic of the Fourier-Jacobi coefficients of Eisenstein series for the group $U(3, 1)$, an explicit description of the exponential map in a well-chosen family with prescribed ramification to obtain integral comparisons of various p -adic L -functions and Selmer modules, Iwasawa theory for the universal zeta elements constructed by K.Nakamura, descent techniques for fundamental lines over the universal regular ring underlying the universal deformation space and ramification properties of the latter over the former.

Contents

1 Introduction	3
1.1 The Iwasawa Main Conjecture for modular motives	3
1.1.1 Historical introduction	3
1.1.2 The main theorems	7
1.2 Outline of the proof	9
1.2.1 Classical points in \mathcal{X}^{sm}	9
1.2.2 From \mathcal{X}^{sm} to all classical points	12
2 Generalities on p-adic Hecke algebras	15
2.1 p -adic Hecke algebras	15
2.2 Commutative algebra properties of $\mathbf{T}_{\mathfrak{m}_p}^\Sigma$	16
2.2.1 Classical primes of the p -adic Hecke algebra	16
2.2.2 Identification with universal deformation rings	17
3 Fundamental lines and zeta elements over deformation rings	21
3.1 Classical Iwasawa theory for modular motives	21
3.1.1 Equivariant modular motives and period maps	21
3.1.2 Zeta morphism for modular motives	23
3.2 Zeta elements over deformation rings	24
3.2.1 Review of the results of Nakamura	24
3.2.2 Iwasawa-suitable specializations	25
3.3 Fundamental lines	31
3.4 The Equivariant Tamagawa Number Conjectures	34
4 The residually irreducible, crystalline case	36
4.1 Review of p -adic Hodge theory	36
4.1.1 Iwasawa Cohomology Groups	36
4.1.2 (φ, Γ) -modules	37
4.1.3 Bloch-Kato exponential maps	40
4.1.4 Co-admissible Λ_∞ -modules	41
4.2 Unramified Iwasawa Theory	42
4.2.1 Boundedness of the exponential map	42
4.2.2 Yager modules	43
4.2.3 Explicit description of the exponential map and Galois cohomology .	44
4.3 Control Theorem of Selmer Groups	45
4.3.1 Notations and definitions	46
4.3.2 No pseudo-null submodules	48
4.4 Beilinson-Flach Elements	49
4.4.1 Hida families of CM representations	49
4.4.2 Analytic families of Beilinson-Flach elements	51
4.5 Selmer Complexes and Iwasawa Main Conjecture	51
4.5.1 Nekovář-Selmer complexes	51
4.5.2 Rankin-Selberg p -adic L -functions	56
4.6 The Iwasawa Main Conjecture in the crystalline case	58
4.6.1 Statement of the Greenberg-Iwasawa Rankin-Selberg Main Conjecture	58
4.6.2 Proof of the Iwasawa Main Conjecture up to powers of p	59
4.6.3 Powers of p	61
4.7 The p -irregular case	66
4.7.1 Geometry of the eigencurve	66
4.7.2 The argument	66
4.8 The ordinary case	68
5 The main theorems	68
5.1 Statements	68
5.2 Proofs of theorems 5.1 and 5.2	70

6 Appendix A: completed cohomology and essential vectors for automorphic representations	74
6.1 Generalities	74
6.1.1 Notations	74
6.1.2 Bernstein-Zelevinsky functors	75
6.1.3 Co-Whittaker modules	76
6.2 Description of co-Whittaker torsion-modules for G_2	77
6.3 Completed cohomology	78
7 Appendix B: Iwasawa-Greenberg Main Conjecture	79
7.1 The Scalar Weight Case: Review	80
7.2 Vector Valued Cases	80
7.3 Unitary groups and Hida Theory for Semi-Ordinary Forms	80
7.3.1 Unitary Groups	80
7.3.2 Shimura varieties for Unitary Similitude Groups	81
7.3.3 Igusa varieties and p -adic automorphic forms	82
7.3.4 Semi-Ordinary Forms	83
7.3.5 Control Theorems	84
7.4 Eisenstein Family	88
7.4.1 Klingen Eisenstein Family	88
7.4.2 Study of the Fourier-Jacobi functional	90

1 Introduction

Let $p > 2$ be a prime. Let $f \in S_k(\Gamma_0(N))$ be a normalized eigencuspform of weight $k \geq 2$. Let $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F})$ be a residual representation attached to f with coefficients in a finite extension \mathbb{F} of \mathbb{F}_p . One of the main results of this article is the following.

Theorem 1.1. *Assume the following three hypotheses.*

1. *The $G_{\mathbb{Q}}$ -representation $\bar{\rho}$ is absolutely irreducible.*
2. *If $\bar{\chi}$ is a character of $G_{\mathbb{Q}_p}$ and $\bar{\chi}_{\mathrm{cyc}}$ is the cyclotomic character modulo p , the semisimplification of $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is equal neither to $\bar{\chi} \oplus \bar{\chi}$ nor to $\bar{\chi} \oplus \bar{\chi}_{\mathrm{cyc}} \bar{\chi}$.*
3. *There exists $\ell \nmid p$ such that $\ell \mid N$ and such that $\dim_{\mathbb{F}} \bar{\rho}^{I_{\ell}} = 1$ and $\dim_{\mathbb{F}} \bar{\rho}^{G_{\mathbb{Q}_{\ell}}} = 0$.*

Then the Iwasawa Main Conjecture ([68, Conjecture 12.10]) is true for for the motive $M(f)$.

Assumption 1 means that the modular form is residually non-Eisenstein: the residually Eisenstein case has been treated in recent and forthcoming works of F.Castella, G.Grossi, J.Lee and C.Skinner ([19]). Both assumption 1 and assumption 2 are required to analyze the universal deformation space of $\bar{\rho}$ and to appeal to the p -adic Langlands correspondence. Assumption 3, which is equivalent to the local automorphic representation $\pi(f)_{\ell}$ being the special Steinberg twisted by the unramified character sending ℓ to $(-1)\ell^{k/2-1}$ and $\bar{\rho}|_{G_{\ell}}$ being ramified, plays a technical role in comparing the periods appearing in various p -adic L -functions. Contrary to most works on the Iwasawa Main Conjecture for modular forms, the eigencuspform f in theorem 1.7 may have arbitrary weight $k \geq 2$ and arbitrary ramification at p . This is allowed by new techniques which enable us to bypass essential difficulties that prevented previous methods to go beyond the ordinary or low-weight crystalline cases.

1.1 The Iwasawa Main Conjecture for modular motives

1.1.1 Historical introduction

The Iwasawa Main Conjecture Let $p > 2$ be a prime. Let $\mathbb{Q}_{\infty}/\mathbb{Q}$ be the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} , let Γ_{Iw} be $\mathrm{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})$ and let $\Lambda_{\mathrm{Iw}} \stackrel{\mathrm{def}}{=} \mathbb{Z}_p[[\Gamma_{\mathrm{Iw}}]]$ be the classical Iwasawa algebra ([112]).

Let M be a pure motive over \mathbb{Q} . The classical Iwasawa Main Conjecture for M ([65, 66]) is a web of conjectures which describe together the p -adic variation of the algebraic part of the special values of the L -function of $M^*(1) \otimes \chi$ as χ ranges over the set of Dirichlet characters of order a power of p in terms of Λ_{Iw} -adic cohomological invariants of M . As such, it can be

understood as a conjectural description of the zeroes of a p -adic L -function (when such an object is known to exist) and hence a p -adic variant of the Generalized Riemann Hypothesis, or as a description of the Galois-module structure of certain cohomological invariants of M by means of congruences modulo p between special values of $L(M^*(1), \chi, 0)$ for various χ .

For the motive $\mathbb{Q}(1)$, the main conjecture was formulated by K.Iwasawa ([63, 90]) as a conjectural equality between the ideal generated by the p -adic L -function L_p of Kubota-Leopoldt ([79]) and the characteristic ideal of $\text{Cl}_\infty[p^\infty]$, the inverse limit on n of the p -torsion part of the class group of the ring $\mathbb{Z}[\zeta_{p^n}]$ seen as a Λ_{Iw} -module. Similar conjectural formulations relating a suitably defined p -adic L -function and the characteristic ideal of a suitably defined Λ_{Iw} -adic Selmer module were found in increasing generality by B.Mazur for ordinary modular abelian varieties ([87, 88]), by R.Greenberg for motives with ordinary reduction at p in the sense of [99] ([47, 48]) and by B.Perrin-Riou for motives with crystalline reduction at p ([100]).

After the work of S.Bloch and K.Kato ([9]) and their reformulation in [42], it has been understood that even for a single motive M , the so-called *motivic fundamental line* $\Delta(M)$ of M should play a crucial role in the general statement of conjectures on special values of L -functions of motives. Let M/\mathbb{Q} be a motive with coefficients in a number field $L \subset \mathbb{C}$, which we may think of in the context of this introduction as a system of realizations $\{M_B, M_{\text{dR}}, \{M_{\text{et},p}\}_{p \in \text{Spec } \mathcal{O}_L}\}$ related by comparison isomorphisms and satisfying the Weight Monodromy Conjecture ([62, 3.13.2]). If $M_{\text{et},p}$ is the p -adic étale realization of M and if $j : \text{Spec } \mathbb{Q} \hookrightarrow \text{Spec } \mathbb{Z}[1/p]$ is the natural morphism, we write $R\Gamma_{\text{et}}(\mathbb{Z}[1/p], -)$ for $R\Gamma_{\text{et}}(\text{Spec } \mathbb{Q}, j_*(-))$. The motivic fundamental line of M is a conjectural free L -vector space $\Delta(M)$ of dimension 1 defined using motivic cohomology and which satisfies the following properties.

1. There is a canonical isomorphism $\text{per}_{\mathbb{C}} : \Delta(M) \otimes_L \mathbb{C} \xrightarrow{\text{can}} \mathbb{C}$.
2. For any prime $p \in \text{Spec } \mathcal{O}_L$, there is a canonical isomorphism

$$\text{per}_p \Delta(M) \otimes_L L_p \xrightarrow{\text{can}} \text{Det}_{L_p}^{-1} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], M_{\text{et},p}) \otimes_{L_p} \text{Det}_{L_p}^{-1} M_{\text{et},p}(-1)^+.$$

To M is attached a partial L -function $L_{\{p\}}(M, s)$ defined as the formal Euler product

$$L_{\{p\}}(M, s) = \prod_{\ell \nmid p} \frac{1}{\det \left(1 - \text{Fr}(\ell)x | M_{\text{et},p}^{I_\ell} \right)_{x=\ell-s}}$$

which is believed to define a complex L -function with an analytic continuation at $s = 0$. Denote by $L_{\{p\}}^*(M, 0)$ the first non-zero term in the Taylor expansion of $L_{\{p\}}(M, s)$ at $s = 0$.

Conjectures on special values of L -functions then become the following compact statement.

Conjecture 1.2. *There exists a basis $\mathbf{z}(M) \in \Delta(M)$ called the zeta element of M such that*

$$\text{per}_{\mathbb{C}}(\mathbf{z}(M) \otimes 1) = L_{\{p\}}^*(M^*(1), 0)$$

and such that

$$\mathcal{O}_{L,p} \cdot \text{per}_p(\mathbf{z}(M) \otimes 1) = \text{Det}_{\mathcal{O}_{L,p}}^{-1} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], T) \otimes_{\mathcal{O}_{L,p}} \text{Det}_{\mathcal{O}_{L,p}}^{-1} T(-1)^+$$

for any $G_{\mathbb{Q}}$ -stable $\mathcal{O}_{L,p}$ -submodule $T \subset M_{\text{et},p}$.

Here we recall that the fact that $\text{per}_{\mathbb{C}}$ and per_p are well-defined is in general an integral part of the conjecture.

Suppose now for simplicity of notations that M/\mathbb{Q} is a pure motive with coefficients in \mathbb{Q} . Suppose also that \mathcal{X} is p -adic family of $G_{\mathbb{Q}}$ -representations containing $M_{\text{et},p}$ (that is to say a rigid analytic p -adic space or a \mathbb{Z}_p -scheme whose $\bar{\mathbb{Q}}_p$ -points are Galois representations and such that one such points is $M_{\text{et},p}$). The advantage of the formulation of conjecture 1.2 is that it is amenable to p -adic interpolation within the family \mathcal{X} . We describe the case of the family of cyclotomic twists of M ([65]). Define as in [66] the Λ_{Iw} -adic fundamental line $\Delta(M)_{\text{Iw}}$ as the free Λ_{Iw} -module of rank 1

$$\Delta(M)_{\text{Iw}} = \text{Det}_{\Lambda_{\text{Iw}}}^{-1} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], T \otimes_{\mathbb{Z}_p} \Lambda_{\text{Iw}}) \otimes_{\Lambda_{\text{Iw}}} \text{Det}_{\Lambda_{\text{Iw}}}^{-1} (T \otimes_{\mathbb{Z}_p} \Lambda_{\text{Iw}})(-1)^+$$

for T any $G_{\mathbb{Q}}$ -stable \mathbb{Z}_p -module inside the p -adic étale realization $M_{\text{et},p}$ of M (the resulting Λ_{Iw} -adic fundamental line is then independent of the choice of T in the sense that there there is a canonical isomorphism between fundamental lines attached to two different such choices). For any finite order character $\chi \in \hat{\Gamma}_{\text{Iw}}$ with values in a finite extension L_{χ}/\mathbb{Q} and any integer $r \in \mathbb{Z}$, there is then a canonical isomorphism

$$\chi_{\text{cyc}}^r \chi : \Delta(M)_{\text{Iw}} \otimes_{\Lambda_{\text{Iw}}} \mathcal{O}_{\chi} \xrightarrow{\text{can}} \text{Det}_{\mathcal{O}_{\chi}}^{-1} R \Gamma_{\text{et}}(\mathbb{Z}[1/p], T(r) \otimes \chi) \otimes_{\mathcal{O}_{\chi}} \text{Det}_{\mathcal{O}_{\chi}}^{-1}(T(r) \otimes \chi)(-1)^+$$

where we denote by \mathcal{O}_{χ} the ring of integers of a finite extension of \mathbb{Q}_p containing L_{χ} and where $T(r) \otimes \chi$ is the \mathcal{O}_{χ} -module $T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\chi}$ with $G_{\mathbb{Q}}$ -action twisted by $\chi_{\text{cyc}}^r \chi$.

The Iwasawa Main Conjecture for the motive M is the following statement, which expresses the fact that there should exist a basis of $\Delta(M)_{\text{Iw}}$ which p -adically interpolates the zeta elements $\mathbf{z}(M(r) \otimes \chi) \in \Delta(M(r) \otimes \chi)$ of the various motives $M(r) \otimes \chi$ as r ranges over \mathbb{Z} and χ ranges over the finite order characters of Γ_{Iw} .

Conjecture 1.3. *There exists a basis $\mathbf{z}(M)_{\text{Iw}} \in \Delta(M)_{\text{Iw}}$ called the Λ_{Iw} -adic zeta element such that for all finite order character $\chi \in \hat{\Gamma}_{\text{Iw}}$ and all integer $r \in \mathbb{Z}$, $\chi_{\text{cyc}}^r \chi(\mathbf{z}(M)_{\text{Iw}})$ is the image of the zeta element $\mathbf{z}(M(r) \otimes \chi)$ of $M(r) \otimes \chi$ through $\text{per}_{\mathfrak{p}}(- \otimes 1)$. Equivalently, for all such χ and r , $\chi_{\text{cyc}}^r \chi(\mathbf{z}(M)_{\text{Iw}})$ is a basis of*

$$\text{Det}_{\mathcal{O}_{\chi}}^{-1} R \Gamma_{\text{et}}(\mathbb{Z}[1/p], T(r) \otimes \chi) \otimes_{\mathcal{O}_{\chi}} \text{Det}_{\mathcal{O}_{\chi}}^{-1}(T(r) \otimes \chi)(-1)^+$$

and

$$\text{per}_{\mathbb{C}}(\text{per}_{\mathfrak{p}}^{-1}(\chi_{\text{cyc}}^r \chi(\mathbf{z}(M)_{\text{Iw}})) \otimes 1) = L_{\{p\}}^*(M^*(1), \chi^{-1}, -r).$$

The case of modular motives Let $f \in S_k(\Gamma_1(N))$ be a normalized eigencuspform of weight $k \geq 2$ and let $M(f)$ be the pure motive attached to f ([111]). Let E/\mathbb{Q}_p be a finite extension containing the eigenvalues of f and let \mathcal{O} be its ring of integers. We fix $T \subset M(f)_{\text{et},\mathfrak{p}}$ a $G_{\mathbb{Q}}$ -stable \mathcal{O} -module and denote by

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\mathcal{O}), \quad \bar{\rho}_f : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\mathbb{F})$$

the corresponding choices of $G_{\mathbb{Q}}$ -representations. In [68], the following remarkable result towards conjecture 1.3 for the motive $M(f)$ was proved.

Theorem 1.4 ([68] Theorem 12.4,12.5). *There exists a non-zero element*

$$\mathbf{z}(f)_{\text{Iw}} \in \Delta(M(f))_{\text{Iw}} \otimes_{\Lambda_{\text{Iw}}} \text{Frac}(\Lambda_{\text{Iw}})$$

satisfying the following two properties.

1. Let $1 \leq r \leq k-1$ be an integer and $\chi \in \hat{\Gamma}_{\text{Iw}}$ be of finite order. If

$$L_{\{p\}}(M(f)^*(1), \chi^{-1}, -r) \neq 0$$

then the fundamental line $\Delta(M(f)(r) \otimes \chi)$ and the period isomorphisms $\text{per}_{\mathbb{C}}$ and $\text{per}_{\mathfrak{p}}$ all exist and satisfy the expected properties. Moreover, $\text{per}_{\mathfrak{p}}^{-1}(\chi_{\text{cyc}}^r \chi(\mathbf{z}(f)_{\text{Iw}}))$ belongs to $\Delta(M(f)(r) \otimes \chi)$ so that

$$\text{per}_{\mathbb{C}}(\text{per}_{\mathfrak{p}}^{-1}(\chi_{\text{cyc}}^r \chi(\mathbf{z}(f)_{\text{Iw}})) \otimes 1)$$

is well-defined and

$$\text{per}_{\mathbb{C}}(\text{per}_{\mathfrak{p}}^{-1}(\chi_{\text{cyc}}^r \chi(\mathbf{z}(f)_{\text{Iw}})) \otimes 1) = L_{\{p\}}(M(f)^*(1), \chi^{-1}, -r).$$

2. If $k > 2$ or if $M(f)$ has potential good reduction and if the image of $\rho_f|G_{\mathbb{Q}(\zeta_p \infty)}$ contains a subgroup conjugated to $\text{SL}_2(\mathbb{Z}_p)$, then there is an inclusion $\Delta(M(f))_{\text{Iw}}^{-1} \subset \Lambda_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}}$ of invertible Λ_{Iw} -modules inside $\Delta(M(f))_{\text{Iw}} \otimes_{\Lambda_{\text{Iw}}} \text{Frac}(\Lambda_{\text{Iw}})$. Equivalently, $\mathbf{z}(f)_{\text{Iw}}$ can then be seen as a class in $H_{\text{et}}^1(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}})$ and there is a divisibility of characteristic ideals

$$\text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^2(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}) \mid \text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^1(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}})/\Lambda_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}}.$$

By the Iwasawa Main Conjecture for the modular motive $M(f)$, we henceforth mean the following statement¹.

Conjecture 1.5 ([68] Conjecture 12.10). *Let $\mathbf{z}(f)_{\text{Iw}} \in \Delta(M(f))_{\text{Iw}} \otimes_{\Lambda_{\text{Iw}}} \text{Frac}(\Lambda_{\text{Iw}})$ be the basis of theorem 1.4. Then $\mathbf{z}(f)_{\text{Iw}}$ is a basis of $\Delta(M(f))_{\text{Iw}}$. Equivalently, there is an equality of characteristic ideals*

$$\text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^2(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}) = \text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^1(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}) / \Lambda_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}}.$$

K.Kato proved the first statement of theorem 1.4 by introducing a family of distinguished elements $\{\mathbf{z}_{M,N} \in K_2(Y(M, N))\}_{M,N}$ where $\mathbf{z}_{M,N}$ is the Steinberg product of two well-defined Siegel units. A delicate reciprocity law links the images of the classes $\mathbf{z}_{M,N}$ through the dual exponential map composed with the Chern class map to products of Eisenstein series and hence to special values of L -functions. The second statement then follows from the method of Euler systems, pioneered by F.Thaine, K.Rubin and especially V.Kolyvagin ([119, 108, 78]) and axiomatized in [101, 67, 109]. In this method, one studies the localization properties of a system of classes $\{\mathbf{z}_m \in H_{\text{et}}^1(\mathbb{Z}[1/p, \zeta_m], T)\}_{m \in \mathbb{N}}$ constructed from the $\mathbf{z}_{M,N}$ which mimics the formal behavior of the L -function of $M^*(1)$ to deduce a crude bound on the length of $H_{\text{et}}^2(\mathbb{Z}[1/p], T/p^nT)$. Then, one obtains an inclusion of fundamental lines or a divisibility between characteristic ideals as in theorem 1.4 above by p -adic interpolation alongside the cyclotomic Iwasawa algebra.

The first proof of the original Iwasawa Main Conjecture for the motive $\mathbb{Q}(1)$ by B.Mazur and A.Wiles ([90]) relied on a completely different technique. The starting point of the proof was the observation that the Main Conjecture predicted that a zero of the p -adic L -function of M (if it exists) should be reflected in the Selmer group. The idea was then to realize this p -adic L -function as the constant term of the Fourier expansion of a p -adic family of automorphic Eisenstein series \mathbf{E} with coefficients in a certain ring A . Whenever the p -adic L -function vanishes modulo a given power \mathcal{P}^n of a prime ideal $\mathcal{P} \in \text{Spec } A$, this automorphic form admits a congruence modulo \mathcal{P}^n with a p -adic family of cuspidal automorphic forms \mathbf{F} with coefficients in A . In that situation, the Galois representation attached to \mathbf{F} modulo \mathcal{P}^n is isomorphic to the Galois representation attached to \mathbf{E} modulo \mathcal{P}^n and is thus reducible, even though the Galois representation attached to \mathbf{F} and with coefficients in A should be irreducible. Ribet's lemma (pioneered in [105]) exploits this reducibility modulo \mathcal{P}^n of an irreducible representation to construct a non-trivial extension between Galois representations of lower dimensions, and hence an element in a certain Selmer A -module, explaining why the vanishing of a p -adic L -function has a cohomological counterpart.

In the case of the motive $\mathbb{Q}(1)$, the Eisenstein series \mathbf{E} is a p -adic family of genuine Eisenstein series for GL_2 and \mathbf{F} is a Hida family of eigencuspforms ([127]). For modular motives, C.Skinner and E.Urban realized the p -adic L -function of an eigencuspform f with good ordinary reduction, or more precisely the two-variable p -adic L -function attached to the automorphic base change of the automorphic representation $\pi(f)$ attached to f to a well-chosen imaginary quadratic number field, as the constant term of a Hida family of Eisenstein series for the reductive group $\text{U}(2, 2)$ ([117, Theorem 3.29]). In this way, they were able to prove the following theorem. (Attentive readers will remark that theorem 1.6 below is slightly different from [117, Theorem 3.29]. First of all, [117] has the supplementary assumptions $k \equiv 2 \pmod{p-1}$, which was lifted by the second named author in [124]. In addition, to deduce the following result from [117], one would need an argument comparing the canonical period and the Gross period, which might not exist in the relevant literature in a satisfying generality for our purpose. In order to provide a self-contained argument, indications on how to reprove theorem 1.6 in a way that bypasses this issue are given in section 4.8.)

Theorem 1.6 ([117] Theorem 3.19). *Let $f \in S_k(\Gamma_0(N))$ with $p \nmid N$ be an eigencuspform of weight $k \geq 2$. Assume the following hypotheses on ρ_f .*

1. *The $G_{\mathbb{Q}}$ -representation $\bar{\rho}_f$ is absolutely irreducible.*
2. *There exists $\ell \mid N$, such that $\dim_{\mathbb{F}} \bar{\rho}_f^{I_\ell} = 1$ and $\dim_{\mathbb{F}} \bar{\rho}_f^{G_\ell} = 0$.*

¹Conjecture 1.5 is weaker than conjecture 1.3, as it predicts nothing on the behavior of the zeta element at the finitely many χ where $L(M(f)^*(1), \chi^{-1}, -r)$ vanishes.

3. The restriction of ρ_f to $G_{\mathbb{Q}_p}$ fits into a short exact sequence of $\mathcal{O}[G_{\mathbb{Q}_p}]$ -modules

$$0 \longrightarrow \chi_1 \longrightarrow \rho_f|G_{\mathbb{Q}_p} \longrightarrow \chi_2 \longrightarrow 0$$

with $\bar{\chi}_1 \neq \bar{\chi}_2$.

Then there is a divisibility of characteristic ideals

$$\text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^1(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}) / \Lambda_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}} \mid \text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^2(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}).$$

Combined with theorem 1.4², this yields an equality

$$\text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^2(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}) = \text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^1(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}) / \Lambda_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}}.$$

and conjecture 1.5 is then true for $M(f)$.

Together, theorems 1.4 and 1.6 establish the Iwasawa Main Conjecture for the modular motive $M(f)$ under some conditions on the image of $\bar{\rho}_f$ and under the assumption that $\rho_f|G_{\mathbb{Q}_p}$ be reducible, or equivalently that f has ordinary reduction at p . In works of the second named author and I.Sprung, the ordinary hypothesis on $\rho_f|G_{\mathbb{Q}_p}$ was relaxed, but only for elliptic curves, that is to say for rational eigencusforms of weight 2.

In this manuscript, we combine the strength of both approaches to obtain results on the Iwasawa Main Conjecture for eigencusforms with arbitrary reduction type at p (either crystalline, semistable or potentially semistable of arbitrary even weights from the point of view of p -adic Hodge theory, or principal series, Steinberg or supercuspidal from the point of view of automorphic theory).

1.1.2 The main theorems

We now state the full version of our main theorems.

Theorem 1.7. *Let $p > 2$ be an odd prime and let \mathbb{F}/\mathbb{F}_p be a finite extension. Let*

$$\bar{\rho} : G_{\mathbb{Q}} \longrightarrow \text{GL}_2(\mathbb{F})$$

be an absolutely irreducible and odd Galois representation. Assume that the following assumptions on $\bar{\rho}$ hold.

1. *There is no character $\bar{\chi} : G_{\mathbb{Q}_p} \longrightarrow \mathbb{F}^{\times}$ such that the semisimplification of $\bar{\rho}|G_{\mathbb{Q}_p}$ is equal to $\bar{\chi} \oplus \bar{\chi}_{\text{cyc}} \bar{\chi}$.*
2. *There exists $\ell \nmid p$ such that $\bar{\rho}|G_{\mathbb{Q}_{\ell}}$ is a ramified extension*

$$0 \longrightarrow \mu \chi_{\text{cyc}}^{1-k/2} \longrightarrow \bar{\rho}|G_{\mathbb{Q}_{\ell}} \longrightarrow \mu \chi_{\text{cyc}}^{-k/2} \longrightarrow 0$$

where $\mu : G_{\mathbb{Q}_{\ell}} \longrightarrow \{\pm 1\}$ is the non-trivial unramified quadratic character.

Let $f \in S_k(\Gamma_0(N))$ be a normalized eigencuspform of weight $k \geq 2$ such that $\bar{\rho}_f \simeq \bar{\rho}$. If $\ell \nmid N$, then the Iwasawa Main Conjecture (conjecture 1.5) holds for $M(f)$.

We draw the attention of the reader on three features of this result. First, note that the three main hypotheses of theorem 1.7 are requirements solely on $\bar{\rho}_f$, or equivalently on the residual representation $\bar{\rho}$. This means that the conclusion of the theorem may be replaced by the statement that the Iwasawa Main Conjecture holds for all eigencusforms of level Γ_0 which are points of a suitably defined universal deformation ring. Second, theorem 1.7 requires very mild hypotheses on $\bar{\rho}_f|G_{\mathbb{Q}_p}$ and none on $\rho_f|G_{\mathbb{Q}_p}$. This means that we may allow arbitrary bad reduction of $M(f)$ at p or, equivalently but from an automorphic point of view, that $\pi(f)_p$ is allowed to be supercuspidal. The generality allowed in the reduction type at p is in contrast with previous works on the topic, which applied only to modular points which were in addition ordinary or crystalline with low weight at p . Given a fixed residual representation $\bar{\rho}$ and a fixed deformation type Σ such that any modular point of $\text{Spec } R_{\Sigma}(\bar{\rho})[1/p]$ belongs to $S_k(\Gamma_0(M))$ for some $M \in \mathbb{N}$ exactly divisible by ℓ , the set of modular points of $\text{Spec } R_{\Sigma}(\bar{\rho})[1/p]$ which are ordinary or crystalline with low weight

²As is recalled in the proof of lemma 3.13, the assumptions of theorem 1.6 are enough to deduce theorem 1.4; see also the discussion after [116, Theorem 1.4].

at p has positive codimension in the deformation space $\text{Spec } R_\Sigma(\bar{\rho})[1/p]$ whereas the set of all modular points is Zariski-dense. In that sense, previous results on the Iwasawa Main Conjecture did not apply to almost all modular points of $\text{Spec } R_\Sigma(\bar{\rho})$. Theorem 1.7 applies to all of them. Finally, we do not make any assumption on the μ -invariant of f . This is in contrast to [36, 92] which contains results similar to theorem 1.7 under the hypothesis that the μ -invariant of f vanishes, an assumption that is expected to always hold in our setting but which at the moment invariably resisted proof (we describe the role of this hypothesis and the method used to avoid it in the description of the strategy of the proof in section 1.2.2 below).

In [66, Conjecture 3.2.2], K.Kato formulated what he called the generalized Iwasawa Main Conjecture on the p -adic variation of Iwasawa Main Conjectures as a motive varies in a p -adic family. Our second main theorem is that this conjecture holds for the universal three-variable power series family of deformations of modular residual representations. In spirit, it says that the Iwasawa Main Conjecture is not only true at each individual modular points of the universal deformation space of a modular residual Galois representation $\bar{\rho}_f$ but also that it varies continuously on that space. Because the statement of [66, Conjecture 3.2.2] for universal families is itself quite complicated, we refer to section 3.4 in the body of the text for the precise meaning of the objects involved in the theorem below.

Theorem 1.8 ([66] Conjecture 3.2.2 for universal p -adic families of modular forms). *Let $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F})$ be a residual Galois representation which satisfies the hypotheses of theorem 1.7. For Σ a finite set of primes, let T_Σ be the universal deformation of $\bar{\rho}$ unramified outside Σ . We view T_Σ as a free module of finite rank over the power-series ring $\Lambda \simeq \mathcal{O}[[X_1, X_2, X_3]]$. Let*

$$\mathbf{z}_\Sigma : T_\Sigma(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_\Sigma)$$

be the zeta morphism of theorem 3.6 below. Then \mathbf{z}_Σ induces an isomorphism

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \xrightarrow{\text{can}} \Lambda.$$

Equivalently, [66, Conjecture 3.2.2] is true for the triple $(\text{Spec } \mathbb{Z}[1/\Sigma], \Lambda, T_\Sigma)$.

In order to help the reader get a better sense of the meaning of this result, we make two remarks. First, if the generalized Iwasawa Main Conjecture is true, then conjecture 1.5 is also true except possibly for modular points in a closed subset of positive codimension. In particular, theorem 1.8 implies theorem 1.7 for most modular points. Second, when the deformation problem for $\bar{\rho}$ is unobstructed, then the universal deformation ring of $\bar{\rho}$ is a power-series ring and theorem 1.8 is the most general statement of the Iwasawa Main Conjecture possible.

We record two corollaries of our main theorem. Write V for $M(f)_{\text{et}, p}(k/2)$ and let $T \subset V$ be a $G_{\mathbb{Q}}$ -stable \mathcal{O} -lattice. For ℓ a rational prime, define $H_f^1(G_{\mathbb{Q}_\ell}, V)$ as in [9, §3] and put

$$\text{Sel}_{\mathbb{Q}}(f) = \ker \left(H^1(G_{\mathbb{Q}}, V/T) \longrightarrow \prod_{\ell} \frac{H^1(G_{\mathbb{Q}_\ell}, V/T)}{\text{Im} \left(H_f^1(G_{\mathbb{Q}_\ell}, V) \right)} \right).$$

Corollary 1.9. *Let $f \in S_k(\Gamma_0(N))$ be a normalized eigencuspform satisfying all the hypotheses of theorem 1.7. Then $\text{Sel}_{\mathbb{Q}}(f)$ is a finite group if and only if $L(f, k/2) \neq 0$.*

Corollary 1.9 is a converse of [68, Theorem], which established that $\text{Sel}_{\mathbb{Q}}(f)$ is a finite group if $L(f, k/2) \neq 0$.

Corollary 1.10. *Let A/\mathbb{Q} be an abelian variety of GL_2 -type of conductor N whose associated weight 2 cusp form is f . Assume that $L(A, 1) \neq 0$ and that the Galois representation $A[p]$ satisfies all the hypotheses of theorem 1.7. Then*

$$v_p(L(A, 1)/\Omega_f) = v_p \left(|\text{III}(A/\mathbb{Q})[p^\infty]| \prod_{q|N} \text{Tam}_q(A/\mathbb{Q}) \right).$$

Equivalently, the p -part of the Birch and Swinnerton-Dyer Conjecture for A holds.

Note that the period appearing in the above corollary is the one for the associated modular form. If A is an elliptic curve over \mathbb{Q} , then this period is the same as the Neron period up to the Manin constant. In an unpublished work of B.Edixhoven, it is shown that the prime divisors of the Manin constant are equal to 2,3,5 or 7. We also note that it would be very interesting to compare and combine the results of this manuscript with the remarkable results obtained in [26].

1.2 Outline of the proof

The proof of theorems 1.7 and 1.8 rely extensively on techniques of p -adic interpolation on various p -adic families of Galois representations, including the universal deformation ring $R_\Sigma(\bar{\rho})$ of the residual representation $\bar{\rho}$.

Under our hypotheses, $R_\Sigma(\bar{\rho}_f)$ is known to be a reduced, complete intersection, local ring of relative dimension 3 over \mathbb{Z}_p isomorphic to a p -adic Hecke algebra $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$. Moreover, $\text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma[1/p]$ contains a Zariski-dense subset \mathcal{X}^{sm} which contains the set of classical points corresponding to classical eigencuspforms satisfying strong supplementary properties at p : for $x \in \mathcal{X}^{\text{sm}}$, $\rho_x|G_{\mathbb{Q}_p}$ is either ordinary or crystalline and short (depending on whether $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible or irreducible). As it is a complete intersection local ring of relative dimension 3, $R_\Sigma(\bar{\rho}) \simeq \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ is a finite, free Λ -module for Λ a power-series ring in three variables. It turns out that the existence of \mathcal{X}^{sm} suggests a preferential choice of such a Λ -structure, that is to say an especially convenient explicit description of the morphism $\Lambda \hookrightarrow \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$. The map

$$\begin{array}{ccc} \text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma & & \\ \downarrow & & \\ \text{Spec } \Lambda & & \end{array}$$

corresponding to this Λ -structure may well be ramified in general, and may be so even at points corresponding to classical eigencuspforms, but it is not ramified at the points of $\text{Spec } \Lambda$ below the points in \mathcal{X}^{sm} .

The general strategy of the proof of conjecture 1.5 for a modular point $\rho_f \in \text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma[1/p]$ is then as follows. First, we prove the two-variable Greenberg-Iwasawa Main Conjecture for the Rankin-Selberg L -function attached to a point in \mathcal{X}^{sm} by realizing the p -adic L -function involved in this conjecture as the constant term of a Hida family of Eisenstein series for the reductive group $U(3,1)$. Second, we show that this two-variable main conjecture for $x \in \mathcal{X}^{\text{sm}}$ implies conjecture 1.5 for the same x (this requires a delicate p -integrality property of p -adic periods, as the p -adic period appearing in the interpolation property of the Greenberg-Iwasawa Main Conjecture is not the same as the one involved in conjecture 1.5 or, to phrase the problem differently, since special values of classical points on the cyclotomic line are not interpolated by the Greenberg Rankin-Selberg p -adic L -function). Third, we use an Euler system argument to show that conjecture 1.5 holds at every point except those in a set of large codimension containing the points where $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ is ramified over Λ (at this stage, the set of points for which conjecture 1.5 has not yet been established may thus very well contain the point corresponding to our ρ_f of interest). Finally, a limit argument relying on the interpolation properties of fundamental lines and zeta elements on the universal deformation ring allows us to conclude that it holds for ρ_f as well.

We now explain each step in more details.

1.2.1 Classical points in \mathcal{X}^{sm}

For simplicity of exposition, we assume in this introduction that $\bar{\rho}|G_{\mathbb{Q}_p}$ is absolutely irreducible and that $x \in \mathcal{X}^{\text{sm}}$ is attached to a classical eigencuspform f such that $\rho_f|G_{\mathbb{Q}_p}$ is a crystalline representation in the image of the Fontaine-Laffaille functor ([40]). The general strategy of the proof here is similar to that of previous works of the second named author in the case of supersingular elliptic curves ([125]): the Iwasawa Main Conjecture for the motive $M(f)$ is deduced from the two-variable Greenberg-Iwasawa Main Conjecture for Rankin-Selberg L -functions as studied in [126].

Choose a quadratic imaginary field \mathcal{K}/\mathbb{Q} such that p splits as $v_0\bar{v}_0$. Let $\mathcal{K}_\infty/\mathcal{K}$ be the \mathbb{Z}_p^2 -extension of \mathcal{K} . Let $\Gamma_{\mathcal{K}}$ be the Galois group $\text{Gal}(\mathcal{K}_\infty/\mathcal{K})$ and let $\Lambda_{\mathcal{K}}$ be the Iwasawa algebra $\mathbb{Z}_p[[\Gamma_{\mathcal{K}}]]$. For simplicity of notations, we assume in this introduction that the relevant objects are defined over the two-variable power-series ring $\Lambda_{\mathcal{K}}$ even though in fact an extension of the coefficients to the ring of integers of the maximal unramified extension of a finite extension of \mathbb{Q}_p might be required. Write Γ^\pm for the rank one over \mathbb{Z}_p submodules of $\Gamma_{\mathcal{K}}$ on which the action of the complex conjugation c is given by ± 1 . If g is a modular form with complex multiplication by \mathcal{K} , then it is ordinary at p . If in addition the weight of g is greater than the weight of f , then the Galois representation attached to $f \otimes g$ satisfies Panchishkin's condition ([49]) so the Iwasawa theory of the Rankin-Selberg product $f \otimes g$ is strongly similar to the theory of Iwasawa theory for ordinary forms. Accordingly, the statement and proof of the Iwasawa Main Conjecture for the seemingly more complicated object $f \otimes g$ is more accessible than their counterparts for f . Moreover, the Rankin-Selberg L -function $L(f \otimes g, s)$ is related to the original L -function of f . In that case, we deduce results on the Iwasawa theory of the original form f from the Iwasawa theory of Rankin-Selberg L -functions developed in [126].

We first recall the setting of the two-variable Iwasawa Main Conjecture for Rankin-Selberg products. Let $\mathcal{K} \subset_f F \subset \mathcal{K}_\infty$ be a finite subextension and let v be a finite place of \mathcal{O}_F . The Greenberg local condition $H^1(G_{F_v}, A) \subset H^1(G_{F_v}, A)$ at v is defined to be

$$\text{Im} \left(H^1(G_{F_v}/I_v, V^{I_v}) \longrightarrow H^1(G_{F_v}/I_v, A^{I_v}) \right) \subset H^1(G_F, A)$$

if $v \nmid p$, to be $H_{\text{Gr}}^1(G_{F_v}, A)$ if $v|v_0$ and to be 0 if $v|\bar{v}_0$. The Greenberg Rankin-Selberg Selmer group of f is the $\Lambda_{\mathcal{K}}$ -module

$$\text{Sel}_{\mathcal{K}}^{\text{Gr}}(f) \stackrel{\text{def}}{=} \varinjlim_{\mathcal{K} \subset_f F \subset \mathcal{K}_\infty} \ker \left(H^1(G_{F,\Sigma}, A) \longrightarrow \bigoplus_{v \in \Sigma} H^1(G_{F_v}, A)/H_{\text{Gr}}^1(G_{F_v}, A) \right).$$

We write

$$X_{\mathcal{K}}^{\text{Gr}}(f) \stackrel{\text{def}}{=} \text{Hom} \left(\text{Sel}_{\mathcal{K}}^{\text{Gr}}(f), \mathbb{Q}_p/\mathbb{Z}_p \right)$$

for the Pontryagin dual of $\text{Sel}_{\mathcal{K}}^{\text{Gr}}(f)$. On the analytic side, the object corresponding to $X_{\mathcal{K}}^{\text{Gr}}(f)$ is the Greenberg Rankin-Selberg p -adic L -function $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f) \in \text{Frac}(\Lambda_{\mathcal{K}})$ constructed in [32] which interpolates certain special values of the Rankin-Selberg L -functions $L(f \otimes \xi, s)$ attached to certain characters of $\Gamma_{\mathcal{K}}$ (see definition 4.34 and proposition 7.19 for details). The two-variable Greenberg-Iwasawa Rankin-Selberg Main Conjecture is then the following statement.

Conjecture 1.11 (Greenberg Main Conjecture). *The $\Lambda_{\mathcal{K}}$ -module $X_{\mathcal{K}}^{\text{Gr}}(f)$ is torsion, $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$ belongs to $\Lambda_{\mathcal{K}}$ and there is an equality of ideals*

$$\text{char}_{\Lambda_{\mathcal{K}}} \left(X_{\mathcal{K}}^{\text{Gr}}(f) \right) = \left(\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f) \right).$$

As we mentioned already, conjecture 1.11 is (relatively) more tractable than conjecture 1.5 when the Galois representation attached to $f \otimes g$ satisfies Panchishkin's condition, that is to say when the weight of g is greater than the weight of f . As conjecture 1.5 corresponds to the choice of g trivial, it *cannot* follow obviously from conjecture 1.11. Nevertheless, when f is of weight 2, it is proved in [126, 20] under some hypotheses that the inclusion of ideals

$$\text{char}_{\Lambda_{\mathcal{K}}} \left(X_{\mathcal{K}}^{\text{Gr}}(f) \right) \subset \left(\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f) \right)$$

holds after inverting p . In [125], the second named author then developed some local \pm -theory in a style similar to S.Kobayashi and B.D.Kim and used the explicit reciprocity law for Beilinson-Flach element and Poitou-Tate exact sequence to deal with the remaining power of p and to deduce conjecture 1.5 from conjecture 1.11.

Our main result regarding that topic is the following generalization to arbitrary even weight (theorem 4.41 in the body of the manuscript).

Theorem 1.12. *Let $f \in S_k(\Gamma_0(N))$ be an eigencuspform of even weight k satisfying the following assumptions.*

1. $\bar{\rho}_f|G_{\mathcal{K}}$ is absolutely irreducible.

2. The local representation $\rho_f|_{G_{\mathbb{Q}_p}}$ is crystalline. Moreover assume either $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible or f is ordinary at p .
3. There exists $q||N$ (in particular $q \nmid p$) which is not split in \mathcal{K} .
4. If $q|N$ is not split in \mathcal{K} , then $q||N$.

Then the following inclusion of ideals of $\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]$ holds

$$\text{char}_{\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]}(X_{\mathcal{K}}^{\text{Gr}}(f) \otimes_{\mathcal{O}} \mathcal{O}^{\text{ur}}) \subseteq (\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f))$$

up to height-one primes which are pullbacks of primes in $\mathcal{O}[[\Gamma^+]]$. Assume in addition that the following assumption holds.

5. If $\ell|N$ is not split in \mathcal{K} , then ℓ is ramified in \mathcal{K} and $\pi(f)_{\ell}$ is a special Steinberg representation twisted by χ_{ur} for χ_{ur} the unramified character sending ℓ to $(-1)\ell^{\frac{k}{2}-1}$.

Then

$$\text{char}_{\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]}(X_{\mathcal{K}}^{\text{Gr}}(f) \otimes_{\mathcal{O}} \mathcal{O}^{\text{ur}}) \subseteq (\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f))$$

holds.

To prove theorem 1.12 for modular forms of general even weight, the difficulty is twofold. First of all, the proof of the Greenberg-Iwasawa Main Conjecture in [125] when f has weight 2 relies on a computation of the Fourier-Jacobi expansion of an Eisenstein series. When f has higher weight, one needs to compute the Fourier-Jacobi expansion for vector valued Eisenstein series, which seems formidable. Secondly we do not have in general an explicit enough local theory as in the \pm case (except in the special case when $a_p = 0$, and for elliptic curves over \mathbb{Q} but $a_p \neq 0$ by I.Sprung), while the work [125] used such a theory in a crucial way.

To prove this theorem, we use the full strength of the joint work [32] of the second named author and E.Eischen. In this work, vector-valued Klingen Eisenstein families on $U(3, 1)$ are constructed from pullbacks of nearly holomorphic Siegel Eisenstein series on $U(3, 3)$. Combining the earlier works [125, 126] on the p -adic property for Fourier-Jacobi coefficients with the general theory of T.Ikeda ([61]) shows that the Fourier-Jacobi coefficient of nearly holomorphic Siegel Eisenstein series can be expressed as finite sums of products of Eisenstein series and theta functions on the Jacobi group containing $U(2, 2)$ (see lemma 7.26 below³). In the scalar valued case the Siegel Eisenstein series on $U(3, 3)$ is holomorphic, and the local Fourier-Jacobi coefficient at the Archimedean place can be explicitly expressed as the product of a Siegel section on $U(2, 2)$ and a Schwartz function. In general, it is difficult to compute Archimedean local Fourier-Jacobi coefficients explicitly. We fix one Archimedean weight and vary the p -adic nebentypus in families. Instead of computing the Archimedean Fourier-Jacobi integral, we can use a conceptual argument to factor out a finite sum of Archimedean integrals out of it and prove the factor is non-zero. After this, we can apply Ichino's triple product formula to prove that a certain Fourier-Jacobi coefficient is co-prime to the p -adic L -function we study. This also allows us to remove the square-free conductor assumption for f in the previous works of the second named author.

With theorem 1.12 in hand, we prove the following theorem, which establishes conjecture 1.5 at points in \mathcal{X}^{sm} .

Theorem 1.13. *Let $f \in S_k(\Gamma_0(N))$ be a normalized eigencuspform of weight $k \geq 2$ satisfying the following hypotheses.*

1. The $G_{\mathbb{Q}_p}$ -representation $\rho_f|_{G_{\mathbb{Q}_p}}$ is crystalline (equivalently $p \nmid N$) and short.
2. The local residual representation $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible.
3. There exists $\ell||N$ such that $\dim_{\mathbb{F}} \bar{\rho}^{I_{\ell}} = 1$ and $\dim_{\mathbb{F}} \bar{\rho}^{G_{\mathbb{Q}_{\ell}}} = 0$.

Then conjecture 1.5 holds for $M(f)$.

The proof of theorem 1.13 is quite different from the one in [125]. We avoid the search for a two-variable integral local theory analogous to the \pm theory. Our main innovation is to study unramified local Iwasawa theory along an appropriately chosen \mathbb{Z}_p -line in $\text{Spec } \Lambda_{\mathcal{K}}$. Suppose first that the image of ρ_f contains $\text{SL}_2(\mathbb{Z}_p)$, in which case we know an upper bound

³We thank T.Ikeda for showing us a simple argument using lowest weight representations giving the proof of this lemma.

on the Selmer group by the works of Kato. After carefully studying the control theorem, proving the full equality amounts to the computation of the cardinality (which is finite) of the Selmer group for $T_f(-\frac{k-2}{2})$ twisted by some appropriately defined generic finite order character of Γ_{Iw} . In order to do so, we consider deformations of the corresponding Galois representation along a one-variable family which corresponds to the \mathbb{Z}_p -extension of \mathcal{K} that is totally ramified at \bar{v}_0 and unramified at v_0 . In this family, we do have a nice integral local theory at v_0 , which we develop in Section 4.2 using p -adic Hodge theory. Roughly speaking, the integral local subspace of interest is generated by the image of the canonical differential class through the exponential map, the crucial point being the boundedness of the denominators involved in the explicit description of the exponential map (proposition 4.12 and corollary 4.13). This local subspaces replaces the \pm -theory and is enough to prove the sought-for bound along this one dimensional family (it seems to us this one-dimensional line is the only choice of p -adic family containing our original point of interest for which the argument works). We note that a deformation-theoretic argument is necessary in this approach: we can not only look at one point on the cyclotomic line as we are unable to rule out the possibility that the Greenberg type p -adic L -function is identically 0 on the cyclotomic line.

The shortcoming of the approach as described is that it does not prove anything without the assumption that the image of ρ_f contains $\text{SL}_2(\mathbb{Z}_p)$ since in this case Kato only proved the upper bound for Selmer groups after inverting p . Nevertheless, we can still prove the Main Conjecture after inverting p . Our idea is to use the analytic Iwasawa theory of J.Pottharst ([103]) and Iwasawa theory for (φ, Γ) -modules (upgraded to a two variable setting), in the context of Nekovář-Selmer complexes ([93]). It turns out that Pottharst's trianguline-ordinary theory and its more flexible analytic Iwasawa theory setting allow similar proofs as in the ordinary case. In the two-variable setting there are subtleties to take care of – for example there is a finite set of height one primes where the regulator map vanishes. Also we need to compare differently constructed analytic p -adic L -functions: although the two p -adic L -functions in question agree on all arithmetic points, this does not suffice to uniquely determine them as analytic functions due to the growth conditions.

We prove the theorem first under the assumption that the Satake parameters of f at p are distinct. This assumption on the Satake parameter is conjecturally automatic. We finally remove this assumption in Section 4.7 by a trick of comparing with the three variable p -adic L -function for Bianchi modular forms constructed by A.Betina and C.Williams ([8]).

Remark: When we do not know that the image of ρ_f contains a subgroup conjugated to $\text{SL}_2(\mathbb{Z}_p)$, our argument actually shows one divisibility for powers of p . More precisely, it shows that the strict Selmer group is bounded below by the index of $\mathbf{z}(f)_{\text{Iw}}$ in the integral Iwasawa cohomology in the case when the zeta element is indeed inside integral Iwasawa cohomology (if not then this statement is empty).

1.2.2 From \mathcal{X}^{sm} to all classical points

We continue our outline of the proof of theorem 1.7. The second main step is to state a version of the Iwasawa Main Conjecture with coefficients in $R_{\Sigma}(\bar{\rho}_f)$. The fundamental idea, which has been known since [66], is to express the Iwasawa Main Conjecture as a description of the image of the fundamental line through a specific zeta morphism. Defining a fundamental line over the full universal deformation space which specializes to the classical fundamental lines at classical points is well-known ([44]), the difficulty therefore lies in the definition of the zeta morphism. Such a zeta morphism has been recently constructed over the universal deformation space by K.Nakamura ([92]) using completed cohomology, the Bernstein derivative functor and the results of V.Paskunas on the explicit description of the so-called Montréal functor of P.Colmez ([25, 96]). The key property of this zeta morphism is that it specializes to the zeta morphism constructed in [68] at classical points of the universal deformation space. For our purposes, we need a slight strengthening of these results which relate the zeta morphism defined over the full universal deformation space to the ones defined only on irreducible components thereof but which keep track of the Euler factors at primes of bad reduction. This requires a technical result on newvectors in p -adic families of automorphic representations of $\text{GL}_2(\mathbb{Q}_{\ell})$ ($\ell \nmid p$) which is certainly well-known to experts but for which we provide a proof in appendix 6 for the convenience of the reader.

Let $\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma$ be the p -adic Hecke algebra isomorphic to the universal deformation ring $R_\Sigma(\bar{\rho}_f)$, let $R(\mathfrak{a})$ be a quotient of $\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma$ by a minimal prime ideal \mathfrak{a} , and let $\lambda(f) : R(\mathfrak{a}) \rightarrow \mathcal{O}_{\text{Iw}}$ be a modular point of $R(\mathfrak{a})$. We write T_Σ for the universal Galois representation with coefficients in $\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma$, $T(\mathfrak{a})$ for $T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma} R(\mathfrak{a})$ and $T(f)_{\text{Iw}}$ for $T(\mathfrak{a}) \otimes_{R(\mathfrak{a}), \lambda(f)} \mathcal{O}_{\text{Iw}}$. Let $\Delta_\Sigma(T_\Sigma), \Delta(T(\mathfrak{a}))$ and $\Delta(f)_{\text{Iw}}$ the fundamental lines of the Galois representations $T_\Sigma, T(\mathfrak{a})$ and $T(f)_{\text{Iw}}$ respectively. Finally, let $Q(\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma)$ be the total quotient ring of the reduced ring $\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma$. The existence of the universal zeta morphism \mathbf{z}_Σ of K.Nakamura can be reformulated as the existence of an isomorphism

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma \xrightarrow{\sim} \frac{x}{y} \mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma \quad (1.2.1)$$

between the fundamental line Δ_Σ and an invertible module $\frac{x}{y} \mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma$ inside $Q(\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma)$ which specializes to the isomorphism

$$\text{triv}_{\mathbf{z}(f)_{\Sigma, \text{Iw}}} : \Delta_\Sigma(f) \xrightarrow{\sim} \frac{x(f)}{y(f)} \mathcal{O}_{\text{Iw}}$$

constructed in [68, Theorem 12.5] (recalled as theorem 3.4 below) at each classical point. The universal Iwasawa Main Conjecture is then the statement that $\text{triv}_{\mathbf{z}_\Sigma}$ is an isomorphism

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma \xrightarrow{\sim} \mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma \subset Q(\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma)$$

or equivalently that the invertible module $\frac{x}{y} \mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma$ of (1.2.1) is $\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma$ itself. Likewise, the existence of the universal morphism $\mathbf{z}(\mathfrak{a})$ with coefficients in $R(\mathfrak{a})$ can be reformulated as the existence of an isomorphism

$$\text{triv}_{\mathbf{z}(\mathfrak{a})} : \Delta(T(\mathfrak{a})) \xrightarrow{\sim} \frac{x(\mathfrak{a})}{y(\mathfrak{a})} R(\mathfrak{a})$$

for some invertible ideal $\frac{x(\mathfrak{a})}{y(\mathfrak{a})} R(\mathfrak{a})$ inside $\text{Frac}(R(\mathfrak{a}))$ and the Iwasawa Main Conjecture with coefficients in $R(\mathfrak{a})$ is the statement that $\text{triv}_{\mathbf{z}(\mathfrak{a})}$ is an isomorphism

$$\text{triv}_{\mathbf{z}(\mathfrak{a})} : \Delta(T(\mathfrak{a})) \xrightarrow{\sim} R(\mathfrak{a}).$$

A crucial step in establishing theorem 1.7 is the proof that, independently of the truth of the various Iwasawa Main Conjectures, there exists a pair of commutative diagrams

$$\begin{array}{ccc} \Delta_\Sigma(T_\Sigma) & \xrightarrow{\text{triv}_{\mathbf{z}_\Sigma}} & \frac{x}{y} \mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma \\ - \otimes_{\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma} R(\mathfrak{a}) \downarrow & & \downarrow \lambda \\ \Delta(T(\mathfrak{a})) & \xrightarrow{\text{triv}_{\mathbf{z}(\mathfrak{a})}} & \frac{x(\mathfrak{a})}{y(\mathfrak{a})} R(\mathfrak{a}) \end{array} \quad (1.2.2)$$

and

$$\begin{array}{ccc} \Delta(T(\mathfrak{a})) & \xrightarrow{\text{triv}_{\mathbf{z}(\mathfrak{a})}} & \frac{x(\mathfrak{a})}{y(\mathfrak{a})} R(\mathfrak{a}) \\ - \otimes_{R(\mathfrak{a}), \lambda_f} \mathcal{O}_{\text{Iw}} \downarrow & & \downarrow \lambda_f \\ \Delta(f)_{\text{Iw}} & \xrightarrow{\text{triv}_{\mathbf{z}(f)_{\text{Iw}}}} & \frac{x(f)}{y(f)} \mathcal{O}_{\text{Iw}}. \end{array} \quad (1.2.3)$$

A way to understand this statement is to interpret it as a perfect control theorem for fundamental lines. In particular, we note that such results seem to be inaccessible for characteristic ideals of Selmer modules, not only because such objects are not defined when the ring of coefficients is not known to be normal (as is the case for $\mathbf{T}_{\mathfrak{m}\bar{\rho}}^\Sigma$ and $R(\mathfrak{a})$) but also because they would require the knowledge that no non-zero pseudo-null submodules may occur in these Selmer modules, which in turn is expected only when the Selmer modules is related to a p -adic L -function, and thus only under strong supplementary hypotheses on the image of the decomposition group at p through the Galois representation. Another candidate in the formulation of the Iwasawa Main Conjectures which is defined more generally is the Fitting

ideal of the Selmer module. The formation of Fitting ideals commutes with arbitrary change of coefficients but unfortunately Fitting ideals are not principal ideals in general, whereas the proof of the commutativity of the diagrams (1.2.2) and (1.2.3) crucially relies on the properties of invertible modules. Consequently, it seems to us that this part of our proof relies on the use of fundamental lines, that is to say the images of perfect complexes through the determinant functor of [76]: only through this choice do we have invertible ideals whose formation commutes with arbitrary base change of coefficients. In that respect, it might be instructive to compare our results with for instance [92, Theorem 1.4]. From the perspective of this manuscript, the proof of that result amounts to the compatibility of the fundamental line with specialization at the maximal ideal under the hypothesis that the μ -invariant does not vanish. The vanishing of the μ -invariant appears however to be a mysterious and hard problem. Working consistently with fundamental lines and relying on the commutativity of the diagrams (1.2.2) and (1.2.3) (and their generalizations to other specializations maps beside $\mathbf{T}_{\mathfrak{m}_\bar{\rho}}^\Sigma \rightarrow R(\mathfrak{a})$ and $R(\mathfrak{a}) \rightarrow \mathcal{O}_{\text{Iw}}$) allows us to remove the hypothesis that the μ -invariant vanishes; indeed this was one our main motivation for working within this framework.

Ideally, the proof of the Iwasawa Main Conjecture would then proceed in the following way: using the fact that the image of $\text{triv}_{z(f)_\text{Iw}}$ is as expected by the Iwasawa Main Conjecture (namely \mathcal{O}_{Iw}) for suitably chosen points in \mathcal{X}^{sm} , we would prove by descent that similarly the images of triv_{z_Σ} and $\text{triv}_{z(\mathfrak{a})}$ are respectively $\mathbf{T}_{\mathfrak{m}_\bar{\rho}}^\Sigma$ and $R(\mathfrak{a})$ as expected, or equivalently that the Iwasawa Main Conjecture holds over these spaces. Then, specializing this time to a classical point outside \mathcal{X}^{sm} , we would conclude that it holds for all classical points. This strategy, however, does not quite work as described. The problem is that its first step is to deduce a universal Iwasawa Main Conjecture from the knowledge of Iwasawa Main Conjectures at given points (here the set of points considered is \mathcal{X}^{sm} but the objection below applies to any set of classical points). The rings $\mathbf{T}_{\mathfrak{m}_\bar{\rho}}^\Sigma$ and $R(\mathfrak{a})$, however, are not known to be factorial. As an invertible module over a non-factorial ring can very well be non-trivial even if it becomes trivial at all classical points, we may not prove a universal main conjecture in this way. To bypass this difficulty, we view T_Σ as a finite, free module of \mathbf{A} and formulate the Iwasawa Main Conjecture over this ring. As \mathbf{A} is regular, the descent argument sketched above can be carried out with coefficients in \mathbf{A} . However, a new problem now arises in that proving the Iwasawa Main Conjecture at a point x now means proving it at point $a \in \text{Spec } \mathbf{A}$ and not anymore at a point of $\text{Spec } \mathbf{T}_{\mathfrak{m}_\bar{\rho}}^\Sigma$. The former is significantly harder than the latter since the contributions to the Iwasawa Main Conjecture at x of various two-dimensional Galois representations corresponding to the various points of $\mathbf{T}_{\mathfrak{m}_\bar{\rho}}^\Sigma$ over x could be entangled in a complicated way. Using the fact that points in \mathcal{X}^{sm} are unramified over \mathbf{A} and the precise structure of $\mathbf{T}_{\mathfrak{m}_\bar{\rho}}^\Sigma$ as \mathbf{A} -algebra, we show first that the Iwasawa Main Conjecture for points of \mathcal{X}^{sm} entails the Iwasawa Main Conjecture with coefficients in \mathbf{A} (theorem 1.8), and then that the Iwasawa Main Conjecture with coefficients in \mathbf{A} entails the Iwasawa Main Conjecture at all classical points except those which are ramified over \mathbf{A} .

This is not quite the end of the proof still, as our original classical point of interest ρ_f might very well be ramified over \mathbf{A} , and in that case it is far from obvious that theorem 1.7 for $M(f)$ can be deduced from 1.8. Using a limit argument, we show that there are many points nearby ρ_f which satisfy the Iwasawa Main Conjecture and which are unramified over \mathbf{A} . In that situation, we can show that even if the Iwasawa Main Conjecture is false over the full deformation space $\mathbf{T}_{\mathfrak{m}_\bar{\rho}}^\Sigma$, it must be true for $M(f)$. The precise working of the proof crucially relies on the fact that both the fundamental lines and the zeta morphisms admit a version with and without Euler factors, and that both commute with specialization at a classical point, even if this classical point is ramified over \mathbf{A} .

Notations

All rings are assumed to be commutative (and unital). The residual field at a prime ideal $\mathfrak{p} \in \text{Spec } A$ is denoted by $\kappa(\mathfrak{p})$. If A is reduced, its total quotient ring is denoted by $Q(R)$. If A is a domain, its fraction field is denoted by $\text{Frac}(A)$. If A is a local ring, we denote by \mathfrak{m}_A its maximal ideal. If F is a field, we denote by G_F the Galois group of a separable closure of F . If F is a number field and Σ is a finite set of finite places of \mathbb{Q} , we denote by F_Σ the maximal Galois extension of F unramified outside $\Sigma \cup \{v|\infty\}$ and by $G_{F,\Sigma}$ the Galois group $\text{Gal}(F_\Sigma/F)$. The ring of integer of F is written \mathcal{O}_F . If v is a finite place of F , we denote by

$\mathcal{O}_{F,v}$ the unit ball of F_v , by ϖ_v a fixed choice of uniformizing parameter of $\mathcal{O}_{F,v}$ and by k_v is the residual field of $\mathcal{O}_{F,v}$. The reciprocity law of local class field theory is normalized so that ϖ_v is sent to (a choice of lift of) the geometric Frobenius morphism $\text{Fr}(v)$. Let E/\mathbb{Q}_p be a finite extension with unit ball \mathcal{O} and residual field \mathbb{F} . We fix a uniformizing parameter ϖ of E .

For G a topological group, a G -representation (T, ρ, A) is an A -module T free of finite rank together with a continuous morphism

$$\rho : G \longrightarrow \text{Aut}_A(T).$$

Let $\mathbb{Q}(\zeta_{p^\infty})$ be the extension of \mathbb{Q} generated by all roots of unity of order a power of p . Then $\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}$ is an abelian extension and $\Gamma \stackrel{\text{def}}{=} \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ is isomorphic to \mathbb{Z}_p^\times through the cyclotomic character χ_{cyc} . We say that a character $\chi : \Gamma \longrightarrow \mathbb{C}_p^\times$ is classical if there exist a finite order character ε and an integer $n \in \mathbb{Z}$ such that $\chi = \varepsilon \chi_{\text{cyc}}^n$ (so the classical characters are Γ are the de Rham characters of $G_{\mathbb{Q}_p}$ seen as characters of Γ). We write Γ_{Iw} for the largest subquotient of Γ isomorphic to \mathbb{Z}_p . Define \mathbb{Q}_n/\mathbb{Q} to be the subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ with Galois group G_n over \mathbb{Q} equal to $\mathbb{Z}/p^n\mathbb{Z}$.

2 Generalities on p -adic Hecke algebras

2.1 p -adic Hecke algebras

We write U^p (resp. U_p) for a compact open subgroup of $\text{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(p\infty)})$ (resp. of $\text{GL}_2(\mathbb{Q}_p)$) and we denote by $U \subset \text{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ a compact open subgroup of the form $U_p U^p$. If $\ell \nmid p$ is a finite prime, we write U_ℓ for the compact open subgroup of $\text{GL}_2(\mathbb{Q}_\ell)$ which is the local component of U . We denote by $\Sigma(U)$ the finite set of finite primes whose elements are p and the rational primes $\ell \nmid p$ such that U_ℓ is not a maximal open compact subgroup. If $U = U_p U^p$, $\Sigma(U)$ depends only on U^p .

For integers M and N , we denote by $U(M, N) \subset \mathbb{A}_{\mathbb{Q}}^{(\infty)}$ the compact open subgroup such that

$$U(M, N)_\ell = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_\ell) \mid a - 1, b \equiv 0 \pmod{\ell^{v_\ell(M)}} \text{ and } c, d - 1 \equiv 0 \pmod{\ell^{v_\ell(N)}} \right\}$$

for all ℓ . Denote by $U(N)$ the compact open subgroup $U(N, N)$ or equivalently such that

$$U(N)_\ell = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_\ell) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\ell^{v_\ell(N)}} \right\}$$

for all ℓ and by $U_1(N)$ the compact open subgroup $U(1, N)$ or equivalently such that

$$U_1(N)_\ell = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_\ell) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\ell^{v_\ell(N)}} \right\}$$

for all ℓ . By construction, the sets $\Sigma(U_p U(N))$ and $\Sigma(U_p U_1(N))$ are both equal to $\{\ell | Np\}$.

For $U \subset \text{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ a compact open subgroup as above, we denote by $Y(U)$ the affine modular curve of level U , that is to say the Shimura curve whose complex points are given by

$$Y(U)(\mathbb{C}) = \text{GL}_2(\mathbb{Q}) \backslash \left(\mathbb{C} - \mathbb{R} \times \text{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)}) \right) / U.$$

For simplicity, we write $Y(M, N)$ for $Y(U(M, N))$, $Y(N)$ for $Y(N, N)$ and $Y_1(N)$ for $Y(U_1(N))$. Note that if N and m are integers dividing M , then there is a covering of $\mathbb{Q}(\zeta_M)$ -schemes from $Y(m, M)$ onto $Y_1(N) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_m)$.

We write $H_{\text{et}}^i(U, -)$ for the i -th étale cohomology group functor $H_{\text{et}}^i(Y(U) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, -)$ and $H_c^i(U, -)$ for the i -th étale cohomology group with compact support functor $H_c^i(Y(U) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, -)$.

For Σ a finite set of finite primes containing $\Sigma(U)$, let $\mathbf{T}^\Sigma(U)$ be the p -adic Hecke algebra of endomorphisms of $H_{\text{et}}^1(U, \mathcal{O})$ generated as \mathcal{O} -algebra by the Hecke operators $T(\ell)$ and $S(\ell)$ for $\ell \notin \Sigma$.

Fix Σ as above and consider

$$\bar{\rho} : G_{\mathbb{Q}, \Sigma} \longrightarrow \text{GL}_2(\mathbb{F})$$

an absolutely irreducible and modular (hence odd) Galois representation unramified outside Σ . We say that a compact open subgroup $U \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ is allowable for $\bar{\rho}$ (or allowable, for short) if there exists a (necessarily unique) maximal ideal $\mathfrak{m}_{\bar{\rho}} \in \mathrm{Spec} \mathbf{T}^{\Sigma}(U)$ such that

$$\begin{cases} T(\ell) \bmod \mathfrak{m}_{\bar{\rho}} = \mathrm{tr} \bar{\rho}(\mathrm{Fr}(\ell)) \\ S(\ell) \bmod \mathfrak{m}_{\bar{\rho}} = \det \bar{\rho}(\mathrm{Fr}(\ell)) \end{cases}$$

holds for all $\ell \notin \Sigma$.

Fix $U' \subset U$ two allowable compact open subgroups. Denote respectively by $\mathfrak{m}'_{\bar{\rho}}$ and $\mathfrak{m}_{\bar{\rho}}$ the maximal ideals of $\mathbf{T}^{\Sigma'}(U')$ and $\mathbf{T}^{\Sigma}(U)$ attached to U' and U . Then the covering $Y(U') \rightarrow Y(U)$ induces by restriction a map $\mathbf{T}^{\Sigma'}(U')_{\mathfrak{m}'_{\bar{\rho}}} \rightarrow \mathbf{T}^{\Sigma}(U)_{\mathfrak{m}_{\bar{\rho}}}$ which is always a surjection and which is an isomorphism if U is sufficiently small. In the following, we always assume that U is allowable and sufficiently small in that sense. We write $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ for the common isomorphism class of Hecke algebras $\{\mathbf{T}^{\Sigma}(U)_{\mathfrak{m}_{\bar{\rho}}}\}_{\Sigma, U} / \simeq$ and $\mathfrak{m}_{\bar{\rho}}$ for the maximal ideal of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$. Even though we suppress this dependence from the notation, note that $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ does depend on the choice of U .

Henceforth, the set Σ is fixed once and for all. We write Σ^p for $\Sigma \setminus \{p\}$.

A specialization of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ is by definition a local ring morphism $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow S$. According to [16], there exists a $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ -module T_{Σ} free of rank 2 endowed with a continuous $G_{\mathbb{Q}, \Sigma}$ -action through the $G_{\mathbb{Q}, \Sigma}$ -representation

$$\rho_{\Sigma} : G_{\mathbb{Q}, \Sigma} \rightarrow \mathrm{Aut}_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}}(T_{\Sigma}) \simeq \mathrm{GL}_2(\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma})$$

uniquely characterized up to isomorphism by the requirement that $\mathrm{tr}(\rho_{\Sigma}(\mathrm{Fr}(\ell))) = T(\ell)$ for all $\ell \notin \Sigma$. To a specialization $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow S$ is attached the $G_{\mathbb{Q}, \Sigma}$ -representation $(T_{\lambda}, \rho_{\lambda}, S)$ defined by $\rho_{\lambda} = \lambda \circ \rho_{\Sigma}$ and characterized uniquely (up to isomorphism) by the fact that $\mathrm{tr}(\rho_{\lambda}(\mathrm{Fr}(\ell))) = \lambda(T_{\ell})$ for all $\ell \notin \Sigma$. If $\mathfrak{a} \in \mathrm{Spec}^{\min} \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ is a minimal prime ideal of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$, let $R(\mathfrak{a})$ be the quotient domain $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}/\mathfrak{a}$, let $\mathrm{Frac}(\mathfrak{a})$ be the field of fractions of $R(\mathfrak{a})$, let $T(\mathfrak{a})$ be the $G_{\mathbb{Q}, \Sigma}$ -representation $T_{\Sigma} \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}} R(\mathfrak{a})$ and let finally $V(\mathfrak{a})$ be the $G_{\mathbb{Q}, \Sigma}$ -representation $T_{\Sigma} \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}} \mathrm{Frac}(\mathfrak{a})$. If $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow A$ is a specialization with values in a domain, let V_{λ} be the representation $T_{\Sigma} \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}, \lambda} \mathrm{Frac}(A)$.

2.2 Commutative algebra properties of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$

2.2.1 Classical primes of the p -adic Hecke algebra

A prime ideal $\mathfrak{p} \subset \mathrm{Spec} \mathbf{T}^{\Sigma}$ with residual field $\mathbf{k}(\mathfrak{p}) \subset \bar{\mathbb{Q}}_p$ is said to be classical if there exists a (necessarily unique) normalized eigencuspform $f \in S_k(U)$ of weight $k \geq 2$ whose system of eigenvalues

$$\lambda_f : \mathbf{T}^{\Sigma} \rightarrow \bar{\mathbb{Q}}_p$$

coincides with the morphism $\mathbf{T}^{\Sigma} \rightarrow \mathbf{k}(\mathfrak{p})$. It is said to be classical up to a twist if there exists a normalized eigencuspform f of weight $k \geq 2$ and a classical character $\chi \in \hat{\Gamma}_{\mathrm{Iw}}$ such that the morphism $\mathbf{T}^{\Sigma} \rightarrow \mathbf{k}(\mathfrak{p})$ is the system of eigenvalues of $f \otimes \chi$. A specialization $\lambda : \mathbf{T}^{\Sigma} \rightarrow S$ is classical (resp. up to a twist) if it factors through a $\mathbf{T}^{\Sigma}/\mathfrak{p}$ for \mathfrak{p} classical (resp. up to a twist). We write f_{λ} or $f_{\mathfrak{p}}$ for the eigencuspform attached to λ . By definition, the $G_{\mathbb{Q}, \Sigma}$ representations ρ_{λ} and $\rho_{f_{\lambda}}$ are isomorphic. Let \mathfrak{p} be a prime of $\mathbf{T}^{\Sigma}[1/p]$ such that the morphism $\lambda : \mathbf{T}^{\Sigma}[1/p] \rightarrow \mathbf{k}(\mathfrak{p})$ is classical up to a twist. Then \mathfrak{p}, λ or ρ_{λ} are said to be crystalline if $D_{\mathrm{cris}}(\rho_{\lambda}|G_{\mathbb{Q}_p})$ is a $\mathbf{k}(\mathfrak{p})$ -vector space of dimension 2. They are said to be distinguished, crystalline if the Frobenius morphism φ acting on $D_{\mathrm{cris}}(\rho_{\lambda}|G_{\mathbb{Q}_p})$ has two distinct eigenvalues. They are said to be crystalline and short if the weight k of f_{λ} satisfies $2 \leq k \leq p$, or equivalently if $\rho|G_{\mathbb{Q}_p}$ is the image by the Fontaine-Laffaille functor of a rank 2 Fontaine-Laffaille module with non-trivial graded piece of the filtration in degree 0 and $k-1$ ([40, Théorème 8.4]). According to [110], if λ is crystalline, then there exist a compact open subgroup $U \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ with U_p a maximal compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$, $f \in S_k(U)$ a newform, $\eta : G_{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_p^{\times}$ a character unramified at p and $i \in \mathbb{Z}$ such that λ is the system of eigenvalues of $f \otimes \eta \chi_{\mathrm{cyc}}^i$.

All the terminology above (classical point, classical prime, crystalline point, crystalline prime...) is extended in the obvious way to $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ and its quotients.

2.2.2 Identification with universal deformation rings

We make the following assumption on $\mathfrak{m}_{\bar{\rho}}$.

Assumption 2.1. *The $G_{\mathbb{Q}, \Sigma}$ -representation $\bar{\rho}$ satisfies the following properties.*

1. If $p^* = (-1)^{(p-1)/2}p$, then $\bar{\rho}|G_{\mathbb{Q}(\sqrt{p^*})}$ is absolutely irreducible.
2. If $\bar{\rho}|G_{\mathbb{Q}_p}$ is an extension

$$0 \longrightarrow \chi_1 \longrightarrow \bar{\rho}|G_{\mathbb{Q}_p} \longrightarrow \chi_2 \longrightarrow 0,$$

$$\text{then } \chi_1^{-1}\chi_2 \notin \{1, \bar{\chi}_{\text{cyc}}\}.$$

If φ is an endomorphism of the $\mathbb{F}[G_{\mathbb{Q}_p}]$ -module of $\bar{\rho}|G_{\mathbb{Q}_p}$, then φ is scalar unless $(\bar{\rho}|G_{\mathbb{Q}_p})^{ss} \simeq \bar{\chi} \oplus \bar{\chi}$. Under assumption 2.1 then, the endomorphisms of $\bar{\rho}|G_{\mathbb{Q}_p}$ are scalar.

Let $D_{\bar{\rho}}$ be the deformation functor from the category of finite, local, $W(\mathbb{F})$ -algebras with residue field \mathbb{F} to the category of sets which attaches to A the set of isomorphism classes of pairs $\{(T, \rho, A), \iota\}$ such that (T, ρ, A) is a $G_{\mathbb{Q}, \Sigma}$ -representation and $\iota : \rho \otimes_A \mathbb{F} \simeq \bar{\rho}$ is an isomorphism of $\mathbb{F}[G_{\mathbb{Q}, \Sigma}]$ -module. It follows from statement 1 of assumption 2.1 that $D_{\bar{\rho}}$ is representable by a complete, local, noetherian ring $R_{\Sigma}(\bar{\rho})$. We denote by $(T_{\Sigma}^u, \rho_{\Sigma}^u, R_{\Sigma}(\bar{\rho}))$ the corresponding universal $G_{\mathbb{Q}, \Sigma}$ -representation. By definition, there is a map $R_{\Sigma}(\bar{\rho}) \rightarrow \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ which is surjective as its image contains the image of $\text{tr}(\rho_{\Sigma}^u(\text{Fr}(\ell)))$, which is $\text{tr}(\rho_{\Sigma}(\text{Fr}(\ell))) = T_{\ell}$ for all $\ell \notin \Sigma$.

The following lemma is well-known.

Lemma 2.2. *Suppose $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible. Then there exists a crystalline and short point $\lambda_f : \mathbf{T}^{\Sigma} \rightarrow \mathcal{O}$ such that $R_{\Sigma}(\bar{\rho}) \simeq R_{\Sigma}(\bar{\rho}_f)$.*

Proof. If $\chi : G_{\mathbb{Q}, \Sigma} \rightarrow W(\mathbb{F})^{\times}$ is a character, then $\rho \mapsto \rho \otimes \chi$ induces an isomorphism of functors $D_{\bar{\rho}} \rightarrow D_{\bar{\rho} \otimes \chi}$. Hence, it is enough to prove that there exists such a character χ and an eigencuspform f such that $\bar{\rho}_f \otimes \bar{\chi} \simeq \bar{\rho}$ and such that ρ_f is crystalline and short.

Let us denote by $I^w \subset I_p$ the wild ramification subgroup of $G_{\mathbb{Q}_p}$ and by $I^t \stackrel{\text{def}}{=} I_p/I^w \simeq \lim_{\leftarrow r} \mathbb{F}_{p^r}^{\times}$ the tame ramification quotient. Recall that a character $\psi : I^t \rightarrow \mathbb{F}^{\times}$ is fundamental if it is the composition of the natural map

$$I^t \simeq \lim_{\leftarrow r} \mathbb{F}_{p^r}^{\times} \longrightarrow \mathbb{F}_{p^n}^{\times}$$

with one of the n embeddings $\mathbb{F}_{p^n}^{\times} \hookrightarrow \mathbb{F}^{\times}$ and that it is of level n if \mathbb{F}_{p^n} is the smallest subfield through which ψ factors. Denote by ψ_i the two fundamental characters of level 2 of I^t . As I^w is a pro- p -group and I^t is commutative, $\bar{\rho}^{ss}|I^w$ is trivial and $\bar{\rho}^{ss}|I_p$ is the sum of two characters $\bar{\chi}_1, \bar{\chi}_2$ of I^t . Since $G_{\mathbb{Q}_p}/I_p$ acts by conjugation on I^t through the monodromy relation, $\bar{\chi}_1$ and $\bar{\chi}_2$ are permuted modulo p by elevation to the p -th power. Hence, the $\bar{\chi}_i$ are of level 1 or 2. This permutation is the identity if and only if the $\bar{\chi}_i$ are both of level 1, if and only if $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible. As we have assumed that $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible, the characters $\bar{\chi}_i$ are both of level 2. Then there is a unique pair (a, b) satisfying $0 \leq a < b \leq p-1$ such that

$$\bar{\rho}|I^t \simeq \begin{pmatrix} \psi_1^a \psi_2^b & 0 \\ 0 & \psi_1^b \psi_2^a \end{pmatrix} \simeq \begin{pmatrix} \psi_2^{b-a} & 0 \\ 0 & \psi_1^{b-a} \end{pmatrix} \otimes \bar{\chi}_{\text{cyc}}^a$$

Accordingly, $\bar{\rho}$ has Serre-weight $k = 1 + pa + b$ ([113, (2.2.4)]). Put $\bar{\rho}' = \bar{\rho} \otimes \bar{\chi}_{\text{cyc}}^{-a}$. Then $\bar{\rho}'|I^t$ may be written

$$\bar{\rho}'|I^t \simeq \begin{pmatrix} \psi_2^{b-a} & 0 \\ 0 & \psi_1^{b-a} \end{pmatrix}$$

and so has Serre-weight $k' = 1 + b - a$. Then $2 \leq k' \leq p$. According to the proof of the weight part of Serre's Conjecture ([31]), there exists an eigencuspform $f \in S_{k'}$ with level prime to p such that $\bar{\rho}_f$ is isomorphic to $\bar{\rho}'$. Then $\lambda_{f \otimes \bar{\chi}_{\text{cyc}}^a}$ is a crystalline point of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}[1/p]$ and so $R_{\Sigma}(\bar{\rho})$ is isomorphic to $R_{\Sigma}(\bar{\rho}_f)$.

□

Lemma 2.3. *There exists an integer $n \geq 4$ and elements (y_4, \dots, y_n) such that the ring $R_{\Sigma}(\bar{\rho})$ admits a presentation*

$$R_{\Sigma}(\bar{\rho}) \simeq \mathcal{O}[[X_1, \dots, X_n]]/(y_4, \dots, y_n).$$

Proof. See [11, Lemma 4.3]. \square

Proposition 2.4. *The universal deformation ring $R_\Sigma(\bar{\rho})$ is a flat \mathcal{O} -algebra which is a complete intersection ring of Krull dimension 4.*

Proof. Let k be the Serre-weight of $\bar{\rho}$. According to [31], there exists a classical prime $\mathfrak{p} \in \text{Spec } \mathbf{T}_{\mathfrak{m}, \bar{\rho}}^\Sigma[1/p]$ attached to a classical eigencuspform f of weight k . Denote by $\chi : G_{\mathbb{Q}} \rightarrow \mathcal{O}^\times$ the character $\det \rho_f$. Let $R_\Sigma^\chi(\bar{\rho})$ be the deformation ring of $\bar{\rho}$ parametrizing deformations ρ of $\bar{\rho}$ with coefficients in complete noetherian local \mathcal{O} -algebras, unramified outside Σ and such that $\det \rho$ is equal to χ (in particular, ρ_f corresponds to a point in $R_\Sigma^\chi(\bar{\rho})$). Let ρ be an arbitrary deformation of $\bar{\rho}$ corresponding to a point of $R_\Sigma(\bar{\rho})$ with coefficients in \mathcal{O}^\times . Then $(\det \rho)^{-1} \chi$ has values in \mathcal{O}^\times and $(\det \rho)^{-1} \chi \equiv (\det \bar{\rho})^{-1} (\det \bar{\rho}_f)^{-1} \equiv 1 \pmod{\varpi}$. So $(\det \bar{\rho})^{-1} \chi$ has values in $1 + \varpi\mathcal{O}$ and corresponds to a point of the universal deformation ring $R_\Sigma(\mathbb{1})$ parametrizing deformations $\psi : G_{\mathbb{Q}, \Sigma} \rightarrow \mathcal{O}^\times$ of the trivial character $\mathbb{1}$ of $G_{\mathbb{Q}, \Sigma}$. As p is odd, the multiplicative group $1 + \varpi\mathcal{O}$ is uniquely 2-divisible so characters with values in $1 + \varpi\mathcal{O}$ admit canonical square roots. Let ψ_ρ be the canonical square root of $(\det \rho)^{-1} \chi$. Then $\rho \otimes \psi_\rho \equiv \bar{\rho} \pmod{\varpi}$ and $\det(\rho \otimes \psi_\rho) = \chi$ so $\rho \otimes \psi_\rho$ corresponds to a point of $R_\Sigma^\chi(\bar{\rho})$. The map $\rho \mapsto \rho \otimes \psi_\rho$ thus induces an isomorphism $R_\Sigma(\bar{\rho}) \simeq R_\Sigma^\chi(\bar{\rho}) \hat{\otimes}_{\mathcal{O}} R_\Sigma(\mathbb{1})$. As $R_\Sigma(\mathbb{1})$ is isomorphic to a power-series ring in one-variable over a complete intersection \mathcal{O} -algebra of relative dimension zero, $R_\Sigma(\bar{\rho})$ is a flat \mathcal{O} -algebra which is a complete intersection ring of relative dimension 3 if and only if $R_\Sigma^\chi(\bar{\rho})$ is a flat \mathcal{O} -algebra which is a complete intersection ring of relative dimension 2. This we now show.

Assume first that $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible. Twisting by a character if necessary, we may then assume that $(\bar{\rho}|G_{\mathbb{Q}_p})^{ss} = \bar{\chi}_1 \oplus \bar{\chi}_2$ with $\bar{\chi}_1(I_p) = \{1\}$. Let $R_\Sigma^{\text{ord}, \chi}(\bar{\rho})$ be the quotient of $R_\Sigma^\chi(\bar{\rho})$ parametrizing deformations ρ which in addition to being points of $R_\Sigma^\chi(\bar{\rho})$ are such that there exists a short exact sequence of non-zero $G_{\mathbb{Q}_p}$ -representations

$$0 \longrightarrow \chi_1 \longrightarrow \rho|G_{\mathbb{Q}_p} \longrightarrow \chi_2 \longrightarrow 0$$

with $\chi_1(I_p) = \{1\}$. According to the main results of [29, 45] (following [128, 118]), the ring $R_\Sigma^{\text{ord}, \chi}(\bar{\rho})$ is isomorphic to a suitable Hecke algebra and hence flat of relative dimension 0 over \mathcal{O} . Suppose now that $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible. Thanks to lemma 2.2, we may then assume that the eigencuspform f above is crystalline of weight $2 \leq k \leq p$. Denote by $R_\Sigma^{\text{crys}, \chi}(\bar{\rho})$ the quotient of $R_\Sigma^\chi(\bar{\rho})$ parametrizing deformations ρ which in addition to being points of $R_\Sigma^\chi(\bar{\rho})$ are such that $\rho|G_{\mathbb{Q}_p}$ is crystalline and short. By the modularity result of [30], the ring $R_\Sigma^{\text{crys}, \chi}(\bar{\rho})$ is isomorphic to a suitable Hecke algebra and hence flat of relative dimension 0 over \mathcal{O} . Both when $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible and $* = \text{ord}$ and when $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible and $* = \text{crys}$ then, the ring $R_\Sigma^{*, \chi}(\bar{\rho})$ is flat of relative dimension 0 over \mathcal{O} .

Next we describe the kernel of the map $R_\Sigma^\chi(\bar{\rho}) \rightarrow R_\Sigma^{*, \chi}(\bar{\rho})$. Assume first that $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible. Assumption 2 of 2.1 and [114, Lemma 2.2] imply that the universal framed ordinary deformation ring $R^{\square, \text{ord}}(\bar{\rho}|G_{\mathbb{Q}_p})$ is a regular ring of relative dimension 3. According to [10, Corollary 7.4], $R^{\square, \text{ord}}(\bar{\rho}|G_{\mathbb{Q}_p})$ is a quotient of $R^\square(\bar{\rho}|G_{\mathbb{Q}_p})$ by a length two regular sequence. As a quotient of a ring by a regular sequence is regular (if and) only if the ring itself was regular, this entails that $R^\square(\bar{\rho}|G_{\mathbb{Q}_p})$ is a regular ring of relative dimension 5 over \mathcal{O} and that kernel of the map $R^\square(\bar{\rho}|G_{\mathbb{Q}_p}) \rightarrow R^{\square, \text{ord}}(\bar{\rho}|G_{\mathbb{Q}_p})$ is generated by a subset of a system of parameters of cardinal 2. Now we assume that $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible. Then [30, Proposition 2.2 and Corollary 2.3] (and previous results of [104]) imply that the universal deformation ring $R(\bar{\rho}|G_{\mathbb{Q}_p})$ is a regular ring of relative dimension 5 and that $R^{\text{crys}}(\bar{\rho}|G_{\mathbb{Q}_p})$ is a regular quotient of relative dimension 2. Let $R^\chi(\bar{\rho}|G_{\mathbb{Q}_p})$ and $R^{\text{crys}, \chi}(\bar{\rho}|G_{\mathbb{Q}_p})$ respectively denote the quotient of the universal deformation ring parametrizing deformations ρ of $\bar{\rho}|G_{\mathbb{Q}_p}$ with coefficients in complete noetherian local \mathcal{O} -algebras and such that $\det \rho$ is equal to χ and the quotient parametrizing such deformations which are in addition crystalline and short. By local class field theory, $R^\chi(\bar{\rho}|G_{\mathbb{Q}_p})$ is then a regular ring of relative dimension 3 and $R^{\text{crys}, \chi}(\bar{\rho}|G_{\mathbb{Q}_p})$ is a regular ring of relative dimension 1. The kernel of $R^\chi(\bar{\rho}|G_{\mathbb{Q}_p}) \rightarrow R^{\text{crys}, \chi}(\bar{\rho}|G_{\mathbb{Q}_p})$ is thus generated by a regular sequence of length 2.

Denoting by $R^\chi(\bar{\rho}|G_{\mathbb{Q}_p})$ and $R^{*, \chi}(\bar{\rho}|G_{\mathbb{Q}_p})$ the quotient of the framed or usual universal deformation ring parametrizing deformations ρ of $\bar{\rho}|G_{\mathbb{Q}_p}$ with coefficients in complete noetherian local \mathcal{O} -algebras and such that $\det \rho$ is equal to χ (resp. which in addition are

of type *), we consequently see that in both the reducible and irreducible case, there is a commutative diagram

$$\begin{array}{ccc} R^X(\bar{\rho}|G_{\mathbb{Q}_p}) & \longrightarrow & R^{*,X}(\bar{\rho}|G_{\mathbb{Q}_p}) \\ \downarrow & & \downarrow \\ R_\Sigma^X(\bar{\rho}) & \longrightarrow & R_\Sigma^{*,X}(\bar{\rho}) \end{array}$$

where the vertical map are induced by restriction from $G_{\mathbb{Q},\Sigma}$ to $G_{\mathbb{Q}_p}$ and where the kernel of the upper horizontal arrow is generated by a regular sequence of length 2. This implies that $R_\Sigma^{*,X}(\bar{\rho})$ is a quotient of $R_\Sigma^X(\bar{\rho})$ by a ideal generated by at most two elements, and thus that $R_\Sigma^{*,X}(\bar{\rho})$ is a quotient of $R_\Sigma(\bar{\rho})$ by an ideal (x_1, x_2, x_3) generated by at most three elements. This entails in particular that $R_\Sigma(\bar{\rho})$ is of Krull dimension at most 4.

According to lemma 2.3, the ring $R_\Sigma(\bar{\rho})$ is of dimension at least 4 so we conclude that it is flat of relative dimension 3 over \mathcal{O} . The same lemma shows that the zero-dimensional ring $R_\Sigma^{*,X}(\bar{\rho})/(\varpi)$ admits a presentation

$$R_\Sigma^{*,X}(\bar{\rho})/(\varpi) \simeq \mathcal{O}[[X_1, \dots, X_n]]/(\varpi, x_1, x_2, x_3, y_4, \dots, y_n)$$

As $R_\Sigma^{*,X}(\bar{\rho})/(\varpi)$ is of Krull dimension zero, this implies that $(\varpi, x_1, x_2, x_3, y_4, \dots, y_n)$ is a regular sequence in $\mathcal{O}[[X_1, \dots, X_n]]$. In particular, $R_\Sigma^{*,X}(\bar{\rho})/(\varpi)$, $R_\Sigma^{*,X}(\bar{\rho})$ and $R_\Sigma(\bar{\rho})$ are complete intersection rings of dimension 0, 1 and 4 respectively. It follows in addition that (x_1, x_2, x_3) is a regular sequence in $R_\Sigma(\bar{\rho})$. \square

We note that the proof above establishes the existence of suitable quotients $R_\Sigma^{*,X}(\bar{\rho})$ even without twisting $\bar{\rho}$: in that case, these rings are classical Hecke algebras acting on spaces of eigencusforms which are ordinary or crystalline and short up to a twist. This entails in particular the following corollary.

Corollary 2.5. *There exists a Zariski-dense, open subset $\mathcal{X}^{\text{sm}} \subset \text{Spec } R_\Sigma(\bar{\rho})$ containing all points ordinary up to a twist if $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible and all points crystalline and short up to a twist if $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible such that the map*

$$\begin{array}{ccc} \text{Spec } R_\Sigma^X(\bar{\rho}) & & \\ \downarrow & & \\ \text{Spec } \mathcal{O} & & \end{array}$$

is formally smooth at $x \in \mathcal{X}^{\text{sm}}$. In particular, the ring $R_\Sigma(\bar{\rho})$ is reduced.

Proof. Let (x_2, x_3) the regular sequence of the proof of proposition 2.4 and let Λ be the power-series ring $\mathcal{O}[[X_2, X_3]]$. We consider the commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{X_i \mapsto x_i} & R_\Sigma^X(\bar{\rho}) \\ \pi \downarrow & & \downarrow \pi' \\ \mathcal{O} & \longrightarrow & R_\Sigma^{*,X}(\bar{\rho}) \end{array}$$

where the vertical maps π and π' are the quotient maps modulo (X_2, X_3) and (x_2, x_3) respectively. Let $\mathfrak{p} \in \text{Spec } R_\Sigma^X(\bar{\rho})$ be a point above $\pi^* : \text{Spec } \mathcal{O} \longrightarrow \text{Spec } \Lambda$. As $R_\Sigma^X(\bar{\rho})_{\mathfrak{p}}$ is flat over $R_\Sigma^X(\bar{\rho})$, the regular sequence (x_2, x_3) of $R_\Sigma^X(\bar{\rho})$ remains a regular sequence of $R_\Sigma^X(\bar{\rho})_{\mathfrak{p}}$. The quotient $R_\Sigma^X(\bar{\rho})_{\mathfrak{p}}/(x_2, x_3)$ is by construction the localization at a minimal prime ideal of the reduced, classical Hecke algebra of appropriate level and weight after extension of scalars to the fraction field E of \mathcal{O} . Hence, it is a separable field extension and $\text{Spec } R_\Sigma^X(\bar{\rho})_{\mathfrak{p}}/(x_2, x_3) \longrightarrow \text{Spec } E$ is an étale morphism. Consequently, $\text{Spec } R_\Sigma^X(\bar{\rho}) \longrightarrow \text{Spec } \Lambda$ is unramified at \mathfrak{p} . As it is also finite and flat by the proof of 2.4, the locus \mathcal{X}^{sm} of étale points of $\text{Spec } R_\Sigma^X(\bar{\rho}) \longrightarrow \text{Spec } \Lambda$ contains \mathfrak{p} and is thus non-empty. Let $U \subset \text{Spec } \Lambda$ be the complement of the support of $\Omega_{R_\Sigma^X(\bar{\rho})/\Lambda}^1$ regarded as Λ -module. By the above, U is non-empty, formally smooth over $\text{Spec } \mathcal{O}$ and \mathcal{X}^{sm} is formally smooth over U . Hence \mathcal{X}^{sm}

is formally smooth over $\text{Spec } \mathcal{O}$. By construction, U is open and non-empty, hence Zariski-dense in $\text{Spec } \Lambda$. So $\text{Spec } \Lambda \setminus U$ is of codimension at least 1. As $\text{Spec } R_\Sigma^\chi(\bar{\rho}) \rightarrow \text{Spec } \Lambda$ is finite, $\text{Spec } R_\Sigma^\chi(\bar{\rho}) \setminus \mathcal{X}^{\text{sm}}$ is also of codimension at least 1. As $R_\Sigma^\chi(\bar{\rho})$ is Cohen-Macaulay, it is equidimensional. Hence \mathcal{X}^{sm} is Zariski-dense in each irreducible component. Let $\{\mathfrak{a}_i | i \in I\} \subset \text{Spec } R_\Sigma^\chi(\bar{\rho})$ be the finite set of minimal prime ideals of $R_\Sigma^\chi(\bar{\rho})$ and write $\mathcal{X}_i \stackrel{\text{def}}{=} \text{Spec } R_\Sigma^\chi(\bar{\rho})/\mathfrak{a}_i$. Then the sets $\mathcal{X}^{\text{sm}} \cap \mathcal{X}_i$ are non-empty and disjoint. As $\mathcal{X}^{\text{sm}} \cap \mathcal{X}_i$ is étale over U , its generic degree is equal to its degree at \mathfrak{p} . Hence, each \mathcal{X}_i contains a point $\mathfrak{p}_i \in \mathcal{X}^{\text{sm}} \cap \mathcal{X}_i$ above π^* and the \mathfrak{p}_i are pairwise distinct.

As $R_\Sigma^\chi(\bar{\rho})$ is a Cohen-Macaulay ring, the set $\{\mathfrak{a}_i | i \in I\}$ is also the set of associated primes of $R_\Sigma^\chi(\bar{\rho})$. By the properties of \mathcal{X}^{sm} just established, for each $i \in I$, there exists an element $a_i \in R_\Sigma^\chi(\bar{\rho})$ such that $a_j \pmod{\mathfrak{a}_i}$ vanishes if and only if $i \neq j$ and such that $\text{Spec } R_\Sigma^\chi(\bar{\rho})_{a_i}$ is smooth over E . Then $a = \sum_{i \in I} a_i$ does not belong to any minimal prime \mathfrak{a}_i , hence does not belong to an associated prime of $R_\Sigma^\chi(\bar{\rho})$. Hence a is not a zero-divisor and there is an embedding $R_\Sigma^\chi(\bar{\rho}) \hookrightarrow R_\Sigma^\chi(\bar{\rho})_a$. As $\text{Spec } R_\Sigma^\chi(\bar{\rho})_a \simeq \coprod_{i \in I} \text{Spec } R_\Sigma^\chi(\bar{\rho})_{a_i}$ and as $\text{Spec } R_\Sigma^\chi(\bar{\rho})_{a_i}$ is smooth, $R_\Sigma^\chi(\bar{\rho})$ is reduced. \square

The following proposition is well-known.

Proposition 2.6. *The set of primes classical up to a twist is Zariski-dense in $\text{Spec } R_\Sigma(\bar{\rho})$. In particular, $R_\Sigma(\bar{\rho})$ and $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ are isomorphic.*

Proof. The first assertion follows from corollary 2.5 as in [11, Proof of theorem 3.7]. As the morphism $\text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \text{Spec } R_\Sigma(\bar{\rho})$ is dominant, the kernel of the natural surjection $R_\Sigma(\bar{\rho}) \rightarrow \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ is included in the nilradical of $R_\Sigma(\bar{\rho})$, which is a reduced ring according to corollary 2.5. \square

Corollary 2.7. *Let Λ be the power-series ring $\mathcal{O}[[X_1, X_2, X_3]]$. Then there is a length 3 regular sequence (x_1, x_2, x_3) inside $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \simeq R_\Sigma(\bar{\rho})$ such that the assignment $X_i \mapsto x_i$ endows $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ with a structure of Λ -algebra for which it is finite and free as Λ -module and verifying in addition the following property. Classical points of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ which induce the same morphisms after restriction to Λ through this specified embedding $\Lambda \hookrightarrow \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ are either all crystalline and short up to a twist (resp. ordinary up to a twist) or none of them are.*

Proof. As $R_\Sigma(1)$ is a power-series ring with coefficients in complete intersection ring of relative dimension zero over \mathcal{O} , there exists elements $x \in \mathfrak{m}_{R_\Sigma(1)} \setminus \mathfrak{m}_{R_\Sigma(1)}^2$ such that (x, ϖ) is a system of parameters of $R_\Sigma(1)$. Let $x_1 \in \mathfrak{m}_{R_\Sigma(1)} \setminus \mathfrak{m}_{R_\Sigma(1)}^2$ be such a regular element which we view as an element of $R_\Sigma(\bar{\rho})$ through the isomorphism $R_\Sigma(\bar{\rho}) \simeq R_\Sigma^\chi(\bar{\rho}) \hat{\otimes}_{\mathcal{O}} R_\Sigma(1)$.

According to the proof of proposition 2.4, the kernel of $R_\Sigma(\bar{\rho})/(x_1) \twoheadrightarrow R_\Sigma^{*,\chi}(\bar{\rho})$ contains a regular sequence (x_2, x_3) which is the image of the regular sequence generating the kernel of $R(\bar{\rho}|G_{\mathbb{Q}_p})/(x_1) \twoheadrightarrow R^{*,\chi}(\bar{\rho}|G_{\mathbb{Q}_p})$. To a crystalline and short (resp. ordinary) point ρ_f is thus attached a Λ -structure on $R_\Sigma(\bar{\rho})$ defined by $X_i \mapsto x_i$ (note that x_2 and x_3 depend on the choice of ρ_f). Given such a Λ -structure on $R_\Sigma(\bar{\rho})$, if two characteristic zero specializations $\psi_i : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow S$ with values in a discrete valuation ring coincide after restriction to Λ , then they have the same value on x_2 and x_3 . So they both factor through the quotient $R_\Sigma^{*,\chi}(\bar{\rho})$ attached to this choice of Λ -structure or neither of them does. \square

Note that the Λ -algebra structure described in the previous proof depends on an initial choice of a classical point ρ_f . In what follows, we omit references to this dependence.

Lemma 2.8. *Let $\lambda_f : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ be a classical point of even weight k . Let $\omega : G_{\mathbb{Q},\Sigma} \rightarrow \mathbb{F}_p^\times \hookrightarrow \mathbb{Z}_p^\times$ be the character giving the action on roots of unity of order p .*

If $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible, then there exists a crystalline and short point $\lambda_g : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ of even weight k' such that either $T(f)(-\frac{k-2}{2})$ and $T(g)(-\frac{k'-2}{2})$ are congruent modulo ϖ or $T(f)(-\frac{k-2}{2})$ and $T(g)(-\frac{k'-2}{2}) \otimes \omega^{\frac{p-1}{2}}$ are congruent modulo ϖ .

If $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible, then there exists a classical point $\lambda_g : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow \mathcal{O}$ of even weight k' with $\rho_g|G_{\mathbb{Q}_p}$ reducible and such that either $T(f)(-\frac{k-2}{2})$ and $T(g)(-\frac{k'-2}{2})$ are congruent modulo ϖ or $T(f)(-\frac{k-2}{2})$ and $T(g)(-\frac{k'-2}{2}) \otimes \omega^{\frac{p-1}{2}}$ are congruent modulo ϖ .

Proof. First assume that $\bar{\rho}|G_{\mathbb{Q}_p}$ is irreducible. According to the lemma 2.2 and its proof, there exist a crystalline point $\lambda_g : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow \mathcal{O}$ of weight $2 \leq k' \leq p$ and an integer a such that $\bar{\rho}_g \otimes \bar{\chi}_{\text{cyc}}^{a-\frac{k-2}{2}}$ and $\bar{\rho} \otimes \bar{\chi}_{\text{cyc}}^{-\frac{k-2}{2}}$ are isomorphic. Hence $\bar{\rho}_g \otimes \bar{\chi}_{\text{cyc}}^{-\frac{k'-2}{2}}$ is isomorphic to $(\bar{\rho} \otimes \bar{\chi}_{\text{cyc}}^{-\frac{k-2}{2}}) \otimes \bar{\chi}^m$ for $m = -a + \frac{k-k'}{2}$. As

$$\bar{\chi}_{\text{cyc}}^{1+2m} = \det \left(\left(\bar{\rho} \otimes \bar{\chi}_{\text{cyc}}^{-\frac{k-2}{2}} \right) \otimes \bar{\chi}^m \right) = \det \left(\bar{\rho}_g \otimes \bar{\chi}_{\text{cyc}}^{-\frac{k'-2}{2}} \right) = \bar{\chi}_{\text{cyc}},$$

the integer m is congruent to 0 or to $(p-1)/2$ modulo p . If $m = 0$, then $T_g(-\frac{k'-2}{2})$ is congruent to $T(f)(-\frac{k-2}{2})$. If $m = (p-1)/2$, then $T_g(-\frac{k'-2}{2}) \otimes \omega^m$ is congruent to $T(-\frac{k-2}{2})$.

Now assume that $\bar{\rho}|G_{\mathbb{Q}_p}$ is reducible. Then there exists a point $\lambda_g : R_{\Sigma}^{\chi, \text{ord}}(\bar{\rho}) \rightarrow \mathcal{O}$ which is classical of a given weight k' . As in the first part of the proof, there exists an integer m such that $T(g)(-\frac{k'-2}{2}) \otimes \omega^m$ is congruent to $T(f)(-\frac{k-2}{2})$. Then, as above, m must be equal to 0 or to $(p-1)/2$ modulo $p-1$.

In both cases, taking determinants (or examining again the proof of 2.2) shows that k' and k are congruent modulo 2 and hence that k' is even. \square

3 Fundamental lines and zeta elements over deformation rings

3.1 Classical Iwasawa theory for modular motives

3.1.1 Equivariant modular motives and period maps

Let

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(U)$$

be an eigencuspform with coefficients in a number field $L \xhookrightarrow{\iota} \mathbb{C}$. Let M be the pure motive over \mathbb{Q} and with coefficients in L attached to f ([111]). We assume that L embeds in E .

Recall that \mathbb{Q}_n/\mathbb{Q} is the subfield of $\mathbb{Q}(\zeta_{p^{n+1}})$ with Galois group G_n over \mathbb{Q} equal to $\mathbb{Z}/p^n\mathbb{Z}$. If $\chi \in \hat{G}_n$ is a character, we denote by L_{χ} the extension of \mathbb{Q} generated by L and the image of χ . Let $\mathfrak{p}|p$ be a prime ideal of $\mathcal{O}_{L_{\chi}}$ and let E_{χ} be the corresponding finite extension of E . We consider $h^0(\text{Spec } \mathbb{Q}_n)$ the Artin motive attached to the regular representation of G_n viewed as a pure motive over \mathbb{Q} with coefficients in L_{χ} and denote by $h^0(\text{Spec } \mathbb{Q}_n)_{\chi}$ the direct summand of $h^0(\text{Spec } \mathbb{Q}_n)$ on which G_n acts through χ .

Let M_{χ} be the motive $M \times_{\mathbb{Q}} h^0(\text{Spec } \mathbb{Q}_n)_{\chi}$. By definition, a realization of M_{χ} is the corresponding realization of M with scalars extended from L to L_{χ} together with an action of G_n through χ . We denote by $V_{\mathbb{C}}$ (resp. $V_{\chi, \mathbb{C}}$) the Betti realization of M (resp. M_{χ}), by V_{dR} (resp. $V_{\chi, \text{dR}}$) the de Rham realization of M (resp. M_{χ}) and by V (resp. V_{χ}) the \mathfrak{p} -adic étale realization of M (resp. M_{χ}).

Definition 3.1. For S a set of finite primes containing $\{p\}$, the S -partial L -function $L_S(M, \chi, s)$ is the holomorphic complex function satisfying

$$L_S(M, \chi, s) \stackrel{\text{def}}{=} \prod_{\ell \notin S} \frac{1}{1 - a_{\ell} \chi(\text{Fr}(\ell)) \ell^{-s} + \varepsilon(\ell) \chi(\text{Fr}(\ell)) \ell^{k-1-2s}}$$

for all $s \in \mathbb{C}$ with $\Re s >> 0$. This is also the S -partial L -function of the motive M_{χ} .

Note that the S -partial L -function $L_S(M, \chi, s)$ depends on the choice of ι though we suppress this dependence from the notation. We fix S a finite set of primes containing p , an integer $1 \leq r \leq k - 1$, an integer $n \in \mathbb{N}$ and a character $\chi \in \hat{G}_n$.

The Betti-de Rham comparison isomorphism of $\mathbb{C}[\text{Gal}(\mathbb{C}/\mathbb{R})]$ -modules

$$V_{\text{dR}}(r) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\text{can}} V_B(r) \otimes_{\mathbb{Q}} \mathbb{C}$$

induces a complex period map

$$\text{per}_{\mathbb{C}} : \text{Fil}^0 V_{\chi, \text{dR}}(r) \otimes_{L_{\chi}} \mathbb{C} \longrightarrow V_{\chi, \mathbb{C}}(r-1)^+ \otimes_{L_{\chi}} \mathbb{C}. \quad (3.1.1)$$

Under our hypothesis on r , $\text{Fil}^0 V_{\text{dR}}(r)$ is equal to the L -vector subspace $S_k(U)(f)$ on which $\mathbf{T}_k(U)$ acts through $\lambda(f)$ and (3.1.1) is an isomorphism ([28]). Besides, the composition of localization at p with the dual exponential map \exp^* of [9]

$$\exp^* : H^1(G_{\mathbb{Q}_p(\zeta_p^n)}, V(r)) \longrightarrow D_{\text{dR}}^0(V(r))$$

induces an inverse p -adic period map of E_{χ} -vector spaces

$$\text{per}_p^{-1} : H_{\text{et}}^1(\mathbb{Z}[1/S], V_{\chi}(r)) \longrightarrow \text{Fil}^0 V_{\chi, \text{dR}}(r) \otimes_L E_{\chi} \quad (3.1.2)$$

which is equivariant under the action of G_n on both sides. According to [107] and [68, Theorems 12.4, 14.5], for all $n \in \mathbb{N}$ and all $\chi \in \hat{G}_n$ except possibly finitely many, $H_{\text{et}}^1(\mathbb{Z}[1/S], V_{\chi}(r))$ is of dimension 1 and (3.1.2) is an isomorphism. We say that $M_{\chi}(r)$ is strictly critical when this holds ([66, Section 3.2.6]).

Suppose that M_{χ} is strictly critical. The determinant functor applied to the p -adic period map (3.1.2) then yields an isomorphism of free E_{χ} -vector spaces of rank 1

$$\begin{array}{c} \text{Det}_{E_{\chi}} H_{\text{et}}^1(\mathbb{Z}[1/S], V_{\chi}(r)) \\ \xrightarrow{\text{can}} \downarrow \text{per}_p^{-1} \\ \text{Fil}^0 V_{\chi, \text{dR}}(r) \otimes_L E_{\chi}. \end{array}$$

Taking the tensor product with the determinant of $V_{\chi}(r-1)^+$ yields an identification

$$\begin{array}{c} \text{Det}_{E_{\chi}} H_{\text{et}}^1(\mathbb{Z}[1/S], V(r)) \otimes_{E_{\chi}} \text{Det}_{E_{\chi}}^{-1} V_{\chi}(r-1)^+ \\ \xrightarrow{\text{can}} \downarrow \text{per}_p^{-1} \\ \text{Det}_{E_{\chi}} (\text{Fil}^0 V_{\chi, \text{dR}}(r) \otimes_L E_{\chi}) \otimes_{E_{\chi}} \text{Det}_{E_{\chi}}^{-1} V_{\chi}(r-1)^+. \end{array} \quad (3.1.3)$$

Similarly, the determinant functor applied to the complex period map (3.1.1) induces an identification

$$\begin{array}{c} [\text{Det}_{\mathbb{C}} \text{Fil}^0 V_{\chi, \text{dR}}(r) \otimes_{L_{\chi}} \mathbb{C}] \otimes [\text{Det}_{\mathbb{C}}^{-1} V_{\chi, \mathbb{C}}(r-1)^+ \otimes_{L_{\chi}} \mathbb{C}] \\ \xrightarrow{\text{can}} \downarrow \text{per}_{\mathbb{C}} \\ \mathbb{C}. \end{array} \quad (3.1.4)$$

Definition 3.2. *The motivic fundamental line $(\Delta_{\text{mot}}(M_{\chi}(r)), \text{per}_p, \text{per}_{\mathbb{C}})$ of the strictly critical motive $M_{\chi}(r)$ is the one-dimensional L_{χ} -vector space*

$$\Delta_{\text{mot}}(M_{\chi}(r)) \stackrel{\text{def}}{=} \text{Det}_{L_{\chi}} \text{Fil}^0 V_{\chi, \text{dR}}(r) \otimes_{L_{\chi}} \text{Det}_{L_{\chi}}^{-1} V_{\chi, \mathbb{C}}(r-1)^+ \quad (3.1.5)$$

together with the two isomorphisms

$$\begin{aligned} \text{per}_p : \Delta_{\text{mot}}(M_{\chi}(r)) \otimes_{L_{\chi}} E_{\chi} &\xrightarrow{\sim} \text{Det}_{E_{\chi}} H_{\text{et}}^1(\mathbb{Z}[1/S], V_{\chi}(r)) \otimes_{E_{\chi}} \text{Det}_{E_{\chi}}^{-1} (V_{\chi}(r-1))^+ \\ \text{per}_{\mathbb{C}} : \Delta_{\text{mot}}(M_{\chi}(r)) \otimes_{L_{\chi}} \mathbb{C} &\xrightarrow{\sim} \mathbb{C}. \end{aligned}$$

Note that motivic fundamental line of $M_{\chi}(r)$ is an L -rational subspace both of the target of (3.1.3) and of the source of (3.1.4). Consequently, to any element \mathbf{z} of the source of (3.1.3) whose image through per_p^{-1} lands in the motivic fundamental line $\Delta_{\text{mot}}(M_{\chi}(r))$ is attached a complex number $\text{per}_{\mathbb{C}}(\text{per}_p^{-1}(\mathbf{z}) \otimes 1)$. Let $\text{per}_{\mathbb{C}}(\text{per}_p^{-1}(-) \otimes 1)$ be the map defined by this composition on the sub- L_{χ} -subspace $\Delta_{\text{mot}}(M_{\chi}(r))_{\mathfrak{p}}$ equal to the image of $\Delta_{\text{mot}}(M_{\chi}(r))$ through per_p . Note also that the motivic fundamental line depends on the choice of S .

Remark: For $n \in \mathbb{N}$, let M_n be the motive $M \times h^0(\mathrm{Spec} \mathbb{Q}_n)$ and denote by $V_{n,\mathbb{C}}$, $V_{n,\mathrm{dR}}$ and V_n its various realizations. It is conjectured ([65]) that there are motivic cohomology groups

$$H_f^1(M_n(r)), H_f^1(M_n^*(1-r))$$

which are finitely generated $L[G_n]$ -modules of finite projective dimension and an equivariant motivic fundamental line $\Delta_{\mathrm{mot}}(M_n(r))$ defined by

$$\begin{aligned} & (\mathrm{Det}_{L[G_n]} H_f^1(M_n(r))) \otimes_{L[G_n]} \left(\mathrm{Det}_{L[G_n]}^{-1} H_f^1(M_n^*(1-r)) \right) \otimes_{L[G_n]} \\ & (\mathrm{Det}_{L[G_n]} \mathrm{Fil}^0 V_{n,\mathrm{dR}}(r)) \otimes_{L[G_n]} \left(\mathrm{Det}_{L[G_n]}^{-1} V_{n,\mathbb{C}}(r-1)^+ \right) \end{aligned}$$

with a canonical isomorphism

$$\Delta_{\mathrm{mot}}(M_n(r)) \otimes_{L[G_n]} L_\chi \xrightarrow{\mathrm{can}} \Delta_{\mathrm{mot}}(M_\chi(r))$$

induced by isomorphisms

$$H_f^1(M_n(r)) \otimes_{L[G_n],\chi} L_\chi \simeq 0, \quad H_f^1(M_n^*(1-r)) \otimes_{L[G_n],\chi} L_\chi \simeq 0$$

whenever M_χ is strictly critical.

3.1.2 Zeta morphism for modular motives

As in the previous subsection, S is a finite set of finite primes containing p , n is an integer and $\chi \in \hat{G}_n$ is such that $M_\chi(r)$ is strictly critical. By definition, the E_χ -vector spaces $V_\chi(r-1)^+$ and $H_{\mathrm{et}}^1(\mathbb{Z}[1/S], V_\chi(r))$ are then isomorphic.

Let

$$Z : V_\chi(r-1)^+ \xrightarrow{\sim} H_{\mathrm{et}}^1(\mathbb{Z}[1/S], V_\chi(r))$$

be an isomorphism between them. Applying the functor Det then yields an identification

$$\mathrm{Det}_{E_\chi}(Z) : \mathrm{Det}_{E_\chi} H_{\mathrm{et}}^1(\mathbb{Z}[1/S], V_\chi(r)) \otimes_{E_\chi} \mathrm{Det}_{E_\chi}^{-1} V_\chi(r-1)^+ \simeq E_\chi.$$

Let $\Delta_{L,Z}(M_\chi)$ be the L_χ -vector space pre-image of $L_\chi \subset E_\chi$ through this isomorphism.

Let $T(f)_{\mathrm{Iw}}$ be $T(f) \otimes_{\mathcal{O}} \mathcal{O}_{\mathrm{Iw}}$ for $T(f)$ a $G_{\mathbb{Q}}$ -stable lattice inside V . Suppose there is a non-zero morphism

$$Z_{\mathrm{Iw}} : T(f)_{\mathrm{Iw}}(-1)^+ \longrightarrow H^1(\mathbb{Z}[1/S], T(f)_{\mathrm{Iw}}) \simeq H^1(\mathbb{Z}[1/p], T(f)_{\mathrm{Iw}}).$$

For all $n \in \mathbb{N}$ and all $\chi \in \hat{G}_n$ except possibly finitely many, Z_{Iw} induces a non-zero morphism

$$Z_\chi : V_\chi(r-1)^+ \longrightarrow H^1(\mathbb{Z}[1/S], V_\chi(r)).$$

Outside of this finite set and of the finite set of characters χ such that M_χ is not strictly critical, the source and target of Z_χ are E_χ -vector spaces of dimension 1 and Z_χ is a non-zero map between them, hence an isomorphism.

Definition 3.3. A morphism

$$Z : V_\chi(r-1)^+ \xrightarrow{\sim} H_{\mathrm{et}}^1(\mathbb{Z}[1/S], V_\chi(r))$$

is motivic if $\Delta_{L,Z}(M_\chi)$ is equal to $\mathrm{per}_p(\Delta_{\mathrm{mot}}(M_\chi))$. If a morphism Z is motivic, we say it is the S -partial zeta morphism of M_χ if $\mathrm{per}_{\mathbb{C}}(\mathrm{per}_p^{-1}(\mathrm{Det}_{E_\chi}(Z)^{-1}(1)) \otimes 1)$ is equal to $L_S(M^*(1), \chi^{-1}, -r) \in \mathbb{C}$. A morphism

$$Z_{\mathrm{Iw}} : T(f)_{\mathrm{Iw}}(-1)^+ \longrightarrow H^1(\mathbb{Z}[1/S], T(f)_{\mathrm{Iw}}).$$

is the S -partial zeta morphism of $T(f)_{\mathrm{Iw}}$ if Z_χ is the S -partial zeta morphism of V_χ for all χ such that M_χ is strictly critical.

Note that the composition $\mathrm{per}_{\mathbb{C}}(\mathrm{per}_p^{-1}(\mathrm{Det}_{E_\chi}(Z)^{-1}(1)) \otimes 1)$ makes sense as the pre-image of $\mathrm{Det}_{E_\chi}(Z)^{-1}(1)$ through per_p belongs to $\Delta_{\mathrm{mot}}(M_\chi(r))$ if (and only if) Z is motivic. A zeta morphism is unique, if it exists.

As recalled in the following theorem, one of the major contribution of K.Kato towards the Equivariant Tamagawa Number Conjecture for motives attached to classical eigencuspforms is that they admit zeta morphisms ([68, Theorem 12.5]).

Theorem 3.4. Let $\lambda_f : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow \mathcal{O}$ be a classical point. For all finite set of primes $S \supset \{p\}$, there exists a zeta morphism

$$Z_S : T(f)_{\text{Iw}}(-1)^+ \rightarrow H_{\text{et}}^1(\mathbb{Z}[1/S], T(f)_{\text{Iw}}).$$

We write $\mathbf{z}(f)_{\Sigma, \text{Iw}}$ for the Σ -partial zeta morphism and $\mathbf{z}(f)_{\text{Iw}}$ for the $\{p\}$ -partial zeta morphism.

3.2 Zeta elements over deformation rings

3.2.1 Review of the results of Nakamura

In this section, we review the work of K.Nakamura ([92]) on the construction of zeta elements over deformation rings. Recently P.Colmez and S.Wang announced the construction of the same object by a rather different (and sophisticated) approach ([26]). We strengthen slightly assumption 2.1.

Assumption 3.5. If $(\bar{\rho}|G_{\mathbb{Q}_p})^{ss} = \bar{\chi}_1 \oplus \bar{\chi}_2$, then $\bar{\chi}_1^{-1}\chi_2 \neq \bar{\chi}_{\text{cyc}}$.

Compared to assumption 2.1, assumption 3.5 adds the slight restriction that $\bar{\rho}|G_{\mathbb{Q}_p}$ cannot be an extension of the trivial character by the cyclotomic character (up to twist).

Theorem 3.6 (K.Nakamura). Under assumptions 2.1 and 3.5, there exists a zeta morphism

$$\mathbf{z}_{\Sigma} : T_{\Sigma}(-1)^+ \rightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\Sigma})$$

such that for all classical point λ_f the following diagram commutes

$$\begin{array}{ccc} T_{\Sigma}(-1)^+ & \xrightarrow{\mathbf{z}_{\Sigma}} & H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\Sigma}) \\ \downarrow & & \downarrow \\ T(f)_{\text{Iw}}(-1)^+ & \xrightarrow{\mathbf{z}(f)_{\Sigma, \text{Iw}}} & H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T(f)_{\text{Iw}}) \end{array}$$

where $\mathbf{z}(f)_{\Sigma, \text{Iw}}$ is the zeta morphism of theorem 3.4.

Proof. This is [92, Theorem 1.1]. □

Remark: In [92], the supplementary assumption $p \geq 5$ is made. In this remark, we briefly explain why we can dispense from this assumption. The assumption $p \geq 5$ is required in the proofs of theorem 4.7 and theorem B.24 of *loc. cit.*. There, it is made in order to appeal respectively to [97, Corollary 6.5] and to [96, Proposition 6.3], which are stated in [96, 97] only under the hypothesis $p \geq 5$. However, the relevant parts of [97, Corollary 6.5] and [96, Proposition 6.3] have been generalized to all p in [98, Proposition 2.13, 2.33].⁴

If $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow S$ is a specialization, we denote by

$$\mathbf{z}_{\Sigma}(\lambda) : T_{\lambda}(-1)^+ \rightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\lambda})$$

the morphism induced by the surjection $T_{\Sigma}(-1)^+ \rightarrow T_{\lambda}(-1)^+$ and the morphism

$$\lambda_* : H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T) \rightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\lambda}).$$

If $\lambda_f : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow \mathcal{O}$ is classical up to a twist, we write $\mathbf{z}(f)_{\Sigma}$ for $\mathbf{z}_{\Sigma}(\lambda_f)$. Suppose λ has values in S and that $x_1 \in \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ belongs to the kernel of λ . To λ is then attached a specialization $\lambda_{\text{Iw}} : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow S_{\text{Iw}}$ by identifying $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ with $(\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}/x_1)[[\Gamma_{\text{Iw}}]]$ and setting λ_{Iw} equal to the identity on Γ_{Iw} . We write $\mathbf{z}_{\Sigma}(\lambda)_{\text{Iw}}$ for the zeta element attached to λ_{Iw} . Theorem 3.6 ensures that $\mathbf{z}(f)_{\Sigma, \text{Iw}}$ is equal to $\mathbf{z}_{\Sigma}(\lambda_f)_{\text{Iw}}$ if $\lambda_f : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow \mathcal{O}$ is a classical point.

⁴We thank K.Nakamura and S-N.Tung for explaining this to us.

3.2.2 Iwasawa-suitable specializations

Henceforth, we consistently assume that $\mathfrak{m}_{\bar{\rho}}$ satisfies the following supplementary assumption.

Assumption 3.7. *There exists $\ell \mid N$, $\ell \neq p$ such that $\dim_{\mathbb{F}} \bar{\rho}_f^{I_\ell} = 1$.*

Let $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow A$ be a specialization with values in a domain A . Recall that $(V_{\lambda}, \rho_{\lambda}, \text{Frac}(A))$ is the $G_{\mathbb{Q}, \Sigma}$ -representation $T_{\lambda} \otimes_A \text{Frac}(A)$.

Definition 3.8. *For $\ell \nmid p$ a prime, the Euler factor $\text{Eul}_{\ell}(T_{\lambda})$ of T_{λ} is $\det(1 - \text{Fr}(\ell)|V_{\lambda}^{I_\ell})$.*

Let $\mathfrak{a} \in \text{Spec}^{\min} \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ be a minimal primer ideal through which λ factors.

Definition 3.9. *A specialization $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow A$ with values in a domain is relatively pure at $\ell \nmid p$ if the A -rank of $H^0(G, V_{\lambda})$ is equal to the $R(\mathfrak{a})$ -rank of $H^0(G, V(\mathfrak{a}))$ for G equal either to $G_{\mathbb{Q}_{\ell}}$ or to I_{ℓ} . It is relatively pure if it is relatively pure at ℓ for all $\ell \nmid p$.*

Note that in particular if λ is a relatively pure specialization at ℓ , then 1 is not an eigenvalue of $\text{Fr}(\ell)$ acting on V^{I_ℓ} .

The following proposition follows from the proof of the Weight-Monodromy Conjecture for modular motives ([110]).

Proposition 3.10. *A classical specialization is relatively pure.*

Proof. See for instance [43, Lemma 3.9]. □

We say that a specialization $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow S$ contains a specialization $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow A$ if there exists a morphism of local rings $\lambda' : S \rightarrow A$ such that the diagram

$$\begin{array}{ccc} \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} & \xrightarrow{\lambda} & A \\ \psi \downarrow & \nearrow \lambda' & \\ S & & \end{array}$$

is commutative.

Lemma 3.11. *Let $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow S$ be a specialization with values in a domain and containing a specialization λ with values in a domain A which is relatively pure (at ℓ). Then ψ is relatively pure (at ℓ), the formation of $\text{Eul}_{\ell}(-)$ commutes with all relatively pure specialization contained in ψ and for all such specialization $\text{Eul}_{\ell}(-)$ does not vanish. This applies in particular to the quotient map $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow R(\mathfrak{a})$ for all $\mathfrak{a} \in \text{Spec}^{\min} \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$.*

Proof. Let $\ell \nmid p$ be a prime. By construction, $\text{rank}_S H^0(G_{\mathbb{Q}_{\ell}}, V_{\psi}) \leq \text{rank}_A H^0(G_{\mathbb{Q}_{\ell}}, V_{\lambda})$ so $H^0(G_{\mathbb{Q}_{\ell}}, V_{\psi}) = 0$. Likewise, $\text{rank}_A H^0(I_{\ell}, V(\mathfrak{a})) \leq \text{rank}_S H^0(I_{\ell}, V_{\psi}) \leq \text{rank}_A H^0(I_{\ell}, V_{\lambda})$ so $\text{rank}_S H^0(I_{\ell}, V_{\psi}) = \text{rank}_A H^0(I_{\ell}, V(\mathfrak{a}))$. So ψ is relatively pure. It is then enough to show that

$$\lambda(\text{Eul}_{\ell}(T(\mathfrak{a}))) = \text{Eul}_{\ell}(T_{\lambda}) \tag{3.2.1}$$

and that

$$\text{Eul}_{\ell}(T_{\lambda}) \neq 0$$

to conclude. Both members of (3.2.1) are equal to $\det(1 - \lambda(\text{Fr}(\ell))|V_{\lambda}^{I_\ell})$. The determinant $\det(1 - \text{Fr}(\ell)|V_{\lambda}^{I_\ell})$ vanishes only if $V_{\lambda}^{G_{\mathbb{Q}_{\ell}}} \neq \{0\}$. A specialization for which this holds cannot be relatively pure by definition.

The last assertion follows from the previous ones since $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow R(\mathfrak{a})$ contains a classical, hence relatively pure, specialization by proposition 2.6. □

Definition 3.12. *A specialization $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}/(x_1) \rightarrow A$ with values in a domain A is Iwasawa-suitable if it contains a specialization $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}/(x_1) \rightarrow \mathcal{O}$ which satisfies the following two properties.*

1. T_{ψ} and $T_{\psi}^*(1)$ are relatively pure.
2. The morphism $\mathbf{z}_{\Sigma}(\psi)_{\text{Iw}}$ is non-zero.

We say that a specialization $\lambda : A \rightarrow S$ is Iwasawa-suitable if there exists an Iwasawa-suitable specialization λ' such that the diagram

$$\begin{array}{ccc} \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} & \xrightarrow{\lambda'} & S \\ \downarrow & \nearrow \lambda & \\ A & & \end{array}$$

commutes.

Note that an Iwasawa-suitable specialization is itself relatively pure.

Lemma 3.13. *Let $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow \mathcal{O}$ be an Iwasawa-suitable specialization. Let $V_{\lambda, \text{Iw}}$ be the $G_{\mathbb{Q}, \Sigma}$ -representation $T_{\lambda, \text{Iw}} \otimes_{\mathcal{O}_{\text{Iw}}} \text{Frac}(\mathcal{O}_{\text{Iw}})$. Then $H_{\text{et}}^1(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}})$ is a free \mathcal{O}_{Iw} -module of rank 1 and the complex*

$$\text{Cone} \left(V_{\lambda, \text{Iw}}(-1)^+ \xrightarrow{\mathbf{z}(\lambda)_{\text{Iw}}} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], V_{\lambda, \text{Iw}}) \right)$$

is acyclic.

Proof. By assumption 2.1, $H_{\text{et}}^0(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}}/x)$ and $H_{\text{et}}^0(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}}/(x, y))$ vanish for all regular sequence $(x, y) \in \mathcal{O}_{\text{Iw}}^2$. The isomorphisms

$$R\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}}) \xrightarrow{\text{L}} \mathcal{O}_{\text{Iw}}/x \simeq R\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}}/x)$$

and

$$R\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}}/x) \xrightarrow{\text{L}} \mathcal{O}_{\text{Iw}}/(x, y) \simeq R\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}}/(x, y))$$

then imply first that $H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}})$ is a torsion-free \mathcal{O}_{Iw} -module, then that there is an embedding of $H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}})/x$ into $H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}}/x)$ and hence that y is a regular element in $H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}})/x$. Hence $H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}})$ is of depth 2 as \mathcal{O}_{Iw} -module and hence an \mathcal{O}_{Iw} -module free of rank 1 by the theorem of Auslander-Buchsbaum and Serre.

Let γ be a generator of the free, rank 1 \mathcal{O}_{Iw} -module $T_{\lambda, \text{Iw}}(-1)^+$. According to [92, Theorem 1.1 (ii)], $\mathbf{z}_{\Sigma}(\lambda)_{\text{Iw}}(\gamma)$ is then the first class of an Euler system, which is necessarily non-zero since λ is Iwasawa-suitable by assumption. As we have just seen, this entails that this Euler system is not \mathcal{O}_{Iw} -torsion. Assumptions 2.1 and 3.7 imply that the image of $\bar{\rho}$ acts irreducibly on $\bar{\mathbb{F}}_p^2$ and has order divisible by p , so contains a subgroup conjugated to $\text{SL}_2(\mathbb{F}_q)$ for some $q = p^n$, so contains a unipotent element $\bar{\sigma} \neq \text{Id}$. Let σ be a lifting of $\bar{\sigma}$ to $\rho_{\lambda}(G_{\mathbb{Q}, \Sigma})$. Then the kernel of $\sigma - 1$ is strictly included inside T_{λ} and its cokernel is of dimension 1 after tensor-product with the residual field k . Hence this kernel is free of rank 1. The representation T_{λ} thus satisfies assumptions (i_{str}), (ii_{str}) and (iv_p) of [67, Theorem 0.8]. As T_{λ} is not abelian, the conclusion of [67, Proposition 8.7] also holds. As obtaining this proposition was the sole use of hypothesis (iii) of [67, Theorem 0.8], this theorem also holds for $T_{\lambda, \text{Iw}}$. Consequently $H_{\text{et}}^2(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}})$ is \mathcal{O}_{Iw} -torsion. The cohomology of the complex

$$C = \text{Cone} \left(V_{\lambda, \text{Iw}}(-1)^+ \xrightarrow{\mathbf{z}_{\Sigma}(\lambda)_{\text{Iw}}} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], V_{\lambda, \text{Iw}}) \right)$$

is concentrated in degree 1 and 2,

$$H^1(C) = H_{\text{et}}^1(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}})/\mathbf{z}_{\Sigma}(\lambda)_{\text{Iw}}(\gamma)$$

and

$$H^2(C) = H_{\text{et}}^2(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}}).$$

The complex $C \xrightarrow{\text{L}} \text{Frac}(\mathcal{O}_{\text{Iw}})$ is thus acyclic. \square

In the following propositions, we extend the results of Nakamura to the case of primitive zeta morphisms over irreducible components of the universal deformation space. We remark that it seems impossible to construct primitive zeta morphisms over the whole universal deformation space. If $p \nmid n$, we denote by σ_n the arithmetic Frobenius morphism in $\text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$ or $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$.

Proposition 3.14. Let $\mathfrak{a} \in \text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ be a minimal prime ideal. Let c, d be integers such that $\ell \nmid cd$ if $\ell \in \Sigma \cup \{2, 3\}$. Under assumptions 2.1 and 3.5, there exists a zeta morphism

$${}_{c,d}\mathbf{z}(\mathfrak{a}) : T(\mathfrak{a})(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a}))$$

such that the following diagram commutes

$$\begin{array}{ccc} T(\mathfrak{a})(-1)^+ & \xrightarrow{{}_{c,d}\mathbf{z}(\mathfrak{a})} & H_{\text{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a})) \\ \downarrow & & \downarrow \\ T(f)_{\text{Iw}}(-1)^+ & \xrightarrow{{}_{c,d}\mathbf{z}(f)_{\text{Iw}}} & H_{\text{et}}^1(\mathbb{Z}[1/p], T(f)_{\text{Iw}}) \end{array}$$

for all classical point $\lambda_f : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \longrightarrow \mathcal{O}$ factoring through $R(\mathfrak{a})$. Here, ${}_{c,d}\mathbf{z}(f)_{\text{Iw}}$ designates any of the morphism appearing in the statement of [68, Theorem 12.6].

Proof. We use the notations and results of section 6.

Let $N(\mathfrak{a})$ be the tame Artin conductor of the representation $\rho(\mathfrak{a})$, or equivalently by relative purity the tame conductor of any classical point factoring through $R(\mathfrak{a})$. Let $U_1(\mathfrak{a})_{\ell}$ be the compact open subgroup

$$U_1(\mathfrak{a})_{\ell} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_{\ell}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\ell^{v_{\ell}(N(\mathfrak{a}))}} \right\}.$$

and let $U_1(\mathfrak{a})$ be the compact open subgroup

$$U_1(\mathfrak{a}) \stackrel{\text{def}}{=} \prod_{\ell \in \Sigma^p} U_1(\mathfrak{a})_{\ell} \subset \prod_{\ell \in \Sigma^p} \text{GL}_2(\mathbb{Q}_{\ell}).$$

We also view $U_1(\mathfrak{a})$ as a compact open subgroup of $\text{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(p\infty)})$, in which case $U_1(\mathfrak{a})_{\ell}$ is taken to be compact open maximal if $\ell \notin \Sigma$.

Let $U_p \subset \text{GL}_2(\mathbb{Q}_p)$ be a compact open subgroup. In [68, Sections 2.2, 5.1] and [46, Sections 2.3.6, 2.4.2], elements

$${}_{c,d}\mathbf{z}_{U_p} \in K_2(Y(U_1(\mathfrak{a})U_p))$$

are constructed (they are denoted there ${}_{c,d}z_{1,Mp^r,p^r}(u, v)$). These elements are compatible for the norm map

$$K_2(Y(U_1(\mathfrak{a})U'_p)) \longrightarrow K_2(Y(U_1(\mathfrak{a})U_p))$$

attached to the covering $Y(U_1(\mathfrak{a})U'_p)) \longrightarrow Y(U_1(\mathfrak{a})U_p)$ if $U'_p \subset U_p$ ([46, Sections 2.4.8]). After localizing at $\mathfrak{m}_{\bar{\rho}}$ and taking the Chern class map, they thus yield morphisms

$${}_{c,d}\tilde{\mathbf{z}} : \tilde{H}_{\text{et}}^1(U_1(\mathfrak{a}), \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}(-1) \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], \tilde{H}_{\text{et}}^1(U_1(\mathfrak{a}), \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}) \quad (3.2.2)$$

from the inverse limit on the level of cohomology groups in the tower of modular curves with full level structure at p and tame level $U_1(\mathfrak{a})$ to the first cohomology group of $\text{Spec } \mathbb{Z}[1/\Sigma]$ with coefficients in this completed cohomology group.

By theorem 6.10, we have the following description of the completed cohomology with full level at all primes in Σ as $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}[\text{GL}_2(\mathbb{Q}_{\Sigma}) \times \text{GL}_2(\mathbb{Q}_p) \times \prod_{\ell \in \Sigma^p} \text{GL}_2(\mathbb{Q}_{\ell})]$ -module

$$\tilde{H}_{\text{et}}^1(\mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \simeq \rho_{\Sigma} \otimes \tilde{\pi}_p(\rho_{\Sigma}|G_{\mathbb{Q}_p}) \hat{\otimes} \bigotimes_{\ell \in \Sigma^p} \tilde{\pi}_{\ell}(\rho_{\Sigma}|G_{\mathbb{Q}_{\ell}}).$$

Taking $R(\mathfrak{a})$ -quotient, $U_1(\mathfrak{a})$ -coinvariants and applying proposition 6.9, we obtain an isomorphism

$$\tilde{H}_{\text{et}}^1(U_1(\mathfrak{a}), \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}} R(\mathfrak{a}) \simeq \rho(\mathfrak{a}) \otimes \tilde{\pi}_p(\rho(\mathfrak{a})|G_{\mathbb{Q}_p}).$$

Hence (3.2.2) induces a morphism of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}[\text{GL}_2(\mathbb{Q}_p)]$ -morphism

$${}_{c,d}\tilde{\mathbf{z}}(\mathfrak{a}) : T(\mathfrak{a})(-1)^+ \otimes \tilde{\pi}_p(\rho(\mathfrak{a})|G_{\mathbb{Q}_p}) \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T(\mathfrak{a})) \otimes \tilde{\pi}_p(\rho(\mathfrak{a})|G_{\mathbb{Q}_p}).$$

Using the description of the (dual of) the so-called Montréal functor ([25, Section IV.2.3]) given in [96], it is shown in [92, Appendix B] that there is an isomorphism

$$\mathrm{End}(\tilde{\pi}_p(\rho_\Sigma|G_{\mathbb{Q}_p})) \simeq \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$$

where endomorphisms are taken in the category of $\mathcal{O}[\mathrm{GL}_2(\mathbb{Q}_p)]$ -modules which are also compact as $\mathcal{O}[[\mathrm{GL}_2(\mathbb{Z}_p)]]$ -modules and Pontryagin dual to a locally admissible $\mathcal{O}[\mathrm{GL}_2(\mathbb{Q}_p)]$ -module.⁵ The base-change properties of Colmez's Montréal functor and the same proof as in [92, Appendix B] then show that

$$\mathrm{End}(\tilde{\pi}_p(\rho(\mathfrak{a})|G_{\mathbb{Q}_p})) \simeq R(\mathfrak{a}).$$

There is thus a bijection between

$$\mathrm{Hom}_{R(\mathfrak{a})[\mathrm{GL}_2(\mathbb{Q}_p)]}(T(\mathfrak{a})(-1)^+ \otimes \tilde{\pi}_p(\rho(\mathfrak{a})|G_{\mathbb{Q}_p}), H_{\mathrm{et}}^1(\mathbb{Z}[1/\Sigma], T(\mathfrak{a})) \otimes \tilde{\pi}_p(\rho(\mathfrak{a})|G_{\mathbb{Q}_p}))$$

and

$$\mathrm{Hom}_{R(\mathfrak{a})[\mathrm{GL}_2(\mathbb{Q}_p)]}(T(\mathfrak{a})(-1)^+, H_{\mathrm{et}}^1(\mathbb{Z}[1/\Sigma], T(\mathfrak{a}))).$$

Let

$${}_{c,d}\mathbf{z}(\mathfrak{a}) : T(\mathfrak{a})(-1)^+ \longrightarrow H_{\mathrm{et}}^1(\mathbb{Z}[1/\Sigma], T(\mathfrak{a}))$$

be the morphism image of ${}_{c,d}\tilde{\mathbf{z}}(\mathfrak{a})$ through this bijection. Because $T(\mathfrak{a}) \simeq (T(\mathfrak{a})/x_1)[[\Gamma_{\mathrm{Iw}}]]$, the morphism ${}_{c,d}\mathbf{z}(\mathfrak{a})$ can be viewed as having values in $H_{\mathrm{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a}))$. Let $\lambda(f) : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \longrightarrow \mathcal{O}$ be a classical point factoring through $R(\mathfrak{a})$. Then $\tilde{\pi}_p(\rho(\mathfrak{a})|G_{\mathbb{Q}_p}) \otimes_{R(\mathfrak{a}), \lambda(f)} \mathcal{O}$ is isomorphic to $\tilde{\pi}_p(\rho_f|G_{\mathbb{Q}_p})$. Hence, the morphism

$${}_{c,d}\mathbf{z}(\mathfrak{a}) \otimes_\lambda 1 : T(f)_{\mathrm{Iw}}(-1)^+ \longrightarrow H_{\mathrm{et}}^1(\mathbb{Z}[1/p], T(f)_{\mathrm{Iw}})$$

induced by ${}_{c,d}\mathbf{z}(\mathfrak{a})$ coincides with the morphism ${}_{c,d}\mathbf{z}(f)_{\mathrm{Iw}}$ (which would be written ${}_{c,d}z_{1,M,p}^\infty$ in [46, Section 3.1.2]). \square

Let $\mu \in \mathrm{Spec} \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$ be the element $(c - \sigma_c)(d - \sigma_d)$. Let

$$\mathbf{z}(\mathfrak{a}) : T(\mathfrak{a})(-1)^+ \longrightarrow H_{\mathrm{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a})) \otimes_{R(\mathfrak{a})} R(\mathfrak{a})[1/\mu] \quad (3.2.3)$$

be the morphism $(c - \sigma_c)^{-1}(d - \sigma_d)^{-1} {}_{c,d}\mathbf{z}(\mathfrak{a})$. For $\lambda : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \longrightarrow A$ factoring through $R(\mathfrak{a})$, we denote by ${}_{c,d}\mathbf{z}(\lambda)_{\mathrm{Iw}}$ the morphism

$${}_{c,d}\mathbf{z}(\lambda)_{\mathrm{Iw}} : T_{\lambda, \mathrm{Iw}}(-1)^+ \longrightarrow H_{\mathrm{et}}^1(\mathbb{Z}[1/p], T_{\lambda, \mathrm{Iw}}).$$

The following proposition, which follows closely the strategy of proof of Nakamura, establishes the integrality of zeta morphisms over irreducible components of $\mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$. Its results are crucial in the proof of proposition 3.29, and hence in the proof of our main result.

Proposition 3.15. *Let $\mathfrak{a} \in \mathrm{Spec} \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$ be a minimal prime ideal. Under assumptions 2.1 and 3.5, the zeta morphism (3.2.3) defines a morphism*

$$\mathbf{z}(\mathfrak{a}) : T(\mathfrak{a})(-1)^+ \longrightarrow H_{\mathrm{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a}))$$

such that there is an equality

$$\mathbf{z}(\mathfrak{a}) = \left(\prod_{\ell \in \Sigma^p} \mathrm{Eul}_\ell(T(\mathfrak{a})^*(1)) \right)^{-1} \mathbf{z}_\Sigma(\mathfrak{a}) \quad (3.2.4)$$

of morphisms $T(\mathfrak{a})(-1)^+ \longrightarrow H_{\mathrm{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a}))$ and such that the following diagram commutes

$$\begin{array}{ccc} T(\mathfrak{a})(-1)^+ & \xrightarrow{\mathbf{z}(\mathfrak{a})} & H_{\mathrm{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a})) \\ \downarrow & & \downarrow \\ T(f)_{\mathrm{Iw}}(-1)^+ & \xrightarrow{\mathbf{z}(f)_{\mathrm{Iw}}} & H_{\mathrm{et}}^1(\mathbb{Z}[1/p], T(f)_{\mathrm{Iw}}) \end{array}$$

⁵Here again, we note that the supplementary assumption $p \geq 5$ appears in [92]. We can dispense from it by modifying the proof exactly in the same way as explained in the remark following the statement of theorem 3.6.

for all classical point $\lambda_f : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \longrightarrow \mathcal{O}$ factoring through $R(\mathfrak{a})$. Here, $\mathbf{z}(f)_{\text{Iw}}$ is the zeta morphism of theorem 3.4. In particular, if $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \longrightarrow S$ is an Iwasawa-suitable specialization factoring through $R(\mathfrak{a})$, then there is an equality

$$\mathbf{z}(\lambda)_{\text{Iw}} = \left(\prod_{\ell \in \Sigma^p} \text{Eul}_{\ell}(T_{\lambda, \text{Iw}}^*(1)) \right)^{-1} \mathbf{z}_{\Sigma}(\lambda_{\text{Iw}})$$

of morphisms $T_{\lambda, \text{Iw}}(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}})$.

Proof. We first show that the morphism

$$\mathbf{z}(\mathfrak{a}) : T(\mathfrak{a})(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a})) \otimes_{R(\mathfrak{a})} \text{Frac}(R(\mathfrak{a}))$$

defined by (3.2.3) has values in $H_{\text{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a}))$.

For $U_p \subset \text{GL}_2(\mathbb{Q}_p)$ a compact open subgroup, we define

$$\mathbf{z}_{U_p} = (c - \sigma_c)^{-1} (d - \sigma_d)^{-1}_{c, d} \mathbf{z}_{U_p}.$$

Let $\mathbf{T}^{\Sigma}(U_1(\mathfrak{a})U_p)_{\mathfrak{m}_{\bar{\rho}}, \text{Iw}}$ be the Hecke algebra $\mathbf{T}^{\Sigma}(U_1(\mathfrak{a})U_p)_{\mathfrak{m}_{\bar{\rho}}}[[\Gamma_{\text{Iw}}]]$. By the previous propositions, we may view \mathbf{z}_{U_p} as a morphism of \mathcal{O}_{Iw} -modules

$$H_{\text{et}}^1(Y(U_1(\mathfrak{a})U_p), \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}, \text{Iw}}(-1) \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], H_{\text{et}}^1(Y(U_1(\mathfrak{a})U_p), \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}, \text{Iw}}) \otimes_{\mathcal{O}_{\text{Iw}}} \mathcal{O}_{\text{Iw}}[1/\mu]$$

equivariant under the action of $\mathbf{T}^{\Sigma}(U_1(\mathfrak{a})U_p)_{\mathfrak{m}_{\bar{\rho}}, \text{Iw}}$ on both sides. For simplicity of notation, we write $M_{U, \bar{\rho}, \text{Iw}}$ for $H_{\text{et}}^1(Y(U_1(\mathfrak{a})U_p), \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}, \text{Iw}}$. As before, the morphisms \mathbf{z}_{U_p} are compatible with the inclusion $U'_p \subset U_p$ in the sense that the diagram

$$\begin{array}{ccc} M_{U'_p, \bar{\rho}, \text{Iw}}(-1)^+ & \xrightarrow{\mathbf{z}_{U'_p}} & H_{\text{et}}^1(\mathbb{Z}[1/p], M_{U'_p, \bar{\rho}, \text{Iw}}[1/\mu]) \\ \downarrow & & \downarrow \\ M_{U_p, \bar{\rho}, \text{Iw}}(-1)^+ & \xrightarrow{\mathbf{z}_{U_p}} & H_{\text{et}}^1(\mathbb{Z}[1/p], M_{U_p, \bar{\rho}, \text{Iw}}[1/\mu]) \end{array}$$

is commutative. In order to show that $\mathbf{z}(\mathfrak{a})$ has values in $H_{\text{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a}))$, it is thus enough to show that \mathbf{z}_{U_p} has values in $H_{\text{et}}^1(\mathbb{Z}[1/p], M_{U_p, \bar{\rho}, \text{Iw}})$ for U_p small enough. As

$$M_{U, \bar{\rho}} \otimes_{\mathcal{O}} E = \bigoplus_f \rho_f \otimes \pi(f)^{U_1(\mathfrak{a})U_p}$$

where the sum is over all $f \in S_2(U_1(\mathfrak{a})U_p)$ attached to classical points of $\mathbf{T}^{\Sigma}(U_1(\mathfrak{a})U_p)_{\mathfrak{m}_{\bar{\rho}}}$, it is enough to show that \mathbf{z}_{U_p} has values in $H_{\text{et}}^1(\mathbb{Z}[1/p], M_{U_p, \bar{\rho}, \text{Iw}})$ after projection to the eigenspace $M_{U_p, \bar{\rho}}[f]$ corresponding to the choice of one such f . As $\pi(f)^{U_1(\mathfrak{a})U_p} \neq 0$, it is an irreducible $\mathbf{T}^{\Sigma}(U_1(\mathfrak{a})U_p)_{\mathfrak{m}_{\bar{\rho}}}[1/p]$ -module ([15, Proposition 4.3]). Hence, it is enough to show that the image of any non-zero vector of $M_{U_p, \bar{\rho}}[f]$ belongs to $H_{\text{et}}^1(\mathbb{Z}[1/p], M_{U_p, \bar{\rho}}[f])$. If v is a vector in $M_{U_p, \bar{\rho}}^{U_1(\mathfrak{a}')U'_p}[f]$ such that f is new of level $U_1(\mathfrak{a}')U'_p$, this is established in [68, Theorem 12.5, 12.6].

If $\lambda_f : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \longrightarrow \mathcal{O}$ is a classical point, then ${}_{c, d} \mathbf{z}_{\Sigma}(f)_{\text{Iw}}$ satisfies

$${}_{c, d} \mathbf{z}_{\Sigma}(f)_{\text{Iw}} = (c - \sigma_c)(d - \sigma_d) \left(\prod_{\ell \in \Sigma^p} \text{Eul}_{\ell}(T(f)_{\text{Iw}}^*(1)) \right) \mathbf{z}(f)_{\text{Iw}}$$

by construction. The remaining assertions of the proposition thus follows by specializations from theorem 3.6, lemma 3.11 and the density of classical points. \square

Proposition 3.16. Let $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \longrightarrow A$ be Iwasawa-suitable. Let $V_{\lambda, \text{Iw}}$ be $T_{\lambda, \text{Iw}} \otimes_A \text{Frac}(A_{\text{Iw}})$. Then the complex

$$\text{Cone} \left(V_{\lambda, \text{Iw}}(-1)^+ \xrightarrow{\mathbf{z}(\lambda)_{\text{Iw}}} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], V_{\lambda, \text{Iw}}) \right)$$

is acyclic. The set of specializations $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}/(x_1) \longrightarrow \mathcal{O}$ which are not Iwasawa-suitable is of codimension at least 1.

Proof. Let \mathfrak{p} be the kernel of an Iwasawa-suitable specialization with values in \mathcal{O} contained in λ . Let $\ell \nmid p$ be a prime. If $H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}})$ vanishes, then so does $H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}}/\mathfrak{p})$ by relative purity at ℓ and

$$H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) \simeq H^0(I_\ell, T_\lambda \otimes_A \kappa(\mathfrak{p})).$$

If $H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}})$ has rank 2, then T_λ is unramified at ℓ , and so is $T_\lambda \otimes_A \kappa(\mathfrak{p})$, so that again

$$H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) \simeq H^0(I_\ell, T_\lambda \otimes_A \kappa(\mathfrak{p})).$$

If finally $H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}})$ is a rank 1 $A_{\mathfrak{p}}$ -module, then $H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p})$ and $H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} \text{Frac}(A_{\mathfrak{p}})$ have the same ranks so $H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}})$ is free of rank 1 by Nakayama's lemma. There is thus an isomorphism of complexes

$$[H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}}) \xrightarrow{1-\text{Fr}(\ell)} H^0(I_\ell, T_\lambda \otimes_A A_{\mathfrak{p}})] \xrightarrow{\text{L}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) \simeq [H^0(I_\ell, T_\lambda \otimes_A \kappa(\mathfrak{p})) \xrightarrow{1-\text{Fr}(\ell)} H^0(I_\ell, T_\lambda \otimes_A \kappa(\mathfrak{p}))]$$

for all $\ell \nmid p$. This family of isomorphisms induces an isomorphism

$$\text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/p], T_\lambda \otimes_A A_{\mathfrak{p}}) \xrightarrow{\text{L}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) \simeq \text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/p], T_\lambda \otimes_A \kappa(\mathfrak{p})).$$

In particular, $H^2(\mathbb{Z}[1/p], T_\lambda)$ is A -torsion and $H^1(\mathbb{Z}[1/p], T_\lambda)$ has A -rank 1. This implies that the complex

$$\text{Cone}\left(V_{\lambda, \text{Iw}}(-1)^+ \xrightarrow{\mathbf{z}(\lambda)_{\text{Iw}}} \text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/p], V_{\lambda, \text{Iw}})\right)$$

is acyclic.

We show that the set of specializations which are not Iwasawa-suitable has large codimension. [43, Lemma 3.9] implies that $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma/(x_1) \longrightarrow \mathcal{O}$ is not relatively pure at ℓ only if λ factors through $R(\mathfrak{a})$ such that $V(\mathfrak{a})$ is generically special Steinberg and if the ℓ -adic monodromy of V_λ is trivial. The set of such specializations has codimension at least 1 in each irreducible component of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$, and the same is then true of the union of these sets. To conclude, it is thus enough to show that the set of specializations $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma/(x_1) \longrightarrow \mathcal{O}$ such that $\mathbf{z}_\Sigma(\lambda)$ is non-zero has codimension at least 1.

Let $\lambda(f) : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \longrightarrow \mathcal{O}$ be a classical point of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$. The isomorphism

$$\text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma) \xrightarrow{\text{L}} \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma, \lambda(f)_{\text{Iw}}} \mathcal{O}_{\text{Iw}} \simeq \text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T(f)_{\text{Iw}})$$

and the fact that both complexes are acyclic outside degree 1 and 2 shows that there is an isomorphism of \mathcal{O}_{Iw} -modules

$$H_{\text{et}}^2(\mathbb{Z}[1/\Sigma], T_\Sigma) \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma, \lambda(f)_{\text{Iw}}} \mathcal{O}_{\text{Iw}} \simeq H_{\text{et}}^2(\mathbb{Z}[1/\Sigma], T(f)_{\text{Iw}}).$$

In particular, $H_{\text{et}}^2(\mathbb{Z}[1/\Sigma], T_\Sigma)$ is $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ -torsion. Let $a \in \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ a regular element such that $aH_{\text{et}}^2(\mathbb{Z}[1/\Sigma], T_\Sigma) = 0$. Let A be the localization of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ at $\{a^n | n \in \mathbb{N}\}$. The complex

$$\text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma) \xrightarrow{\text{L}} \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A \simeq \text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A)$$

is then a perfect complex of A -modules acyclic outside degree 1. As the cohomology of

$$\text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma) \xrightarrow{\text{L}} \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A/\mathfrak{m}_A \simeq \text{R}\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A/\mathfrak{m}_A)$$

vanishes in degree 0 and 2 (in degree 0 by assumption 2.1, in degree 2 by Nakayama's lemma), the A -module $H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A)$ is a free A -module, necessarily of rank 1. Let u be the morphism

$$u : (T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A)(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A)$$

which sends a generator of the free A -module $(T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A)(-1)^+$ to a generator of the free A -module $H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A)$. Then the morphism

$$\mathbf{z}_\Sigma \otimes 1 : (T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A)(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma} A)$$

may be written $\mathbf{z}_\Sigma \otimes 1 = \alpha u$ for some $\alpha \in A$. Let $\psi : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \rightarrow \mathcal{O}$ be such that $\psi(a) \neq 0$ and let $\psi' : A \rightarrow E$ be the morphism through which ψ factors by the universal property of localization. Then

$$R\Gamma_{et}(\mathbb{Z}[1/\Sigma], T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_\rho}^\Sigma} A) \xrightarrow{L} \otimes_{A, \psi'} E \simeq R\Gamma_{et}(\mathbb{Z}[1/\Sigma], T_\psi \otimes_{\mathcal{O}} E)$$

so $H_{et}^2(\mathbb{Z}[1/\Sigma], T_\psi \otimes_{\mathcal{O}} E)$ vanishes, $H_{et}^1(\mathbb{Z}[1/\Sigma], T_\psi \otimes_{\mathcal{O}} E)$ is free of rank 1 and

$$H_{et}^1(\mathbb{Z}[1/\Sigma], T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_\rho}^\Sigma} A) \otimes_{A, \psi'} E \simeq H_{et}^1(\mathbb{Z}[1/\Sigma], T_\psi \otimes_{\mathcal{O}} E).$$

This means that $\psi'_*(\mathbf{z}_\Sigma \otimes 1) = \mathbf{z}_\Sigma(\psi)$ is non-zero. The set of specializations $\lambda : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \rightarrow \mathcal{O}$ such that $\mathbf{z}_\Sigma(\lambda)$ vanishes is thus of codimension at least 1. This implies that the set of specializations $\lambda : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma/(x_1) \rightarrow \mathcal{O}$ such that $\mathbf{z}(\lambda)_{Iw}$ vanishes is also of codimension at least 1. \square

3.3 Fundamental lines

Let A be a quotient of $\mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$ or Λ . In particular, A could be $\mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$, $R(\mathfrak{a})$, Λ , \mathcal{O}_{Iw} , or \mathcal{O} . Let T be $T_\Sigma \otimes_{\mathbf{T}_{\mathfrak{m}_\rho}^\Sigma} A$ or $T_\Sigma \otimes_{\Lambda} A$ accordingly.

To T is attached an étale sheaf over $\text{Spec } \mathbb{Z}[1/\Sigma]$. The following lemma is well-known.

Lemma 3.17. *The complexes $R\Gamma_{et}(\mathbb{Z}[1/\Sigma], T)$ and $R\Gamma(G_{\mathbb{Q}_\ell}, T)$, for all ℓ , are perfect and commute with arbitrary derived base-change of coefficients.*

Proof. See for example [93, (4.2.9) Proposition]. \square

Definition 3.18. *The Σ -partial fundamental line $\Delta_\Sigma(T)$ is the graded invertible A -module*

$$\Delta_\Sigma(T) \stackrel{\text{def}}{=} \text{Det}_A^{-1} R\Gamma_{et}(\mathbb{Z}[1/\Sigma], T) \otimes_A \text{Det}_A^{-1} T(-1)^+.$$

Suppose in addition that A is a domain. For $\ell \in \Sigma$ prime to p , the graded invertible module $\mathcal{X}_\ell(T)$ is

$$\mathcal{X}_\ell(T) = \begin{cases} \text{Det}_A R\Gamma(G_{\mathbb{Q}_\ell}/I_\ell, T^{I_\ell}) & \text{if } \text{rank}_A T^{I_\ell} \neq 1, \\ \text{Det}_A[A \xrightarrow{1-\alpha_\ell} A] & \text{if } \text{rank}_A T^{I_\ell} = 1. \end{cases}$$

Here, the complex $[A \xrightarrow{1-\alpha_\ell} A]$ is placed in degree 0, 1. The fundamental line $\Delta_A(T)$ is the graded invertible A -module

$$\Delta(T) \stackrel{\text{def}}{=} \text{Det}_A^{-1} R\Gamma_{et}(\mathbb{Z}[1/\Sigma], T) \otimes_A \text{Det}_A^{-1} T(-1)^+ \otimes_A \bigotimes_{\ell \in \Sigma} (\mathcal{X}_\ell(T) \otimes_A \text{Det}_A^{-1} R\Gamma(G_{\mathbb{Q}_\ell}, T)).$$

Note that the previous definition makes sense in all cases thanks to lemma 3.17. It follows from the same lemma that any morphism $\psi : A \rightarrow A'$ induces a canonical isomorphism

$$\Delta_A(T) \otimes_A A' \xrightarrow{\text{can}} \Delta_{A'}(T').$$

Definition 3.19. *A specialization $\lambda : \Lambda/(x_1) \rightarrow A$ with values in a domain A is Iwasawa-suitable if it contains a specialization $\psi : \Lambda/(x_1) \rightarrow \mathcal{O}$ such that all specialization $\psi' : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma/(x_1) \rightarrow \mathcal{O}$ above ψ are Iwasawa-suitable. We say that a specialization $\lambda : A \rightarrow S$ is Iwasawa-suitable if there exists an Iwasawa-suitable specialization λ' such that the diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\lambda'} & S \\ \downarrow & \nearrow \lambda & \\ A & & \end{array}$$

commutes.

Proposition 3.20. *Let A be a domain. The fundamental line $\Delta_A(T)$ is canonically isomorphic to*

$$\text{Det}_A^{-1} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], T) \otimes_A \text{Det}_A^{-1} T(-1)^+$$

when both are defined. Suppose that $\psi : A \rightarrow A'$ is an Iwasawa-suitable specialization which is contained in λ . Then the natural map

$$\Delta_A(T) \otimes_{A,\psi} A' \rightarrow \Delta_{A'}(T')$$

is an isomorphism.

Proof. If $\text{Det}_A^{-1} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], T)$ is defined, then $R\Gamma_{\text{et}}(\mathbb{Z}[1/p], T)$ is a perfect complex of A -modules, so T^{I_ℓ} is a perfect complex of A -modules for all $\ell \in \Sigma$. In that case, taking a projective resolution of T^{I_ℓ} induces a canonical isomorphism between $\text{Det}_A[T^{I_\ell}] \xrightarrow{1-\text{Fr}(\ell)} T^{I_\ell}$ and $\mathcal{X}_\ell(T)$. The properties of the Det functor and the collection of these canonical isomorphisms for all $\ell \nmid p$ induce a canonical isomorphism between

$$\text{Det}_A^{-1} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], T) \otimes_A \text{Det}_A^{-1} T(-1)^+$$

first with

$$\text{Det}_A^{-1} \text{Cone} \left(R\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T) \oplus \bigoplus_{\ell \in \Sigma} R\Gamma(G_{\mathbb{Q}_\ell}, T) \longrightarrow \bigoplus_{\ell \in \Sigma} R\Gamma(G_{\mathbb{Q}_\ell}/I_\ell, T^{I_\ell}) \right) [-1] \otimes_A \text{Det}_A^{-1} T(-1)^+.$$

then with

$$\text{Det}_A^{-1} R\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], T) \otimes_A \text{Det}_A^{-1} T(-1)^+ \otimes_A \bigotimes_{\ell \in \Sigma} (\mathcal{X}_\ell(T) \otimes_A \text{Det}_A^{-1} R\Gamma(G_{\mathbb{Q}_\ell}, T)).$$

Let ψ be as in the lemma. Then $H^0(I_\ell, T)$ and $H^0(I_\ell, T')$ have the same rank and $\psi(\alpha_\ell) = \alpha'_\ell$ for all $\ell \nmid p$. Hence $\mathcal{X}_\ell(T) \otimes_{A,\psi} A'$ is canonically isomorphic to $\mathcal{X}_\ell(T')$. As all other constituents of $\Delta_A(T)$ commute with $- \otimes_{A,\psi} A'$ for general reasons, the collection of the canonical isomorphisms $\mathcal{X}_\ell(T) \otimes_{A,\psi} A' \xrightarrow{\text{can}} \mathcal{X}_\ell(T')$ induce a canonical isomorphism $\Delta_A(T) \otimes_{A,\psi} A' \xrightarrow{\text{can}} \Delta_{A'}(T')$. \square

It follows from lemma 3.13 that the fundamental lines of Iwasawa-suitable specializations come with canonical isomorphisms to the trivial graded invertible module.

Definition 3.21. *Let $\lambda : \mathbf{T}_{\mathfrak{m}_p}^\Sigma \rightarrow A$ be an Iwasawa-suitable specialization with zeta morphism*

$$\mathbf{z}(\lambda)_{\text{Iw}} : V_{\lambda, \text{Iw}}(-1)^+ \xrightarrow{\sim} H_{\text{et}}^1(\mathbb{Z}[1/p], V_{\lambda, \text{Iw}}).$$

The trivialization map

$$\text{triv}_{\mathbf{z}(\lambda)_{\text{Iw}}} : \Delta_{A_{\text{Iw}}}(T_{\lambda, \text{Iw}}) \longrightarrow \text{Frac}(A_{\text{Iw}})$$

is the composition

$$\Delta_{A_{\text{Iw}}}(T_{\lambda, \text{Iw}}) \hookrightarrow \Delta_{\text{Frac}(A_{\text{Iw}})}(V_{\lambda, \text{Iw}}) \xrightarrow{\text{can}} \text{Det}_{\text{Frac}(A_{\text{Iw}})} \text{Cone}(\mathbf{z}(\lambda)_{\text{Iw}})[-1] \xrightarrow{\text{can}} \text{Frac}(A_{\text{Iw}}).$$

where the first canonical isomorphism is

$$R\Gamma(\mathbb{Z}[1/p], V_{\lambda, \text{Iw}}) \simeq H_{\text{et}}^1(\mathbb{Z}[1/p], V_{\lambda, \text{Iw}})[-1]$$

and where the second follows from the acyclicity of the complex

$$\text{Cone} \left(V_{\lambda, \text{Iw}}(-1)^+ \xrightarrow{\mathbf{z}(\lambda)_{\text{Iw}}} R\Gamma_{\text{et}}(\mathbb{Z}[1/p], V_{\lambda, \text{Iw}}) \right).$$

If more generally Λ is $\mathbf{T}_{\mathfrak{m}_p}^\Sigma/x_1$ or Λ/x_1 and if $\lambda : \Lambda \rightarrow S$ is an Iwasawa suitable specialization with values in a reduced ring and if $V_{\lambda, \text{Iw}}$ denotes $T_{\lambda, \text{Iw}} \otimes_{S_{\text{Iw}}} Q(S_{\text{Iw}})$, the trivialization map

$$\text{triv}_{\mathbf{z}_\Sigma(\lambda_{\text{Iw}})} : \Delta_\Sigma(T_{\lambda, \text{Iw}}) \longrightarrow Q(S_{\text{Iw}})$$

is the composition

$$\Delta_\Sigma(T_{\lambda, \text{Iw}}) \hookrightarrow \Delta_\Sigma(V_{\lambda, \text{Iw}}) \xrightarrow{\text{can}} \text{Det}_\Sigma \text{Cone}(\mathbf{z}_\Sigma(\lambda_{\text{Iw}}))[-1] \xrightarrow{\text{can}} Q(S_{\text{Iw}}). \quad (3.3.1)$$

Note that if $\lambda : \Lambda \rightarrow S$ is Iwasawa-suitable, then the isomorphism

$$R\Gamma_{et}(\mathbb{Z}[1/\Sigma], T_{\lambda, Iw}) \xrightarrow{L} \otimes_{S_{Iw}, \psi} \mathcal{O}_{Iw}$$

for ψ an Iwasawa-suitable specialization shows that $H^2_{et}(\mathbb{Z}[1/\Sigma], T_{\lambda, Iw})$ is S_{Iw} -torsion so that the last isomorphism in (3.3.1) is well defined.

Proposition 3.22. *Let $\lambda : T_{m, \bar{\rho}}^\Sigma \rightarrow A$ be an Iwasawa-suitable specialization and let $\psi : A \rightarrow S$ be an Iwasawa-suitable specialization contained in λ . The canonical isomorphism*

$$\Delta_{A_{Iw}}(T_{\lambda, Iw}) \otimes_{A_{Iw}, \psi} S_{Iw} \xrightarrow{\sim} \Delta_{S_{Iw}}(T_{\psi, Iw})$$

of proposition 3.20 fits into a commutative diagram

$$\begin{array}{ccc} \Delta_{A_{Iw}}(T_{\lambda, Iw}) & \xrightarrow{\text{triv}_{z(\lambda)_{Iw}}} & \frac{x}{y} A_{Iw} \\ - \otimes_{A_{Iw}, \psi} S_{Iw} \downarrow & & \downarrow \psi \\ \Delta_{S_{Iw}}(T_{\psi, Iw}) & \xrightarrow{\text{triv}_{z(\psi)_{Iw}}} & \frac{x'}{y'} S_{Iw} \end{array} .$$

In particular, the morphism ψ extends to a morphism $\frac{x}{y} A_{Iw} \rightarrow \frac{x'}{y'} S_{Iw}$.

Proof. Let $\mathfrak{p} \subset \text{Spec } A_{Iw}$ be the kernel of $\psi : A_{Iw} \rightarrow S_{Iw}$. We have seen in the proof of proposition 3.16 that $R\Gamma_{et}(\mathbb{Z}[1/p], T_{\lambda, Iw} \otimes_{A_{Iw}} A_{Iw, \mathfrak{p}})$ is a perfect complex which commutes with $- \otimes_{A_{Iw, \mathfrak{p}}} \kappa(\mathfrak{p})$ and that the complex

$$\text{Cone} \left((T_{\lambda, Iw} \otimes_{A_{Iw}} A_{Iw, \mathfrak{p}})(-1)^+ \xrightarrow{\text{z}(\lambda)_{Iw}} R\Gamma_{et}(\mathbb{Z}[1/p], T_{\lambda, Iw} \otimes_{A_{Iw}} A_{Iw, \mathfrak{p}}) \right)$$

is acyclic. By functoriality of Det, there is a commutative diagram

$$\begin{array}{ccc} \Delta_{A_{Iw}}(T_{\lambda, Iw}) & \hookrightarrow & \Delta_{A_{Iw, \mathfrak{p}}}(T_{\lambda, Iw} \otimes_{A_{Iw}} A_{Iw, \mathfrak{p}}) & \xrightarrow{\sim} & A_{Iw, \mathfrak{p}} \\ - \otimes_{A_{Iw}} S_{Iw} \downarrow & & - \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) \downarrow & & \downarrow - \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) \\ \Delta_{S_{Iw}}(T_{\psi, Iw}) & \hookrightarrow & \Delta_{\kappa(\mathfrak{p})}(V_{\psi, Iw}) & \xrightarrow{\sim} & \kappa(\mathfrak{p}) \end{array}$$

whose vertical arrows are induced by ψ . In particular, if $\text{triv}_{z(\lambda)_{Iw}}(\Delta_{A_{Iw}}(T_{\lambda, Iw}))$ is generated by x/y , then y may be chosen so that it does not belong to \mathfrak{p} and $\psi(x)/\psi(y)$ generates $\text{triv}_{z(\psi)_{Iw}}(\Delta_{S_{Iw}}(T_{\psi, Iw}))$. \square

The preceding lemma admits a variant which applies to Σ -partial fundamental lines.

Proposition 3.23. *Let $\lambda : T_{m, \bar{\rho}}^\Sigma \rightarrow A$ be an Iwasawa-suitable specialization and let $\psi : A \rightarrow S$ be an Iwasawa-suitable specialization contained in λ . The canonical isomorphism*

$$\Delta_\Sigma(T_{\lambda, Iw}) \otimes_{A_{Iw}, \psi} S_{Iw} \xrightarrow{\sim} \Delta_\Sigma(T_{\psi, Iw})$$

fits into a commutative diagram

$$\begin{array}{ccc} \Delta_\Sigma(T_{\lambda, Iw}) & \xrightarrow{\text{triv}_{z_\Sigma(\lambda)_{Iw}}} & \frac{x}{y} A_{Iw} \\ - \otimes_{A_{Iw}, \psi} S_{Iw} \downarrow & & \downarrow \psi \\ \Delta_\Sigma(T_{\psi, Iw}) & \xrightarrow{\text{triv}_{z_\Sigma(\psi)_{Iw}}} & \frac{x'}{y'} S_{Iw} \end{array} .$$

In particular, the morphism ψ extends to a morphism $\frac{x}{y} A_{Iw} \rightarrow \frac{x'}{y'} S_{Iw}$.

Proof. As $R\Gamma_{et}(\mathbb{Z}[1/\Sigma], -)$ commutes with arbitrary change of coefficients and sends perfect complexes to perfect complexes, the proof is the same as that of proposition 3.22 only easier. \square

3.4 The Equivariant Tamagawa Number Conjectures

The following conjecture is Kato's statement of the Iwasawa Main Conjecture of classical Iwasawa theory ([65]).

Conjecture 3.24. *Let $\lambda(f)_{\text{Iw}} : \mathbf{T}_{\mathfrak{m}, \bar{\rho}}^{\Sigma} \rightarrow \mathcal{O}_{\text{Iw}}$ be the specialization attached to a classical point $\lambda(f)$. Then the trivialization morphism*

$$\text{triv}_{\mathbf{z}(f)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T(f)_{\text{Iw}}) \hookrightarrow \text{Frac}(\mathcal{O}_{\text{Iw}})$$

given by the zeta morphism

$$\mathbf{z}(f)_{\text{Iw}} : T(f)_{\text{Iw}}(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/p], T(f)_{\text{Iw}})$$

induces an isomorphism

$$\text{triv}_{\mathbf{z}(f)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T(f)_{\text{Iw}}) \xrightarrow{\text{can}} \mathcal{O}_{\text{Iw}}.$$

In [66], the preceding conjecture was extended to arbitrary Λ -adic families of motives. In our situation of interest, we obtain the following conjecture.

Conjecture 3.25. *Let $\lambda : \mathbf{T}_{\mathfrak{m}, \bar{\rho}}^{\Sigma} \rightarrow S$ be an Iwasawa-suitable specialization. Then the trivialization morphism*

$$\text{triv}_{\mathbf{z}_{\Sigma}(\lambda)_{\text{Iw}}} : \Delta_{\Sigma}(T_{\lambda, \text{Iw}}) \hookrightarrow Q(S_{\text{Iw}})$$

given by the zeta morphism

$$\mathbf{z}_{\Sigma}(\lambda)_{\text{Iw}} : T_{\lambda, \text{Iw}}(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\lambda, \text{Iw}})$$

induces an isomorphism

$$\text{triv}_{\mathbf{z}_{\Sigma}(\lambda)} : \Delta_{\Sigma}(T_{\lambda}) \xrightarrow{\text{can}} S_{\text{Iw}}.$$

Assume moreover that S is a domain. Then the trivialization morphism

$$\text{triv}_{\mathbf{z}(\lambda)_{\text{Iw}}} : \Delta_{S_{\text{Iw}}}(T_{\lambda, \text{Iw}}) \hookrightarrow \text{Frac}(S_{\text{Iw}})$$

given by the zeta morphism

$$\mathbf{z}(\lambda)_{\text{Iw}} : T_{\lambda, \text{Iw}}(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}})$$

induces an isomorphism

$$\text{triv}_{\mathbf{z}(\lambda)_{\text{Iw}}} : \Delta_{S_{\text{Iw}}}(T_{\lambda, \text{Iw}}) \xrightarrow{\text{can}} S_{\text{Iw}}.$$

Note in particular the following special cases of interests.

Conjecture 3.26. *The trivialization morphism*

$$\text{triv}_{\mathbf{z}_{\Sigma}} : \Delta_{\Sigma}(T_{\Sigma}) \hookrightarrow Q(\mathbf{T}_{\mathfrak{m}, \bar{\rho}}^{\Sigma})$$

given by the zeta morphism

$$\mathbf{z}_{\Sigma} : T_{\Sigma}(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\Sigma})$$

induces an isomorphism

$$\text{triv}_{\mathbf{z}_{\Sigma}} : \Delta_{\Sigma}(T_{\Sigma}) \xrightarrow{\text{can}} \mathbf{T}_{\mathfrak{m}, \bar{\rho}}^{\Sigma}.$$

In view of proposition 2.6, conjecture 3.26 is the most general statement on the p -adic variation of special values of L -functions that can be formulated for modular motives.

Conjecture 3.27. *Let $\mathbf{z}(\mathfrak{a})$ be the morphism $\mathbf{z}(\lambda)$ for $\lambda : \mathbf{T}_{\mathfrak{m}, \bar{\rho}}^{\Sigma} \rightarrow R(\mathfrak{a})$. The trivialization morphism*

$$\text{triv}_{\mathbf{z}(\mathfrak{a})} : \Delta_{R(\mathfrak{a})}(T_{\lambda}) \hookrightarrow \text{Frac}(\mathfrak{a})$$

given by the zeta morphism

$$\mathbf{z}(\mathfrak{a}) : T(\mathfrak{a})(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/p], T(\mathfrak{a}))$$

induces an isomorphism

$$\text{triv}_{\mathbf{z}(\mathfrak{a})} : \Delta_{R(\mathfrak{a})}(T(\mathfrak{a})) \xrightarrow{\text{can}} R(\mathfrak{a}).$$

We record the following compatibility property between the conjectures stated so far.

Proposition 3.28. *Let $\lambda : \mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma} \rightarrow S$ be an Iwasawa-suitable specialization (resp. with values in a domain). Assume that conjecture 3.26 is true (resp. conjecture 3.27 for $R(\mathfrak{a})$ an irreducible component through which λ factors). Then conjecture 3.25 is true for λ (resp. the second half of conjecture 3.25 is true for λ).*

Proof. According to proposition 3.23, if $\text{triv}_{\mathbf{z}_{\Sigma}} : \Delta_{\Sigma}(T_{\Sigma}) \xrightarrow{\text{can}} \mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma}$, then $\text{triv}_{\mathbf{z}_{\Sigma}(\lambda)} : \Delta_{\Sigma}(T_{\lambda}) \simeq \Lambda$. Hence, the first half of conjecture 3.25 is true. Next, we establish the second half of this conjecture when λ is the quotient map $\mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma} \rightarrow R(\mathfrak{a})$. First, we record the commutative diagram

$$\begin{array}{ccc} \Delta_{\Sigma}(T_{\Sigma}) & \xrightarrow{\text{triv}_{\mathbf{z}_{\Sigma}}} & \mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma} \\ - \otimes_{\mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma}} R(\mathfrak{a}) \downarrow & & \downarrow \lambda \\ \Delta_{\Sigma}(T(\mathfrak{a})) & \xrightarrow{\text{triv}_{\mathbf{z}_{\Sigma}(\mathfrak{a})}} & R(\mathfrak{a}). \end{array}$$

Let \mathcal{X} be $\Delta_{R(\mathfrak{a})}(T(\mathfrak{a})) \otimes_{R(\mathfrak{a})} \Delta_{\Sigma}(T(\mathfrak{a}))^{-1}$. By definition of $R\Gamma_{\text{et}}(\mathbb{Z}[1/\Sigma], -)$ and $R\Gamma_{\text{et}}(\mathbb{Z}[1/p], -)$, we have

$$\mathcal{X} \simeq \bigotimes_{\ell \in \Sigma} \mathcal{X}_{\ell}(T(\mathfrak{a})) \otimes \text{Det}_{R(\mathfrak{a})}^{-1} R\Gamma(G_{\mathbb{Q}_{\ell}}, T(\mathfrak{a})).$$

Hence

$$\mathcal{X} \otimes_{R(\mathfrak{a})} \text{Frac}(\mathfrak{a}) \simeq \bigotimes_{\ell \in \Sigma} \mathcal{X}_{\ell}(V(\mathfrak{a})^*(1))^{-1}$$

by Tate's local duality. Hence $\text{triv}_{\mathbf{z}_{\Sigma}(\mathfrak{a})}$ sends the $R(\mathfrak{a})$ -submodule $\Delta_{R(\mathfrak{a})}(T(\mathfrak{a}))$ of $\Delta_{\Sigma}(T(\mathfrak{a})) \otimes_{R(\mathfrak{a})} \text{Frac}(\mathfrak{a})$ to $\prod_{\ell} \text{Eul}_{\ell}(V(\mathfrak{a})^*(1))^{-1} R(\mathfrak{a})$. As

$$\mathbf{z}(\mathfrak{a}) = \left(\prod_{\ell} \text{Eul}_{\ell}(V(\mathfrak{a})^*(1))^{-1} \right) \mathbf{z}_{\Sigma}(\mathfrak{a}),$$

there is an isomorphism

$$\text{triv}_{\mathbf{z}(\mathfrak{a})} : \Delta_{R(\mathfrak{a})}(T(\mathfrak{a})) \simeq R(\mathfrak{a}).$$

Hence conjecture 3.25 is true in this case.

Now let $\lambda : \mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma} \rightarrow S$ be an arbitrary Iwasawa-suitable specialization with values in a domain. Then $\mathbf{z}(\lambda)_{\text{Iw}}$ is the image of $\mathbf{z}(\mathfrak{a})$ through λ and $\Delta_{R(\mathfrak{a})}(T(\mathfrak{a})) \otimes_{R(\mathfrak{a})} S_{\text{Iw}} \simeq \Delta_{S_{\text{Iw}}}(T_{\lambda, \text{Iw}})$ so there is an isomorphism

$$\text{triv}_{\mathbf{z}(\lambda)_{\text{Iw}}} : \Delta_{S_{\text{Iw}}}(T_{\lambda, \text{Iw}}) \simeq S_{\text{Iw}}$$

and conjecture 3.25 is thus true for λ . □

The method of Euler system ([101, 109, 67]) yields the following partial result towards conjecture 3.25 when $\lambda : \mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma} \rightarrow \mathcal{O}$ is an Iwasawa-suitable specialization.

Proposition 3.29. *Let $\lambda : \mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma} \rightarrow \mathcal{O}$ be an Iwasawa-suitable specialization. Then the image of*

$$\text{triv}_{\mathbf{z}(\lambda)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T_{\lambda, \text{Iw}})^{-1} \hookrightarrow \text{Frac}(\mathcal{O}_{\text{Iw}})$$

is included inside \mathcal{O}_{Iw} .

Proof. This proof relies crucially on the fact that $\mathbf{z}(\lambda)_{\text{Iw}}$ has values in $H_{\text{et}}^1(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}})$ (rather than $H_{\text{et}}^1(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}}) \otimes_{\mathcal{O}_{\text{Iw}}} \text{Frac}(\mathcal{O}_{\text{Iw}})$) when $\lambda : \mathbf{T}_{\mathfrak{m}\bar{\rho}}^{\Sigma} \rightarrow \mathcal{O}$ is an Iwasawa-suitable specialization, as was proved in proposition 3.15. As recalled in the proof of lemma 3.13, the representation $T_{\lambda, \text{Iw}}$ then satisfies all hypotheses of [67, Theorem 0.8] except possibly (iii) whose role is replaced here by the statement of [67, Proposition 8.7]. We thus have

$$\text{char}_{\mathcal{O}_{\text{Iw}}} H_{\text{et}}^2(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}}) \mid \text{char}_{\mathcal{O}_{\text{Iw}}} H_{\text{et}}^1(\mathbb{Z}[1/p], T_{\lambda, \text{Iw}}) / \text{Im } \mathbf{z}(\lambda)_{\text{Iw}}.$$

After localization at a height-one prime, the structure theorem of modules of discrete valuation ring shows that this divisibility is equivalent to the statement that the image of

$$\text{triv}_{\mathbf{z}(\lambda)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T_{\lambda, \text{Iw}})^{-1} \hookrightarrow \text{Frac}(\mathcal{O}_{\text{Iw}})$$

is included inside \mathcal{O}_{Iw} . □

We are interested in the following weaker version of conjecture 3.26 in which we view T_Σ as a Λ -module.

Conjecture 3.30. *The trivialization morphism*

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \hookrightarrow \text{Frac}(\Lambda)$$

given by the zeta morphism

$$\mathbf{z}_\Sigma : T_\Sigma(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_\Sigma)$$

induces an isomorphism

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \xrightarrow{\text{can}} \Lambda.$$

4 The residually irreducible, crystalline case

In this section, we prove conjecture 3.24 for the $G_{\mathbb{Q}}$ -representation $(T(f), \rho_f, \mathcal{O})$ when $f \in S_k(\Gamma_0(N))$ is an eigencuspform of even weight k with trivial central character, with coefficients in a number field $F \subset E$, such that $\rho_f|G_{\mathbb{Q}_p}$ is a crystalline and short $G_{\mathbb{Q}_p}$ -representation and such that $\bar{\rho}_f|G_{\mathbb{Q}_p}$ is an irreducible $G_{\mathbb{Q}_p}$ -representation. We write (T_f, ρ, \mathcal{O}) for the $G_{\mathbb{Q}}$ -representation $(T(f)(k/2), \rho_f(k/2), \mathcal{O})$ and let V_f be $T_f \otimes_{\mathcal{O}} E$.

We record once and for all the following assumptions, that will be crucial in all the section.

Assumption 4.1. *The $G_{\mathbb{Q}_p}$ -representation $\rho|G_{\mathbb{Q}_p}$ is crystalline and short with irreducible residual representation.*

We write $\mathbb{Q}_{p,\infty}$ for the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q}_p , $\mathbb{Q}_p^{p,\text{ur}}$ for the unramified \mathbb{Z}_p -extension of \mathbb{Q}_p and $\mathbb{Z}_p^{p,\text{ur}}$ for the unit ball of $\mathbb{Q}_p^{p,\text{ur}}$. Let $\mathbb{Q}_{p,\infty}^{p,\text{ur}}$ be the unramified \mathbb{Z}_p -extension of $\mathbb{Q}_{p,\infty}$. As before, we let Γ_{Iw} be $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and fix one of its topological generator γ . We write $\Gamma_{p,\infty}^{p,\text{ur}} \simeq \mathbb{Z}_p^2$ for the Galois group $\text{Gal}(\mathbb{Q}_{p,\infty}^{p,\text{ur}}/\mathbb{Q}_p)$. Let \mathcal{K} be a quadratic imaginary extension of \mathbb{Q} where p splits as $v_0 \bar{v}_0$. Let \mathcal{K}_∞ be the \mathbb{Z}_p^2 -extension of \mathcal{K} . We write $\Gamma_{\mathcal{K}}$ for the Galois group $\text{Gal}(\mathcal{K}_\infty/\mathcal{K})$. Define $\Lambda_{\mathcal{K}} = \mathcal{O}[[\Gamma_{\mathcal{K}}]]$. Let \mathcal{K}^{v_0} and $\mathcal{K}^{\bar{v}_0}$ be respectively the \mathbb{Z}_p -extension of \mathcal{K} unramified outside v_0 and unramified outside \bar{v}_0 . Define $\Gamma_{v_0} = \text{Gal}(\mathcal{K}^{v_0}/\mathcal{K})$ and $\Gamma_{\bar{v}_0} = \text{Gal}(\mathcal{K}^{\bar{v}_0}/\mathcal{K})$. We fix topological generators $\gamma_{v_0} \in \Gamma_{v_0}$ and $\gamma_{\bar{v}_0} \in \Gamma_{\bar{v}_0}$.

4.1 Review of p -adic Hodge theory

We introduce necessary material in Iwasawa cohomology (including over affinoid rings) and recall standard notions of p -adic Hodge theory ([37, 39]). In this section, (T, ρ, \mathcal{O}) and (V, ρ, E) denote $G_{\mathbb{Q}_p}$ -representations.

4.1.1 Iwasawa Cohomology Groups

The classical Iwasawa cohomology $H_{\text{cl}, \text{Iw}}^1(\mathbb{Q}_{p,\infty}, T)$ is the inverse limit with respect to the co-restriction map.

$$\varprojlim_{\mathbb{Q}_p \subseteq \mathbb{Q}_{p,n} \subset \mathbb{Q}_{p,\infty}} H^1(G_{\mathbb{Q}_{p,n}}, T).$$

More generally, if $K \subset K_n \subset K_\infty$ is a tower of abelian separable extensions, we define $H_{\text{cl}, \text{Iw}}^i(K_\infty, T)$ to be the i -th cohomology group of the complex $R\Gamma_{\text{cl}, \text{Iw}}(K_\infty, T)$ defined to be the image in the derived category of

$$\varprojlim_n C_{\text{cont}}^\bullet(G_K, T \otimes_{\mathcal{O}} \mathcal{O}[\text{Gal}(K_n/K)])$$

(see [93, Section 8] for details). The Iwasawa cohomology complex $R\Gamma_{\text{cl}, \text{Iw}}(K_\infty, V)$ is the image in the derived category of

$$\varprojlim_n C_{\text{cont}}^\bullet(G_K, T \otimes_{\mathcal{O}} \mathcal{O}[\text{Gal}(K_n/K)][1/p])$$

for T any G_K -stable \mathcal{O} -lattice inside V .

Lemma 4.2. Assume that $\bar{\rho}$ is irreducible. For A equal to $\mathbb{F}, \mathcal{O}, \mathcal{O}_{\text{Iw}}$ or $\mathcal{O}[[\Gamma_1]]$, the complex $R\Gamma(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} A)$ is a perfect complex of A -modules whose cohomology is concentrated in degree 1. Moreover, the A -module $H^1(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} A)$ is free of rank 2. In particular,

$$H_{\text{cl}, \text{Iw}}^1(\mathbb{Q}_{p, \infty}, T) \simeq \mathcal{O}_{\text{Iw}}^2$$

and

$$H_{\text{cl}, \text{Iw}}^1(\mathbb{Q}_{p, \infty}^{p, \text{ur}}, T) \simeq \mathcal{O}[[\Gamma_{p, \infty}^{p, \text{ur}}]]^2.$$

Proof. As $R\Gamma(G_{\mathbb{Q}_p}, - \otimes_{\mathcal{O}} A)$ sends perfect complexes to perfect complexes and commutes with $- \otimes_A S$ for arbitrary S , the statements for $A = \mathcal{O}, \mathcal{O}_{\text{Iw}}$ and $\mathcal{O}[[\Gamma_{p, \infty}^{p, \text{ur}}]]$ follow from the statement for $A = \mathbb{F}$ by Nakayama's lemma. In that case, $H^0(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathbb{F}) = H^2(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathbb{F}) = 0$ by the assumption and Tate local duality. Hence, the local Euler characteristic formula

$$\frac{|H^0(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathbb{F})| \cdot |H^2(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathbb{F})|}{|H^1(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathbb{F})|} = |\mathbb{F}|^{-2}$$

reduces to $|H^1(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathbb{F})| = |\mathbb{F}|^2$ and hence to $H^1(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathbb{F}) \simeq \mathbb{F}^2$. \square

4.1.2 (φ, Γ) -modules

Let \mathbb{C}_p be the p -adic completion of $\bar{\mathbb{Q}}_p$ and let $\mathcal{O}_{\mathbb{C}_p}$ be its unit ball, that is to say the set of elements with p -adic norm less than or equal to 1. Let \tilde{E}^+ be the perfect characteristic p ring

$$\tilde{E}^+ \stackrel{\text{def}}{=} \{x = (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}, \forall n \in \mathbb{N}, x_{n+1}^p = x_n\}.$$

If $x = (x_n)_{n \in \mathbb{N}}$ belongs to \tilde{E}^+ and if for all $n \in \mathbb{N}$, \hat{x}_n is a lifting of x_n to $\mathcal{O}_{\mathbb{C}_p}$, then for all $n \in \mathbb{N}$ the limit

$$x^{(n)} = \lim_k (\hat{x}_{n+k})^{p^k} \in \mathcal{O}_{\mathbb{C}_p}$$

does not depend on the choices of the \hat{x}_{n+k} . We identify \tilde{E}^+ with

$$\mathcal{O}_{\mathbb{C}_p}^b \stackrel{\text{def}}{=} \left\{ x = (x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in \mathcal{O}_{\mathbb{C}_p}, \forall n \in \mathbb{N}, (x^{(n+1)})^p = x^{(n)} \right\}.$$

through the map $x \mapsto (x^{(n)})_{n \in \mathbb{N}}$. If v_p is the valuation on \mathbb{C}_p normalized so that $v_p(p) = 1$, then the valuation

$$v(x) = v_p(x^{(0)}) = \lim_n p^n v_p(\hat{x}_n)$$

makes \tilde{E}^+ a complete valuation ring (in particular a domain). Let \tilde{E} be the fraction field of \tilde{E}^+ .

Fix once and for all non-trivial p^n -th roots of unity ζ_{p^n} with $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ and write ε for the element $(1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{\mathbb{C}_p}^b$. Then

$$v(\varepsilon - 1) = \lim_n p^n v_p(\zeta_{p^n} - 1) = \lim_n p^n \frac{1}{(p-1)p^{n-1}} = \frac{p}{p-1} > 1.$$

We write \tilde{A}^+, \tilde{A} and $[.]$ respectively for the ring of Witt vectors of \tilde{E}^+ or \tilde{E} and for the Teichmüller lift from \tilde{E}^+ to \tilde{A}^+ or from \tilde{E} to \tilde{A} . Then $x \in \tilde{A}^+$ may be written in a unique way

$$x = \sum_{m=0}^{\infty} p^m [x_m]$$

with $x_m \in \mathcal{O}_{\mathbb{C}_p}^b$. Let φ be the Frobenius map on \tilde{E}^+ . We also denote by φ the functorial extension of φ to \tilde{A}^+ , that is to say the map

$$\varphi(x) = \sum_{m=0}^{\infty} p^m [x_m^p].$$

J-M.Fontaine observed that the map

$$\theta : \tilde{A}^+ \longrightarrow \mathcal{O}_{\mathbb{C}_p}$$

defined either by

$$\theta((x_n)_{n \in \mathbb{N}}) = \sum_{n=0}^{\infty} p^n x_n^{(n)} \quad (4.1.1)$$

or by

$$\theta \left(\sum_{m=0}^{\infty} p^m [b_m] \right) = \sum_{m=0}^{\infty} p^m b_m^{(0)} \quad (4.1.2)$$

is a surjective ring homomorphism with principal kernel. The ring B_{dR}^+ is the completion

$$B_{\text{dR}}^+ \stackrel{\text{def}}{=} \varprojlim_{n \geq 0} \tilde{A}^+[1/p]/(\ker(\theta))^n$$

of \tilde{A}^+ with respect to $\ker \theta$. Let $\pi \in \tilde{A}^+$ be $[\varepsilon] - 1$. Then $\theta(\pi) = 0$. We define

$$t = \log(\varepsilon) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} ([\varepsilon] - 1)^n \in B_{\text{dR}}^+.$$

Then B_{dR}^+ a discrete valuation ring with maximal ideal (t) and residue field \mathbb{C}_p .

Lemma 4.3. *Let $[a]$ be the lift of $a \in \mathbb{F}_p$ in \mathbb{Z}_p . Put*

$$q = \varphi^{-1} \left(\sum_{a \in \mathbb{F}_p} [\varepsilon]^{|a|} \right) \in \tilde{A}^+.$$

Then $q \in \ker \theta$ but $q/\pi \notin \ker \theta$.

Proof. By (4.1.1)

$$\theta(q) = \sum_{a \in \mathbb{F}_p} \zeta_p^a = 0.$$

Choose $x \in \tilde{E}^+$ with $x^{(0)} = -p$. Let $\xi \in \tilde{A}^+$ be the element $[x] + p$. Then

$$\theta(\xi) = -p + p = 0$$

by (4.1.2) and ξ is a generator of the principal ideal $\ker \theta$ by [38, Proposition 2.4]. If $\alpha \in \tilde{A}^+$ satisfies $\theta(\alpha/\pi) = 0$, then there exists $\lambda \in \tilde{A}^+$ such that $\alpha = \xi^2 \lambda$. So the valuation of the image of α in the residual field \mathbb{C}_p is at least 2. The valuation of the class of q in \mathbb{C}_p is 1. \square

Definition 4.4. *Let A_{crys}^0 be the divided power envelop of \tilde{A}^+ with respect to $\ker \theta$, that is to say the set obtained by adding all elements $a^m/m!$ for all $a \in \ker \theta$. Let A_{crys} and B_{crys} be the rings $A_{\text{crys}} \stackrel{\text{def}}{=} \varprojlim_n A_{\text{crys}}^0/p^n A_{\text{crys}}^0$ and $B_{\text{crys}}^+ \stackrel{\text{def}}{=} A_{\text{crys}}[1/p]$.*

For $n \in \mathbb{N}$, we write $\gamma_n(x) = \frac{x^n}{n!}$. Let $\text{Fil}^r A_{\text{crys}}$ be $A_{\text{crys}} \cap \text{Fil}^r B_{\text{dR}}$ and let $\text{Fil}_p^r A_{\text{crys}}$ be $\{x \in \text{Fil}^r A_{\text{crys}} \mid \varphi x \in p^r A\}$. Then we have the following lemma.

Lemma 4.5. *Let a be the largest integer such that $(p-1)a < r$. Then for every $x \in \text{Fil}^r A_{\text{crys}}$, $p^a \cdot a!x$ belongs to $\text{Fil}_p^r A_{\text{crys}}$. Moreover, $\text{Fil}_p^r A_{\text{crys}}$ is the associated sub- \tilde{A}^+ -module of A_{crys} generated by $q^j \gamma_b(p^{-1} t^{p-1})$ for all integers j, b such that $j + (p-1)b \geq r$.*

Proof. This is [41, Proposition 6.24]. \square

For $0 \leq r \leq s \leq \infty$, $r, s \in \mathbb{Q}$, let $\tilde{A}^{[r,s]}$ be the p -adic completion of

$$\tilde{A}^+ \left[\frac{p}{[\varepsilon - 1]^r}, \frac{[\varepsilon - 1]^s}{p} \right]$$

(see [6, Corollaire 2.2]). The Frobenius morphism φ on \tilde{A}^+ extends to a map $\varphi : \tilde{A}^{[r,s]} \rightarrow \tilde{A}^{[pr,ps]}$. For $0 \leq r_1 \leq r_2 \leq s_2 \leq s_1 \leq +\infty$, the natural inclusion

$$\tilde{A}^+ \left[\frac{p}{[\varepsilon - 1]^{r_1}}, \frac{[\varepsilon - 1]^{s_1}}{p} \right] \hookrightarrow \tilde{A}^+ \left[\frac{p}{[\varepsilon - 1]^{r_2}}, \frac{[\varepsilon - 1]^{s_2}}{p} \right]$$

extends to an injective morphism $\tilde{A}^{[r_1, s_1]} \hookrightarrow \tilde{A}^{[r_2, s_2]}$. Let $\tilde{B}^{[r, s]}$ be $\tilde{A}^{[r, s]}[1/p]$ and let $\tilde{B}_{\text{rig}}^{\dagger, r}$ be

$$\tilde{B}_{\text{rig}}^{\dagger, r} \stackrel{\text{def}}{=} \bigcap_{r \leq s \leq \infty} \tilde{B}^{[r, s]}.$$

The Frobenius map φ on $\tilde{A}^{[r, s]}$ extends to bijective maps

$$\varphi : \tilde{B}^{[r, s]} \longrightarrow \tilde{B}^{[pr, ps]}, \varphi : \tilde{B}_{\text{rig}}^{\dagger, r} \longrightarrow \tilde{B}_{\text{rig}}^{\dagger, pr}.$$

Let $\tilde{B}_{\text{rig}}^{\dagger}$ be

$$\tilde{B}_{\text{rig}}^{\dagger} \stackrel{\text{def}}{=} \bigcup_{r \geq 0} \tilde{B}_{\text{rig}}^{\dagger, r}.$$

The ring $\tilde{B}_{\text{rig}}^{\dagger}$ comes naturally with an action of the Frobenius morphism φ . For all $n \in \mathbb{N}$, the Frobenius morphism induces bijection

$$\varphi^{-n} : \tilde{A}^{[p^{n-1}(p-1), p^{n-1}(p-1)]} \xrightarrow{\sim} \tilde{A}^{[(p-1)/p, (p-1)/p]}.$$

According to [6, Proposition 2.11], there is a natural injection

$$\tilde{B}^{[\frac{p-1}{p}, \frac{p-1}{p}]} \hookrightarrow B_{\text{dR}}^+$$

and consequently injections

$$\iota_n : \tilde{B}^{\dagger, p^{n-1}(p-1)} \rightarrow \tilde{B}_{\text{rig}}^{\dagger, \frac{p-1}{p}} \hookrightarrow \tilde{B}^{[\frac{p-1}{p}, \frac{p-1}{p}]} \hookrightarrow B_{\text{dR}}^+ \quad (4.1.3)$$

for each $n \geq 0$.

Let K/\mathbb{Q}_p be a finite extension and let K_0 be its largest unramified subextension. Let $H_K \stackrel{\text{def}}{=} \text{Gal}(\bar{K}/K(\zeta_{p^\infty}))$ be the absolute Galois group of the cyclotomic extension of K . For $0 \leq r \leq \infty$, we write $\tilde{B}_{\text{rig}, K}^{\dagger, r}$ and $\tilde{B}_{\text{rig}, K}^{\dagger}$ for the H_K -invariants of $\tilde{B}_{\text{rig}}^{\dagger, r}$ and $\tilde{B}_{\text{rig}}^{\dagger}$ respectively. Let $B_{\text{rig}}^{\dagger, r}$ and B_{rig}^{\dagger} be the closure of $K_0[\pi, \pi^{-1}]$ inside $\tilde{B}_{\text{rig}}^{\dagger, r}$ and $\tilde{B}_{\text{rig}}^{\dagger}$ respectively. There exists $r(K) \in \mathbb{R}_+$ such that for all $r \geq r(K)$, there exists a unique étale extension $B_{\text{rig}, K}^{\dagger, r}$ of $B_{\text{rig}}^{\dagger, r}$ such that the natural map $B_{\text{rig}, K}^{\dagger, r} \otimes_{B_{\text{rig}}^{\dagger, r}} \tilde{B}_{\text{rig}}^{\dagger, r} \longrightarrow \tilde{B}_{\text{rig}, K}^{\dagger, r}$ is an isomorphism. The Robba ring $B_{\text{rig}, K}^{\dagger}$ is defined to be

$$B_{\text{rig}, K}^{\dagger} \stackrel{\text{def}}{=} \varinjlim_{r \geq r(K)} B_{\text{rig}, K}^{\dagger, r}.$$

If K/\mathbb{Q}_p is unramified, then $B_{\text{rig}, K}^{\dagger, r}$ is isomorphic to the ring of power-series converging in the annulus $[p^{-1/r}, 1[$

$$\left\{ f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \mid a_n \in K, \forall \rho \in [p^{-1/r}, 1[, \lim_{n \rightarrow \pm\infty} |a_n| \rho^n = 0 \right\}$$

through the evaluation morphism at π ([6, Proposition 2.31]) and so $B_{\text{rig}, K}^{\dagger}$ is isomorphic to the ring \mathcal{R} of power-series which converge in some non-empty annulus $]p_0, 1[$

$$\left\{ f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \mid a_n \in K, \forall \rho \in]p_0, 1[, \lim_{n \rightarrow \pm\infty} |a_n| \rho^n = 0 \right\}.$$

In that case, the action of φ on $B_{\text{rig}, K}^{\dagger, r}$ induces the action $\varphi(z) = (1+z)^p - 1$ on power-series. Let \mathcal{E} be the field of Laurent series $\sum_{n \in \mathbb{Z}} a_n z^n$ with coefficients in K , such that $(v_p(a_n))_{n \in \mathbb{Z}}$ is bounded below and such that $\lim_{n \rightarrow -\infty} v_p(a_n) = +\infty$ together with the action of φ defined above. Then \mathcal{E} is an extension of $\varphi(\mathcal{E})$ of degree p . For $f \in \mathcal{E}$, let $\psi(f)$ be $p^{-1}\varphi^{-1}(\text{Tr}_{\mathcal{E}/\varphi(\mathcal{E})} f)$. Then

$$\psi = \mathcal{E} \longrightarrow \mathcal{E}$$

is a left-inverse of φ which extends by continuity to $\psi : \mathcal{R} \longrightarrow \mathcal{R}$. Any $x \in \mathcal{R}$ may then be written

$$x = \sum_{i=0}^{p-1} (1+z)^i \varphi(x_i).$$

The map ψ then satisfies

$$\psi(x) = x_0.$$

Moreover, if K_n denotes the extension $K(\zeta_{p^n})$, then the maps ι_n defined in (4.1.3) above satisfy

$$\iota_n(B_{\text{rig},K}^{\dagger,p^{n-1}(p-1)}) \rightarrow K_n[[t]].$$

Definition 4.6. We say a $B_{\text{rig},K}^\dagger$ -module D is a (φ, Γ_K) -module of rank d over $B_{\text{rig},K}^\dagger$ if

- D is a finite free $B_{\text{rig},K}^\dagger$ -module of rank d ;
- D is equipped with a φ -semilinear map $\varphi : D \rightarrow D$ such that

$$\varphi^*(D) : B_{\text{rig},K}^\dagger \otimes_{\varphi, B_{\text{rig},K}^\dagger} D \rightarrow D : a \otimes x \mapsto a\varphi(x)$$

is an isomorphism;

- D is equipped with a continuous semilinear action of Γ_K which commutes with φ .

In practice, we only need the notion of (φ, Γ_K) -module for K an extension of \mathbb{Q}_p included inside E (enlarging E if necessary). In the following, we always make this hypothesis.

Write $r_n = p^{n-1}(p-1)$. According to [23], any (φ, Γ_K) -module D is overconvergent, i.e. there exists $n(D) > 0$ and a unique finite free $B_{\text{rig},K}^{\dagger, r_n(D)}$ -submodule $D^{n(D)} \subseteq D$ of rank d with $B_{\text{rig},K}^\dagger \otimes_{B_{\text{rig},K}^{\dagger, r_n(D)}} D^{n(D)} = D$.

For any $n > n(D)$ we define

$$D_{\text{dif}}^+(D) \stackrel{\text{def}}{=} K_n[[t]] \otimes_{\iota_n, B_{\text{rig},K}^{\dagger, r_n}} D^{(n)}$$

and

$$D_{\text{dif}}(D) \stackrel{\text{def}}{=} K_n((t)) \otimes_{\iota_n, B_{\text{rig},K}^{\dagger, r_n}} D^{(n)}.$$

We also define

$$D_{\text{dR}}^K(D) = D_{\text{dif}}(D)^{\Gamma_K=1}, D_{\text{crys}}^K(D) = D[1/t]^{\Gamma_K=1}.$$

The filtration on $D_{\text{dR}}^K(D)$ is given by

$$\text{Fil}^i D_{\text{dR}}^K(D) = D_{\text{dR}}^K(D) \cap t^i D_{\text{dif}}^+(D), i \in \mathbb{Z}.$$

We define a (φ, Γ) -module D to be crystalline (de Rham) if the rank D is equal to the \mathbb{Z}_p -rank of D_{crys} (D_{dR}). If V is a representation over some finite extension L of \mathbb{Q}_p , we make all these definitions by regarding it as a \mathbb{Q}_p -representation.

4.1.3 Bloch-Kato exponential maps

Recall the fundamental exact sequence in p -adic Hodge theory ([9])

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{\text{crys}}^{\varphi=1} \oplus B_{\text{dR}}^+ \xrightarrow{\iota} B_{\text{dR}} \longrightarrow 0. \quad (4.1.4)$$

Tensoring with the Galois representation V and taking G_K -cohomology yields an exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(G_K, V) \longrightarrow D_{\text{crys}}(V) &\xrightarrow{1-\varphi, \iota} D_{\text{crys}}(V) \oplus D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V) \xrightarrow{\delta} H^1(G_K, V) \\ &\xrightarrow{\delta^*} D_{\text{crys}}(V^*(1))^* \oplus \text{Fil}^0 D_{\text{dR}}(V) \longrightarrow D_{\text{crys}}(V^*(1))^* \longrightarrow H^2(G_K, V) \longrightarrow 0 \end{aligned} \quad (4.1.5)$$

in which $(-)^*$ denotes $\text{Hom}(-, E)$. The exponential and dual exponential maps \exp and \exp^* of [9, 66] are respectively the maps

$$\exp : \frac{D_{\text{dR}}(V)}{\text{Fil}^0 D_{\text{dR}}(V)} \longrightarrow H^1(G_K, V)$$

and

$$\exp^* : H^1(G_K, V) \longrightarrow \text{Fil}^0 D_{\text{dR}}(V)$$

deduced from the previous exact sequence by considering only the second component of the maps δ and δ^* . The exact sequence (4.1.5) and consequently the maps \exp and \exp^* are generalized to de Rham (φ, Γ_K) -modules over $B_{\text{rig}, K}^\dagger$ ([91]).

As in [9], we define the subspace $H_f^1(G_K, V)$ of $H^1(G_K, V)$ to be the image of δ , which is also the kernel of the map

$$H^1(G_K, V) \longrightarrow H^1(G_K, V \otimes_{\mathbb{Q}_p} B_{\text{crys}}).$$

By definition, $H_f^1(G_K, V)$ belongs to the kernel of \exp^* . If T is an $\mathcal{O}[G_K]$ -submodule inside V , we define $H_f^1(G_K, T)$ to be the intersection of $H^1(G_K, T)$ with $H_f^1(G_K, V)$. Note in particular that $H_f^1(G_K, T)$ is by construction a saturated \mathcal{O} -submodule, and hence is a direct summand inside $H^1(G_K, T)$.

4.1.4 Co-admissible Λ_∞ -modules

We recall necessary some facts and definitions introduced by J.Pottharst in his studies of analytic families of Selmer complexes ([102, 103]).

Definition 4.7. ([91, Definition 3.1]) Let Λ_n be the p -adic completion of the ring $\mathcal{O}_{\text{Iw}}[\mathfrak{m}^n/p]$. The analytic Iwasawa algebra Λ_∞ is the inverse limit $\varprojlim_n \Lambda_n[1/p]$. If D is a (φ, Γ_K) -module over $B_{\text{rig}, K}^\dagger$, the analytic Iwasawa cohomology is the Λ_∞ -module

$$H_{\text{Iw}}^q(G_K, D) \stackrel{\text{def}}{=} \varprojlim_n H^q(G_K, D \hat{\otimes}_K \Lambda_n^\iota)$$

where Γ_{Iw} acts on Λ_n^ι through the inverse of the natural action by multiplication on Λ_n .

The ring Λ_∞ is the ring of rigid analytic functions on the open unit disc and is a Bezout domain. If V is a p -adic $G_{\mathbb{Q}_p}$ -representation, we write $H_{\text{Iw}}^q(G_K, V)$ for $H_{\text{Iw}}^q(G_K, D_{\text{rig}}^\dagger(V))$. Then the natural map

$$H_{\text{cl}, \text{Iw}}^q(G_K, V) \otimes_{\Lambda} \Lambda_\infty \longrightarrow H_{\text{Iw}}^q(G_K, V)$$

is an isomorphism ([103, Theorem 1.9]).

Definition 4.8. A Λ_∞ -module M is said to be co-admissible if there exists an inverse system $(M_n)_n$ of finitely generated $\Lambda_n[1/p]$ -modules such that the map $M_{n+1} \rightarrow M_n$ induces isomorphisms $M_{n+1} \otimes_{\Lambda_{n+1}[1/p]} \Lambda_n[1/p] \simeq M_n$ for all $n \in \mathbb{N}$ and such that

$$M = \varprojlim_n M_n.$$

Let W be the generic fiber of the formal spectrum $\text{Spf } \Lambda_{\text{Iw}}$ of Λ_{Iw} (the weight space). A Λ_∞ -module M is co-admissible if it is the module of global sections of a coherent analytic sheaf on W . Even though the ring Λ_∞ is not Noetherian, co-admissible Λ_∞ -module admit characteristic ideals ([102]).

Proposition 4.9. 1. The torsion submodule M_{tors} of a co-admissible Λ_∞ -module M is also co-admissible.

2. A co-admissible Λ_∞ -module is torsion if and only if there exists a collection $\{\mathfrak{p}_\alpha\}_{\alpha \in I}$ of closed points of $\bigcup_n \text{Spec } \Lambda_n[1/p]$ such that the subset

$$I_n = \{\alpha \in I \mid \mathfrak{p}_\alpha \in \text{Spec } \Lambda_n[1/p]\}$$

is finite for all $n \in \mathbb{N}$ and a function $n : I \longrightarrow \mathbb{N}$ such that there is an isomorphism

$$M \simeq \prod_{\alpha \in I} \Lambda_\infty / \mathfrak{p}_\alpha^{n_\alpha}$$

Proof. For all $n \in \mathbb{N}$, the ring $\Lambda_n[1/p]$ is a principal ideal domain and the natural map $\Lambda_{n+1}[1/p] \hookrightarrow \Lambda_n[1/p]$ induces an injection $\text{Spec } \Lambda_n[1/p] \longrightarrow \text{Spec } \Lambda_{n+1}[1/p]$ which sends the generic point to itself. The structure theorem for finitely generated module over principal ideal domain then yields both claims. \square

In fact, the quotient of a co-admissible Λ_∞ -module M by its torsion submodule M_{tors} is a free Λ_∞ -module, but we will not make use of this fact.

Definition 4.10. Let M be a co-admissible, torsion Λ_∞ -module such that there is an isomorphism

$$M \simeq \prod_{\alpha \in I} \Lambda_\infty / \mathfrak{p}_\alpha^{n_\alpha}.$$

The analytic characteristic ideal $\text{char}_{\Lambda_\infty} M$ of M is the principal ideal

$$\text{char}_{\Lambda_\infty} M \stackrel{\text{def}}{=} \prod_{\alpha \in I} \mathfrak{p}_\alpha^{n_\alpha}.$$

Here, the fact that $\text{char}_{\Lambda_\infty} M$ is a principal ideal follows from the fact that it is a closed ideal and the property that closed ideals of Λ_∞ are principal ([81]).

4.2 Unramified Iwasawa Theory

In this subsection, we prove some key facts about the uniform boundedness of Bloch-Kato logarithm map along unramified field extensions. These results are of crucial importance in our later study of \mathbb{Z}_p -extensions of quadratic imaginary extensions of \mathbb{Q} totally ramified at exactly one prime above p and unramified at the other.

In this subsection, we assume that (T, ρ, \mathcal{O}) is a two-dimensional $G_{\mathbb{Q}_p}$ -representation such that $V \stackrel{\text{def}}{=} T \otimes_{\mathcal{O}} E$ is de Rham with Hodge-Tate weights (r, s) such that $r > 0$ and $s \leq 0$.

4.2.1 Boundedness of the exponential map

Let v_1, v_2 be a basis of T . The E -vector space $D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V)$ is one-dimensional. Let ω_V be a fixed generator $D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V)$ over E . Then $t^r \omega_V \in B_{\text{dR}}^+ \otimes T$. By density of $\tilde{A}^+[\frac{1}{p}]$ in its completion B_{dR}^+ , there is an element $z \in \tilde{A}^+[\frac{1}{p}] \otimes T$ such that

$$t^r \omega_V - z \in \text{Fil}^{r+1} B_{\text{dR}} \otimes T. \quad (4.2.1)$$

Fix an n such that $p^n z \in \tilde{A}^+ \otimes_{\mathcal{O}} T$.

Write $U^1 = \{x \in \tilde{E} | v(x-1) \geq 1\}$. We define U_r^1 to be the p -adic closure of the \mathbb{Z}_p -submodule of $\text{Fil}^{-r} B_{\text{crys}}^{\varphi=1}$ generated by elements of the form $\frac{a_1 \cdots a_i}{t^i}$ where a_i are elements in $\log(U^1)$, $i \leq r$ (See [41, 6.1.3] for details about the log map).

Lemma 4.11. Let $\text{Im}(U_r^1)$ be the image of U_r^1 in $B_{\text{dR}}/B_{\text{dR}}^+$. As in lemma 4.5, let a be the largest integer such that $(p-1)a < r$. Let m_r be an integer large enough so that $p^{mr} \nmid p^a \cdot a! b p^b / \theta(q/\pi)^j$ for all $b(p-1) \leq r$ and all $j + (p-1)b \geq r$. Suppose m is greater than $r^2 + (r-1)m_r + n$. Then $\text{Im}(U_r^1) \otimes_{\mathbb{Z}_p} T$ contains $p^m w \omega_V$ for all $w \in W(\bar{\mathbb{F}}_p)$.

Proof. We argue by descending induction. Let w be in $W(\bar{\mathbb{F}}_p)$.

Choose \tilde{v} such that $\theta(\tilde{v})$ is equal to p . As $\theta(\log(U^1)) \supseteq p\mathcal{O}_{\mathbb{C}_p}$, there exists $\tilde{b}_{r-1} \in \log U^1$ such that

$$\theta(\tilde{b}_{r-1}) \cdot \theta(\tilde{v})^{r-1} \cdot t^{-r} \equiv p^{r+n} w \omega_V \pmod{\text{Fil}^{1-r} B_{\text{dR}} \otimes T}.$$

Suppose we found $\tilde{b}_{r-1}, \dots, \tilde{b}_i \in \log(U^1)$ and $\tilde{c}_{r-1}, \dots, \tilde{c}_i \in \log(U^1)$ such that

$$p^{r(r-i)+m_r r(r-i-1)+n} w \cdot \omega_V$$

and

$$\left(\frac{\tilde{b}_{r-1}}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^{r-1} + \cdots + \frac{\tilde{b}_i}{t} \left(\frac{\tilde{v}}{t} \right)^i \right) v_1 + \left(\frac{\tilde{c}_{r-1}}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^{r-1} + \cdots + \frac{\tilde{c}_i}{t} \left(\frac{\tilde{v}}{t} \right)^i \right) v_2$$

are equal modulo $\text{Fil}^{-i} B_{\text{dR}} \otimes T$ (here we recall that (v_1, v_2) is our fixed basis of T). Then the element

$$t^r \left(p^{r(r-i)+m_r r(r-i-1)+n} w \cdot \omega_V - \left(\sum_{j=r-1}^i \frac{\tilde{b}_j}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^j \right) v_1 - \left(\sum_{j=r-1}^i \frac{\tilde{c}_j}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^j \right) v_2 \right)$$

belongs to $(\text{Fil}^{r-i} A_{\text{crys}} + \text{Fil}^{r+1} B_{\text{dR}}^+) \otimes T$ by (4.2.1). By lemma 4.5, we know that both the image under θ of the coefficient of v_1 in

$$t^i \left(p^{r(r-i)+m_r(r(r-i)-1)+n} w \cdot \omega_V - \left(\frac{\tilde{b}_{r-1}}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^{r-1} + \cdots + \frac{\tilde{b}_i}{t} \left(\frac{\tilde{v}}{t} \right)^i \right) v_1 \right)$$

and the image under θ of the coefficient of v_2 in

$$t^i \left(p^{r(r-i)+m_r(r(r-i)-1)+n} w \cdot \omega_V - \left(\frac{\tilde{c}_{r-1}}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^{r-1} + \cdots + \frac{\tilde{c}_i}{t} \left(\frac{\tilde{v}}{t} \right)^i \right) v_2 \right)$$

belong to $p^{-mr} \mathcal{O}_{\mathbb{C}_p}$. For $i \leq j \leq r$, define \tilde{b}'_j and \tilde{c}'_j to be respectively \tilde{b}_j and \tilde{c}_j multiplied by p^{r+m_r} . Then there is a choice of \tilde{b}'_{i-1} and \tilde{c}'_{i-1} such that

$$p^{r(r-i+1)+m_r(r-i)+n} w \cdot \omega_V - \left(\sum_{j=r-1}^{i-1} \frac{\tilde{b}_j}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^j \right) v_1 - \left(\sum_{j=r-1}^{i-1} \frac{\tilde{c}_j}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^j \right) v_2$$

belongs to $\text{Fil}^{1-i} B_{\text{dR}} \otimes T$. Continuing this process we can find $\tilde{b}_{r-1}, \dots, \tilde{b}_0$ and $\tilde{c}_{r-1}, \dots, \tilde{c}_0$ such that

$$p^{r^2+m_r(r-1)+n} w \cdot \omega_V - \left(\sum_{j=r-1}^0 \frac{\tilde{b}_j}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^j \right) v_1 - \left(\sum_{j=r-1}^0 \frac{\tilde{c}_j}{t} \cdot \left(\frac{\tilde{v}}{t} \right)^j \right) v_2$$

is in $B_{\text{dR}}^+ \otimes T$. This proves the claim. \square

The following proposition follows immediately.

Proposition 4.12. *We consider the co-boundary map*

$$\exp : \frac{(B_{\text{dR}} \otimes V)^{I_p}}{(B_{\text{dR}}^+ \otimes V)^{I_p} + (B_{\text{crys}} \otimes V)^{I_p, \varphi=1}} \longrightarrow H^1(I_p, V).$$

Then there is an $m > 0$ such that for any $w \in W(\bar{\mathbb{F}}_p)$ we have $\exp(w \cdot \omega_V) \in p^{-m} H^1(I_p, T)$.

Proof. Let C be in the intersection of U_r^1 with the kernel of the map

$$B_{\text{crys}}^{\varphi=1} \longrightarrow B_{\text{dR}} / B_{\text{dR}}^+.$$

Then $t^r C$ belongs to $\text{Fil}^r A_{\text{crys}}$ so $\theta(C)$ is in $p^{-mr} \mathcal{O}_{\mathbb{C}_p}$ by lemma 4.5. Combining this with lemma 4.11 and the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker & \longrightarrow & U_r^1 \oplus B_{\text{dR}}^+ & \longrightarrow & U_r^1 + B_{\text{dR}}^+ \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Q}_p & \longrightarrow & B_{\text{crys}}^{\varphi=1} \oplus B_{\text{dR}}^+ & \longrightarrow & B_{\text{dR}} \longrightarrow 0 \end{array}$$

yields the statement. \square

4.2.2 Yager modules

We mainly follow [84, Section 3.2] to present the theory of Yager modules. Let K/\mathbb{Q}_p be a finite unramified extension. Let y_{K/\mathbb{Q}_p} be the map

$$y_{K/\mathbb{Q}_p} : \mathcal{O}_K \longrightarrow \mathcal{O}_K[\text{Gal}(K/\mathbb{Q}_p)]$$

sending $x \in \mathcal{O}_K$ to

$$y_{K/\mathbb{Q}_p}(x) = \sum_{\sigma \in \text{Gal}(K/\mathbb{Q}_p)} x^\sigma [\sigma] \in \mathcal{O}_K[\text{Gal}(K/\mathbb{Q}_p)] \quad (4.2.2)$$

(note that our convention is slightly different from [84]). Recall that $\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p$ is the unramified \mathbb{Z}_p -extension and that $U = \text{Gal}(\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p)$. For $m \in \mathbb{N}$, we write U_m for the quotient group of U of cardinal p^m .

On the power-series ring $\hat{\mathbb{Z}}_p^{p,\text{ur}}[[U]]$, we consider the following two actions of U : the multiplication action $[\cdot]$ in which $u \in U$ acts as the group-like element $u^{-1} \in \hat{\mathbb{Z}}_p^{p,\text{ur}}[[U]]^\times$ and the Galois action \cdot of U on $\hat{\mathbb{Z}}_p^{p,\text{ur}}$ extended to $(\hat{\mathbb{Z}}_p^{p,\text{ur}})^\mathbb{N}$. Then the map (4.2.2) induces an isomorphism of $\mathbb{Z}_p[[U]]$ -modules

$$y_{\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p} : \varprojlim_{\mathbb{Q}_p \subseteq K \subseteq \mathbb{Q}_p^{p,\text{ur}}} \mathcal{O}_K \simeq \mathcal{S}_{\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p} = \left\{ f \in \hat{\mathbb{Z}}_p^{p,\text{ur}}[[U]] \mid \forall u \in U, f^u = [u]f \right\}.$$

The module $\mathcal{S}_{\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p}$ is called the Yager module in [84].

Denote by $\mathbb{Q}_p \subset K_m \subset \mathbb{Q}_p^{p,\text{ur}}$ the sub-extension with Galois group U_m over \mathbb{Q}_p . As K_m/\mathbb{Q}_p is unramified, the trace map $\text{Tr}_{K_m/\mathbb{Q}_p}$ is surjective, \mathcal{O}_K is a free $\mathbb{Z}_p[U_m]$ -module of rank 1 and $x \in \mathcal{O}_{K_m}$ is a $\mathbb{Z}_p[U_m]$ -generator if and only if $\text{Tr}_{K_m/\mathbb{Q}_p} x \in \mathbb{Z}_p^\times$. Consider $\varprojlim_m \mathcal{O}_{K_m}$ the inverse system with respect to the trace maps and let $(d_m)_{m \in \mathbb{N}} \in \varprojlim_m \mathcal{O}_{K_m}$ be an element with $d_0 \in \mathbb{Z}_p^\times$. Then

$$d \stackrel{\text{def}}{=} \varprojlim_m d_m \in \varprojlim_{\mathbb{Q}_p \subseteq K_m \subseteq \mathbb{Q}_p^{p,\text{ur}}} \mathcal{O}_{K_m}$$

is a generator of $\varprojlim_m \mathcal{O}_{K_m}$ as $\mathbb{Z}_p[[U]]$ -module and $y_{\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p}(d)$ is a generator of $\mathcal{S}_{\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p}$ as $\mathbb{Z}_p[[U]]$ -module, which is thus free of rank 1 as $\mathbb{Z}_p[[U]]$ -module.

Let ρ on \mathcal{F} be a one-dimensional representation of U . Mapping u to $\text{Aut}(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{p,\text{ur}})$ and extending by linearity defines a map

$$\rho : \hat{\mathbb{Z}}_p^{p,\text{ur}}[[U]] \rightarrow \text{Aut}(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{p,\text{ur}}).$$

Identifying d with its image in $\mathcal{S}_{\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p} \subset \hat{\mathbb{Z}}_p^{p,\text{ur}}[[U]]$, we may thus define $\rho(d) \in \text{Aut}(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{p,\text{ur}})$. The defining property of d ensures that image $\rho(d) \cdot x$ belongs to $(\mathcal{F} \otimes \hat{\mathbb{Z}}_p^{p,\text{ur}})^{G_{\mathbb{Q}_p}}$ for all $x \in \mathcal{F}$. Similarly, we define $\rho(d)^\vee$ to be the image under ρ of the element in $\hat{\mathbb{Z}}_p^{p,\text{ur}}[[U]]$ which is the inverse of

$$\varprojlim_m \sum_{\sigma \in U_m} d_m^\sigma [\sigma^{-1}]$$

in this group ring.

4.2.3 Explicit description of the exponential map and Galois cohomology

The existence of d and proposition 4.12 imply the following corollary.

Corollary 4.13. *Let $V(\rho)$ be the twist of V by an unramified p -adic character ρ of $G_{\mathbb{Q}_p}$ such that $\rho(\text{Frob}_p) = 1 + x \in \mathcal{O}_{\mathbb{C}_p}$ with $v_p(x) > 0$. Then the map*

$$\exp : \left(\frac{B_{\text{dR}}}{B_{\text{dR}}^+} \otimes V(\rho) \right)^{G_{\mathbb{Q}_p}} \rightarrow H^1(G_{\mathbb{Q}_p}, V(\rho))$$

can be constructed as

$$\exp \left(\lim_n \sum_{\sigma \in U_n} d_n^\sigma \rho(\sigma) \cdot \omega_V \right) = \lim_n \sum_{\sigma \in U_n} \rho(\sigma) \exp(d_n^\sigma \cdot \omega_V).$$

The right-hand side is well-defined thanks to proposition 4.12.

Proof. Consider the natural unramified rank-one Galois representation of $U = \text{Gal}(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_p)$ over the Iwasawa algebra $\mathcal{O}[[U]]$. We consider the map from $\mathcal{O}[[U]]$ to ρ mapping u to $\rho(u)$. Tensoring this map with the short exact sequence (4.1.4) and taking the long exact sequence of Galois cohomology, we get the required formula. \square

Corollary 4.14. Assume that for every specialization T' of $T \otimes_{\mathcal{O}} \mathcal{O}[[U]]$, the representation $V' \stackrel{\text{def}}{=} T'[1/p]$ satisfies

$$D_{\text{crys}}^{\varphi=1}(V') = 0.$$

Fix $m \in \mathbb{N}$ as proposition 4.12. Identifying it with its image through $y_{\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p}$, we write d for our fixed generator of the $\mathbb{Z}_p[[U]]$ -module $\mathcal{S}_{\mathbb{Q}_p^{p,\text{ur}}/\mathbb{Q}_p}$. Let

$$\exp : (\varprojlim_n \mathcal{O}_{F_n}) \cdot \omega_V \rightarrow p^{-m} H^1(G_{\mathbb{Q}_p}, T \otimes \mathcal{O}[[U]])$$

be the inverse limit of the maps

$$\exp_{F_n} : \mathcal{O}_{F_n} \cdot \omega_V \rightarrow p^{-m} H^1(G_{F_n}, T)$$

where $\mathbb{Q}_p \subseteq F_n \subset \mathbb{Q}_p^{p,\text{ur}}$ runs over finite extensions. Then $\exp(d \cdot \omega_V)$ generates a $\mathcal{O}[[U]] \otimes_{\mathcal{O}} E$ -direct summand of

$$H^1(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathcal{O}[[U]]) \otimes_{\mathcal{O}} E \simeq (\mathcal{O}[[U]] \otimes_{\mathcal{O}} E)^2.$$

Proof. As $\mathcal{O}[[U]] \otimes_{\mathcal{O}} E$ is a Bezout domain, the module generated by $\exp(d \cdot \omega_V)$ is a direct summand if the specialization of $\exp(d \cdot \omega_V)$ through any $\phi \in \text{Spec}(\mathcal{O}[[U]] \otimes_{\mathcal{O}} E)$ with u mapping to $1+x$ with $v_p(x) > 0$ is non-zero. Since the exp map for $V(\rho)$ is injective on $\frac{D_{\text{dR}}(V(\rho))}{D_{\text{dR}}^+(V(\rho))}$, this follows from corollary 4.13. \square

Remark: In our later applications, the representation V will be the representation V_f twisted by a finite order ramified character. In that case, the first assumption is satisfied.

The previous corollary ensures that there exists $b \in E^\times$ such that $b \exp(d \cdot \omega_V)$ is an $\mathcal{O}[[U]]$ -submodule of $H^1(G_{\mathbb{Q}_p}, T \otimes \mathcal{O}[[U]])$ with torsion-free quotient. For any such, the localization of $b \exp(d \cdot \omega_V)$ at any height-one prime $\mathcal{P} \in \text{Spec } \mathcal{O}[[U]]$ is a rank one direct summand of the rank two, free module $H^1(G_{\mathbb{Q}_p}, T \otimes \mathcal{O}[[U]])_{\mathcal{P}}$. Equivalently, this means that if we write $b \exp(d \cdot \omega_V)$ as linear combination of an $\mathcal{O}[[U]]$ -basis, then the coefficients are co-prime to each other.

We define two non-zero elements $\mathbf{b}_p, \mathbf{d}_p$ in E which are closely related to Tamagawa numbers.

Definition 4.15. Let the assumptions be the same as in the previous corollary. Choose $\mathbf{b}_p \in E^\times$ such that $\mathbf{b}_p \exp(d \cdot \omega_V) \in H^1(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathcal{O}[[U]])$ and such that the torsion submodule of the quotient

$$\frac{H^1(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathcal{O}[[U]])}{\mathbf{b}_p \mathcal{O}[[U]] \cdot \exp(d \cdot \omega_V)}$$

is zero. We define

$$H_f^1(G_{\mathbb{Q}_p}, T \otimes_{\mathcal{O}} \mathcal{O}[[U]]) \stackrel{\text{def}}{=} \mathbf{b}_p \mathcal{O}[[U]] \cdot \exp(d \cdot \omega_V). \quad (4.2.3)$$

Let $\mathbf{d}_p \in E^\times$ be such that the two \mathcal{O} -submodules $\mathbf{d}_p H_f^1(G_{\mathbb{Q}_p}, T)$ and $\mathbf{b}_p \exp(\omega_V)$ coincide in $H_f^1(G_{\mathbb{Q}_p}, V)$ (this specifies \mathbf{d}_p up to an element of \mathcal{O}^\times).

When $T = T_f$, we may pick the dual of ω_f as ω_V .

4.3 Control Theorem of Selmer Groups

In this subsection, we study the descent properties of Selmer groups for T_f alongside the cyclotomic extension. We show in particular that for an appropriate choice of local condition at p , these Selmer modules have non non-trivial pseudo-null submodules and satisfy a perfect control theorem at suitable points of the classical Iwasawa algebra.

4.3.1 Notations and definitions

Galois representations Put $T_{f,\text{Iw}} \stackrel{\text{def}}{=} T_f \otimes_{\mathcal{O}} \mathcal{O}_{\text{Iw}}$, $V_{f,\text{Iw}} \stackrel{\text{def}}{=} T_{f,\text{Iw}} \otimes_{\mathcal{O}_{\text{Iw}}} \text{Frac}(\mathcal{O}_{\text{Iw}})$. Let $A_{f,\text{Iw}}^*(1)$ be the Pontrjagin dual

$$A_{f,\text{Iw}}^*(1) \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Z}_p}(T_{f,\text{Iw}}, \mathbb{Q}_p/\mathbb{Z}_p(1))$$

of $T_{f,\text{Iw}}$ (here $\text{Hom}_{\mathbb{Z}_p}$ means continuous morphisms).

If x is an element in a system of parameter of \mathcal{O}_{Iw} such that $\mathcal{O}_x \stackrel{\text{def}}{=} \mathcal{O}_{\text{Iw}}/x$ is of characteristic zero, we write \mathcal{P}_x for the height-one prime $x\mathcal{O}_{\text{Iw}} \in \text{Spec } \mathcal{O}_{\text{Iw}}$, E_x for the fraction field of \mathcal{O}_x and ϖ_x for a choice of uniformizing parameter of \mathcal{O}_x . We write $(T_x, \rho_x, \mathcal{O}_x)$ (resp. (V_x, ρ_x, E_x) , resp. $(A_x, \rho_x, E_x/\mathcal{O}_x)$, resp. $A_x^*(1)$) for the $G_{\mathbb{Q},\Sigma}$ -representation $T_{f,\text{Iw}} \otimes_{\mathcal{O}_{\text{Iw}}} \mathcal{O}_x$ (resp. $T_x \otimes_{\mathcal{O}_x} E_x$, resp. $T_x \otimes_{\mathcal{O}_x} E_x/\mathcal{O}_x$, resp. $\text{Hom}_{\mathbb{Z}_p}(T_x, \mathbb{Q}_p/\mathbb{Z}_p(1))$).

If T is equal to $T_{f,\text{Iw}}$ or T_x , if $A^*(1)$ is equal to $A_{f,\text{Iw}}^*(1)$ or $A_x^*(1)$, if \mathcal{H} is $\text{Frac}(\mathcal{O}_x)/\mathcal{O}_x$ and if $M \subset H^1(G_{\mathbb{Q}_p}, T)$ is an \mathcal{O}_{Iw} or \mathcal{O}_x -submodule, we denote by $M^\vee \subset H^1(G_{\mathbb{Q}_p}, A)$ the orthogonal complement of M under Tate's local duality

$$H^1(G_{\mathbb{Q}_p}, T) \times H^1(G_{\mathbb{Q}_p}, A^*(1)) \longrightarrow \mathcal{H}(1).$$

We recall that $R\Gamma(G_{\mathbb{Q}_p}, T_{f,\text{Iw}})$ and $R\Gamma(G_{\mathbb{Q}_p}, T_x)$ are perfect complexes of \mathcal{O}_{Iw} and \mathcal{O}_x -modules respectively whose cohomology is concentrated in degree 1, that $H^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}})$ and $H^1(G_{\mathbb{Q}_p}, T_x)$ are free modules of rank 2 and that the natural map

$$H^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}})/xH^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}}) \longrightarrow H^1(G_{\mathbb{Q}_p}, T_x)$$

is an isomorphism (all these assertions follow from lemma 4.2). We also recall that $H^1(G_{\mathbb{Q},\Sigma}, T_{f,\text{Iw}})$ is an \mathcal{O}_{Iw} -module free of rank 1 containing the non-zero element $\mathbf{z}(f)_{\text{Iw}}$ (lemma 3.13). According to [68, Theorem 12.5], there exist infinitely many $x \in \mathcal{O}_{\text{Iw}}$ such that the image of $\mathbf{z}(f)_{\text{Iw}} \in H^1(G_{\mathbb{Q},\Sigma}, T_{f,\text{Iw}})$ in $H^1(G_{\mathbb{Q}_p}, T_x)/H^1_f(G_{\mathbb{Q}_p}, T_x)$ is non-zero. In particular, the localization map

$$\text{loc}_p : H^1(G_{\mathbb{Q},\Sigma}, T_{f,\text{Iw}}) \longrightarrow H^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}})$$

is not identically zero. As its source is a free module of rank 1 and its target is a free module, it is injective. Let $H_{\text{glob}}^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}})$ denote the image of loc_p .

According to [68, Theorem 12.5], the image of $\mathbf{z}(f)_{\text{Iw}}$ under the map

$$H_{\text{glob}}^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}})/xH_{\text{glob}}^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}}) \hookrightarrow H^1(G_{\mathbb{Q}_p}, T_x) \longrightarrow H^1(G_{\mathbb{Q}_p}, T_x)/H^1_f(G_{\mathbb{Q}_p}, T_x) \quad (4.3.1)$$

is non-zero for infinitely many $x \in \mathcal{O}_{\text{Iw}}$. For such an x , the composition (4.3.1) is a map between \mathcal{O}_x -modules free of rank 1 which is not identically zero, hence injective. We fix $x_0 \in \mathcal{O}_{\text{Iw}}$ such an element and let \mathbf{v} be a pre-image inside $H^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}})$ of a generator of $H^1_f(G_{\mathbb{Q}_p}, T_{x_0})$. Then $H_{\text{glob}}^1(G_{\mathbb{Q}_p}, V_{f,\text{Iw}})$ and $\text{Frac}(\mathcal{O}_{\text{Iw}}) \cdot \mathbf{v}$ are in direct sum inside $H^1(G_{\mathbb{Q}_p}, V_{f,\text{Iw}}) \simeq \text{Frac}(\mathcal{O}_{\text{Iw}})^2$. In particular, the submodule they generate together inside $H^1(G_{\mathbb{Q}_p}, V_{f,\text{Iw}})$ is rank 2.

For $x \in \mathcal{O}_{\text{Iw}}$ such that $\mathcal{O}_{\text{Iw}}/(x)$ is a characteristic zero discrete valuation ring, let $\bar{\mathbf{v}}_x$ denote the reduction of \mathbf{v} modulo $x\mathcal{O}_{\text{Iw}}$ (by definition, $\bar{\mathbf{v}}_{x_0}$ is thus equal to $H^1_f(G_{\mathbb{Q}_p}, T_{x_0})$). As $H_{\text{glob}}^1(G_{\mathbb{Q}_p}, V_{f,\text{Iw}})$ and $\text{Frac}(\mathcal{O}_{\text{Iw}}) \cdot \mathbf{v}$ generate a rank 2 submodule, the composition

$$H^1(G_{\mathbb{Q},\Sigma}, T_{f,\text{Iw}})/xH^1(G_{\mathbb{Q},\Sigma}, T_{f,\text{Iw}}) \hookrightarrow H^1(G_{\mathbb{Q},\Sigma}, T_x) \longrightarrow H^1(G_{\mathbb{Q}_p}, T_x)/\mathcal{O}_x \cdot \bar{\mathbf{v}}_x \quad (4.3.2)$$

is injective for all x except possibly finitely many. Henceforth, we say that $x \in \mathcal{O}_{\text{Iw}}$ is suitable if $\mathcal{O}_{\text{Iw}}/(x)$ is a characteristic zero discrete valuation ring and if the map (4.3.2) is injective. Observe that if x is suitable, then the image of the map (4.3.2) is a free \mathcal{O}_x -module and so has trivial intersection with the torsion submodule of its target.

Selmer modules and Selmer complexes Under Tate's local duality, the orthogonal complement of $\mathcal{O}_{\text{Iw}} \cdot \mathbf{v} \subset H^1(G_{\mathbb{Q}_p}, T_{f,\text{Iw}})$ is a submodule $H^1(G_{\mathbb{Q},\Sigma}, A_{f,\text{Iw}}^*(1)^*(1))$. Hence, our choice of \mathbf{v} allows us to define in our setting an analogue of the Greenberg or \pm -Selmer conditions in the ordinary case or supersingular elliptic curves case respectively. In particular, we show that the Pontrjagin duals of \mathbf{v}^\vee -Selmer modules, that is to say Selmer modules for the Selmer condition obtained by setting the usual Selmer condition at primes $\ell \nmid p$ and the orthogonal complement under Tate's local duality of $\mathcal{O}_{\text{Iw}} \cdot \mathbf{v}$ at p , have no non-trivial pseudo-null submodules and satisfy a control theorem.

Definition 4.16. Let $x \in \mathcal{O}_{\text{Iw}}$ be suitable. For $A^*(1)$ equal to $A_{f,\text{Iw}}^*(1)$ or $A_x^*(1)$, S equal to \mathcal{O}_{Iw} or \mathcal{O}_x , $\bar{\mathbf{v}}$ equal to \mathbf{v} or $\bar{\mathbf{v}}_x$ and ℓ a prime, define $H_{\text{str},\mathbf{v}}^1(G_{\mathbb{Q}_\ell}, A^*(1)) \subset H^1(G_{\mathbb{Q}_\ell}, A^*(1))$ by

$$H_{\text{str},\mathbf{v}}^1(G_{\mathbb{Q}_\ell}, A) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \ell \nmid p, \\ (S \cdot \bar{\mathbf{v}})^\vee & \text{otherwise.} \end{cases}$$

We define the strict \mathbf{v}^\vee -Selmer group as follows. Let $\text{Sel}_{\mathbf{v}^\vee}(A^*(1))$ be the \mathcal{O}_{Iw} -module

$$\text{Sel}_{\mathbf{v}^\vee}(A^*(1)) \stackrel{\text{def}}{=} \ker \left(H^1(G_{\mathbb{Q},\Sigma}, A^*(1)) \longrightarrow \bigoplus_{\ell \in \Sigma} H^1(G_{\mathbb{Q}_\ell}, A^*(1)) / H_{\text{str},\mathbf{v}}^1(G_{\mathbb{Q}_\ell}, A^*(1)) \right).$$

We write $X_{\mathbf{v}^\vee}(A^*(1))$ for the Pontryagin dual of $\text{Sel}_{\mathbf{v}^\vee}(A^*(1))^\iota$ where ι indicates that the Γ_{Iw} -action on $\text{Sel}_{\mathbf{v}^\vee}(A^*(1))^\iota$ is the inverse of the Γ_{Iw} -action on $\text{Sel}_{\mathbf{v}^\vee}(A^*(1))$.

Tamagawa numbers Let $x \in \mathcal{O}_{\text{Iw}}$ such that \mathcal{P}_x is a height-one prime. Then the \mathcal{O}_x -module $H^1(G_{\mathbb{Q}_p}, T_x)/H_f^1(G_{\mathbb{Q}_p}, T_x)$ is free of rank 1. Choose ω_x a basis of this module. Then $\exp^* \omega_x$ belongs to $\text{Fil}^0 D_{\text{dR}}(V_x)$ and the one-dimensional vector spaces it generates inside $\text{Fil}^0 D_{\text{dR}}(V_x)$ coincides with the E_x -vector space generated by the image ω_f of f in $\text{Fil}^0 D_{\text{dR}}(V_x)$ through the comparison isomorphism.

Definition 4.17. Let $\ell \nmid p$ be a prime. The Tamagawa number $c_{x,\ell}$ or $c_{\mathcal{P}_x,\ell}$ at ℓ is

$$c_{x,\ell} \stackrel{\text{def}}{=} \varpi_x^{\text{length}_{\mathcal{O}_x} H^1(I_\ell, T_x)^{\text{Fr}(\ell)=1}}.$$

Choose an \mathcal{O}_x -basis ω_x of $H^1(G_{\mathbb{Q}_p}, T_x)/H_f^1(G_{\mathbb{Q}_p}, T_x)$. The Tamagawa number $c_{x,p}$ or $c_{\mathcal{P}_x,p}$ is such that

$$\exp^* \omega_x = c_{x,p} \omega_f.$$

Remark: If in definition 4.15, we choose ω_V to be the element pairing to 1 with ω_f under the duality between $D_{\text{dR}}/\text{Fil}^0 D_{\text{dR}}$ and $\text{Fil}^0 D_{\text{dR}}$, then $c_{x,p}$ is equal up to a p -adic unit to the quotient $\mathfrak{d}_p/\mathfrak{b}_p$. When f is ordinary at p , or f corresponds to a supersingular elliptic curve with $a_p(f) = 0$, $c_{x,p}$ may be computed explicitly using local theory, see for instance [125]. If $2 \leq k < p$, $c_{x,p}$ is actually a local number, as can be seen by using the integral comparison theorem between crystalline and de Rham cohomology (see [68, Section 14.17]).

Let $x^\iota \in \text{Spec}\Lambda$ be the point x composed with the involution $\gamma \mapsto \gamma^{-1}$ on Λ for each $\gamma \in \Gamma$. We have from definition, $\langle \exp^* a, \log b \rangle = (a, b)$ ((,) is the local Tate pairing). Also the pairing of cup product

$$(,): \frac{H^1(\mathbb{Q}_p, T_x)}{H_f^1(\mathbb{Q}_p, T_x)} \times H^1(\mathbb{Q}_p, T_{x^\iota}) \rightarrow \mathcal{O}_L$$

is surjective. Thus for some \mathcal{O}_{x^ι} basis $\bar{\mathcal{V}}$ of $H_f^1(\mathbb{Q}_p, T_{x^\iota})$, we have

$$\log \bar{\mathcal{V}} = \frac{1}{c_{x,p}} \cdot \omega_f^\vee. \quad (4.3.3)$$

We consider the control theorem for \mathbf{v}^\vee -Selmer groups. We look at the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Sel}_{\mathbf{v}^\vee}(A_x^*(1)) & \longrightarrow & H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_x^*(1)) & \longrightarrow & \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_x^*(1)) \\ & & \downarrow s & & \downarrow h & & \downarrow g \\ 0 & \longrightarrow & \text{Sel}_{\mathbf{v}^\vee}(A_{f,\text{Iw}}^*(1))^{\mathcal{P}_x} & \longrightarrow & H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1))^{\mathcal{P}_x} & \longrightarrow & \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1))^{\mathcal{P}_x} \end{array}$$

where $\mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1)) = \prod_{\ell \nmid p} H^1(G_\ell, A_{f,\text{Iw}}^*(1)) \times \frac{H^1(G_p, A_{f,\text{Iw}}^*(1))}{(\mathcal{O}_x \cdot \text{Im}(\mathbf{v}))^\vee}$ and

$$\mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_x^*(1)) = \prod_{\ell \nmid p} H^1(G_\ell, A_x^*(1)) \times \frac{H^1(G_p, A_x^*(1))}{((\mathcal{O}_x) \cdot \text{Im}(\mathbf{v}))^\vee}.$$

A standard argument using Poitou-Tate exact sequence, as in [64, Section 3.3.2] implies that if $\#\text{Sel}_{\mathbf{v}^\vee}(A_x^*(1)) < \infty$ and $H_{\mathbf{v}^\vee}^1(\mathbb{Q}^S/\mathbb{Q}, T_x) = 0$, then

$$\prod_{\ell \neq p} c_{x,\ell} \text{Fitt} \text{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, A_x^*(1)) = \text{Fitt} X_{\mathbf{v}^\vee} / \mathcal{P}_x X_{\mathbf{v}^\vee}. \quad (4.3.4)$$

4.3.2 No pseudo-null submodules

Lemma 4.18. *The cardinality $\#(H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}}))[\mathcal{P}_x] < \infty$ for all but finitely many m 's and $x = \gamma - (1+p)^m$.*

Proof. The $H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}})$ is a finitely generated Λ -module. Then the lemma follows from the well known structure theorem of finitely generated Λ -modules. \square

Lemma 4.19. *The $\frac{H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1))}{x H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1))} = 0$ for all but finitely many m 's and $x = \gamma - (1+p)^m$.*

Proof. We have

$$\frac{H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1))}{x H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1))} \hookrightarrow H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, A_x^*(1)).$$

It is easy to see by Tate local duality that for all places v and all but finitely many x ,

$$H^2(\mathbb{Q}_v, A_x^*(1)) = 0.$$

From the Global duality for these x , the $H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, A_x^*(1))$ is dual to

$$\ker\{H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_x) \rightarrow \prod_{v \in \Sigma} H^1(\mathbb{Q}_v, T_x)\}.$$

We claim this term is 0 for all but finitely many m . Indeed $H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_x)$ is p -torsion free by (Irred). Moreover we have exact sequence

$$\frac{H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}})}{x H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}})} \hookrightarrow H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_x) \rightarrow H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}})[x].$$

The last term is finite for all but finitely many m by lemma 4.18. The $H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}})$ is a torsion-free rank one Λ -module such that the localization map $H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}}) \rightarrow H^1(\mathbb{Q}_p, T_{f,\text{Iw}})$ is injective. (Because by [107] the image of $\mathbf{z}(f)_{\text{Iw}}$ under this map is non-zero). Now it is easy to see that

$$\ker\{H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_x) \rightarrow H^1(\mathbb{Q}_p, T_x)\}$$

is 0 for all but finitely many m (for example we take a nonzero element a of $H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}})$, and let x to avoid the points where the image of a in $H^1(\mathbb{Q}_p, T_x)$ is zero). The lemma follows readily. \square

Proposition 4.20. *The $X_{\mathbf{v}^\vee}$ has no pseudo-null submodules.*

Proof. Let $x = \gamma - (1+p)^m$ for some integer m be a suitable point. Then we claim for all but finitely many integers m we have surjection

$$H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_x^*(1)) \twoheadrightarrow \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_x^*(1)). \quad (4.3.5)$$

We first look at the exact sequence

$$H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}}) \xrightarrow{\times x} H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}}) \longrightarrow H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_x) \longrightarrow H^2(\mathbb{Q}^\Sigma/\mathbb{Q}, T_{f,\text{Iw}})[x].$$

From lemma 4.18 the last term is torsion for all but finitely many m . By our assumption that x is suitable, we get

$$H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_x) = 0$$

for these x . From Poitou-Tate exact sequence

$$H_\mathbf{v}^1(\mathbb{Q}^\Sigma/\mathbb{Q}, T_x) \rightarrow \text{Hom}_{\mathbb{Z}_p}(\mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_x^*(1)), \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow \text{Hom}_{\mathbb{Z}_p}(H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_x^*(1)), \mathbb{Q}_p/\mathbb{Z}_p).$$

It is also clear that the map $H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_x^*(1)) \rightarrow H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1))[x]$ is an isomorphism, and that the map

$$\mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_x^*(1)) \rightarrow \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1))[x]$$

is surjective. These altogether imply

$$H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1))[x] \rightarrow \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1))[x]$$

is surjective.

Then consider the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1)) & \longrightarrow & H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1)) & \longrightarrow & \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1)) \\ & & \downarrow x & & \downarrow x & & \downarrow x \\ 0 & \longrightarrow & \text{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1)) & \longrightarrow & H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, A_{f,\text{Iw}}^*(1)) & \longrightarrow & \mathcal{P}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1)) \end{array}$$

By Snake lemma and lemma 4.19 the $\frac{\text{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1))}{x \text{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1))} = 0$ for all but finitely many m and $x = \gamma - (1+p)^m$. By Nakayama's lemma, there is no quotient of $\text{Sel}_{\mathbf{v}^\vee}(\mathbb{Q}, A_{f,\text{Iw}}^*(1))$ of finite cardinality. Thus $X_{\mathbf{v}^\vee}$ has no pseudo-null submodules. \square

Definition 4.21. Write \mathcal{F} for the characteristic polynomial of $X_{\mathbf{v}^\vee}$. We also write $X_{\mathbf{v}^\vee,x}$ for the dual Selmer group for $A_x^*(1)$.

The control theorem as before implies that

$$\prod_{\ell \nmid p} c_{x,\ell}(f) \text{char}(X_{\mathbf{v}^\vee,x}) = \text{char}(X_{\mathbf{v}^\vee}/\mathcal{P}_x X_{\mathbf{v}^\vee}). \quad (4.3.6)$$

Note for any finitely generated torsion \mathcal{O}_{Iw} -module M , if (x) is a prime ideal of \mathcal{O}_{Iw} with $\#(\frac{M}{xM}) < \infty$, and \mathcal{F} is a generator of $\text{char}_{\mathcal{O}_{\text{Iw}}}(M)$, then

$$\text{char}_{\mathcal{O}_{\text{Iw}}} \left(\frac{M}{xM} \right) \subseteq \text{char}_{\mathcal{O}_{\text{Iw}}} \left(\frac{\Lambda}{(\mathcal{F}, x)} \right).$$

If M has no pseudo-null submodule then the above inclusion is an equality. The control theorem argument proved above thus implies that

$$\prod_{\ell \nmid p} c_{x,\ell}(f) \text{char}_{\mathcal{O}_x}(X_{\mathbf{v}^\vee,x}) = \text{char}_{\mathcal{O}_x}(\mathcal{O}_{\text{Iw}}/(\mathcal{F}, x)). \quad (4.3.7)$$

Equality (4.3.7) is main result of this section and is used later to prove Kato's Iwasawa main conjecture.

4.4 Beilinson-Flach Elements

In this section, we recall constructions and results from [125] and then use the theory of Beilinson-Flach elements ([82, 73]) to prove conjecture 4.40 for f .

4.4.1 Hida families of CM representations

Let $\Lambda_{\mathbf{g}}$ be the completed group algebra $\mathbb{Z}_p[[\text{Gal}(\mathcal{K}_\infty/\mathcal{K}_{\text{cyc}})]] \simeq \mathbb{Z}_p[[\Gamma_{\bar{v}_0}]]$, which we identify through the choice of a topological generator of $\text{Gal}(\mathcal{K}_\infty/\mathcal{K}_{\text{cyc}})$ with the power-series ring in one variable $\mathbb{Z}_p[[Y]]$. We write $\mathcal{L}_{\mathbf{g}}$ for the fraction field of $\Lambda_{\mathbf{g}}$.

There exists a unique Hida family \mathbf{g} parametrized by $\Lambda_{\mathbf{g}}$ of normalized CM (and hence necessarily ordinary) forms attached to characters of $\Gamma_{\mathcal{K}}$ which passes through the CM form corresponding to the trivial character (we refer to [55, Theorem 6.2] for the definition and details). To \mathbf{g} is attached a morphism $\lambda_{\mathbf{g}}$ of the ordinary p -adic Hida-Hecke algebra $\mathbf{T}_\infty^{\text{ord}}$, or equivalently a system of Hecke eigenvalues $\lambda_{\mathbf{g}} : \mathbf{T}_\infty^{\text{ord}} \rightarrow \Lambda_{\mathbf{g}}$, as well as a unique maximal ideal $\mathfrak{m}_{\mathbf{g}} \in \text{Spec } \mathbf{T}_\infty^{\text{ord}}$ of the Hida-Hecke algebra. To an arithmetic specialization $\phi : \Lambda_{\mathbf{g}} \rightarrow E$ is attached a CM character ψ_ϕ and an automorphic representation $\pi(\psi_\phi)$ of $\text{GL}_2(\mathbb{A}_{\mathbb{Q}})$ which is the base-change of ψ_ϕ from \mathcal{K} to \mathbb{Q} . Up to conjugation, there is a unique character $\Psi_{\mathbf{g}} : G_{\mathcal{K}} \rightarrow \Lambda_{\mathbf{g}}$ which interpolates the ψ_ϕ as ϕ ranges over arithmetic points of $\Lambda_{\mathbf{g}}$. To \mathbf{g} is attached a rank two $G_{\mathbb{Q}}$ -representation $(V(\mathbf{g}), \rho_{\mathbf{g}}, \mathcal{L}_{\mathbf{g}})$, which may be constructed either geometrically or using induction. As we need both constructions in the following, we briefly recall them.

From the geometric point of view, let $D_{\mathcal{K}}$ be the discriminant of \mathcal{K} . Let $ES_p(D_{\mathcal{K}})$ and $GES_p(D_{\mathcal{K}})$ be the $\mathbb{Z}_p[G_{\mathbb{Q}}]$ -modules

$$ES_p(D_{\mathcal{K}}) \stackrel{\text{def}}{=} \varprojlim_r H^1_{\text{et}}(X_1(D_{\mathcal{K}}p^r) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p)$$

and

$$GES_p(D_{\mathcal{K}}) \stackrel{\text{def}}{=} \varprojlim_r H^1_{\text{et}}(Y_1(D_{\mathcal{K}}p^r) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p).$$

Let e^* be the ordinary idempotent attached to the covariant Hecke operator U_p acting on $H^1_{\text{et}}(X_1(D_{\mathcal{K}}p^r) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p)$ and $H^1_{\text{et}}(Y_1(D_{\mathcal{K}}p^r) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_p)$ ([94]). Then $e^*ES_p(D_{\mathcal{K}})$ is a $\mathbf{T}_{\infty}^{\text{ord}}$ -module of finite type. Let $T(\mathbf{g})$ be the quotient of $e^*ES_p(D_{\mathcal{K}})_{\mathbf{m}_{\mathbf{g}}}$ by its $\Lambda_{\mathbf{g}}$ -torsion submodule. Then $V(\mathbf{g})$ is $T(\mathbf{g}) \otimes_{\Lambda_{\mathbf{g}}} \mathcal{L}_{\mathbf{g}}$. We record for further use the fact that the modules of I_p -invariants $e^*ES_p(D_{\mathcal{K}})^{I_p}$ and $e^*GES_p(D_{\mathcal{K}})^{I_p}$ are equal ([94, Theorem]) and denote them both by \mathfrak{A}_{∞}^* . Let \mathfrak{B}_{∞}^* (resp. $\tilde{\mathfrak{B}}_{\infty}^*$) be the quotient of $e^*ES_p(D_{\mathcal{K}})$ (resp. $e^*GES_p(D_{\mathcal{K}})$) by \mathfrak{A}_{∞}^* . The proof of [94, Corollary 2.3.6] shows that there is a natural isomorphism

$$\mathfrak{A}_{\infty}^* \otimes_{\mathbb{Z}_p[[Y]]} \hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]] \simeq \text{Hom}_{\hat{\mathbb{Z}}_p^{p,\text{ur}}}(S^{\text{ord}}(D_{\mathcal{K}}, \chi_{\mathcal{K}}, \hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]]), \hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]]) \quad (4.4.1)$$

where $S^{\text{ord}}(D_{\mathcal{K}}, \chi_{\mathcal{K}}, \hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]])$ is the $\hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]]$ -module of ordinary eigenforms with level $D_{\mathcal{K}}p^{\infty}$, central character $\chi_{\mathcal{K}}$ and coefficients in $\hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]]$.

From the automorphic point of view, there is by construction an isomorphism

$$\text{Ind}_{G_{\mathcal{K}}}^{G_{\mathbb{Q}}} (\Psi_{\mathbf{g}} \otimes_{\Lambda_{\mathbf{g}}} \mathcal{L}_{\mathbf{g}}) \simeq V(\mathbf{g}) \quad (4.4.2)$$

of $\mathcal{L}[G_{\mathbb{Q}}]$ -modules. Burungale-Skinner-Tian proved the following strengthening of (4.4.2), which identifies $T(\mathbf{g})$ with the induction of $\Psi_{\mathbf{g}}$ and which is crucial to our purpose.

Proposition 4.22 ([14]). *There is an isomorphism*

$$\text{Ind}_{G_{\mathcal{K}}}^{G_{\mathbb{Q}}} \Psi_{\mathbf{g}} \simeq T(\mathbf{g}) \quad (4.4.3)$$

of $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}}]$ -modules.

After restriction to $G_{\mathbb{Q}_p}$, $V(\mathbf{g})$ fits in a short exact sequence

$$0 \longrightarrow \mathcal{F}_{\mathbf{g}}^+ \longrightarrow \rho_{\mathbf{g}}|G_{\mathbb{Q}_p} \longrightarrow \mathcal{F}_{\mathbf{g}}^- \longrightarrow 0$$

of non-zero $\mathcal{L}[G_{\mathbb{Q}_p}]$ -modules which is split as p splits in \mathcal{K} . We take the convention that $\mathcal{F}_{\mathbf{g}}^-$ is an unramified $\mathcal{L}[G_{\mathbb{Q}_p}]$ -module. Similarly, since p splits as $v_0 \bar{v}_0$ in \mathcal{K} , there is an identification

$$\left(\text{Ind}_{G_{\mathcal{K}}}^{G_{\mathbb{Q}}} \Psi_{\mathbf{g}} \right) |G_{\mathbb{Q}_p} \simeq \Psi_{\mathbf{g}}|G_{\mathcal{K}_{v_0}} \oplus \Psi_{\mathbf{g}}|G_{\mathcal{K}_{\bar{v}_0}}. \quad (4.4.4)$$

We choose the convention that the isomorphism (4.4.3) sends $\Psi_{\mathbf{g}}|G_{\mathcal{K}_{v_0}}$ to $\mathcal{F}_{\mathbf{g}}^-$. We also fix a $\Lambda_{\mathbf{g}}$ -basis of $\left(\text{Ind}_{G_{\mathcal{K}}}^{G_{\mathbb{Q}}} \Psi_{\mathbf{g}} \right) |G_{\mathbb{Q}_p}$ of the form $(v, c \cdot v)$ for some v (here we recall that c denotes complex conjugation).

Let $\chi_{\mathbf{g}}$ be the central character of \mathbf{g} . Then $\mathcal{F}_{\mathbf{g}}^+(\chi_{\mathbf{g}}^{-1})$ and $\mathcal{F}_{\mathbf{g}}^-$ are in addition unramified at p , they may be identified with quotients of \mathfrak{A}_{∞}^* and \mathfrak{B}_{∞}^* respectively. Let $\omega_{\mathbf{g}}^{\vee} \in \mathcal{F}_{\mathbf{g}}^+(\chi_{\mathbf{g}}^{-1}) \otimes \text{Frac}(\hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]])$ be the functional which maps \mathbf{g} to 1 in the isomorphism (4.4.1). The product of local root numbers of g_{ϕ} at places prime to p moves p -adic analytically and is a unit when ϕ ranges over the arithmetic point of $\Lambda_{\mathbf{g}}$. Hence, the product over places prime to p of the local root numbers of \mathbf{g} at places prime to p product is well-defined element $\varepsilon_{\mathbf{g}} \in \Lambda_{\mathbf{g}}$. Let $\eta_{\mathbf{g}}^{\vee} \in \mathfrak{B}_{\infty}^*$ be the element which pairs with $\omega_{\mathbf{g}}^{\vee}$ to the $\varepsilon_{\mathbf{g}}$ under the pairing of [94, Theorem 2.3.5] (classes dual to $\omega_{\mathbf{g}}^{\vee}$ and $\eta_{\mathbf{g}}^{\vee}$ are studied in [73, Section 10]).

Suppose (v^+, v^-) is a $\text{Frac}(\Lambda_{\mathbf{g}})$ -basis of the $\text{Frac}(\Lambda_{\mathbf{g}})$ -vector space generated by the $\hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]]$ -lattice $\mathcal{F}_{\mathbf{g}}^+(\chi_{\mathbf{g}}^{-1}) \otimes_{\Lambda_{\mathbf{g}}} \hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]] \oplus \mathcal{F}_{\mathbf{g}}^- \otimes_{\Lambda_{\mathbf{g}}} \hat{\mathbb{Z}}_p^{p,\text{ur}}[[Y]]$ with respect to which $\omega_{\mathbf{g}}^{\vee}$ and $\eta_{\mathbf{g}}^{\vee}$ are equal to $\rho(d)^{\vee} v^+$ and $\rho(d)v^-$ respectively (here $\rho(d)$ is as in section 4.2.2 applied to the unramified representations $\mathcal{F}_{\mathbf{g}}^+(\chi_{\mathbf{g}}^{-1})$ and $\mathcal{F}_{\mathbf{g}}^-$). According to our choices of isomorphisms (4.4.3) and (4.4.4), the family $(v^+, c \cdot v^+)$ generates a free $\Lambda_{\mathbf{g}}$ -module which we identify with $T(\mathbf{g})$. Note the important fact that the $\Lambda_{\mathbf{g}}$ lattices $\Lambda_{\mathbf{g}}v^+ \oplus \Lambda_{\mathbf{g}}c \cdot v^+$ and $\Lambda_{\mathbf{g}}v^+ \oplus \Lambda_{\mathbf{g}}v^-$ are *not* the same.

Finally, let $\alpha_{\mathbf{g}}$ be the U_p -eigenvalue on \mathbf{g} .

4.4.2 Analytic families of Beilinson-Flach elements

Let f be an eigencuspform as in the introduction of this section (in particular, $\rho_f|G_{\mathbb{Q}_p}$ satisfies hypothesis 4.1). If $\phi : \Lambda_{\mathbf{g}} \longrightarrow E$ is an arithmetic specialization, then the Rankin-Selberg representation $G_{\mathbb{Q}}$ -representation $V(f) \otimes_E \rho_{g_\phi}$ is isomorphic to $\text{Ind}_{G_K}^{G_{\mathbb{Q}}}(V(f)|G_K \otimes_E \psi_\phi)$. Let

$$\eta_f \in D_{\text{dR}}(V(f))/\text{Fil}^0 D_{\text{dR}}(V(f))$$

be the element which pairs to 1 with ω_{f^*} under the de Rham pairing.

As in the convention of [74] (see also [73, Proposition 10.1.2]) we identify the f -component of the cohomology of the modular curve $X_0(N)$ with the quotient of the f -component of the cohomology of the modular curve $X_0(Np)$ on which U_p acts through the eigenvalue α and write $(\text{Pr}^\alpha)^*$ for the corresponding map. We have

$$(\text{Pr}^\alpha)^* \omega_f = \omega_{f_\alpha}.$$

Definition 4.23. Let $U \simeq \mathbb{Z}_p$ be the Galois group $\text{Gal}(\mathbb{Q}_p^{\text{p},\text{ur}}/\mathbb{Q}_p)$. For $r > 0$ sufficiently large, let \mathcal{A} be the affinoid algebra $\mathcal{O}\langle p^{-r}U\rangle$ and let $\Lambda_{\mathcal{A},\infty}$ be $\Lambda_\infty \hat{\otimes} \mathcal{A}$. The analytic Beilinson-Flach class

$$\text{BF}_\alpha = \text{BF}_{f_\alpha, \mathbf{g}} \in H_{\text{cl},\text{Iw}}^1(G_{\mathbb{Q}_\infty}, T_f \otimes T(\mathbf{g}) \otimes \Lambda_{\mathcal{A},\infty})$$

on \mathcal{A} is the class constructed in [85, Theorem A]. If $\phi : \Lambda_{\mathcal{A},\infty} \longrightarrow S$ is a point of $\Lambda_{\mathcal{A},\infty}$, then we write $\text{BF}_{\alpha,\phi}$ for the image of BF_α through ϕ . In particular, the one-dimensional Beilinson-Flach class

$$\text{BF}_{\alpha,\phi} \in H^1(G_{\mathbb{Q},\Sigma}, T_f \otimes \chi_\phi \otimes T(\mathbf{g})).$$

attached to a classical point $\phi \in \text{Spec } \mathcal{O}_{\text{Iw}}$ is the classe constructed in [82, Section 6.9]⁶. Finally, we denote by $\text{BF}_\alpha^{\text{cyc}}$ the specialization of BF_α to the cyclotomic deformation.

Before continuing, we explain how we choose the quadratic imaginary field \mathcal{K} .

Definition 4.24. The quadratic imaginary field \mathcal{K} is henceforth chosen so that it satisfies the following properties.

1. Let ℓ be in the assumption of theorem 1.7. If the ℓ is not 2, then we take \mathcal{K} which is split at 2 and at any prime divisor of N except ℓ , and is ramified in ℓ . If ℓ is 2, then by assumption $2||N$, and we take \mathcal{K} to be ramified at 2 and split at all other primes divisors of N .
2. The prime p is split in \mathcal{K} .
3. All other primes dividing N are split in \mathcal{K} .
4. The $G_{\mathcal{K},\Sigma}$ -representation $\bar{\rho}_f$ is irreducible.

4.5 Selmer Complexes and Iwasawa Main Conjecture

4.5.1 Nekovář-Selmer complexes

In this subsection, we study Nekovář-Selmer complexes of analytic families Galois representations following [103]. In this section, (T, ρ, \mathcal{O}) and (V, ρ, E) denote $G_{\mathbb{Q},\Sigma}$ -representations. As our actual goal is the study of T_f and V_f , we will come to assume that T and V satisfy the essential properties of T_f and V_f , that is to say that they are of rank 2, that $\rho|G_{\mathbb{Q}_p}$ is crystalline with two distinct eigenvalues and that $\bar{\rho}|G_{\mathbb{Q}_p}$ is absolutely irreducible.

Let A, M and G be as in [103, Section 1.1] (in particular G is a topological group and M is a continuous $A[G]$ -module). We write $C_{\text{cont}}^\bullet(G, M)$ for the complex of continuous cochains of the G -module M . Suppose G is equal to $G_{F,\Sigma}$ for F/\mathbb{Q} a finite extension. For $\ell \in \Sigma$ and $v|\ell$ a finite place of a finite extension F/\mathbb{Q}_ℓ , a local condition at v for M is a pair (U_v^\bullet, i_v) formed of a bounded complex of finite type A -modules U_v^\bullet together with a morphism

$$i_v : U_v^\bullet \longrightarrow C_{\text{cont}}^\bullet(G_{F_v}, M).$$

This definition applies in particular to the $\Lambda_{\mathcal{A},\infty}[G_{\mathcal{K},\Sigma}]$ -module $T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}$ and more generally to the $A[G_{\mathcal{K},\Sigma}]$ -module $T_\phi \stackrel{\text{def}}{=} T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty} \otimes_{\Lambda_{\mathcal{A},\infty}, \phi} S$ when $\phi : \Lambda_{\mathcal{A},\infty} \longrightarrow A$ is a map of separated, flat \mathcal{O} -algebras complete with respect to a proper ideal containing p .

⁶There, a supplementary hypothesis is put on α in the definition of $\text{BF}_{\alpha,\phi}$. We thank D.Loeffler for informing us that this condition is not necessary if we do not impose local conditions on $\text{BF}_{\alpha,\phi}$ at primes in Σ .

Definition 4.25 ([93, 103]). Let F/\mathbb{Q} be a finite extension. We identify Σ with the finite set of finite places of F above places in Σ . Suppose that for all $v \in \Sigma$, we are given a local condition (U_v^{\bullet}, i_v) . The Nekovář-Selmer complex $R\tilde{\Gamma}_f(G_{F,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty})$ of the analytic family $T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}$ is the image in the derived category of the mapping cone

$$\text{Cone} \left(C_{\text{cont}}^{\bullet}(G_{F,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) \oplus \bigoplus_{v \in \Sigma} U_v^{\bullet} \xrightarrow{v \in \Sigma} \bigoplus_{v \in \Sigma} C_{\text{cont}}^{\bullet}(G_{F_v}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) \right) [-1] \quad (4.5.1)$$

where d_v is equal to $\text{loc}_v - i_v$ for all $v \in \Sigma$. More generally, if there is a map $\phi : \Lambda_{\mathcal{A},\infty} \rightarrow A$, we write $R\tilde{\Gamma}_f(G_{F,\Sigma}, T_{\phi})$ for the image in the derived category of the mapping cone of (4.5.1) but with $T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}$ replaced with $T_{\phi} = T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty} \otimes_{\Lambda_{\mathcal{A},\infty}, \phi} A$.

As in [93, 103], if $v \in \Sigma$ does not divide p , we always assume that (U_v^{\bullet}, i_v) is the unramified local condition $(C_{\text{cont}}^{\bullet}(G_{F_v}/I_v, (-)^{I_v}), i_v)$ where i_v is the inflation map

$$i_v : C_{\text{cont}}^{\bullet}(G_{F_v}/I_v, (-)^{I_v}) \longrightarrow C_{\text{cont}}^{\bullet}(G_{F_v}, -)$$

Local conditions at p Write \mathcal{R} for the Robba ring $B_{\text{rig}, \mathbb{Q}_p}^{\dagger}$ over \mathbb{Q}_p , \mathcal{R}^+ for $B_{\text{rig}, \mathbb{Q}_p}^+$ and \mathcal{R}_E for $\mathcal{R} \otimes_{\mathbb{Q}_p} E$. We recall the notions of triangulation of a (φ, Γ) -module D and of refinement of a crystalline $G_{\mathbb{Q}_p}$ -representation ([5, Definitions 2.3.2, 2.4.1]).

Definition 4.26. Let D be a rank-two (φ, Γ) -modules D over \mathcal{R}_E . A triangulation of D over \mathcal{R}_E is a short exact sequence

$$0 \longrightarrow \mathcal{F}^+ D \longrightarrow D \longrightarrow \mathcal{F}^- D \longrightarrow 0$$

where $\mathcal{F}^{\pm} D$ are (φ, Γ) -submodules which are free and direct summands as \mathcal{R}_E -submodules. Let V be a two-dimensional crystalline representation of $G_{\mathbb{Q}_p}$. A refinement of V is a full φ -stable E -filtration of $D_{\text{crys}}(V)$

$$\mathcal{F}_0 = 0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 = D_{\text{crys}}(V).$$

When V is a crystalline $G_{\mathbb{Q}_p}$ -representation with distinct eigenvalues, a refinement of V is equivalent to a choice of ordering of eigenvalues. More generally, according to [5, Proposition 2.4.1], there is a one-to-one correspondence between triangulations of $D_{\text{crys}}(V)$ and refinements of V , given by $\mathcal{F}^+ D = \mathcal{R}[1/t] \mathcal{F}_1 \cap D$ and $\mathcal{F}_1 = \mathcal{F}^+ D[1/t]^{\Gamma}$.

Let the Robba ring over \mathcal{A} be $\mathcal{R}_{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{R} \hat{\otimes} \mathcal{A}$. We also write $\mathcal{R}_{\mathcal{A}}^+ = \mathcal{R}^+ \hat{\otimes} \mathcal{A}$. There is a natural action $U \hookrightarrow \mathcal{A}^{\times}$. Then we can define a (φ, Γ) -module $D_{\mathcal{A}}$ over $\mathcal{R}_{\mathcal{A}}$ by pulling back the action of Γ on D but twisting the action of φ on D by the Frobenius action of U as above. One can define triangulation for this family of $(\varphi, \Gamma_{\mathcal{K}})$ -modules over $\mathcal{R}_{\mathcal{A}}$ in an obvious way, and define the analytic Iwasawa cohomology for $D_{\mathcal{A}}$ in the same way as in section 4.7.

Lemma 4.27. The Iwasawa cohomology groups $H_{\text{Iw}}^i(G_{\mathbb{Q}_p}, D_{\mathcal{A}})$ are the cohomology groups of the complex

$$R\Gamma_{\text{Iw}}(G_{\mathbb{Q}_p}, D_{\mathcal{A}}) \stackrel{\text{def}}{=} [D_{\mathcal{A}} \xrightarrow{\psi-1} D_{\mathcal{A}}]$$

concentrated in degrees 1 and 2. □

Proof. See [69, Theorem 4.4.8].

Let $\mathcal{R}(\alpha^{-1})$ be the unramified rank one (φ, Γ) -module with Frobenius action given by the scalar α^{-1} where α is a Weil number of weight $k-1$.

Proposition 4.28. Let D be $\mathcal{R}(\alpha^{-1})$. Then, for N sufficiently large, there is an exact sequence

$$0 \longrightarrow \bigoplus_{m=0}^{\infty} (t^m D_{\text{crys}}(D_{\mathcal{A}}))^{\varphi=1} \longrightarrow (\mathcal{R}_{\mathcal{A}}^+ \otimes D)^{\psi=1} \xrightarrow{\varphi-1} (\mathcal{R}_{\mathcal{A}}^+ \otimes D)^{\psi=0} \longrightarrow \bigoplus_{m=0}^N \frac{t^m \otimes D_{\text{crys}}(D_{\mathcal{A}})}{(1-\varphi)(t^m \otimes D_{\text{crys}}(D_{\mathcal{A}}))}.$$

in which the term $\bigoplus_{m=0}^{\infty} (t^m D_{\text{crys}}(D_{\mathcal{A}}))^{\varphi=1}$ vanishes.

Proof. This is the family version of [91, Lemma 3.17,3.18] proved in [22, Section 2]. The vanishing of $(t^m D_{\text{crys}}(D_{\mathcal{A}}))^{\varphi=1}$ follows from the fact that $k-1$ is odd. \square

The Galois group $\Gamma_{p,\infty}^{p,\text{ur}}$ is isomorphic to the product $\Gamma_{\text{Iw}} \times \Gamma_{\text{ur}}$ where Γ_{ur} is the Galois group of the unramified \mathbb{Z}_p -extension of \mathbb{Q}_p and where each factor is isomorphic to \mathbb{Z}_p . Let $Y' + 1 \in \Gamma_{\text{ur}}$ be the Frobenius element. For $m \in \mathbb{N}$ sufficiently large, $1 - \alpha p^{-m}$ is not a power of p so $Y' + 1 - \alpha p^{-m}$ is a unit in $\mathcal{O}[[\Gamma_{p,\infty}^{p,\text{ur}}]]$. For D the rank one (φ, Γ) -module $D = \mathcal{R}(\alpha^{-1})$ above, we consider the set of height-one prime ideals $S(D)$ of $\mathcal{O}[[\Gamma_{p,\infty}^{p,\text{ur}}]]$ of the form $(Y' + 1 - \alpha p^{-m})$. Then the set $S(D)$ is finite, as we just observed. According to proposition 4.28, for any height-one prime $\mathcal{P} \in \text{Spec } \mathcal{O}[[\Gamma_{p,\infty}^{p,\text{ur}}]]$ not in $S(D)$, the map

$$\varphi - 1 : (\mathcal{R}_{\mathcal{A}}^+ \otimes D)^{\psi=1} \longrightarrow (\mathcal{R}_{\mathcal{A}}^+ \otimes D)^{\psi=0}$$

localized at \mathcal{P} is an isomorphism.

Now suppose D is the (φ, Γ) -module attached to the $G_{\mathbb{Q}_p}$ -representation V_f . We fix a triangulation of D by asking that $\mathcal{F}^- \stackrel{\text{def}}{=} D/\mathcal{F}^+$ be the (φ, Γ) -module $\mathcal{R}(\alpha^{-1})$ and extend this triangulation to a triangulation

$$0 \longrightarrow \mathcal{F}^+(D_{\mathcal{A}}) \longrightarrow D_{\mathcal{A}} \longrightarrow \mathcal{F}^-(D_{\mathcal{A}}) \longrightarrow 0 \quad (4.5.2)$$

of $D_{\mathcal{A}}$ in the obvious way. For simplicity, we occasionally write the corresponding modules as $\mathcal{F}_{\mathcal{A}}^{\pm} = \mathcal{F}^{\pm}(D_{\mathcal{A}})$. If $\phi : \mathcal{A} \longrightarrow S$ is a map of flat \mathcal{O} -algebra, we define $\mathcal{F}^+(D_{\mathcal{A}} \otimes_{\mathcal{A}, \phi} S)$ to be $\mathcal{F}^+(D_{\mathcal{A}}) \otimes_{\mathcal{A}, \phi} S$. To (4.5.2) is attached a long exact sequence in cohomology which simplifies under assumption 4.1 to a short exact sequence

$$0 \longrightarrow H_{\text{Iw}}^1(G_{\mathbb{Q}_p}, \mathcal{F}^+(D_{\mathcal{A}})) \longrightarrow H_{\text{Iw}}^1(G_{\mathbb{Q}_p}, D_{\mathcal{A}}) \longrightarrow H_{\text{Iw}}^1(G_{\mathbb{Q}_p}, \mathcal{F}^-(D_{\mathcal{A}})) \longrightarrow 0 \quad (4.5.3)$$

where exactness on the right follows by local duality. The inclusion

$$H_{\text{Iw}}^1(G_{\mathbb{Q}_p}, \mathcal{F}^+(D_{\mathcal{A}})) \longrightarrow H_{\text{Iw}}^1(G_{\mathbb{Q}_p}, D_{\mathcal{A}})$$

of the previous short exact sequence may be viewed as a morphism between cohomology complexes of (φ, Γ) -modules and hence as a morphism between cohomology complexes of $G_{\mathbb{Q}_p}$ -representations with coefficients in the affinoid algebra \mathcal{A} through the functorial isomorphism of [103, Theorem 2.8].

Definition 4.29. Let $v|p$ be a finite place. Let $\phi : \Lambda_{\mathcal{A}, \infty} \longrightarrow A$ be a map of flat \mathcal{O} -algebras as above. The α local condition at v is defined by the inclusion

$$i_p : H_{\text{Iw}}^1(G_{F_v}, \mathcal{F}^+(D_{\mathcal{A}})) \longrightarrow H_{\text{Iw}}^1(G_{F_v}, D_{\mathcal{A}})$$

seen as a morphism in the derived category of cohomology complexes of $G_{\mathbb{Q}_p}$ -representations with coefficients in \mathcal{A} . The relaxed local condition at v is the local condition $(C_{\text{cont}}^{\bullet}(G_{F_v}, D_{\mathcal{A}}), \text{Id}_v)$. The strict local condition at v is the local condition $(0, 0)$. We define $H_{\alpha}^1(G_{\mathbb{Q}_p}, T_{\phi})$ to be the kernel of the map

$$H^1(G_{\mathbb{Q}_p}, T_{\phi}) \longrightarrow H_{\text{Iw}}^1(G_{\mathbb{Q}_p}, (D_{\mathcal{A}} \otimes A)/\mathcal{F}^-(D_{\mathcal{A}} \otimes A)).$$

Note that the exactness of (4.5.3) shows that the α condition is also defined by the morphism

$$i_p : R\Gamma_{\text{Iw}}(G_{\mathbb{Q}_p}, \mathcal{F}^+(D_{\mathcal{A}})) \longrightarrow R\Gamma_{\text{Iw}}(G_{\mathbb{Q}_p}, D_{\mathcal{A}}).$$

As $\mathcal{F}^+(D_{\mathcal{A}} \otimes_{\mathcal{A}, \phi} S)$ is equal to $\mathcal{F}^+(D_{\mathcal{A}}) \otimes_{\mathcal{A}, \phi} S$ when $\phi : \mathcal{A} \longrightarrow A$ is a map of flat \mathcal{O} -algebra, we see that the local conditions α , str and rel all commute with $- \overset{L}{\otimes}_{\mathcal{A}, \phi} A$ in the sense that the diagram

$$\begin{array}{ccc} R\Gamma_{\text{Iw}}(G_{\mathbb{Q}_p}, \mathcal{F}^+(D_{\mathcal{A}})) & \xrightarrow{i_p} & R\Gamma_{\text{Iw}}(G_{\mathbb{Q}_p}, D_{\mathcal{A}}) \\ \downarrow - \overset{L}{\otimes}_{\mathcal{A}, \phi} A & & \downarrow - \overset{L}{\otimes}_{\mathcal{A}, \phi} A \\ R\Gamma_{\text{Iw}}(G_{\mathbb{Q}_p}, \mathcal{F}^+(D_{\mathcal{A}} \otimes_{\mathcal{A}, \phi} A)) & \xrightarrow{i_p} & R\Gamma_{\text{Iw}}(G_{\mathbb{Q}_p}, D_{\mathcal{A}} \otimes_{\mathcal{A}, \phi} A) \end{array} \quad (4.5.4)$$

and its obvious counterpart for the two other conditions are commutative.

For $\phi : \Lambda_{\mathcal{A},\infty} \longrightarrow A$, let $R\tilde{\Gamma}_{?,?}(G_{K,\Sigma}, T_\phi)$ be the Nekovář-Selmer complex attached to the analytic family of $G_{K,\Sigma}$ -representations T_ϕ with the unramified condition at $v \in \Sigma$ prime to p , the condition $?_1$ at v_0 and the condition $?_2$ at \bar{v}_0 . In particular, $R\tilde{\Gamma}_{\alpha,\alpha}(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty})$ is the Nekovář-Selmer complex attached to the analytic family of $G_{K,\Sigma}$ -representations $T \otimes \Lambda_{\mathcal{A},\infty}$ with the unramified condition at $v \in \Sigma$ prime to p and the condition of definition 4.29 when $v|p$, $R\tilde{\Gamma}_{\text{str,rel}}(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty})$ (resp. $R\tilde{\Gamma}_{\alpha,\text{rel}}(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty})$) is the Nekovář-Selmer complex attached to the strict condition at v_0 and the relaxed condition at \bar{v}_0 (resp. the alpha condition at v_0 and the relaxed condition at \bar{v}_0). We write $\tilde{H}_{?,?}^i(G_{K,\Sigma}, -)$ or sometimes more simply $\tilde{H}_{?,?}^i(-)$ for the i -th cohomology module of $R\tilde{\Gamma}_{\alpha,\alpha}(G_{K,\Sigma}, -)$.

Proposition 4.30. *For $?_i \in \{\text{str, rel, } \alpha\}$ and $\phi : \Lambda_{\mathcal{A},\infty} \longrightarrow A$ a map of flat \mathcal{O} -algebras, the natural map*

$$R\tilde{\Gamma}_{?,?}(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty}) \xrightarrow{L} \otimes_{\Lambda_{\mathcal{A},\infty}, \phi} A \longrightarrow R\Gamma_{?,?}(G_{K,\Sigma}, T_\phi)$$

is an isomorphism. In particular, there is an isomorphism

$$R\tilde{\Gamma}_{\alpha,\alpha}(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty}) \xrightarrow{L} \otimes_{\Lambda_{\mathcal{A},\infty}} \Lambda_\infty \simeq R\tilde{\Gamma}_{\alpha,\alpha}(G_{K,\Sigma}, T \otimes \Lambda_\infty)$$

which induces short exact sequences

$$0 \longrightarrow \tilde{H}_{\alpha,\alpha}^i(T \otimes \Lambda_{\mathcal{A},\infty}) \otimes_{\Lambda_{\mathcal{A},\infty}} \Lambda_\infty \longrightarrow \tilde{H}_{\alpha,\alpha}^i(T \otimes \Lambda_\infty) \longrightarrow \text{Tor}_1^{\Lambda_{\mathcal{A},\infty}}(\tilde{H}_{\alpha,\alpha}^{i+1}(T \otimes \Lambda_{\mathcal{A},\infty}), \Lambda_{\mathcal{A},\infty}) \longrightarrow 0$$

for all $i \in \mathbb{Z}$ and $\tilde{H}_{?,?}^1(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty})$ is torsion-free.

Proof. As noted above, the diagram (4.5.4) commutes for each local conditions α, str and rel commute with $- \xrightarrow{L} \otimes_{\Lambda_{\mathcal{A},\infty}} \Lambda_{\mathbb{Q},\infty}$ and all $v \in \Sigma$. The results then follow formally. \square

It follows from the definition of the relevant Selmer complexes that there are long exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{H}_{\alpha,\text{rel}}^1(G_{K,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) & \longrightarrow & H^1(G_{K,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) & \xrightarrow{A} & H^1(G_{K,v_0}, \mathcal{F}^-(D_{\mathcal{A}})) \\ & & & & & & \downarrow \\ & & \bigoplus_{v \in \Sigma} H^2(G_{K_v}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) & \longleftarrow & H^2(G_{K,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) & \longleftarrow & \tilde{H}_{\alpha,\text{rel}}^2(G_{K,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) \end{array} \quad (4.5.5)$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{H}_{\text{str,rel}}^1(G_{K,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) & \longrightarrow & H^1(G_{K,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) & \xrightarrow{B} & H^1(G_{K,v_0}, D_{\mathcal{A}}) \\ & & & & & & \downarrow \\ & & \bigoplus_{v \in \Sigma} H^2(G_{K_v}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) & \longleftarrow & H^2(G_{K,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) & \longleftarrow & \tilde{H}_{\text{str,rel}}^2(G_{K,\Sigma}, T \otimes_{\mathcal{O}} \Lambda_{\mathcal{A},\infty}) \end{array} \quad (4.5.6)$$

As in the diagram above, denote the third arrows of (4.5.5) and (4.5.6) by A and B respectively. Define Ker by the exact sequence

$$0 \longrightarrow \text{Ker} \longrightarrow \frac{H^1(G_{K,v_0}, D_{\mathcal{A}})}{\text{Im}(B)} \longrightarrow \frac{H^1(G_{K,v_0}, \mathcal{F}^-(D_{\mathcal{A}}))}{\text{Im}(A)} \longrightarrow 0$$

Then

$$\text{Ker} = \frac{\text{Im}(B) + H^1(G_{K,v_0}, \mathcal{F}_{\mathcal{A}}^+)}{\text{Im}(B)} \simeq \frac{H^1(G_{K,v_0}, \mathcal{F}_{\mathcal{A}}^+)}{\text{Im}(B) \cap H^1(G_{K,v_0}, \mathcal{F}_{\mathcal{A}}^+)} \simeq \frac{H^1(G_{K,v_0}, \mathcal{F}_{\mathcal{A}}^+)}{\text{Im}(\tilde{H}_{\alpha,\text{rel}}^1(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty}))}.$$

Combining this with (4.5.5) and (4.5.6) we obtain

$$0 \rightarrow \frac{H^1(G_{K,v_0}, \mathcal{F}_{\mathcal{A}}^+)}{\text{Im}(\tilde{H}_{\alpha,\text{rel}}^1(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty}))} \rightarrow \tilde{H}_{\text{str,rel}}^2(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty}) \rightarrow \tilde{H}_{\alpha,\text{rel}}^2(G_{K,\Sigma}, T \otimes \Lambda_{\mathcal{A},\infty}) \rightarrow 0. \quad (4.5.7)$$

Similarly we get

$$0 \rightarrow \frac{H^1(G_{\mathcal{K}_{\bar{v}_0}}, \mathcal{F}_{\mathcal{A}})}{\text{Im}(\tilde{H}_{\alpha, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty}))} \rightarrow \tilde{H}_{\alpha, \alpha}^2(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty}) \rightarrow \tilde{H}_{\alpha, \text{rel}}^2(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty}) \rightarrow 0. \quad (4.5.8)$$

Lemma 4.31. *Let χ be a finite order character of Γ_{Iw} . Then*

$$H_{\alpha}^1(G_{\mathbb{Q}_p}, T \otimes \chi) = H_f^1(G_{\mathbb{Q}_p}, T \otimes \chi) \quad (4.5.9)$$

and

$$\tilde{H}_{\alpha, \alpha}^1(G_{\mathcal{K}, \Sigma}, T \otimes \chi) = H_f^1(G_{\mathcal{K}, \Sigma}, T \otimes \chi). \quad (4.5.10)$$

In particular $\tilde{H}_{\alpha, \alpha}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty})$ is a torsion $\Lambda_{\mathcal{A}, \infty}$ -module and $\tilde{H}_{\alpha, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty})$ is a torsion-free $\Lambda_{\mathcal{A}, \infty}$ -module of rank one.

Proof. By [4, Proposition 5], there is an equality $H_{\alpha}^1(G_{\mathbb{Q}_p}, V \otimes \chi) = H_f^1(G_{\mathbb{Q}_p}, V \otimes \chi)$. As $H_{\alpha}^1(G_{\mathbb{Q}_p}, T \otimes \chi)$ and $H_f^1(G_{\mathbb{Q}_p}, T \otimes \chi)$ are free \mathbb{Z}_p -modules by lemma 4.2, (4.5.9) holds. Equation (4.5.10) then follows as the local condition at $v \nmid p$ are the same for $\tilde{H}_{\alpha, \alpha}^1(G_{\mathcal{K}, \Sigma}, T \otimes \chi)$ and $H_f^1(G_{\mathcal{K}, \Sigma}, T \otimes \chi)$. It follows from [68, Theorem 12.5] that the \mathcal{O} -modules $H_f^1(G_{\mathbb{Q}, \Sigma}, T \otimes \chi)$ and $H_f^1(G_{\mathbb{Q}, \Sigma}, T \otimes \chi \otimes \chi_{\mathcal{K}})$ have rank zero, and hence vanish, for all finite order characters $\chi \in \hat{\Gamma}_{\text{Iw}}$ except possibly finitely many (here $\chi_{\mathcal{K}}$ is the quadratic character attached to the extension \mathcal{K}/\mathbb{Q}). As

$$\tilde{H}_{\alpha, \alpha}^1(G_{\mathcal{K}, \Sigma}, T \otimes \chi) \simeq H_f^1(G_{\mathbb{Q}, \Sigma}, T \otimes \chi) \oplus H_f^1(G_{\mathbb{Q}, \Sigma}, T \otimes \chi \otimes \chi_{\mathcal{K}})$$

and as $\tilde{H}_{\alpha, \alpha}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty}) \otimes_{\Lambda_{\mathcal{A}, \infty}, \chi} E$ maps injectively into $\tilde{H}_{\alpha, \alpha}^1(G_{\mathcal{K}, \Sigma}, V \otimes \chi)$ for all $\chi \in \hat{\Gamma}_{\text{Iw}}$ with values in \mathcal{O} by proposition 4.30, we obtain that $\tilde{H}_{\alpha, \alpha}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty})$ is a torsion $\Lambda_{\mathcal{A}, \infty}$ -module. As the cokernel of

$$\tilde{H}_{\alpha, \alpha}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty}) \longrightarrow \tilde{H}_{\alpha, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty})$$

is included inside $H^1(G_{\mathcal{K}_{\bar{v}_0}}, T \otimes \Lambda_{\mathcal{A}, \infty})/H^1(G_{\mathcal{K}_{\bar{v}_0}}, \mathcal{F}^- D_{\mathcal{A}})$, it is of rank at most 1. It follows that $\tilde{H}_{\alpha, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty})$ is of rank at most 1. According to [74, Section 7], the class $\text{loc}_{v_0} \text{BF}_{\alpha}$ belongs to $H^1(G_{\mathbb{Q}_p}, \mathcal{F}^+(D_{\mathcal{A}}))$ and is not torsion. This entails that the family of Beilinson-Flach elements over \mathcal{A} is a non-torsion element of $\tilde{H}_{\alpha, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T \otimes \Lambda_{\mathcal{A}, \infty})$, hence that this module is of rank at least 1. Hence, it is of rank exactly 1. \square

The regulator map and duality For any (φ, Γ) -module of the form $\mathcal{R}_{\mathcal{A}}(\alpha^{-1})$ for some $\alpha \in \mathcal{A}^*$, we define a regulator map as in [74, (6.2.1)]

$$\begin{aligned} \text{Reg}_{\mathcal{R}_{\mathcal{A}}(\alpha^{-1})} : H_{\text{Iw}}^1(G_{\mathbb{Q}_p}, \mathcal{R}_{\mathcal{A}}(\alpha^{-1})) &\xrightarrow{\sim} \mathcal{R}_{\mathcal{A}}(\alpha^{-1})^{\psi=1} \xrightarrow{\sim} \mathcal{R}_{\mathcal{A}}^+(\alpha^{-1})^{\psi=1} \\ &\xrightarrow{\varphi^{-1}} \mathcal{R}_{\mathcal{A}}^+(\alpha^{-1})^{\psi=0} \xrightarrow{\sim} \mathcal{A} \hat{\otimes} \Lambda_{\infty}. \end{aligned}$$

Here, the last map is the Mellin transform. By construction, this applies in particular to $\mathcal{F}^-(D_{\mathcal{A}})$. As we observed, the map $\text{Reg}_{\mathcal{F}^-}$ is an isomorphism after localization at a height-one prime ideal not in $S(\mathcal{F}_{\mathcal{A}}^-)$. The (φ, Γ) -module $\mathcal{F}_{\mathcal{A}}^+$ being a twist of $\mathcal{R}(\beta^{-1})$ by a character of $\Gamma_{p, \infty}^{p, \text{ur}}$ which factors through Γ_{Iw} , after a reparametrization of the weight space, we may by the same method define also a regulator map

$$\text{Reg}_{\mathcal{F}^+} : H_{\text{Iw}}^1(G_{\mathbb{Q}_p}, \mathcal{F}^+(D_{\mathcal{A}})) \hookrightarrow \mathcal{A} \hat{\otimes} \Lambda_{\infty}.$$

Then $\text{Reg}_{\mathcal{F}^+}$ is an isomorphism after localization outside the finite set $S(\mathcal{R}_{\mathcal{A}}(\beta^{-1}))$, which we denote $S(\mathcal{F}^+)$ in a slight abuse of notation.

For $(*, ?) \in \{+, -\}^2$, we define

$$\mathcal{F}^{*, ?}(D \hat{\otimes} D_{\mathcal{A}}(\mathbf{g})) \stackrel{\text{def}}{=} \mathcal{F}^*(D) \hat{\otimes} \mathcal{F}^?(D_{\mathcal{A}}(V(\mathbf{g}))).$$

Then the $G_{\mathbb{Q}_p}$ -representation $\mathcal{F}^{-, +}(D_{\mathcal{A}} \hat{\otimes} D_{\mathcal{A}}(\mathbf{g})(\chi_{\mathbf{g}}^{-1}))$ is unramified. There is then a map

$$H^1(G_{\mathbb{Q}_p}, \mathcal{F}^{-, +}(D_{\mathcal{A}} \hat{\otimes} D_{\mathcal{A}}(\mathbf{g})(\chi_{\mathbf{g}}^{-1}))) \longrightarrow (\mathcal{F}^{-, +}(D_{\mathcal{A}} \hat{\otimes} D(\mathbf{g})(\chi_{\mathbf{g}}^{-1})))^{G_{\mathbb{Q}_p}} \hat{\otimes} (\mathcal{A} \hat{\otimes} \Lambda_{\infty})$$

by [74, Theorem 8.2.3]. We denote the previous map by $\text{Reg}_{\mathcal{F}^-, +}$.

4.5.2 Rankin-Selberg p -adic L -functions

Definitions In this subsection, we introduce three p -adic L -functions attached to f and state their fundamental Iwasawa-theoretic properties.

Definition 4.32. *The cyclotomic p -adic L -function $\mathcal{L}_\alpha(f)$ of f_α is the unique locally analytic function on $\text{Hom}_{\text{cont}}(\text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}), \mathbb{C}_p^\times)$ satisfying*

$$\mathcal{L}_\alpha(f)(\chi_{\text{cyc}}^r \chi^{-1}) = \frac{(r + \frac{k-2}{2} - 1)! p^{n(r + \frac{k-2}{2})} \alpha^{-n} G(\chi)^{-1}}{(2\pi i)^{\frac{k-2}{2} + r} \Omega_f^{(-1)\frac{k}{2} - r}} L_{\{p\}} \left(f, \chi, r + \frac{k-2}{2} \right)$$

for all character $\chi \in \hat{\Gamma}_{\text{Iw}}$ of finite order p^n and all integer r such that $1 \leq r + \frac{k-2}{2} \leq k-1$.

As f_α has non-critical slope, existence and unicity of $\mathcal{L}_\alpha(f)$ follow from [86, 122, 89].

Recall that \mathbf{g} is the Hida family of section 4.4.1 and write $\pi(\psi_\phi)$ an arithmetic specialization thereof. In [83]), it is shown that the special values of the L -function of $f \otimes \pi(\psi_\phi)$ admit a p -adic interpolation when the weight of $\pi(\psi_\phi)$ is lower than the weight of f . According to [85, Theorem 7.5.1], these special values define a 3-variables p -adic L -function.

Definition 4.33. *The Rankin-Selberg p -adic L -function $\mathcal{L}_{\alpha,\alpha}(f \otimes \mathbf{g})$ attached to f and to the Hida family of CM forms \mathbf{g} is the p -adic L -function of [85, Theorem 7.5.1].*

The last p -adic L -function attached to f is the Rankin-Selberg p -adic L -function constructed as constant terms of p -adic families of Klingen Eisenstein series for $\text{GU}(2,0)$ in [32].

Definition 4.34. *The Greenberg Rankin-Selberg p -adic L -function $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f) \in \Lambda_{\mathcal{K}}$ is the p -adic L -function of [32, Theorem 1.2].*

The construction and properties of $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$ are recalled in appendix 7 below.

The regulator map and duality Crucially for our purpose, the Rankin-Selberg p -adic L -function $\mathcal{L}_{\alpha,\alpha}(f \otimes \mathbf{g})$ is related to the image of Beilinson-Flach elements through the regulator map by the explicit reciprocity law ([75, Theorem 6.5.9],[85, Theorem 7.5.1]). In particular, choosing a suitable specialization of \mathbf{g} shows that $\mathcal{L}_{\alpha,\alpha}(f \otimes \mathbf{g})$ specializes to a p -adic L -function closely related to $\mathcal{L}_\alpha(f)$ while exchanging the role of f and \mathbf{g} in $\mathcal{L}_{\alpha,\alpha}$, one may also obtain a p -adic L -function which is closely related to $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$. There are however two difficulties worth noting. The first is that different construction of the p -adic L -function corresponds to different choices of p -adic periods. The second is that exchanging the role of f and \mathbf{g} replaces them with their dual while the explicit reciprocity law involves Poincarduality, which does not induce the self-duality of the $\mathcal{O}[G_{\mathbb{Q},\Sigma}]$ -module T_f . In this paragraph, we deal with these subtleties

Let $\pi : E \rightarrow Y(N)$ be the universal elliptic curve over the open modular curve $Y(N)$ and let \mathbb{L}_{k-2} be the local system $\text{Sym}^{k-2} R^1 \pi_* \mathcal{O}$, which is of rank $k-1$ over \mathcal{O} . Consider the Poincarduality pairing

$$\langle \cdot, \cdot \rangle : H_{\text{et}}^1(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{L}_{k-2})_{\mathfrak{m}_f} \times H_c^1(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{L}_{k-2})_{\mathfrak{m}_f} \longrightarrow E. \quad (4.5.11)$$

Note that as $\bar{\rho}_f$ is absolutely irreducible, we have

$$H_{\text{et}}^1(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{L}_{k-2})_{\mathfrak{m}_f} = H_c^1(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{L}_{k-2})_{\mathfrak{m}_f} = H_{\text{et},!}^1(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{L}_{k-2})_{\mathfrak{m}_f}$$

(the last cohomology group being interior cohomology). Inverting p and applying Faltings comparison map, Poincarduality induces Serre duality

$$H_{\text{dR}}^1(X(N), \omega_{-k} \otimes \Omega_{X(N)}^1)_{\mathfrak{m}_f} \times H_{\text{dR}}^0(X, \omega_k)_{\mathfrak{m}_f} \longrightarrow E$$

on the (graded piece of) algebraic de Rham cohomology. As $\mathcal{O}[G_{\mathbb{Q},\Sigma}]$ -module, T_f may be identified with $H_{\text{et}}^1(Y(N) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{L}_{k-2})_{\mathfrak{m}_f}[\lambda_f](k/2)$ and this latter $\mathcal{O}[G_{\mathbb{Q},\Sigma}]$ -module naturally comes with the pairing induced from (4.5.11). However, this pairing does not induce the self-duality of T_f . To be more precise, let $\{x, y\}$ be an \mathcal{O} -basis of the free \mathcal{O} -module

$H_{\text{et}}^1(Y, \mathbb{L}_{k-2} \otimes_{\mathbb{Z}_p} \mathcal{O})_{\mathfrak{m}_f}[\lambda_f]$ and let $\langle f, f \rangle_{X(N)}$ be the Petersson inner product. Up to a p -adic unit, we then have

$$\langle x, y \rangle = \frac{\langle f, f \rangle_{X(N)}}{\Omega_f^+ \Omega_f^-} \quad (4.5.12)$$

under (4.5.11) by [27, Lemma 4.17 to Theorem 4.20]. Denoting by $\mathfrak{C}_f \in \mathbb{C}_p$ the right-hand side of (4.5.12), the self-duality pairing (\cdot, \cdot) on T_f then satisfies

$$(\cdot, \cdot) = \frac{1}{\mathfrak{C}_f} \langle \cdot, \cdot \rangle. \quad (4.5.13)$$

In particular, the vector which pairs to 1 with ω_f under (4.5.11) is paired to $\frac{1}{\mathfrak{C}_f}$ under the perfect self-duality pairing of T_f .

The following proposition, which is due to [75, 85], provides the precise relation between our three p -adic L -functions and Beilinson-Flach elements.

Proposition 4.35. *Let $\mathcal{L}_K^{\text{Katz}}$ be Katz p -adic L -function as in [55, (8.2)]. Write $\mathcal{E}(f)$ and $\mathcal{E}^*(f)$ for $1 - \frac{\beta}{p^\alpha}$ and $1 - \frac{\beta}{\alpha}$ respectively and denote the Atkin-Lehner pseudo-eigenvalue of f by λ_N (see [74, Corollary 6.4.3]). Finally, recall that ω_f and η_f are identified with $(\text{Pr}^\alpha)^*(\omega_f)$ and $(\text{Pr}^\alpha)^*(\eta_f)$ respectively. Under the convention at the end of section 4.2.2, the following equalities then hold*

$$\langle \text{Reg}_{\bar{v}_0, \mathcal{F}^-}(\text{loc}_{\bar{v}_0} \text{BF}_\alpha), \eta_f \rangle = \mathcal{E}(f) \mathcal{E}^*(f) \lambda_N \mathcal{L}_{\alpha, \alpha}(f \otimes \mathbf{g}), \quad (4.5.14)$$

$$(\eta_f, \omega_f) = \frac{1}{\mathfrak{C}_f} \quad (4.5.15)$$

and

$$\langle \text{Reg}_{v_0, \mathcal{F}^+}(\text{loc}_{v_0} \text{BF}_\alpha), \omega_f \rangle = \mathcal{L}_K^{\text{Gr}}(f) \cdot \frac{1}{h_K \mathcal{L}_K^{\text{Katz}}} \cdot \frac{v^-}{c \cdot v^+} \quad (4.5.16)$$

up to p -adic units.

Proof. Equation (4.5.14) follows from [75, Theorem 6.5.9] and the discussion immediately before the statement of the proposition. After exchanging the role f and g using [85, Theorem 7.5.1], [75, Theorem 6.5.9] describes the interpolation property of the left-hand side of equation (4.5.16) and thus shows that $\langle \text{Reg}_{v_0, \mathcal{F}^+}(\text{loc}_{v_0} \text{BF}_\alpha), \omega_f \rangle$ is equal to the Rankin-Selberg p -adic L -function of [56, Theorem I] multiplied by the scalar ratio comparing v^- and $c \cdot v^+$. After multiplying by $h_K \mathcal{L}_K^{\text{Katz}}$, [126, Equation (7.5)] then shows that up to a p -adic unit, the coefficient x such that

$$\langle \text{Reg}_{v_0, \mathcal{F}^+}(\text{loc}_{v_0} \text{BF}_\alpha), \omega_f \rangle \left(h_K \mathcal{L}_K^{\text{Katz}} \right) = x \frac{v^-}{c \cdot v^+}$$

satisfies the same interpolation properties as $\mathcal{L}_K^{\text{Gr}}(f)$. Equation (4.5.16) follows. Finally, equation (4.5.15) is a restatement of the definition of η_f taking into account (4.5.13). \square

The following integrality proposition is needed later.

Lemma 4.36. *Recall that we have identified the coefficient ring $\Lambda_{\mathbf{g}}$ of \mathbf{g} with the power-series ring $\mathbb{Z}_p[[Y]]$ by setting $Y = \gamma_{\bar{v}_0-1}$ for $\gamma_{\bar{v}_0}$ a topological generator of $\Gamma_{\bar{v}_0}$. Then*

$$\frac{h_K \mathcal{L}_K^{\text{Katz}} c \cdot v^+}{v^-} \quad (4.5.17)$$

is in $\mathbb{Z}_p^{\text{ur}}[[Y]]$ up to some powers of Y .

Proof. See [126, Proposition 8.3]. \square

The following lemma relates $\mathcal{L}_\alpha(f)$ to the restriction $\mathcal{L}_{\alpha, \alpha}^{\text{cyc}}(f)$ of $\mathcal{L}_{\alpha, \alpha}(f)$ to the cyclotomic line.

Lemma 4.37. *There is an equality*

$$\mathcal{L}_{\alpha, \alpha}^{\text{cyc}}(f) = \mathcal{L}_\alpha(f) \cdot \mathcal{L}_\alpha(f \otimes \chi_K)$$

up to a non-zero constant. In particular, $\mathcal{L}_{\alpha, \alpha}^{\text{cyc}}(f)$ is not identically zero.

Proof. We view (f, α) as a non-critical point x_f on the eigencurve $E(1, M)$ of [24, 33] (of some tame level M). According to [33, Theorem 4.5.7] and [85, Theorem 7.5.1], there exists an affinoid neighborhood U of x and p -adic L -functions $\mathcal{L}_\alpha^U(f)$, $\mathcal{L}_\alpha^U(f \otimes \chi_K)$ and $\mathcal{L}_\alpha^U(f \otimes \mathbf{g})$ on $\text{Hom}_{\text{cont}}(\text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}), \mathbb{C}_p^\times) \times U$ and $\text{Hom}_{\text{cont}}(\text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}), \mathbb{C}_p^\times) \times U \times \text{Spec } \mathbb{Z}_p[[Y]]$ respectively. Denote by U^{crys} the subset of points such that the underlying $G_{\mathbb{Q}_p}$ -representation is crystalline. Then U^{crys} is a Zariski-dense subset. All three p -adic L -functions satisfy an interpolation property at the points of the subset $W \subset \text{Hom}_{\text{cont}}(\text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}), \mathbb{C}_p^\times) \times U \times \text{Spec } \mathbb{Z}_p[[Y]]$ of points of the form (χ, x, y) with $x \in U^{\text{crys}}$ and $y \in \text{Spec } \mathbb{Z}_p[[Y]]$ equal to the augmentation ideal. Let us denote by $\mathcal{L}^{\{x\}}(f_x \otimes \mathbf{g}_y)$ (resp. $\mathcal{L}_\alpha^{\{x\}}(f_x)$, $\mathcal{L}_\alpha^{\{x\}}(f_x \otimes \chi_K)$) the specialization of $\mathcal{L}_\alpha^U(f \otimes \mathbf{g})$ at $(x, y) \in U \times \{y\}$ (resp. $x \in U$). In particular, $\mathcal{L}^{\{x_f\}}(f_{x_f} \otimes \mathbf{g}_y)$ is $\mathcal{L}_{\alpha, \alpha}^{\text{cyc}}(f)$ and $\mathcal{L}_\alpha^{\{x\}}(f_x) \cdot \mathcal{L}_\alpha^{\{x\}}(f_x \otimes \chi_K)$ is $\mathcal{L}_\alpha(f) \cdot \mathcal{L}_\alpha(f \otimes \chi_K)$ and we may compare the special values interpolated by $\mathcal{L}^{\{x\}}(f_x \otimes \mathbf{g}_y)$ and $\mathcal{L}_\alpha^{\{x\}}(f_x) \cdot \mathcal{L}_\alpha^{\{x\}}(f_x \otimes \chi_K)$ if $x \in U^{\text{crys}}$.

As $\mathcal{L}^{\{x\}}(f_x \otimes \mathbf{g}_y)$ is non-zero for $x \in U^{\text{crys}}$, the quotient

$$x \mapsto \frac{\mathcal{L}_\alpha^{\{x\}}(f_x) \cdot \mathcal{L}_\alpha^{\{x\}}(f_x \otimes \chi_K)}{\mathcal{L}^{\{x\}}(f_x \otimes \mathbf{g}_y)} \quad (4.5.18)$$

is defined on U^{crys} and extends to a unique rigid analytic function defined outside the closed locus of vanishing of $\mathcal{L}^U(f \otimes \mathbf{g}_y)$. Comparing the interpolation properties of $\mathcal{L}_\alpha^{\{x\}}(f_x) \cdot \mathcal{L}_\alpha^{\{x\}}(f_x \otimes \chi_K)$ and $\mathcal{L}^{\{x\}}(f_x \otimes \mathbf{g}_y)$, we see that (4.5.18) coincides with

$$x \mapsto \frac{\langle f_x, f_x \rangle}{\Omega_{f_x}^+ \Omega_{f_x \otimes \chi_K}^+}$$

on U^{crys} . Specializing to x_f , we get the statement of the lemma (with the non-zero constant $\langle f, f \rangle / \Omega_f^+ \Omega_{f \otimes \chi_K}^+$). \square

4.6 The Iwasawa Main Conjecture in the crystalline case

In this subsection, we state the two-variable Greenberg-Iwasawa Rankin-Selberg Main Conjecture for the eigencuspform f and show that it implies the Iwasawa Main Conjecture f . In particular, we prove conjecture 1.5 up to powers of p when our ongoing assumptions on f and the assumptions of 4.41 of appendix 7 are satisfied. More precisely, we prove the following theorem.

Theorem 4.38. *Let $f \in S_k(\Gamma_0(N))$ be a normalized eigencuspform of weight $k \geq 2$ satisfying the following hypotheses.*

1. *The $G_{\mathbb{Q}_p}$ -representation $\rho_f|G_{\mathbb{Q}_p}$ is crystalline and short with irreducible residual representation.*
2. *There exists $\ell \mid N$, $\ell \neq p$ such that $\bar{\rho}_f|G_{\mathbb{Q}_\ell}$ is a ramified extension*

$$0 \longrightarrow \mu \chi_{\text{cyc}}^{1-k/2} \longrightarrow \bar{\rho}_f|G_{\mathbb{Q}_\ell} \longrightarrow \mu \chi_{\text{cyc}}^{-k/2} \longrightarrow 0$$

where $\mu : G_{\mathbb{Q}_\ell} \longrightarrow \{\pm 1\}$ is the non-trivial unramified character.

Then the Iwasawa Main Conjecture (conjecture 1.5) holds for f .

4.6.1 Statement of the Greenberg-Iwasawa Rankin-Selberg Main Conjecture

Let $\mathcal{K} \subset_f F \subset \mathcal{K}_\infty$ be a finite subextension and let v be a finite place of \mathcal{O}_F . The Greenberg local condition $H^1(G_{F_v}, A) \subset H^1(G_{F_v}, A)$ at v is defined to be

$$\text{Im} \left(H^1(G_{F_v}/I_v, V^{I_v}) \longrightarrow H^1(G_{F_v}/I_v, A^{I_v}) \right) \subset H^1(G_F, A)$$

if $v \nmid p$, to be $H_{\text{Gr}}^1(G_{F_v}, A)$ if $v|v_0$ and to be 0 if $v|\bar{v}_0$.

Definition 4.39. *The Greenberg Rankin-Selberg Selmer group of f is the Λ_K -module*

$$\text{Sel}_K^{\text{Gr}}(f) \stackrel{\text{def}}{=} \varinjlim_{\mathcal{K} \subset_f F \subset \mathcal{K}_\infty} \ker \left(H^1(G_{F, \Sigma}, A) \longrightarrow \bigoplus_{v \in \Sigma} H^1(G_{F_v}, A)/H_{\text{Gr}}^1(G_{F_v}, A) \right).$$

We write

$$X_{\mathcal{K}}^{\text{Gr}}(f) \stackrel{\text{def}}{=} \text{Hom}\left(\text{Sel}_{\mathcal{K}}^{\text{Gr}}(f), \mathbb{Q}_p/\mathbb{Z}_p\right)$$

for the Pontryagin dual of $\text{Sel}_{\mathcal{K}}^{\text{Gr}}(f)$.

We recall the statement of the two-variable Greenberg-Iwasawa Rankin-Selberg Main Conjecture.

Conjecture 4.40. (Greenberg Main Conjecture) *The $\Lambda_{\mathcal{K}}$ -module $X_{\mathcal{K}}^{\text{Gr}}(f)$ is torsion and there is an equality of ideals of $\Lambda_{\mathcal{K}}$*

$$\text{char}_{\Lambda_{\mathcal{K}}} \left(X_{\mathcal{K}}^{\text{Gr}}(f) \right) = (\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)).$$

In appendix 7, we prove the following results towards conjecture 4.40.

Theorem 4.41. *Let $f \in S_k(\Gamma_0(N))$ be an eigencuspform of even weight k satisfying the following assumptions.*

1. $\bar{\rho}_f|G_{\mathcal{K}}$ is absolutely irreducible.
2. $\rho_f|G_{\mathbb{Q}_p}$ is a crystalline representation. Moreover either $\bar{\rho}_f|G_{\mathbb{Q}_p}$ is absolutely irreducible, or f is ordinary at p .
3. There exists $q||N$ (in particular $q \nmid p$) which is not split in \mathcal{K} .
4. If $\ell|N$ is not split in \mathcal{K} , then $\ell||N$. Moreover if 2 is non-split in \mathcal{K} , then $2||N$.

Then the following inclusion of ideals of $\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]$ holds

$$\text{char}_{\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]}(X_{\mathcal{K}}^{\text{Gr}}(f) \otimes_{\mathcal{O}} \mathcal{O}^{\text{ur}}) \subseteq (\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f))$$

up to height-one primes which are pullbacks of primes in $\mathcal{O}[[\Gamma^+]]$. Assume in addition that the following assumption holds.

5. If $\ell|N$ is not split in \mathcal{K} , then ℓ is ramified in \mathcal{K} and $\pi(f)_{\ell}$ is a special Steinberg representation twisted by χ_{ur} for χ_{ur} the unramified character sending ℓ to $(-1)\ell^{\frac{k}{2}-1}$.

Then

$$\text{char}_{\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]}(X_{\mathcal{K}}^{\text{Gr}}(f) \otimes_{\mathcal{O}} \mathcal{O}^{\text{ur}}) \subseteq (\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f))$$

holds.

4.6.2 Proof of the Iwasawa Main Conjecture up to powers of p

In this subsection, we prove the following proposition.

Proposition 4.42. *The following equalities hold.*

$$\text{char}_{\Lambda_{\infty}} \tilde{H}_{\alpha,\alpha}^2(G_{\mathbb{Q},\Sigma}, T \otimes \Lambda_{\infty}) = (\mathcal{L}_{\alpha}(f)) \Lambda_{\infty} \quad (4.6.1)$$

$$\text{char}_{\Lambda_{\text{Iw}}[1/p]} H_{\text{et}}^2(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}) = \text{char}_{\Lambda_{\text{Iw}}[1/p]} H_{\text{et}}^1(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}})/\Lambda_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}}. \quad (4.6.2)$$

Extending slightly the notations of definition 4.7, we let $\Lambda_{\mathcal{K},n}$ be the p -adic completion of $\Lambda_{\mathcal{K}}[\mathfrak{m}^n/p]$ and $\Lambda_{\mathcal{K},\infty}$ be the inverse limit on n of the $\Lambda_{\mathcal{K},n}$. We let $\Lambda_{\mathcal{A}}$ (resp. $\Lambda_{\mathcal{A},n}$) be the analytic ring $\Lambda_{\infty} \hat{\otimes} \mathcal{A}$ (resp. $\Lambda_n \hat{\otimes} \mathcal{A}$) and similarly for $\Lambda_{\mathcal{K},\mathcal{A}}$ and $\Lambda_{\mathcal{K},\mathcal{A},n}$.

By [93, 5.1.6] and since $\Lambda_{\mathcal{K},\mathcal{A},\infty}$ is a flat $\Lambda_{\mathcal{K}}$ -module, there is an isomorphism of co-admissible $\Lambda_{\mathcal{K},\mathcal{A},\infty}$ -modules

$$\tilde{H}_{\text{str},\text{rel}}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},\infty}) \simeq X_{\mathcal{K}}^{\text{Gr}}(f) \hat{\otimes}_{\Lambda_{\mathcal{K}}} \Lambda_{\mathcal{K},\mathcal{A},\infty}$$

and hence an equality of characteristic ideals in the sense of 4.10 if they are torsion co-admissible $\Lambda_{\mathcal{K},\mathcal{A},\infty}$ -modules.

Proposition 4.43. *For $* \in \{v_0, \bar{v}_0\}$, let $S_*(\mathcal{F}^{\pm}(D_{\mathcal{A}}))$ be the set of height-one primes $S(\mathcal{F}^{\pm}(D_{\mathcal{A}}))$, in which we identify \mathcal{K}_* with \mathbb{Q}_p . Let $\mathcal{P} \in \text{Spec } \Lambda_{\mathcal{K},\mathcal{A},\infty}$ be a height-one prime which is neither in $S_{v_0}(\mathcal{F}^+(D_{\mathcal{A}}))$ nor in $S_{\bar{v}_0}(\mathcal{F}^-(D_{\mathcal{A}}))$. Let $n \in \mathbb{N}$ be an integer and let \mathcal{P}_n be the intersection of \mathcal{P} with $\Lambda_{\mathcal{K},\mathcal{A},n}$. Assume the hypotheses of theorem 4.41. Then*

$$\text{ord}_{\mathcal{P}_n} \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}} \left(\tilde{H}_{\alpha,\alpha}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n}) \right) \geq \text{ord}_{\mathcal{P}_n} (\mathcal{L}_{\alpha,\alpha}(f)). \quad (4.6.3)$$

Proof. According to (4.5.8) and the multiplicativity of characteristic ideals in short exact sequences, there is an equality

$$\begin{aligned} \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\alpha,\alpha}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right) &= \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\alpha,\text{rel}}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right) \quad (4.6.4) \\ &\times \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\frac{H^1(G_{\mathcal{K}_{\bar{v}_0}}, \mathcal{F}_{\mathcal{A}}^-)}{\text{Im}(\tilde{H}_{\alpha,\text{rel}}^1(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n}))}\right). \end{aligned}$$

Similarly, there is an equality

$$\begin{aligned} \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\alpha,\text{rel}}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right) &= \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\text{str},\text{rel}}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right) \quad (4.6.5) \\ &\times \left(\text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\frac{H^1(G_{\mathcal{K}_{v_0}}, \mathcal{F}_{\mathcal{A}}^+)}{\text{Im}(\tilde{H}_{\alpha,\text{rel}}^1(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n}))}\right)\right)^{-1} \end{aligned}$$

according to (4.5.7). Under the hypothesis of the proposition, theorem 4.41 of appendix 7 applies so

$$(\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)) \mid \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\text{str},\text{rel}}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right).$$

Let $\mathcal{P}_n \in \text{Spec } \Lambda_{\mathcal{K},\mathcal{A},n}$ be as in the statement of the proposition. Assume in addition that $Y \notin \mathcal{P}_n$. Then

$$\text{ord}_{\mathcal{P}_n}\left(\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)\right) = \text{ord}_{\mathcal{P}_n} \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\frac{H^1(G_{\mathcal{K}_{v_0}}, \mathcal{F}_{\mathcal{A}}^+)}{\text{Im}(\text{BF}_{\alpha})}\right)$$

by equation (4.5.16) in proposition 4.35 and lemma 4.36. Then (4.6.5) implies that the \mathcal{P}_n -adic valuation of

$$\left(\text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\frac{H^1(G_{\mathcal{K}_{v_0}}, \mathcal{F}_{\mathcal{A}}^+)}{\text{Im}(\text{BF}_{\alpha})}\right)\right) \left(\text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\frac{H^1(G_{\mathcal{K}_{v_0}}, \mathcal{F}_{\mathcal{A}}^+)}{\text{Im}(\tilde{H}_{\alpha,\text{rel}}^1(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n}))}\right)\right)^{-1} \quad (4.6.6)$$

is less than the \mathcal{P}_n -adic valuation of

$$\text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\alpha,\text{rel}}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right).$$

Then, (4.6.4) further implies that the \mathcal{P}_n -adic valuation of the product of (4.6.6) with

$$\text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\frac{H^1(G_{\mathcal{K}_{\bar{v}_0}}, \mathcal{F}_{\mathcal{A}}^-)}{\text{Im}(\tilde{H}_{\alpha,\text{rel}}^1(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n}))}\right)$$

is less than the \mathcal{P}_n -adic valuation of

$$\text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\alpha,\alpha}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right).$$

Using again the multiplicativity of characteristic ideals in short exact sequences, we find that

$$\text{ord}_{\mathcal{P}_n} \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\frac{H^1(G_{\mathcal{K}_{\bar{v}_0}}, \mathcal{F}_{\mathcal{A}}^-)}{\text{Im}(\text{BF}_{\alpha})}\right) \leq \text{ord}_{\mathcal{P}_n} \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\alpha,\alpha}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right).$$

Equation (4.5.14) of proposition 4.35 and the fact that \mathfrak{C}_f is a unit in $\Lambda_{\mathcal{K},\mathcal{A},n}$ then shows that

$$\text{ord}_{\mathcal{P}_n} \text{char}_{\Lambda_{\mathcal{K},\mathcal{A},n}}\left(\tilde{H}_{\alpha,\alpha}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_{\mathcal{K},\mathcal{A},n})\right) \geq \text{ord}_{\mathcal{P}_n} (\mathcal{L}_{\alpha,\alpha}(f))$$

as desired. If now $\mathcal{P} = (Y)$, then $\mathcal{L}_{\alpha,\alpha}(f) \bmod \mathcal{P}$ is $\mathcal{L}_{\alpha,\alpha}^{\text{cyc}}(f)$, which is not identically zero by lemma 4.37. As the right-hand side is equal to zero, (4.6.3) also holds in that case. \square

Recall that $\mathcal{L}_{\alpha,\alpha}^{\text{cyc}}(f)$ is the specialization of $\mathcal{L}_{\alpha,\alpha}(f)$ to the cyclotomic line.

Corollary 4.44. *Let $n \in \mathbb{N}$ be an integer. Under the assumptions of proposition 4.43,*

$$\text{ord}_{\mathcal{P}} \text{char}_{\Lambda_n}\left(\tilde{H}_{\alpha,\alpha}^2(G_{\mathcal{K},\Sigma}, T \otimes \Lambda_n)\right) \geq \text{ord}_{\mathcal{P}} (\mathcal{L}_{\alpha,\alpha}^{\text{cyc}}(f)) \Lambda_n \quad (4.6.7)$$

for all height-one prime $\mathcal{P} \in \text{Spec } \Lambda_n$.

Proof. By proposition 4.30 (and its proof), there is an isomorphism

$$R\tilde{\Gamma}_{\alpha,\alpha}(G_{K,\Sigma}, T \otimes \Lambda_{K,A,n}) \overset{L}{\otimes}_{\Lambda_{K,A,\infty}} \Lambda_n \simeq R\tilde{\Gamma}_{\alpha,\alpha}(G_{K,\Sigma}, T \otimes \Lambda_n)$$

and hence an isomorphism

$$\tilde{H}_{\alpha,\alpha}^2(G_{K,\Sigma}, T \otimes \Lambda_{K,A,\infty}) \otimes_{\Lambda_{K,A,\infty}} \Lambda_n \simeq \tilde{H}_{\alpha,\alpha}^2(G_{K,\Sigma}, T \otimes \Lambda_n) \quad (4.6.8)$$

as $\tilde{H}_{\alpha,\alpha}^3(G_{K,\Sigma}, T \otimes \Lambda_{K,A,\infty})$ vanishes. Let $\mathcal{P} \notin S_{v_0}(\mathcal{F}^+(D_A)) \cup S_{\bar{v}_0}(\mathcal{F}^-(D_A))$ be a height-one prime of $\Lambda_{K,A,\infty}$ and denote by $\mathcal{P}_{cyc,n}$ its image through the natural map to Λ_n . Let $\mathcal{L}_{\alpha,\alpha}^{cyc,n}(f)$ the image of $\mathcal{L}_{\alpha,\alpha}^{cyc}(f)$ in Λ_n . Then

$$\text{ord}_{\mathcal{P}_{cyc,n}}(\mathcal{L}_{\alpha,\alpha}^{cyc}(f)) \leq \text{ord}_{\mathcal{P}_{cyc,n}}\left(\left(\text{char}_{\Lambda_{K,A,\infty}} \tilde{H}_{\alpha,\alpha}^2(G_{K,\Sigma}, T \otimes \Lambda_{K,A,\infty})\right) \otimes_{\Lambda_{K,A,\infty}} \Lambda_n\right)$$

by (4.6.3) and so

$$\text{ord}_{\mathcal{P}_{cyc,n}}(\mathcal{L}_{\alpha,\alpha}^{cyc}(f)) \leq \text{ord}_{\mathcal{P}_{cyc,n}} \text{char}_{\Lambda_n}\left(\tilde{H}_{\alpha,\alpha}^2(G_{K,\Sigma}, T \otimes \Lambda_{K,A,\infty}) \otimes_{\Lambda_{K,A,\infty}} \Lambda_n\right)$$

as characteristic ideals can only shrink by specialization. Combining this with (4.6.8), we obtain

$$\text{ord}_{\mathcal{P}_{cyc,n}}(\mathcal{L}_{\alpha,\alpha}^{cyc}(f)) \leq \text{ord}_{\mathcal{P}_{cyc,n}} \text{char}_{\Lambda_n}\left(\tilde{H}_{\alpha,\alpha}^2(G_{K,\Sigma}, T \otimes \Lambda_n)\right)$$

Suppose that \mathcal{P} now belongs to $S_{v_0}(\mathcal{F}^+(D_A)) \cup S_{\bar{v}_0}(\mathcal{F}^-(D_A))$. Then its intersection $\mathcal{P}_{cyc,n}$ with Λ_∞ is the unit ideal and so the inequality (4.6.7) still holds. \square

Once corollary 4.44 is known, theorem 4.38 follows from [102, Theorem 5.4]. As this reference might not be widely available, we sum up the argument below.

Proof of theorem 4.38. Letting n go to infinity in corollary 4.44 yields the divisibility

$$(\mathcal{L}_\alpha(f)) \Lambda_\infty \mid \text{char}_{\Lambda_\infty} \tilde{H}_{\alpha,\alpha}^2(G_{\mathbb{Q},\Sigma}, T \otimes \Lambda_\infty).$$

According to theorem 1.4, there is on the other hand a divisibility

$$\text{char}_{\Lambda_{\text{Iw}}[1/p]} H_{\text{et}}^2(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}[1/p]) \mid \text{char}_{\Lambda_{\text{Iw}}[1/p]} H_{\text{et}}^1(\mathbb{Z}[1/p], T \otimes \Lambda_{\text{Iw}}[1/p]) / \Lambda_{\text{Iw}}[1/p] \cdot \mathbf{z}(f)_{\text{Iw}}.$$

After extension of coefficients to Λ_∞ , localization at p and projection to \mathcal{F}^- , the regulator map sends Kato's zeta element to \mathcal{L}_α , thus just as before [68, Theorem 16.6] yields

$$\text{char}_{\Lambda_\infty} \tilde{H}_{\alpha,\alpha}^2(G_{\mathbb{Q},\Sigma}, T \otimes \Lambda_\infty) \mid (\mathcal{L}_\alpha(f)) \Lambda_\infty.$$

Equations (4.6.1) and (4.6.2) follow. \square

4.6.3 Powers of p

In this subsection, we provide the ingredient which remains missing after theorem 4.38 to establish conjecture 1.5 for the modular motive attached to f , namely we show that the power of p dividing $\text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^2(\mathbb{Z}[1/p], T(f)_{\text{Iw}})$ is at least as large as predicted by the Iwasawa Main Conjecture. The main idea is as follows: as we already know that $\text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^2(\mathbb{Z}[1/p], T(f)_{\text{Iw}})$ divides $\text{char}_{\Lambda_{\text{Iw}}} H_{\text{et}}^1(\mathbb{Z}[1/p], T(f)_{\text{Iw}}) / \Lambda_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}}$, it is enough to show (as we have seen at the end of the control theorem section) that that there is an arithmetic point of $\text{Spec } \Lambda_{\text{Iw}}$ where the orders of a certain Selmer group is as predicted by Kato's Iwasawa main conjecture.

Definition 4.45. A cyclotomic arithmetic point $\tilde{\phi} \in \text{Spec } \mathcal{O}_{\text{Iw}}$ of conductor p^n is a $\bar{\mathbb{Q}}_p$ -point mapping $(1+X)$ to $\zeta(1+p)^{1+j}$ for ζ a root of unity of order p^n and $0 \leq j \leq k-2$ an integer. If $\tilde{\phi}$ has values in a discrete valuation ring $\mathcal{O}_{\tilde{\phi}}$, we write $T_{\tilde{\phi}}$ for $T_{f,\text{Iw}} \otimes_{\mathcal{O}_{\text{Iw}},\tilde{\phi}} \mathcal{O}_{\tilde{\phi}}$.

If $\tilde{\phi}$ is an arithmetic point of conductor p^n and if $\chi_{\tilde{\phi}} : (\mathbb{Z}/p^n\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ is a character, then the specialization to $\tilde{\phi}$ of the p -adic L -function $\mathcal{L}_\alpha(f)$ interpolates the L -value $L(f, \chi_{\tilde{\phi}}^{-1}, 1+j)$ up to normalization factors. We also write $\tilde{\phi}^{-1}$ for the arithmetic point which is symmetric to $\tilde{\phi}$ with respect to the central point of the functional equation. At $\tilde{\phi}^{-1}$, the L -values interpolated are at the critical integers $k-1-j$.

We recall that $H^1(G_{\mathcal{K}_{\bar{v}_0}}, T_{\tilde{\phi}})$ is a free $\mathcal{O}_{\tilde{\phi}}[G_n]$ -module of rank 2 and write (v_1, v_2) for an $\mathcal{O}_{\tilde{\phi}}[G_n]$ -basis of $H^1(G_{\mathcal{K}_{\bar{v}_0}}, T_{\tilde{\phi}})$ such that v_1 is a generator of $H_f^1(G_{\mathcal{K}_{\bar{v}_0}}, T_{\tilde{\phi}})$. We sometimes write them as $v_{1,\bar{v}_0}, v_{2,\bar{v}_0}$. Write $\Gamma_{\mathcal{K}} = \Gamma_{\text{cyc}} \times \Gamma_{\bar{v}_0}$. Let \mathcal{Y}, \mathcal{X} and pr_{cyc} be respectively $\text{Spec } \Lambda_{\mathcal{K}}$, $\text{Spec } \Lambda_{\text{Iw}}$ and the natural projection

$$\text{pr}_{\text{cyc}} : \mathcal{Y} \longrightarrow \mathcal{X}.$$

Define the fiber $\mathcal{Y}_{\tilde{\phi}}$ at $\tilde{\phi}$ to be $\mathcal{Y} \times_{\mathcal{X}, \tilde{\phi}} \text{Spec } \mathcal{O}_{\tilde{\phi}}$ and denote by $\text{BF}_{\alpha, \tilde{\phi}}^{\mathcal{Y}_{\tilde{\phi}}}$ the one variable family $\text{BF}_{\alpha, \tilde{\phi}}$ of Beilinson-Flach element in Definition 4.23.

Definition 4.46. An arithmetic point $\tilde{\phi} \in \mathcal{X}$ is generic if $L(f, \chi_{\tilde{\phi}}, 1+j) \neq 0$ and if the restriction of $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$ to $\mathcal{Y}_{\tilde{\phi}}$ is not identically 0.

It is clear that all but finite many arithmetic points are generic. We fix an arithmetic point $\tilde{\phi}$ of conductor p^r and equal to $\chi_{\text{cyc}}^{1+j-\frac{k}{2}}$ times a finite order $\chi_{\tilde{\phi}}$ and such that $\tilde{\phi}$ and $\tilde{\phi}^{-1}$ are both generic. We freely consider $(v_{1,\bar{v}_0}, v_{2,\bar{v}_0})$ as a basis of the free rank-two $\mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]]$ -module $H^1(G_{\mathbb{Q}_p}, T_{\tilde{\phi}} \otimes_{\mathcal{O}_{\tilde{\phi}}} \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}})$.

Remark: The family of p -adic Galois representation deforming T alongside $\mathcal{Y}_{\tilde{\phi}}$ and $\mathcal{Y}_{\tilde{\phi}^{-1}}$ is a family of twists of T_f which are unramified as $G_{\mathbb{Q}_p}$ -representations (where we identify $G_{\mathbb{Q}_p}$ with $G_{\mathcal{K}_{v_0}}$).

Define an element $\mathcal{L}_{\tilde{\phi}}^1 \in \mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]]$ such that along $\mathcal{Y}_{\tilde{\phi}}$,

$$\text{BF}_{\alpha}^{\mathcal{Y}_{\tilde{\phi}}} \equiv (k-2-j)! \left(\mathcal{L}_{\tilde{\phi}}^1 \right) \left(G(\chi_{\tilde{\phi}}^{-1}) \left(\frac{\beta_f}{p^{1+j}} \right)^r \right) v_{2,\bar{v}_0} \bmod v_{1,\bar{v}_0}. \quad (4.6.9)$$

Then $\mathcal{L}_{\tilde{\phi}}^1(0) \neq 0$ and

$$\alpha_f^{2r} \cdot \frac{\tilde{\phi}(\mathcal{L}_{\alpha}^{\text{cyc}}(f)\mathcal{L}_{\alpha}^{\text{cyc}}(f \otimes \chi_{\mathcal{K}}))}{G(\chi_{\tilde{\phi}})^2 p^{2rj}} = \frac{(j!)^2 L_{\mathcal{K}}(f, \chi_{\tilde{\phi}}^{-1}, 1+j)}{(2\pi i)^{2+2j} \Omega_f^+ \Omega_f^-}.$$

From our assumption on $\tilde{\phi}$, we deduce

$$\alpha_f^{2r} \cdot \frac{\tilde{\phi}(\mathcal{L}_{\alpha}^{\text{cyc}}(f)\mathcal{L}_{\alpha}^{\text{cyc}}(f \otimes \chi_{\mathcal{K}}))}{G(\chi_{\tilde{\phi}})^2 p^{2rj}} = \frac{(j!)^2 L_{\mathcal{K}}(f, \chi_{\tilde{\phi}}^{-1}, 1+j)}{(2\pi i)^{2+2j} \Omega_f^+ \Omega_f^-}.$$

Let

$$\mathcal{B} \stackrel{\text{def}}{=} \frac{h_{\mathcal{K}} \mathcal{L}_{\mathcal{K}}^{\text{Kat}} c \cdot v^+}{v^-}$$

be the displayed element (4.5.17) in lemma 4.36. Let $\phi \in \text{Spec } \mathbb{Z}_p[[Y]]$ be a generic point. Then

$$\text{Reg}_{v_0, \mathcal{F}^+}(\text{loc}_{v_0} \text{BF}_{\alpha, \phi}) = \frac{1}{\mathfrak{C}_f} \phi(\mathcal{B}) (k-2-j)! \phi \left(\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f) \right) G(\chi_{\tilde{\phi}}^{-1}) \cdot \left(\frac{\phi(\alpha_g) \beta_f}{p^{1+j}} \right)^r \omega_f^{\vee} \quad (4.6.10)$$

according to [74, Theorem 7.1.4]. Note that β_f, j, r only depend on $\tilde{\phi}$ and that α_g is an element in $\mathbb{Z}_p[[\Gamma_{v_0}]]$ such that $\alpha_g(0) = 1$. If \mathfrak{C}_f is the number defined after (4.5.12), we have

$$\exp_{\bar{v}_0}^* \text{BF}_{\alpha, \tilde{\phi}} = \frac{1}{j! \mathfrak{C}_f} \tilde{\phi}(\mathcal{L}_{\alpha}^{\text{cyc}}(f)\mathcal{L}_{\alpha}^{\text{cyc}}(f \otimes \chi_{\mathcal{K}})) \cdot G(\chi_{\tilde{\phi}}^{-1}) \cdot \left(\frac{\alpha_f \beta_g}{p^{1+j}} \right)^r \cdot \omega_f.$$

Hence

$$\begin{aligned} \exp_{\bar{v}_0}^* \text{BF}_{\alpha, \tilde{\phi}} &= \frac{L_{\mathcal{K}}(f, \chi_{\tilde{\phi}}^{-1}, 1+j) \cdot G(\chi_{\tilde{\phi}}^{-1})}{j! \mathfrak{C}_f (2\pi i)^{2+2j} \Omega_f^+ \Omega_f^-} G^2(\chi_{\tilde{\phi}}) \left(\frac{\alpha_f}{p^{1+j}} \right)^r p^{2rj} \alpha_f^{-2r} \cdot \omega_f \\ &= \frac{L_{\mathcal{K}}(f, \chi_{\tilde{\phi}}^{-1}, 1+j)}{j! \mathfrak{C}_f (2\pi i)^{2+2j} \Omega_f^+ \Omega_f^-} G(\chi_{\tilde{\phi}}) \alpha_f^{-r} p^{rj} \cdot \omega_f. \end{aligned}$$

Note here that the factor $\mathcal{E}(f)\mathcal{E}(f^*)$ in the interpolation formula for Rankin-Selberg p -adic L -function is cancelled by the factor $(1 - \frac{\beta}{\alpha})(1 - \frac{\beta}{p\alpha})$ in [74, Corollary 6.4.3]. Thus

$$\exp_{\bar{v}_0}^* \tilde{\phi}(\text{BF}_{\alpha}) = \frac{L_{\mathcal{K}}(f, \chi_{\tilde{\phi}}^{-1}, 1+j)}{j! \mathfrak{C}_f \Omega_f^+ \Omega_f^- (2\pi i)^{2+2j}} G(\chi_{\tilde{\phi}}) \left(\frac{\beta_f}{p^{1+j}} \right)^r \cdot p^{r(2j+2-k)} \cdot \omega_f. \quad (4.6.11)$$

Now we repeat these constructions for the arithmetic point $\tilde{\phi}^{-1}$. We define a basis $(v_{1, \bar{v}_0}, v_{2, \bar{v}_0})$ at $\tilde{\phi}^{-1}$ as at $\tilde{\phi}$. To emphasize the dependence on the point we use $\tilde{\phi}(v_{i, \bar{v}_0})$ or $\tilde{\phi}^{-1}(v_{i, \bar{v}_0})$ to denote them. Similarly as for $\tilde{\phi}$, we may compute the image through the regulator map of the specialization of the Beilinson-Flach class at $\tilde{\phi}^{-1}$ and their images though arithmetic points ϕ and ϕ^{-1} . As above, we obtain

$$\text{Reg}_{v_0, \mathcal{F}^+}(\text{loc}_{v_0} \text{BF}_{\alpha, \phi^{-1}}) = \frac{1}{\mathfrak{C}_f} \phi^{-1}(\mathcal{B}) j! \phi^{-1}(\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)) G(\chi_{\phi}) \left(\frac{\phi^{-1}(\alpha_g) \beta_f}{p^{1+j}} \right)^r \omega_f^{\vee} \quad (4.6.12)$$

and

$$\exp_{\bar{v}_0}^* \tilde{\phi}^{-1}(\text{BF}_{\alpha}) = \frac{L_{\mathcal{K}}(f, \chi_{\tilde{\phi}^{-1}}^{-1}, k-1-j)}{(k-2-j)! \mathfrak{C}_f (2\pi i)^{2k-2-2j} \Omega_f^+ \Omega_f^-} \left(G(\chi_{\tilde{\phi}}) \left(\frac{\beta_f}{p^{1+j}} \right)^r \right) (p^{2r(k-2-j)+r}/p^{(k-1)r}) \omega_f. \quad (4.6.13)$$

Putting everything together, we obtain the equation

$$\mathcal{L}_{\phi}^1(0) = \frac{j!}{(k-2-j)!} \frac{G(\chi_{\tilde{\phi}})}{G(\chi_{\tilde{\phi}}^{-1})} \frac{L_{\mathcal{K}}(f, 1+j, \chi_{\tilde{\phi}}^{-1})(p^{2jr+r}/p^{(k-1)r})}{\mathfrak{C}_f (2\pi i)^{2+2j} \Omega_f^+ \Omega_f^- \exp^* \tilde{\phi}(v_{2, \bar{v}_0})}. \quad (4.6.14)$$

In definition 4.15, we defined

$$H_f^1(G_{\mathcal{K}_{v_0}}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]])$$

such that for any height one prime $\mathcal{P} \in \text{Spec } \mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]]$, the $H_f^1(G_{\mathcal{K}_{v_0}}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]])_{\mathcal{P}}$ is an $\mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]]_{\mathcal{P}}$ -direct summand of

$$H^1(G_{\mathcal{K}_{v_0}}, T_{\tilde{\phi}} \otimes \mathcal{O}[[\Gamma_{\bar{v}_0}]])_{P'} \simeq \mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]]_{\mathcal{P}}^2.$$

Definition 4.47. *The unramified local Selmer condition at v_0 is the subspace*

$$H_f^1(G_{\mathcal{K}_{v_0}}, T_{\tilde{\phi}} \otimes_{\mathcal{O}_{\tilde{\phi}}} \mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]])$$

along the one-variable family over $\mathcal{O}_{\tilde{\phi}}[[\Gamma_{\bar{v}_0}]]$.

Let $\mathcal{K} \subset_f F \subset \mathcal{K}_{\infty}$ be a finite subextension and let v be a finite place of \mathcal{O}_F . The Greenberg local condition $H^1(G_{F_v}, A) \subset H^1(G_{F_v}, A)$ at v is defined to be

$$\text{Im} \left(H^1(G_{F_v}/I_v, V^{I_v}) \longrightarrow H^1(G_{F_v}/I_v, A^{I_v}) \right) \subset H^1(G_F, A)$$

if $v \nmid p$, to be $H_{\text{Gr}}^1(G_{F_v}, A)$ if $v|v_0$ and to be 0 if $v|\bar{v}_0$. Recall that $\Gamma_{\bar{v}_0} = \text{Gal}(\mathcal{K}^{\bar{v}_0}/\mathcal{K})$ is the Galois group of the maximal subextension of \mathcal{K}_{∞} unramified outside \bar{v}_0 . Using the identification of $\mathcal{O}[[Y]]$ with $\mathcal{O}[[\Gamma_{\bar{v}_0}]]$ through the map $Y \mapsto \gamma_{\bar{v}_0} - 1$, we identify the $\mathcal{O}_{\tilde{\phi}}[[Y]]$ -modules

$$H^1(G_{\mathcal{K}, \Sigma}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}), \quad \lim_{\substack{\longleftarrow \\ \mathcal{K} \subset F \subset \mathcal{K}^{\bar{v}_0}}} H^1(G_{F, \Sigma}, T_{\tilde{\phi}})$$

The following proposition is the payoff of our working alongside the unramified family $\mathcal{Y}_{\tilde{\phi}}$ passing through the classical point $\tilde{\phi}$ of the cyclotomic Iwasawa algebra.

Proposition 4.48.

$$\prod_v c_{\mathcal{K}, \tilde{\phi}, v}(f) \prod_v c_{\mathcal{K}, \tilde{\phi}^{-1}, v}(f) \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}}(X_{\text{BK}, v_1, \tilde{\phi}}) \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}}(X_{\text{BK}, v_1, \tilde{\phi}^{-1}})$$

is contained in

$$\left(\frac{L_{\mathcal{K}}(f, 1+j, \chi_{\tilde{\phi}}^{-1})}{(2\pi i)^{2+2j} \Omega_f^+ \Omega_f^-} \right) \left(\frac{L_{\mathcal{K}}(f, k-1-j, \chi_{\tilde{\phi}^{-1}}^{-1})}{(2\pi i)^{2k-2-2j} \Omega_f^+ \Omega_f^-} \right).$$

Proof. Recall that $H^1(G_{\mathcal{K}, \bar{v}_0}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}})$ is a free rank two module over $\mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}$, and that we lifted v_1 to a generator of a rank one direct summand of this module, which we still denote as v_1 . Now we look at the following exact sequences of $\mathcal{O}_{\tilde{\phi}}[[U]]$ -modules

$$0 \rightarrow H_{\text{ur}, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}) \rightarrow H_{\text{ur}}^1(G_{\mathcal{K}, \bar{v}_0}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}) \rightarrow X_{\text{rel}, \text{str}} \rightarrow X_{\text{ur}, \text{str}} \rightarrow 0 \quad (4.6.15)$$

$$0 \rightarrow H_{\text{ur}, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}) \rightarrow \frac{H^1(\mathcal{K}_{\bar{v}_0}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}})}{\mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}} v_1} \rightarrow X_{\text{ur}, v_1} \rightarrow X_{\text{ur}, \text{str}} \rightarrow 0. \quad (4.6.16)$$

Recall we wrote $\mathcal{O}[[\Gamma_{\bar{v}_0}]] = \mathcal{O}[[Y]]$ for the variable $Y = \gamma_{\bar{v}_0} - 1$ and definition 4.15. By Corollaries 4.13 and 4.14 there is an isomorphism

$$H_{\text{ur}}^1(G_{\mathcal{K}, \bar{v}_0}, T \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}}) \simeq \mathcal{O}_{\tilde{\phi}}[[Y]]$$

which interpolates $1/b_{\tilde{\phi}, p}$ times the regulator map, divided by the specialization of $\rho(d)$ there. Theorem 7.32 establishes an inclusion in the Greenberg-Iwasawa main conjecture. Combined with the definition of Tamagawa numbers in definition 4.17, equation (4.6.9) describing the interpolation properties of the Beilinson-Flach classes, equations (4.6.10), (4.6.11), (4.6.12), (4.6.13) computing the image of $\text{BF}_{\alpha, \tilde{\phi}}$ through the regulator and dual exponential maps, the integrality of \mathcal{B} in lemma 4.36 and the short exact sequence (4.6.15), it therefore implies that

$$\frac{1}{b_{\tilde{\phi}, p}} \cdot \frac{1}{b_{\tilde{\phi}^{-1}, p}} \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}[[Y]]}(X_{\text{ur}, \text{str}, \tilde{\phi}}) \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}[[Y]]}(X_{\text{ur}, \text{str}, \tilde{\phi}^{-1}})$$

is contained in

$$\operatorname{char}_{\mathcal{O}_{\tilde{\phi}}[[Y]]} \left(\frac{H_{\text{ur}, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}})}{\frac{\mathfrak{e}_f}{(k-2-j)!(G(x_{\tilde{\phi}}^{-1})(\frac{\beta_f}{p^{1+j}})^r)} \mathcal{O}_{\tilde{\phi}} \cdot [[Y]] \text{BF}_{\alpha}^{\mathcal{Y}_{\tilde{\phi}}}} \right) \cdot \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}[[Y]]} \left(\frac{H_{\text{ur}, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T_{\tilde{\phi}^{-1}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}^{-1}}})}{\frac{\mathfrak{e}_f}{j!(G(x_{\tilde{\phi}})(\frac{\beta_f}{p^{k-1-j}})^r)} \mathcal{O}_{\tilde{\phi}} \cdot [[Y]] \text{BF}_{\alpha}^{\mathcal{Y}_{\tilde{\phi}^{-1}}}} \right)$$

up to powers of Y (as fractional ideals for the right hand side). Here, the containments

$$\frac{1}{b_{\tilde{\phi}, p}} \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}[[Y]]}(X_{\text{ur}, \text{str}, \tilde{\phi}}) \subseteq \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}[[Y]]} \left(\frac{H_{\text{ur}, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T_{\tilde{\phi}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}}})}{\frac{\mathfrak{e}_f}{(k-2-j)!(G(x_{\tilde{\phi}}^{-1})(\frac{\beta_f}{p^{1+j}})^r)} \mathcal{O}_{\tilde{\phi}} \cdot [[Y]] \text{BF}_{\alpha}^{\mathcal{Y}_{\tilde{\phi}}}} \right)$$

and

$$\frac{1}{b_{\tilde{\phi}^{-1}, p}} \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}[[Y]]}(X_{\text{ur}, \text{str}, \tilde{\phi}^{-1}}) \subseteq \operatorname{char}_{\mathcal{O}_{\tilde{\phi}}[[Y]]} \left(\frac{H_{\text{ur}, \text{rel}}^1(G_{\mathcal{K}, \Sigma}, T_{\tilde{\phi}^{-1}} \otimes \mathcal{O}_{\mathcal{Y}_{\tilde{\phi}^{-1}}})}{\frac{\mathfrak{e}_f}{(k-2-j)!(G(x_{\tilde{\phi}})(\frac{\beta_f}{p^{k-1-j}})^r)} \mathcal{O}_{\tilde{\phi}} \cdot [[Y]] \text{BF}_{\alpha}^{\mathcal{Y}_{\tilde{\phi}^{-1}}}} \right)$$

are both deduced from the containment

$$\operatorname{char}(X_{\mathcal{K}}^{\text{Gr}}(f)) \subseteq (\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f))$$

of theorem 7.32 by specializing to $\mathcal{Y}_{\tilde{\phi}}$ and $\mathcal{Y}_{\tilde{\phi}^{-1}}$ and using the fact that characteristic ideals can only shrink by specialization (in fact, any prime v of \mathcal{K} not dividing p is finitely decomposed in the \mathbb{Z}_p -extension $\mathcal{K}^{\bar{v}_0}/\mathcal{K}$ and is completely split in $\mathcal{K}_{\infty}/\mathcal{K}^{\bar{v}_0}$ so the characteristic

ideal of $X_{\mathcal{K}}^{\text{Gr}}(f)$ specialized to $\mathcal{O}_{\tilde{\phi}}[[Y]]$ and $\mathcal{O}_{\tilde{\phi}^{-1}}[[Y]]$ is exactly the characteristic ideal of $X_{\text{ur}, \text{str}, \tilde{\phi}}$ and $X_{\text{ur}, \text{str}, \tilde{\phi}^{-1}}$ respectively).

Just as before applying the Poitou-Tate exact sequences (4.6.16), we get the lower bound for the (ur, v_1) Selmer groups over $\mathcal{O}[[\Gamma_{v_0}]]$ by $\mathcal{L}_{\tilde{\phi}}^1$ up to powers of Y . Recall also that Y is not a divisor of $\mathcal{L}_{\tilde{\phi}}^1$ and $\mathcal{L}_{\tilde{\phi}^{-1}}^1$ since $\mathcal{L}_{\tilde{\phi}}^1(0)$ and $\mathcal{L}_{\tilde{\phi}^{-1}}^1(0)$ are nonzero. Specializing Y to zero, we obtain that

$$\prod_v c_{\mathcal{K}, \tilde{\phi}, v}(f) \prod_v c_{\mathcal{K}, \tilde{\phi}^{-1}, v}(f) \text{char}_{\mathcal{O}_{\tilde{\phi}}}(X_{\text{BK}, v_1, \tilde{\phi}}) \text{char}_{\mathcal{O}_{\tilde{\phi}}}(X_{\text{BK}, v_1, \tilde{\phi}^{-1}})$$

is contained in

$$\left(\frac{L_{\mathcal{K}}(f, 1+j, \chi_{\tilde{\phi}}^{-1})}{(2\pi i)^{2+2j} \Omega_f^+ \Omega_f^-} \right) \left(\frac{L_{\mathcal{K}}(f, k-1-j, \chi_{\tilde{\phi}^{-1}}^{-1})}{(2\pi i)^{2k-2-2j} \Omega_f^+ \Omega_f^-} \right).$$

Above, the v may or may not divide p . Pay attention that in the formula (4.6.14), the $p^{2jr+r}/p^{(k-1)r}$ and $\frac{j!}{(k-2-j)!}$ at $\tilde{\phi}$ and $\tilde{\phi}^{-1}$ cancel out. Here the subscript \mathcal{K} in $c_{\mathcal{K}}$ means the local Tamagawa numbers over \mathcal{K} . It is known that $c_{\mathcal{K}, \ell, \phi}(f) = c_{\ell, \phi}(f) \cdot c_{\ell, \phi}(f \otimes \chi_{\mathcal{K}})$. The subscript BK stands for the Bloch-Kato Selmer condition at v_0 . \square

Finally, we record the following.

Lemma 4.49. *The period $\Omega_{f \otimes \chi_{\mathcal{K}}}^{\mp}$ is an \mathcal{O} -multiple of Ω_f^{\pm} .*

Proof. This is a much easier variant of [115, Lemma 9.6]. As our running assumptions are different from those of [115] (and our result correlative much weaker), we recall the proof.

The eigencusforms f and $f \otimes \chi_{\mathcal{K}}$ are new of level N and of level M for some M dividing $ND_{\mathcal{K}}^2$. Consider the map

$$H^1(\Gamma_0(N), L_{/\mathcal{O}}^{k-2})_{\mathfrak{m}_f} \longrightarrow H^1(\Gamma_0(ND_{\mathcal{K}}^2), L_{/\mathcal{O}}^{k-2})_{\mathfrak{m}_f}$$

constructed in [115, Proof of Lemma 9.6]. As \mathfrak{m}_f is non-Eisenstein, this is a map between free modules which sends the canonical differentials ω_f^{\pm} are mapped to $\omega_{f \otimes \chi_{\mathcal{K}}}^{\mp}$ multiplied by the Gauss sums attached to $\chi_{\mathcal{K}}$ and the elements γ_f^+, γ_f^- forming a basis of the left-hand side to some \mathcal{O} -multiple of $\gamma_{f \otimes \chi_{\mathcal{K}}}^-, \gamma_{f \otimes \chi_{\mathcal{K}}}^+$. Hence $\Omega_{f \otimes \chi_{\mathcal{K}}}^{\mp}$ is an \mathcal{O} -multiple of Ω_f^{\pm} divided by the Gauss sum attached to $\chi_{\mathcal{K}}$, which is a p -adic unit as p is prime to $D_{\mathcal{K}}$. \square

Let \mathcal{F}_1 be the image of $\mathbf{z}(f)_{\text{Iw}}$ in $\frac{H^1(\mathbb{Q}^S/\mathbb{Q}, \mathbf{T})}{\Lambda_{\mathbf{V}}^{\vee}}$, and \mathcal{F} be the characteristic polynomial of the \mathbf{v}^{\vee} -Selmer group $X_{\mathbf{v}^{\vee}}(f)$ as in Section 4.3. We have the following

Lemma 4.50. *Conjecture 1.5 for $M(f)$ is equivalent to the equality*

$$(\mathcal{F}_1) = (\mathcal{F})$$

as principal ideals of Λ_{Iw} .

Proof. This is proved using Poitou-Tate duality in the same way as in the proof of [77, Theorem 7.4], once it is observed that the role \mathcal{L}_p^{\pm} in loc.cit is played here by \mathcal{F}_1 . \square

We finally obtain the following theorem.

Theorem 4.51. *Let $f \in S_k(\Gamma_0(N))$ be a normalized eigencuspform of weight $k \geq 2$ satisfying the following hypotheses.*

1. The $G_{\mathbb{Q}_p}$ -representation $\rho_f|G_{\mathbb{Q}_p}$ is crystalline (equivalently $p \nmid N$) and short.
2. The local residual representation $\bar{\rho}_f|G_{\mathbb{Q}_p}$ is absolutely irreducible.
3. There exists $\ell \mid N$ such that $\dim_{\mathbb{F}} \bar{\rho}^{I_{\ell}} = 1$ and $\dim_{\mathbb{F}} \bar{\rho}^{G_{\mathbb{Q}_{\ell}}} = 0$.

Then conjecture 1.5 holds for $M(f)$.

Proof. Similarly to the proof of [68, Theorem 17.4] and [77, 1.3], theorem 1.4 and lemma 4.50 shows that \mathcal{F}_1 is divisible by \mathcal{F} . Hence, it is enough to check that $\tilde{\phi}(\mathcal{F})$ is divisible by $\tilde{\phi}(\mathcal{F}_1)$ for the arithmetic point $\tilde{\phi}$. Let P be the height one prime corresponding to $\tilde{\phi}$.

We have $\tilde{\phi}(\mathcal{F}_1) = \exp^*(\tilde{\phi}(\mathbf{z}(f)_{\text{Iw}}))/c'_{P,p}$. On the other hand Kato proved that

$$\exp^* \tilde{\phi}(\mathbf{z}(f)_{\text{Iw}}) = \frac{L(f, 1+j, \chi_{\tilde{\phi}}^{-1})}{(2\pi i)^{1+j} \Omega_f^{(-1)^j}} \omega_f.$$

The theorem then follows from proposition 4.48, equality (4.3.7) and lemma 4.49. \square

4.7 The p -irregular case

Previously, we have assumed the Satake parameters at p satisfy $\alpha \neq \beta$. Conjecturally this is always true, however only known when $k = 2$. For completeness of the result, we treat the case $\alpha = \beta$ as well⁷. Note that since k is even, this can happen only when $k \geq 4$. In this section we establish (4.7.5), which is the only missing ingredient for proving the Iwasawa main conjecture. (Loeffler-Zerbes assumed $\alpha \neq \beta$ in [74].) This replaces equation 4.5.14 in proposition 4.35 (equation 4.5.16 remains unchanged).

4.7.1 Geometry of the eigencurve

We start by briefly discussing the local geometry of the eigencurve near the p -irregular point, using freely the notions in [2] (more details are given in [8]). Let U be a small affinoid neighborhood in the weight space for the point k . We write D_k (D_U) for the coefficient sheaf of weight k (over U) on the modular curve $Y_0(N)$ used to define modular symbols (following the notation of Ash-Stevens [2]). We consider the long exact sequence

$$\cdots H^0(Y_0(N), D_k) \rightarrow H^1(Y_0(N), D_U) \rightarrow H^1(Y_0(N), D_U) \rightarrow H^1(Y_0(N), D_k) \rightarrow H^2(Y_0(N), D_U) \cdots$$

induced from

$$0 \rightarrow D_U \rightarrow D_U \rightarrow D_k \rightarrow 0,$$

where the second arrow is multiplying by a uniformizer m_k at k . Ash-Stevens proved that one can do slope decomposition for the cohomology for slope h of f_α . This slope is non-critical, *i.e.* the p -adic valuation of α is less than $k - 1$. According to Coleman inequality ([2]), the cohomology of slope $\leq h$ and weight k is then classical. On the other hand the Hecke eigensystem for f does not appear in $H^0(Y_0(N), V_k)$, so the localization $H^0(Y_0(N), D_k)_f^{\leq h}$ of $H^0(Y_0(N), D_k)^{\leq h}$ at f vanishes. Thus the multiplying by m_k map on $H^1(Y_0(N), D_U)_f^{\leq h}$ is injective. So $H^1(Y_0(N), D_U)_f^{\leq h}$ is a torsion-free module over the discrete valuation ring $\mathcal{O}(U)_k$, thus free. Now by [3, Lemma 2.15], one can lift this to obtain freeness of an appropriate localization of $H^1(Y_0(N), D_U)^{\leq h}$ over a neighborhood of k in U . From Coleman inequality and multiplicity one for GL_2 , there is a Hecke operator t outside p over U , such that the $tH^1(Y_0(N), D_U)^{\leq h}$ is free of rank two over $\mathcal{O}(U)$, which specializes to the 2-dimensional fixed space by $\Gamma_0(p)$ of $\pi_{f,p}$ (we simply choose it to kill all forms at weight k outside the eigensystem of f). Thus locally the U_p is acting on this rank two module, and satisfies a quadratic equation over the weight ring. It is easy to show that the local coefficient ring of the eigencurve is obtained from joining the U_p operator (on this rank two module), U_q operators for all $q|N$ and the T_q operators for $q \nmid N$ to the weight space (*i.e.* a neighborhood of k). It is of degree 2 over the weight ring. We denote the normalization of the local eigencurve as \mathcal{C} . The \mathcal{C} is a smooth curve with one or two irreducible components. There is a rigid analytic function Z on \mathcal{C} giving the U_p eigenvalue of the corresponding eigenform. Now we take base change extension of the coefficient ring to \mathcal{C} . We have the following easy lemma.

Lemma 4.52. *The subspace of $tH^1(Y_0(N), D_C)^{\leq h}$ killed by $U_p - Z$ is free of rank one over \mathcal{C} .*

Proof. This submodule is clearly generically of rank one. We just need to note that the α -eigenspace for the U_p operator in $\pi_{f,p}$ is 1-dimensional to get the lemma. \square

4.7.2 The argument

Now we consider a Coleman family \mathcal{C} as above in a small neighborhood of f_α , and denote the form as \mathbf{f} with coefficient ring a normal domain \mathbb{I} , and let \mathbf{g} be a Hida family of CM forms with respect to the field \mathcal{K} . In [8] it is constructed a 3-variable p -adic L -function interpolating critical values of Rankin-Selberg L -values for \mathbf{f} and \mathbf{g} using modular symbols, which we denote as $\mathcal{L}_{\mathbf{f}, \mathbf{g}, \mathrm{Symb}}$. There also constructed the two-variable p -adic L -function of \mathbf{f} over \mathbb{Q} using modular symbols. We refer the details of the notion of Coleman family \mathbf{f} and interpolation properties of $\mathcal{L}_{\mathbf{f}, \mathbf{g}, \mathrm{Symb}}$ to *loc.cit..*

⁷We thank David Loeffler and Chris Williams for discussions on this part, and thank Betina-Williams for writing up [8] for us.

Remark: Strictly speaking the construction in [8] gives L -values for \mathbf{f} over \mathcal{K} twisted by CM characters corresponding to forms in \mathbf{g} , which are nothing but Rankin-Selberg L -values for \mathbf{f} and \mathbf{g} as in [74, Theorem 7.1.5], except for periods used.

Note that when $\alpha = \beta$ we are not in the “noble eigenform” case as defined in [75, Definition 4.6.3]. As a consequence, the $\eta_{\mathbf{f}}$ (our \mathbf{f} is \mathcal{F} there) defined in [75, Corollary 6.4.3] is only a meromorphic section instead of a basis of the free rank one module there.

Proposition 4.53. *We use the same notations as in [75, Theorem 6.5.9]. Then the right hand side of identity of loc.cit. equals*

$$(-1)^{k'-j+1} (k')! \binom{k}{j} L(f, g, 1+j).$$

Note that if $\alpha \neq \beta$, and thus $\mathcal{E}^*(f)$ is nonzero, then the statement is the same as [75, Theorem 6.5.9]. When $\alpha = \beta$ the same argument as the proof of [75, Theorem 6.5.9], using the computations in [82, Proposition 5.3.5] actually gives the above proposition.

We denote $\mathcal{L}_{\mathbf{f}, \mathbf{g}}$ for the p -adic L -function of [74, Theorem 7.1.5]. The $\mathcal{L}_{\mathbf{f}, \mathbf{g}, \text{Symb}}$ has the same interpolation formula as $\mathcal{L}_{\mathbf{f}, \mathbf{g}}$, except that the period is a nonzero constant $\Omega_{\Pi_f, \mathcal{K}}$ depending on f , while the period factor for $\mathcal{L}_{\mathbf{f}, \mathbf{g}}$ is $\langle f, f \rangle \mathcal{E}(f_\alpha) \mathcal{E}^*(f_\alpha)$ (here f is the normalized eigenform of level N whose α -stabilization is f_α). Note that the latter period takes 0 in the $\alpha = \beta$ case, which is a key difficulty. Note also that the $\mathcal{E}(f_\alpha)$ and $\mathcal{E}^*(f_\alpha)$ are not p -adically rigid analytic. We write \mathcal{G} for the meromorphic function $\frac{\mathcal{L}_{\mathbf{f}, \mathbf{g}, \text{Symb}}}{\mathcal{L}_{\mathbf{f}, \mathbf{g}}}$.

Remark: An automorphic construction of the meromorphic function \mathcal{G} is done in [72]. It is also explained there that it annihilates the congruence module up to a fixed power of p . We do not need these facts here.

From the interpolation formulas, we see that \mathcal{G} involves only the variable corresponding to the Coleman family \mathbf{f} , namely it is a nonzero element in $\text{Frac}(\mathbb{I})$. Now we consider an arithmetic point ϕ' where $f_{\phi'}$ takes the form f_α we start with, and choose $\mathbf{g}_{\phi'}$ is a CM form with trivial character whose conductor is prime to p and whose weight is lower than that of f , and $L(f_{\phi'}, g_{\phi'}, 1 + j_{\phi'})$ is a critical value which is nonzero. Note that by our assumption that the weight of f is at least 4, this is always possible (e.g. by taking the critical value to be non-central).

We also consider another arithmetic point ϕ'_0 , where again $f_{\phi'_0}$ is our f_α , but the $g_{\phi'_0}$ corresponds to a finite order character of $\Gamma_{\mathbb{Q}}$. We require the $L(f_{\phi'_0}, g_{\phi'_0}, 1 + j_{\phi'_0})$ to correspond to the nonzero L -value $L_{\mathcal{K}}(f, \chi_\phi, \frac{k}{2})$, where χ_ϕ is the finite order character of $\Gamma_{\mathbb{Q}}$ in the previous section. (Our convention on the notation is the $j_{\phi'_0}$ denotes a character corresponding to an arithmetic point – it incorporates an integer together with a finite order Hecke character). Note also that we allow that the ϕ'_0 and ϕ' are not in the same irreducible component of CM families.

Remark: If we can prove Proposition 4.53 at these points ϕ'_0 then we are done. However the $g_{\phi'_0}$ has weight one and is not covered in the work of Loeffler-Zerbes [74]. Nevertheless we can still achieve the formula (4.7.5) by the argument below.

Let $\langle \cdot, \cdot \rangle$ and \mathcal{L} be the pairing and regulator map as in [74, Theorem 7.1.5]. We also use the notations $\eta_{\mathbf{f}}$ and $\omega_{\mathbf{g}}$ as in *loc.cit.*. Let $\text{BF}_{\mathbf{f}, \mathbf{g}}$ be the Beilinson-Flach element constructed for \mathbf{f} and \mathbf{g} as in [74]. Its construction, and its relation to the Rankin-Selberg p -adic L -function can be done in the same way as in *loc.cit.*, thanks to our previous discussion on the geometry of the eigencurve (especially that it is locally of degree 2 over the weight space). In particular we have

$$\langle \mathcal{L}(\text{BF}_{\mathbf{f}, \mathbf{g}}), \eta_{\mathbf{f}} \mathcal{G} \otimes \omega_{\mathbf{g}} \rangle = \mathcal{L}_{\mathbf{f}, \mathbf{g}, \text{Symb}}.$$

However we need some additional work to prove the following (4.7.4) and (4.7.5). Let η_f^α be as in [75, Theorem 6.5.9]. Using proposition 4.53, the argument to prove [74, Theorem 7.1.5] (note that the weight of $\mathbf{g}_{\phi'}$ is at least 2) gives

$$\langle \phi'(\mathcal{L}(\text{BF}_{\mathbf{f}, \mathbf{g}})), \eta_f^\alpha \otimes \phi'(\omega_{\mathbf{g}}) \rangle = \phi'(\mathcal{L}_{\mathbf{f}, \mathbf{g}, \text{Symb}}) \frac{\Omega_{\Pi_f, \mathcal{K}}}{\langle f, f \rangle}. \quad (4.7.1)$$

By taking specializations, we have

$$\langle \phi'(\mathcal{L}(\text{BF}_{\mathbf{f},\mathbf{g}})), \phi'(\eta_{\mathbf{f}}\mathcal{G} \otimes \omega_{\mathbf{g}}) \rangle = \phi'(\mathcal{L}_{\mathbf{f},\mathbf{g},\text{Symb}}). \quad (4.7.2)$$

Similarly we have

$$\langle \phi'_0(\mathcal{L}(\text{BF}_{\mathbf{f},\mathbf{g}})), \phi'_0(\eta_{\mathbf{f}}\mathcal{G}) \otimes \phi'_0(\omega_{\mathbf{g}}) \rangle = \phi'_0(\mathcal{L}_{\mathbf{f},\mathbf{g},\text{Symb}}). \quad (4.7.3)$$

By comparing the equations (4.7.1), (4.7.2) and (4.7.3), we see that $\phi'(\eta_{\mathbf{f}}\mathcal{G})$ (or equivalently $\phi'_0(\eta_{\mathbf{f}}\mathcal{G})$) is a finite nonzero multiple of η_f^α , and that

$$\langle \phi'_0(\mathcal{L}(\text{BF}_{\mathbf{f},\mathbf{g}})), \eta_f^\alpha \otimes \phi'_0(\omega_{\mathbf{g}}) \rangle = \phi'_0(\mathcal{L}_{\mathbf{f},\mathbf{g},\text{Symb}}) \frac{\Omega_{\Pi_{f,\mathcal{K}}}}{\langle f, f \rangle}, \quad (4.7.4)$$

By the same argument we can also get that

$$\langle \mathcal{L}(\text{BF}_{\mathbf{f},\mathbf{g}})|_{f_\alpha}, \eta_f^\alpha \otimes \omega_{\mathbf{g}} \rangle = \mathcal{L}_{\mathbf{f},\mathbf{g},\text{Symb}}|_{f_\alpha} \cdot \frac{\Omega_{\Pi_{f,\mathcal{K}}}}{\langle f, f \rangle}. \quad (4.7.5)$$

These are precisely the results we need on the explicit reciprocity laws for Beilinson-Flach elements in the previous section to prove the Iwasawa main conjecture.

Remark: To summarize, the key in the argument above is the comparison between the specialization of $\eta_{\mathbf{f}}\mathcal{G}$ to f_α and the η_f^α . This is made possible thanks to the construction in [8].

4.8 The ordinary case

As mentioned in the introduction, the available literature might not contain a full argument for the proof of theorem 1.6. Here, we briefly indicate how the methods of section 4.6 carry out to this case.

Proof of theorem 1.6. The argument is similar as that of section 4.6 but much easier. We first choose an auxiliary quadratic field \mathcal{K} in the same way and prove the two-variable Iwasawa-Greenberg Rankin-Selberg Main Conjecture over \mathcal{K} involving the p -adic L -function, by reducing it to the Greenberg type main conjecture (theorem 7.32) via explicit reciprocity law for Beilinson-Flach elements and Poitou-Tate exact sequence. The details of this argument are given in [21, Theorem 3.8] (there the result is proved for weight 2. However the argument goes through in general by replacing the Greenberg's main conjecture there by Theorem 7.32 here). Then completely as above, we combine it with Kato's result to get full equality for the Iwasawa main conjecture over \mathbb{Q} . \square

5 The main theorems

5.1 Statements

Theorem 5.1. Let $p \geq 3$ be a prime. Let $f \in S_k(\Gamma_0(Np^r))$ be a normalized eigencuspform with $k \geq 2$. Assume that $\bar{\rho}_f$ satisfies the following properties.

1. The $G_{\mathbb{Q},\Sigma}$ -representation $\bar{\rho}_f$ is absolutely irreducible.
2. The semisimplification of $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is not equal to $\bar{\chi} \oplus \bar{\chi}_{\text{cyc}} \bar{\chi}$.
3. There exists $\ell \nmid p$ such that $\bar{\rho}_f|_{G_{\mathbb{Q}_\ell}}$ is a ramified extension

$$0 \longrightarrow \mu\chi_{\text{cyc}}^{1-k/2} \longrightarrow \bar{\rho}|_{G_{\mathbb{Q}_\ell}} \longrightarrow \mu\chi_{\text{cyc}}^{-k/2} \longrightarrow 0$$

where $\mu : G_{\mathbb{Q}_\ell} \longrightarrow \{\pm 1\}$ is an unramified character. If moreover $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is irreducible, then μ is not trivial.

If $\ell \mid N$, then the zeta morphism is an isomorphism

$$\text{triv}_{\mathbf{z}(f)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T(f)_{\text{Iw}}) \xrightarrow{\text{can}} \mathcal{O}_{\text{Iw}}.$$

Equivalently, conjecture 3.24 is true.

Remark: We note that under the hypothesis of the theorem, the determinant of ρ_f is an odd power of χ_{cyc} . So the assumption that the semisimplification of $\bar{\rho}_f|G_{\mathbb{Q}_p}$ be different from $\bar{\chi} \oplus \bar{\chi}$ is automatically satisfied.

Theorem 5.2. *Under the same assumption as theorem 5.1, the trivialization morphism*

$$\text{triv}_{\mathbf{z}_{\Sigma}} : \Delta_{\Sigma}(T_{\Sigma}) \hookrightarrow \text{Frac}(\Lambda)$$

given by the zeta morphism

$$\mathbf{z}_{\Sigma} : T_{\Sigma}(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/\Sigma], T_{\Sigma})$$

induces an isomorphism

$$\text{triv}_{\mathbf{z}_{\Sigma}} : \Delta_{\Sigma}(T_{\Sigma}) \xrightarrow{\text{can}} \Lambda.$$

Equivalently, conjecture 3.30 is true.

We state and prove the following corollaries.

Corollary 5.3. *Let $f \in S_k(\Gamma_0(N))$ be a normalized eigencuspform satisfying all the hypotheses of theorem 5.1. Then $\text{Sel}_{\mathbb{Q}}(f)$ is a finite group if and only if $L(f, k/2) \neq 0$.*

Proof. If $L(f, k/2) \neq 0$, then $\text{Sel}_{\mathbb{Q}}(f)$ is finite by [68, Theorem] so we may and do assume that $L(f, k/2)$ is equal to zero. Let γ be a topological generator of Γ_{Iw} and write ψ for the point in $\text{Spec } \mathcal{O}_{\text{Iw}}$ corresponding to $\gamma - 1 \mapsto 0$. If the specialization of $\mathbf{z}(f)_{\text{Iw}}$ through ψ is 0, then $\gamma - 1$ belongs to $\text{char}_{\mathcal{O}_{\text{Iw}}}(H_{\text{et}}^1(\mathbb{Z}[1/p], T(f)_{\text{Iw}})/\mathcal{O}_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}})$. As conjecture 3.24 is true for $M(f)$,

$$\text{char}_{\mathcal{O}_{\text{Iw}}}(H_{\text{et}}^2(\mathbb{Z}[1/p], T(f)_{\text{Iw}})) = \text{char}_{\mathcal{O}_{\text{Iw}}}(H_{\text{et}}^1(\mathbb{Z}[1/p], T(f)_{\text{Iw}})/\mathcal{O}_{\text{Iw}} \cdot \mathbf{z}(f)_{\text{Iw}}).$$

Hence $\gamma - 1$ also belongs to $\text{char}_{\mathcal{O}_{\text{Iw}}}(H_{\text{et}}^2(\mathbb{Z}[1/p], T(f)_{\text{Iw}}))$ and the \mathcal{O} -corank of $\text{Sel}_{\mathbb{Q}}(f)$ is strictly positive. If the specialization of $\mathbf{z}(f)_{\text{Iw}}$ through ψ does not vanish, then our assumption that $L(f, k/2)$ vanishes implies that the image of $\mathbf{z}(f)$ generates a non-zero line in $H_f^1(G_{\mathbb{Q}, \Sigma}, V(f)(k/2))$. As before, this means that $\text{corank } \text{Sel}_{\mathbb{Q}}(f) \geq 1$. \square

The following corollary contributes to the study of the Birch and Swinnerton-Dyer Conjecture for abelian varieties of GL_2 -type whose L -function does not vanish at 1.

Corollary 5.4. *Let A/\mathbb{Q} be an abelian variety of GL_2 -type of conductor N . According to [106] and the proof of Serre's conjecture [70, 71], A is modular. Let f be the associated weight two cusp form. Assume that $L(A, 1) \neq 0$ and that f satisfies the assumptions of theorem 5.1. Then*

$$v_p(L(A, 1)/\Omega_f) = v_p \left(|\text{III}(A/\mathbb{Q})[p^\infty]| \prod_{q|N} \text{Tam}_q(A/\mathbb{Q}) \right).$$

Equivalently, the p -part of the Birch and Swinnerton-Dyer Conjecture for A holds.

Proof. According to theorem 5.1 conjecture 3.24 holds and the zeta morphism

$$\text{triv}_{\mathbf{z}(A)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T_p A \otimes \mathcal{O}_{\text{Iw}}) \hookrightarrow \text{Frac}(\mathcal{O}_{\text{Iw}})$$

is an isomorphism

$$\text{triv}_{\mathbf{z}(A)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T_p A \otimes \mathcal{O}_{\text{Iw}}) \xrightarrow{\text{can}} \mathcal{O}_{\text{Iw}}.$$

is an isomorphism. As $L(A, 1) \neq 0$, $\mathbf{z}(A)$ does not vanish. If $\psi : \mathcal{O}_{\text{Iw}} \longrightarrow \mathcal{O}$ is the quotient map by the augmentation ideal, the diagram

$$\begin{array}{ccc} \Delta_{\mathcal{O}_{\text{Iw}}}(T_p A \otimes \mathcal{O}_{\text{Iw}}) & \xrightarrow{\text{triv}_{\mathbf{z}(A)_{\text{Iw}}}} & \mathcal{O}_{\text{Iw}} \\ \downarrow - \otimes_{\mathcal{O}_{\text{Iw}}, \psi} \mathcal{O} & & \downarrow - \otimes_{\mathcal{O}_{\text{Iw}}, \psi} \mathcal{O} \\ \Delta_{\mathcal{O}}(T_p A) & \xrightarrow{\text{triv}_{\mathbf{z}(A)}} & \mathcal{O} \end{array}$$

is thus commutative and its horizontal arrows are isomorphisms. Under our hypotheses, $\text{III}(A/\mathbb{Q})[p^\infty]$ is finite by [68, Theorem]. Consequently, the isomorphism

$$\text{triv}_{\mathbf{z}(A)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T_p A \otimes \mathcal{O}_{\text{Iw}}) \xrightarrow{\text{can}} \mathcal{O}_{\text{Iw}}$$

implies the Birch and Swinnerton-Dyer Conjecture at p by [13, Proposition 1.55]. \square

5.2 Proofs of theorems 5.1 and 5.2

Proposition 5.5. *The trivialization map $\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \hookrightarrow \text{Frac}(\Lambda)$ of conjecture 3.30 sends $\Delta_\Sigma(T_\Sigma)^{-1}$ inside Λ .*

Proof. By way of contradiction, we assume that $\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \hookrightarrow \text{Frac}(\Lambda)$ does not send $\Delta_\Sigma(T_\Sigma)^{-1}$ inside Λ . Let $\lambda : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \longrightarrow \mathcal{O}$ be an Iwasawa-suitable specialization and let $T_{\lambda, \text{Iw}}$ be $T_\Sigma \otimes_{\Lambda, \lambda} \mathcal{O}_{\text{Iw}}$ (note that $T_{\lambda, \text{Iw}}$ is a \mathcal{O}_{Iw} -module of rank $2d$ where d is the rank of $\mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$ as Λ -module). In order to distinguish them from their counterparts with coefficients in $\mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$, we write $\mathbf{z}_{\Sigma, \Lambda}$ and $\mathbf{z}_{\Sigma, \Lambda}(\lambda)$ for the morphisms of theorem 3.6 but where all objects are regarded as having coefficients in Λ (in particular λ is then seen as a morphism $\lambda : \Lambda \longrightarrow \mathcal{O}$ by restriction). According to proposition 3.23, there is then a commutative diagram

$$\begin{array}{ccc} \Delta_\Sigma(T_\Sigma)^{-1} & \xrightarrow{\text{triv}_{\mathbf{z}_{\Sigma, \Lambda}}} & \frac{x}{y}\Lambda \\ \downarrow - \otimes_{\Lambda} \mathcal{O}_{\text{Iw}} & & \downarrow \lambda \\ \Delta_\Sigma(T_{\lambda, \text{Iw}})^{-1} & \xrightarrow{\text{triv}_{\mathbf{z}_{\Sigma, \Lambda}(\lambda_{\text{Iw}})}} & \frac{x'}{y'}\mathcal{O}_{\text{Iw}} \end{array}$$

where T_Σ is viewed as a Λ -module. Our hypothesis is that x/y does not belong to Λ . As Λ is a factorial ring, there exists $\mathfrak{p} \in \text{Spec } \Lambda$ a height-one prime containing y but not x . As Λ is regular, \mathfrak{p} is principal. Let y_0 be one of its generator. Let n be a sufficiently large integer. According to propositions 2.6 and 3.16, the set of specialization $\lambda' : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \longrightarrow \mathcal{O}$ such that one of the specialization in the fiber above the point of Λ below λ' is not Iwasawa-suitable or such that $\text{Spec } \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$ is not étale over Λ at λ' is of large codimension. Hence, it does not contain the set of specialization $\lambda : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \longrightarrow \mathcal{O}_{\text{Iw}}$ such that $\lambda(y_0)$ belongs to $\mathfrak{m}_{\mathcal{O}_{\text{Iw}}}^n$. For n large enough and such a λ , $\lambda(x)$ does not belong to $\mathfrak{m}_{\mathcal{O}_{\text{Iw}}}^n$. Hence, there exists a specialization $\lambda_{\text{Iw}} : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \longrightarrow \mathcal{O}_{\text{Iw}}$ such that $\text{Spec } \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$ is étale over Λ at λ_{Iw} , such that all specializations in the fiber above the point of Λ below are Iwasawa-suitable and such that $\text{triv}_{\mathbf{z}_{\Sigma, \Lambda}(\lambda_{\text{Iw}})}(\Delta_\Sigma(T_{\lambda, \text{Iw}})^{-1})$ does not belong to \mathcal{O}_{Iw} .

Because $\text{Spec } \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma$ is étale over Λ at λ_{Iw} , enlarging \mathcal{O} if necessary, we may assume that $T_{\lambda, \text{Iw}}$ is a lattice inside a direct sum

$$\bigoplus_{r=1}^d V_{\lambda, r, \text{Iw}}$$

of $G_{\mathbb{Q}, \Sigma}$ -representations $V_{\lambda, r, \text{Iw}}$ which are free modules of rank 2 over $\text{Frac}(\mathcal{O}_{\text{Iw}})$. There is thus a short exact sequence

$$0 \longrightarrow T_{\lambda, \text{Iw}} \longrightarrow \bigoplus_{r=1}^d T_{\lambda, r, \text{Iw}} \longrightarrow C \longrightarrow 0 \tag{5.2.1}$$

where $T_{\lambda, r, \text{Iw}}$ are specializations of T_Σ attached to specializations $\lambda_r : \mathbf{T}_{\mathfrak{m}_\rho}^\Sigma \longrightarrow \mathcal{O}$ and where C a torsion \mathcal{O}_{Iw} -module. The short exact sequence (5.2.1) induces a canonical isomorphism

$$\bigotimes_{r=1}^d \Delta_\Sigma(T_{\lambda, r, \text{Iw}}) \xrightarrow{\text{can}} \Delta_\Sigma(T_{\lambda, \text{Iw}}) \otimes_{\mathcal{O}_{\text{Iw}}} \Delta_\Sigma(C). \tag{5.2.2}$$

According to [13, Proposition 1.20] (taking into account the duality between $R\Gamma_c$ and $R\Gamma_{\text{et}}$, see for instance [120, Appendix]), $\Delta_\Sigma(C)$ is the unit object in the category of graded invertible modules. Hence

$$\text{triv}_{\mathbf{z}_{\Sigma, \Lambda}(\lambda_{\text{Iw}})} \left(\bigotimes_{r=1}^d \Delta_\Sigma(T_{\lambda, r, \text{Iw}})^{-1} \right) \not\subset \mathcal{O}_{\text{Iw}}.$$

By construction,

$$\text{triv}_{\mathbf{z}_{\Sigma, \Lambda}(\lambda_{\text{Iw}})} = \bigotimes_{r=1}^d \text{triv}_{\mathbf{z}_\Sigma(\lambda_{r, \text{Iw}})}$$

so there must exists an r such that

$$\text{triv}_{\mathbf{z}_\Sigma(\lambda_{r,\text{Iw}})} (\Delta_\Sigma(T_{\lambda,r,\text{Iw}})^{-1}) \not\subset \mathcal{O}_{\text{Iw}}.$$

By our choice of λ , $\lambda_{r,\text{Iw}}$ is Iwasawa-suitable . Consequently, proposition 3.29 yields the inclusion

$$\text{triv}_{\mathbf{z}_\Sigma(\lambda_{r,\text{Iw}})} (\Delta_\Sigma(T_{\lambda,r,\text{Iw}})^{-1}) \subset \mathcal{O}_{\text{Iw}}.$$

This is a contradiction. \square

Proposition 5.6. *Suppose that $\psi(f) : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ is a classical point which is either crystalline and short, or crystalline and short up to the quadratic twist $\omega^{\frac{p-1}{2}}$ (recall ω is the Teichmuller character), or good ordinary, or good ordinary up to a quadratic twist. Then the trivialization morphism*

$$\text{triv}_{\mathbf{z}(f)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T(f)_{\text{Iw}}) \hookrightarrow \text{Frac}(\mathcal{O}_{\text{Iw}})$$

given by the zeta morphism

$$\mathbf{z}(f)_{\text{Iw}} : V(f)_{\text{Iw}}(-1)^+ \longrightarrow H_{\text{et}}^1(\mathbb{Z}[1/p], V(f)_{\text{Iw}})$$

induces an isomorphism

$$\text{triv}_{\mathbf{z}(f)_{\text{Iw}}} : \Delta_{\mathcal{O}_{\text{Iw}}}(T(f)_{\text{Iw}}) \xrightarrow{\text{can}} \mathcal{O}_{\text{Iw}}.$$

Equivalently, conjecture 3.24 is true for $\psi(f)$.

Proof. If $\rho_f|G_{\mathbb{Q}_p}$ is crystalline and short or crystalline and short, this is theorem 4.51. If $\rho_f|G_{\mathbb{Q}_p}$ is ordinary or ordinary up to a quadratic twist, this is theorem 1.6 (proved in the end of the last section). The cases involving quadratic twists are proved in the completely same way. \square

Proposition 5.7. *Suppose that there exists a point $x : \Lambda \rightarrow \mathcal{O}$ at which*

$$\begin{array}{ccc} \text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma & & \\ \downarrow & & \\ \text{Spec } \Lambda & & \end{array}$$

is étale and such that all points $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ above x satisfy conjecture 3.24. Then the trivialization map

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \hookrightarrow \text{Frac}(\Lambda)$$

of conjecture 3.30 is an isomorphism

$$\Delta_\Sigma(T_\Sigma) \simeq \Lambda.$$

Proof. According to proposition 5.5, there exists $\alpha \in \Lambda$ such that the image of $\Delta_\Sigma(T_\Sigma)^{-1}$ through $\text{triv}_{\mathbf{z}_\Sigma}$ is $\alpha\Lambda$. Let S_x be the set of points $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ above x . Then $\text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \text{Spec } \Lambda$ is étale at x and all λ_{Iw} are Iwasawa-suitable specializations. According to proposition 3.23, there thus exists $\alpha \in \Lambda$ and a commutative diagram

$$\begin{array}{ccc} \Delta_\Sigma(T_\Sigma)^{-1} & \xrightarrow{\mathbf{z}_{\Sigma,\Lambda}} & \alpha\Lambda \\ \downarrow - \otimes_{\Lambda,x_{\text{Iw}}} \mathcal{O}_{\text{Iw}} & & \downarrow \lambda \\ \bigotimes_{\lambda \in S_x} \Delta_\Sigma(T_{\lambda,\text{Iw}})^{-1} & \xrightarrow{\bigotimes_{\lambda \in S_x} \mathbf{z}_{\Sigma}(\lambda_{\text{Iw}})} & \frac{x'}{y'} \mathcal{O}_{\text{Iw}} \end{array}$$

According to our hypothesis, for all $\lambda \in S_x$, $\text{triv}_{\mathbf{z}_\Sigma(\lambda_{\text{Iw}})}(\Delta_\Sigma(T_{\lambda, \text{Iw}})^{-1}) = \mathcal{O}_{\text{Iw}}$. Hence, the commutative diagram above may be written

$$\begin{array}{ccc} \Delta_\Sigma(T_\Sigma)^{-1} & \xrightarrow{\mathbf{z}_\Sigma, \Lambda} & \alpha \Lambda \\ \downarrow - \otimes_{\Lambda, x_{\text{Iw}}} \mathcal{O}_{\text{Iw}} & & \downarrow \lambda \\ \bigotimes_{\lambda \in S_x} \Delta_\Sigma(T_{\lambda, \text{Iw}})^{-1} & \xrightarrow{\bigotimes_{\lambda \in S_x} \mathbf{z}_\Sigma(\lambda_{\text{Iw}})} & \mathcal{O}_{\text{Iw}} \end{array}$$

This implies that α is a unit and thus that the trivialization map

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \hookrightarrow \text{Frac}(\Lambda)$$

of conjecture 3.30 is an isomorphism

$$\Delta_\Sigma(T_\Sigma)^{-1} \simeq \Lambda.$$

□

Proof of theorem 5.2. According to lemma 2.8, there exists a point of $\text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ which is crystalline and short, or crystalline and short up to a quadratic twist, or good ordinary, or good ordinary up to a quadratic twist. By proposition 5.6, there then exists a point $x : \Lambda \rightarrow \mathcal{O}$ at which

$$\begin{array}{ccc} \text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma & & \\ \downarrow & & \\ \text{Spec } \Lambda & & \end{array}$$

is étale and such that all points $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ above x satisfy conjecture 3.24. By proposition 5.7, the trivialization map

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \hookrightarrow \text{Frac}(\Lambda)$$

of conjecture 3.30 is an isomorphism

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \simeq \Lambda$$

and conjecture 3.30 is thus true. □

Proposition 5.8. *Suppose that the trivialization map*

$$\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \hookrightarrow \text{Frac}(\Lambda)$$

of conjecture 3.30 is an isomorphism

$$\Delta_\Sigma(T_\Sigma) \simeq \Lambda.$$

Let $\lambda(f) : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ be a classical point. Then there is an isomorphism

$$\text{triv}_{\mathbf{z}(f)_\text{Iw}} : \Delta_\Sigma(T(f)_\text{Iw}) \simeq \mathcal{O}_{\text{Iw}}.$$

Equivalently, conjecture 3.24 is true for $\lambda(f)$.

Proof. We know that there is an inclusion $\text{triv}_{\mathbf{z}(f)_\text{Iw}}(\Delta_\Sigma(T(f)_\text{Iw})^{-1}) \subset \mathcal{O}_{\text{Iw}}$. Hence, there exists $\gamma \in \mathcal{O}_{\text{Iw}}$ such that

$$\text{triv}_{\mathbf{z}(f)_\text{Iw}} : \Delta_\Sigma(T(f)_\text{Iw}) \simeq \frac{1}{\gamma} \mathcal{O}_{\text{Iw}}.$$

Moreover, there is a commutative diagram

$$\begin{array}{ccc}
\Delta_\Sigma(T_\Sigma) & \xrightarrow{\mathbf{z}_\Sigma} & \frac{\alpha}{\beta} \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \\
\downarrow - \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma, \lambda(f)_\text{Iw}} \mathcal{O}_\text{Iw} & & \downarrow \lambda(f)_\text{Iw} \\
\Delta_\Sigma(T(f)_\text{Iw}) & \xrightarrow{\mathbf{z}_\Sigma(f)_\text{Iw}} & \frac{1}{\gamma} \mathcal{O}_\text{Iw}
\end{array}$$

According to proposition 3.23, we may choose β such that $\lambda(f)_\text{Iw}(\beta) \neq 0$.

Let Z be the set of specialization sending β to zero. According to propositions 2.6 and 3.16, there exists a specialization $\lambda : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ which is not in Z above a point x of Λ such that $\text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \text{Spec } \Lambda$ is étale over x and such that all specializations $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \rightarrow \mathcal{O}$ over x are Iwasawa-suitable. By our assumption, there are then elements $\gamma_\psi \in \mathcal{O}_\text{Iw}$ for all ψ over x and a commutative diagram

$$\begin{array}{ccc}
\Delta_\Sigma(T_\Sigma) & \xrightarrow{\mathbf{z}_\Sigma, \Lambda} & \Lambda \\
\downarrow - \otimes_{\Lambda, x_\text{Iw}} \mathcal{O}_\text{Iw} & & \downarrow x \\
\bigotimes_{\psi \in S_x} \Delta_\Sigma(T_{\psi, \text{Iw}}) & \xrightarrow{\otimes \mathbf{z}_\Sigma(\psi)_\text{Iw}} & \left(\prod_{\psi \in S_x} \frac{1}{\gamma_\psi} \right) \mathcal{O}_\text{Iw}
\end{array}$$

This shows that all the γ_ψ are units and that the diagram above may be written

$$\begin{array}{ccc}
\Delta_\Sigma(T_\Sigma) & \xrightarrow{\mathbf{z}_\Sigma, \Lambda} & \Lambda \\
\downarrow - \otimes_{\Lambda, x_\text{Iw}} \mathcal{O}_\text{Iw} & & \downarrow x \\
\bigotimes_{\psi \in S_x} \Delta_\Sigma(T_{\psi, \text{Iw}}) & \xrightarrow{\otimes \mathbf{z}_\Sigma(\psi)_\text{Iw}} & \mathcal{O}_\text{Iw}
\end{array}$$

Because this holds in particular for λ and because $\lambda \notin Z$, we have a commutative diagram

$$\begin{array}{ccc}
\Delta_\Sigma(T_\Sigma) & \xrightarrow{\mathbf{z}_\Sigma} & \frac{\alpha}{\beta} \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \\
\downarrow - \otimes_{\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma, \lambda_\text{Iw}} \mathcal{O}_\text{Iw} & & \downarrow \lambda_\text{Iw} \\
\Delta_\Sigma(T_{\lambda, \text{Iw}}) & \xrightarrow{\mathbf{z}_\Sigma(\lambda)_\text{Iw}} & \mathcal{O}_\text{Iw}.
\end{array}$$

Hence, $\lambda_\text{Iw}(\alpha/\beta)$ is a unit while $\lambda(f)_\text{Iw}(\alpha/\beta)$ is $1/\gamma$. Taking λ close to $\lambda(f)$, we see that $1/\gamma$ is a unit. \square

Proof of theorem 5.1. Let $\bar{\rho}$ be the residual representation attached to $M(f)(-\frac{k-2}{2})$. Then by proposition 5.5, the trivialization map $\text{triv}_{\mathbf{z}_\Sigma} : \Delta_\Sigma(T_\Sigma) \hookrightarrow \text{Frac}(\Lambda)$ of conjecture 3.30 sends $\Delta_\Sigma(T_\Sigma)^{-1}$ inside Λ . According to lemma 2.8, there exists a point of $\text{Spec } \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma$ which is crystalline and short, or crystalline and short up to a quadratic twist, or good ordinary,

or good ordinary up to a quadratic twist. By proposition 5.6, there then exists a point $x : \Lambda \rightarrow \mathcal{O}$ at which

$$\begin{array}{ccc} \mathrm{Spec} \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} & & \\ \downarrow & & \\ \mathrm{Spec} \Lambda & & \end{array}$$

is étale and such that all points $\psi : \mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma} \rightarrow \mathcal{O}$ above x satisfy conjecture 3.24. By proposition 5.7, the trivialization map

$$\mathrm{triv}_{\mathbf{z}_{\Sigma}} : \Delta_{\Sigma}(T_{\Sigma}) \hookrightarrow \mathrm{Frac}(\Lambda)$$

of conjecture 3.30 is an isomorphism

$$\Delta_{\Sigma}(T_{\Sigma}) \simeq \Lambda.$$

By proposition 5.8, the classical point of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^{\Sigma}$ corresponding to $M(f)(-\frac{k-2}{2})$ then satisfies conjecture 3.24. \square

6 Appendix A: completed cohomology and essential vectors for automorphic representations

6.1 Generalities

6.1.1 Notations

In this section, A is a complete, local, reduced, noetherian ring with residue field. Let $\ell \nmid p$ be a rational prime. Let F be a finite extension of \mathbb{Q}_{ℓ} with ring of integers \mathcal{O}_F , uniformizing parameter ϖ and residual field \mathbb{F} of cardinal q . For $n \geq 0$, we denote by \mathbf{G}_n the group $\mathrm{GL}_n(F)$ (so that \mathbf{G}_0 is the trivial group). We consider \mathbf{G}_n as a subgroup of \mathbf{G}_{n+1} through the embedding

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

where the two zeroes 0 indicate that the last line and column of $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ have all their coefficients equal to zero.

Let K_n be the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$ of \mathbf{G}_n . If $m \geq 0$, let $K_n(m)$ be the compact subgroup

$$K_n(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_n \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varpi^m} \right\}.$$

For $n \geq 2$, we denote by U_n, P_n and N_n the following subgroups of \mathbf{G}_2 :

$$U_n = \left\{ \begin{pmatrix} I_{n-1} & v \\ 0 & 1 \end{pmatrix} \mid v \in F^{n-1} \right\}$$

The mirabolic subgroup P_n of \mathbf{G}_n is the subgroup $\mathbf{G}_{n-1}U_n$ and N_n is the subgroup of upper-triangular, unipotent matrices. The abstract group U_n is isomorphic to the additive group F^{n-1} . If $n = 1$, we set all these subgroups equal to the trivial subgroup of F^{\times} .

For H equal to any of the group $\mathbf{G}_n, K_n(m), U_n, P_n$ or N_n , we denote by $\mathrm{Rep}_A H$ the category of smooth $A[H]$ -modules.

If H is a closed subgroup of \mathbf{G}_n and if (σ, W) is an $A[H]$ -module, we denote by $\mathrm{Ind}_H^{\mathbf{G}_n} \sigma$ the unnormalized induction, that is to say the \mathbf{G}_n -representation (π, V) where V is the A -module of functions $f : \mathbf{G}_n \rightarrow W$ satisfying

1. For all $(g, h) \in \mathbf{G}_2 \times H$, $f(hg) = \sigma(h)f(g)$.
2. There is a compact open subgroup K of \mathbf{G}_n (which depends possibly on f) such that $f(gx) = f(g)$ for all $(g, x) \in \mathbf{G}_n \times K$.

and where $\pi(g)f = f(\cdot g)$. The sub-representation $c\text{-Ind}_H^{\mathbf{G}_n}$ is the sub- A -module of functions which satisfy in addition the property that their support is included in HK for some compact subset K of \mathbf{G}_n .

Let \tilde{k} be a Galois extension of \mathbb{F} containing all ℓ -power roots of unity and let \tilde{A} be $A \otimes_{W(\mathbb{F})} W(\tilde{k})$. We fix $\psi : F \rightarrow W(\tilde{k})^\times$ an additive character of F whose kernel is equal to $\varpi\mathcal{O}_F$ and extend it to a character of N_n by setting $\psi(n) = \psi(n_{1,2} + n_{2,3} + \dots + n_{n-1,n})$ for $n \in N_n$.

If $H \subset \mathbf{G}_n$ is a subgroup, if (ψ, H) is a $W(\tilde{k})[H]$ -character and if V is an $\tilde{A}[H]$ -module, we write $V(H, \psi)$ for the \tilde{A} -submodule of V generated by elements of the form $hv - \psi(h)v$ and we write $V_{H, \psi}$ for the quotient $V/V(H, \psi)$, which is the largest quotient of V on which H acts through ψ . In particular, if M is a Levi subgroup of \mathbf{G}_n with unipotent radical U and if $\mathbb{1}$ is the constant character 1, then $J_U : V \mapsto V_{U, \mathbb{1}}$ is the Jacquet functor ([15]).

6.1.2 Bernstein-Zelevinsky functors

In [35], the following functors first introduced in [7] are extended to our setting.

1.

$$\begin{aligned} \Phi^+ : \quad \text{Rep}_{\tilde{A}} P_{n-1} &\longrightarrow \text{Rep}_{\tilde{A}} P_n \\ V &\longmapsto c\text{-Ind}_{P_{n-1} U_n}^{P_n} V \end{aligned}$$

Here U_n acts on $\Phi^+(V)$ via ψ .

2.

$$\begin{aligned} \Phi^- : \quad \text{Rep}_{\tilde{A}} P_n &\longrightarrow \text{Rep}_{\tilde{A}} P_{n-1} \\ V &\longmapsto V_{U_n, \psi} \end{aligned}$$

3.

$$\begin{aligned} \Psi^+ : \quad \text{Rep}_{\tilde{A}} \text{GL}_{n-1} &\longrightarrow \text{Rep}_{\tilde{A}} P_n \\ V &\longmapsto V \end{aligned}$$

Here U_n acts on $\Phi^+(V)$ trivially.

4.

$$\begin{aligned} \Psi^- : \quad \text{Rep}_{\tilde{A}} P_n &\longrightarrow \text{Rep}_{\tilde{A}} \text{GL}_{n-1} \\ V &\longmapsto V_{U_n, \mathbb{1}} \end{aligned}$$

5.

$$\begin{aligned} (-)^{(r)} : \quad \text{Rep}_{\tilde{A}} \text{GL}_n &\longrightarrow \text{Rep}_{\tilde{A}} \text{GL}_{n-r} \\ V &\longmapsto \Psi^-(\Phi^-)^{r-1} \text{Res}_{\mathbf{G}_n}^{P_n} V \end{aligned}$$

Proposition 6.1. *If*

$$\tilde{\Theta} : \text{Rep}_{\tilde{A}} G \longrightarrow \text{Rep}_{\tilde{A}} H$$

is any of the functor $\Phi^+, \Phi^-, \Psi^+, \Psi^-$ or $(-)^{(r)}$, then $\tilde{\Theta}$ descends to a functor

$$\Theta : \text{Rep}_A G \longrightarrow \text{Rep}_A H$$

in the sense that for all $V \in \text{Rep}_A G$

$$\Theta(V) \otimes_A \tilde{A} = \tilde{\Theta}(V \otimes_A \tilde{A}).$$

Proof. See [35, Proposition 3.1.4]. □

We use the same notations $\Phi^+, \Phi^-, \Psi^+, \Psi^-$ and $(-)^{(r)}$ for the functors on $\text{Rep}_A H$ whose existence is asserted in proposition 6.1. All these functors are exact and commute with arbitrary base-change of ring of coefficients. The functor Φ^+ is left-adjoint to Φ^- . The functor Ψ^- is left-adjoint to Ψ^+ . The functors $\Psi^- \Psi^+$ and $\Phi^- \Phi^+$ are isomorphic to the identity functor.

Definition 6.2. *The Bernstein-Zelevinsky derivative of order r is the functor $(-)^{(r)}$.*

By construction, the Bernstein-Zelevinsky derivative of order n of $V \in \text{Rep}_A \mathbf{G}_n$ is simply an A -module and the first Bernstein-Zelevinsky derivative of a character is a free A -module of rank 1. We record the following fact.

Proposition 6.3. *The Bernstein-Zelevinsky derivative of order n is multiplicative with respect to parabolic induction in the following sense. If V and W are respectively an admissible $A[\mathbf{G}_n]$ -module and an admissible $A[\mathbf{G}_m]$ -module, then the parabolic induction $\text{Ind}_P^{\mathbf{G}_{n+m}} V \otimes W$ of the representation $V \otimes W$ or the parabolic subgroup*

$$P \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{G}_n & * \\ 0 & \mathbf{G}_m \end{pmatrix}$$

of \mathbf{G}_{n+m} satisfies $(\text{Ind}_P^{\mathbf{G}_{n+m}} V \otimes W)^{(n+m)} \simeq V^{(n)} \otimes W^{(m)}$. Suppose that A is a domain and that $V \in \text{Rep}_A \mathbf{G}_n$ is an absolutely irreducible, admissible $A[\mathbf{G}_n]$ -module. Then $\text{rank}_A V^{(n)} \leq 1$ and $\text{rank}_A V^{(n)} = 1$ if V is cuspidal.

Proof. The first assertion is [121, Lemme 1.10]. After extension of scalars to the field of fractions of A , the second assertion is [121, III.5.10] as slightly extended in [35, Theorem 3.1.7]. \square

By definition

$$\begin{aligned} (\Phi^+)^{n-1} \Psi^+ (V^{(n)}) &= (\Phi^+)^{n-1} \Psi^+ \Psi^- (\Phi^-)^{n-1} (V) \\ &= (\Phi^+)^{n-1} (\Phi^-)^{n-1} (V) \end{aligned}$$

As the functor Φ^+ is left adjoint to Φ^- , there is thus a natural injective map of $A[P_n]$ -modules

$$(\Phi^+)^{n-1} \Psi^+ (V^{(n)}) \hookrightarrow V. \quad (6.1.1)$$

The image of this injection is a sub- $A[P_n]$ -module denoted by $\mathcal{I}(V)$ and called the space of Schwartz function. Notice that by construction, the sub- $A[\mathbf{G}_n]$ -module $A[\mathbf{G}_n]\mathcal{I}(V)$ generated by $\mathcal{I}(V)$ satisfies $(A[\mathbf{G}_n]\mathcal{I}(V))^{(n)} = V^{(n)}$.

6.1.3 Co-Whittaker modules

In this subsection, we collect the results we need on the theory of co-Whittaker modules and the integral Bernstein center. All the results are due to D.Helm ([52, 53]) and D.Helm and M.Emerton ([35]).

Let (π, V) be an $A[\mathbf{G}_n]$ -module.

Proposition 6.4. *Suppose that A is a field and that V is smooth, admissible. Let C be the cosocle of V , that is to say the largest semisimple quotient of V . Then the following assertions are equivalent.*

1. The $A[\mathbf{G}_n]$ -module V has finite length, C is an absolutely irreducible $A[\mathbf{G}_n]$ -module and $V^{(n)} = C^{(n)} \neq 0$.
2. $V^{(n)} \simeq A$ and for all quotient W of V , $W^{(n)} = 0$ if and only if $W = 0$.
3. $V^{(n)} \simeq A$ and $\mathcal{I}(V)$ generates V as $A[\mathbf{G}_n]$ -module.

Proof. This is a reformulation of comparable results in [35, Section 6.3]. \square

Definition 6.5. *When A is a field, we say that a smooth, admissible $A[\mathbf{G}_n]$ -module V has essentially absolutely irreducible and generic dual if it satisfies one of the equivalent properties of proposition 6.4.*

We return to the general case where A is a complete, local, reduced noetherian $W(\mathbb{F})$ -algebra.

Definition 6.6. *An $A[\mathbf{G}_n]$ -module (π, V) is co-Whittaker if it satisfies the following conditions.*

1. The $A[\mathbf{G}_n]$ -module (π, V) is smooth, admissible.
2. The top Bernstein-Zelevinsky derivative $V^{(n)}$ is a free A -module of rank 1.
3. If \mathfrak{p} is a prime ideal in $\text{Spec } A$ with residual field $\kappa(\mathfrak{p})$, then $V \otimes_A \kappa(\mathfrak{p})$ has essentially absolutely irreducible and generic dual in the sense of definition 6.5.

We recall the following unicity result ([35, Theorem 6.2.1])

Proposition 6.7. *Let A be a reduced, complete, Noetherian, local p -torsion free $W(\mathbb{F})$ -algebra, with residue field \mathbb{F} , and let (V, ρ, A) be an $A[G_F]$ -module. Then there is, up to isomorphism, at most one admissible $A[G]$ -module ρ such that:*

1. The $A[\mathbf{G}_n]$ -module $\pi(\rho)$ is co-Whittaker and torsion-free as A -module.
2. For each minimal prime $\mathfrak{a} \in \text{Spec } A$, the $\kappa(\mathfrak{a})[\mathbf{G}_n]$ -module attached to $\rho \otimes_A \kappa(\mathfrak{a})$ through the generic Local Langlands Correspondence of [12] is isomorphic to $\pi(\rho) \otimes_A \kappa(\mathfrak{a})$.

Proof. Suppose $\pi(\rho)$ is such an $A[\mathbf{G}_n]$ -module. Then $\pi(\rho) \otimes_A A/\mathfrak{m}$ has essentially absolutely irreducible and generic dual so $\pi(\rho)$ satisfies all the conditions of [35, Theorem 6.2.1]. \square

6.2 Description of co-Whittaker torsion-modules for \mathbf{G}_2

Let A be a complete, local, noetherian domain of residual characteristic zero with fraction field K . Let

$$\rho : G_F \longrightarrow \text{GL}_2(A)$$

be a continuous Galois representation such that the attached Weil-Deligne representation is semisimple. Let \mathcal{N} be the monodromy operator acting on the Weil-Deligne representation attached to $\rho \otimes_A K$. Let $\lambda : A \longrightarrow \mathcal{O}$ be a local morphism of rings and let ρ_λ be the G_F -representation $\lambda \circ \rho$. Let \mathcal{N}_λ be the monodromy operator acting on the Weil-Deligne representation attached to $\rho_\lambda \otimes_{\mathcal{O}} E$. We assume that the monodromy filtration on $\rho_\lambda \otimes_{\mathcal{O}} E$ is equal to the monodromy filtration on $\rho \otimes_A K$ (this means that ρ is a minimal lift of ρ_λ in the sense of [35, Definition 4.5.9]).

Proposition 6.8. *There exist co-Whittaker $A[\mathbf{G}_2]$ modules $\pi(\rho)$ and $\pi(\rho_\lambda)$ attached to ρ and ρ_λ through the Local Langlands Correspondance in the sense of [35, Theorem 1.2.1]. Moreover, $\pi(\rho)$ and $\pi(\rho_\lambda)$ admit explicit descriptions directly parallel to the case of characteristic zero field coefficients. Namely, if S is either A or \mathcal{O} and π is either $\pi(\rho)$ or $\pi(\rho_\lambda)$, then:*

1. Either π is in the principal series: there exist two characters $\chi_i : F^\times \longrightarrow S^\times$ such that

$$\pi \simeq \text{Ind}_B^{\mathbf{G}_2}(\chi_1 \otimes \chi_2).$$

2. Or it is generically special Steinberg: there exists a character $\mu : F^\times \longrightarrow S^\times$ and a short exact sequence of $S[\mathbf{G}_2]$ -modules

$$0 \longrightarrow \mu \circ \det \longrightarrow \text{Ind}_B^{\mathbf{G}_2}(\mu \otimes \mu) \longrightarrow \pi \longrightarrow 0.$$

3. Or it is supercuspidal: there exists a subgroup J of \mathbf{G} which is compact modulo center, an $S[J]$ -module (σ, Λ) and a character (ϕ, S) of F^\times such that

$$\pi \simeq (\text{c-Ind}_J^{\mathbf{G}_2} \Lambda) \otimes (\phi \circ \det).$$

Moreover, $\pi(\rho) \otimes_{A,\lambda} \mathcal{O} \simeq \pi(\rho_\lambda)$.

Proof. This follows in much greater generality from the Local Langlands Correspondence in p -adic families conjectured in [35] and proven by D.Helm, and Helm-Moss in [52, 53, 54] (note that under our hypotheses, the automorphic representations attached by the Local Langlands Correspondence in p -adic families must be irreducible). However, under our specific setting, the proof is easy and presumably well-known. We briefly recall it.

Let L denote $\text{Frac}(S)$ and let $\pi \otimes L$ be the $L[\mathbf{G}_2]$ -module attached by the Local Langlands Correspondance to $\rho \otimes_A K$ if $S = A$ and to $\rho_\lambda \otimes_{\mathcal{O}} E$ if $S = \mathcal{O}$. Then $\pi \otimes L$ falls in exactly one of the three categories of the proposition and our hypothesis on the monodromy action on ρ_λ ensures that the category of $\pi \otimes L$ does not depend on whether S is equal to A

or to \mathcal{O} . Each of the $S[\mathbf{G}_2]$ -module (π, V) defined in the statement of the proposition is then by construction a smooth, admissible, $S[\mathbf{G}_2]$ -module which is torsion-free as S -module and such that $\pi \otimes_S L$ is irreducible and isomorphic to $\pi \otimes L$. As $V \otimes_S L$ is an irreducible $S[\mathbf{G}_2]$ -module and since $\mathcal{I}(V \otimes_S L) \simeq (V \otimes_S L)^{(2)} \simeq V^{(2)} \otimes_S L$, $\mathcal{I}(V \otimes_S L)$ generates V as $S[\mathbf{G}_2]$ -module if $V^{(2)}$ has positive S -rank. We compute $V^{(2)}$.

In the principal series case, $V^{(2)} \simeq \chi_1^{(1)} \otimes \chi_2^{(1)} \simeq S$ by proposition 6.3. In the generically special Steinberg case, $V^{(2)}$ is equal to $\Psi^- \circ \Phi^-(V)$. As ψ is a non-trivial character while the action of N_2 on $\mu \circ \det$ is trivial, $\Phi^-(\mu \circ \det)$ vanishes. As Φ^- is exact, $\Phi^-(\text{St } \mu)$ is equal to $\Phi^-(\text{Ind}_B^{\mathbf{G}_2} \mu \otimes \mu)$. Consequently, we may use again proposition 6.3 again to obtain

$$(\text{St } \mu)^{(2)} \simeq (\text{Ind}_B^{\mathbf{G}_2} \mu \otimes \mu)^{(2)} \simeq \mu^{(1)} \otimes_S \mu^{(1)} \simeq S.$$

Finally, if (π, V) is an $S[\mathbf{G}_2]$ -module as in the supercuspidal case, then $V \otimes_A \mathbb{F}$ is cuspidal, in particular absolutely irreducible. Then proposition 6.3 shows that $(V \otimes_S L)^{(2)}$ and $(V \otimes_S \mathbb{F})^{(2)}$ are vector spaces of dimension 1 over L and \mathbb{F} respectively. As $(-)^{(r)}$ commutes with arbitrary change of coefficients, $V^{(2)}$ is of dimension 1 after scalar extension to L and \mathbb{F} . It then follows from Nakayama's lemma that $V^{(2)}$ is a free A -module of rank 1.

In all three cases then, (π, V) is a co-Whittaker $S[\mathbf{G}_2]$ -module compatible with the Local Langlands Correspondence after extension of scalars to L . According to proposition 6.7, it must be the unique such $S[\mathbf{G}_2]$ -module. As $\pi(\rho) \otimes_{A,\lambda} \mathcal{O}$ also satisfies all these properties, there is an isomorphism $\pi(\rho) \otimes_{A,\lambda} \mathcal{O} \simeq \pi(\rho_\lambda)$. \square

Let $N(\rho)$ be the Artin conductor of ρ . By our hypothesis, this is also the Artin conductor of ρ_λ .

Proposition 6.9. *Let m be the ϖ -adic valuation of $N(\rho)$ and let $U \subset \mathbf{G}_2$ be the compact open subgroup*

$$U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_F) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\varpi^m} \right\}.$$

The A -module $\pi(\rho)_U$ and the \mathcal{O} -module $\pi(\rho_\lambda)_U$ are free of rank 1 and the natural map

$$\pi(\rho)_U \otimes_{A,\lambda} \mathcal{O} \longrightarrow \pi(\rho_\lambda)_U$$

is an isomorphism.

Proof. As the representation $\pi(\rho_\lambda)$ is co-Whittaker and has coefficients in \mathcal{O} , the $\mathcal{O}[\mathbf{G}_2]$ -module $\pi(\rho_\lambda^*) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{O}}(\pi(\rho_\lambda), \mathcal{O})$ is smooth, admissible and essentially absolutely irreducible and generic. By the theory of essential vectors ([18]), $(\pi(\rho_\lambda^*) \otimes_{\mathcal{O}} E)^U$ is free of dimension 1. As $\pi(\rho_\lambda^*)^U$ is \mathcal{O} -torsion free, it is a free \mathcal{O} -module of rank 1. By duality, we get that $\pi(\rho_\lambda)_U$ is \mathcal{O} -free of rank 1. As

$$\pi(\rho)_U \otimes_{A,\lambda} \mathcal{O} \simeq (\pi(\rho) \otimes_{A,\lambda} \mathcal{O})_U \simeq \pi(\rho_\lambda)_U,$$

and $\pi(\rho)_U \otimes_A \text{Frac}(A)$ is of rank 1, the A -module $\pi(\rho)_U$ is A -free of rank 1. \square

6.3 Completed cohomology

We return to the setting of the main text and in particular assume that $\mathfrak{m}_{\bar{\rho}}$ satisfies assumption 2.1.

In the following, the letter A stands for a ring which is equal either to \mathcal{O}/ϖ^s for some $s \geq 1$, or to \mathcal{O} or to E . Let U be an allowable subgroup attached to the maximal ideal $\mathfrak{m}_{\bar{\rho}}$. We assume that $U = U_p U_\Sigma U^\Sigma$ with U_p a compact open subgroup of $\text{GL}_2(\mathbb{Q}_p)$, U_Σ a compact open subgroup of $\prod_{\ell \in \Sigma^p} \text{GL}_2(\mathbb{Q}_\ell)$ and U^Σ a maximal compact open subgroup of $\text{GL}_2(\mathbb{A}_\mathbb{Q}^{(\Sigma^\infty)})$.

Then $H_{\text{et}}^i(U, A)_{\mathfrak{m}_{\bar{\rho}}}$ is an $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma(U) \otimes_{\mathcal{O}} A$ -module which we denote by $H_{\text{et}}^i(U, A)_{\mathfrak{m}_{\bar{\rho}}}$ for simplicity.

For $s \geq 1$, the completed cohomology $H_c^1(U^p, \mathcal{O}/\varpi^s)$ with compact support, tame level U_Σ and finite coefficients is defined as

$$\tilde{H}_c^1(U_\Sigma, \mathcal{O}/\varpi^s)_{\mathfrak{m}_{\bar{\rho}}} \stackrel{\text{def}}{=} \lim_{\substack{\longrightarrow \\ U_p}} H_c^1\left(U^\Sigma U_\Sigma U_p, \mathcal{O}/\varpi^s\right)_{\mathfrak{m}_{\bar{\rho}}}.$$

The completed cohomology with compact support, tame level U_Σ and coefficients in \mathcal{O} is defined as

$$\tilde{H}_c^1(U_\Sigma, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \stackrel{\text{def}}{=} \varprojlim_s \varinjlim_{U_p} H_c^1\left(U^\Sigma U_\Sigma U_p, \mathcal{O}/\varpi^s\right)_{\mathfrak{m}_{\bar{\rho}}}$$

and the completed cohomology with compact support, tame level U_Σ and coefficients in E is defined as

$$\tilde{H}_c^1(U_\Sigma, E)_{\mathfrak{m}_{\bar{\rho}}} \stackrel{\text{def}}{=} \left(\varprojlim_s \varinjlim_{U_p} H_c^1\left(U^\Sigma U_\Sigma U_p, \mathcal{O}/\varpi^s\right)_{\mathfrak{m}_{\bar{\rho}}} \right) \otimes_{\mathcal{O}} E.$$

In all three cases, $\tilde{H}_c(U_\Sigma, A)_{\mathfrak{m}_{\bar{\rho}}}$ is naturally a faithful $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma \otimes_{\mathcal{O}} A$ -modules endowed with an action of $G_{\mathbb{Q}, \Sigma}$, with an admissible action of $\mathrm{GL}_2(\mathbb{Q}_p)$ and with an action of $\mathrm{GL}_2(\mathbb{R})$ through the natural action of this latter group on $\mathbb{C} - \mathbb{R}$ (this action factors through the sign of the determinant).

The completed cohomology with compact support and coefficients in A is then the direct limit on all sufficiently small compact open subgroups

$$\tilde{H}_c^1(A)_{\mathfrak{m}_{\bar{\rho}}} \stackrel{\text{def}}{=} \varinjlim_{K \subset U_\Sigma} \tilde{H}_c^1(K, A)_{\mathfrak{m}_{\bar{\rho}}}.$$

Then $\tilde{H}_c^1(A)_{\mathfrak{m}_{\bar{\rho}}}$ is an admissible $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma [\prod_{\ell \in \Sigma^p} \mathrm{GL}_2(\mathbb{Q}_\ell)]$ -module and

$$\tilde{H}_c^1(A)_{\mathfrak{m}_{\bar{\rho}}}^K = \tilde{H}_c^1(K, A)_{\mathfrak{m}_{\bar{\rho}}}$$

for all compact open subgroup $K \subset \prod_{\ell \in \Sigma^p} \mathrm{GL}_2(\mathbb{Q}_\ell)$ contained in U_Σ ([33, Theorem 2.2.16]).

For all such $K \subset U_\Sigma$, we also consider

$$\tilde{H}_{\mathrm{et}}^1(\mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} = \varprojlim_{U' \subset U} H_{\mathrm{et}}^1(U', \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}, \quad \tilde{H}_{\mathrm{et}}^1(K, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} = \varprojlim_{U' \subset U_p} H_{\mathrm{et}}^1(U' K U^\Sigma, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}$$

where the inverse limits are respectively over all compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ which are maximal at all primes not in Σ and other all compact open subgroups of U_p . The \mathcal{O} -modules $\tilde{H}_{\mathrm{et}}^1(\mathcal{O})$ and $\tilde{H}_c^1(\mathcal{O})$ are then related by Poincarduality

$$\tilde{H}_{\mathrm{et}}^1(\mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \times \tilde{H}_c^1(\mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \xrightarrow{\sim} \mathcal{O}(-1). \tag{6.3.1}$$

In particular, $\tilde{H}_{\mathrm{et}}^1(K, \mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}$ is the module of K -coinvariants of $\tilde{H}_{\mathrm{et}}^1(\mathcal{O})_{\mathfrak{m}_{\bar{\rho}}}$.

The following combines the proof of the Local Langlands Correspondance in p -adic families ([52, 53, 54]) and [34, Theorem 6.2.13].

Theorem 6.10. [M.Emerton] Let $\pi_\ell(\rho_\Sigma|G_{\mathbb{Q}_\ell})$ be the continuous \mathcal{O} -dual of the co-Whittaker module $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma[\mathrm{GL}_2(\mathbb{Q}_\ell)]$ -module $\tilde{\pi}_\ell(\rho_\Sigma|G_{\mathbb{Q}_\ell})$ attached to $\rho_\Sigma|\mathbf{G}_{\mathbb{Q}_\ell}$ through the Local Langlands Correspondence. There is an isomorphism of $\mathbf{T}_{\mathfrak{m}_{\bar{\rho}}}^\Sigma[G_{\mathbb{Q}, \Sigma} \times \mathrm{GL}_2(\mathbb{Q}_p) \times \prod_{\ell \in \Sigma^p} \mathrm{GL}_2(\mathbb{Q}_\ell)]$ -modules

$$\tilde{H}_c^1(\mathcal{O})_{\mathfrak{m}_{\bar{\rho}}} \simeq \rho_\Sigma^*(1) \otimes \pi_p(\rho_\Sigma|G_{\mathbb{Q}_p}) \hat{\otimes} \bigotimes_{\ell \in \Sigma^p} \pi_\ell(\rho_\Sigma|G_{\mathbb{Q}_\ell}).$$

7 Appendix B: Iwasawa-Greenberg Main Conjecture

In this appendix, we prove theorem 4.41. We strongly advise the reader to look at [126, Introduction] for a concise outline of the argument proving this Greenberg main conjecture.

7.1 The Scalar Weight Case: Review

The idea in proving [20, Theorem 8.2.1] when f has weight two is roughly summarized as follows. We first construct families of Klingen Eisenstein series E_{Kling} on the unitary group $U(3, 1)$ using [32]. This corresponds to Eisenstein series induced from the Klingen parabolic subgroup of $U(3, 1)$. (The motivation for Klingen Eisenstein family is as an automorphic object corresponding to reducible family of Galois representations containing ρ_π as a direct summand.) The Hida theory developed in [20, Sections 2-4] (especially the fundamental exact sequence there) for semi-ordinary forms enables us to construct a family of cusp forms, which is congruent to the Klingen Eisenstein family modulo $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$. Then we proved there is a functional (constructed via Fourier-Jacobi expansion map) acting on the space of families of semi-ordinary forms on $U(3, 1)$, which maps E_{Kling} to an element which is a unit of the coefficient ring $\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]$, up to multiplying by an element in $\bar{\mathbb{Q}}_p^\times$. (*i.e.* Proposition 7.11.3 of *loc.cit.* This is the hard part of the whole argument). With this in hand, this functional and the cuspidal family we mentioned above gives a map from the cuspidal Hecke algebra to $\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]$ which, modulo $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$ gives the Hecke eigenvalues acting on the Klingen Eisenstein family (*i.e.* a congruence of Hecke eigenvalues between Eisenstein family and cusp forms). Passing to the Galois side, such congruence enables us to construct enough elements in the Selmer groups from the “lattice construction”, proving the lower bound of the Selmer group. (See the proof of [20, Theorem 8.2.1] and [126, Section 9.3]).

7.2 Vector Valued Cases

Now we return to the situation in this paper (*i.e.* general weight). All ingredients are available, except that we need some new idea in the vector valued case to construct and study the corresponding functional on semi-ordinary forms on $U(3, 1)$ using Fourier-Jacobi expansion map, so that its value on the Klingen Eisenstein family is an element in $\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]^\times$, up to multiplying by a non-zero constant in $\bar{\mathbb{Q}}_p^\times$ (*i.e.* primitivity of the Klingen Eisenstein family constructed). In this section we present the whole argument for the entirety of the logic, be brief and refer to the specific part of [20] or [126] for parts which are completely the same as *loc.cit.*, and explain full details for the new ingredients (Archimedean argument involving Ikeda’s theory), which are mainly in Subsection 7.4.2.

7.3 Unitary groups and Hida Theory for Semi-Ordinary Forms

In [20] we developed Hida theory assuming the weight of f is two for ease of presentation (as only scalar weight forms are needed there and this was enough for the application there). See Proposition 2.9.1 and Remark 2.9.2 of *op.cit.*. Here for completeness we briefly develop the Hida theory needed here for general weight.

7.3.1 Unitary Groups

Define $G_n = \text{GU}(n, n)$ for the unitary similitude group for the skew-Hermitian matrix $\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$ and $U(n, n)$ for the corresponding unitary group.

Let $\delta \in \mathcal{K}$ be a totally imaginary element such that $-i\delta$ is positive, and $d = \text{Nm}(\delta)$ is a p -adic unit. Let $U(2) = U(2, 0)$ (resp. $\text{GU}(2) = \text{GU}(2, 0)$) be the unitary group (resp. unitary similitude group) associated to the skew-Hermitian matrix $\zeta = \begin{pmatrix} \mathfrak{s}\delta & \\ & \delta \end{pmatrix}$ for some $\mathfrak{s} \in \mathbb{Z}_+$ prime to p . More precisely $\text{GU}(2)$ is the group scheme over \mathbb{Z} defined by: for any \mathbb{Z} -algebra A ,

$$\text{GU}(2)(A) = \{g \in \text{GL}_2(A \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{K}}) \mid {}^t \bar{g} \zeta g = \lambda(g)\zeta, \lambda(g) \in A^\times\}.$$

In application below we are going to choose the groups as in [20, Section 5.4]. The map $\lambda : \text{GU}(2) \rightarrow \mathbb{G}_m$, $g \mapsto \lambda(g)$ is called the similitude character and $U(2) \subseteq \text{GU}(2)$ is the kernel of μ . Let W be the corresponding Hermitian space over \mathcal{K} and fix a lattice $L \subset W$ over $\mathcal{O}_{\mathcal{K}}$ such that $\text{Tr}_{\mathcal{K}/\mathbb{Q}}(L, L) \subset \mathbb{Z}$. Let $G = \text{GU}(3, 1)$ (resp. $U(3, 1)$) be the similarly defined unitary similitude group (resp. unitary group) over \mathbb{Z} associated to the skew-Hermitian

matrix $\begin{pmatrix} & & 1 \\ & \zeta & \\ -1 & & \end{pmatrix}$. Let $P \subseteq G$ be the parabolic subgroup of $\mathrm{GU}(3, 1)$ consisting of those matrices in G of the form $\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & \times \end{pmatrix}$. Let N_P be the unipotent radical of P . Then

$$M_P \stackrel{\text{def}}{=} \mathrm{GL}(X_{\mathcal{K}}) \times \mathrm{GU}(2) \hookrightarrow \mathrm{GU}(V), \quad (a, g_1) \mapsto \mathrm{diag}(a, g_1, \mu(g_1)\bar{a}^{-1})$$

is the Levi subgroup. Let $G_P \stackrel{\text{def}}{=} \mathrm{GU}(2)(\subseteq M_P)$ be $\mathrm{diag}(1, g_1, \lambda(g_1))$. Let δ_P be the modulus character for P . We usually use a more convenient character δ' such that $\delta'^3 = \delta_P$.

Since p splits as $v_0\bar{v}_0$ in \mathcal{K} , $\mathrm{GL}_4(\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{v_0}}) \times \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{\bar{v}_0}})$. Here $\mathrm{U}(3, 1)(\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{v_0}}) = \mathrm{GL}_4(\mathbb{Z}_p)$ with the projection onto the first factor. Let B and N be the upper triangular Borel subgroup of GL_4 and its unipotent radical, respectively. Let $K_p = \mathrm{GU}(3, 1)(\mathbb{Z}_p) \simeq \mathrm{GL}_4(\mathbb{Z}_p)$, and for any $n \geq 1$ let K_0^n be the subgroup of K consisting of matrices which are upper-triangular modulo p^n . Let $K_1^n \subset K_0^n$ be the subgroup of matrices whose diagonal elements are 1 modulo p^n .

Definition 7.1. A weight \underline{k} is defined to be an $(r+s)$ -tuple

$$\underline{k} = (a_1, \dots, a_r; b_1, \dots, b_s) \in \mathbb{Z}^{r+s}$$

with $a_1 \geq \dots \geq a_r \geq -b_1 \geq \dots \geq -b_s$.

We refer to [57, Section 3.1] for the definition of the algebraic representation $L_{\underline{k}}$ of H with the action denoted by $\rho_{\underline{k}}$ and define a model $L_{\underline{k}}^k$ of the representation H with the highest weight \underline{k} as follows. The underlying space of $L_{\underline{k}}^k$ is $L_{\underline{k}}$ and the group action is defined by

$$\rho_{\underline{k}}^k(h) = \rho_{\underline{k}}({}^t h^{-1}), \quad h \in H.$$

In [57, Section 3.1] also defined a distinguished functional $l_{\underline{k}} : L_{\underline{k}}(R) \rightarrow R$ for any ring R . We refer to [57, Section 3.4] for the notion of holomorphic automorphic forms of weight \underline{k} , and to [57, Definition 3.2] for the automorphic sheaf $\omega_{\underline{k}}$ of weight \underline{k} .

7.3.2 Shimura varieties for Unitary Similitude Groups

In the following we follow closely [57, Section 2, 3] and refer to some of the details there. We consider the group $\mathrm{GU}(3, 1)$. For any open compact subgroup $K = K_p K^p$ of $\mathrm{GU}(3, 1)(\mathbb{A}_f)$ whose p -component is $K_p = \mathrm{GU}(3, 1)(\mathbb{Z}_p)$ and whose prime-to- p component is K^p , we refer to [57, Section 2.1] for the definition and arithmetic models of the associated Shimura variety, which we denote as $S_G(K)_{/\mathcal{O}_{\mathcal{K}, (v_0)}}$. The scheme $S_G(K)$ represents the following functor \underline{A} : for any $\mathcal{O}_{\mathcal{K}, (v_0)}$ -algebra R , $\underline{A}(R) = \{(A, \bar{\lambda}, \iota, \bar{\eta}^p)\}$ where A is an abelian scheme over R with CM by $\mathcal{O}_{\mathcal{K}}$ given by ι , $\bar{\lambda}$ is an orbit of prime-to- p polarizations and $\bar{\eta}^p$ is an orbit of prime-to- p level structures. We denote $\bar{S}_G(K)$ a smooth toroidal compactification and $S_G^*(K)$ the minimal compactification. We refer to [57, Section 2.7] for details. The boundary components of $S_G^*(K)$ are in one-to-one correspondence with the set of cusp labels defined below. For $K = K_p K^p$ as above we define the set of cusp labels to be:

$$C(K) \stackrel{\text{def}}{=} (\mathrm{GL}(X_{\mathcal{K}}) \times G_P(\mathbb{A}_f)) N_P(\mathbb{A}_f) \backslash G(\mathbb{A}_f) / K.$$

This is a finite set. We denote by $[g]$ the class represented by $g \in G(\mathbb{A}_f)$. For each such g whose p -component is 1 we define $K_P^g = G_P(\mathbb{A}_f) \cap gKg^{-1}$ and denote $S_{[g]} \stackrel{\text{def}}{=} S_{G_P}(K_P^g)$ the corresponding Shimura variety for the group G_P with level group K_P^g . By strong approximation we can choose a set $\underline{C}(K)$ of representatives of $C(K)$ consisting of elements $g = pk^0$ for $p \in P(\mathbb{A}_f^\Sigma)$ and $k^0 \in K^0$ for K^0 the maximal compact subgroup of $G(\mathbb{A}_f)$ defined in [57, Section 1.10].

7.3.3 Igusa varieties and p -adic automorphic forms

Now we recall briefly the notion of Igusa varieties in [57, Section 2.3]. Let M be the standard lattice of V and $M_p = M \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let $\text{Pol}_p = \{N^{-1}, N^0\}$ be a polarization of M_p . Recall this means that if N^{-1} and N^0 are maximal isotropic $\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p$ -submodules in M_p , that they are dual to each other with respect to the Hermitian metric on V , and also that:

$$\text{rank}_{\mathbb{Z}_p} N_{v_0}^{-1} = \text{rank}_{\mathbb{Z}_p} N_{\bar{v}_0}^0 = 3, \quad \text{rank}_{\mathbb{Z}_p} N_{\bar{v}_0}^{-1} = \text{rank}_{\mathbb{Z}_p} N_{v_0}^0 = 1.$$

We mainly follow [57, Section 2.3] in this subsection. The Igusa variety of level p^n and tame level K is the scheme over $\mathcal{O}_{\mathcal{K},(v_0)}$ representing the quadruple $\underline{\mathcal{A}}(R) = \{(A, \bar{\lambda}, \iota, \bar{\eta}^p)\}$ for the Shimura variety of $\text{GU}(3, 1)$ as above, together with an injection of group schemes

$$j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$$

over R which is compatible with the $\mathcal{O}_{\mathcal{K}}$ -action on both sides. Note that the existence of j implies that A must be ordinary along the special fiber. There is also a theory of Igusa varieties over $\bar{S}_G(K)$. As in *loc.cit.* let $\bar{H}_{p-1} \in H^0(S_G(K)/\bar{\mathbb{F}}, \det(\underline{\omega})^{p-1})$ be the Hasse invariant. Over the minimal compactification some power (say the t -th) of the Hasse invariant can be lifted to \mathcal{O}_{v_0} . We denote such a lift by E . By the Koecher principle we can regard E as in $H^0(\bar{S}_G(K), \det(\underline{\omega}^{t(p-1)}))$. Let $\mathcal{O}_m \stackrel{\text{def}}{=} \mathcal{O}_{\mathcal{K},v_0}/p^m \mathcal{O}_{\mathcal{K},v_0}$. Set $T_{0,m} \stackrel{\text{def}}{=} \bar{S}_G(K)[1/E]_{/\mathcal{O}_m}$. For any positive integer n define $T_{n,m} \stackrel{\text{def}}{=} I_G(K^n)_{/\mathcal{O}_m}$ and $T_{\infty,m} = \varprojlim_n T_{n,m}$. Then $T_{\infty,m}$ is a Galois cover over $T_{0,m}$ with Galois group $\mathbf{H} \simeq \text{GL}_3(\mathbb{Z}_p) \times \text{GL}_1(\mathbb{Z}_p)$. Let $\mathbf{N} \subset \mathbf{H}$ be the upper triangular unipotent radical. Define:

$$V_{n,m} = H^0(T_{n,m}, \mathcal{O}_{T_{n,m}}).$$

Let $V_{\infty,m} = \varinjlim_n V_{n,m}$ and $V_{\infty,\infty} = \varprojlim_m V_{\infty,m}$ be the space of p -adic automorphic forms on $\text{GU}(3, 1)$ with level group K . We also define $W_{n,m} = V_{n,m}^{\mathbf{N}}$, $W_{\infty,m} = V_{\infty,m}^{\mathbf{N}}$ and $\mathcal{W} = \varinjlim_m \varinjlim_n W_{n,m}$. We define $V_{n,m}^0$, etc, to be the cuspidal part of the corresponding spaces.

We can make similar definitions for the definite unitary similitude groups G_P as well and define $V_{n,m,P}, V_{\infty,m,P}, V_{\infty,\infty,P}, V_{n,m,P}^{\mathbf{N}}, \mathcal{W}_P$, etc.

Let K_0^n and K_1^n be the subgroup of \mathbf{H} consisting of matrices which are in $B_3 \times {}^t B_1$ or $N_3 \times {}^t N_1$ modulo p^n . (These notations are already used for level groups of automorphic forms. The reason for using the same notation here is that automorphic forms with level group K_{\bullet}^n are p -adic automorphic forms of level group K_{\bullet}^n). We sometimes denote $I_G(K_1^n) = I_G(K^n)^{K_1^n}$ and $I_G(K_0^n) = I_G(K^n)^{K_0^n}$. We define

$$M_{\underline{k}}(K_{\bullet}^n, R) \stackrel{\text{def}}{=} H^0(I_G(K_{\bullet}^n)_{/R}, \omega_{\underline{k}}).$$

We can define Igusa varieties for G_P as well. For $\bullet = 0, 1$ we let $K_{P,\bullet}^{g,n} \stackrel{\text{def}}{=} g K_{\bullet}^n g^{-1} \cap G_P(\mathbb{A}_f)$ and let $I_{[g]}(K_{\bullet}^n) \stackrel{\text{def}}{=} I_{G_P}(K_{P,\bullet}^{g,n})$ be the corresponding Igusa variety over $S_{[g]}$. We denote the coordinate ring of $I_{[g]}(K_1^n)$ over \mathcal{O}_m by $A_{[g],m}^n$. Let $A_{[g],m}^{\infty} = \varinjlim_n A_{[g],m}^n$ and let $\hat{A}_{[g]}^{\infty}$ be the p -adic completion of $A_{[g],m}^{\infty}$. This is the space of p -adic automorphic forms for the group $\text{GU}(2, 0)$ of level group $g K g^{-1} \cap G_P(\mathbb{A}_f)$.

For Unitary Groups

Assume the tame level group K is neat. For any c an element in $\mathbb{Q}_+ \setminus \mathbb{A}_{\mathbb{Q},f}^{\times} / \mu(K)$, we refer to [57, 2.5] for the notion of c -Igusa schemes $I_{\text{U}(2)}^0(K, c)$ for the unitary groups $\text{U}(2, 0)$ (not the similitude group). It parameterizes quintuples $(A, \lambda, \iota, \bar{\eta}^{(p)}, j)_{/S}$ similar to the Igusa schemes for unitary similitude groups but requires λ to be a prime to p c -polarization of A such that $(A, \bar{\lambda}, \iota, \bar{\eta}^{(p)}, j)$ is a quintuple as in the definition of Shimura varieties for $\text{GU}(2)$. Let g_c be such that $\mu(g_c) \in \mathbb{A}_{\mathbb{Q}}^{\times}$ is in the class of c . Let ${}^c K = g_c K g_c^{-1} \cap U(2)(\mathbb{A}_{\mathbb{Q},f})$. Then the space $I_{\text{U}(2)}^0(K, c)$ is isomorphic to the space of forms on $I_{\text{U}(2)}^0({}^c K, 1)$ (see *loc.cit.*).

Fourier-Jacobi Expansions

Define $N_H^1 \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & 0 \\ * & 1_2 \end{pmatrix} \right\} \times \{1\} \subset H$. For an automorphic form or p -adic automorphic form F on $\mathrm{GU}(3, 1)$ we refer to [32, Section 2.8] for the notion of analytic Fourier-Jacobi expansions

$$FJ_P(g, f) = a_0(g, f) + \sum_{\beta} a_{\beta}(y, g, f)q^{\beta}$$

at $g \in \mathrm{GU}(3, 1)(\mathbb{A}_{\mathbb{Q}})$ for $a_{\beta}(-, g, f) : \mathbb{C}^2 \rightarrow L_{\mathbb{k}}(\mathbb{C})$ being theta functions with complex multiplication. Also there is an algebraic Fourier-Jacobi expansion

$$FJ_{[g]}^h(f)_{N_H^1} = \sum_{\beta} a_{[g]}^h(\beta, f)q^{\beta},$$

at a p -adic cusp $([g], h)$, and $a_{[g]}^h(\beta, f) \in L_{\mathbb{k}}(A_{[g]}^{\infty})_{N_H^1} \otimes_{A_{[g]}} H^0(\mathcal{Z}_{[g]}, \mathcal{L}(\beta))$ (see [57, (3.9)]. Note that the subscript N_H^1 is important to take it out of the H^0). We define the Siegel operator to be taking the 0-th Fourier-Jacobi coefficient as in *loc.cit.*. Over \mathbb{C} the analytic Fourier-Jacobi expansion for a holomorphic automorphic form f is given by:

$$FJ_{\beta}(f, g) = a_{\beta}(y, g, f) = \int_{\mathbb{Q} \setminus \mathbb{A}} f\left(\begin{pmatrix} 1 & & n \\ & 1_2 & \\ & & 1 \end{pmatrix} g\right) e_{\mathbb{A}}(-\beta n) dn.$$

7.3.4 Semi-Ordinary Forms

In this subsection we develop a theory for families of “semi-ordinary” forms over a two dimensional weight space (the whole weight space for $\mathrm{U}(3, 1)$ is three dimensional). The idea goes back to the work of Hida (also Tilouine-Urban for $\mathrm{GSp}(4)$) who they defined the concept of being ordinary with respect to general parabolic subgroups (the usual definition of ordinary is with respect to the Borel subgroup), except that we are working with coherent cohomology while Hida and Tilouine-Urban use cohomology of arithmetic groups. In our case it means being ordinary with respect to the parabolic subgroup of $\mathrm{GL}_4(\mathbb{Q}_p) \cong \mathrm{U}(3, 1)(\mathbb{Q}_p)$

consisting of matrices of the form $\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$. The crucial point is that our families

are defined over the two dimensional *Iwasawa algebra*, as Hida theory for ordinary forms instead of Coleman-Mazur theory for finite slope forms, which is over some affinoid domain. Our argument here will sometimes be an adaption of the argument in the ordinary case in [57] and we will sometimes be brief and refer to *loc.cit.* for some computations so as not to introduce too many notations.

We always use the identification $\mathrm{U}(3, 1)(\mathbb{Q}_p) \simeq \mathrm{GL}_4(\mathbb{Q}_p)$. Define $\alpha_i = \mathrm{diag}(1_{4-i}, p \cdot 1_i)$. We

let $\alpha = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p^2 \end{pmatrix}$ and refer to [57, 3.7, 3.8] for the notion of Hida’s U_{α} and U_{α_i}

operators associated to α or α_i . We define $e_{\alpha} = \lim_{n \rightarrow \infty} U_{\alpha}^{n!}$. (That this is well-defined follows as in [57, Section 4.3].) We are going to study forms and families invariant under e_{α} and call them “semi-ordinary” forms. Suppose π is an irreducible automorphic representation on $\mathrm{U}(3, 1)$ with weight \underline{k} and suppose that π_p is an unramified principal series representation. If we write $\kappa_1 = b_1$ and $\kappa_i = -a_{5-i} + 5 - i$ for $2 \leq i \leq 4$, then there is a semi-ordinary vector in π if and only if we can re-order the Satake parameters as $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$\mathrm{val}_p(\lambda_3) = \kappa_3 - \frac{3}{2}, \quad \mathrm{val}_p(\lambda_4) = \kappa_4 - \frac{3}{2}.$$

7.3.5 Control Theorems

We define $K_0(p, p^n) = \prod_{\ell \neq p} K_\ell \times K_0(p, p^n)_p$, for $K_0(p, p^n)_p$ consisting of matrices which are of the form $\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$ modulo p and are of the form $\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$ modulo p^n .

We are going to prove some control theorems for the level group $K_0(p, p^n)$. We also define a GL_2 level group $K'_0(p) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ to be the set of matrices congruent to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ modulo p . Let N' be the set of matrices $\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}$. For the definition of the automorphic sheaves $\omega_{\underline{k}}$ of weight \underline{k} we refer to [57, section 3.2]. There also defined a subsheaf $\omega_{\underline{k}}^b$ in Section 4.1 of *loc.cit.* as follows. Let $\mathcal{D} = \bar{S}_G(K) - S_G(K)$ be the boundary of the toroidal compactification and $\underline{\omega}$ the pullback to identity of the relative differential of the Raynaud extension of the universal Abelian variety. Let $\underline{k}'' = (a_1 - a_3, a_2 - a_3)$. Let \mathcal{B} be the abelian part of the Mumford family of the boundary. Its relative differential is identified with a subsheaf of $\underline{\omega}|_{\mathcal{D}}$. The $\omega_{\underline{k}}^b \subset \omega_{\underline{k}}$ is defined to be $\{s \in \omega_{\underline{k}}, s|_{\mathcal{D}} \in \mathcal{F}_{\mathcal{D}}\}$ for $\mathcal{F}_{\mathcal{D}} \stackrel{\text{def}}{=} \det(\underline{\omega}|_{\mathcal{D}})^{a_3} \otimes \underline{\omega}_{\mathcal{B}}^{k''}$, where the last term means the automorphic sheaf of weight \underline{k}'' for $\mathrm{GU}(2, 0)$ (see [57, Section 4.1]).

Weight Space

Let $H = \mathrm{GL}_3 \times \mathrm{GL}_1$ and T be the diagonal torus. Then $\mathbf{H} = H(\mathbb{Z}_p)$. We let $\Lambda_{3,1} = \Lambda$ be the completed group algebra $\mathbb{Z}_p[[T(1 + p\mathbb{Z}_p)]]$. This is a formal power series ring with four variables. There is an action of $T(\mathbb{Z}_p)$ given by the action on the $j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$. (see [57, 3.4]) This gives the space of p -adic modular forms a structure of Λ -algebra. A \mathbb{Q}_p -point ϕ of $\mathrm{Spec}\Lambda$ is called arithmetic if it is determined by a character $[\underline{k}].[\zeta]$ of $T(1 + p\mathbb{Z}_p)$ where \underline{k} is a weight and $\zeta = (\zeta_1, \zeta_2, \zeta_3; \zeta_4)$ for $\zeta_i \in \mu_{p^\infty}$. Here $[\underline{k}]$ is the character of $T(1 + \mathbb{Z}_p)$ by $[\underline{k}](t_1, t_2, t_3, t_4) = (t_1^{a_1} t_2^{a_2} t_3^{a_3} t_4^{-b_1})$ and $[\zeta]$ is the finite order character given by mapping $(1 + p\mathbb{Z}_p)$ to ζ_i at the corresponding entry t_i of $T(\mathbb{Z}_p)$. We often write this point \underline{k}_ζ . We also define $\omega^{[\underline{k}]}$ a character of the torsion part of $T(\mathbb{Z}_p)$ (isomorphic to $(\mathbb{F}_p^\times)^4$) given by $\omega^{[\underline{k}]}(t_1, t_2, t_3, t_4) = \omega(t_1^{a_1} t_2^{a_2} t_3^{a_3} t_4^{-b_1})$.

Definition 7.2. We fix $\underline{k}' = (a_1, a_2)$ and $\rho = L_{\underline{k}'}$. Let \mathcal{X}_ρ be the set of arithmetic points $\phi \in \mathrm{Spec}\Lambda_{3,1}$ corresponding to the weight $(a_1, a_2, a_3; b_1)$ such that $a_1 \geq a_2 \geq a_3 \geq -b_1 + 4$. (The ζ -part is trivial). Let $\mathrm{Spec}\tilde{\Lambda} = \mathrm{Spec}\tilde{\Lambda}_{(a_1, a_2)}$ be the Zariski closure of \mathcal{X}_ρ .

We define for $q = 0, b$

$$V_{\underline{k}}^q(K_0(p, p^n), \mathcal{O}_m) \stackrel{\text{def}}{=} H^0(T_{n,m}, \omega_{\underline{k}}^q)^{K_0(p, p^n) \cap \mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)}.$$

As in [57, 3.3] we have a canonical isomorphism given by taking the “ p -adic avartar”

$$H^0(T_{n,m}, \omega_{\underline{k}}) \simeq V_{n,m} \otimes L_{\underline{k}}, f \mapsto \hat{f}$$

and $\beta_{\underline{k}} : V_{\underline{k}}(K_1^n, \mathcal{O}_m) \rightarrow V_{n,m}^N$ by $f \mapsto \beta_{\underline{k}}(f) \stackrel{\text{def}}{=} l_{\underline{k}}(\hat{f})$. The following lemma is [57, lemma 4.2].

Lemma 7.3. Let $q \in \{0, b\}$ and let $V_{\underline{k}}^q(K_0(p, p^n), \mathcal{O}_m) \stackrel{\text{def}}{=} H^0(T_{n,m}, \omega_{\underline{k}}^q)^{K_0(p, p^n)}$. Then we have

$$H^0(I_G(K_0(p, p^n))[1/E], \omega_{\underline{k}}^q) \otimes \mathcal{O}_m = V_{\underline{k}}^q(K_0(p, p^n), \mathcal{O}_m).$$

We record a contraction property for the operator U_α .

Lemma 7.4. If $n > 1$, then we have

$$U_\alpha \cdot V_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) \subset V_{\underline{k}}(K_0(p, p^{n-1}), \mathcal{O}_m).$$

The proof is the same as [57, Proposition 4.4]. The following proposition follows from the contraction property for e_α :

Proposition 7.5.

$$e_\alpha V_{\underline{k}}^q(K_0(p, p^n), \mathcal{O}_m) = e_\alpha V_{\underline{k}}(K_0(p), \mathcal{O}_m).$$

The following lemma tells us that to study semi-ordinary forms one only needs to look at the sheaf $\omega_{\underline{k}}^{\flat}$.

Lemma 7.6. *Let $n \geq m > 0$, then*

$$e_{\alpha} \cdot V_{\underline{k}}^{\flat}(K_0(p, p^n), \mathcal{O}_m) = e_{\alpha} \cdot V_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m).$$

Proof. Same as [57, lemma 4.10]. \square

Similar to the $\beta_{\underline{k}}$ we define a more general $\beta_{\underline{k}, \rho}$ as follows: Let ρ be the algebraic representation $L_{\rho} = L_{\underline{k}'}$ of GL_2 with lowest weight $-\underline{k}' = (-a_1, -a_2)$. We identify $L_{\underline{k}}$ with the algebraically induced representation $\mathrm{Ind}_{\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1}^{\mathrm{GL}_3 \times \mathrm{GL}_1} \rho \otimes \chi_{a_3} \otimes \chi_{b_1}$ (χ_a means the algebraic character defined by taking the $(-a)$ -th power). We define the functional $l_{\underline{k}, \rho} : L_{\underline{k}} \rightarrow L_{\underline{k}'}$ by evaluating at identity (similar to the definition of $l_{\underline{k}}$). We define $\beta_{\underline{k}, \rho}$ similar to $\beta_{\underline{k}}$ but replacing $l_{\underline{k}}$ by $l_{\underline{k}, \rho}$.

Proposition 7.7. *If $n \geq m > 0$, then the morphism*

$$\beta_{\underline{k}, \rho} : V_{\underline{k}}(K_1(p^n), \mathcal{O}_m) \rightarrow (V_{n, m} \otimes L_{\rho})^{N'}$$

is U_{α} -equivariant (here $N' \subset \mathrm{GL}_2(\mathbb{Z}_p)$ is embedded into $\mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$ as $\mathrm{diag}(N', 1, 1)$), and there is a Hecke-equivariant homomorphism $s_{\underline{k}, \rho} : (V_{n, m} \otimes L_{\rho})^{N'} \rightarrow V_{\underline{k}}(K_1(p^n), \mathcal{O}_m)$ such that $\beta_{\underline{k}, \rho} \circ s_{\underline{k}, \rho} = U_{\alpha}^m$ and $s_{\underline{k}, \rho} \circ \beta_{\underline{k}, \rho} = U_{\alpha}^m$. So the kernel and the cokernel of $\beta_{\underline{k}, \rho}$ are annihilated by U_{α}^m .

Proof. We follow [57, Proposition 4.7]. Our $s_{\underline{k}, \rho}$ is defined as follows: for (\underline{A}, \bar{j}) over a \mathcal{O}_m -algebra R ,

$$s_{\underline{k}, \rho}(\alpha^m) f(\underline{A}, \bar{j}) \stackrel{\text{def}}{=} \sum_{v_{\chi'} \in \rho \otimes \chi_{a_3} \otimes \chi_{b_1}} \sum_u \frac{1}{\chi_{r,1}(\alpha^m)} \cdot \mathrm{Tr}_{R_0^{\alpha^m u}/R}(f(\underline{A}_{\alpha^m u} \cdot j_{\alpha^m u})) \rho_{\underline{k}}(u) v_{\chi'}.$$

Here the character $\chi_{r,1}$ is defined by

$$\chi_{r,1}(\mathrm{diag}(a_1, a_2, a_3; d)) \stackrel{\text{def}}{=} (a_1 a_2 a_3)^{-1} d.$$

The $v_{\chi'}$'s form a basis of the representation $\rho \otimes \chi_{a_3} \otimes \chi_{b_1}$ which are eigenvectors for the diagonal torus action with eigenvalues χ' 's (the eigenvalues appear with multiplicity one so we use the subscript χ' to denote the corresponding vector). The u runs over a set of representatives of

$$\alpha^{-m} N_H(\mathbb{Z}_p) \alpha^m \cap N_H(\mathbb{Z}_p) \setminus N_H(\mathbb{Z}_p).$$

The $(\underline{A}_{\alpha u}, j_{\alpha u})$ is a certain pair with $\underline{A}_{\alpha u}$ an abelian variety admitting an isogeny to \underline{A} of type α (see [57, 3.7.1] for details) and $R_0^{\alpha u}/R$ being the coordinate ring for $(\underline{A}_{\alpha u}, j_{\alpha u})$ (see 3.8.1 of *loc.cit.*). Note that the twisted action of

$$\tilde{\rho}_{\underline{k}}(\alpha^{-1}) v_{\chi'} \stackrel{\text{def}}{=} p^{-\langle \mu, \underline{k} + \chi' \rangle} v_{\chi'}$$

satisfies $\tilde{\rho}_{\underline{k}}(\alpha^{-1}) v_{\chi'} = 1$ for all the χ' above. Write χ for $\chi_{a_3} \otimes \chi_{b_1}$. Note also that for any eigenvector $v_{\chi'} \in \mathrm{Ind}_{\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1}^{\mathrm{GL}_3 \times \mathrm{GL}_1} \rho \otimes \chi$ for the torus action such that $v_{\chi'} \notin \rho \otimes \chi$, and $\mu \in X_*(T)$ (the co-character group) with $\mu(p) = \alpha$, we have $\langle \mu, \underline{k} + \chi' \rangle < 0$. By the definition of $U_{\alpha}^m = U_{\alpha^m}$, if $f = \sum_{\chi} g_{\chi} \otimes v_{\chi}$, then

$$\begin{aligned} U_{\alpha^m} \cdot f(\underline{A}, j) &= \sum_{v_{\chi'} \in \rho \otimes \chi} s_{\underline{k}, \rho}(\alpha^m) g_{\chi'}(\underline{A}, j) \\ &\quad + \sum_{v_{\chi'} \notin \rho \otimes \chi} p^{-\langle m\mu, \underline{k} + \chi' \rangle} \frac{1}{\chi_{r,1}(\alpha^m)} \mathrm{Tr}_{R_0^{\alpha^m u}/R}(g_{\chi'}(\underline{A}_{\alpha^m u}, j)) \otimes \rho_{\underline{k}}(u) v_{\chi'}. \end{aligned}$$

For the notation $R_0^{\alpha^m u}$ see [57, 3.8.1] for an explanation. So $\beta_{\underline{k}, \rho} \circ s_{\underline{k}, \rho}(\alpha^m) = U_{\alpha^m}$ and $s_{\underline{k}, \rho}(\alpha^m) \circ \beta_{\underline{k}, \rho} = U_{\alpha^m}$. Taking $s_{\underline{k}, \rho} \stackrel{\text{def}}{=} s_{\underline{k}, \rho}(\alpha^m)$, then we proved the proposition. \square

The next proposition follows from the above one as [57, Proposition 4.9]. Let \underline{k} and ρ be as before.

Proposition 7.8. *If $n \geq m > 0$, then there is an isomorphism*

$$\beta_{\underline{k}, \rho} : e_\alpha \cdot V_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) \simeq e_\alpha(V_{n,m} \otimes L_\rho)^{K'_0(p)}[\underline{k}].$$

Here the k in $[\underline{k}]$ is regarded as a character of $\text{diag}(1, 1, \mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$.

We are going to prove some control theorems and fundamental exact sequence for semi-ordinary forms along this two-dimensional weight space $\text{Spec } \tilde{\Lambda}$. The following proposition follows from Lemma 7.3 and Proposition 7.5 in the same way as [57, Lemma 4.10, Proposition 4.11], noting that the level group is actually in $K_0(p)$ by the contraction property.

Proposition 7.9. *Let $e_\alpha \cdot \mathcal{V}_{\underline{k}}(K_0(p, p^n)) \stackrel{\text{def}}{=} \varinjlim_m e_\alpha \cdot V_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m)$. Then $e_\alpha \cdot \mathcal{V}(K_0(p, p^n))$ is p -divisible and*

$$e_\alpha \cdot \mathcal{V}_{\underline{k}}(K_0(p, p^n))[p^m] = e_\alpha \cdot V_{\underline{k}}(K_0(p, p^n), \mathcal{O}_m) = e_\alpha \cdot H^0(I_G(K_1^n)[1/E], \omega_{\underline{k}}) \otimes \mathcal{O}_m.$$

The next proposition is crucial to prove control theorems for semi-ordinary forms along the weight space $\text{Spec } \tilde{\Lambda}$.

Proposition 7.10. *The dimensions of the spaces $e_\alpha M_{\underline{k}}(K_0(p, p^n), \mathbb{C})$ are uniformly bounded for all $\underline{k} \in \mathcal{X}_\rho$.*

Proof. This is [20, Proposition 3.0.5]. □

The following theorem says that all semi-ordinary forms of sufficiently regular weights are classical, and can be proved in the same way as [57, Theorem 4.19] using Proposition 7.10.

Theorem 7.11. *For each weight $\underline{k} = (a_1, a_2, a_3; b_1) \in \mathcal{X}_\rho$, there is a positive integer $A(\underline{a})$ depending on $\underline{a} = (a_1, a_2, a_3)$ such that if $b_1 > A(\underline{a}, n)$ then the natural restriction map*

$$e_\alpha M_{\underline{k}}(K_0(p), \mathcal{O}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \simeq e_\alpha \cdot \mathcal{V}_{\underline{k}}(K_0(p))$$

is an isomorphism.

For $q = 0$ define \mathcal{W}^q as \mathcal{W} but with the structure sheaf replaced by its cuspidal part. For $q = 0$ or \emptyset define the space of $\tilde{\Lambda}$ -adic semi-ordinary forms

$$\begin{aligned} V_{\text{so}}^q &\stackrel{\text{def}}{=} \text{Hom}(e_\alpha \cdot (\mathcal{W}^q \otimes L_\rho)^{K'_0(p)}, \mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\Lambda_{3,1}} \tilde{\Lambda} \\ \mathcal{M}_{\text{so}}^q(K, \tilde{\Lambda}) &\stackrel{\text{def}}{=} \text{Hom}_{\tilde{\Lambda}}(V_{\text{so}}^q, \tilde{\Lambda}). \end{aligned}$$

Thus from the finiteness results above Proposition 7.9, we get a form of Hida's control theorem.

Theorem 7.12. *Let $q = 0$ or \emptyset . Then*

- (1) V_{so}^q is a free $\tilde{\Lambda}$ -module of finite rank.
- (2) For any $k \in \mathcal{X}_\rho$ satisfying the assumption of Theorem 7.11 we have $\mathcal{M}_{\text{so}}^q(K, \tilde{\Lambda}) \otimes \tilde{\Lambda}/P_{\underline{k}} \simeq e_\alpha \cdot M_{\underline{k}}^q(K, \mathcal{O})$.

Proof. Same as [57, Theorem 4.21] using propositions 7.5, 7.8, theorem 7.11 and proposition 7.9. □

Descent to Prime to p -Level

The following proposition is needed to apply the description of local Galois representations at p of semi-ordinary forms with prime to p level above.

Proposition 7.13. *Suppose \underline{k} is such that*

$$a_1 = a_2 = 0, a_3 \equiv b_1 \equiv 0 \pmod{p-1}, a_2 - a_3 \gg 0, a_3 + b_1 \gg 0.$$

Suppose $F \in e_\alpha M_{\underline{k}}^0(K_0(p), \mathbb{C})$ is an eigenform with trivial nebentypus at p such that $\bar{\rho}_F$ can be written $\bar{\rho}_f \oplus \psi_1 \oplus \psi_2$ for ψ_i a character of G_K . Let π_F be the associated automorphic representation. Then $\pi_{F,p}$ is an unramified principal series representation.

Proof. Similar to [57, proposition 4.17]. As F is semi-ordinary at p , it has a $\pi_{F,p}$ has a fixed vector for $K_0(p)$. By the classification of admissible representations with $K_0(p)$ -fixed vector (see e.g. [17, Theorem 3.7]) we know $\pi_{F,p}$ has to be a subquotient of $\text{Ind}_B^{\text{GL}_4} \chi$ for χ an unramified character of $T_n(\mathbb{Q}_p)$. If this induced representation is irreducible then we are done. Suppose this is not the case. As $a_2 - a_3 > 0$, $a_3 + b_1 > 0$ and as F is semi-ordinary, the character χ may be written as $\chi = \chi_1 \otimes \chi_2 \otimes \chi_3 \otimes \chi_4$ with $\chi_1 = \chi_2 |\cdot|$ and χ_3, χ_4 having p -adic weights $\kappa_1 = b_1$ and $\kappa_2 = 3 - a_3$ respectively. This implies that F is in fact ordinary and so that the local representation $\bar{\rho}_F$ at p is reducible. By our assumption, $\bar{\rho}_F^{ss}$ is the direct sum of $\bar{\rho}_f$ with two characters. So $\bar{\rho}_f|G_{\mathbb{Q}_p}$ must itself be reducible. This contradicts our ongoing hypothesis that $\bar{\rho}_f|G_{\mathbb{Q}_p}$ is irreducible. Thus $\pi_{F,p}$ must be unramified. \square

A Definition Using Fourier-Jacobi Expansion

We can define a $\tilde{\Lambda}$ -adic Fourier-Jacobi expansion map for families of semi-ordinary families as in [57, 4.6.1] by taking the $\tilde{\Lambda}$ -dual of the Pontryagin dual of the usual Fourier-Jacobi expansion map (replacing the e 's in *loc.cit.* by e_α 's). We also define the Λ -adic Siegel operators $\Phi_{[g]}^h$'s by taking the 0-th Fourier-Jacobi coefficient.

Definition 7.14. Let A be a finite torsion free Λ -algebra. Let $\mathcal{N}_{\text{so}}(K, A)$ be the set of formal Fourier-Jacobi expansions:

$$F = \left\{ \sum_{\beta \in \mathcal{S}_{[g]}} a(\beta, F) q^\beta, a(\beta, F) \in (A \hat{\otimes} \hat{A}_{[g]}^\infty)^\Lambda \otimes H^0(\mathcal{Z}_{[g]}^\circ, \mathcal{L}(\beta)) \right\}_{g \in X(K)}$$

such that for a Zariski dense set $\mathcal{X}_F \subseteq \mathcal{X}_\rho$ of points $\phi \in \text{Spec } A$ where the induced point in $\text{Spec } \Lambda$ is some arithmetic weight \underline{k}_ζ , the specialization F_ϕ of F is the highest weight vector of the Fourier-Jacobi expansion of a semi-ordinary modular form with tame level $K^{(p)}$, weight \underline{k} and nebentype at p given by $[\underline{k}] [\zeta] \omega^{-[\underline{k}]}$ as a character of $K_0(p)$.

Then we have the following

Theorem 7.15.

$$\mathcal{M}_{\text{so}}(K, A) = \mathcal{N}_{\text{so}}(K, A).$$

The proof is the same as [57, Theorem 4.25].

Fundamental Exact Sequence

Now we prove a fundamental exact sequence for semi-ordinary forms. Let $w'_3 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}$.
Lemma 7.16. Let $\underline{k} \in \mathcal{X}_\rho$ and $F \in e_\alpha M_{\underline{k}}(K_0(p, p^n), R)$ and $R \subset \mathbb{C}$. Let $W_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cup$

Id be the Weyl group for $G_P(\mathbb{Q}_p)$. There is a constant A such that for any $\underline{k} \in \mathcal{X}_\rho$ such that $a_2 - a_3 > A, a_3 + b_1 > A$, for each $g \in G(\mathbb{A}_f^{(p)})$, $\Phi_{P,wg}(F) = 0$ for any $w \notin W_2 w'_3$.

The lemma can be proved using the computations in the proof of [57, lemma 4.14]. Note that by semi-ordinarity and the contraction property the level group at p for F is actually $K_0(p)$.

The following is a semi-ordinary version of [57, Theorem 4.16], noting that e_α induces identity after the Siegel operator $\hat{\Phi}^{w'_3}$ by [20, Proposition 4.5.2]. The proof is also similar (even easier since the level group at p is in fact in $K_0(p)$ by the contraction property).

Theorem 7.17. For $\underline{k} \in \mathcal{X}_\rho$, we have

$$0 \rightarrow e_\alpha \mathcal{M}_{\underline{k}}^0(K, A) \rightarrow e_\alpha \mathcal{M}_{\underline{k}}(K, A) \xrightarrow{\hat{\Phi}^{w'_3} = \oplus \hat{\Phi}_{[g]}^{w'_3}} \oplus_{g \in C(K)} \mathcal{M}_{\underline{k}'}(K_{P,0}^g(p), A)$$

is exact.

We need a family version of the fundamental exact sequence

Theorem 7.18. *The following short sequence is exact :*

$$0 \longrightarrow e_\alpha \mathcal{M}^0(K, A) \longrightarrow e_\alpha \mathcal{M}(K, A) \xrightarrow{\hat{\Phi}^{w'_3} = \oplus \hat{\Phi}_{[g]}^{w'_3}} \oplus_{g \in C(K)} \mathcal{M}(K_{P,0}^g(p), A) \longrightarrow 0.$$

Proof. The minimal compactification of the Igusa variety is affine, by [57, Theorem 4.16]. After choosing a weight \underline{k} and specializing \underline{k} , the result thus follows from theorems 7.11, 7.12 and 7.17. \square

7.4 Eisenstein Family

7.4.1 Klingen Eisenstein Family

In this section we recall the construction in [32] of the Klingen Eisenstein series, using the pullback formula for

$$\mathrm{U}(3, 1) \times \mathrm{U}(0, 2) \hookrightarrow \mathrm{U}(3, 3)$$

from some nearly holomorphic Siegel Eisenstein series E_{sieg} on $\mathrm{U}(3, 3)$. We refer to [32, Section 3.2-3.3] for backgrounds on Siegel Eisenstein series and pullback formula. We suppose ξ_0 is a Hecke character of Archimedean type $(\kappa/2, -\kappa/2)$ for $\kappa \equiv 0 \pmod{2(p-1)}$. Suppose also the p -adic avatar of $\xi'_0 \stackrel{\text{def}}{=} \xi_0 \cdot (\epsilon^{-1} \circ \mathrm{Nm})$ factors through Γ_K .

Proposition 7.19. [32, Theorem 1.2] *Let $\pi = \pi_f$ be the unitary automorphic representation generated by the weight k form f . Let $\tilde{\pi}$ be the dual representation of π . Let Σ be a finite set of primes containing all the bad primes*

- (i) *There is an element $\mathcal{L}_{f, \mathcal{K}, \xi_0}^\Sigma \in \Lambda_{\mathcal{K}, \mathcal{O}^{\text{ur}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ such that for any character ξ_ϕ of Γ_K , which is the avatar of a Hecke character of conductor p , infinite type $(\frac{\kappa_\phi}{2} + m_\phi, -\frac{\kappa_\phi}{2} - m_\phi)$ with κ_ϕ an even integer which is at least 6, $m_\phi \geq \frac{k-2}{2}$, we have*

$$\phi(\mathcal{L}_{f, \mathcal{K}, \xi_0}^\Sigma) = \frac{L^\Sigma(\tilde{\pi}, \xi_\phi, \frac{\kappa_\phi-1}{2}) \Omega_p^{4m_\phi+2\kappa_\phi}}{\Omega_\infty^{4m_\phi+2\kappa_\phi}} c'_\phi \cdot p^{\kappa_\phi-3} \mathfrak{g}(\xi_{\phi,2})^2 \prod_{i=1}^2 (\chi_i^{-1} \xi_{\phi,2}^{-1})(p)$$

where c'_ϕ is a constant coming from an Archimedean integral.

- (ii) *There is a set of formal q -expansions $\mathbf{E}_{f, \xi_0} \stackrel{\text{def}}{=} \{\sum_\beta a_{[g]}^t(\beta) q^\beta\}_{([g], t)}$ for $\sum_\beta a_{[g]}^t(\beta) q^\beta \in \Lambda_{\mathcal{K}, \mathcal{O}^{\text{ur}}} \otimes_{\mathbb{Z}_p} \mathcal{R}_{[g], \infty}$, $([g], t)$ are p -adic cusp labels (we refer to [32, Section 2.8.2] for the notation and the notion of p -adic cusps), such that for a Zariski dense set of arithmetic points $\phi \in \mathrm{Spec}_{\mathcal{K}, \mathcal{O}}$, $\phi(\mathbf{E}_{f, \xi_0})$ is the Fourier-Jacobi expansion of the highest weight vector of the holomorphic Klingen Eisenstein series constructed by pullback formula which is an eigenvector for U_{t+} with non-zero eigenvalue. The weight $L_{\underline{k}}$ for $\phi(\mathbf{E}_{f, \xi_0})$ is $(m_\phi + \frac{k-2}{2}, m_\phi - \frac{k-2}{2}, 0; \kappa_\phi)$.*
- (iii) *The $a_{[g]}^t(0)$'s are divisible by $\mathcal{L}_{f, \mathcal{K}, \xi_0}^\Sigma \cdot \mathcal{L}_{\tilde{\pi}}^\Sigma$, where $\mathcal{L}_{\tilde{\pi}}^\Sigma$ is the p -adic L -function of a Dirichlet character as in [32].*

We assumed in [32] that the $\pi_{f,p}$ has distinct Satake parameters, which turns out to be unnecessary in our $\mathrm{U}(2)$ case. We also refer to [32, Section 2.4] for the convention of weights of automorphic forms on $\mathrm{U}(2)$ and $\mathrm{U}(3, 1)$. This is just a translation of the main theorem of [32] to the situation here.

To be compatible with arithmetic applications to our main results, we re-parameterize the family by a translation given below. We define an automorphism of the Iwasawa algebra Λ_K as the following composition

$$t_{\xi'_0}^{-c} : \Lambda_K \rightarrow \Lambda_K \rightarrow \Lambda_K$$

where the first map is given by $\gamma' \mapsto \gamma' \xi'_0(\gamma'^{-1})$ for each $\gamma' \in \Gamma_K$, and the second map is determined by the map $\gamma' \rightarrow \gamma'^{-c}$ ($-c$ means inverse composed with complex conjugation) for each $\gamma' \in \Gamma_K$. For each $\phi \in \mathrm{Spec}\Lambda_K$ corresponding to a character χ_ϕ of Γ_K , the composition of ϕ with $t_{\xi'_0}^{-c}$ corresponds to the character $(\chi_\phi \xi'_0^{-1})^{-c}$. We write $\mathcal{L}_{f, \mathcal{K}}^{\mathrm{Gr}, \Sigma} \in \mathrm{Frac}(W(\bar{\mathbb{F}}_p))[[\Gamma_K]]$ for the p -adic L -function $t_{\xi'_0}^{-c}(\mathcal{L}_{f, \mathcal{K}, \xi_0})$ we constructed above (here $W(R)$ means the ring of Witt vectors of R and we dropped the subscript ξ as this p -adic L -function is distinguished corresponding to the trivial character). We can recover the full

p -adic L -function $\mathcal{L}_{f,\mathcal{K}}$ by putting back the Euler factors at primes in Σ . We write this p -adic L -function as

$$\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f). \quad (7.4.1)$$

In proposition 7.19, an Archimedean constant c'_ϕ appears. It is perhaps not easy to compute this constant directly. Nevertheless, comparing the above p -adic L -function and Hida's Rankin-Selberg p -adic L -function, we can make the constant c'_ϕ precise.

Lemma 7.20. *The constant c'_ϕ of proposition 7.19 is given by*

$$\Gamma(\kappa_\phi + m_\phi - \frac{k}{2}) \Gamma(\kappa_\phi + m_\phi + \frac{k}{2} - 1) 2^{-3\kappa_\phi - 4m_\phi + 1} \pi^{1-2\kappa_\phi - 2m_\phi} i^{k-\kappa_\phi - 2m_\phi - 1}.$$

Proof. Let \mathbf{g} be the Hida family corresponding to the family of characters of $\Gamma_{\mathcal{K}}$ (see [126, Definition 7.8]). We pick an auxiliary Hida family of ordinary forms \mathbf{f}' (for example using CM forms) and compare

- The product $\mathcal{L}_{\mathbf{f}' \otimes \mathbf{g}}^{\text{Hida}} \cdot \mathcal{L}_{\mathcal{K}}^{\text{Katz}} h_{\mathcal{K}}$, where $\mathcal{L}_{\mathcal{K}}^{\text{Katz}} h_{\mathcal{K}}$ is the class number $h_{\mathcal{K}}$ of \mathcal{K} times the Katz p -adic L -function, which interpolates the Petersson inner product of specializations of \mathbf{g}_ϕ (we refer to [126, Section 7.5] for details). The $\mathcal{L}_{\mathbf{f}' \otimes \mathbf{g}}^{\text{Hida}}$ is the Rankin-Selberg p -adic L -function constructed by Hida in [56] interpolating algebraic part of the critical values of Rankin-Selberg L -functions for specializations of \mathbf{f}' and \mathbf{g} , where the specializations of \mathbf{g} has higher weight.
- The p -adic L -function $\mathcal{L}_{\mathbf{f}',\mathcal{K}}$ constructed using the doubling method as above.

We first look at the interpolation formulas at arithmetic points where the Siegel Eisenstein series are of scalar weight. The computations are done in [123] (although the ramifications in *loc.cit* is slightly different, however those assumptions are put for constructing the family of Klingen Eisenstein series. The computations in the doubling method construction of the p -adic L -function carries out in the same way in the situation here). We see that the above two items have the same value at these points. As these arithmetic points are Zariski dense, the $\mathcal{L}_{\mathbf{f}' \otimes \mathbf{g}}^{\text{Hida}} \cdot \mathcal{L}_{\mathcal{K}}^{\text{Katz}} h_{\mathcal{K}}$ and $\mathcal{L}_{\mathbf{f}',\mathcal{K}}$ should be equal. Then we look at the arithmetic points considered in the above proposition. Comparing the interpolation formulas here and in [56, Theorem I], we get the formulas for c'_ϕ (note that the critical L -value is not zero since it is away from center). \square

For convenience of application we normalize our Klingen Eisenstein series by twisting by an anticyclotomic character so that the specializations are of the weight $L_{\underline{k}} = (\frac{k-2}{2}, -\frac{k-2}{2}, -m_\phi; \kappa_\phi + m_\phi)$. Now we still write $\mathcal{L}_{f \otimes g}^{\text{Hida}}$ for the Rankin-Selberg Hida p -adic L -function interpolating critical values of the Rankin-Selberg L -function for f and specializations of g whose weight is higher than f (see [126, Definition 7.8]). Since the higher weight form g is ordinary, Hida's construction works in the same way even though f is not ordinary. We have the following corollary by comparing interpolation formulas (see [126, (7-2), (7-5)] for details).

Corollary 7.21.

$$\mathcal{L}_{f \otimes g}^{\text{Hida}} \cdot \mathcal{L}_{\mathcal{K}}^{\text{Katz}} h_{\mathcal{K}} = \mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f).$$

The corollary follows from above lemma and the interpolation formulas on both hand sides.

We prove the following

Lemma 7.22. *The p -adic L -function $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$ belongs to $\mathcal{O}^{\text{ur}}[[\Gamma_{\mathcal{K}}]]$.*

Proof. From the construction the denominator of $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$ can only be powers of p times the product of the Euler factors of a finite number of primes of $\mathcal{L}_{\mathcal{K}}^{\text{Gr}}(f)$. But by the argument in [125, Proposition 8.3] we know the denominator can at most be powers of Y if we take $\mathbb{Z}_p[[Y]]$ as the coefficient ring of \mathbf{g} . These two sets are disjoint. Thus the denominator must be a unit. \square

7.4.2 Study of the Fourier-Jacobi functional

It remains to study the p -adic property of the Klingen Eisenstein family. Before continuing we refer to [126, Introduction] for an outline of the strategy to construct and study the Fourier-Jacobi functional (this is very helpful for understanding the argument below). It uses the pullback formula of E_{Sieg} on $\text{U}(3, 3)$ via

$$\text{U}(3, 1) \times \text{U}(0, 2) \hookrightarrow \text{U}(3, 3)$$

with

$$E_{\text{Sieg}}|_{\text{U}(3, 1) \times \text{U}(0, 2)} = E_{\text{Kling}} \boxtimes f.$$

We first compute the Fourier-Jacobi coefficient for E_{Sieg} and pair it with the form f on $\text{U}(0, 2)$, and thus obtain the Fourier-Jacobi coefficient of E_{Kling} (detailed below). We refer to [126, Section 6D] for the computation of Fourier-Jacobi expansion for E_{Sieg} and the notion of local Fourier-Jacobi integrals.

Idea Here

The main difficulty here is that the explicit local Fourier-Jacobi computation at Archimedean place is complicated if the weight is not scalar. Our idea is not to compute such local Fourier-Jacobi integrals at ∞ . Instead we fix the weight (k, κ_ϕ, m_ϕ) (notation as before) and vary its nebentypus of the Klingen Eisenstein series at p . Such arithmetic points are Zariski dense in $\text{Spec } \mathcal{O}[[\Gamma_K]]$. We construct (as in previous works) a Fourier-Jacobi functional $\text{FJ}_{\beta, \theta_1}^h$ (in Definition 7.25) on families on $\text{U}(3, 1)$, and show that we can factor out a number C_∞ depending only on the Archimedean data (and is thus the same number for all arithmetic points), which can be proved to be non-zero, and an element $\mathcal{L} \in \mathcal{O}^{\text{ur}}[[\Gamma_K]]^\times \cdot \bar{\mathbb{Q}}_p^\times$, such that at each of such arithmetic point we have

$$\text{FJ}_{\beta, \theta^*}^h(\phi(E_{\text{Kling}})) = C_\infty \cdot \phi(\mathcal{L}).$$

The key idea is to employ Ikeda's theory on Fourier-Jacobi coefficients to get lemma 7.26, which says that the Fourier-Jacobi coefficient of a nearly holomorphic Siegel Eisenstein series on $\text{U}(3, 3)$ is a *finite sum* of products of Siegel Eisenstein series on $\text{U}(2, 2)$ and theta functions.⁸

Interpolating Inner Products

We first recall a construction of Hsieh [59, Definition 4.3] which is important in [126, Section 7]. Let $\Lambda_{\text{U}(2)} = \mathcal{O}_L[[T_1, T_2]]$ be the weight algebra for $\text{U}(2)$ (*i.e.* the complete group ring of the diagonal torus of $\text{U}(2)(1 + p\mathbb{Z}_p)$). Write $\mathcal{M}_{\text{ord}}(K, \Lambda_{\text{U}(2)})$ for the space of ordinary Hida families on $\text{U}(2)$ with respect to a tame level group $K \subset \text{U}(2)(\mathbb{A}_f)$. We similarly write $\check{\mathcal{M}}_{\text{ord}}(K, \Lambda_{\text{U}(2)})$ for the space of such Hida families with the weight map being the inverse of that of $\mathcal{M}_{\text{ord}}(K, \Lambda_{\text{U}(2)})$.

Definition 7.23. *For a neat tame level group $K \subset \text{U}(2)(\mathbb{A}^{(p\infty)})$ we define a $\Lambda_{\text{U}(2)}$ -pairing $\mathbf{B}_K \langle -, - \rangle$*

$$\mathbf{B}_K : \mathcal{M}_{\text{ord}}(K, \Lambda_{\text{U}(2)}) \times \check{\mathcal{M}}_{\text{ord}}(K, \Lambda_{\text{U}(2)}) \rightarrow \Lambda_{\text{U}(2)}.$$

For any $\mathbf{f} \in \mathcal{M}_{\text{ord}}(K, \Lambda_{\text{U}(2)})$, $\mathbf{g} \in \check{\mathcal{M}}_{\text{ord}}(K, \Lambda_{\text{U}(2)})$, $\phi \in \text{Spec } \Lambda_{\text{U}(2)}(\mathbb{C}_p)$ a weight two point (in other words, the specializations of \mathbf{f}_ϕ and \mathbf{g}_ϕ have scalar weights), and any $n > 0$ we define

$$\begin{aligned} \mathbf{B}_{K,n} \langle \mathbf{g}, \mathbf{f} \rangle &\stackrel{\text{def}}{=} \sum_{[x_i] \in \text{U}(2)(\mathbb{Q}) \setminus \text{U}(2)/KU_0(p^n)} U_p^{-n} \mathbf{f}(x_i) \mathbf{g}(x_i) \binom{1}{p^n} \\ &\quad (\text{mod}(1 + T_1)^{p^n} - 1, (1 + T_2)^{p^n} - 1). \end{aligned}$$

Then Hsieh proves that

$$\mathbf{B}_{K,n+1} \equiv \mathbf{B}_{K,n} (\text{mod}(1 + T_1)^{p^n} - 1, (1 + T_2)^{p^n} - 1).$$

We define

$$\mathbf{B}_K \langle \mathbf{g}, \mathbf{f} \rangle = \lim_n \mathbf{B}_{K,n} \langle \mathbf{g}, \mathbf{f} \rangle.$$

⁸We are thankful to Ikeda for showing us the argument.

By definition we have

$$\phi(\mathbf{B}_K \langle \mathbf{g}, \mathbf{f} \rangle) = \sum_{[x_i] \in \mathrm{U}(2)(\mathbb{Q}) \setminus \mathrm{U}(2)/KU_0(p^n)} U_p^{-n} \mathbf{f}_\phi(x_i) \mathbf{g}_\phi(x_i \begin{pmatrix} & 1 \\ p^n & \end{pmatrix})$$

and hence

$$\begin{aligned} \phi(\mathbf{B}_K \langle \mathbf{g}, \mathbf{f} \rangle) &= \mathrm{vol}(KU_0(p^n))^{-1} \int_{[\mathrm{U}(2)]} U_p^{-n} \mathbf{f}_\phi(h) \mathbf{g}_\phi(h \begin{pmatrix} & 1 \\ p^n & \end{pmatrix}) dh \\ &= \mathrm{vol}(KU_0(p^n))^{-1} \int_{[\mathrm{U}(2)]} \mathbf{f}_\phi(h \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}) \mathbf{g}_\phi(h) dh \end{aligned}$$

if ϕ corresponds to an ordinary form whose p -part conductor is p^n .

In the below we fix and suppress the tame level group K and write the pairing as $\mathbf{B}\langle \cdot, \cdot \rangle$. Hsieh also proved ([59, Lemma 4.4]) when specializing the pairing to arithmetic points Q of conductor p^α (allowed to have vector valued weight), this gives

$$\langle U_p^{-\alpha} \mathbf{f}_Q, \mathbf{f}_Q \rangle.$$

(Petersson inner product of vector valued forms in *loc.cit*).

We recall some explicit constructions in [20, Section 7.7] (also in [126, Section 8B]). These are important in our study of the Fourier-Jacobi functional.

Definition 7.24. • Hida families $\boldsymbol{\theta}$, $\tilde{\boldsymbol{\theta}}_3$, \mathbf{h} , $\tilde{\mathbf{h}}_3$ of CM forms on $\mathrm{U}(2)$ constructed in [125, Section 8B], using the Hecke characters chosen in *loc.cit*.

- Elements g_1, g_2, g_3 and g_4 defined in [125, Definition 8.20] in $\mathrm{GU}_2(\mathbb{A}_{\mathbb{Q}})$ whose p components are 1.
- Some p -adic L -functions $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_6$, where $\mathcal{L}_1, \mathcal{L}_5$ and \mathcal{L}_6 are units in the Iwasawa algebra, while \mathcal{L}_2 is a nonzero constant fixed throughout the family.

Note that the families \mathbf{h} and $\tilde{\mathbf{h}}_3$ are constructed via interpolating at weight two arithmetic points. By standard facts about Hida control theorems, we know the families \mathbf{h} and $\tilde{\mathbf{h}}_3$ also interpolate the highest weight vectors of CM forms at arithmetic points of weight $(\frac{k-2}{2}, -\frac{k-2}{2})$ and $(-\frac{k-2}{2}, \frac{k-2}{2})$ respectively. From the computations in [126, Section 8D], we know $\mathbf{B}\langle \mathbf{h}, \mathbf{h}_3 \rangle$ and $\mathbf{B}\langle \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}_3 \rangle$ (Hsieh's Λ -adic pairing) are interpolated by p -adic L -functions \mathcal{L}_3 and \mathcal{L}_4 . So Hsieh's theory in [59] also implies the Petersson inner products at weight k arithmetic points are also interpolated by \mathcal{L}_3 and \mathcal{L}_4 respectively.

Geometry and Fourier-Jacobi Functional

Now we look at the Fourier-Jacobi expansion theory for forms on $\mathrm{GU}(3, 1)$. Recall [126, Section 3.6] that for $\beta \in \mathbb{Q}^\times$ there is a line bundle $\mathcal{L}(\beta)$ on the boundary component $\mathcal{Z}_{[g]}$ ($[g]$ is some cusp label) of the Shimura variety for $\mathrm{GU}(3, 1)$. We refer to *loc.cit.* and [57, Section 3.6] for the theory of Fourier-Jacobi coefficients for forms on $\mathrm{GU}(3, 1)$. Some points are worth pointing out when working with vector valued forms. We refer to [80] for the comparison between algebraic and analytic Fourier-Jacobi coefficients. As noted in [57, Section 3.6], the algebraic Fourier-Jacobi coefficient takes values only in the N_H^1 -coinvariants (see 3.6.2 of *loc.cit* for the notation) of the representation $L_{\underline{k}}$, due to the ambiguity in choosing a basis for the differentials of the Mumford family. Thus this β -th Fourier-Jacobi expansion takes values in the space of forms on $\mathrm{U}(2)$ tensored with the space of global sections of $\mathcal{L}(\beta)$. We only look at the $L_{\underline{k}'} \stackrel{\text{def}}{=} L_{(\frac{k-2}{2}, -\frac{k-2}{2})}$ -components (regarded as a sub-representation of the restriction of the representation $L_{\underline{k}}$ to GL_2 , which clearly appears with multiplicity one in this restriction). In fact the ambiguity on the choice above does not make any differences when looking at this $L_{\underline{k}'}$ -component, by the description of the N_H^1 -coinvariants in the proof of [57, Lemma 3.12] (this corresponds to the $L_{\underline{k}'}$ -component there). According to the description in [80, Section 5.3] this corresponds to looking at a quotient of the $\mathcal{E}_{M, \text{an}}^{\Phi_{\mathcal{H}}^{(g)}, \delta_{\mathcal{H}}^{(g)}}(W)$ in *loc.cit.* which is the pullback to $\mathcal{Z}_{[g]}$ of an automorphic vector bundle of weight \underline{k}' on the Igusa variety for the definite unitary group $\mathrm{U}(2)$, tensoring with the line bundle $\mathcal{L}(\beta)$. We look at theta functions θ^* on the Klingen parabolic subgroup $P(\mathbb{A}) \subseteq \mathrm{GU}(3, 1)(\mathbb{A})$, which are exactly the ones considered in [126, Section 6.10] and [125, Section 5], *i.e.* corresponds to the dual of

the space of global sections of $\mathcal{L}(\beta)$. It is defined as in [126, Definition 6.45] corresponding to the Schwartz function $\phi_1 = \phi_{1,\infty} \times \prod_{v < \infty} \phi_{1,v}$ there, and define the functional l_{θ^*} as in [126, Section 8E] (this corresponds to inner product integral over $N_P(\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}})$ with the theta function θ_{ϕ_1} corresponding to ϕ_1 where N_P is the unipotent radical of the Klingen parabolic subgroup P), such that $l_{\theta^*} \in \text{Hom}(H^0(\mathcal{Z}_{[g]}, \mathcal{L}(\beta)), \mathcal{O})$.

Definition 7.25. *We define the Fourier-Jacobi functional on $U(3, 1)$ families by*

$$FJ_{\beta, \theta^*}^{\mathbf{h}} : F \mapsto \mathbf{B}\langle e^{\text{ord}} l_{\theta^*} FJ_{\beta}(F), \pi(g_2) \mathbf{h} \rangle.$$

We note that since the FJ_{β} map takes the N_H^1 covariant quotient, so it takes values in the $L_{(\frac{k}{2}, -\frac{k}{2})}$ -component.

Remark: To see that

$$F \mapsto \mathbf{B}\langle e^{\text{ord}} l_{\theta^*} (FJ_{\beta}(F)), \pi(g_2) \mathbf{h} \rangle$$

indeed gives a functional on the space of semi-ordinary families (over the two-dimensional weight space) of forms on $U(3, 1)$, we note that the Archimedean weight is fixed throughout the two-dimensional family. In the theory of p -adic semi-ordinary families (as developed in [125]) we are interpolating the highest weight vector of the automorphic forms. Thus each component of the $L_{k'}$ -projection of $FJ_{\beta}(F)$ is interpolated p -adic analytically. Note also that in the ordinary case when $k > 2$ the family of functionals constructed here is different from the two-variable specializations of the construction in [126]. (The highest weight vector is not in the $(\frac{k-2}{2}, -\frac{k-2}{2})$ -component above).

We now compute this functional on Klingen Eisenstein series at arithmetic points. We first need some preparations.

Ikeda Theory

We refer to [126, Section 6.1] for the background of Siegel Eisenstein series and write $I_n(\tau)$ for the corresponding space of (local or global) Siegel sections defined using a Hecke or local character τ . Write $E = E_{\text{Sieg}}$ for the Siegel Eisenstein series on $U(3, 3)$ that we use in the pullback formula. Now we put ourselves in the context of [61]. We are in the $m = 1$ and $n = 2$ of [61, Section 2, case 2] (see the definitions of X, Y, Z, V there). Let ψ be an additive character of \mathbb{A} and ω_{ψ} be the Weil representation there.

We first make some observations. For $\beta \in \mathbb{Q}^{\times}$, let $\psi = \psi_{\beta}$ be the additive character corresponding to β . Let π be the representation of the Jacobi group $N' U(2, 2)$ (see [126, Introduction] for the notations) generated by the β -th Fourier-Jacobi coefficient of a nearly holomorphic Siegel Eisenstein series on $U(3, 3)$. Then by [61, Proposition 1.3], there is a map

$$\omega_{\psi} \otimes I \hookrightarrow \pi$$

with dense image, where ω_{ψ} is the Weil representation of the Jacobi group defined there (determined by a splitting character λ we fix throughout), and I is the representation generated by the integral of *loc.cit.* By [61, Theorem 3.2] this is the sub-representation of the automorphic representation corresponding to Siegel Eisenstein series on $U(2, 2)$. Let \mathfrak{g} be the Lie algebra for $U(2, 2)(\mathbb{R})$ and K be a maximal compact subgroup of it. Let \mathfrak{k} be its Lie algebra. Write the Harish-Chandra decomposition of the complex Lie algebra by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

By a *lowest weight representation* we mean a (\mathfrak{g}, K) -module generated by elements which are killed by some finite power of \mathfrak{p}^- . Then as E_{Sieg} is nearly holomorphic by construction, we know π is a lowest weight representation. Also as ω_{ψ} is a Weil representation, it is also a lowest weight representation (see [1]). By the Leibnitz rule

$$X^-(v_1 \otimes v_2) = X^- v_1 \otimes v_2 + v_1 \otimes X^- v_2$$

for $v_1 \in \omega_{\psi}$, $v_2 \in I$ and $X^- \in \mathfrak{p}^-$, we see I is also a lowest weight representation. Note that under the Archimedean theta correspondence of $U(1)(\mathbb{R}) \times U(2, 2)(\mathbb{R})$, any component of the Weil representation of $U(2, 2)(\mathbb{R})$ with given central character must be irreducible (since $U(1)(\mathbb{R})$ is compact, see *e.g.* [1]). Recall that the space I is contained in the space of Siegel Eisenstein series on $U(2, 2)$ with fixed character ([61, Theorem 3.2]). We also note the well known fact that the representation of $U(2, 2)(\mathbb{R})$ admissibly induced from a character

of the Siegel parabolic subgroup is of finite length. Now we claim that for a fixed K -type σ of π , there are only finitely many K -types of ω_ψ and I , whose tensor product contains σ . This can be seen, for example by noting that in any lowest weight representation generated by a single vector, the $U(1)(\mathbb{R})$ -weights are monotonically going up with finite dimensional eigenspaces for each eigencharacter. It follows that the map

$$\omega_\psi \otimes I \hookrightarrow \pi \quad (7.4.2)$$

is surjective for K -finite vectors, from the density of the image of the above \hookrightarrow .

Notation: Write w_1, w_2, \dots for a basis of $L_{(\frac{k-2}{2}, -\frac{k-2}{2})}$. To save notation, in the following we also write E_{sieg} for the w_1 -component adelic Siegel Eisenstein series in the $L_{(\frac{k-2}{2}, -\frac{k-2}{2})}$ -component of E_{sieg} written out using this basis.

We refer to [61, Theorem 3.2] for the definition of an integral operator $R(f, \phi)$, where f is a Siegel Eisenstein section on $U(3, 3)$ and ϕ is a Schwartz function. It also makes sense to talk about local versions of these integrals. From the above discussion we get the following key lemma from [61, Lemma 1.1, Proposition 1.3], which states that the $\text{FJ}_\beta(E_{\text{sieg}})$ is a finite sum of products of theta functions and Siegel Eisenstein series on $U(2, 2)$.

Lemma 7.26. *There are a finite number of Archimedean Schwartz functions $\phi'_{4,i,\infty}, \phi'_{2,i,\infty}$ of $X(\mathbb{R})$ for X in the Introduction of [61], such that $\text{FJ}_\beta(E_{\text{sieg}})$ can be written as a finite sum of expressions as*

$$\Theta_{\phi'_{4,i,\infty} \otimes \prod_{v < \infty} \phi_{i,v}}(nh) E(R(f_\infty, \phi'_{2,i,\infty})) \cdot \prod_{v < \infty} f'_{v,i}$$

for some Siegel sections $\prod_{v < \infty} f'_{v,i} \in I_2(\tau)$ and Schwartz functions $\prod_{v < \infty} \phi_{i,v}$.

Note with the formula on page 620 of [61] that the inner product of theta functions is equal to the inner product of the kernel Schwartz function. We choose $\phi_{4,i,\infty}$ to be a basis contributing to the surjection (7.4.2) on the K -type of ω_ψ , and the $\phi_{2,i,\infty}$ a set of dual basis (finite dimensional) with respect to the pairing in *loc.cit.*. These $\phi_{4,i,\infty}$ and $\phi_{2,i,\infty}$ depend only on k ($i = 1, 2, \dots$).

Define the local β -th Fourier-Jacobi integral for $f_v \in I_3(\tau_v)$ as in [126, Definition 6.4] by

$$\text{FJ}_\beta(f_v, nh) = \int_{\mathbb{Q}_v} f_v \left(\begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix} \begin{pmatrix} S_v & 0 \\ 0 & 0_2 \\ & 1_3 \end{pmatrix} nh \right) e(-\beta S_v) dS_v.$$

Note the relation between this integral and the integral in [61, Theorem 3.2].

Note also the computations in [126, Section 6] imply for each $v < \infty$ and f_v the Siegel section on $U(3, 3)$ that we use to define the Siegel Eisenstein series E_{sieg} , the local Fourier-Jacobi integral can be written in the form

$$\text{FJ}_\beta(f_v) = \sum_{j_v=1}^{n_v} f_{j_v} \phi_{j_v}$$

for $f_{j_v} \in I_2(\tau_v)$ Siegel sections on $U(2, 2)$ and ϕ_{j_v} local Schwartz functions. Then from lemma 7.26 and the computation in [61, Page 628] on

$$\langle \text{FJ}_\beta(E_{\text{sieg}}), \Theta_\phi \rangle$$

and choosing the test Schwartz function ϕ properly, noting the duality between the $\phi_{2,i,\infty}$ and $\phi_{4,i,\infty}$'s (and take appropriate Schwartz functions at non-Archimedean places as well), we obtain that

$$\text{FJ}_\beta(E_{\text{sieg}}) = \sum_i (\prod_v \sum_{j_v}) E(R(f_\infty, \phi_{2,i,\infty})) \cdot \prod_v f_{j_v} \cdot \Theta_{\phi_{4,i,\infty} \otimes \prod_{v < \infty} \phi_{j_v}}, \quad (7.4.3)$$

A choice for β

We can ensure that the $L_{(\frac{k-2}{2}, -\frac{k-2}{2})}$ -component of $l_{\theta^*}(\text{FJ}_\beta(E_{\text{Kling}}))$ is non-zero for some

θ^* and $\beta \in \mathbb{Q}^\times \cap \mathbb{Z}_p^\times$. If for all nonzero β this component is zero, then the $L_{(\frac{k-2}{2}, -\frac{k-2}{2})}$ -component of E_{Kling} is a constant function on the Shimura variety of $\text{GU}(3, 1)$. This contradicts the description of the boundary restriction of E_{Kling} , namely the $L_{(\frac{k-2}{2}, -\frac{k-2}{2})}$ -component is non-zero at some cusp (namely at w_3 in [126, Section 6.8]) while is zero at other cusps.

Thus there must be a $\beta' \neq 0$ such that $\text{proj}_{(\frac{k-2}{2}, -\frac{k-2}{2})}(\text{FJ}_{\beta'}(E_{\text{Kling}})) \neq 0$. Let $\beta' = p^n \beta''$ for $\beta'' \in \mathbb{Z}_p^\times$ and $n \in \mathbb{Z}$. Let y be an element in \mathcal{K}^\times which is very close to $(p, 1)$ in the p -adic topology of \mathcal{K}_p . Then $\text{diag}(y\bar{y}, y, y, 1)^n \in \text{U}(3, 1)(\mathbb{Q})$. Set $\beta = \beta'(y\bar{y})^{-n} \in \mathbb{Z}_p^\times \cap \mathbb{Q}$ then

$$\text{proj}_{(\frac{k-2}{2}, -\frac{k-2}{2})}(\text{FJ}_\beta \rho(\text{diag}(y\bar{y}, y, y, 1)_p^n) E_{\text{Kling}})$$

is not the zero function. So there must be a choice of θ^* such that the weight $(\frac{k-2}{2}, -\frac{k-2}{2})$ -component of $l_{\theta^*}(\text{FJ}_\beta(E_{\text{Kling}}))$ above is non-zero. (The reason of making sure that $\beta \in \mathbb{Z}_p^\times$ is that only for those β we did the Fourier-Jacobi coefficient computation at p for the Klingen Eisenstein series in [126, Section 6H].) Without loss of generality we assume that the w_1 -component of it is nonzero using the basis w_1, w_2, \dots we fixed before.

Compute the Fourier-Jacobi Functional

Suppose we are doing our computations at an arithmetic point \mathbf{z} of conductor p^t (notation as in [126]) and write $\mathbf{h}_\mathbf{z}$ for the specialization of \mathbf{h} (the CM family in Definition 7.24) at \mathbf{z} . By the doubling method for $\mathbf{h}_\mathbf{z}$ under $\iota : \text{U}(2) \times \text{U}(2) \hookrightarrow \text{U}(2, 2)$ with the Siegel Eisenstein series on $\text{U}(2, 2)$

$$E(R(f_\infty, \phi_{2,i,\infty}) \cdot \prod_v f_{j_v}, -)$$

above (see [126, Proposition 6.1] for details), we know by the pullback formula (see [32, Proposition 3.4]) expressing the Klingen Eisenstein series as pullback of Siegel Eisenstein series, and (7.4.3) (for detail of this argument, see [126, Proposition 8.24], which uses lemma 6.46 and Corollary 6.47 there), there is a constant $C_{i,\infty}$ such that the Fourier-Jacobi functional in Definition 7.25 computed as (recall we noted that the $L_{(\frac{k-2}{2}, -\frac{k-2}{2})}$ appears with multiplicity one in the restriction of representations of GL_3 to GL_2),

$$\begin{aligned} & \int_{[N_P/Z(N_P)]} \int_{[\text{U}(2)]} \text{FJ}_\beta(E_{\text{Kling}})(ng) \mathbf{h}_\mathbf{z}(g) \theta_{\phi_1}(n) dndg \\ &= p^t \mathbf{z}(\mathcal{L}_5 \mathcal{L}_6) C_{i,\infty} \int_{[\text{U}(2)]} \mathbf{h}_\mathbf{z}(g) \cdot \boldsymbol{\theta}_\mathbf{z}^{\text{low}}(g) f(g) dg \\ &= p^t \mathbf{z}(\mathcal{L}_5 \mathcal{L}_6) C_{i,\infty} \int_{[\text{U}(2)]} \mathbf{h}_\mathbf{z}(g) \boldsymbol{\theta}_\mathbf{z}^{\text{low}}(g) f(g) dg. \end{aligned} \tag{7.4.4}$$

Here we write $[G]$ to denote $G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})$ for a group G , and denote $Z(G)$ as the center of G . The N_P is the unipotent radical of the Klingen parabolic subgroup P of $\text{U}(3, 1)$. The $f(g)$ is the form on $\text{U}(2)$ given by the GL_2 modular form we study extended using the trivial character via $\text{GU}(2) = \text{GL}_2 \times_{\mathbb{Q}^\times} \mathcal{K}^\times$. The \mathcal{L}_5 and \mathcal{L}_6 are as in Definition 7.24, and come from the pullback formula for $\mathbf{h}_\mathbf{z}$ above. The $\boldsymbol{\theta}_\mathbf{z}^{\text{low}}(g) \stackrel{\text{def}}{=} \theta_\mathbf{z}(g \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}_p)$ as in [126]. The $C_{i,\infty}$ comes from the product of

- The local pullback integral at ∞ of the Siegel section $R(f_\infty, \phi_{2,i,\infty})$ with respect to specializations $\mathbf{h}_\mathbf{z}$ under $\text{U}(2) \times \text{U}(2) \hookrightarrow \text{U}(2, 2)$. (See e.g. [126, Section 6E] the pullback integral. This is fixed since we fix the Archimedean datum).
- We pair the $\Theta_{\phi_{4,i,\infty}} \Pi_{v<\infty} \phi_{j_v}$ with the θ^* as in [126, Lemma 6.46, Corollary 6.47] and obtain a theta function on $1 \times \text{U}(2)(\mathbb{A}_\mathbb{Q})$. The corresponding Schwartz function at ∞ is with respect to the theta correspondence between compact unitary groups $\text{U}(1)(\mathbb{R})$ and $\text{U}(2)(\mathbb{R})$. By considering the central character in the above triple product integral (7.4.4), the “ $\boldsymbol{\theta}_\mathbf{z}$ ” part contributing non-trivially to the triple product is the eigen-component of trivial central character (the Archimedean weights of f and of $\mathbf{h}_\mathbf{z}$ are dual to each other). This is a multiple of the Archimedean kernel function of the $\phi_{2,\infty}$ in *loc.cit.*. This multiple is the second factor contributing to $C_{i,\infty}$.

We write

$$C_\infty = \sum_i C_{i,\infty}.$$

Note that the C_∞ only depends on our Archimedean datum as the $C_{i,\infty}$'s are. We see

$$C_\infty \neq 0$$

as by our choices that the w_1 -component of $l_{\theta^*}(\mathrm{FJ}_\beta(E_{\mathrm{Kling}}))$ is nonzero for appropriate local choices at finite places for E_{Kling} .

Now we turn to evaluate the integral (7.4.4) using Ichino's triple product formula. We refer to [126, Section 8D] for the details of Ichino's formula. As in [126, Section 8E] we need to pair (7.4.4) with the following

$$\mathbf{z}(\mathbf{B}\langle \tilde{\mathbf{h}}_3 \tilde{f}, \tilde{\boldsymbol{\theta}}_3 \rangle), \quad (7.4.5)$$

(recall the specializations of $\tilde{\mathbf{h}}_3$, \tilde{f} , $\tilde{\boldsymbol{\theta}}_3$ are in the dual automorphic representation space for the corresponding specializations of \mathbf{h} , f and $\boldsymbol{\theta}$, respectively). Here by the product $\tilde{\mathbf{h}}_3 \tilde{f}$ we mean the scalar valued form obtained using the natural pairing between the coefficient representations $(L_{(\frac{k-2}{2}, -\frac{k-2}{2})})$ and $(L_{(-\frac{k-2}{2}, \frac{k-2}{2})})$ of $\tilde{\mathbf{h}}_3$ and \tilde{f} respectively. As in [126, Sections 8D, 8E] we appeal to Ichino's formula to evaluate the product of the two triple product integrals above (namely the product of (7.4.4) and (7.4.5)). The local triple product computations are already done in *loc.cit.*, except the following two cases which are different from their.

Archimedean triple product integral

Note that at the Archimedean place, the representation L^{k-2} for π_f has dimension $k-1$. We note also the following for the vector-valued case here: for a basis (v_1, v_2, \dots, v_n) of L^{k-2} and a basis $(v_1^\vee, v_2^\vee, \dots, v_n^\vee)$ of its dual representation. If we write L^{k-2} -valued and $(L^{k-2})^\vee$ -valued forms

$$\begin{aligned} h &= h_1 v_1 + h_2 v_2 + \dots + h_n v_n \\ f &= f_1 v_1^\vee + f_2 v_2^\vee + \dots + f_n v_n^\vee, \end{aligned}$$

then the vector valued pairing $\langle h, f \rangle$ can be expressed as

$$(k-1) \int_{[\mathrm{U}(2)]} h_i((g) f_i(g) dg$$

for any $i = 1, 2, \dots, k-1$. So we can choose one i and do the Ichino triple product computation. It follows easily from Schur orthogonality that the local triple product integral at the Archimedean place (see [126, Section 8.4]) is equal to $\frac{1}{k-1}$ (a fixed number). Thus the product of (7.4.4) and (7.4.5)) turns out to be some constant $C \in \bar{\mathbb{Q}}_p^\times$ times a product of several p -adic L -functions (see [126, between Definition 8.25 to Lemma 8.26]), which are units in $\mathcal{O}^{\mathrm{ur}}[[\Gamma_K]]$ (see [126, Proposition 8.27]) by our choices for $\boldsymbol{\theta}$ and \mathbf{h} (similar as in [126, Section 8.2]). Here to remove the square-free conductor assumption in [126] for prime divisors of N split in K , we compute here a local triple product integral below for supercuspidals which is not studied in [126, Section 8D].

Supercuspidal Triple Product Integral

Proposition 7.27. *Suppose π_ℓ is supercuspidal representation with trivial character and conductor p^t , $t \geq 2$ and $\varphi_\ell \in \pi_\ell$ is a new vector. Let $\tilde{\pi}_\ell$ be the contragradient representation of π_ℓ and $\tilde{\varphi}_\ell \in \tilde{\pi}_\ell$ be the new vector. Consider the matrix coefficient*

$$\Phi = \Phi_{\varphi_\ell, \tilde{\varphi}_\ell}(g) \stackrel{\mathrm{def}}{=} \langle \pi(g) \varphi_\ell, \tilde{\varphi}_\ell \rangle$$

normalized such that $\Phi_{\varphi_\ell, \tilde{\varphi}_\ell}(1) = 1$. Then for $g \in \mathrm{diag}(\ell^n, 1) \begin{pmatrix} 1 & \ell^{-n} \mathbb{Z}_\ell \\ & 1 \end{pmatrix} K_t$, $\Phi(g) \neq 0$ only when $n = 0$. In that case $\Phi(g) = 1$. For $g \in \mathrm{diag}(1, \ell^n) \begin{pmatrix} 1 & \ell^{t-n} \mathbb{Z}_\ell \\ \ell^{t-n} \mathbb{Z}_\ell & 1 \end{pmatrix} K_t$, $\Phi(g) \neq 0$ only when $n = 0$. In this case $\Phi(g) = 1$.

This is an easy consequence of [60, Proposition 3.1]. The following corollary follows immediately from the above proposition and the computations in [126, Section 8D, especially Lemma 8.13].

Corollary 7.28. Let K_t be the level group defined before [126, Lemma 8.13]. Let π_ℓ be a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ with trivial character and conductor ℓ^t . Let $\varphi_\ell \in \pi_\ell$ and $\tilde{\varphi}_\ell \in \tilde{\pi}_\ell$ be as above. Let $\pi_h = \pi(\chi_{h,1}, \chi_{h,2})$, $\pi_\theta(\chi_{\theta,1}, \chi_{\theta,2})$ and $\chi_{h,1}, \chi_{h,2}, \chi_{\theta,1}, \chi_{\theta,2}$ be characters of \mathbb{Q}_ℓ^\times with conductors ℓ^{t_1} and $t_1 > t$ such that $\chi_{h,1}\chi_{\theta,1}$ and $\chi_{h,2}\chi_{\theta,2}$ are both unramified. Suppose $f_{\chi_\theta}, f_{\chi_h}, \tilde{f}_{\tilde{\chi}_\theta}$ and $\tilde{f}_{\tilde{\chi}_h}$ are as in [126, Section 8D]. Then Ichino's local triple product integral

$$I_\ell(\varphi_\ell \otimes f_{\chi_\theta} \otimes f_{\chi_h}, \tilde{\varphi}_\ell \otimes \tilde{f}_{\tilde{\chi}_\theta} \otimes \tilde{f}_{\tilde{\chi}_h}) = \mathrm{Vol}(K_{t_1}).$$

Now we arrived at the

Proposition 7.29. The

$$\mathrm{FJ}_{\beta, \theta^*}^h(E_{\mathrm{Kling}}) = \langle e^{\mathrm{ord}} l_{\theta^*}(\mathrm{FJ}_\beta(E_{\mathrm{Kling}})), h \rangle$$

is a product of an element in $\mathcal{O}^{\mathrm{ur}}[[\Gamma_K]]^\times$ and an element in $\bar{\mathbb{Q}}_p^\times$.

Proof. We compute the ratio of the product of (7.4.4) and (7.4.5) over the p -adic L -functions $\mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_5 \mathcal{L}_6$. From (7.4.4) and the computation of the Ichino's triple product integral (and that the $\mathbf{B}(\mathbf{h}, \tilde{\mathbf{h}}_3)$ and $\mathbf{B}'(\mathbf{h}, \tilde{\mathbf{h}}_3)$ are interpolated by \mathcal{L}_3 and \mathcal{L}_4 are used here), we see that the ratio is a fixed nonzero constant times the above $C_\infty \neq 0$ (see [126, Proposition 8.29]). Note also that $\mathcal{L}_1, \mathcal{L}_5$ and \mathcal{L}_6 are units, while \mathcal{L}_2 is a fixed nonzero constant. These altogether imply the $\mathrm{FJ}_{\beta, \theta^*}^h(E_{\mathrm{Kling}})$ is a divisor of a product of an element in $\mathcal{O}^{\mathrm{ur}}[[\Gamma_K]]^\times$ and an element in $\bar{\mathbb{Q}}_p^\times$, and thus itself also is. \square

Now we state the main theorem of this section. We start with the following definition.

Definition 7.30. We let \mathbb{T}_D be the reduced Hecke algebra generated by the Hecke operators at unramified primes acting on the space of the two variable family of semi-ordinary cusp forms with level group K_D , the U_α operator at p , and then take the reduced quotient. Let the Eisenstein ideal I_D of \mathbb{T}_D be generated by $\{t - \lambda(t)\}_t$ for t in the abstract Hecke algebra where $\lambda(t)$ is the Hecke eigenvalue of t acting on E_{Kling} . Let the Eisenstein ideal \mathcal{E}_D be the inverse image of I_D in $\mathcal{O}[[\Gamma_K]] \subset \mathbb{T}_D$.

In [126, Lemma 9.1], it is proved that if \mathcal{P} is a height 1 prime ideal of $\mathcal{O}^{\mathrm{ur}}[[\Gamma_K]]$ different from (p) , then the order of divisibility of $\mathcal{L}_{f,K}^{\mathrm{Gr}, \Sigma}$ by \mathcal{P} is less or equal to the order of divisibility of \mathcal{E}_D by \mathcal{P} . This is generalized to our setting as follows.

Lemma 7.31. Let \mathcal{P} be a height 1 prime of $\mathcal{O}^{\mathrm{ur}}[[\Gamma_K]]$ which is not (p) . Then

$$\mathrm{ord}_{\mathcal{P}}(\mathcal{L}_{f,K}^{\mathrm{Gr}, \Sigma}) \leq \mathrm{ord}_{\mathcal{P}}(\mathcal{E}_D).$$

Proof. The proof is entirely similar to that [126, Lemma 9.1] using proposition 7.29 and theorem 7.18 above in place of Theorem 3.6 and the functional constructed in section 8.5 of [126]. \square

Theorem 7.32. Let f be a normalized cuspidal eigenform of even weight k , trivial character and conductor N prime to p , such that either $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible, or f is ordinary at p . Suppose for each prime divisor q of N non-split in \mathcal{K} , we have $q||N$, and that there is at least one such prime. Suppose 2 is non-split in \mathcal{K} , then $2||N$. Suppose also that $\bar{\rho}_f$ is absolutely irreducible over G_K . Then we have

$$\mathrm{char}_{\mathcal{O}^{\mathrm{ur}}[[\Gamma_K]]}(X_K^{\mathrm{Gr}}(f) \otimes_{\mathcal{O}} \mathcal{O}^{\mathrm{ur}}) \subseteq (\mathcal{L}_K^{\mathrm{Gr}}(f))$$

up to height one primes which are pullbacks of primes in $\mathcal{O}[[\Gamma^+]]$. Moreover, if for each $\ell|N$ non-split in \mathcal{K} , we have ℓ is ramified in \mathcal{K} and π_ℓ is the Steinberg representation twisted by χ_{ur} for χ_{ur} being the unramified character sending ℓ to $(-1)\ell^{\frac{k}{2}-1}$, then the whole containment above is true.

Proof. As in [20, Theorem 8.2.1], the main theorem can be proven from Proposition 7.29 in almost the same way as the proof at the end of [126, Section 9.2] and [117, Theorem 7.5]. First prove the Σ -primitive version: we use the lattice construction (Proposition 9.2 in [126, Section 9.2]) to show that \mathcal{E}_D contains the characteristic ideal of the dual Selmer group. The only difference is to check the condition (9) in the “Set up” of *loc.cit.*. As in [117,

Theorem 7.5], let R be the four dimensional $\tilde{\Lambda}$ -valued pseudo-character of $G_{\mathcal{K}}$ corresponding to the space of semi-ordinary cuspidal families on $U(3, 1)$. We suppose for contradiction our pseudo-character $R = R_1 + R_2 + R_3$ where R_1 and R_2 are 1-dimensional and R_3 is 2-dimensional. Then by residual irreducibility of $\bar{\rho}_f$ we can associate to R_3 a 2-dimensional $\mathbb{T}_{\mathcal{D}}$ -coefficient Galois representation. Take an arithmetic point x in the absolute convergence region for Klingen Eisenstein series of sufficiently regular weight (in the sense that $a_2 - a_3 \gg 0$ and $a_3 + b_1 \gg 0$) and consider the specialization of the Galois representation to x (the specialization of R to x corresponds to a classical cuspidal automorphic representation of $U(3, 1)$ unramified at p). First of all as in [117, Theorem 7.5] a twist of this specialization of R_3 descends to a Galois representation of $G_{\mathbb{Q}}$ which we denote as $R_{3,x}$. We know that $R_{3,x}$ has Hodge-Tate weight 0, $k-1$ and is deRham (by the corresponding property for the semi-ordinary representation $R_x = R_1 + R_2 + R_3$). The $R_{3,x}$ is modular by the modularity lifting result of Pan in [95, Theorem 1.0.4] which states that any odd Galois representation

$$\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O})$$

unramified outside a finite set of primes such that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible and de Rham with distinct Hodge-Tate weights is modular. These imply that R_x is CAP (see [117, proof of Theorem 7.5]), that is, it has the same system of Hecke eigenvalues as a Klingen-type Eisenstein series), contradicting the result of [51, Theorem 2.5.6].

We first get the divisibility for $\mathcal{L}_{f,\mathcal{K}}^{\Sigma}$, up to height one primes which are pullbacks of height one primes of $\hat{\mathcal{O}}^{\mathrm{ur}}[[\Gamma^+]]$, the corresponding result for $\mathcal{L}_{f,\mathcal{K}}$ follows by putting back local Euler factors at Σ using [50, Proposition 2.4] (note that \mathcal{K}_{∞} contains the cyclotomic \mathbb{Z}_p -extension). Finally we use the last assumption of the theorem to apply [58] on the vanishing of the anticyclotomic μ -invariant of the p -adic L -function in our theorem and see that it is not contained in any height one prime of $\mathcal{O}[[\Gamma^+]]$ (and thus is co-prime to the Dirichlet p -adic L -function showing up in the constant terms of the Klingen Eisenstein family). \square

References

- [1] Jeffrey Adams. The theta correspondence over \mathbb{R} . In *Harmonic analysis, group representations, automorphic forms and invariant theory*, volume 12 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 1–39. World Sci. Publ., Hackensack, NJ, 2007.
- [2] Avner Ash and Glenn Stevens. p -adic deformations of cohomology classes of subgroups of $\mathrm{GL}(n, \mathbf{Z})$. *Collect. Math.*, 48(1-2):1–30, 1997. Journées Arithmétiques (Barcelona, 1995).
- [3] Daniel Barrera, Mladen Dimitrov, and Andrei Jorza. p -adic l -functions of Hilbert cusp forms and the trivial zero conjecture. Preprint.
- [4] Joël Bellaïche. Ranks of Selmer groups in an analytic family. *Trans. Amer. Math. Soc.*, 364(9):4735–4761, 2012.
- [5] Joël Bellaïche and Gaëtan Chenevier. Families of Galois representations and Selmer groups. *Astérisque*, (324):xii+314, 2009.
- [6] Laurent Berger. Représentations p -adiques et équations différentielles. *Invent. Math.*, 148(2):219–284, 2002.
- [7] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive p -adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977.
- [8] Adel Betina and Chris Williams. Arithmetic of p -irregular modular forms: families and p -adic l -functions. Preprint, 2020.
- [9] Spencer Bloch and Kazuya Kato. L -functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.
- [10] Gebhard Böckle. Demuškin groups with group actions and applications to deformations of Galois representations. *Compositio Math.*, 121(2):109–154, 2000.

- [11] Gebhard Böckle. On the density of modular points in universal deformation spaces. *Amer. J. Math.*, 123(5):985–1007, 2001.
- [12] Christophe Breuil and Peter Schneider. First steps towards p -adic Langlands functoriality. *J. reine angew. Math.*, 610:149–180, 2007.
- [13] David Burns and Matthias Flach. Motivic L -functions and Galois module structures. *Math. Annalen*, 305:65–102, 1996.
- [14] Ashay Burungale, Chris Skinner, and Yichao Tian. Elliptic curves and Beilinson-Kato elements: rank one aspects. Preprint, 2019.
- [15] Colin J. Bushnell and Guy Henniart. *The local Langlands conjecture for $\mathrm{GL}(2)$* , volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
- [16] Henri Carayol. Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet. In *p -adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, volume 165 of *Contemp. Math.*, pages 213–237. Amer. Math. Soc., Providence, RI, 1994.
- [17] Pierre Cartier. Representations of p -adic groups: a survey. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.
- [18] William Casselman. On some results of Atkin and Lehner. *Math. Ann.*, 201:301–314, 1973.
- [19] Francesc Castella, Grossi Giada, Jaehoon Lee, and Christopher Skinner. On the anti-cyclotomic iwasawa theory of rational elliptic curves at Eisenstein primes. *Inventiones Mathematicae*, to appear, 2021.
- [20] Francesc Castella, Zheng Liu, and Xin Wan. Iwasawa–Greenberg main conjecture for non-ordinary modular forms and eisenstein congruences on $\mathrm{GU}(3, 1)$. Preprint, arXiv. 2109.08375.
- [21] Francesc Castella and Xin Wan. The Iwasawa main conjectures for GL_2 and derivatives of p -adic L -functions. to appear in *Adv. Math.*
- [22] Gaëtan Chenevier. Sur la densité des représentations cristallines de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. *Math. Ann.*, 355(4):1469–1525, 2013.
- [23] Frédéric Cherbonnier and Pierre Colmez. Théorie d’Iwasawa des représentations p -adiques d’un corps local. *J. Amer. Math. Soc.*, 12(1):241–268, 1999.
- [24] Robert Coleman and Barry Mazur. The eigencurve. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge, 1998.
- [25] Pierre Colmez. Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (ϕ, Γ) -modules. *Astérisque*, (330):281–509, 2010.
- [26] Pierre Colmez and Shanwen Wang. Une factorisation du système de Beilinson-Kato. Preprint, 2021.
- [27] Henri Darmon, Fred Diamond, and Richard Taylor. *Fermat’s Last Theorem*. Number 41 in Current Developments in Mathematics. 2000.
- [28] Pierre Deligne. Valeurs de fonctions L et périodes d’intégrales. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 313–346. Amer. Math. Soc., Providence, R.I., 1979. With an appendix by N. Koblitz and A. Ogus.
- [29] Fred Diamond. On deformation rings and Hecke rings. *Ann. of Math. (2)*, 144(1):137–166, 1996.
- [30] Fred Diamond, Matthias Flach, and Li Guo. The Tamagawa number conjecture of adjoint motives of modular forms. *Ann. Sci. École Norm. Sup. (4)*, 37(5):663–727, 2004.

- [31] Bas Edixhoven. The weight in Serre's conjectures on modular forms. *Invent. Math.*, 109(3):563–594, 1992.
- [32] Ellen Eischen and Xin Wan. p -adic Eisenstein series and L -functions of certain cusp forms on definite unitary groups. *J. Inst. Math. Jussieu*, 15(3):471–510, 2016.
- [33] Matthew Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.*, 164(1):1–84, 2006.
- [34] Matthew Emerton. Local-Global compatibility in the p -adic Langlands program $\mathrm{GL}_2 / \mathbf{Q}$. 2011.
- [35] Matthew Emerton and David Helm. The local Langlands correspondence for GL_n in families. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(4):655–722, 2014.
- [36] Matthew Emerton, Robert Pollack, and Tom Weston. Variation of Iwasawa invariants in Hida families. *Invent. Math.*, 163(3):523–580, 2006.
- [37] Jean-Marc Fontaine. Sur certains types de représentations p -adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate. *Ann. of Math. (2)*, 115(3):529–577, 1982.
- [38] Jean-Marc Fontaine. Sur certains types de représentations p -adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate. *Ann. of Math. (2)*, 115(3):529–577, 1982.
- [39] Jean-Marc Fontaine. Le corps des périodes p -adiques. *Astérisque*, (223):59–111, 1994. With an appendix by Pierre Colmez, Périodes p -adiques (Bures-sur-Yvette, 1988).
- [40] Jean-Marc Fontaine and Guy Laffaille. Construction de représentations p -adiques. *Ann. Sci. École Norm. Sup. (4)*, 15(4):547–608 (1983), 1982.
- [41] Jean-Marc Fontaine and Yi Ouyang. Theory of p -adic Galois representations. Preprint.
- [42] Jean-Marc Fontaine and Bernadette Perrin-Riou. Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L . In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 599–706. Amer. Math. Soc., Providence, RI, 1994.
- [43] Olivier Fouquet. Dihedral Iwasawa theory of nearly ordinary quaternionic automorphic forms. *Compos. Math.*, 149(3):356–416, 2013.
- [44] Olivier Fouquet. p -adic properties of motivic fundamental lines. *J. Éc. polytech. Math.*, 4:37–86, 2017.
- [45] Kazuhiro Fujiwara. Deformation rings and Hecke algebras in the totally real case, 1999. Preprint, 99pp.
- [46] Takako Fukaya and Kazuya Kato. On conjectures of Sharifi. Preprint, 2012.
- [47] Ralph Greenberg. Iwasawa theory for p -adic representations. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 97–137. Academic Press, Boston, MA, 1989.
- [48] Ralph Greenberg. Iwasawa theory for motives. In *L -functions and arithmetic (Durham, 1989)*, volume 153 of *London Math. Soc. Lecture Note Ser.*, pages 211–233. Cambridge Univ. Press, Cambridge, 1991.
- [49] Ralph Greenberg. Iwasawa theory and p -adic deformations of motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 193–223. Amer. Math. Soc., Providence, RI, 1994.
- [50] Ralph Greenberg and Vinayak Vatsal. On the Iwasawa invariants of elliptic curves. *Invent. Math.*, 142(1):17–63, 2000.
- [51] Michael Harris. Eisenstein series on Shimura varieties. *Ann. of Math. (2)*, 119(1):59–94, 1984.
- [52] David Helm. The Bernstein center of the category of smooth $W(k)[\mathrm{GL}_n(F)]$ -modules. *Forum Math. Sigma*, 4:Paper No. e11, 98, 2016.
- [53] David Helm. Whittaker models and the integral Bernstein center for GL_n . *Duke Math. J.*, 165(9):1597–1628, 2016.

- [54] David Helm and Gilbert Moss. Converse theorems and the Local Langlands Correspondance in families. 2016.
- [55] H. Hida and J. Tilouine. Anti-cyclotomic Katz p -adic L -functions and congruence modules. *Ann. Sci. École Norm. Sup. (4)*, 26(2):189–259, 1993.
- [56] Haruzo Hida. p -adic ordinary Hecke algebras for $\mathrm{GL}(2)$. *Ann. Inst. Fourier (Grenoble)*, 44(5):1289–1322, 1994.
- [57] Ming-Lun Hsieh. Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields. *J. Amer. Math. Soc.*, 27(3):753–862, 2014.
- [58] Ming-Lun Hsieh. Special values of anticyclotomic Rankin-Selberg L -functions. *Doc. Math.*, 19:709–767, 2014.
- [59] Ming-Lun Hsieh. Hida families and p -adic triple product L -functions. *Amer. J. Math.*, 143(2):411–532, 2021.
- [60] Yueke Hu. Triple product formula and mass equidistribution on modular curves of level n . Preprint, 2014.
- [61] Tamotsu Ikeda. On the theory of Jacobi forms and Fourier-Jacobi coefficients of Eisenstein series. *J. Math. Kyoto Univ.*, 34(3):615–636, 1994.
- [62] Luc Illusie. Autour du théorème de monodromie locale. *Astérisque*, (223):9–57, 1994. Périodes p -adiques (Bures-sur-Yvette, 1988).
- [63] Kenkichi Iwasawa. Analogies between number fields and function fields. In *Some Recent Advances in the Basic Sciences, Vol. 2 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1965–1966)*, pages 203–208. Belfer Graduate School of Science, Yeshiva Univ., New York, 1969.
- [64] Dimitar Jetchev, Christopher Skinner, and Xin Wan. The Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one. *Camb. J. Math.*, 5(3):369–434, 2017.
- [65] Kazuya Kato. Iwasawa theory and p -adic Hodge theory. *Kodai Math. J.*, 16(1):1–31, 1993.
- [66] Kazuya Kato. Lectures on the approach to Iwasawa theory for Hasse-Weil L -functions via B_{dR} . I. In *Arithmetic algebraic geometry (Trento, 1991)*, volume 1553 of *Lecture Notes in Math.*, pages 50–163. Springer, Berlin, 1993.
- [67] Kazuya Kato. Euler systems, Iwasawa theory, and Selmer groups. *Kodai Math. J.*, 22(3):313–372, 1999.
- [68] Kazuya Kato. p -adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):ix, 117–290, 2004. Cohomologies p -adiques et applications arithmétiques. III.
- [69] Kiran S. Kedlaya, Jonathan Pottharst, and Liang Xiao. Cohomology of arithmetic families of (φ, Γ) -modules. *J. Amer. Math. Soc.*, 27(4):1043–1115, 2014.
- [70] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre’s modularity conjecture. I. *Invent. Math.*, 178(3):485–504, 2009.
- [71] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre’s modularity conjecture. II. *Invent. Math.*, 178(3):505–586, 2009.
- [72] Walter Kim. *Ramification points on the eigencurve and the two variable symmetric square p -adic L -function*. PhD thesis, University of California, Berkeley, 2006.
- [73] Guido Kings, David Loeffler, and Sarah Livia Zerbes. Rankin-Eisenstein classes and explicit reciprocity laws. *Camb. J. Math.*, 5(1):1–122, 2017.
- [74] Guido Kings, David Loeffler, and Sarah Livia Zerbes. Rankin-Eisenstein classes and explicit reciprocity laws. *Camb. J. Math.*, 5(1):1–122, 2017.
- [75] Guido Kings, David Loeffler, and Sarah Livia Zerbes. Rankin-Eisenstein classes for modular forms. *Amer. J. Math.*, 142(1):79–138, 2020.
- [76] Finn Faye Knudsen and David Mumford. The projectivity of the moduli space of stable curves. I. Preliminaries on “det” and “Div”. *Math. Scand.*, 39(1):19–55, 1976.

- [77] Shin-ichi Kobayashi. Iwasawa theory for elliptic curves at supersingular primes. *Invent. Math.*, 152(1):1–36, 2003.
- [78] Viktor Kolyvagin. Euler systems. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 435–483. Birkhäuser Boston, Boston, MA, 1990.
- [79] Tomio Kubota and Heinrich-Wolfgang Leopoldt. Eine p -adische Theorie der Zetawerte. I. Einführung der p -adischen Dirichletschen L -Funktionen. *J. reine angew. Math.*, 214/215:328–339, 1964.
- [80] Kai-Wen Lan. Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties. *J. Reine Angew. Math.*, 664:163–228, 2012.
- [81] Michel Lazard. Les zéros des fonctions analytiques d'une variable sur un corps valué complet. *Inst. Hautes Études Sci. Publ. Math.*, (14):47–75, 1962.
- [82] Antonio Lei, David Loeffler, and Sarah Livia Zerbes. Euler systems for Rankin-Selberg convolutions of modular forms. *Ann. of Math. (2)*, 180(2):653–771, 2014.
- [83] David Loeffler. A note on p -adic Rankin-Selberg L -functions. *Canad. Math. Bull.*, 61(3):608–621, 2018.
- [84] David Loeffler and Sarah Livia Zerbes. Iwasawa theory and p -adic L -functions over \mathbb{Z}_p^2 -extensions. *Int. J. Number Theory*, 10(8):2045–2095, 2014.
- [85] David Loeffler and Sarah Livia Zerbes. Rankin-Eisenstein classes in Coleman families. *Res. Math. Sci.*, 3:Paper No. 29, 53, 2016.
- [86] Ju. I. Manin. Non-Archimedean integration and p -adic Jacquet-Langlands L -functions. *Uspehi Mat. Nauk*, 31(1(187)):5–54, 1976.
- [87] Barry Mazur. Rational points of abelian varieties with values in towers of number fields. *Invent. Math.*, 18:183–266, 1972.
- [88] Barry Mazur and Peter Swinnerton-Dyer. Arithmetic of Weil curves. *Invent. Math.*, 25:1–61, 1974.
- [89] Barry Mazur, J. Tate, and J. Teitelbaum. On p -adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent. Math.*, 84(1):1–48, 1986.
- [90] Barry Mazur and Andrew Wiles. Class fields of abelian extensions of \mathbf{Q} . *Invent. Math.*, 76(2):179–330, 1984.
- [91] Kentaro Nakamura. Iwasawa theory of de Rham (φ, Γ) -modules over the Robba ring. *J. Inst. Math. Jussieu*, 13(1):65–118, 2014.
- [92] Kentaro Nakamura. Zeta morphisms for rank two universal deformations. Preprint, 2020.
- [93] Jan Nekovář. Selmer complexes. *Astérisque*, (310):559, 2006.
- [94] Masami Ohta. Ordinary p -adic étale cohomology groups attached to towers of elliptic modular curves. II. *Math. Ann.*, 318(3):557–583, 2000.
- [95] Lue Pan. The Fontaine-Mazur conjecture in the residually reducible case. Preprint, 2019.
- [96] Vytautas Paškūnas. The image of Colmez's Montreal functor. *Publ. Math. Inst. Hautes Études Sci.*, 118:1–191, 2013.
- [97] Vytautas Paškūnas. On the Breuil-Mézard conjecture. *Duke Math. J.*, 164(2):297–359, 2015.
- [98] Vytautas Paškūnas. On 2-dimensional 2-adic Galois representations of local and global fields. *Algebra Number Theory*, 10(6):1301–1358, 2016.
- [99] Bernadette Perrin-Riou. Représentations p -adiques ordinaires. Number 223, pages 185–220. 1994. With an appendix by Luc Illusie, Périodes p -adiques (Bures-sur-Yvette, 1988).
- [100] Bernadette Perrin-Riou. Fonctions L p -adiques des représentations p -adiques. *Astérisque*, (229):198, 1995.

- [101] Bernadette Perrin-Riou. Systèmes d’Euler p -adiques et théorie d’Iwasawa. *Ann. Inst. Fourier (Grenoble)*, 48(5):1231–1307, 1998.
- [102] Jay Pottharst. Cyclotomic Iwasawa Theory of Motives. Preprint, 2012.
- [103] Jonathan Pottharst. Analytic families of finite-slope Selmer groups. *Algebra Number Theory*, 7(7):1571–1612, 2013.
- [104] Ravi Ramakrishna. On a variation of Mazur’s deformation functor. *Compositio Math.*, 87(3):269–286, 1993.
- [105] Kenneth A. Ribet. A modular construction of unramified p -extensions of $\mathbb{Q}(\mu_p)$. *Invent. Math.*, 34(3):151–162, 1976.
- [106] Kenneth A. Ribet. Abelian varieties over \mathbb{Q} and modular forms. In *Algebra and topology 1992 (Taejön)*, pages 53–79. Korea Adv. Inst. Sci. Tech., Taejön, 1992.
- [107] David E. Rohrlich. Nonvanishing of L -functions for $\mathrm{GL}(2)$. *Invent. Math.*, 97(2):381–403, 1989.
- [108] Karl Rubin. The “main conjectures” of Iwasawa theory for imaginary quadratic fields. *Invent. Math.*, 103(1):25–68, 1991.
- [109] Karl Rubin. *Euler systems*, volume 147 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.
- [110] Takeshi Saito. Weight-monodromy conjecture for l -adic representations associated to modular forms. A supplement to: “Modular forms and p -adic Hodge theory” [Invent. Math. **129** (1997), no. 3, 607–620; MR1465337 (98g:11060)]. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 427–431. Kluwer Acad. Publ., Dordrecht, 2000.
- [111] A. J. Scholl. Motives for modular forms. *Invent. Math.*, 100(2):419–430, 1990.
- [112] Jean-Pierre Serre. Classes des corps cyclotomiques (d’après K. Iwasawa). In *Séminaire Bourbaki, Vol. 5*, pages Exp. No. 174, 83–93. Soc. Math. France, Paris, 1958.
- [113] Jean-Pierre Serre. Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Duke Math. J.*, 54(1):179–230, 1987.
- [114] C. M. Skinner and A. J. Wiles. Residually reducible representations and modular forms. *Inst. Hautes Études Sci. Publ. Math.*, (89):5–126 (2000), 1999.
- [115] Chris Skinner and Wei Zhang. Indivisibility of Heegner points in the multiplicative case. Preprint, 2014.
- [116] Christopher Skinner. Multiplicative reduction and the cyclotomic main conjecture for GL_2 . *Pacific J. Math.*, 283(1):171–200, 2016.
- [117] Christopher Skinner and Eric Urban. The Iwasawa main conjecture for GL_2 . *Invent. Math.*, 195(1):1–277, 2014.
- [118] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.
- [119] Francisco Thaine. On the ideal class groups of real abelian number fields. *Ann. of Math. (2)*, 128(1):1–18, 1988.
- [120] Otmar Venjakob. From the Birch and Swinnerton-Dyer Conjecture to non-commutative Iwasawa theory via the Equivariant Tamagawa Number Conjecture—a survey. In *L -functions and Galois representations (Durham, July 2004)*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 333–380. Cambridge Univ. Press, Cambridge, 2004.
- [121] Marie-France Vignéras. *Représentations l -modulaires d’un groupe réductif p -adique avec $l \neq p$* , volume 137 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1996.
- [122] M. M. Višik. Nonarchimedean measures associated with Dirichlet series. *Mat. Sb. (N.S.)*, 99(141)(2):248–260, 296, 1976.
- [123] Xin Wan. Families of nearly ordinary Eisenstein series on unitary groups. *Algebra Number Theory*, 9(9):1955–2054, 2015. With an appendix by Kai-Wen Lan.

- [124] Xin Wan. The Iwasawa main conjecture for Hilbert modular forms. *Forum Math. Sigma*, 3:e18, 95, 2015.
- [125] Xin Wan. Iwasawa Main Conjecture for Supersingular Elliptic curves. 2015.
- [126] Xin Wan. Iwasawa main conjecture for Rankin-Selberg p -adic L -functions. *Algebra Number Theory*, 14(2):383–483, 2020.
- [127] A. Wiles. The Iwasawa conjecture for totally real fields. *Ann. of Math. (2)*, 131(3):493–540, 1990.
- [128] Andrew Wiles. Modular elliptic curves and Fermat’s last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

Olivier Fouquet

DÉPARTEMENT DE MATHÉMATIQUES
16, ROUTE DE GRAY 25000 BESANÇON
FRANCE
E-mail address: olivier.fouquet@univ-fcomte.fr
Telephone number: +33622645039

Xin Wan

MORNINGSIDE CENTER OF MATHEMATICS
ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCE
BEIJING, 100190
CHINA
E-mail Address: xwan@math.ac.cn