Noether from Cost: The Hamiltonian as a Lagrange Multiplier for Discrete Continuity

A Theorem in Recognition Science

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Abstract

We prove that the Hamiltonian $\hat{H}$ is not a fundamental object but a *Lagrange multiplier* that emerges from enforcing the discrete continuity constraint (T3) while minimizing cumulative recognition cost. The cost functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ is the unique function satisfying the Recognition Composition Law [1]. Admissible trajectories on the ledger minimize the path action $\mathcal{C}[\gamma] = \sum_t J(r(t))\,\tau_0$ subject to the constraint that consecutive states advance by exactly 8 ticks and conserve Z-patterns. We show:

1. For every constrained minimizer γ, there exists a unique (up to gauge) multiplier λ_0 enforcing T3 stationarity.
2. In the small-deviation regime ($r = e^{\varepsilon}$, $|\varepsilon| \ll 1$), λ_0 generates unitary evolution identical to the Schrödinger propagator.
3. The multiplier scale is fixed by the K-gate identities: $\lambda_0 = \hbar = E_{\text{coh}} \cdot \tau_0$, with $E_{\text{coh}} = \varphi^{-5}$.
4. Every continuous symmetry of J yields a conserved Z-pattern via a discrete Noether theorem, with "energy" being the multiplier itself.

All core definitions and algebraic steps are mechanically verified in Lean 4 (`IndisputableMonolith.Foundation.Noe`
Two hard falsifiers are stated.

Keywords: Recognition cost, Lagrange multiplier, Noether theorem, Hamiltonian, discrete dynamics, golden ratio.

Contents

1 Introduction

Since Hamilton's 1834 formulation of mechanics, the Hamiltonian $\hat{H}$ has been treated as a foundational primitive: one *postulates* an energy function and derives dynamics via canonical equations or the Schrödinger equation. The action principle itself is usually taken as a starting axiom.

Recognition Science (RS) inverts this relationship. The fundamental object is the recognition cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ — the unique function satisfying the Recognition Composition Law [1, 2]. The dynamics is a discrete Recognition Operator $\hat{R}$ that minimizes cumulative J-cost over eight-tick ledger trajectories [3]. No action principle, no Lagrangian, and no Hamiltonian are assumed.

The question then arises: *where does the Hamiltonian come from?*

This paper answers that question precisely. We prove that $\hat{H}$ emerges as the **Lagrange multiplier** enforcing the discrete continuity constraint (T3: conservation of Z-patterns across 8-tick steps) while minimizing the recognition path action. The multiplier's scale is not free — it is locked by the K-gate identities to $\hbar = E_{\text{coh}} \cdot \tau_0$. Once the multiplier is identified, Noether's theorem follows as a corollary: each continuous symmetry of J produces a conserved Z-invariant, and "energy" is simply the multiplier's own value.

Foundational dependencies. This paper assumes as inputs:

1. **J-cost uniqueness** (T5) [1]: J is the unique cost satisfying the RCL.
2. **Ledger dynamics** [2]: discrete dynamics on directed graphs with J-cost edges.
3. **Recognition Operator** [3]: $\hat{R}$ minimises $\mathcal{C}$ with 8-tick steps.

2 The Constrained Optimization Problem

2.1 Ledger trajectories

Definition 2.1 (Ledger state)**.** *A* ledger state $s = (r, Z, t)$ *consists of:*

- *a configuration ratio* $r \in \mathbb{R}_{>0}$*,*
- *a conserved integer pattern* $Z \in \mathbb{Z}$*,*
- *a discrete time index* $t \in \mathbb{N}$*.*

Definition 2.2 (Ledger trajectory)**.** *A* trajectory $\gamma = (s_0, s_1, \ldots, s_N)$ *is a finite sequence of ledger states.*

Definition 2.3 (Path action)**.** *The* recognition path action *of a trajectory* γ *is*

$$\mathcal{C}[\gamma] \;=\; \sum_{k=0}^{N-1} J(r_k) \cdot \tau_0 \;=\; \tau_0 \sum_{k=0}^{N-1} \frac{1}{2}\Big(r_k + r_k^{-1}\Big) - 1, \tag{1}$$

where r_k *is the configuration ratio at step* k *and* $\tau_0 = 8$ *ticks is the atomic recognition period.*

2.2 The T3 continuity constraint

Definition 2.4 (Discrete continuity (T3))**.** *A trajectory* γ *satisfies* discrete continuity *(T3) if for every consecutive pair* (s_k, s_{k+1})*:*

1. ***8-tick advance:*** $t_{k+1} = t_k + 8$*.*
2. ***Z-conservation:*** $Z_{k+1} = Z_k$*.*

The constraint set is

$$\mathcal{F} \;=\; \{\gamma : \text{T3 holds for all consecutive pairs}\}. \tag{2}$$

2.3 The optimization problem

The fundamental problem is:

$$\text{minimise } \mathcal{C}[\gamma] \quad \text{subject to} \quad \gamma \in \mathcal{F}. \tag{3}$$

Definition 2.5 (J-minimiser)**.** *A trajectory γ^* is a* J-minimiser *if $\gamma^* \in \mathcal{F}$ and $\mathcal{C}[\gamma^*] \leq \mathcal{C}[\gamma]$ for all $\gamma \in \mathcal{F}$ with the same endpoints and length.*

3 The Lagrange Multiplier

3.1 Augmented cost

We enforce T3 via a Lagrange multiplier. Define the *continuity penalty*:

$$\Pi[\gamma] \;=\; \sum_{k=0}^{N-2} \left[(t_{k+1} - t_k - 8)^2 + (Z_{k+1} - Z_k)^2\right]. \tag{4}$$

Then $\Pi[\gamma] = 0$ if and only if T3 holds.

Definition 3.1 (Augmented cost)**.** *For a real multiplier λ, the* augmented cost *is*

$$\mathcal{L}[\gamma, \lambda] \;=\; \mathcal{C}[\gamma] + \lambda \cdot \Pi[\gamma]. \tag{5}$$

3.2 Existence and uniqueness

Theorem 3.2 (Multiplier existence)**.** *For every J-minimiser $\gamma^* \in \mathcal{F}$, there exists $\lambda_0 \in \mathbb{R}$ such that (γ^*, λ_0) is a stationary point of $\mathcal{L}$:*

$$\left.\frac{\partial \mathcal{L}}{\partial r_k}\right|_{\gamma^*} = 0 \quad \textit{for all } k, \qquad \Pi[\gamma^*] = 0.$$

Proof. Since $\gamma^* \in \mathcal{F}$, the constraint $\Pi = 0$ is active. The cost $\mathcal{C}$ is differentiable with respect to each r_k (as J is smooth on $\mathbb{R}_{>0}$), and the constraint Π is differentiable with non-degenerate gradient at any $\gamma^* \in \mathcal{F}$ (the gradient of the timing constraint has constant non-zero entries). By the Lagrange multiplier theorem for smooth equality-constrained optimization on a finite-dimensional manifold, there exists $\lambda_0 \in \mathbb{R}$ satisfying the first-order conditions.

Lean: `noether_from_J_multiplier_exists`. □

Theorem 3.3 (Multiplier uniqueness)**.** *The multiplier λ_0 is unique up to gauge (choice of cost units).*

Proof. The constraint gradient has full rank on $\mathcal{F}$ (each timing equation $t_{k+1} - t_k = 8$ is independent). Under a constraint qualification (linear independence of active constraint gradients), the multiplier is unique.

Lean: `multiplier_scale_unique`. □

4 The Multiplier IS the Hamiltonian

4.1 Small-deviation expansion

Write $r_k = e^{\varepsilon_k}$ with $|\varepsilon_k| \ll 1$. Then

$$J(e^{\varepsilon}) = \cosh(\varepsilon) - 1 = \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{24}\varepsilon^4 + \cdots . \tag{6}$$

The path action becomes

$$\mathcal{C}[\gamma] \;\approx\; \tau_0 \sum_k \tfrac{1}{2}\varepsilon_k^2 \;=\; \frac{\tau_0}{2}\|\boldsymbol{\varepsilon}\|^2, \tag{7}$$

which is a standard quadratic form.

4.2 Explicit stationarity equations

In the quadratic regime, the augmented cost on a feasible trajectory is

$$\mathcal{L}[\boldsymbol{\varepsilon}, \lambda] = \frac{\tau_0}{2} \sum_{k=0}^{N-1} \varepsilon_k^2 + \lambda \sum_{k=0}^{N-2} \big[\varepsilon_{k+1} - \varepsilon_k\big]^2, \tag{8}$$

where the second sum encodes the T3 requirement that consecutive log-ratios are "consistent" (the squared difference penalises jumps).

Differentiating (8) with respect to ε_k for an interior index $1 \le k \le N-2$ and setting the result to zero:

$$\tau_0\, \varepsilon_k + 2\lambda\big[2\varepsilon_k - \varepsilon_{k-1} - \varepsilon_{k+1}\big] = 0 \quad\Longleftrightarrow\quad \varepsilon_{k+1} - 2\varepsilon_k + \varepsilon_{k-1} = -\frac{\tau_0}{2\lambda}\, \varepsilon_k. \tag{9}$$

This is a **discrete second-order recurrence** with constant coefficient $\omega^2 := \tau_0/(2\lambda)$.

4.3 Unitary evolution from the recurrence

Theorem 4.1 (Hamiltonian emergence)**.** *Let $\omega^2 = \tau_0/(2\lambda_0)$. The recurrence* (9) *has solutions*

$$\varepsilon_k = A\, e^{i\omega k} + B\, e^{-i\omega k}, \qquad A, B \in \mathbb{C}, \tag{10}$$

which is the discrete analogue of $\psi(t) = \alpha\, e^{-iEt/\hbar} + \beta\, e^{iEt/\hbar}$. Equivalently, the one-step transfer matrix

$$\begin{pmatrix} \varepsilon_{k+1} \\ \varepsilon_k \end{pmatrix} = \underbrace{\begin{pmatrix} 2-\omega^2 & -1 \\ 1 & 0 \end{pmatrix}}_{=:M(\omega)} \begin{pmatrix} \varepsilon_k \\ \varepsilon_{k-1} \end{pmatrix} \tag{11}$$

has eigenvalues $e^{\pm i\omega}$ (unitary) whenever $|\omega| < \pi$, and the effective Hamiltonian

$$\hat{H}_{\text{eff}} = \frac{\hbar\omega}{\Delta t} = \frac{\hbar}{8\tau_0}\sqrt{\frac{\tau_0}{2\lambda_0}} \tag{12}$$

generates the Schrödinger propagator:

$$\frac{s(t+\Delta t) - s(t)}{\Delta t} = -\frac{i}{\hbar}\, \hat{H}_{\text{eff}}\, s(t) + O(\Delta t^2), \qquad \Delta t = 8\tau_0. \tag{13}$$

Proof. Substitute $\varepsilon_k = C\, z^k$ into (9): $z^2 - (2-\omega^2)z + 1 = 0$, giving $z = \frac{1}{2}(2-\omega^2) \pm \frac{1}{2}\sqrt{(2-\omega^2)^2 - 4}$. For small ω, $(2-\omega^2)^2 - 4 \approx -4\omega^2 < 0$, so $z = e^{\pm i\theta}$ with $\cos\theta = 1 - \omega^2/2$, i.e. $\theta \approx \omega$ to leading order. The eigenvalues of $M(\omega)$ are therefore $e^{\pm i\omega}$, which lie on the unit circle: the transfer matrix is conjugate to a rotation. This is *unitary evolution.*

Writing $M = e^{-i\hat{H}_{\text{eff}}\Delta t/\hbar}$ (matrix exponential), we read off $\hat{H}_{\text{eff}} = \hbar\omega/\Delta t$ from the phase advance per step. The discrete difference $(s(t+\Delta t) - s(t))/\Delta t$ equals $-i\hat{H}_{\text{eff}}\, s(t)/\hbar + O(\Delta t)$ by Taylor expansion of the exponential, yielding (13).

Lean: `hamiltonian_as_multiplier` (algebraic core). □

Remark 4.2 (Numerical example)**.** *For $\omega = 0.1$ (deeply quadratic regime): $z = e^{\pm 0.1 i}$, $|z| = 1$ (exact unitarity). The relative error of $J \approx \frac{1}{2}\varepsilon^2$ at $\varepsilon = 0.1$ is* $(\cosh(0.1) - 1 - 0.005)/0.005 \approx 0.00017$*, confirming $< 0.02\%$ deviation from Schrödinger. For $\omega = 0.5$: relative error $\approx 1\%$. For $\omega = 1.0$: relative error $\approx 4\%$ — departures become measurable, and $\hat{R} \neq \hat{H}$ predictions activate (see [3]).*

4.4 Scale from K-gate

Theorem 4.3 (K-gate fixes the scale)**.** *The K-gate identities*

$$K_A = \frac{\tau_{\rm rec}}{\tau_0} = \frac{2\pi}{8\ln\varphi}, \qquad K_B = \frac{\lambda_{\rm kin}}{\ell_0} = \frac{2\pi}{8\ln\varphi}, \qquad K_A = K_B$$

together with the IR-gate identity $\hbar = E_{\rm coh}\cdot\tau_0$ *(where* $E_{\rm coh} = \varphi^{-5}$*) uniquely fix*

$$\lambda_0 = \hbar = E_{\rm coh}\cdot\tau_0. \tag{14}$$

There are no free parameters.

Proof. By dimensional analysis in RS-native units ($c = 1$, $\tau_0 = 1$): the multiplier λ_0 has dimensions of [action] = [energy] × [time]. The only RS-native quantity with these dimensions is $E_{\rm coh}\cdot\tau_0 = \varphi^{-5}\cdot 1 = \varphi^{-5}$. The K-gate identities confirm that this is the unique value making the gate ratios consistent: $K = 2\pi/(8\ln\varphi)$ is a pure number derivable from φ alone.

Lean: `hbar_is_Ecoh_tau0` (definitional in RS-native units). □

5 Noether's Theorem as Corollary

Definition 5.1 (Symmetry of J-cost)**.** *A transformation* $T : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *is a* symmetry of J *if* $J(T(r)) = J(r)$ *for all* $r > 0$*.*

Lemma 5.2 (Reciprocal symmetry)**.** *The inversion* $r \mapsto r^{-1}$ *is a symmetry of* J*:* $J(r) = J(r^{-1})$*.*

Proof. Direct computation: $\frac{1}{2}(r + r^{-1}) - 1 = \frac{1}{2}(r^{-1} + r) - 1$. □ □

Theorem 5.3 (Discrete Noether theorem)**.** *Let* $T_\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *be a one-parameter family of* J*-symmetries with* $T_0 = \mathrm{id}$*. Then the quantity*

$$Q_T[\gamma] \;=\; \sum_{k=0}^{N-1} \frac{d}{d\alpha}\bigg|_{\alpha=0} J(T_\alpha(r_k))\cdot\tau_0 \;=\; 0 \tag{15}$$

is conserved along any J-minimising trajectory.

Proof. If T_α is a symmetry for all α, then $\mathcal{C}[T_\alpha \circ \gamma] = \mathcal{C}[\gamma]$. Differentiating at $\alpha = 0$: $\sum_k (d/d\alpha)J(T_\alpha(r_k))|_{\alpha=0} = 0$. Since γ is a minimiser, the first variation of $\mathcal{C}$ vanishes, and the Noether current $q_k = (d/d\alpha)J(T_\alpha(r_k))|_{\alpha=0}$ satisfies the discrete conservation law $\sum_k q_k = 0$ over any 8-tick window. □

Corollary 5.4 (Energy is the multiplier)**.** *Time-translation symmetry (*$s_k \mapsto s_{k+n}$*,* $n \in 8\mathbb{Z}$*) gives the conserved quantity "energy," which equals the Lagrange multiplier* $\lambda_0 = \hbar$*.*

Proof. Time translation leaves $J(r_k)$ invariant (the cost depends on the ratio, not the time index). The associated Noether charge is the total action per period, whose stationarity is precisely the multiplier condition. Hence $Q_{\rm time} = \lambda_0$. □

Corollary 5.5 (Z-pattern conservation)**.** *Z-pattern conservation is enforced by T3. Noether's theorem confirms: any symmetry of the path action that commutes with T3 produces a conserved integer* Z*.*

5.1 Explicit Noether charges

We compute the conserved charge for three concrete symmetries.

Example 5.6 (Reciprocal symmetry → parity). *Let* $T_\alpha(r) = r^{1-2\alpha}$ *for* $\alpha \in [0,1]$. *Then* $T_0 = \mathrm{id}$ *and* $T_{1/2}(r) = 1$ *(collapse to unity), and* $T_1(r) = r^{-1}$ *(inversion). Since* $J(r) = J(r^{-1})$, T_α *is a symmetry for* $\alpha \in \{0,1\}$ *and interpolates between them. The infinitesimal generator is*

$$\left.\frac{d}{d\alpha}\right|_{\alpha=0} T_\alpha(r) = -2r\ln r,$$

giving the Noether current $q_k = -2r_k \ln r_k \cdot J'(r_k) \cdot \tau_0$. *On a J-minimiser this sums to zero:* $\sum_k q_k = 0$. *The conserved charge is the* ***parity*** *of the ledger — the symmetry between debit and credit that forces double-entry structure (T3).*

Example 5.7 (Scaling symmetry → momentum). *Let* $T_\alpha(r) = e^\alpha \cdot r$ *(multiplicative shift). Then* $J(e^\alpha r) = \frac{1}{2}(e^\alpha r + e^{-\alpha}/r) - 1 \neq J(r)$ *in general, so this is* not *an exact symmetry. However, in the quadratic regime* $J(e^{\varepsilon+\alpha}) \approx \frac{1}{2}(\varepsilon+\alpha)^2$, *and the shift* $\varepsilon \mapsto \varepsilon + \alpha$ *is an exact symmetry of the quadratic action. The associated Noether charge is*

$$Q_{shift} = \tau_0 \sum_k \varepsilon_k \approx \tau_0 \sum_k \ln r_k,$$

which is the ***total log-momentum*** *of the trajectory. This is the discrete analogue of total momentum* $p = \sum m_i v_i$.

6 Comparison with Classical Noether

Feature	Classical (Lagrangian)	RS (Cost)
Primitive	Lagrangian $L(q, \dot{q}, t)$	$J(r) = \frac{1}{2}(r + r^{-1}) - 1$
Action	$S = \int L\, dt$ (postulated)	$\mathcal{C} = \sum J \cdot \tau_0$ (forced)
Constraint	None (free boundary)	T3: 8-tick advance + Z-conservation
Hamiltonian	Legendre transform of L	Lagrange multiplier for T3
Noether	Symmetry of L → conservation	Symmetry of J → conservation
Scale	Free (choice of units)	Fixed by K-gate: $\hbar = E_{\mathrm{coh}} \cdot \tau_0$

Remark 6.1. *The critical difference is that in RS no action principle is* postulated. *The path action arises from the unique cost functional, and the Hamiltonian emerges as a constraint-enforcement device. The scale of* $\hbar$ *is not a measured input but a consequence of* φ*-structure.*

7 Falsification Criteria

Falsification Criterion 7.1 (Alternative cost). *If any convex symmetric cost* $F \neq J$ *on* $\mathbb{R}_{>0}$ *satisfying the d'Alembert equation, normalization* $F(1) = 0$, *and calibration* $F''(1) = 1$ *is exhibited, the uniqueness theorem (T5) is falsified and the multiplier argument collapses.*

Falsification Criterion 7.2 (Hamiltonian without T3). *If a physical system is found where unitary evolution (Schrödinger equation) holds but no discrete conservation law analogous to T3 operates at the fundamental scale, then the "Hamiltonian as multiplier" thesis is falsified.*

8 Lean Formalization

The core algebraic steps are formalized in `IndisputableMonolith.Foundation.NoetherFromJ`:

Lean symbol	**Content**
`continuityT3`	T3 constraint on trajectories
`augmentedCost`	$\mathcal{L}[\gamma, \lambda]$
`isJMinimizer`	J-minimiser definition
`isLagrangeMultiplier`	Stationarity of $\mathcal{L}$
`noether_from_J_multiplier_exists`	Theorem 3.2
`multiplier_scale_unique`	Theorem 3.3
`hbar_is_Ecoh_tau0`	$\hbar = E_{\text{coh}} \cdot \tau_0$
`hamiltonian_as_multiplier`	Theorem 4.1 + scale

The module compiles with zero `sorry` obligations for the algebraic identities. The continuum-limit analysis (Theorem 4.1) involves analytic hypotheses stated explicitly; the algebraic core is verified.

9 Discussion

The result has three consequences:

1. **The Hamiltonian is derived, not postulated.** Four centuries of physics assumed $\hat{H}$ as primitive. We show it is a consequence of cost minimisation under a discrete conservation law.

2. **The scale of $\hbar$ is fixed.** In standard physics, $\hbar$ is a measured constant. In RS, $\hbar = \varphi^{-5} \cdot \tau_0$ is algebraic in φ. This removes one fundamental constant from the list of "unexplained inputs."

3. **Noether's theorem is deeper than the action principle.** The standard derivation assumes an action principle and derives conservation laws. Here the action principle is itself derived (from J-minimisation), and conservation laws follow from symmetries of the unique cost functional. Noether's theorem gains a new foundation.

Comparison with existing work

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Feature	Standard (Lagrangian)	RS (this paper)
Noether [6]	Symmetry of $L \to$ conservation law	Symmetry of $J \to$ conservation law
Hamiltonian	Legendre transform of L (postulated)	Lagrange multiplier for T3 (derived)
$\hbar$	Measured constant	$E_{\text{coh}} \cdot \tau_0 = \varphi^{-5}$ (algebraic)
Action principle	Postulated	Derived from J-minimisation
Discrete symmetry	Requires lattice field theory [7]	Native (8-tick is fundamental)

Limitations. The continuum-limit passage from discrete stationarity to the Schrödinger equation involves smoothness and coarse-graining assumptions stated explicitly in Section 4. The algebraic core (existence, uniqueness, scale) is rigorous. The identification of the multiplier with $\hbar$ is definitional in RS-native units; connecting to SI values requires the external calibration seam.

Open problems.

(Q1) Extend the discrete Noether theorem to spatial symmetries (momentum conservation) and internal symmetries (gauge charges).

(Q2) Connect the multiplier structure to the EFE emergence module [4].

(Q3) Does the multiplier structure have a categorical formulation (e.g. as a natural transformation)?

(Q4) Can the discrete transfer matrix $M(\omega)$ be used for numerical quantum mechanics (bypassing Schrödinger)?

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