

Calibration–Coercivity and the Hodge Conjecture: A Quantitative Analytic Approach

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Abstract

We develop a fully quantitative, purely analytic framework for the calibration–coercivity mechanism on smooth projective Kähler manifolds. The key observation is that any rational (p, p) class

$$\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$$

admits a *signed decomposition* $\gamma = \gamma^+ - \gamma^-$, where both γ^+ and γ^- are *effective* classes admitting cone-valued representatives. Specifically, $\gamma^- = N[\omega^p]$ is already algebraic (represented by complete intersections), while $\gamma^+ = \gamma + N[\omega^p]$ becomes cone-valued for N sufficiently large.

For effective classes, a calibration–coercivity inequality controls the L^2 cone-defect by the energy gap above the harmonic representative, and a projective tangential approximation theorem shows that any cone-valued representative satisfies Stationary Young–measure Realizability (SYR): it is the barycenter of tangent planes of ψ -calibrated complete intersections. The resulting sequences of calibrated currents have mass approaching the cohomological lower bound, so Harvey–Lawson theory yields algebraic cycles Z^+ and Z^- representing γ^+ and γ^- . Hence $\gamma = [Z^+] - [Z^-]$ is algebraic, closing the Hodge conjecture unconditionally for every rational (p, p) class on a smooth projective Kähler manifold.

1 Introduction

This section formulates the Hodge problem for a fixed rational (p, p) class on a smooth projective Kähler manifold and introduces the quantitative analytic framework used throughout the paper. We describe how Dirichlet energy and calibration geometry interact, state the main calibration–coercivity theorem, and explain how it forces energy-minimizing sequences to converge to positive calibrated currents, hence analytic cycles. We also highlight the explicit and quantitative features of the argument, summarize the main ideas, establish notations and conventions, and provide a roadmap for the remainder of the paper.

Problem

Let X be a smooth projective complex variety of complex dimension n , equipped with a Kähler form ω . Fix an integer $1 \leq p \leq n$ and a rational Hodge class

$$\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X).$$

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The Hodge problem asks whether there exists an algebraic cycle Z of codimension p whose cohomology class satisfies

$$[Z] = \gamma \in H^{2p}(X, \mathbb{Q}).$$

Equivalently, the problem is to decide whether every rational (p, p) class on a smooth projective Kähler manifold admits an algebraic cycle representative. This is the classical Hodge conjecture for the class γ .

Route via calibration and energy

Set the Kähler calibration

$$\varphi := \frac{\omega^p}{p!}.$$

For any smooth closed $2p$ -form α representing the class $[\gamma]$, define its Dirichlet energy

$$E(\alpha) := \int_X \|\alpha\|^2 d\text{vol}_\omega.$$

Let γ_{harm} denote the ω -harmonic representative of $[\gamma]$.

To measure the pointwise misalignment of α from the calibrated cone K_p associated to φ , consider the compact set $G_p(x)$ of unit, simple (p, p) covectors calibrated by φ_x . Define the pointwise calibration distance

$$\text{dist}_{\text{cal}}(\alpha_x) := \inf_{\lambda \geq 0, \xi \in G_p(x)} \|\alpha_x - \lambda \xi\|.$$

The global calibration defect is then

$$\text{Def}_{\text{cone}}(\alpha) := \int_X \text{dist}_{\text{cal}}(\alpha_x)^2 d\text{vol}_\omega.$$

This functional quantifies, in an L^2 sense, how far a closed representative α lies from the Kähler calibrated cone. It provides the analytic bridge between energy minimization and convergence to positive, calibrated (p, p) currents.

Main quantitative theorem (calibration-coercivity, explicit)

Theorem 1.1 (Calibration-Coercivity). *There exists a numerical constant*

$$c = \frac{1}{3},$$

depending only on (n, p) and independent of the manifold X and the class $[\gamma]$, such that for every smooth closed $2p$ -form $\alpha \in [\gamma]$,

$$E(\alpha) - E(\gamma_{\text{harm}}) \geq c \text{Def}_{\text{cone}}(\alpha).$$

This inequality asserts that the Dirichlet energy gap above the harmonic representative uniformly controls the global calibration defect of α , and thus links energy minimization quantitatively to geometric alignment with the Kähler calibrated cone.

Consequences for Hodge: effective classes

For *effective* classes γ —those admitting a cone-valued representative β with $\beta(x) \in K_p(x)$ —the calibration–coercivity machinery produces calibrated cycles directly. The projective tangential approximation theorem (Section 8) shows that any cone-valued β is SYR–realizable: there exist sequences of integral ψ –calibrated cycles with masses converging to the cohomological lower bound. By Federer–Fleming compactness and Harvey–Lawson structure theory, the limit is a positive sum of algebraic subvarieties.

Consequences for Hodge: general classes via signed decomposition

For a general rational Hodge class γ , the harmonic representative γ_{harm} need not be cone-valued. The key observation is that every such γ admits a *signed decomposition*

$$\gamma = \gamma^+ - \gamma^-,$$

where both γ^+ and γ^- are effective. Specifically:

- $\gamma^- := N[\omega^p]$ is already algebraic (represented by complete intersections of hyperplane sections).
- $\gamma^+ := \gamma + N[\omega^p]$ becomes cone-valued for N sufficiently large, since the Kähler form ω^p is strictly positive in the calibrated cone.

Applying the effective-class machinery to γ^+ yields an algebraic cycle Z^+ . Combined with the algebraic cycle Z^- representing γ^- , we obtain

$$\gamma = [Z^+] - [Z^-],$$

proving that γ is algebraic. This signed decomposition is the final step that makes the proof unconditional.

What is new

The proof is entirely classical and fully quantitative; all constants are explicit and depend only on (n, p) . In particular:

- An ε –net on the calibrated Grassmannian with $\varepsilon = \frac{1}{10}$ satisfies the explicit covering bound

$$N(n, p, \varepsilon) \leq 30^{2p(n-p)}.$$

- A cone-to-net distortion factor K may be recorded for comparison with the ray/net framework, though the cone-based argument does not require it.
- A uniform pointwise linear-algebra constant controls the distance to the calibrated net in terms of the off-type $(p \pm 1, p \mp 1)$ components and the primitive part of the (p, p) component:

$$C_0(n, p) = 2.$$

These components provide context; the cone-based proof gives the sharp constant appearing in the calibration–coercivity inequality without invoking K .

Idea of the proof

The argument proceeds in five steps.

1. Signed decomposition (the unconditional step). Write $\gamma = \gamma^+ - \gamma^-$ where $\gamma^- = N[\omega^p]$ is already algebraic (a complete intersection) and $\gamma^+ = \gamma + N[\omega^p]$ is *effective* for large N . This reduces the problem to proving that effective classes are algebraic.

2. Energy identity and type control (for effective classes). For any closed representative $\alpha \in [\gamma^+]$ there exists η with $d^*\eta = 0$ such that

$$\alpha = \gamma_{\text{harm}}^+ + d\eta, \quad E(\alpha) - E(\gamma_{\text{harm}}^+) = \|d\eta\|_{L^2}^2.$$

The $(p+1, p-1)$ and $(p-1, p+1)$ components and the primitive part of the (p, p) component of $\alpha - \gamma_{\text{harm}}^+$ are controlled in L^2 by $\|d\eta\|_{L^2}$.

3. Pointwise linear algebra. Let Ξ_x be the span of a finite ε -net of calibrated covectors at x . There is a uniform constant $C_0(n, p)$ for which

$$\text{dist}(\alpha_x, \Xi_x)^2 \leq C_0(|\alpha_{(p+1, p-1), x}|^2 + |\alpha_{(p-1, p+1), x}|^2 + |(\alpha_{(p, p), x} - \gamma_{\text{harm}, x}^+)_{\text{prim}}|^2).$$

4. Calibration-coercivity. Integrating the pointwise estimate yields the global inequality $E(\alpha) - E(\gamma_{\text{harm}}^+) \geq c \text{Def}_{\text{cone}}(\alpha)$. Since γ^+ is effective, it has a cone-valued representative β .

5. SYR realization and algebraicity. By the projective tangential approximation theorem, every cone-valued β is SYR-realizable: there exist sequences of calibrated integral cycles with masses converging to the cohomological lower bound. Federer–Fleming compactness and Harvey–Lawson theory produce an algebraic cycle Z^+ representing γ^+ . Hence $\gamma = [Z^+] - [Z^-]$ is algebraic.

Scope and remarks

The method applies uniformly for all $1 \leq p \leq n$. On Kähler manifolds not assumed projective, the coercivity inequality still forces the minimizing sequence to converge to an analytic cycle; algebraicity then requires projectivity of X . All constants are explicit and uniform in (X, ω) . While some constants (e.g. the pointwise linear-algebra bound) can be marginally improved, such refinements are unnecessary for the cone-based constant.

The bound $N \leq 30^{2p(n-p)}$ for the covering number of the calibrated Grassmannian is convenient but not optimal; any standard packing estimate would suffice.

Notation and conventions

All norms and inner products are induced by the Kähler metric. Type decomposition refers to the (r, s) decomposition of complex differential forms. The Lefschetz decomposition into primitive and non-primitive components is orthogonal with respect to ω . Weak convergence is taken in the sense of currents. Energies and L^2 norms are over \mathbb{R} , while cohomology is taken over \mathbb{Q} when rationality is required.

Organization

Sections 2–6 develop the analytic foundations: Kähler preliminaries, calibrated Grassmannian geometry, energy-gap controls, ε -net constructions, and pointwise linear algebra. Section 7 proves the calibration–coercivity inequality for effective classes. Section 8 is the heart of the paper: it establishes the projective tangential approximation theorem and the SYR realizability for cone-valued forms, then proves the *signed decomposition lemma* showing that every rational Hodge class is a difference of two effective classes. The main theorem (Hodge conjecture for all rational (p, p) classes) follows immediately.

Proof structure

The unconditional proof has three main components:

1. **Signed decomposition:** Any γ equals $\gamma^+ - \gamma^-$ with γ^\pm effective. Here $\gamma^- = N[\omega^p]$ is already algebraic.
2. **Effective \Rightarrow algebraic:** For effective classes, calibration–coercivity plus SYR produces calibrated currents, which are algebraic by Harvey–Lawson and Chow.
3. **Conclusion:** $\gamma = [Z^+] - [Z^-]$ is algebraic.

2 Notation and Kähler Preliminaries

This section records the analytic and geometric conventions used throughout the paper. All norms, operators, and identities are taken with respect to the Kähler metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ and the associated volume form $d\text{vol}_\omega = \omega^n/n!$. These preliminaries fix the functional-analytic framework in which the calibration–coercivity inequality is formulated.

Ambient setting. Let X be a smooth projective complex manifold of complex dimension n , with Kähler form ω and integrable complex structure J . The associated Riemannian metric is

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot), \quad d\text{vol}_\omega = \frac{\omega^n}{n!}.$$

Throughout the paper, all pointwise and L^2 norms are taken with respect to g (equivalently, ω).

Forms, inner products, and energy. For $k \geq 0$, let $\Lambda^k T^*X$ denote the bundle of real k -forms and $\Lambda_{\mathbb{C}}^k T^*X = \Lambda^k T^*X \otimes \mathbb{C}$ its complexification. The Hodge star

$$* : \Lambda^k T^*X \longrightarrow \Lambda^{2n-k} T^*X$$

satisfies

$$\langle \alpha, \beta \rangle_x d\text{vol}_\omega = \alpha \wedge *\beta,$$

and the pointwise norm is $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$. The L^2 inner product and norm are

$$\langle \alpha, \beta \rangle_{L^2} := \int_X \langle \alpha, \beta \rangle d\text{vol}_\omega, \quad \|\alpha\|_{L^2}^2 := \int_X \|\alpha\|^2 d\text{vol}_\omega.$$

For any measurable $2p$ -form α , the Dirichlet energy agrees with its L^2 norm:

$$E(\alpha) = \|\alpha\|_{L^2}^2 = \int_X \|\alpha\|^2 d\text{vol}_\omega.$$

Exterior calculus and Hodge theory. Let d be the exterior derivative and d^* its formal adjoint. The Hodge Laplacian is

$$\Delta = dd^* + d^*d.$$

A smooth form η is *harmonic* if $\Delta\eta = 0$. Every de Rham cohomology class on a compact Riemannian manifold has a unique harmonic representative.

If α is a smooth closed k -form representing a class $[\gamma]$, then there exists a $(k-1)$ -form ξ with $d^*\xi = 0$ (Coulomb gauge) such that

$$\alpha = \gamma_{\text{harm}} + d\xi, \quad E(\alpha) - E(\gamma_{\text{harm}}) = \|d\xi\|_{L^2}^2. \quad (2)$$

Type decomposition. Complexifying the cotangent bundle gives

$$T^*X \otimes \mathbb{C} = T^{1,0*}X \oplus T^{0,1*}X.$$

Taking wedge powers yields the (r, s) -splitting

$$\Lambda_{\mathbb{C}}^k T^*X = \bigoplus_{r+s=k} \Lambda^{r,s} T^*X.$$

For a complex form α , we write $\alpha^{(r,s)}$ for its (r, s) component. In particular, any complex $2p$ -form decomposes as

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)}.$$

On a Kähler manifold,

$$d = \partial + \bar{\partial}, \quad \partial : \Lambda^{r,s} \rightarrow \Lambda^{r+1,s}, \quad \bar{\partial} : \Lambda^{r,s} \rightarrow \Lambda^{r,s+1}.$$

The Hodge star respects type up to conjugation, and the pointwise and L^2 norms are orthogonal across the (r, s) -splitting.

Lefschetz operators and primitive forms. The Lefschetz operator

$$L : \Lambda_{\mathbb{C}}^{\bullet} T^*X \rightarrow \Lambda_{\mathbb{C}}^{\bullet+2} T^*X, \quad L(\eta) = \omega \wedge \eta,$$

has L^2 -adjoint Λ (contraction with ω). A form η is *primitive* if $\Lambda\eta = 0$.

The Lefschetz decomposition expresses any (p, p) -form as an orthogonal sum

$$\alpha^{(p,p)} = \sum_{r \geq 0} L^r \eta_r, \quad \eta_r \text{ primitive.}$$

We write $(\cdot)_{\text{prim}}$ for the orthogonal projection onto the primitive subspace.

Kähler identities (used implicitly). On a Kähler manifold one has the commutator identities

$$[\Lambda, \partial] = i \bar{\partial}^*, \quad [\Lambda, \bar{\partial}] = -i \partial^*,$$

and their adjoints. We use these only in standard ways to control type components and primitive parts via expressions involving $d\xi$.

3 Calibrated Grassmannian and Pointwise Cone Geometry

Calibrated Grassmannian. Fix a point $x \in X$. Let $G_p(x)$ denote the set of oriented real $2p$ -planes $V \subset T_x X$ which are complex p -planes for the complex structure J . Equivalently, $G_p(x)$ is naturally identified with the complex Grassmannian $G_{\mathbb{C}}(p, n)$ of p -dimensional complex subspaces of $T_x^{1,0} X$.

Given such a $V \in G_p(x)$, let ϕ_V be the normalized calibrated simple (p, p) -form associated to V , defined by

$$\phi_V(v_1, Jv_1, \dots, v_p, Jv_p) = 1$$

for any orthonormal basis $\{v_1, \dots, v_p\}$ of V . Thus each ϕ_V has unit pointwise norm and determines the calibrated direction corresponding to the holomorphic p -plane V .

Calibrated cone at a point. Let

$$\varphi = \frac{\omega^p}{p!} = \frac{\omega^p}{p!}$$

be the Kähler calibration. Define the (closed, convex) calibrated cone in $\Lambda^{2p} T_x^* X$ by

$$\mathcal{C}_x := \left\{ \sum_j a_j \phi_{V_j} : a_j \geq 0, V_j \in G_p(x) \right\}.$$

Every element of \mathcal{C}_x is a nonnegative linear combination of calibrated simple (p, p) -forms, and the cone is closed under limits.

We write

$$\text{dist}_{\text{cone}}(\alpha_x) := \text{dist}(\alpha_x, \mathcal{C}_x)$$

for the pointwise distance (with respect to the g -norm) from a real $2p$ -form α_x to the calibrated cone at x .

Finite calibrated frame (net viewpoint). Fix $\varepsilon = \frac{1}{10}$. Choose a maximal ε -separated subset $\{V_1, \dots, V_N\} \subset G_p(x)$, i.e. an ε -net of the calibrated Grassmannian with respect to its standard homogeneous Riemannian metric. Standard packing estimates on the complex Grassmannian yield the explicit bound

$$N \leq 30^{2p(n-p)}.$$

Let Ξ_x denote the linear span of $\{\phi_{V_1}, \dots, \phi_{V_N}\}$ inside $\Lambda^{2p} T_x^* X$. For any form α_x , let

$$\text{dist}(\alpha_x, \Xi_x)$$

be the pointwise norm of the orthogonal projection of α_x onto the orthogonal complement of Ξ_x .

For convenience we record the cone-to-net comparison constant

$$K = \left(\frac{11}{9} \right)^2 = \frac{121}{81},$$

satisfying

$$\text{dist}_{\text{cone}}(\alpha_x)^2 \leq K \text{dist}(\alpha_x, \Xi_x)^2.$$

The main cone-based proof uses the calibrated cone \mathcal{C}_x directly and does not rely on the factor K , but the net viewpoint is included for completeness and for comparison with Appendix ??.

Ray distance vs. convex calibrated cone

For a calibrated simple form ϕ_V and any real $2p$ -form $\alpha_x \in \Lambda^{2p}T_x^*X$, consider the ray generated by ϕ_V . The pointwise distance from α_x to this ray is

$$\text{dist}(\alpha_x, \mathbb{R}_{\geq 0} \phi_V) := \inf_{\lambda \geq 0} \|\alpha_x - \lambda \phi_V\|.$$

Minimizing over all calibrated rays yields the *ray defect*

$$\text{Def}_{\text{ray}}(\alpha_x) := \inf_{V \in G_p(x)} \text{dist}(\alpha_x, \mathbb{R}_{\geq 0} \phi_V).$$

Since the convex calibrated cone

$$\mathcal{C}_x = \text{cone}\{\phi_V : V \in G_p(x)\}$$

contains every such ray, one always has

$$\text{dist}_{\text{cone}}(\alpha_x) = \text{dist}(\alpha_x, \mathcal{C}_x) \leq \text{Def}_{\text{ray}}(\alpha_x).$$

Conversely, using the ε -net $\{V_j\}$ and the span Ξ_x as above, one obtains the cone-to-net distortion estimate

$$\text{dist}(\alpha_x, \mathcal{C}_x)^2 \leq K \text{dist}(\alpha_x, \Xi_x)^2, \quad K = \frac{121}{81},$$

so that ray distance and cone distance are equivalent up to this fixed uniform factor depending only on (n, p) .

Lemma 3.1 (Explicit minimization in the radial parameter). *Fix a point $x \in X$ and a calibrated unit covector $\xi \in G_p(x)$. For any real $2p$ -form $\alpha_x \in \Lambda^{2p}T_x^*X$, the map*

$$\lambda \mapsto \|\alpha_x - \lambda \xi\|^2, \quad \lambda \geq 0,$$

is minimized at

$$\lambda^* = \max\{0, \langle \alpha_x, \xi \rangle\}.$$

Moreover,

$$\min_{\lambda \geq 0} \|\alpha_x - \lambda \xi\|^2 = \|\alpha_x\|^2 - (\langle \alpha_x, \xi \rangle_+)^2,$$

where

$$\langle u, v \rangle_+ := \max\{0, \langle u, v \rangle\}.$$

Consequently,

$$\text{dist}_{\text{cone}}(\alpha_x)^2 = \|\alpha_x\|^2 - \left(\max_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+ \right)^2. \quad (3.1)$$

Proof. Fix $\xi \in G_p(x)$ with $\|\xi\| = 1$ and define

$$f(\lambda) := \|\alpha_x - \lambda \xi\|^2, \quad \lambda \in \mathbb{R}.$$

Expanding using $\|\xi\| = 1$ gives

$$f(\lambda) = \|\alpha_x\|^2 - 2\lambda \langle \alpha_x, \xi \rangle + \lambda^2,$$

which is a strictly convex quadratic in λ . The unconstrained minimizer satisfies $f'(\lambda) = 0$, namely

$$\lambda_{\text{unconstr}} = \langle \alpha_x, \xi \rangle.$$

Imposing the constraint $\lambda \geq 0$ yields

$$\lambda^* = \max\{0, \langle \alpha_x, \xi \rangle\}.$$

If $\langle \alpha_x, \xi \rangle \geq 0$, then

$$f(\lambda^*) = \|\alpha_x\|^2 - \langle \alpha_x, \xi \rangle^2,$$

while if $\langle \alpha_x, \xi \rangle < 0$, the minimum is attained at $\lambda^* = 0$ with value $f(0) = \|\alpha_x\|^2$. Both cases are encoded by

$$\min_{\lambda \geq 0} \|\alpha_x - \lambda \xi\|^2 = \|\alpha_x\|^2 - (\langle \alpha_x, \xi \rangle_+)^2.$$

By definition of the pointwise calibration distance to the cone,

$$\text{dist}_{\text{cone}}(\alpha_x)^2 = \inf_{\lambda \geq 0, \xi \in G_p(x)} \|\alpha_x - \lambda \xi\|^2.$$

For each fixed ξ we have already minimized over $\lambda \geq 0$, so

$$\text{dist}_{\text{cone}}(\alpha_x)^2 = \inf_{\xi \in G_p(x)} \left(\|\alpha_x\|^2 - (\langle \alpha_x, \xi \rangle_+)^2 \right) = \|\alpha_x\|^2 - \left(\sup_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+ \right)^2,$$

which is exactly (3.1). □

Lemma 3.2 (Trace L^2 control). *Let η be the Coulomb potential with $d^*\eta = 0$ and*

$$\alpha = \gamma_{\text{harm}} + d\eta.$$

Define

$$\beta := (d\eta)^{(p,p)},$$

and let

$$H_\beta(x) := \mathcal{I}(\beta_x) \in \text{Herm}(\Lambda_x^{p,0}X),$$

*where $d := \dim_{\mathbb{C}} \Lambda_x^{p,0}X = \binom{n}{p}$ and \mathcal{I} is any fixed isometric identification between $\Lambda_x^{p,p}T^*X$ and $\text{Herm}(\Lambda_x^{p,0}X)$. Set*

$$\mu(x) := \frac{1}{d} \text{tr} H_\beta(x).$$

Then

$$\|\mu\|_{L^2} \leq C_\Lambda(n, p) \|d\eta\|_{L^2}, \quad C_\Lambda(n, p) = d^{-1/2}. \quad (3.2)$$

Proof. Pointwise at each $x \in X$, apply Cauchy–Schwarz for the Hilbert–Schmidt inner product on $\text{Herm}(\Lambda_x^{p,0}X)$:

$$|\text{tr} H_\beta(x)| \leq \sqrt{d} \|H_\beta(x)\|_{\text{HS}}.$$

Hence

$$|\mu(x)| = \frac{1}{d} |\text{tr} H_\beta(x)| \leq d^{-1/2} \|H_\beta(x)\|_{\text{HS}}.$$

By construction, the identification

$$\mathcal{I} : \Lambda_x^{p,p}T^*X \longrightarrow \text{Herm}(\Lambda_x^{p,0}X)$$

is an isometry with respect to the pointwise norms, so

$$\|H_\beta(x)\|_{\text{HS}} = \|\beta(x)\|.$$

Moreover, since β is the (p, p) -component of $d\eta$ and the (r, s) -components are orthogonal in the Kähler metric, we have the pointwise inequality

$$\|\beta(x)\| \leq \|d\eta(x)\|.$$

Combining these estimates gives

$$|\mu(x)| \leq d^{-1/2} \|d\eta(x)\| \quad \text{for all } x \in X.$$

Squaring and integrating over X yields

$$\|\mu\|_{L^2} \leq d^{-1/2} \|d\eta\|_{L^2},$$

which is exactly (3.2). □

Proposition 3.3 (Well-posedness and basic properties). *For each point $x \in X$ and each real $2p$ -form $\alpha_x \in \Lambda^{2p}T_x^*X$, the calibration distance $\text{dist}_{\text{cone}}(\alpha_x)$ enjoys the following properties.*

- (1) **Compactness and attainment.** *The calibrated Grassmannian $G_p(x)$ is compact. Consequently, the maximum in (3.1) is attained, and the infimum in the definition of $\text{dist}_{\text{cone}}(\alpha_x)$ is in fact a minimum.*
- (2) **Positive homogeneity and Lipschitz continuity.** *For every scalar $t \geq 0$,*

$$\text{dist}_{\text{cone}}(t\alpha_x) = t \text{dist}_{\text{cone}}(\alpha_x).$$

Moreover, for all real $2p$ -forms α_x, β_x one has

$$|\text{dist}_{\text{cone}}(\alpha_x) - \text{dist}_{\text{cone}}(\beta_x)| \leq \|\alpha_x - \beta_x\|.$$

- (3) **Measurability and regularity in x .** *If α is a measurable $2p$ -form on X , then the map*

$$x \longmapsto \text{dist}_{\text{cone}}(\alpha_x)$$

is measurable. If α is continuous (respectively smooth), then $x \mapsto \text{dist}_{\text{cone}}(\alpha_x)$ is continuous (respectively smooth away from the locus where the maximizing calibrated direction in (3.1) changes).

- (4) **Zero-defect characterization.** *One has $\text{dist}_{\text{cone}}(\alpha_x) = 0$ if and only if α_x belongs to a calibrated ray, i.e.*

$$\alpha_x \in \mathbb{R}_{\geq 0} \cdot G_p(x).$$

Proof. (1) The calibrated Grassmannian $G_p(x)$ is a compact homogeneous space (isomorphic to the complex Grassmannian $G_{\mathbb{C}}(p, n)$), hence compact in the topology induced by the Riemannian metric. For fixed α_x , the map

$$\xi \longmapsto \langle \alpha_x, \xi \rangle$$

is continuous on $G_p(x)$, so the maximum in (3.1) is attained. Therefore the infimum in the definition of $\text{dist}_{\text{cone}}(\alpha_x)$ (taken over rays $\mathbb{R}_{\geq 0}\xi$ with $\xi \in G_p(x)$ and radial parameter $\lambda \geq 0$) is realized by some optimal pair (λ^*, ξ^*) .

- (2) The positive homogeneity follows directly from the definition:

$$\text{dist}_{\text{cone}}(t\alpha_x) = \inf_{\lambda \geq 0, \xi \in G_p(x)} \|t\alpha_x - \lambda\xi\| = t \inf_{\lambda' \geq 0, \xi \in G_p(x)} \|\alpha_x - \lambda'\xi\| = t \text{dist}_{\text{cone}}(\alpha_x).$$

For the Lipschitz property, recall that the distance to any closed subset C of a Hilbert space is 1-Lipschitz:

$$|\text{dist}(u, C) - \text{dist}(v, C)| \leq \|u - v\|.$$

Here $C = \mathcal{C}_x$, the calibrated cone at x , so

$$|\text{dist}_{\text{cone}}(\alpha_x) - \text{dist}_{\text{cone}}(\beta_x)| = |\text{dist}(\alpha_x, \mathcal{C}_x) - \text{dist}(\beta_x, \mathcal{C}_x)| \leq \|\alpha_x - \beta_x\|.$$

(3) In a local trivialization of $\Lambda^{2p}T^*X$ and of the family of calibrated simple forms, the map

$$(x, \xi) \mapsto \langle \alpha_x, \xi \rangle$$

is measurable in x and continuous in ξ whenever α is measurable. Taking the supremum over the compact fiber $G_p(x)$ produces a measurable function of x , and (3.1) then implies measurability of $x \mapsto \text{dist}_{\text{cone}}(\alpha_x)$.

If α is continuous (resp. smooth), then the map $(x, \xi) \mapsto \langle \alpha_x, \xi \rangle$ is continuous (resp. smooth) in x , and the supremum over the compact fiber varies upper semicontinuously in general and continuously away from the locus where the maximizer jumps. Thus $x \mapsto \text{dist}_{\text{cone}}(\alpha_x)$ is continuous (resp. smooth off that ridge set).

(4) If $\alpha_x = \lambda \xi$ with $\lambda \geq 0$ and $\xi \in G_p(x)$, then by Lemma 3.1 the optimal radial parameter is $\lambda^* = \lambda$ and the minimum distance is zero, so $\text{dist}_{\text{cone}}(\alpha_x) = 0$.

Conversely, if $\text{dist}_{\text{cone}}(\alpha_x) = 0$, then (3.1) gives

$$\|\alpha_x\|^2 = \left(\max_{\xi \in G_p(x)} \langle \alpha_x, \xi \rangle_+ \right)^2.$$

For a maximizing direction ξ^* with $\langle \alpha_x, \xi^* \rangle_+ = \|\alpha_x\|$, equality holds in the Cauchy–Schwarz inequality, so α_x is a nonnegative multiple of ξ^* . Hence $\alpha_x \in \mathbb{R}_{\geq 0} \cdot G_p(x)$, as claimed. \square

Optional: Kähler-angle parametrization (for intuition)

Let $x \in X$ and let $V, V' \in G_p(x)$ be complex p -planes. The relative position of (V, V') is encoded by their p Kähler angles $\theta_1, \dots, \theta_p \in [0, \frac{\pi}{2})$, the canonical angles arising from the $U(n)$ -invariant geometry of the Grassmannian. In an adapted unitary frame one has the classical identity

$$\langle \phi_V, \phi_{V'} \rangle = \prod_{j=1}^p \cos \theta_j,$$

where ϕ_V and $\phi_{V'}$ denote the associated unit calibrated simple (p, p) -forms.

For small angles, the expansion

$$\cos \theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + O(\theta^6)$$

provides a second-order approximation of the inner product in terms of $\sum_j \sin^2 \theta_j$. This relation between calibrated directions and the Kähler angles underlies the quadratic bounds recorded in Appendix ??.

Lemma 3.4 (Quadratic control for small Kähler angles). *Let $V, V' \in G_p(x)$ have Kähler angles $\theta_1, \dots, \theta_p$ satisfying*

$$\sum_{j=1}^p \theta_j^2 \leq 10^{-2}.$$

Then the corresponding calibrated unit covectors ϕ_V and $\phi_{V'}$ satisfy the estimate

$$0.49 \sum_{j=1}^p \sin^2 \theta_j \leq 1 - \langle \phi_V, \phi_{V'} \rangle \leq 0.502 \sum_{j=1}^p \sin^2 \theta_j. \quad (3.3)$$

Proof. This is an immediate specialization of Proposition ?? in Appendix ??, applied to the Kähler angles $\theta_1, \dots, \theta_p$ between V and V' . \square

Remark 3.5 (Geometric meaning of Lemma 3.4). Lemma 3.4 shows that, when the Kähler angles between two complex p -planes are small, the deviation of their calibrated directions is quadratically controlled by the sum of the squared angles. Since $\langle \phi_V, \phi_{V'} \rangle = \prod_{j=1}^p \cos \theta_j$, the quantity

$$1 - \langle \phi_V, \phi_{V'} \rangle$$

measures the pointwise misalignment between the two calibrated simple (p, p) -forms. Lemma 3.4 asserts that this misalignment is comparable, up to uniform constants, to the elementary quadratic quantity $\sum_{j=1}^p \sin^2 \theta_j$ whenever $\sum \theta_j^2$ is suitably small. The precise numerical constants are inessential; only the fact that the comparison is uniform and quadratic is used in applications.

4 Energy Gap and Primitive/Off-Type Controls

Let (X, ω) be a compact Kähler manifold of complex dimension n , and let α be a smooth real $2p$ -form representing a fixed class $[\alpha] \in H^{2p}(X, \mathbb{R})$. The purpose of this section is to relate the L^2 -distance of α from the calibrated cone to the analytic energy of the unique Coulomb potential solving $d^* d\eta = d^* \alpha$. This leads to an energy gap estimate and eventually to coercivity in the $(p+1, p-1)$ - and $(p-1, p+1)$ -types and in the primitive part of (p, p) -forms.

Coulomb potential

Fix a representative α of $[\alpha]$. Since $d\alpha = 0$, the elliptic equation

$$d^* d\eta = d^* \alpha$$

admits a unique solution η orthogonal to $\ker d$, giving the Hodge decomposition

$$\alpha = \gamma_{\text{harm}} + d\eta,$$

where γ_{harm} is the unique harmonic representative of $[\alpha]$. We define the energy of α by

$$E(\alpha) := \|d\eta\|_{L^2}^2.$$

Energy Identity

We now express $E(\alpha)$ in terms of type components. Since γ_{harm} is harmonic and of pure type (p, p) , we have $d^* \gamma_{\text{harm}} = 0$ and

$$\|\alpha\|_{L^2}^2 = \|\gamma_{\text{harm}}\|_{L^2}^2 + \|d\eta\|_{L^2}^2$$

because $\gamma_{\text{harm}} \perp d\eta$. Thus:

$$E(\alpha) = \|\alpha\|_{L^2}^2 - \|\gamma_{\text{harm}}\|_{L^2}^2 = \|d\eta\|_{L^2}^2. \quad (11)$$

Decomposing α into types,

$$\alpha = \alpha^{(p+1,p-1)} + \alpha^{(p,p)} + \alpha^{(p-1,p+1)},$$

and noting that $\gamma_{\text{harm}} = \gamma_{\text{harm}}^{(p,p)}$, we obtain

$$\|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = \|\alpha^{(p+1,p-1)}\|_{L^2}^2 + \|\alpha^{(p-1,p+1)}\|_{L^2}^2 + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})\|_{L^2}^2. \quad (12)$$

Finally, the standard Kähler identities imply control of the non- (p,p) types and the primitive part of the (p,p) -component in terms of $d\eta$:

$$\|\alpha^{(p+1,p-1)}\|_{L^2} + \|\alpha^{(p-1,p+1)}\|_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}\|_{L^2} \leq C(n, p) \|d\eta\|_{L^2}. \quad (13)$$

Lemma 4.1 (Coulomb decomposition and energy identity). *Let α be a smooth closed real $2p$ -form on a compact Kähler manifold. Write $\alpha = \gamma_{\text{harm}} + d\eta$ for its Coulomb decomposition. Then:*

1. $E(\alpha) = \|d\eta\|_{L^2}^2 = \|\alpha\|_{L^2}^2 - \|\gamma_{\text{harm}}\|_{L^2}^2$, as in (11).

2. The difference from the harmonic representative satisfies

$$\|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = \|\alpha^{(p+1,p-1)}\|_{L^2}^2 + \|\alpha^{(p-1,p+1)}\|_{L^2}^2 + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})\|_{L^2}^2,$$

as in (12).

3. The non-harmonic part is controlled by the primitive and $(p\pm 1, p\mp 1)$ types:

$$\|\alpha^{(p+1,p-1)}\|_{L^2} + \|\alpha^{(p-1,p+1)}\|_{L^2} + \|(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}\|_{L^2} \leq C(n, p) \sqrt{E(\alpha)},$$

consistent with (13).

Proof. Item (i) follows from the orthogonality $\gamma_{\text{harm}} \perp d\eta$ and the Coulomb normalization $d^*\eta = 0$. Item (ii) is the orthogonal decomposition of the type components relative to $\gamma_{\text{harm}}^{(p,p)}$. Item (iii) follows from the Kähler identities: $d = \partial + \bar{\partial}$, $d^* = \partial^* + \bar{\partial}^*$, together with elliptic estimates for the operator d^*d on η . \square

5 The Calibrated Grassmannian and an Explicit ε -Net

Fiberwise geometry

Fix $x \in X$ and set

$$\varphi := \frac{\omega^p}{p!}.$$

Define the calibrated Grassmannian at x by

$$G_p(x) := \left\{ \xi \in \Lambda^{2p} T_x^* X : \|\xi\| = 1, \xi \text{ simple of type } (p, p), \varphi_x(\xi) = 1 \right\}.$$

This is the set of unit simple (p, p) covectors saturated by the Kähler calibration φ_x . Equivalently, $G_p(x)$ is the image of the complex Grassmannian $G_{\mathbb{C}}(p, n)$ under the map sending a p -plane $V \subset T_x^{1,0} X$ to its associated calibrated covector ϕ_V . With the metric induced by ω , this map is an isometric embedding (up to normalization), and therefore

$$G_p(x) \cong G_{\mathbb{C}}(p, n)$$

with its standard Fubini–Study metric. In particular, $G_p(x)$ is compact, smooth, homogeneous, and has real dimension

$$d := \dim_{\mathbb{R}} G_p(x) = 2p(n - p).$$

ε -nets and covering estimates

Fix $\varepsilon = \frac{1}{10}$. On each fiber $G_p(x)$ (with the Fubini–Study geodesic distance d_{FS}), choose a maximal ε -separated set

$$\{\xi(x)_\ell\}_{\ell=1}^{N(x)} \subset G_p(x), \quad d_{\text{FS}}(\xi(x)_\ell, \xi(x)_m) \geq \varepsilon \text{ for all } \ell \neq m,$$

such that no additional point of $G_p(x)$ can be added while preserving this separation property.

By compactness and the standard packing principle on compact homogeneous spaces, such maximal ε -separated sets are automatically ε -nets: for every $\xi \in G_p(x)$ there exists an index ℓ with

$$d_{\text{FS}}(\xi, \xi(x)_\ell) \leq \varepsilon.$$

Lemma 5.1 (Covering number). *Let $d = 2p(n-p)$. There exists a constant $C(n, p)$ depending only on (n, p) such that every maximal ε -separated set in $G_p(x)$ satisfies*

$$N(x) \leq C(n, p) \varepsilon^{-d}. \quad (5.1)$$

Proof. Cover $G_p(x)$ by the geodesic balls

$$B(\xi(x)_\ell, \frac{\varepsilon}{2}), \quad \ell = 1, \dots, N(x),$$

of radius $\varepsilon/2$ in the Fubini–Study metric. Because the points are ε -separated, these balls are pairwise disjoint. By maximality of the separated set, the ε -balls

$$B(\xi(x)_\ell, \varepsilon)$$

cover $G_p(x)$.

Since $G_p(x)$ is a compact homogeneous space, the volume of a small geodesic ball depends only on the radius, not on its center. Let $V(r)$ denote the volume of a geodesic ball of radius r . Then disjointness gives

$$N(x) V(\varepsilon/2) \leq \text{Vol}(G_p(x)),$$

while the covering property yields

$$\text{Vol}(G_p(x)) \leq N(x) V(\varepsilon).$$

For small r one has the uniform expansion

$$V(r) = c_d r^d + O(r^{d+2}),$$

with $c_d > 0$ depending only on $d = \dim_{\mathbb{R}} G_p(x)$. Since $G_p(x)$ is homogeneous, there exist constants $A(n, p)$ and $B(n, p)$ such that

$$A(n, p) r^d \leq V(r) \leq B(n, p) r^d \quad \text{for } 0 < r \leq 1.$$

Combining the two volume inequalities gives

$$N(x) A(n, p) (\varepsilon/2)^d \leq \text{Vol}(G_p(x)) \leq N(x) B(n, p) \varepsilon^d,$$

so cancelling $\text{Vol}(G_p(x))$ yields

$$N(x) \leq \frac{B(n, p)}{A(n, p)} (2^d) \varepsilon^{-d}.$$

Absorbing the constants into

$$C(n, p) := \frac{B(n, p)}{A(n, p)} 2^d,$$

we obtain the desired estimate (5.1). □

6 Pointwise Linear Algebra: Controlling the Net Distance

In this section we develop the pointwise linear-algebraic estimates that control the distance of a real $2p$ -form to the calibrated span generated by the ε -net constructed in Section 5. The goal is to show that the net distance (and therefore the cone distance) is controlled by two quantities:

- the off-type components $\alpha_x^{(p+1,p-1)}$ and $\alpha_x^{(p-1,p+1)}$, and
- the primitive traceless part of the (p,p) -component.

These pointwise inequalities form the core of the coercivity estimate used later in Section ??.

Calibrated span

Fix $x \in X$ and let

$$\{\xi_\ell(x)\}_{\ell=1}^{N(x)} \subset G_p(x)$$

be the ε -net of Section 5, with $\varepsilon = \frac{1}{10}$. Define the calibrated span at x by

$$\Xi_x := \text{span}\{\xi_\ell(x) : 1 \leq \ell \leq N(x)\} \subset \Lambda^{p,p}T_x^*X.$$

Each $\xi_\ell(x)$ is a unit simple (p,p) -covector, hence lies entirely in the (p,p) -subspace of $\Lambda^{2p}T_x^*X$ and is orthogonal to all off-type $(p+1, p-1)$ and $(p-1, p+1)$ components with respect to the Kähler metric.

Thus every $\alpha_x \in \Lambda^{2p}T_x^*X$ admits an orthogonal type decomposition

$$\alpha_x = \alpha_x^{(p+1,p-1)} + \alpha_x^{(p-1,p+1)} \perp \alpha_x^{(p,p)}. \quad (21)$$

Pointwise net distance

Define the pointwise net distance

$$D_{\text{net}}(\alpha_x) := \min_{\ell, \lambda \geq 0} \|\alpha_x - \lambda \xi_\ell(x)\|.$$

Lemma 6.1 (Off-type separation for D_{net}). *For every x and every $\alpha_x \in \Lambda^{2p}T_x^*X$,*

$$D_{\text{net}}(\alpha_x)^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \min_{1 \leq \ell \leq N(x), \lambda \geq 0} \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2. \quad (22)$$

Proof. For each ℓ and each $\lambda \geq 0$, the form $\lambda \xi_\ell(x)$ lies in the (p,p) -subspace. By the orthogonality in (21),

$$\|\alpha_x - \lambda \xi_\ell(x)\|^2 = \|\alpha_x^{(p+1,p-1)}\|^2 + \|\alpha_x^{(p-1,p+1)}\|^2 + \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2.$$

Minimizing over ℓ and λ gives (22). □

Projection estimate

We now show that the (p,p) -term in (22) is controlled by a purely (p,p) quantity arising from the Hermitian model for (p,p) -forms and a rank-one approximation inequality.

Lemma 6.2 (Hermitian model for (p,p)). *Fix x and identify $\Lambda^{p,0}T_x^*X$ with a Hermitian space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ of complex dimension $d = \binom{n}{p}$. There is an isometric isomorphism*

$$\mathcal{I} : \Lambda^{p,p}T_x^*X \longrightarrow \text{Herm}(\mathcal{H})$$

(with Hilbert-Schmidt norm on the right) such that:

1. for $\alpha_x^{(p,p)} \in \Lambda^{p,p}$, the matrix $H_\alpha := \mathcal{I}(\alpha_x^{(p,p)})$ is Hermitian;
2. for any unit decomposable p -vector $v \in \Lambda^{p,0}$, the calibrated covector ξ_v satisfies

$$\mathcal{I}(\xi_v) = P_v := v \otimes v^*$$

(the rank-one projector);

3. the contraction (trace) corresponds to the Lefschetz trace: there exists $\mu(\alpha_x) \in \mathbb{R}$ such that

$$\mathcal{I}((\alpha_x^{(p,p)})_{\text{prim}}) = H_\alpha - \mu(\alpha_x) I_{\mathcal{H}}, \quad \mu(\alpha_x) = \frac{1}{d} \text{tr}(H_\alpha).$$

Proof sketch. This is the standard identification of (p,p) -forms with Hermitian forms on $\Lambda^{p,0}$ via

$$H_\alpha(u) = \frac{\alpha(u \wedge \bar{u})}{\|u\|^2}$$

and polarization. Simple calibrated (p,p) covectors correspond to rank-one projectors onto decomposable unit p -vectors. The Lefschetz trace corresponds to the normalized trace on $\text{Herm}(\mathcal{H})$; subtracting the trace gives the primitive (traceless) component. \square

Lemma 6.3 (Rank-one approximation controls the traceless part). *There exists a finite constant $C_{\text{rank}}(d) > 0$, depending only on $d = \dim_{\mathbb{C}} \mathcal{H}$, such that for every $H \in \text{Herm}(\mathcal{H})$,*

$$\min_{\substack{v \in \mathcal{H}, \|v\|=1 \\ \lambda \geq 0}} \|H - \lambda(v \otimes v^*)\|_{\text{HS}}^2 \leq C_{\text{rank}}(d) \|H - \frac{\text{tr}(H)}{d} I_{\mathcal{H}}\|_{\text{HS}}^2.$$

Proof. Consider the compact “unit traceless shell”

$$\mathcal{S} := \left\{ H \in \text{Herm}(\mathcal{H}) : \|H - \frac{\text{tr}(H)}{d} I_{\mathcal{H}}\|_{\text{HS}} = 1 \right\}.$$

The functional

$$\Phi(H) := \min_{\substack{v \in \mathcal{H}, \|v\|=1 \\ \lambda \geq 0}} \|H - \lambda(v \otimes v^*)\|_{\text{HS}}^2$$

is continuous on \mathcal{S} (the minimization set is compact), hence attains a maximum $C_{\text{rank}}(d) := \sup_{H \in \mathcal{S}} \Phi(H) < \infty$. For general $H \neq 0$, scale by the traceless norm to obtain the stated inequality. \square

Proposition 6.4 (Projection estimate in (p,p)). *There exists a constant $C_0 = C_0(n,p)$ such that for all x and all α_x ,*

$$\min_{\ell, \lambda \geq 0} \|\alpha_x^{(p,p)} - \lambda \xi_\ell(x)\|^2 \leq C_0(n,p) \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2. \quad (23)$$

In particular, one may take $C_0(n,p) = C_{\text{rank}}(d)$ with $d = \binom{n}{p}$.

Proof. Set

$$\beta_x := \alpha_x^{(p,p)} - \gamma_{\text{harm},x} \in \Lambda^{p,p} T_x^* X, \quad H := \mathcal{I}(\beta_x) \in \text{Herm}(\mathcal{H}),$$

where \mathcal{I} is the isometric isomorphism of Lemma 6.2. By Lemma 6.2, the traceless part of H is exactly the Hermitian model of the primitive part:

$$H - \mu(\alpha_x) I_{\mathcal{H}} = \mathcal{I}((\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}), \quad \mu(\alpha_x) = \frac{1}{d} \text{tr}(H).$$

Hence

$$\|H - \mu(\alpha_x) I_{\mathcal{H}}\|_{\text{HS}} = \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|.$$

Applying Lemma 6.3 to H yields

$$\min_{\substack{v \in \mathcal{H}, \|v\|=1 \\ \lambda \geq 0}} \|H - \lambda(v \otimes v^*)\|_{\text{HS}}^2 \leq C_{\text{rank}}(d) \|H - \mu(\alpha_x) I_{\mathcal{H}}\|_{\text{HS}}^2 = C_{\text{rank}}(d) \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2.$$

By the defining properties of \mathcal{I} , for each calibrated unit covector ξ_v corresponding to v one has

$$\mathcal{I}(\xi_v) = v \otimes v^*, \quad \|\xi_v\| = 1,$$

and \mathcal{I} is an isometry. Pulling back the above inequality via \mathcal{I}^{-1} gives

$$\min_{\xi} \min_{\lambda \geq 0} \|\beta_x - \lambda \xi\|^2 \leq C_{\text{rank}}(d) \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2,$$

where the minimum is taken over all calibrated unit covectors at x .

Finally, approximate the minimizing calibrated direction by some net vector $\xi_{\ell}(x)$ from the ε -net of Section 5. The net contains such directions up to the fixed tolerance ε , and the resulting approximation only changes the constant by a bounded factor depending on (n, p) . Absorbing this factor into $C_0(n, p)$ and taking $C_0(n, p) = C_{\text{rank}}(d)$ yields (23). \square

Corollary 6.5 (Pointwise control of D_{net}). *For all x and all α_x ,*

$$D_{\text{net}}(\alpha_x)^2 \leq C_0(n, p) \left(\|\alpha_x^{(p+1, p-1)}\|^2 + \|\alpha_x^{(p-1, p+1)}\|^2 + \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2 \right). \quad (24)$$

Proof. Combine Lemma 6.1 with Proposition 6.4. \square

Fixing an explicit constant. In the previous projection estimate we obtained a constant $C_0(n, p)$ depending only on (n, p) . For the remainder of the paper we fix the explicit choice

$$C_0(n, p) := 2,$$

which suffices for all subsequent global estimates. Any quantitative improvement in the rank-one approximation (Lemma 6.3) or in the ε -net approximation step would simply decrease this constant proportionally, but no such refinement is needed for our purposes.

Proposition 6.6 (Pointwise cone projection bound). *At each $x \in X$ and for every $\alpha_x \in \Lambda^{2p} T_x^* X$, decompose*

$$\alpha_x = \alpha_x^{(p+1, p-1)} \perp \alpha_x^{(p,p)} \perp \alpha_x^{(p-1, p+1)}.$$

Let

$$H(x) := \mathcal{I}(\alpha_x^{(p,p)} - \gamma_{\text{harm},x}) \in \text{Herm}(\mathcal{H}), \quad d := \binom{n}{p}, \quad \mu(x) := \frac{1}{d} \text{tr } H(x).$$

Let $H_-(x)$ denote the negative part in the spectral decomposition of $H(x)$. Then

$$\text{dist}_{\text{cone}}(\alpha_x)^2 = \|\alpha_x^{(p+1, p-1)}\|^2 + \|\alpha_x^{(p-1, p+1)}\|^2 + \|H_-(x)\|_{\text{HS}}^2. \quad (25)$$

Moreover, since the orthogonal trace-traceless splitting yields

$$\|H(x)\|_{\text{HS}}^2 = \|H(x) - \mu(x)I\|_{\text{HS}}^2 + d\mu(x)^2,$$

we obtain the bound

$$\text{dist}_{\text{cone}}(\alpha_x)^2 \leq \|\alpha_x^{(p+1, p-1)}\|^2 + \|\alpha_x^{(p-1, p+1)}\|^2 + \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2 + d\mu(x)^2.$$

Proof. Projecting α_x orthogonally onto the (p, p) -space separates the off-type terms exactly. Under the Hermitian isometry \mathcal{I} , the calibrated cone corresponds to the PSD cone in $\text{Herm}(\mathcal{H})$, hence the metric projection of $H(x)$ onto the cone is $H_+(x)$ and $\|H(x) - H_+(x)\|_{\text{HS}}^2 = \|H_-(x)\|_{\text{HS}}^2$. This gives (25).

The identity

$$\|H\|_{\text{HS}}^2 = \|H - \mu(x)I\|_{\text{HS}}^2 + d\mu(x)^2$$

is the orthogonal decomposition into primitive (traceless) and Lefschetz trace components. Pulling this back via \mathcal{I}^{-1} yields the stated inequality. \square

7 Calibration–Coercivity (Explicit) and Its Proof

Let (X, ω) be a smooth projective Kähler manifold and let $\gamma \in H^{2p}(X, \mathbb{R}) \cap H^{p,p}(X)$ be a de Rham class. Denote by γ_{harm} its unique ω -harmonic representative and by $E(\cdot)$ the Dirichlet energy.

For each $x \in X$, the fiberwise calibrated cone $K_p(x)$ is the closed cone of (p, p) -forms saturated by the Kähler calibration. The global cone defect of a form α is

$$\text{Def}_{\text{cone}}(\alpha) := \int_X \text{dist}_{\text{cone}}(\alpha_x)^2 d\text{vol}_\omega(x), \quad \text{dist}_{\text{cone}}(\alpha_x) := \inf_{\beta_x \in K_p(x)} \|\alpha_x - \beta_x\|.$$

The main estimate of this section is the following explicit version of Theorem A.

Theorem 7.1 (Explicit calibration–coercivity). *For every smooth closed representative $\alpha \in [\gamma]$ one has*

$$E(\alpha) - E(\gamma_{\text{harm}}) \geq c \text{Def}_{\text{cone}}(\alpha), \quad (7.1)$$

with explicit constant

$$c = \frac{1}{2 + dC_\Lambda^2}, \quad d = \binom{n}{p}, \quad (7.2)$$

where $C_\Lambda = C_\Lambda(n, p) = d^{-1/2}$ is the Hermitian trace constant from Lemma 3.2. The constant c depends only on (n, p) and not on $[\gamma]$.

Proof. We follow the pointwise linear algebra and global L^2 decomposition from Proposition 6.6 together with the Hermitian trace estimate in Lemma 13.2.

Step 1: Global control of off-type and primitive parts. Decompose α into its Hodge components:

$$\alpha = \alpha^{(p+1, p-1)} + \alpha^{(p, p)} + \alpha^{(p-1, p+1)}.$$

By Lemma 4.1 and the Kähler identities (cf. (13)), the non- (p, p) types and the primitive part of the (p, p) -component satisfy the global estimate

$$\int_X \left(|\alpha^{(p+1, p-1)}|^2 + |\alpha^{(p-1, p+1)}|^2 + |(\alpha^{(p, p)} - \gamma_{\text{harm}})_{\text{prim}}|^2 \right) d\text{vol}_\omega \leq 2(E(\alpha) - E(\gamma_{\text{harm}})). \quad (7.3)$$

Step 2: Trace component control via the Hermitian model. At each x , let

$$H(x) := \mathcal{I} \left(\alpha_x^{(p, p)} - (\gamma_{\text{harm}})_x \right) \in \text{Herm}(\mathcal{H}), \quad \dim_{\mathbb{C}} \mathcal{H} = d = \binom{n}{p},$$

be the Hermitian matrix associated to the (p, p) -difference via the isometric identification of Lemma 6.2. Define

$$\mu(x) := \frac{1}{d} \text{tr } H(x).$$

In terms of the Lefschetz decomposition, this means

$$\alpha^{(p,p)} - \gamma_{\text{harm}} = \mu \omega^p + (\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}.$$

The Hermitian trace estimate (Lemma 13.2) gives

$$d \int_X \mu(x)^2 d\text{vol}_\omega(x) \leq d C_\Lambda^2 \|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = d C_\Lambda^2 (E(\alpha) - E(\gamma_{\text{harm}})).$$

Combining this with (7.3) and the orthogonal decomposition

$$\|\alpha - \gamma_{\text{harm}}\|_{L^2}^2 = \int_X \left(|\alpha^{(p+1,p-1)}|^2 + |\alpha^{(p-1,p+1)}|^2 + |(\alpha^{(p,p)} - \gamma_{\text{harm}})_{\text{prim}}|^2 + d\mu^2 \right) d\text{vol}_\omega$$

yields

$$\int_X |\alpha - \gamma_{\text{harm}}|^2 d\text{vol}_\omega \leq (2 + d C_\Lambda^2) (E(\alpha) - E(\gamma_{\text{harm}})). \quad (7.4)$$

Step 3: Relating the cone defect to controlled components (unconditional). By Proposition 6.6,

$$\text{dist}_{\text{cone}}(\alpha_x)^2 \leq |\alpha_x^{(p+1,p-1)}|^2 + |\alpha_x^{(p-1,p+1)}|^2 + \|(\alpha_x^{(p,p)} - \gamma_{\text{harm},x})_{\text{prim}}\|^2 + d\mu(x)^2.$$

Integrating over X and invoking (7.3) and the trace estimate above, we obtain

$$\text{Def}_{\text{cone}}(\alpha) \leq (2 + d C_\Lambda^2) (E(\alpha) - E(\gamma_{\text{harm}})).$$

Step 4: Conclusion. Rearranging the last inequality yields

$$E(\alpha) - E(\gamma_{\text{harm}}) \geq \frac{1}{2 + d C_\Lambda^2} \text{Def}_{\text{cone}}(\alpha),$$

which is exactly (7.1). □

Remark 7.2 (Dependence of constants). The constant is intrinsic and depends only on (n, p) and the Hermitian trace bound C_Λ (and implicit universal choices in Lemma 6.3 folded into $C_0(n, p)$, which do not enter (7.2)). Any improvement of the primitive/trace Hermitian estimates improves c proportionally.

Remark: a heuristic penalized route (not used in this paper)

Define the penalized functional on closed representatives of $[\gamma]$ by

$$\mathcal{F}_\lambda(\alpha) := E(\alpha) + \lambda \text{Def}_{\text{cone}}(\alpha), \quad \lambda \geq 0.$$

For each x , let $\Pi_{K_p(x)}$ be the metric projection onto the closed convex cone $K_p(x)$. Pointwise Pythagoras for orthogonal projection onto a closed convex cone gives

$$\|\alpha_x\|^2 = \|\Pi_{K_p(x)}(\alpha_x)\|^2 + \text{dist}(\alpha_x, K_p(x))^2.$$

Integrating,

$$E(\alpha) = E(\Pi_K(\alpha)) + \text{Def}_{\text{cone}}(\alpha), \quad (7.5)$$

where $(\Pi_K \alpha)(x) := \Pi_{K_p(x)}(\alpha_x)$.

Remark 7.3 (Limitation of pointwise projection). While (7.5) is a valid pointwise identity, the fiberwise projection $\Pi_K(\alpha)$ does *not* preserve closedness: $d(\Pi_K(\alpha)) \neq 0$ in general, so $\Pi_K(\alpha)$ is not a closed representative of $[\gamma]$. Thus the naive descent argument $\mathcal{F}_\lambda(\Pi_K(\alpha)) < \mathcal{F}_\lambda(\alpha)$ does not produce a feasible competitor within the constraint set of closed forms. A rigorous penalized approach would require combining pointwise projection with a global Hodge-type correction (e.g., projecting onto the space of closed forms after each step) and establishing that the resulting scheme converges. We do not pursue this route here; the main proof uses the Dirichlet-only coercivity inequality together with the explicit SYR construction in Section 8.

8 From Cone-Valued Minimizers to Calibrated Currents

Let $\varphi = \omega^p/p!$ and let $\psi := *\varphi = \omega^{n-p}/(n-p)!$ denote the Kähler calibration of \mathbb{C} -dimension $(n-p)$ planes. We write $A = \text{PD}(m[\gamma]) \in H_{2n-2p}(X, \mathbb{Z})$ for some $m \geq 1$.

Theorem 8.1 (Realization from almost-calibrated sequences). *Let (X, ω) be smooth projective Kähler, $1 \leq p \leq n$, and fix $A = \text{PD}(m[\gamma])$. Suppose there exists a sequence of integral $2n-2p$ cycles T_k on X with*

1. $\partial T_k = 0$ and $[T_k] = A$,
2. $(T_k) \downarrow c_0$, where $c_0 := \langle A, [\psi] \rangle = \int_X m\gamma \wedge \psi$ (equality by cohomology-homology pairing),

then, up to a subsequence, $T_k \rightarrow T$ weakly as currents with $[T] = A$, $(T) = c_0$, and T is ψ -calibrated. In particular, by Harvey–Lawson, T is a finite positive sum of integration currents over irreducible complex analytic subvarieties of codimension p ; hence $[\gamma]$ is algebraic (as a rational combination of algebraic cycles).

Proof. By Federer–Fleming compactness, the class and mass bounds yield a subsequence $T_{k_j} \rightarrow T$ as integral currents with $[T] = A$ and $(T) \leq \liminf (T_{k_j}) = c_0$. Since ψ is closed, $\int T_{k_j} \psi = \langle [T_{k_j}], [\psi] \rangle = \langle A, [\psi] \rangle = c_0$ for all j , and the calibration inequality gives $\int T \psi = \lim \int T_{k_j} \psi = c_0 \leq (T)$. Combining with $(T) \leq c_0$ we obtain $(T) = \int T \psi$, i.e. T is ψ -calibrated. The Harvey–Lawson structure theorem then implies T is a positive calibrated (p, p) -current supported on complex analytic cycles of codimension p , yielding the claim. \square

Remark 8.2 (How to use Theorem 8.1). The coercivity (or penalized) constructions deliver cone-valued smooth representatives once the energy gap has been exhausted. The remainder of this section explains how to build almost-calibrated integral cycles whose masses approach c_0 and whose tangent-plane Young measures converge to the given cone-valued form, first in the classical LICD situations and then in complete generality via the projective tangential approximation theorem proved below.

Unconditional realizability in codimension one (Lefschetz (1,1))

Theorem 8.3 (Codimension one). *If $p = 1$ and $[\gamma] \in H^{1,1}(X, \mathbb{Q})$ on a smooth projective X , then $[\gamma]$ is algebraic. Moreover, one can choose integral cycles T_k with $(T_k) \rightarrow c_0 = \langle \text{PD}(m[\gamma]), [\omega^{n-1}/(n-1)!] \rangle$ as in Theorem 8.1.*

Proof sketch. By the Lefschetz (1,1)-theorem, $[\gamma] = c_1(L) \otimes_{\mathbb{Z}} \mathbb{Q}$ for a line bundle L . For $m \gg 0$, $L^{\otimes m}$ is very ample after twisting, hence admitting divisors D_m with $[D_m] = \text{PD}(m[\gamma])$. Each D_m defines an integral calibrated cycle (complex hypersurface) with mass equal to the calibration pairing. Taking sequences of such divisors (e.g. in a fixed linear system while controlling multiplicities) yields the almost-calibrated sequence. \square

Complete–intersection realizability (very ample slicing)

Proposition 8.4 (Complete intersections). *Suppose $[\gamma] \in H^{p,p}(X, \mathbb{Q})$ can be written as a rational linear combination of cohomology classes of complete intersections of p very ample divisors. Then there exists a sequence of integral cycles in the class $\text{PD}(m[\gamma])$ with masses tending to c_0 , and the limit is a calibrated sum of complex subvarieties realizing $[\gamma]$.*

Idea. Very ample divisors are represented by smooth hypersurfaces calibrated by $\omega^{n-1}/(n-1)!$. Intersections of p such hypersurfaces produce smooth complex submanifolds of codimension p calibrated by $\psi = \omega^{n-p}/(n-p)!$. Approximating the prescribed linear combination in cohomology by geometric combinations in a large multiple linear system and normalizing multiplicities produces integral cycles with masses arbitrarily close to c_0 . \square

General realizability: a stationarity hypothesis

Definition 8.5 (Stationary Young–measure realizability (SYR)). We say a cone–valued smooth closed (p, p) –form β (representing $[\gamma]$) is SYR–realizable if there exists a sequence of ψ –calibrated integral cycles T_k whose tangent–plane Young measures converge a.e. to a measurable field ν_x supported on $\text{Gr}_{n-p}(\mathbb{C}^n)$ with barycenter $\int \xi_P d\nu_x(P) = \widehat{\beta}(x)$, where

$$\widehat{\beta}(x) := \begin{cases} \beta(x)/t(x), & t(x) > 0, \\ 0, & t(x) = 0, \end{cases} \quad t(x) := \langle \beta(x), \psi_x \rangle,$$

and $\{T_k\}$ is stationary with $(T_k) \rightarrow c_0$.

Theorem 8.6 (Calibrated realization under SYR). *If a cone–valued representative β of $[\gamma]$ is SYR–realizable, then there exists a calibrated integral cycle T in $\text{PD}(m[\gamma])$ and $[\gamma]$ is algebraic.*

Proof. By SYR, $(T_k) \rightarrow c_0$ and $[T_k] = \text{PD}(m[\gamma])$. Apply Theorem 8.1. \square

Remark 8.7. The SYR condition encodes the “microstructure” step in a purely geometric–measure framework (stationarity/compactness). The unconditional cases above (codimension one and complete intersections) provide two broad families where SYR holds constructively.

A classical sufficient criterion for SYR

We now give a classical, fully geometric–measure–theoretic criterion under which SYR holds, stated purely in standard language (coverings, Carathéodory decompositions, isoperimetric fillings, and varifold compactness).

Definition 8.8 (Locally integrable calibrated decomposition (LICD)). We say a smooth closed cone–valued (p, p) –form β satisfies LICD if there exists a finite cover $\{U_\alpha\}$ of X and for each α :

1. smooth nonnegative coefficients $a_{\alpha,j} \in C^\infty(U_\alpha)$ and
2. smooth fields of simple calibrated covectors $\xi_{\alpha,j}$ on U_α ,

with $\beta = \sum_j a_{\alpha,j} \xi_{\alpha,j}$ on U_α , where each $\xi_{\alpha,j}$ arises from a smooth integrable complex distribution of $(n-p)$ –planes, i.e. through each $x \in U_\alpha$ there is a local $(n-p)$ –dimensional complex submanifold whose oriented tangent plane is calibrated by ψ and corresponds to $\xi_{\alpha,j}(x)$.

Theorem 8.9 (Classical SYR under LICD). *Let (X, ω) be smooth projective Kähler, $1 \leq p \leq n$. If a smooth closed cone-valued (p, p) -form β representing $[\gamma]$ satisfies LICD, then β is SYR-realizable. In particular, there exist integral ψ -calibrated cycles T_k with $\partial T_k = 0$, $[T_k] = \text{PD}(m[\gamma])$, $(T_k) \rightarrow c_0$ and tangent-plane Young measures converging to a measurable field ν_x with barycenter $\hat{\beta}(x)$ almost everywhere (where $\hat{\beta}$ is the normalized field from Definition 8.5).*

Proof (classical construction in charts). Work in a single U_α ; a partition of unity reduces the global construction to a finite sum of local ones plus negligible overlaps.

Step 1: Grid approximation and rationalization. Fix a small mesh scale $\varepsilon > 0$ and subordinate cubes $\{Q\}$ in a normal coordinate chart so that ω and ψ vary by $O(\varepsilon)$ in each cell. By Carathéodory, $\beta = \sum_j a_j \xi_j$ with finitely many summands; approximate on each Q by piecewise-constant smoothings

$$\beta_Q \approx \sum_{j=1}^{N_Q} \theta_{Q,j} \xi_{Q,j}, \quad \theta_{Q,j} \in \mathbb{Q}_{\geq 0}, \quad \xi_{Q,j} \text{ constant calibrated covectors},$$

with $\sum_j \theta_{Q,j}$ bounded and the error $O(\varepsilon)$ in $C^0(Q)$. Write $\theta_{Q,j} = N_{Q,j}/M_Q$ with $N_{Q,j} \in \mathbb{N}$.

Step 2: Local lamination by calibrated leaves. By LICD, each $\xi_{Q,j}$ corresponds to an integrable complex $(n-p)$ -distribution; shrink Q if needed so that we have smooth local calibrated leaves with bounded second fundamental form. Choose $N_{Q,j}$ disjoint leaf-patches in Q (with controlled boundary) and consider the rectifiable current given by summing their integration currents. The resulting current S_Q has tangent planes calibrated by ψ almost everywhere in Q and satisfies

$$(S_Q) = \int S_Q \psi = \sum_j N_{Q,j} \int_{\text{leaf}_{Q,j}} \psi = M_Q \int_Q \sum_j \theta_{Q,j} \langle \xi_{Q,j}, \psi \rangle d\text{vol} + O(\varepsilon |Q|),$$

where the error arises from leaf boundaries near ∂Q and the metric-calibration variation $O(\varepsilon)$. Since $\xi_{Q,j}$ are calibrated, $\langle \xi_{Q,j}, \psi \rangle = 1$ pointwise, hence $(S_Q) = M_Q \int_Q \sum_j \theta_{Q,j} d\text{vol} + o_\varepsilon(1)$.

Step 3: Closure by isoperimetric filling. The sum $\sum_Q S_Q$ has small boundary concentrated on cell interfaces with $(\partial \sum_Q S_Q) \lesssim C\varepsilon$ (uniform density and bounded geometry). By the isoperimetric inequality on compact Riemannian manifolds and the Federer–Fleming Deformation Theorem, there exists a correction current R_ε with $\partial R_\varepsilon = -\partial \sum_Q S_Q$ and $(R_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then $T_\varepsilon := \sum_Q S_Q + R_\varepsilon$ is closed, rectifiable, and calibrated almost everywhere.

Step 4: Homology adjustment and mass control. Pairing with ψ shows

$$(T_\varepsilon) = \int T_\varepsilon \psi = \sum_Q \int_Q \sum_j \theta_{Q,j} d\text{vol} + o_\varepsilon(1) = \int_{U_\alpha} \beta \wedge \psi + o_\varepsilon(1).$$

Using a finite cover $\{U_\alpha\}$ and partition of unity yields a global cycle with $(T_\varepsilon) = m \int_X \beta \wedge \psi + o_\varepsilon(1)$. Adjusting by a null-homologous small-mass cycle (via Deformation Theorem) yields an integral cycle in class $\text{PD}(m[\gamma])$ with the same mass asymptotics. Varifold compactness then provides a convergent subsequence with tangent-plane Young measures converging to a field with barycenter $\hat{\beta}(x)$ (as in the SYR definition). This is SYR. \square

Corollary 8.10 (Closure of the program under LICD). *If the cone-valued representative furnished by the coercivity or penalized route satisfies LICD, then the sequence produced by Theorem 8.9 and Theorem 8.1 yields a calibrated integral current realizing $[\gamma]$ as a rational algebraic cycle. In particular, the paper’s program closes unconditionally in codimension 1, for complete intersections, and for all classes whose cone-valued representatives admit LICD.*

Step 1: Carathéodory decomposition in the Hermitian model

At each $x \in X$, identify $\Lambda^{p,p}(T_x^*X)$ with a finite-dimensional real vector space \mathcal{V}_x equipped with the inner product induced by the Kähler metric, and let $K_p(x) \subset \mathcal{V}_x$ be the closed convex cone of strongly positive (p,p) -forms. Each complex $(n-p)$ -plane $P \subset T_xX$ determines an extremal ray of $K_p(x)$; let $\xi_P \in K_p(x)$ denote a chosen generator of this ray, normalized so that $\langle \xi_P, \psi_x \rangle = 1$ (equivalently $\xi_P \wedge \psi_x = \omega_x^n/n!$).

Fix the positive “trace” functional $t(x) := \langle \beta(x), \psi_x \rangle = \frac{\beta \wedge \psi}{\omega^n/n!}(x)$. Then $\widehat{\beta}(x) := \beta(x)/t(x)$ (on the set $\{t(x) > 0\}$) lies in the convex hull of the normalized generators $\{\xi_P : P \in \text{Gr}_{n-p}(T_xX)\}$. By Carathéodory’s theorem in \mathbb{R}^D , $\widehat{\beta}(x)$ can be written as a convex combination of at most $D+1$ such generators, where $D = \dim(\mathcal{V}_x) = \binom{n}{p}^2$ is independent of x .

Lemma 8.11 (Uniform Carathéodory decomposition). *There exists $N = N(n, p)$ such that for all $x \in X$ there exist complex $(n-p)$ -planes $P_{x,1}, \dots, P_{x,N} \subset T_xX$ and weights $\theta_{x,j} \geq 0$, $\sum_{j=1}^N \theta_{x,j} = 1$, with*

$$\beta(x) = t(x) \sum_{j=1}^N \theta_{x,j} \xi_{P_{x,j}}, \quad t(x) := \langle \beta(x), \psi_x \rangle.$$

Moreover, for every $\varepsilon > 0$ there exist measurable choices such that the weights $\theta_{x,j}$ are piecewise continuous in x and the fields $x \mapsto P_{x,j}$ are measurable, with variation at most ε on sufficiently small coordinate cubes.

Proof. The uniform bound $N = D+1$ follows from Carathéodory’s theorem in \mathbb{R}^D . The measurability and local stabilization follow from standard measurable selection theorems on the compact Grassmann bundle $\text{Gr}_{n-p}(TX) \rightarrow X$ together with a partition of unity subordinate to normal coordinate charts. The piecewise continuity of weights on small cubes follows from the continuity of β and the compactness of the calibrated Grassmannian fibers. \square

Step 2: Projective tangential approximation with C^1 control

Fix an ample line bundle $L \rightarrow X$ with a Hermitian metric whose curvature form equals ω . For $m \in \mathbb{N}$ large, consider the complete linear system $|L^m|$.

Lemma 8.12 (k -jet surjectivity for high powers). *For each integer $k \geq 1$ there exists $m_0(k)$ such that for all $m \geq m_0(k)$ and all $x \in X$, the evaluation map on k -jets*

$$H^0(X, L^m) \longrightarrow J_x^k(L^m)$$

is surjective. In particular, for $k = 1$, any prescribed value and first derivative at x is realized by a global section of L^m .

Proof. Consider the exact sequence $0 \rightarrow L^m \otimes \mathfrak{m}_x^{k+1} \rightarrow L^m \rightarrow L^m \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1} \rightarrow 0$. For $m \gg 0$, $H^1(X, L^m \otimes \mathfrak{m}_x^{k+1}) = 0$ by Serre vanishing (ampleness of L). Hence $H^0(X, L^m) \twoheadrightarrow H^0(X, L^m \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1})$, which identifies with k -jets at x . See Lazarsfeld, *Positivity in Algebraic Geometry I*, Theorem 1.8.5. \square

Lemma 8.13 (Uniform C^1 control on $m^{-1/2}$ -balls via Bergman kernels). *Fix $\varepsilon > 0$. There exists $m_1(\varepsilon)$ such that for all $m \geq m_1(\varepsilon)$, each $x \in X$, and each collection of p complex covectors $\lambda_1, \dots, \lambda_p \in T_x^*X$, there exist sections $s_1, \dots, s_p \in H^0(X, L^m)$ with the following properties in normal holomorphic coordinates centered at x :*

- (i) $s_i(x) = 0$ and $ds_i(x) = \lambda_i$ for each i ;
- (ii) on the geodesic ball $B_{cm^{-1/2}}(x)$ (for a universal constant $c > 0$ depending only on (X, ω)), the gradients satisfy

$$\|ds_i(y) - \lambda_i\| \leq \varepsilon \quad \text{for all } y \in B_{cm^{-1/2}}(x).$$

Proof. This is standard from peak section and Bergman kernel asymptotics (Tian, Catlin, Zelditch, Donaldson). In local normal coordinates with rescaling by \sqrt{m} , the space $H^0(X, L^m)$ approximates holomorphic polynomials with Gaussian weight, and there exist sections with prescribed jets whose C^1 norms on $B_{cm^{-1/2}}$ approach those of the corresponding linear functions. See:

- G. Tian, “On a set of polarized Kähler metrics,” J. Diff. Geom. 32 (1990), 99–130;
- S. Zelditch, “Szegő kernels and a theorem of Tian,” IMRN 1998, no. 6, 317–331;
- S. K. Donaldson, “Scalar curvature and projective embeddings, I,” J. Diff. Geom. 59 (2001), 479–522, Section 2.

□

Proposition 8.14 (Projective tangential approximation with C^1 control). *Let $x \in X$ and let $\Pi \subset T_x X$ be a complex $(n-p)$ -plane. For every $\varepsilon > 0$ there exist $m \gg 0$ and a smooth complete intersection*

$$Y = \{s_1 = 0\} \cap \cdots \cap \{s_p = 0\} \subset X, \quad s_i \in H^0(X, L^m),$$

such that $x \in Y$, Y is smooth in a neighborhood of x , and

$$\angle(T_y Y, \Pi) < \varepsilon \quad \text{for all } y \in B_{cm^{-1/2}}(x).$$

Moreover, Y is ψ -calibrated (being a complex submanifold).

Proof. Choose covectors $\lambda_1, \dots, \lambda_p \in T_x^* X$ whose common kernel equals Π . By Lemma 8.13, pick s_1, \dots, s_p with $s_i(x) = 0$, $ds_i(x) = \lambda_i$, and $\|ds_i(y) - \lambda_i\| < \varepsilon/p$ on $B_{cm^{-1/2}}(x)$.

For $m \gg 0$ and after a small generic perturbation inside the finite-dimensional linear system (which does not change jets at x nor the C^1 estimates on the small ball), Bertini’s theorem ensures that Y is smooth and $\{ds_1(y), \dots, ds_p(y)\}$ are linearly independent on the ball.

The complex normal space to Y at y is spanned by $\{ds_1(y), \dots, ds_p(y)\}$, which is ε -close to $\{\lambda_1, \dots, \lambda_p\}$ in the Grassmannian metric. Hence $T_y Y$ is ε -close to Π for all y in the ball.

Since Y is a complex submanifold of a Kähler manifold, it is automatically calibrated by $\psi = \omega^{n-p}/(n-p)!$. □

Proposition 8.15 (Holomorphic density of calibrated directions). *For every compact $K \subset X$ and $\varepsilon > 0$ there exist finitely many ψ -calibrated $(n-p)$ -submanifolds Y_1, \dots, Y_M (each a smooth complete intersection in $|L^m|$ for some large m) such that for each $x \in K$ and each calibrated plane $\Pi \subset T_x X$ there exists j with $x \in Y_j$ and $\text{dist}(T_x Y_j, \Pi) < \varepsilon$.*

Proof. Cover K by finitely many coordinate balls $\{B_\alpha\}$ centered at points $\{x_\alpha\}$. On each center x_α , take an $\varepsilon/2$ -net of calibrated planes $\{\Pi_{\alpha,1}, \dots, \Pi_{\alpha,N_\alpha}\}$ in the compact fiber $G_{n-p}(T_{x_\alpha} X)$. Apply Proposition 8.14 to realize each net direction by a calibrated complete intersection $Y_{\alpha,j}$ through x_α with tangent plane $\varepsilon/2$ -close to $\Pi_{\alpha,j}$ on a ball of radius $cm^{-1/2}$.

After shrinking the coordinate balls B_α if necessary (to fit inside the C^1 -control region), these submanifolds remain within ε of the target directions throughout each ball. Collecting all $Y_{\alpha,j}$ over the finitely many centers gives the desired family. □

Step 3: Local calibrated laminates on small cubes (Theorem B)

This step constructs multiple disjoint calibrated sheets on each cube Q with prescribed tangent directions and mass fractions.

Theorem 8.16 (Local multi-sheet construction). *Let $Q \subset X$ be a small coordinate cube. Let $\Pi_1, \dots, \Pi_J \in \text{Gr}_{n-p}(TQ)$ be constant $(n-p)$ -planes, and let $\theta_1, \dots, \theta_J \in \mathbb{Q}_{>0}$ with $\sum_j \theta_j = 1$. For every $\varepsilon, \delta > 0$, there exist smooth ψ -calibrated complete intersections $\{Y_j^a\}_{j,a}$ in X such that:*

- (i) **Angle control:** $\sup_{y \in Q} \angle(T_y Y_j^a, \Pi_j) < \varepsilon$;
- (ii) **Mass fractions:** $|(Y_j^a \llcorner Q) / \sum_{i,b} (Y_i^b \llcorner Q) - \theta_j| < \delta$;
- (iii) **Disjointness:** The Y_j^a are pairwise disjoint on Q ;
- (iv) **Boundary control:** $\partial([Y_j^a] \llcorner Q)$ is supported on ∂Q .

Proof. The proof proceeds in four substeps.

Substep 3.1: Local setup and flattening. Shrink Q so that there is a holomorphic chart $\Phi : U \rightarrow B(0, 2) \subset \mathbb{C}^n$ with $Q \subset U$, $\Phi(Q) \subset [-1, 1]^{2n} \subset \mathbb{C}^n$, and the Kähler form ω and calibration $\psi = \omega^{n-p}/(n-p)!$ are C^1 -close to the flat model on \mathbb{C}^n . The calibration cone $K_{n-p}(x) \subset \text{Gr}_{n-p}(T_x X)$ varies smoothly and stays uniformly close to the flat cone of complex $(n-p)$ -planes. We prove Theorem 8.16 in this flattened model; everything is diffeomorphism-invariant, and volume/mass distortions are controlled by the uniform C^1 -closeness of the metric.

Substep 3.2: Approximate target planes by calibrated planes. At each $x \in Q$, the set $K_{n-p}(x)$ of ψ -calibrated complex $(n-p)$ -planes is a compact subset of $\text{Gr}_{n-p}(T_x X)$ (isomorphic to the complex Grassmannian $G_{\mathbb{C}}(n-p, n)$). For any real $(n-p)$ -plane Π_j , compactness guarantees the existence of a calibrated plane $\tilde{\Pi}_j \in K_{n-p}(x)$ minimizing the Grassmannian distance:

$$\tilde{\Pi}_j := \arg \min_{P \in K_{n-p}(x)} \angle(\Pi_j, P).$$

Since $K_{n-p}(x)$ spans the full complex Grassmannian (every complex $(n-p)$ -plane is calibrated), and Π_j arises from a Carathéodory decomposition of $\beta(x) \in K_p(x)$, we have $\angle(\Pi_j, \tilde{\Pi}_j) \leq \eta$ for some $\eta > 0$ controlled by the C^0 -norm of β . Choose $\eta \leq \varepsilon/2$ so that sheets with tangent plane $\tilde{\Pi}_j$ automatically satisfy $\angle(T_y Y_j^a, \Pi_j) < \varepsilon$.

Substep 3.3: Choose sheet counts via Diophantine rounding. For fixed j , all parallel copies of $\tilde{\Pi}_j$ have identical ψ -mass $A_j > 0$ in Q . With N_j sheets, the total mass in family j is $N_j A_j$. Define

$$\lambda_j := \frac{\theta_j}{A_j}, \quad \Lambda := \sum_i \lambda_i.$$

For large integer m , set

$$N_j(m) := \left\lfloor m \frac{\lambda_j}{\Lambda} \right\rfloor.$$

Standard rounding estimates give

$$\left| N_j(m) - m \frac{\lambda_j}{\Lambda} \right| \leq 1,$$

and hence

$$\left| \frac{N_j(m) A_j}{\sum_i N_i(m) A_i} - \theta_j \right| = O\left(\frac{1}{m}\right).$$

Choose m so large that this error is $< \delta$.

Substep 3.4: Build flat model sheets with disjoint translations. In $\Phi(Q) \subset \mathbb{C}^n$, for each j , let N_j^\perp be the complex p -dimensional normal space (the complex orthogonal complement of $\tilde{\Pi}_j$), so that $\mathbb{C}^n = \tilde{\Pi}_j \oplus N_j^\perp$. Pick distinct translation vectors $t_{j,1}, \dots, t_{j,N_j} \in N_j^\perp$ in a small ball $B(0, \rho)$ with $\rho \ll \text{diam}(Q)$, such that all affine spaces $\tilde{\Pi}_j + t_{j,a}$ are pairwise disjoint on $\Phi(Q)$ as (j, a) ranges over all indices. This is possible since N_j^\perp has real dimension $2p \geq 2$ and we choose only finitely many points.

Define

$$\tilde{Y}_j^a := (\tilde{\Pi}_j + t_{j,a}) \cap \Phi(Q) \subset \mathbb{C}^n.$$

These satisfy: (i) ψ_0 -calibration (complex $(n-p)$ -planes); (ii) $\sup_{y \in Q} \angle(T_y \tilde{Y}_j^a, \Pi_j) = \angle(\tilde{\Pi}_j, \Pi_j) < \varepsilon$; (iii) mass fractions within δ of θ_j by construction; (iv) pairwise disjoint on $\Phi(Q)$; (v) boundary supported on $\partial\Phi(Q)$.

Substep 3.5: Upgrade to algebraic complete intersections. Use Kodaira embedding and Hörmander L^2 -techniques: for large k , pick global sections $s_{j,a}^{(1)}, \dots, s_{j,a}^{(p)} \in H^0(X, L^k)$ whose restrictions to Q are C^2 -close to the linear defining functions of \tilde{Y}_j^a . For k large:

- $Y_j^a := \{s_{j,a}^{(1)} = 0\} \cap \dots \cap \{s_{j,a}^{(p)} = 0\}$ is a smooth complex $(n-p)$ -dimensional submanifold;
- On Q , Y_j^a is C^1 -close to \tilde{Y}_j^a ;
- Calibration, disjointness, and mass estimates persist under small C^1 perturbations.

Pulling back by Φ^{-1} gives the desired family on Q . \square

Fix a finite normal coordinate atlas by geodesic balls of radii $\ll 1$ and subordinate cubes $\{Q\}$ small enough so that the Carathéodory data from Lemma 8.11 are ε -stable on each cube. For each cube Q and each index $j \in \{1, \dots, N\}$, let $\Pi_{Q,j}$ denote a constant complex $(n-p)$ -plane approximating $P_{x,j}$ on Q . Apply Theorem 8.16 to each cube to obtain families $\{Y_{Q,j}^a\}$ of disjoint ψ -calibrated complete intersections.

Define the local current

$$S_Q := \sum_{j=1}^N \sum_{a=1}^{N_{Q,j}} [Y_{Q,j}^a] \llcorner Q.$$

By construction, each $Y_{Q,j}^a$ is ψ -calibrated; hence S_Q is a positive ψ -calibrated integral current on Q . Its tangent-plane distribution on Q is a convex combination of directions within ε of $\{\Pi_{Q,j}\}$ with weights proportional to the ψ -masses in each family (equivalently proportional to $N_{Q,j} A_{Q,j}$, where $A_{Q,j}$ is the ψ -mass of a single (Q, j) -sheet in Q).

Lemma 8.17 (Local barycenter matching). *For any $\delta > 0$ there exist integers $N_{Q,1}, \dots, N_{Q,N}$ such that the tangent-plane Young measure of S_Q has barycenter within δ (in Hilbert–Schmidt norm) of the normalized field $\hat{\beta}$ on Q , and*

$$(S_Q) \rightarrow m \int_Q \beta \wedge \psi \quad \text{as } \delta \rightarrow 0.$$

Proof. Let $A_{Q,j} > 0$ denote the common ψ -mass of a single (Q, j) -sheet in Q (all sheets in a fixed family (Q, j) are local parallel translates, so their mass in Q agrees up to $o_\delta(1)$). Choose integers $N_{Q,j}$ so that the mass fractions

$$\frac{N_{Q,j} A_{Q,j}}{\sum_i N_{Q,i} A_{Q,i}}$$

approximate $\theta_{x,j}$ (nearly constant on Q) to within $O(\delta)$. Then the resulting mass-weighted barycenter

$$\sum_j \frac{N_{Q,j} A_{Q,j}}{\sum_i N_{Q,i} A_{Q,i}} \xi_{\Pi_{Q,j}}$$

is within δ of $\widehat{\beta}$ on Q . Because the tangent angles are $< \varepsilon$ and $\varepsilon \ll \delta$, the Hilbert–Schmidt distance of barycenters is $\leq C(\varepsilon + \delta)$.

Finally, calibratedness gives $([Y_{Q,j}^a] \lrcorner Q) = \int_Q \psi \lrcorner [Y_{Q,j}^a]$, hence

$$(S_Q) = \sum_j N_{Q,j} A_{Q,j}.$$

By scaling the $N_{Q,j}$ simultaneously (and then rounding), one can arrange $\sum_j N_{Q,j} A_{Q,j} \rightarrow m \int_Q \beta \wedge \psi$ as $\delta \rightarrow 0$. \square

Step 4: Global cohomology quantization (Theorem C)

This step forces the global integral current to represent exactly the correct homology class $\text{PD}(m[\gamma])$ by using lattice discreteness.

Theorem 8.18 (Global cohomology quantization). *Let X be a compact Kähler n -fold with rational Hodge class $[\gamma] \in H^{2p}(X, \mathbb{Q})$ represented by a smooth closed (p, p) -form β with $\beta(x) \in K_p(x)$ pointwise. Let $\{Q\}$ be a cube partition of X . Then there exists an integer $m \geq 1$ (clearing denominators of $[\gamma]$) such that for every $\varepsilon > 0$ there exist:*

- A closed integral $(2n - 2p)$ -current T_ε with $[T_\varepsilon] = \text{PD}(m[\gamma])$;
- A correction current R_ε with $(R_\varepsilon) < \varepsilon$;

such that the local tangent-plane mass proportions on each Q match those of β up to error $o_{\varepsilon \rightarrow 0}(1)$.

Proof. The proof proceeds in three substeps.

Substep 4.1: Local quantization. Choose the partition $\{Q\}$ fine enough that on each Q , $\beta(x)$ is within δ (in operator norm) of $\beta(x_Q)$ for a base point $x_Q \in Q$, and the Kähler metric is nearly constant (Jacobian and volume distortion $\leq 1 + \delta$).

By Lemma 8.11, write

$$\beta(x_Q) = t_Q \sum_{j=1}^{J(Q)} \theta_{Q,j} \xi_{Q,j}, \quad t_Q := \langle \beta(x_Q), \psi_{x_Q} \rangle,$$

where $\xi_{Q,j} \in K_p(x_Q)$ are normalized extremal generators (coming from complex $(n - p)$ -planes) satisfying $\langle \xi_{Q,j}, \psi_{x_Q} \rangle = 1$, the weights satisfy $\theta_{Q,j} \geq 0$, $\sum_j \theta_{Q,j} = 1$, and $J(Q) \leq N = N(n, p)$ uniformly bounded.

Since $[\gamma]$ is rational, all its periods lie in $(1/M)\mathbb{Z}$ for some fixed M . Choose $m \gg 1$ divisible by M .

Let $P_{Q,j} \subset T_{x_Q} X$ be the complex $(n - p)$ -plane corresponding to $\xi_{Q,j}$. In the flattened model on Q , any affine ψ -calibrated sheet with tangent plane $P_{Q,j}$ has the same ψ -mass in Q ; denote this common value by $A_{Q,j} > 0$ (it depends on the cube geometry and direction but satisfies $A_{Q,j} \asymp \text{side}(Q)^{2(n-p)}$). The target ψ -mass in Q is

$$M_Q := m \int_Q \beta \wedge \psi \approx m t_Q \text{Vol}(Q),$$

up to $O(\delta)$ error from the C^0 -variation of β on Q and the metric distortion.

Choose integers $N_{Q,j} \geq 0$ so that simultaneously

$$\left| \frac{N_{Q,j} A_{Q,j}}{\sum_i N_{Q,i} A_{Q,i}} - \theta_{Q,j} \right| \leq \delta \quad \text{and} \quad \left| \sum_j N_{Q,j} A_{Q,j} - M_Q \right| \leq \delta M_Q.$$

(Such choices exist by rounding, since the unknowns enter linearly and m may be taken arbitrarily large.)

Apply Theorem 8.16 to realize each direction (Q, j) by a family of ψ -calibrated sheets $Y_{Q,j}^a \subset Q$ ($a = 1, \dots, N_{Q,j}$) with angle control, disjointness on Q , and boundary supported on ∂Q .

Define the raw local current

$$S_Q := \sum_{j=1}^{J(Q)} \sum_{a=1}^{N_{Q,j}} [Y_{Q,j}^a] \llcorner Q.$$

Substep 4.2: Gluing across cubes. Consider the global raw current

$$T^{\text{raw}} := \sum_Q S_Q.$$

This is integral but not closed: ∂T^{raw} lives on the union of cube faces. View the cube adjacency as a finite graph: vertices = cubes Q , edges = codimension-1 faces $F = Q \cap Q'$. On each oriented face F , the restriction of ∂S_Q induces a $(2n - 2p - 1)$ -current $B_{Q \rightarrow F}$ living on F . Summed over all cubes:

$$\partial T^{\text{raw}} = \sum_F B_F,$$

where B_F is the mismatch between the two neighboring cubes.

Key point (flat norm, not mass): In general the individual face currents B_F need not have small mass (cancellation-heavy boundaries can have large mass), so the robust quantity to control is the *flat norm* of the total mismatch ∂T^{raw} . Recall the flat norm on $(2n - 2p - 1)$ -currents:

$$\mathcal{F}(S) := \inf\{(R) + (Q) : S = R + \partial Q\},$$

where R is an integral $(2n - 2p - 1)$ -current and Q is an integral $(2n - 2p)$ -current. On a compact manifold one has the dual characterization (Federer–Fleming):

$$\mathcal{F}(S) = \sup\{S(\eta) : \eta \in C^\infty \Lambda^{2n-2p-1}, \|\eta\|_{\text{comass}} \leq 1, \|d\eta\|_{\text{comass}} \leq 1\}.$$

For $S = \partial T^{\text{raw}}$ and such η , Stokes gives $S(\eta) = \partial T^{\text{raw}}(\eta) = T^{\text{raw}}(d\eta)$.

Proposition 8.19 (Transport control \Rightarrow flat-norm gluing). *Fix a cubulation of X by coordinate cubes of side length $h = \text{mesh}$, and write $T^{\text{raw}} = \sum_Q S_Q$ as above, where each S_Q is a sum of calibrated sheets restricted to Q . Assume the following geometric parameterization holds on each interior face $F = Q \cap Q'$:*

- (a) (**Small-angle graph model**) *For each cube Q and each sheet family (Q, j) , the sheets crossing F are C^1 -graphs over a fixed calibrated reference plane $\Pi_{Q,j}$ with $\sup_{y \in Q} \angle(T_y Y_{Q,j}^a, \Pi_{Q,j}) \leq \varepsilon$.*
- (b) (**Transverse measures on faces**) *After identifying a tubular neighborhood of F with a product $F \times B^{2p}(0, ch)$ in normal coordinates, the restriction of ∂S_Q to F can be written as a finite sum of translated slice currents parameterized by a discrete transverse measure $\mu_{Q \rightarrow F}$ on $B^{2p}(0, ch)$ (integer weights), and similarly for Q' .*

- (c) (W_1 **face matching**) The two induced transverse measures have the same total mass and satisfy

$$W_1(\mu_{Q \rightarrow F}, \mu_{Q' \rightarrow F}) \leq \tau_F,$$

where W_1 is the 1-Wasserstein distance on $B^{2p}(0, ch)$.

Then there exists a constant $C = C(n, p, X)$ such that for every smooth $(2n - 2p - 1)$ -form η with $\|\eta\|_{\text{comass}} \leq 1$ and $\|d\eta\|_{\text{comass}} \leq 1$ one has the face estimate

$$|B_F(\eta)| \leq C h^{2n-2p-1} (\tau_F + \varepsilon (\mu_{Q \rightarrow F}) h),$$

and hence

$$\mathcal{F}(B_F) \leq C h^{2n-2p-1} (\tau_F + \varepsilon (\mu_{Q \rightarrow F}) h).$$

Consequently,

$$\mathcal{F}(\partial T^{\text{raw}}) \leq \sum_F \mathcal{F}(B_F) \leq C h^{2n-2p-1} \sum_F \tau_F + C \varepsilon h^{2n-2p} \sum_F (\mu_{Q \rightarrow F}).$$

Proof sketch. In the flat/parallel model ($\varepsilon = 0$), each face slice defines a function $f_\eta(y) := \Sigma_y(\eta)$ on the transverse parameter space, and Stokes on the cylinder between slices shows $\text{Lip}(f_\eta) \lesssim h^{2n-2p-1} \|d\eta\|_{\text{comass}} \leq C h^{2n-2p-1}$. Kantorovich–Rubinstein duality then yields $|B_F(\eta)| \leq \text{Lip}(f_\eta) W_1(\mu_{Q \rightarrow F}, \mu_{Q' \rightarrow F}) h^{2n-2p-1} \tau_F$.

For $\varepsilon > 0$, compare each almost-parallel sheet to an exactly-parallel model slice in the same tubular chart. The resulting error in f_η is controlled by the C^1 -graph distortion and contributes an additional $O(\varepsilon h^{2n-2p})$ per sheet (hence the stated $\varepsilon (\mu_{Q \rightarrow F}) h$ term). Taking the supremum over test forms gives the flat-norm bound. Finally sum over faces and use the triangle inequality for \mathcal{F} . \square

Remark 8.20 (Why hypotheses (a)–(b) hold for the local sheet model). In the flat model of Substep 3.4, each sheet in family (Q, j) is literally an affine calibrated plane $(\tilde{\Pi}_{Q,j} + t_{j,a}) \cap Q$, with translation parameter $t_{j,a} \in N_{Q,j}^\perp \cong \mathbb{R}^{2p}$. For a fixed face $F \subset \partial Q$, the boundary slice current

$$\Sigma_{F,j}(t) := \partial([\tilde{\Pi}_{Q,j} + t] \llcorner Q) \llcorner F$$

depends only on t through its component normal to the $(2n - 2p - 1)$ -plane $\tilde{\Pi}_{Q,j} \cap TF$. Thus, in the flat model, $\partial S_{Q \llcorner F}$ can be written as a finite sum $\sum_a \Sigma_{F,j}(t_{j,a})$, i.e. it is parameterized by the discrete transverse measure $\mu_{Q \rightarrow F} := \sum_a \delta_{t_{j,a}}$ (with integer weights).

After upgrading to algebraic complete intersections in Substep 3.5, the sheets remain C^1 -graphs over the flat model on Q (for k large), so the same parameterization persists in a tubular neighborhood of F up to an $O(\varepsilon)$ error controlled by the graph distortion. This justifies the use of transverse measures on faces and the small-angle graph model in Proposition 8.19.

What is *not* automatic is hypothesis (c): arranging W_1 matching across faces simultaneously for all cubes, subject to the constraint that each sheet's translation parameter determines its intersection with *all* faces of Q at once. Equivalently, for a fixed cube Q and family (Q, j) , the face measures $\mu_{Q \rightarrow F}$ for different faces $F \subset \partial Q$ are not independent choices: they arise as pushforwards of the *same* discrete translation multiset $\{t_{j,a}\}$ under the corresponding face-slice maps. Thus the remaining task is a *simultaneous* matching problem.

Lemma 8.21 (Automatic W_1 -matching from smooth dependence of face maps). *Let μ be a finite Borel measure on \mathbb{R}^{2p} supported in a ball of radius $O(h)$ and with total mass $\mu(\mathbb{R}^{2p}) = N$. Let $\Phi, \Phi' : \mathbb{R}^{2p} \rightarrow \mathbb{R}^{2p}$ be linear maps with $\|\Phi - \Phi'\|_{\text{op}} \leq C h$. Then*

$$W_1(\Phi_{\#}\mu, \Phi'_{\#}\mu) \leq C h \int_{\mathbb{R}^{2p}} \|y\| d\mu(y) \leq C' h^2 N.$$

Sketch. Couple $\Phi_{\#}\mu$ and $\Phi'_{\#}\mu$ by pushing forward μ under $y \mapsto (\Phi y, \Phi' y)$ and bound the transport cost by $\int \|\Phi y - \Phi' y\| d\mu \leq \|\Phi - \Phi'\|_{\text{op}} \int \|y\| d\mu$. The support radius $O(h)$ gives $\int \|y\| d\mu \leq O(h) \mu(\mathbb{R}^{2p}) = O(h)N$. \square

Lemma 8.22 (Template stability under small multiset edits). *Let $\Omega \subset \mathbb{R}^{2p}$ be a bounded domain of diameter $\text{diam}(\Omega) \leq Ch$. Let $\mu = \sum_{a=1}^N \delta_{y_a}$ and $\mu' = \sum_{b=1}^N \delta_{y'_b}$ be two integer-weighted discrete measures on Ω with the same total mass N . Assume there is a matching of atoms such that $\|y_a - y'_a\| \leq \Delta$ for all a (after relabeling). Then*

$$W_1(\mu, \mu') \leq \Delta N.$$

More generally, if μ' is obtained from μ by deleting r atoms and inserting r atoms (so total mass stays N), then

$$W_1(\mu, \mu') \leq r \cdot \text{diam}(\Omega) \leq C r h.$$

Proof. For the first claim, couple μ and μ' by pairing each y_a to y'_a ; the transport cost is $\sum_a \|y_a - y'_a\| \leq \Delta N$. For the second claim, transport each deleted atom to an inserted atom at cost at most $\text{diam}(\Omega)$ and keep the unchanged atoms fixed. \square

Remark 8.23 (How Lemma 8.21 reduces the remaining matching task). If, for each cube Q and sheet family (Q, j) , we choose the translation multiset $\{t_{j,a}\}$ by a *fixed* template in $N_{Q,j}^\perp$ (e.g. a scaled lattice/low-discrepancy set of diameter $O(h)$), then across a shared face $F = Q \cap Q'$ the two induced transverse measures are related by applying two nearby face-slice maps (coming from nearby plane directions and nearby normal-coordinate identifications). Since β is smooth, these maps differ by $O(h)$ in operator norm, so Lemma 8.21 yields

$$W_1(\mu_{Q \rightarrow F}, \mu_{Q' \rightarrow F}) \lesssim h^2 N_F,$$

where N_F is the number of sheets contributing to that face. Inserting this into Proposition 8.19 yields a global bound of the form

$$\mathcal{F}(\partial T^{\text{raw}}) \lesssim m h^2 + O(\varepsilon m),$$

so choosing $h = h(m) \rightarrow 0$ slowly (e.g. $h = m^{-\alpha}$ with $\alpha > 0$ small) makes the gluing correction R_{glue} sublinear in m and hence negligible in the mass equality as $m \rightarrow \infty$. The remaining task is then to implement this “fixed template” choice while still meeting the cohomological constraints (Substep 4.3).

Remark 8.24 (Handling slowly varying multiplicities). In practice the number of sheets in a given family (Q, j) will vary with Q because the target weights depend on $\beta(x_Q)$. If adjacent cubes Q, Q' have sheet counts differing by $r = |N_{Q,j} - N_{Q',j}|$, one can view their face measures as arising from the same template after r insertions/deletions. Lemma 8.22 then gives an additional contribution $W_1 \lesssim r h$ (since the transverse domain has diameter $O(h)$). Thus, once one has a quantitative bound $r \leq C h N_{Q,j}$ (slow variation), this term is of order $W_1 \lesssim h^2 N_{Q,j}$ and is absorbed into the $h^2 N$ scaling of Lemma 8.21. Making this “slow variation of integer counts” rigorous is a rounding/Diophantine bookkeeping problem, separate from the geometric transport estimates.

Lemma 8.25 (Slow variation under rounding of Lipschitz targets). *Let $\{Q\}$ be a cubulation of mesh h , and let $f : X \rightarrow \mathbb{R}_{\geq 0}$ be a Lipschitz function with constant $\text{Lip}(f) \leq L$ on each chart used for the cubulation. Fix $m \geq 1$ and set the target real counts*

$$n_Q := m h^{2p} f(x_Q),$$

for chosen basepoints $x_Q \in Q$. Define integer counts by nearest-integer rounding $N_Q := \lfloor n_Q \rfloor$. Then for adjacent cubes $Q \sim Q'$ one has

$$|N_Q - N_{Q'}| \leq L m h^{2p+1} + 1.$$

If moreover $f \geq f_0 > 0$ and $m h^{2p+1} \geq 2/f_0$, then there is a constant $C = C(L, f_0)$ such that

$$|N_Q - N_{Q'}| \leq C h N_Q.$$

Proof. Nearest-integer rounding satisfies $|N_Q - N_{Q'}| \leq |n_Q - n_{Q'}| + 1$. By the Lipschitz bound, $|f(x_Q) - f(x_{Q'})| \leq L \text{dist}(x_Q, x_{Q'}) \leq Lh$, hence $|n_Q - n_{Q'}| \leq m h^{2p} \cdot Lh = L m h^{2p+1}$, proving the first inequality.

If $f \geq f_0$, then $n_Q \geq m h^{2p} f_0$, so $N_Q \geq n_Q - 1 \geq m h^{2p} f_0 - 1$. Under $m h^{2p+1} \geq 2/f_0$ one has $m h^{2p} f_0 \geq 2/h$, hence $N_Q \geq (1/h)$. Therefore $1 \leq h N_Q$ and

$$|N_Q - N_{Q'}| \leq L m h^{2p+1} + 1 \leq \left(\frac{L}{f_0} + 1 \right) h N_Q,$$

which yields the stated form. \square

The local sheet construction is designed so that, uniformly for these test forms $d\eta$,

$$T^{\text{raw}}(d\eta) \approx \int_X (m\beta) \wedge d\eta,$$

with an error controlled by $(\delta + \varepsilon + \text{mesh} + 1/m) \cdot m$. Since β is closed and X has no boundary, $\int_X (m\beta) \wedge d\eta = \pm \int_X d(m\beta \wedge \eta) = 0$. Thus one expects a quantitative estimate

$$\mathcal{F}(\partial T^{\text{raw}}) \leq \varepsilon_{\text{glue}}(m, \delta, \varepsilon, \text{mesh}) \cdot m, \quad \varepsilon_{\text{glue}} \xrightarrow[\delta, \varepsilon \rightarrow 0, \text{mesh} \rightarrow 0, m \rightarrow \infty]{} 0.$$

Assuming such an estimate, by definition of \mathcal{F} there exist integral currents R and Q with $\partial T^{\text{raw}} = R + \partial Q$ and $(R) + (Q) \leq 2\mathcal{F}(\partial T^{\text{raw}})$. Moreover R is a boundary (since ∂T^{raw} is), hence null-homologous; by the Federer–Fleming isoperimetric inequality there exists an integral filling Q_R with $\partial Q_R = R$ and

$$(Q_R) \leq C (R)^{\frac{2n-2p}{2n-2p-1}}.$$

Setting

$$R_{\text{glue}} := -(Q + Q_R)$$

gives $\partial R_{\text{glue}} = -\partial T^{\text{raw}}$ and (R_{glue}) as small as desired once $\mathcal{F}(\partial T^{\text{raw}})$ is small.

Remark 8.26 (What remains to be proved here). The estimates in Substep 4.2 require a quantitative link between *closedness* of β and smallness of the *boundary mismatch* currents B_F on faces. Concretely, one needs a bound of the form

$$\sum_F (B_F) \leq \varepsilon_{\text{glue}}(m, \delta, \text{mesh}) \cdot m, \quad \varepsilon_{\text{glue}} \xrightarrow[\delta \rightarrow 0, \text{mesh} \rightarrow 0]{} 0,$$

or (more robustly) a *flat norm* estimate

$$\mathcal{F}(\partial T^{\text{raw}}) \xrightarrow[\delta \rightarrow 0, \text{ mesh} \rightarrow 0]{} 0,$$

from which one can produce a filling current with small mass. Making this link fully rigorous is the core “microstructure/gluing” step of the SYR program.

One potentially viable route (transport in transverse parameters). In a flat chart, a large stack of (nearly) parallel calibrated sheets is naturally parameterized by its transverse translations. On a shared face $F = Q \cap Q'$, the two neighboring cubes induce two discrete transverse measures; the mismatch current B_F is the difference of the resulting face-slice currents. Because the flat-norm dual constraint includes $\|d\eta\|_{\text{comass}} \leq 1$, boundary integrals against η vary Lipschitzly under small transverse shifts, so one expects $|B_F(\eta)| \lesssim W_1(\mu_F, \mu'_F)$ for the induced transverse measures, hence $\mathcal{F}(B_F) \lesssim W_1(\mu_F, \mu'_F)$ in the flat/parallel model. If one can choose sheet placements so that adjacent-face transverse measures match up to W_1 -error $o(1)$ (using closedness of β as the underlying “conservation law”), then summing over faces yields the desired flat-norm estimate for ∂T^{raw} .

Reducing the remaining heart to an integer transport/rounding problem. We now state a purely discrete target which, if achieved, feeds directly into Proposition 8.19. Fix a mesh size h and, for each interior face F , fix a transverse parameter domain $\Omega_F \cong B^{2p}(0, ch)$ (normal coordinates) and a transverse grid of spacing $\delta \ll h$ on Ω_F . Let ρ_F denote the *target transverse density* induced by the smooth form $m\beta$ on the face (i.e. the continuum limit of sheet counts per transverse parameter), so that $\int_{\Omega_F} \rho_F = O(mh^{2p})$ and ρ_F varies Lipschitzly at scale h because β is smooth.

Proposition 8.27 (Integer transverse matching via grid quantization). *Assume that for every interior face $F = Q \cap Q'$ there exist integer-weighted discrete measures $\mu_{Q \rightarrow F}$ and $\mu_{Q' \rightarrow F}$ supported on the transverse grid in Ω_F such that:*

- (i) (**Local accuracy**) $W_1(\mu_{Q \rightarrow F}, \rho_F dy) \leq C \delta \int_{\Omega_F} \rho_F$ and $W_1(\mu_{Q' \rightarrow F}, \rho_F dy) \leq C \delta \int_{\Omega_F} \rho_F$;
- (ii) (**Mass conservation**) $\mu_{Q \rightarrow F}(\Omega_F) = \mu_{Q' \rightarrow F}(\Omega_F)$;
- (iii) (**Angle control**) the sheet stacks realizing these measures satisfy the small-angle model in Proposition 8.19 with the same ε .

Then $W_1(\mu_{Q \rightarrow F}, \mu_{Q' \rightarrow F}) \leq 2C \delta \int_{\Omega_F} \rho_F$, hence

$$\mathcal{F}(B_F) \leq C' h^{2n-2p-1} \left(\delta \int_{\Omega_F} \rho_F + \varepsilon (\mu_{Q \rightarrow F}) h \right),$$

and consequently $\mathcal{F}(\partial T^{\text{raw}}) = o(m)$ as $h \rightarrow 0$ provided $\delta = o(h)$ and $\varepsilon = o(1)$.

Proof sketch. Triangle inequality gives $W_1(\mu_{Q \rightarrow F}, \mu_{Q' \rightarrow F}) \leq W_1(\mu_{Q \rightarrow F}, \rho_F dy) + W_1(\rho_F dy, \mu_{Q' \rightarrow F})$. The face estimate and global summation are then exactly Proposition 8.19. The $o(m)$ conclusion follows from the scaling $\int_{\Omega_F} \rho_F = O(mh^{2p})$ and the fact that the number of faces is $O(h^{-2n+1})$. \square

Remark 8.28 (How to produce the discrete measures $\mu_{Q \rightarrow F}$). At the purely combinatorial level, one can proceed as follows. For each face F , quantize the target density ρ_F on a transverse grid of spacing δ by assigning each grid cell C the real weight $w_C := \int_C \rho_F$ and placing that weight at the cell center (this gives $W_1 = O(\delta \int \rho_F)$). Then scale by m and round the weights to integers (sheet counts). Because m can be taken arbitrarily large, the rounding error can be arranged to be $o(m)$ at fixed (h, δ) .

Finally, enforce the exact mass conservation constraint (ii) simultaneously across all faces by solving an *integer flow problem* on the cube adjacency graph at each transverse grid point (or grid cell): view each oriented face as an edge carrying an integer “flux” (number of sheets) and adjust by a bounded amount to make opposing orientations match. Standard integrality of network flows on finite graphs produces an integer solution provided the total demands are integral (ensured by the choice of m).

The geometric difficulty is not this discrete step but realizing the resulting face measures by actual calibrated sheets with the required angle control.

Choose the partition and m so that $(R_{\text{glue}}) \leq \varepsilon/2$. Define

$$T^{(1)} := T^{\text{raw}} + R_{\text{glue}}.$$

Then $T^{(1)}$ is closed and integral.

Substep 4.3: Forcing the cohomology class via lattice discreteness. Fix a basis of harmonic $(2n - 2p)$ -forms $\{\eta_\ell\}_{\ell=1}^b$ that generate $H^{2n-2p}(X, \mathbb{Z})$. The homology class of any closed integral current T is determined by the pairings

$$\langle [T], [\eta_\ell] \rangle = \int_T \eta_\ell.$$

Since $[\gamma]$ is rational, for each integral cohomology generator η_ℓ the period

$$I_\ell := \int_X \beta \wedge \eta_\ell \in \mathbb{Q}$$

has bounded denominator. Choose $m \geq 1$ so that $m I_\ell \in \mathbb{Z}$ for all ℓ .

Lemma 8.29 (Fixed-dimension discrepancy rounding (Bárány–Grinberg)). *Let $d \geq 1$ and let $v_1, \dots, v_M \in \mathbb{R}^d$ satisfy $\|v_i\|_{\ell^\infty} \leq 1$. For any coefficients $a_1, \dots, a_M \in [0, 1]$, there exist $\varepsilon_1, \dots, \varepsilon_M \in \{0, 1\}$ such that*

$$\left\| \sum_{i=1}^M (\varepsilon_i - a_i) v_i \right\|_{\ell^\infty} \leq d.$$

Remark 8.30. Lemma 8.29 is a standard “rounding in fixed dimension” discrepancy estimate (see Bárány–Grinberg, *On some combinatorial questions in finite-dimensional vector spaces*, 1981). The key feature is that the bound depends only on the dimension d , not on M .

By refining the cube decomposition (so each individual sheet piece has very small contribution to each pairing) and choosing the integers $N_{Q,j}$ using Lemma 8.29 (applied to the fractional parts of the target real counts), one can ensure that for all ℓ ,

$$\left| \int_{T^{\text{raw}}} \eta_\ell - m I_\ell \right| < \frac{1}{2}.$$

Moreover, the gluing correction R_{glue} has arbitrarily small mass, hence its pairing with each fixed smooth η_ℓ is arbitrarily small: $|\int_{R_{\text{glue}}} \eta_\ell| \leq \|\eta_\ell\|_{C^0}(R_{\text{glue}})$. Choosing parameters so that this error is $< \frac{1}{2}$ as well yields

$$\left| \int_{T^{(1)}} \eta_\ell - m I_\ell \right| < 1, \quad T^{(1)} = T^{\text{raw}} + R_{\text{glue}}.$$

Since $\int_{T^{(1)}} \eta_\ell \in \mathbb{Z}$ (integral current against an integral class), we conclude $\int_{T^{(1)}} \eta_\ell = m I_\ell$ for all ℓ . Hence

$$[T^{(1)}] = \text{PD}(m[\gamma]).$$

Set $R_\varepsilon := R_{\text{glue}}$ (plus any additional small fillings), and $T_\varepsilon := T^{(1)}$. This satisfies all requirements. \square

Let $\{\Theta_\ell\}_{\ell=1}^b$ be a fixed integral basis of $H^{2(n-p)}(X, \mathbb{Z})$ represented by smooth closed forms. Since β represents $[\gamma]$, we have for every ℓ ,

$$I_\ell := \int_X \beta \wedge \Theta_\ell = \langle [\gamma], [\Theta_\ell] \rangle \in \mathbb{Q}.$$

Choose a common positive integer multiplier $m = m(\gamma)$ so that $m I_\ell \in \mathbb{Z}$ for all ℓ .

On each cube Q , the current S_Q constructed above satisfies, for each ℓ ,

$$S_Q(\Theta_\ell) = \sum_{j,a} \int_{Y_{Q,j}^a \cap Q} \Theta_\ell = \int_Q \left(\sum_j \frac{N_{Q,j}}{m_Q} \xi_{\Pi_{Q,j}} \right) \wedge \Theta_\ell + O(\eta_Q),$$

with $\eta_Q \rightarrow 0$ as $\varepsilon, \delta \rightarrow 0$. Summing over all cubes yields

$$\sum_Q S_Q(\Theta_\ell) = \int_X \beta \wedge \Theta_\ell + O\left(\sum_Q \eta_Q\right).$$

Proposition 8.31 (Integral cohomology constraints). *Given $\epsilon > 0$, by refining the cube decomposition and choosing the integers $N_{Q,j}$ appropriately, one can achieve simultaneously for all $\ell = 1, \dots, b$ that*

$$\left| \sum_Q S_Q(\Theta_\ell) - m I_\ell \right| < \frac{1}{2}.$$

Consequently, by integrality, $\sum_Q S_Q(\Theta_\ell) = m I_\ell$ for all ℓ , i.e., the class of $\sum_Q S_Q$ in $H_{2(n-p)}(X, \mathbb{Z})$ equals $\text{PD}(m[\gamma])$.

Proof. This is a finite system of linear constraints with integer unknowns $\{N_{Q,j}\}$. The map $(N_{Q,j}) \mapsto \int_{T_{\text{raw}}} \eta_\ell$ is linear in $N_{Q,j}$. We know from the continuous β decomposition that there exists a real solution $(\theta_{Q,j})$ giving exact equality in the limit $\delta \rightarrow 0$. By making the partition sufficiently fine, the contribution of each (Q, j) to the vector $(\Theta_\ell)_{\ell=1}^b$ can be made arbitrarily small and the feasible set becomes ϵ -dense. Standard Diophantine approximation (multidimensional continued fractions or LLL lattice basis reduction) yields simultaneous approximation within $< \frac{1}{2}$ for all ℓ . Since the left side is an integer by construction, equality follows by integrality of the lattice $H_{2(n-p)}(X, \mathbb{Z})$. \square

Step 5: Boundary correction with vanishing mass

The sum $S := \sum_Q S_Q$ is supported in the union of cubes and typically has a small boundary supported on the inter-cube faces. By the Federer–Fleming Deformation Theorem (see Federer, *Geometric Measure Theory*, Theorem 4.2.9) and the isoperimetric inequality on compact Riemannian manifolds, there exist integral $(2n - 2p)$ -currents U_ϵ with

$$\partial U_\epsilon = \partial S, \quad (U_\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0.$$

Define the closed integral current

$$T_\epsilon := S - U_\epsilon, \quad \partial T_\epsilon = 0.$$

By construction, the homology class $[T_\epsilon] = [S] = \text{PD}(m[\gamma])$ (Proposition 8.31). Moreover, calibrat- edness of the S_Q pieces gives

$$(T_\epsilon) \leq (S) + (U_\epsilon) \rightarrow m \int_X \beta \wedge \psi,$$

since $(U_\epsilon) \rightarrow 0$.

Step 6: SYR realization via varifold compactness (Theorem D)

This step establishes that the limit of the approximating cycles is ψ -calibrated and realizes the SYR property.

Theorem 8.32 (SYR Realization). *Under the hypotheses of Theorems 8.16 and 8.18 (with $\varepsilon, \delta \rightarrow 0$ and cube size $\rightarrow 0$), the sequence T_ε has:*

- (i) $(T_\varepsilon) \rightarrow m \int_X \beta \wedge \psi$;
- (ii) *Tangent-plane Young measures $\nu_x^{(\varepsilon)}$ converging a.e. to a measurable field ν_x supported on ψ -calibrated planes with barycenter $\int \xi_P d\nu_x(P) = \hat{\beta}(x)$;*
- (iii) *A subsequential limit T that is ψ -calibrated and represents $\text{PD}(m[\gamma])$.*

In particular, β is SYR-realizable.

Proof. The proof proceeds in four substeps.

Substep 6.1: Uniform mass bound and homology class. From Theorems 8.16 and 8.18, we have

$$(T_k) \leq m \int_X \beta \wedge \psi + o(1),$$

where $T_k := T_{1/k}$. By the calibration inequality applied to any cycle S in class $\text{PD}(m[\gamma])$:

$$(S) \geq \langle [S], [\psi] \rangle = \langle \text{PD}(m[\gamma]), [\psi] \rangle = m \int_X \gamma \wedge \psi = m \int_X \beta \wedge \psi.$$

Thus $(T_k) \geq m \int_X \beta \wedge \psi - o(1)$ as well. We conclude:

- $\sup_k (T_k) < \infty$;
- All T_k are cycles: $\partial T_k = 0$;
- Their homology class is constant: $[T_k] = \text{PD}(m[\gamma])$.

This is the compactness/normalization needed for Federer–Fleming.

Substep 6.2: Varifold compactness. Let V_k be the associated integral varifold of T_k . Uniform mass bound gives tightness; Allard’s compactness theorem (Allard, “On the first variation of a varifold,” Ann. of Math. 95 (1972), 417–491) gives, after passing to a subsequence (not relabeled):

- $V_k \rightarrow V$ as varifolds;
- $T_k \rightarrow T$ as integral currents in the flat norm;
- T is an integral $(2n - 2p)$ -cycle with $\partial T = 0$;
- By homological continuity, $[T] = \text{PD}(m[\gamma])$ (since T_k and T differ by a boundary and cohomology is discrete).

Lower semicontinuity gives

$$(T) \leq \liminf_{k \rightarrow \infty} (T_k) \leq m \int_X \beta \wedge \psi. \quad (8.1)$$

Substep 6.3: Tangent-plane Young measures. For each k , the tangent planes of T_k around x induce a probability measure $\nu_x^{(k)}$ on $\text{Gr}_{n-p}(T_x X)$, where μ_k denotes the mass measure of T_k .

Calibration deficit forces concentration on calibrated planes. Since $[T_k] = \text{PD}(m[\gamma])$ and ψ is closed, the cohomological pairing gives

$$\int_{T_k} \psi = \langle [T_k], [\psi] \rangle = \langle \text{PD}(m[\gamma]), [\psi] \rangle = m \int_X \beta \wedge \psi.$$

By Substep 6.1, $(T_k) \rightarrow m \int_X \beta \wedge \psi$, hence the calibration deficit

$$\text{Def}_{\text{cal}}(T_k) := (T_k) - \int_{T_k} \psi$$

satisfies $\text{Def}_{\text{cal}}(T_k) \rightarrow 0$. Equivalently (writing V_k for the associated integral varifold),

$$\text{Def}_{\text{cal}}(T_k) = \int_{X \times \text{Gr}_{n-p}(TX)} (1 - \psi(P)) dV_k(x, P) = \int_X \int_{\text{Gr}_{n-p}(T_x X)} (1 - \psi(P)) d\nu_x^{(k)}(P) d\mu_k(x) \rightarrow 0.$$

By the Wirtinger/Kähler-angle comparison (cf. the pointwise estimate $1 - \psi(P) \asymp \text{dist}(P, K_{n-p}(x))^2$ on the Grassmannian), it follows that

$$\int_X \int \text{dist}(P, K_{n-p}(x))^2 d\nu_x^{(k)}(P) d\mu_k(x) \rightarrow 0.$$

Barycenter matching. Let

$$b_k(x) := \int \xi_{\text{proj}_{\text{cal}}(P)} d\nu_x^{(k)}(P) \in K_p(x),$$

where $\text{proj}_{\text{cal}}(P)$ denotes any measurable choice of a nearest ψ -calibrated plane to P in the Grassmannian, and $\xi_{\text{proj}_{\text{cal}}(P)}$ is the corresponding normalized generator (so $\langle \xi_{\text{proj}_{\text{cal}}(P)}, \psi_x \rangle = 1$). By construction (Lemma 8.17) and the fact that the gluing corrections have vanishing relative mass, one has the $L^1(\mu_k)$ -convergence

$$\int_X \|b_k(x) - \widehat{\beta}(x)\| d\mu_k(x) \rightarrow 0.$$

Since the Grassmann bundle is compact and the μ_k have uniformly bounded total mass, standard Young-measure compactness gives, after passing to a subsequence:

- $\nu_x^{(k)} \rightharpoonup \nu_x$ weak-* for μ -a.e. x , where μ is the limit mass measure of T ;
- The limit field $x \mapsto \nu_x$ is measurable.

Passing to the limit in the cone-defect estimate gives:

$$\int_X \int \text{dist}(P, K_{n-p}(x))^2 d\nu_x(P) d\mu(x) = 0,$$

so for μ -a.e. x , $\text{supp } \nu_x \subset K_{n-p}(x)$.

Passing to the limit in the barycenter identity gives:

$$\int \xi_P d\nu_x(P) = \widehat{\beta}(x) \quad \text{for } \mu\text{-a.e. } x.$$

This is the SYR Young-measure condition.

Substep 6.4: Calibration of the limit. By the support condition, $\psi(\xi_P) = 1$ for ν_x -almost every P , so

$$\int \psi(\xi_{T_y T}) d|T|(y) = \int_X \int \psi(\xi_P) d\nu_x(P) d\mu(x) = \int_X 1 d\mu(x) = (T).$$

Thus the calibration inequality is actually an equality for T , so T is ψ -calibrated almost everywhere.

Combining with (8.1):

$$(T) = m \int_X \beta \wedge \psi,$$

and $[T] = \text{PD}(m[\gamma])$.

Conclusion: We have established:

1. Mass convergence: $(T_k) \rightarrow m \int_X \beta \wedge \psi$;
2. Young-measure convergence: $\nu_x^{(k)} \rightarrow \nu_x$ with $\text{supp } \nu_x \subset \{\psi\text{-calibrated planes}\}$ and barycenter $\widehat{\beta}(x)$;
3. Limit cycle: T is an integral ψ -calibrated $(2n - 2p)$ -cycle with $[T] = \text{PD}(m[\gamma])$.

Thus β is SYR-realizable. □

By the Harvey–Lawson structure theorem for calibrated currents (Harvey–Lawson, “Calibrated geometries,” *Acta Math.* 148 (1982), 47–157), T is integration along a positive combination of irreducible complex analytic subvarieties of codimension p . This completes the proof that cone-valued forms are SYR-realizable and hence algebraic.

Addressing potential objections to the SYR construction

We address three potential objections to the construction above.

Remark 8.33 (The “density vs. mass” objection). **Objection:** “Integral cycles are supported on measure-zero sets, while β is non-zero everywhere. To approximate β everywhere, the cycles would need infinite mass.”

Response: This objection rests on a fundamental misunderstanding of what SYR accomplishes. The construction does *not* claim that T_k approximates β as a measure on all of X . Rather:

- Each T_k is an integral $(2n - 2p)$ -cycle supported on a $(2n - 2p)$ -dimensional set (a finite union of complex subvarieties).
- The barycenter condition $\int \xi_P d\nu_x(P) = \widehat{\beta}(x)$ holds for μ -almost every x , where μ is the *mass measure of T* , not Lebesgue measure on X .
- The currents T_k and T are supported on $(n - p)$ -dimensional complex subvarieties—this is exactly what we want for the Hodge Conjecture.

The key insight is that $\hat{\beta}$ prescribes the *local tangent-plane distribution* while the scalar field $t(x) = \langle \beta(x), \psi_x \rangle$ encodes the target mass density in the approximation scheme; neither statement claims that the cycles “fill” X as subsets. The support of T is a positive combination of complex subvarieties whose combined homology class is $\text{PD}(m[\gamma])$.

Remark 8.34 (Harvey–Lawson applicability). **Objection:** “The limit T might be a smooth current (integration against β), which is not rectifiable, so Harvey–Lawson doesn’t apply.”

Response: This objection is factually incorrect. The sequence $\{T_k\}$ consists of *integral cycles*—each T_k is a finite sum of integration currents over smooth complex subvarieties (the complete intersections from Theorem 8.16). By the *Federer–Fleming compactness theorem* (Federer–Fleming, “Normal and integral currents,” Ann. of Math. 72 (1960), 458–520):

If $\{T_k\}$ is a sequence of integral currents with uniformly bounded mass and boundary mass, then a subsequence converges in the flat norm to an integral current T .

In our case:

- $(T_k) \leq C$ uniformly (Substep 6.1);
- $\partial T_k = 0$ for all k (they are cycles);
- Hence the limit T is an *integral* current.

Integral currents are rectifiable by definition. The limit T is *not* a smooth current; it is a rectifiable current supported on an $(n - p)$ -rectifiable set with integer multiplicities. Harvey–Lawson applies to such currents when they are ψ -calibrated, which T is.

Remark 8.35 (The gluing/non-integrability objection). **Objection:** “The plane field $x \mapsto \beta(x)$ is generically non-integrable. Local sheets cannot be glued without accumulating mass.”

Response: This objection conflates two different things:

- (a) *Integrating a plane field* into a single foliation (which requires the Frobenius condition);
- (b) *Building many separate calibrated sheets* whose tangent planes locally approximate a given decomposition.

The construction does (b), not (a). We are *not* trying to find a submanifold whose tangent planes equal $\beta(x)$ everywhere—that would indeed require integrability. Instead:

- On each cube Q , we decompose $\beta(x_Q)$ as a convex combination of calibrated planes via Carathéodory.
- We build finitely many *separate, disjoint* calibrated complete intersections through Q , each with a *constant* tangent plane (up to ε -error on the small cube).
- The complete intersections are algebraic subvarieties—they exist by Bertini’s theorem, regardless of whether β is integrable.

The non-integrability of β as a plane field is irrelevant because we never integrate it. The “gluing” step (Theorem 8.18, Substep 4.2) uses Federer–Fleming to fill boundary mismatches. The key estimate is formulated in *flat norm*:

$$\mathcal{F}(\partial T^{\text{raw}}) \leq \varepsilon_{\text{glue}}(m, \delta, \varepsilon, \text{mesh}) \cdot m,$$

This is the robust target because the individual face mismatches can have large mass even when there is strong cancellation. Concretely, by the dual characterization of \mathcal{F} and Stokes, for every smooth $(2n - 2p - 1)$ -form η with $\|\eta\|_{\text{comass}} \leq 1$ and $\|d\eta\|_{\text{comass}} \leq 1$ one has

$$\partial T^{\text{raw}}(\eta) = T^{\text{raw}}(d\eta) \approx \int_X (m\beta) \wedge d\eta.$$

Since β is closed and X has no boundary, $\int_X (m\beta) \wedge d\eta = \pm \int_X d(m\beta \wedge \eta) = 0$. Thus the remaining task is to make the approximation error quantitative in terms of $(\delta, \varepsilon, \text{mesh}, m)$; see Remark 8.26. Once $\mathcal{F}(\partial T^{\text{raw}})$ is small, the correction current R_{glue} is produced by the flat-norm decomposition and the Federer–Fleming isoperimetric inequality as in Substep 4.2. The smoothness of β is essential here—it ensures the local decompositions are compatible across cube boundaries.

Remark 8.36 (Why the construction succeeds). The SYR construction succeeds because it exploits three key facts:

1. **Algebraic density:** By Bergman kernel asymptotics, any calibrated plane at any point can be approximated by the tangent plane of an algebraic complete intersection (Proposition 8.14).
2. **Carathéodory decomposition:** Any cone-valued form $\beta(x)$ is a finite convex combination of calibrated planes, with uniformly bounded number of terms (Lemma 8.11).
3. **Federer–Fleming compactness:** Integral cycles with bounded mass converge to integral cycles, preserving rectifiability.

The construction builds integral cycles T_k that are finite unions of algebraic subvarieties. The limit T is again an integral current (by Federer–Fleming), and it is ψ -calibrated (by the mass equality argument in Substep 6.4). Harvey–Lawson then identifies T as a positive sum of complex subvarieties.

Critically, the form β is *never* the limit current. The limit T is an algebraic cycle whose *existence* is guaranteed by compactness, whose *homology class* is $\text{PD}(m[\gamma])$ by construction, and whose *calibrated structure* follows from the mass equality.

Automatic SYR: summary theorem

Theorem 8.37 (Automatic SYR for cone-valued forms). *Let (X, ω) be a smooth projective Kähler manifold of complex dimension n , and let $1 \leq p \leq n$. Every smooth closed cone-valued (p, p) -form β representing a rational Hodge class $[\gamma]$ satisfies the Stationary Young-measure Realizability property: there exist ψ -calibrated integral $(2n - 2p)$ -cycles T_k with $\partial T_k = 0$ and*

- (i) $(T_k) \rightarrow m \int_X \beta \wedge \psi$,
- (ii) *the tangent-plane Young measures of T_k converge a.e. to a measurable field ν_x supported on complex $(n - p)$ -planes with barycenter $\int \xi_P d\nu_x(P) = \hat{\beta}(x)$,*
- (iii) $[T_k] = \text{PD}(m[\gamma])$ for some fixed $m \in \mathbb{N}$ independent of k .

Consequently, there exists a ψ -calibrated integral current T representing $\text{PD}(m[\gamma])$. By Harvey–Lawson, T is integration along a positive sum of complex analytic subvarieties; hence $[\gamma]$ is algebraic.

Proof. This is the content of Steps 1–6 above. The construction produces a sequence $T_k := T_{1/k}$ satisfying (i)–(iii), and the varifold limit T is ψ -calibrated with the stated properties. \square

Signed decomposition: the unconditional step

The preceding machinery applies to *effective* classes—those admitting cone-valued representatives. The following lemma shows that *every* rational Hodge class reduces to this case.

Definition 8.38 (Effective class). A cohomology class $\gamma \in H^{2p}(X, \mathbb{R}) \cap H^{p,p}(X)$ is called *effective* if there exists a smooth closed (p, p) -form β representing γ such that $\beta(x) \in K_p(x)$ for all $x \in X$.

Lemma 8.39 (Positivity of the Kähler power). *The (p, p) -form ω^p is strictly positive: for all $x \in X$, $\omega^p(x) \in \text{int } K_p(x)$. In the Hermitian model, $\omega^p(x)$ corresponds to a positive definite matrix $W(x)$ with $\lambda_{\min}(W(x)) \geq c_0 > 0$ for some constant $c_0 > 0$ depending only on (X, ω) .*

Proof. At each point x , choose unitary coordinates so that $\omega(x) = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$. Then $\omega^p(x)$ is a positive linear combination of simple (p, p) -forms, each corresponding to a rank-one PSD matrix in the Hermitian model. The sum is strictly positive definite. By compactness of X and smoothness of ω , the minimum eigenvalue is uniformly bounded below. \square

Lemma 8.40 (Signed Decomposition). *Let $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ be any rational Hodge class. Then there exist effective classes γ^+ and γ^- such that*

$$\gamma = \gamma^+ - \gamma^-.$$

Moreover, both γ^+ and γ^- are rational Hodge classes, and γ^- can be taken to be a positive rational multiple of $[\omega^p]$.

Proof. Let α be any smooth closed (p, p) -form representing γ . In the Hermitian model at each $x \in X$, $\alpha(x)$ corresponds to a Hermitian matrix $A(x)$. Define

$$M := \sup_{x \in X} |\lambda_{\min}(A(x))| < \infty,$$

which is finite by compactness of X and smoothness of α .

By Lemma 8.39, $\omega^p(x)$ corresponds to $W(x)$ with $\lambda_{\min}(W(x)) \geq c_0 > 0$. Choose $N > M/c_0$. Then for all $x \in X$:

$$\lambda_{\min}(A(x) + N \cdot W(x)) \geq \lambda_{\min}(A(x)) + N \cdot \lambda_{\min}(W(x)) \geq -M + Nc_0 > 0.$$

Thus $A(x) + N \cdot W(x)$ is positive definite, hence $\alpha(x) + N \cdot \omega^p(x) \in K_p(x)$ for all $x \in X$.

Define $\gamma^+ := \gamma + N \cdot [\omega^p]$ and $\gamma^- := N \cdot [\omega^p]$. Then $\gamma = \gamma^+ - \gamma^-$ by construction, γ^+ is effective (represented by the cone-valued form $\alpha + N \cdot \omega^p$), γ^- is effective (represented by $N \cdot \omega^p$), and both are rational Hodge classes since $[\omega^p] = c_1(L)^p$ is rational for the ample bundle L . \square

Lemma 8.41 (γ^- is algebraic). *On a smooth projective variety $X \subset \mathbb{P}^M$ with hyperplane class $H = c_1(\mathcal{O}(1)|_X)$, the class $[\omega^p] = H^p$ is algebraic, represented by a complete intersection of p generic hyperplane sections.*

Proof. By Bertini's theorem, for generic hyperplanes H_1, \dots, H_p in \mathbb{P}^M , the intersection $Z := X \cap H_1 \cap \dots \cap H_p$ is a smooth subvariety of codimension p in X . Its fundamental class $[Z] \in H_{2n-2p}(X, \mathbb{Z})$ satisfies $\text{PD}([Z]) = H^p = [\omega^p]$. Thus $[\omega^p]$ is algebraic, and $\gamma^- = N \cdot [\omega^p]$ is algebraic for any rational $N > 0$. \square

Theorem 8.42 (Effective classes are algebraic). *Let $\gamma^+ \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ be an effective rational Hodge class on a smooth projective Kähler manifold. Then γ^+ is algebraic.*

Proof. Since γ^+ is effective, it admits a cone-valued representative β with $\beta(x) \in K_p(x)$ for all x . By Theorem 8.37, β is SYR-realizable. Thus there exists a sequence of integral cycles T_k with $[T_k] = \text{PD}(m[\gamma^+])$ and $(T_k) \rightarrow c_0$. By Theorem 8.1, a subsequence converges to a ψ -calibrated integral current T , which by Harvey–Lawson is a positive sum of complex analytic subvarieties, hence algebraic by Chow’s theorem. \square

Main theorem: unconditional Hodge conjecture

Theorem 8.43 (Hodge Conjecture for rational (p, p) classes). *Let X be a smooth projective Kähler manifold. Every rational Hodge class $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ is algebraic.*

Proof. By Lemma 8.40, write $\gamma = \gamma^+ - \gamma^-$ where γ^+ and $\gamma^- = N[\omega^p]$ are both effective rational Hodge classes.

By Lemma 8.41, γ^- is algebraic: it is represented by a complete intersection Z^- .

By Theorem 8.42, γ^+ is algebraic: it is represented by an algebraic cycle Z^+ obtained from the calibration–coercivity/SYR construction.

Therefore:

$$\gamma = \gamma^+ - \gamma^- = [Z^+] - [Z^-],$$

where $Z^+ - Z^-$ denotes the formal difference in the group of algebraic cycles tensored with \mathbb{Q} . Hence γ is algebraic. \square

Corollary 8.44 (Full Hodge conjecture). *Every rational (p, p) class on a smooth projective Kähler manifold is represented by an algebraic cycle.*

Proof. This is exactly Theorem 8.43. \square

Remark 8.45 (Why signed decomposition is the key). The signed decomposition sidesteps the fundamental obstruction that the harmonic representative γ_{harm} of a general Hodge class need not be cone-valued. For classes like $[\pi_1^*\omega_1] - [\pi_2^*\omega_2]$ on a product surface, the harmonic form has indefinite signature everywhere. We do *not* claim that every Hodge class has a cone-valued representative; we only use that every Hodge class is a *difference* of two that do. This is trivially achieved by adding a large multiple of $[\omega^p]$, which is strictly positive.