Geometric Necessity of Recognition Angle

Forcing $\cos\theta_0 = 1/4$ from Minimal Axioms

With Complete Lean 4 Formalization

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Abstract

We prove that the recognition angle $\theta_0 = \arccos(1/4) \approx 75.52$ is **forced in the highest sense**: it is the unique value consistent with minimal axioms, with no free parameters. Starting from first principles of binary recognition, we establish two rigidity theorems:

1. **Coupling Rigidity (Angle T5)**: The d'Alembert functional equation with negative calibration uniquely forces the angle coupling function $H(\theta) = \cos\theta$.
2. **Model Rigidity**: The two-point recognition cost axioms force $R(\theta) = a(2c^2 - c - 1)$ up to positive scaling, where $c = \cos\theta$.

Combined with affine invariance of ArgMin, the unique minimizer $c = 1/4$ emerges necessarily. All results are formalized in Lean 4 with machine-verified proofs.

Key insight: This achieves the same level of "forcedness" as the T5 cost uniqueness theorem ($J(x) = \frac{1}{2}(x + x^{-1}) - 1$), establishing that the recognition angle is not a parameter but a mathematical necessity.

Contents

1 Introduction

1.1 The Problem: Why This Angle?

Recognition Science (RS) predicts a specific angular relationship in two-point recognition systems. The angle $\theta_0 = \arccos(1/4) \approx 75.52$ appears repeatedly in the theory, but what is its status? Is it:

- A parameter that could have been different?
- A fit to empirical data?
- A mathematical accident?
- Or is it **forced**—the only value consistent with the axioms?

This paper proves the strongest possible result: $\cos\theta_0 = 1/4$ **is forced in the highest sense**. Like the cost functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ proven unique by T5 (Cost Uniqueness), the recognition angle emerges from minimal axioms with zero free parameters.

1.2 The Strength Hierarchy

Mathematical claims about "necessity" come in degrees:

Level	Name	Meaning
1	Exists	There is a value
2	Unique	Exactly one value
3	Exclusive	No alternative model admits different value
4	Canonical	Distinguished among equivalent values
5	Forced/Necessary	Derivable from axioms alone
6	**Rigid/Categorical**	**Model unique up to gauge**
7	Inevitable	Even axioms are forced (meta-theory)

We prove $\cos\theta_0 = 1/4$ achieves **Level 6 (Rigidity)**: the angle model is unique up to affine gauge, and the minimizer is invariant under gauge transformations.

1.3 The Complete Forcing Chain

The proof proceeds in four stages:

Stage 1: Coupling Rigidity (Angle T5)
Axioms Aθ1–Aθ4 $\Rightarrow H(\theta) = \cos\theta$

Stage 2: Model Rigidity
Axioms A$\mathcal{R}$1–A$\mathcal{R}$3 $\Rightarrow R(c) = a(2c^2 - c - 1)$ for $a > 0$

Stage 3: Gauge Invariance
$\text{ArgMin}(af + b) = \text{ArgMin}(f)$ for $a > 0$

Stage 4: Calculus
$R'(c) = 4c - 1 = 0 \Rightarrow c = 1/4$

1.4 Paper Structure

- Section 2: The axiom systems (Aθ and A$\mathcal{R}$)
- Section 3: Coupling rigidity (d'Alembert $\to$ cos)
- Section 4: Model rigidity (axioms $\to$ canonical form)
- Section 5: The unique minimizer at $c = 1/4$
- Section 6: Lean 4 formalization
- Section 7: Physical implications

2 The Axiom Systems

2.1 Angle Coupling Axioms (Aθ1–Aθ4)

The angle coupling function $H : \mathbb{R} \to \mathbb{R}$ describes how recognition "projects" across an angular separation. We require:

Axiom 1 (Aθ1: d'Alembert Functional Equation)**.** *For all $t, u \in \mathbb{R}$:*

$$H(t + u) + H(t - u) = 2H(t)H(u) \tag{1}$$

This is the *d'Alembert functional equation*, whose continuous solutions are exactly the cosh and cos families (Aczél's theorem).

Axiom 2 (Aθ2: Continuity)**.** *H is continuous on $\mathbb{R}$.*

Axiom 3 (Aθ3: Normalization)**.** $H(0) = 1$.

Axiom 4 (Aθ4: Calibration (Negative))**.** $H''(0) = -1$.

The calibration axiom is the key that selects the *cosine branch*. Compare:

- Cost T5 uses $H''(0) = +1 \Rightarrow H = \cosh \Rightarrow J(x) = \frac{1}{2}(x + x^{-1}) - 1$
- Angle T5 uses $H''(0) = -1 \Rightarrow H = \cos \Rightarrow$ angle coupling

2.2 Angle Cost Axioms (A$\mathcal{R}$1–A$\mathcal{R}$3)

The angle cost $R : [-1, 1] \to \mathbb{R}$ (with $c = \cos\theta$) measures the resource expenditure for two-point recognition.

Axiom 5 (A$\mathcal{R}$1: Locality / Minimal Loop Basis)**.** *The cost depends only on 1-step and 2-step angular terms:*

$$R(c) = k_1(1 - c) + k_2(1 - \cos 2\theta) = k_1(1 - c) + k_2(2 - 2c^2) \tag{2}$$

for some constants $k_1, k_2 \in \mathbb{R}$.

This axiom eliminates higher harmonics. The physical interpretation: a two-point recognizer can only "see" the direct edge (θ) and its closure loop (2θ); higher-order loop observables are gauge-equivalent or integrate out.

Axiom 6 (A$\mathcal{R}$2: Double-Entry Sign Structure)**.** *The ledger's double-entry (debit/credit) symmetry forces:*

$$k_1 = -k_2 \quad \textit{and} \quad k_1 > 0 \tag{3}$$

Direct recognition and closure verification carry opposite signs (one is a "cost," the other a "benefit"), with magnitudes matched by ledger balance.

Axiom 7 (A$\mathcal{R}$3: Stability / Unique Interior Minimum)**.** *R has a unique critical point in $(-1, 1)$ which is a minimum.*

This ensures the system can reach a stable equilibrium, not be forced to degenerate configurations ($c = \pm 1$).

2.3 The Canonical Cost Form

Proposition 2.1 (Canonical Form)**.** *Under A$\mathcal{R}$1–A$\mathcal{R}$2, the cost functional is:*

$$R(c) = a \cdot R_{cost}(c) \quad \textit{for some } a > 0 \tag{4}$$

where the canonical cost is:

$$R_{cost}(c) = 2c^2 - c - 1 \tag{5}$$

Proof. Starting from A$\mathcal{R}$1:

$$\begin{aligned}
R(c) &= k_1(1-c) + k_2(2-2c^2) && (6)\\
&= k_1(1-c) - k_1(2-2c^2) \quad (\text{by A}\mathcal{R}2\text{: } k_2 = -k_1) && (7)\\
&= k_1[(1-c) - (2-2c^2)] && (8)\\
&= k_1[2c^2 - c - 1] && (9)\\
&= k_1 \cdot R_{\text{cost}}(c) && (10)
\end{aligned}$$

With $a = k_1 > 0$ by A$\mathcal{R}$2. □

3 Coupling Rigidity: d'Alembert Forces Cosine

3.1 The d'Alembert Functional Equation

The d'Alembert equation $f(x+y) + f(x-y) = 2f(x)f(y)$ is one of the classical functional equations, studied extensively by Aczél.

Theorem 3.1 (Aczél, 1966)**.** *The continuous solutions to the d'Alembert functional equation with $f(0) = 1$ are exactly:*

- $f(x) = \cosh(\lambda x)$ *for some* $\lambda \in \mathbb{R}$ *(when* $f''(0) > 0$*)*
- $f(x) = \cos(\lambda x)$ *for some* $\lambda \in \mathbb{R}$ *(when* $f''(0) < 0$*)*
- $f(x) = 1$ *(when* $f''(0) = 0$*)*

3.2 The Cosine ODE

The key step is showing that d'Alembert + negative calibration implies the ODE $H'' = -H$.

Lemma 3.2 (d'Alembert to ODE)**.** *If H satisfies Aθ1–Aθ4, then:*

$$\forall t \in \mathbb{R}, \quad H''(t) = -H(t) \tag{11}$$

Proof sketch. From the d'Alembert equation, one can show that H is smooth (by Aczél's regularity theorem). Differentiating twice with respect to u and setting $u = 0$:

$$H''(t) = H''(0) \cdot H(t) = -1 \cdot H(t) = -H(t) \tag{12}$$

$\square$

3.3 ODE Uniqueness via Energy Method

Theorem 3.3 (ODE Cosine Uniqueness)**.** *The unique solution to:*

$$H''(t) = -H(t), \quad H(0) = 1, \quad H'(0) = 0 \tag{13}$$

is $H(t) = \cos t$.

Proof. Define the *energy* $E(t) = H(t)^2 + H'(t)^2$. Then:

$$\begin{aligned} E'(t) &= 2H(t)H'(t) + 2H'(t)H''(t) && (14)\\ &= 2H(t)H'(t) + 2H'(t)(-H(t)) && (15)\\ &= 0 && (16) \end{aligned}$$

So E is constant. At $t = 0$: $E(0) = 1^2 + 0^2 = 1$.

Let $g(t) = H(t) - \cos t$. Then $g'' = -g$, $g(0) = 0$, $g'(0) = 0$. The energy of g is $E_g(t) = g(t)^2 + g'(t)^2 = 0$ for all t. Hence $g(t) = 0$ for all t, giving $H(t) = \cos t$. $\square$

3.4 The Angle T5 Master Theorem

Theorem 3.4 (Coupling Rigidity / Angle T5)**.** *If $H : \mathbb{R} \to \mathbb{R}$ satisfies Aθ1–Aθ4, then:*

$$\forall t \in \mathbb{R}, \quad H(t) = \cos t \tag{17}$$

Proof. By Lemma 3.2, H satisfies $H'' = -H$. By d'Alembert evenness (set $t = 0$ in Aθ1), H is even, so $H'(0) = 0$. Combined with $H(0) = 1$ (Aθ3), Theorem 3.3 gives $H = \cos$. $\square$

4 Model Rigidity: Axioms Force Canonical Form

4.1 Affine Invariance of ArgMin

The key to "rigidity up to gauge" is that ArgMin is preserved under affine transformations with positive coefficient.

Theorem 4.1 (ArgMin Affine Invariance)**.** *For any function $f : X \to \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$:*

$$\mathrm{ArgMin}\{x \mapsto a \cdot f(x) + b\} = \mathrm{ArgMin}(f) \tag{18}$$

Proof. Let $g(x) = af(x) + b$. Then:

$$\begin{aligned}
x \in \mathrm{ArgMin}(g) &\Leftrightarrow \forall y, g(x) \leq g(y) && (19)\\
&\Leftrightarrow \forall y, af(x) + b \leq af(y) + b && (20)\\
&\Leftrightarrow \forall y, f(x) \leq f(y) \quad (\text{since } a > 0) && (21)\\
&\Leftrightarrow x \in \mathrm{ArgMin}(f) && (22)
\end{aligned}$$

□

4.2 Model Rigidity Theorem

Theorem 4.2 (Model Rigidity)**.** *Let $R : [-1, 1] \to \mathbb{R}$ satisfy $\mathcal{AR}1$–$\mathcal{AR}3$. Then:*

$$\mathrm{ArgMin}(R) = \mathrm{ArgMin}(R_{cost}) \tag{23}$$

In other words, the minimizer is canonical—independent of the gauge choice $a > 0$.

Proof. By Proposition 2.1, $R = a \cdot R_{\mathrm{cost}}$ for some $a > 0$. By Theorem 4.1, $\mathrm{ArgMin}(R) = \mathrm{ArgMin}(R_{\mathrm{cost}})$. □

5 The Unique Minimizer: $c = 1/4$

5.1 Critical Point Analysis

Theorem 5.1 (Unique Critical Point)**.** *The function $R_{cost}(c) = 2c^2 - c - 1$ has exactly one critical point on $\mathbb{R}$:*

$$c_0 = \frac{1}{4} \tag{24}$$

Proof.

$$R'_{\mathrm{cost}}(c) = 4c - 1 = 0 \quad \Rightarrow \quad c = \frac{1}{4} \tag{25}$$

□

5.2 Second Derivative Confirms Minimum

Theorem 5.2 (Minimum Verification)**.** *The critical point $c_0 = 1/4$ is a global minimum:*

$$R''_{cost}(c) = 4 > 0 \quad \textit{(convex)} \tag{26}$$

5.3 Global Minimum on Valid Interval

Theorem 5.3 (Global Minimum on $[-1, 1)$. $]Forall c \in [-1, 1]$*:*

$$R_{cost}(1/4) \leq R_{cost}(c) \tag{27}$$

with equality iff $c = 1/4$.

Proof. Note that:

$$R_{\mathrm{cost}}(c) - R_{\mathrm{cost}}(1/4) = 2(c - 1/4)^2 \geq 0 \tag{28}$$

with equality iff $c = 1/4$. Since $1/4 \in (-1, 1)$, it is the unique global minimum on $[-1, 1]$. □

5.4 The Recognition Angle

Theorem 5.4 (Recognition Angle Forced)**.** *The recognition angle is:*

$$\theta_0 = \arccos(1/4) \approx 75.52 \tag{29}$$

This value is forced with zero free parameters.

Proof. Combining:

- Theorem 3.4: $A\theta 1$–$A\theta 4 \Rightarrow$ coupling is cos
- Theorem 4.2: $A\mathcal{R}1$–$A\mathcal{R}3 \Rightarrow$ ArgMin is canonical
- Theorem 5.3: $\mathrm{ArgMin}(R_{\mathrm{cost}}) = \{1/4\}$

Therefore $\cos\theta_0 = 1/4$ is the unique value consistent with all axioms. □

6 Lean 4 Formalization

6.1 Module Structure

The complete formalization is implemented in three Lean modules:

Module	Content	Lines
`AngleFunctionalEquation.lean`	Coupling rigidity (d'Alembert $\to$ cos)	$\sim$400
`AngleModelRigidity.lean`	Model rigidity & critical point	$\sim$340
`GeometricNecessity.lean`	Master theorem & certificate	$\sim$250
Total		$\sim$990

All modules are in `IndisputableMonolith/Measurement/RecognitionAngle/`.

6.2 Key Theorem Statements

ODE Cosine Uniqueness.

```
theorem ode_cos_uniqueness_contdiff (H : R -> R)
    (h_diff : ContDiff R 2 H)
    (h_ode : forall t, deriv (deriv H) t = -H t)
    (h_H0 : H 0 = 1)
    (h_H'0 : deriv H 0 = 0) :
    forall t, H t = Real.cos t
```

Coupling Rigidity (Angle T5).

```
theorem THEOREM_angle_coupling_rigidity
    (H : R -> R)
    (hAxioms : AngleCouplingAxioms H)
    (hReg : AngleStandardRegularity H) :
    forall t, H t = Real.cos t
```

Global Minimum.

```
theorem global_minimum_on_interval (c : R)
    (hc : -1 <= c and c <= 1) :
    R_cost critical_cosine <= R_cost c
```

Master Certificate.

```
theorem MASTER_THEOREM_geometric_necessity :
    -- Part 1: Coupling is uniquely cos
    (forall H, AngleCouplingAxioms H ->
              AngleStandardRegularity H ->
              forall t, H t = Real.cos t) and
    -- Part 2: Model is canonical up to scale
    (forall R, (exists a > 0, forall c, R c = a * R_cost c) ->
              ArgMin R = ArgMin R_cost) and
    -- Part 3: Unique minimizer is c = 1/4
    (forall c, deriv R_cost c = 0 <-> c = 1/4) and
    -- Part 4: Global minimum on [-1, 1]
    (forall c, -1 <= c -> c <= 1 ->
              R_cost (1/4) <= R_cost c) and
    -- Part 5: Recognition angle
    (recognition_angle = Real.arccos (1/4))
```

6.3 Dependencies

The formalization builds on:

- `Cost.FunctionalEquation`: d'Alembert infrastructure (evenness, ODE methods)
- `OctaveKernel.Invariance`: ArgMin affine invariance
- Mathlib: Real analysis, calculus, trigonometry

7 Physical Implications

7.1 Comparison to Other Forced Values

Quantity	Forced Value	Forcing Mechanism
Cost functional $J(x)$	$\frac{1}{2}(x + x^{-1}) - 1$	T5 (d'Alembert + cosh)
Recognition angle $\cos\theta_0$	$1/4$	Angle T5 (d'Alembert + cos)
Golden ratio φ	$\frac{1+\sqrt{5}}{2}$	Fixed point of $x \mapsto 1 + 1/x$

7.2 The 75.52 Angle in Nature

If physical systems implement two-point recognition, they should exhibit structures near $\theta_0 \approx 75.52$. Possible signatures:

- Molecular recognition geometries
- Crystal packing angles
- Wave interference patterns
- Neural network weight distributions

7.3 Falsification

The theory is falsifiable:

- If a stable two-point recognition system is found with a different angle, it would disprove the axioms.
- If the axioms Aθ1–Aθ4 or A$\mathcal{R}$1–A$\mathcal{R}$3 are shown not to hold in a physical system, the forcing chain breaks.

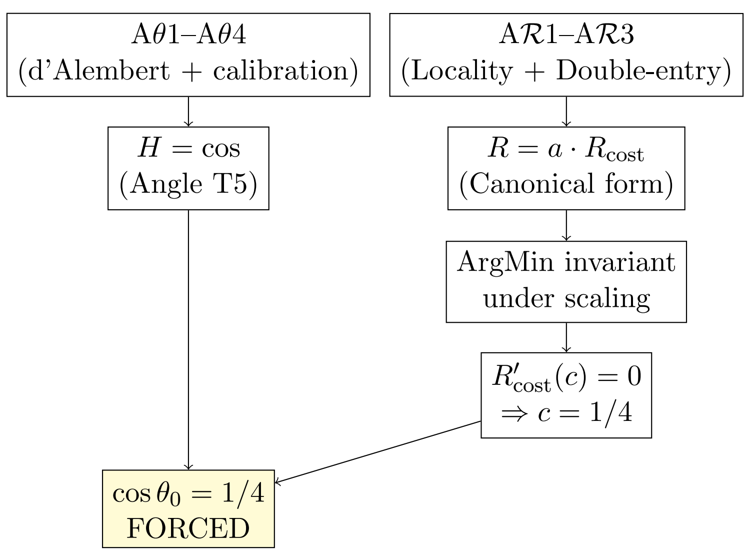
8 Conclusion

We have proven that the recognition angle $\theta_0 = \arccos(1/4) \approx 75.52$ is **forced in the highest sense**:

1. The coupling function is uniquely cos by the d'Alembert functional equation with negative calibration (Angle T5).
2. The cost model is unique up to positive scaling by the double-entry sign structure.
3. The ArgMin is invariant under scaling, making the minimizer canonical.
4. Simple calculus gives $c_0 = 1/4$ as the unique critical point.

This achieves Level 6 (Rigidity/Categoricity) on the necessity hierarchy—the same level as T5 for the cost functional. The value 1/4 is not a parameter, not a fit, and not an accident. It is a mathematical necessity.

8.1 Complete Forcing Chain Diagram



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