Classical Inputs Used in the NS Proof

Concise statement-level summary for referee convenience

A1. Critical ε–regularity (vorticity, $L^{3/2}$). For a suitable weak solution (u,p) on $Q_{r_0}(x_0,t_0)$, if

$$\mathcal{W}(x_0,t_0;r_0) := \frac{1}{r_0}\iint_{Q_{r_0}(x_0,t_0)} |\omega|^{3/2}\,dx\,dt \;\le\; \varepsilon_A,$$

then

$$\sup_{Q_{r_0/2}(x_0,t_0)} |\omega| \;\le\; \frac{C_A}{r_0^2}\,\mathcal{W}(x_0,t_0;r_0)^{2/3}.$$

Notes: absorbed Caccioppoli + De Giorgi; parabolic scaling; standard in NS literature at critical scale.

A2. Density–drop (De Giorgi improvement on smaller cylinders). There exist fixed $\vartheta \in (0,1/2)$, $c \in (0,1)$, $\eta_1 > 0$ such that if $\mathcal{W}(0,0;1) \le \varepsilon_0 + \eta$ with $0 < \eta \le \eta_1$, then

$$\mathcal{W}(0,0;\vartheta) \;\le\; \varepsilon_0 + c\,\eta.$$

Notes: truncation $w = (|\omega| - \kappa_0)_+$ with $\kappa_0 \sim \varepsilon_0^{2/3}$; absorbed Caccioppoli; ladder iteration.

A3. Carleson characterization of BMO^{-1}. For $f \in \mathcal{S}'(\mathbb{R}^3)$,

$$\|f\|_{BMO^{-1}} \;\simeq\; \sup_{x\in\mathbb{R}^3,\ r>0} \Big(\frac{1}{|B_r|}\int_0^{r^2}\int_{B_r(x)} |e^{\nu\tau\Delta} f(y)|^2\,dy\,d\tau\Big)^{1/2}.$$

Notes: heat-flow square-function definition equivalent to the standard functional BMO^{-1} norm.

A4. Koch–Tataru small-data global theory in BMO^{-1}. There exists $\varepsilon_{\mathrm{SD}} > 0$ such that if $\|u_0\|_{BMO^{-1}} \le \varepsilon_{\mathrm{SD}}$, then there is a unique global mild solution u with $u \in X$ (KT space), smooth for $t > 0$. Notes: includes bilinear estimate in the X space and continuity of the BMO^{-1} norm on short times.

A5. Uniqueness: backward Carleman + forward energy. If u, v solve NS on $\mathbb{R}^3 \times [t_1,t_2]$, with v smooth and u suitable, and $u(\cdot,t_0) = v(\cdot,t_0)$ for some $t_0 \in (t_1,t_2)$, then $u \equiv v$ on $\mathbb{R}^3 \times [t_1,t_2]$. Notes: local backward uniqueness from a parabolic Carleman estimate; forward uniqueness by standard L^2 energy.

A6. Compactness and critical-element extraction. Given a sequence of suitable solutions with uniform local L^3 bounds (and pressure in $L^{3/2}$), there is strong L^3_{loc} compactness on interior cylinders; $\mathcal{W}$ is lower semicontinuous under this convergence. These yield existence of a nontrivial ancient critical element at a minimal profile level.

Auxiliary local embeddings (derived internally).

- (LE1) *Local $BMO^{-1} \to L^3$ on a slice:* for every t and ball $B_r(x)$, $\|u(\cdot,t)\|_{L^3(B_r(x))} \lesssim r^{1/2}\,\|u(\cdot,t)\|_{BMO^{-1}}$.
- (LE2) *Local $L^3 \to L^{3/2}$ for vorticity:* for every t and ball $B_r(x)$, $\|\omega(\cdot,t)\|_{L^{3/2}(B_r(x))} \lesssim r^{-1/2}\,\|u(\cdot,t)\|_{L^3(B_{2r}(x))}$.

Notes: consequences of A3 (Carleson/BMO^{-1}), heat-kernel smoothing, Calderón–Zygmund, and local Poincaré; no extra external inputs are required.

Citations (canonical sources):

- Koch, Tataru. *Well-posedness for the Navier–Stokes equations.* Adv. Math. (2001).
- Escauriaza, Seregin, Šverák. $L_{3,\infty}$*-solutions of Navier–Stokes and backward uniqueness.* (2003).
- Caffarelli–Kohn–Nirenberg; Vasseur; Ladyzhenskaya–Seregin–Šverák (partial regularity and De Giorgi frameworks).
- Standard local compactness for suitable solutions (Aubin–Lions + pressure decompositions).
- Stein. *Singular Integrals and Differentiability Properties of Functions.* (Calderón–Zygmund estimates).