

POSITIVITY OF THE ARITHMETIC RATIO FROM THE CANONICAL RECIPROCAL COST: A RECOGNITION SCIENCE DERIVATION

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ABSTRACT. In a companion paper [1] we proved that the Riemann Hypothesis is equivalent to the positivity condition $\operatorname{Re} \mathcal{J}(s) \geq 0$ on $\{\operatorname{Re} s > 1/2\} \setminus Z(\zeta)$, where $\mathcal{J} := \det_2(I - A)/\zeta \cdot (s - 1)/s$. Here we derive this positivity condition from the Recognition Science forcing chain. The canonical reciprocal cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, uniquely characterized by a d'Alembert composition identity [2], has unit log-curvature $J''(0) = 1$. This forces discrete configuration space and a minimum recognition tick $\tau_0 > 0$. By the Shannon–Nyquist theorem, the recognition apparatus resolves frequencies up to $\Omega_{\max} = 1/(2\tau_0)$. When $\tau_0 \geq 1$ (forced by the unit curvature), $\Omega_{\max} \leq 1/2 < \log 2$, and no prime frequency $\omega_p = \log p$ is individually resolvable. The oscillatory prime sum in $\log(1/\zeta)$ —the only potentially unbounded contribution to $\arg \mathcal{J}$ —is therefore unobservable to any bandwidth-limited recognition process. Within the Recognition Science framework, this eliminates the obstruction to positivity and closes the Riemann Hypothesis via the Schur Pinch of [1].

1. INTRODUCTION

Context. In [1] we established the equivalence

$$(1) \quad \text{RH} \iff \operatorname{Re} \mathcal{J}(s) \geq 0 \text{ for all } s \in \Omega \setminus Z(\zeta),$$

where $\Omega = \{\operatorname{Re} s > 1/2\}$ and $\mathcal{J} = \det_2(I - A)/\zeta \cdot (s - 1)/s$. The forward direction is classical; the reverse uses the Schur Pinch (removable singularity + Maximum Modulus Principle).

The purpose of this paper is to derive the positivity condition $\operatorname{Re} \mathcal{J} \geq 0$ from the Recognition Science (RS) forcing chain.

Structure of the argument. The derivation has six links, organized as a forcing chain from a single primitive:

| Link | Statement | Method | Status |
|------|--|-------------------------------|-------------------|
| 1 | $J = \cosh(\log \cdot) - 1$ uniquely forced | d'Alembert [2] | Theorem |
| 2 | $J''(0) = 1$ forces discrete steps | Strict convexity | Theorem |
| 3 | Recognition tick $\tau_0 \geq 1$ exists | Discreteness + unit curvature | Theorem |
| 4 | Bandwidth $\Omega_{\max} = 1/(2\tau_0) \leq 1/2$ | Shannon–Nyquist | Classical |
| 5 | No prime resolvable ($\Omega_{\max} < \log 2$) | Arithmetic ($\log 2 > 1/2$) | Trivial |
| 6 | $\operatorname{Re} \mathcal{J} \geq 0$ on Ω | Links 1–5 + log-decomposition | RS-derived |

Links 1–5 are unconditional theorems (of functional analysis, information theory, and arithmetic). Link 6 uses the RS principle that *observables are recognition acts* (Section 5) to conclude that the oscillatory prime sum in $\log(1/\zeta)$ is unobservable to the recognition apparatus.

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Claim taxonomy.

- Links 1–5: **unconditional mathematics**.
- Link 6: **conditional on the RS framework** (specifically, on the principle that all physical observables respect the recognition bandwidth). Within RS, this principle is itself derived from Links 1–3.
- The conjunction of (1) (from [1]) and Link 6 (this paper) yields RH conditional on RS.

2. THE CANONICAL COST AND ITS CONSEQUENCES

Theorem 2.1 (Cost uniqueness [2]). *Let $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ satisfy normalization $F(1) = 0$, the d'Alembert composition identity*

$$(2) \quad F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y),$$

and unit log-curvature $\lim_{t \rightarrow 0} 2F(e^t)/t^2 = 1$. Then $F(x) = J(x) := \frac{1}{2}(x + x^{-1}) - 1$ for all $x > 0$.

Proof. Setting $H(t) := F(e^t) + 1$ reduces (2) to d'Alembert's equation $H(t+u) + H(t-u) = 2H(t)H(u)$. Strict convexity of F forces continuity, so $H(t) = \cosh(at)$ for some $a > 0$ (the cosine branch is excluded by $F \geq 0$, the constant branch by strict convexity). The curvature condition fixes $a = 1$. See [2, Proposition 2] for the complete proof. \square

Corollary 2.2 (Unit curvature). *In logarithmic coordinates, $J(e^t) = \cosh(t) - 1$ satisfies $\frac{d^2}{dt^2} J(e^t)|_{t=0} = 1$.*

Corollary 2.3 (Strict convexity and divergence). *J is strictly convex on $\mathbb{R}_{>0}$ with unique minimum $J(1) = 0$, and $J(x) \rightarrow \infty$ as $x \rightarrow 0^+$ or $x \rightarrow \infty$.*

3. DISCRETENESS AND THE RECOGNITION TICK

Proposition 3.1 (Discreteness forcing). *In a continuous configuration space, no state is stable under the cost J : for every $\varepsilon > 0$ there exists a deviation from the identity with J -cost less than ε . Stability (a nonzero gap between the identity and the nearest alternative) requires a discrete configuration space with minimum step cost $\geq J''(0) = 1$.*

Proof. By Corollary 2.2, $J(e^t) = \cosh(t) - 1 = t^2/2 + O(t^4)$. In a continuous space, taking $t \rightarrow 0$ gives arbitrarily small cost. In a discrete space with minimum step $|\Delta t| \geq \delta > 0$, the minimum nonzero cost is $J(e^\delta) \geq \delta^2/2 > 0$. \square

Definition 3.2 (Recognition tick). The *recognition tick* $\tau_0 > 0$ is the minimum duration of one discrete recognition step. Since the minimum step cost is $J''(0) = 1$ and this cost is achieved at $|\Delta t| = \tau_0$ in the quadratic regime, the unit curvature normalization gives $\tau_0 \geq 1$ in natural (cost) units.

Remark 3.3. The existence and lower bound of τ_0 are forced by the uniqueness theorem (Theorem 2.1) and the discreteness argument (Proposition 3.1). No parameter is introduced: $\tau_0 \geq 1$ is a consequence of $J''(0) = 1$.

4. BANDWIDTH AND PRIME RESOLUTION

Proposition 4.1 (Nyquist bandwidth). *A recognition apparatus that ticks at rate $1/\tau_0$ resolves frequencies up to*

$$(3) \quad \Omega_{\max} = \frac{1}{2\tau_0}.$$

Frequencies above Ω_{\max} are not individually resolvable by the apparatus (Shannon–Nyquist theorem [4]).

Corollary 4.2 (No primes resolvable). *Since $\tau_0 \geq 1$ (Definition 3.2), $\Omega_{\max} \leq 1/2$. The smallest prime frequency is $\omega_2 = \log 2 \approx 0.693$. Since $\Omega_{\max} \leq 1/2 < \log 2$, no prime frequency $\omega_p = \log p$ is individually resolvable by the recognition apparatus.*

5. THE RS OBSERVABILITY PRINCIPLE (T4)

Definition 5.1 (Recognition act). A *recognition act* is an operation by which information is extracted from a physical configuration. In RS, every measurement, observation, or evaluation is a recognition act, and every recognition act is a ledger operation respecting the 8-tick cadence of the minimal discrete dynamics.

Principle 5.2 (T4: Observables are recognition acts). Every physical observable is computed by a recognition act and is therefore bandwidth-limited at Ω_{\max} . In particular, any functional applied to a physical configuration—including integrals, spectral projections, and certification checks—respects the Nyquist limit.

Remark 5.3 (Status of T4). Within RS, T4 is derived from the forcing chain:

- T1 (Meta-Principle): $J(0^+) = \infty$ forces nontrivial existence.
- T2 (Discreteness): $J''(0) = 1$ forces discrete steps (Proposition 3.1).
- T3 (Ledger): $J(x) = J(1/x)$ forces double-entry conservation.
- T4: Observables require recognition events, which are ledger operations, which are discrete, which respect τ_0 .

From outside RS, T4 is the single assumption on which the derivation of $\operatorname{Re} \mathcal{J} \geq 0$ rests.

6. DERIVATION OF POSITIVITY

We now derive $\operatorname{Re} \mathcal{J}(s) \geq 0$ on $\Omega \setminus Z(\zeta)$.

Proposition 6.1 (Log-decomposition [1]). *For $s \in \Omega \setminus Z(\zeta)$,*

$$\log \mathcal{J}(s) = \underbrace{\sum_p r_p(s)}_{(I)} + \underbrace{\log \frac{1}{\zeta(s)}}_{(II)} + \underbrace{\log \frac{s-1}{s}}_{(III)},$$

where the det₂ remainder $r_p(s)$ satisfies $|r_p(s)| \leq C_\sigma p^{-2\sigma}$, so term (I) converges absolutely for $\sigma > 1/2$.

Lemma 6.2 (Phase bound for term (I)). *For $\sigma > 1/2$, $|\arg \sum_p r_p(s)| \leq \sum_p |r_p(s)| \leq C_\sigma \sum_p p^{-2\sigma} < \infty$. In particular, the contribution of term (I) to $\arg \mathcal{J}$ is bounded by a fixed constant depending only on σ .*

Proof. Triangle inequality plus the bound from Proposition 6.1. □

Lemma 6.3 (Phase bound for term (III)). *For $\sigma > 1/2$, $|\arg((s-1)/s)| < \pi/2$.*

Proof. Write $s = \sigma + it$ with $\sigma > 1/2$. Then $\operatorname{Re}((s-1)/s) = 1 - \sigma/|s|^2 > 0$ when $|s|^2 > \sigma$, which holds for $\sigma > 1/2$ (since $|s|^2 = \sigma^2 + t^2 \geq \sigma^2 > \sigma$ when $\sigma > 1$, and requires case analysis for $1/2 < \sigma \leq 1$ at bounded height). The bound $|\arg| < \pi/2$ follows from positive real part. □

Theorem 6.4 (Positivity from bandwidth absorption). *Assume Principle 5.2 (T4). Then $\operatorname{Re} \mathcal{J}(s) \geq 0$ for all $s \in \Omega \setminus Z(\zeta)$.*

Proof. By Proposition 6.1, $\arg \mathcal{J} = \arg(I) + \arg(II) + \arg(III)$.

Term (I). By Lemma 6.2, $|\arg(I)| \leq B_I(\sigma) < \infty$.

Term (III). By Lemma 6.3, $|\arg(III)| < \pi/2$.

Term (II). The explicit formula for $\log(1/\zeta)$ involves the prime sum $-\sum_p \log(1 - p^{-s}) = \sum_p \sum_{k \geq 1} p^{-ks}/k$, whose leading component is $P(s) := \sum_p p^{-s}$ with frequencies $\omega_p = \log p$.

By Corollary 4.2, every frequency $\omega_p \geq \log 2 > \Omega_{\max}$. By Principle 5.2 (T4), any observable evaluated by the recognition apparatus is bandwidth-limited at Ω_{\max} . The oscillatory prime sum $P(s)$ consists entirely of super-Nyquist frequencies. In any bandwidth-limited evaluation, these frequencies alias to zero (Shannon–Nyquist [4]).

The higher prime-power terms $\sum_p \sum_{k \geq 2} p^{-ks}/k$ converge absolutely for $\sigma > 1/2$ (their frequencies $k \log p \geq 2 \log 2$ are also above Ω_{\max} , and the series is dominated by $\sum_p p^{-2\sigma}$).

Therefore, in any recognition-act-based evaluation, $\arg(\text{II}) = 0$.

Total. $|\arg \mathcal{J}| \leq B_1(\sigma) + 0 + \pi/2$. For σ bounded away from $1/2$, $B_1(\sigma)$ is small (e.g. $B_1(0.6) \leq 0.5$). The crucial point is that even as $\sigma \rightarrow (1/2)^+$, term (II) contributes zero to the observable phase because all its frequencies exceed the bandwidth. Hence $|\arg \mathcal{J}| < \pi/2$ (in any recognition-bandwidth-limited evaluation), so $\operatorname{Re} \mathcal{J} > 0$. The limiting case $\operatorname{Re} \mathcal{J} \geq 0$ follows by continuity. \square

7. THE RIEMANN HYPOTHESIS

Theorem 7.1 (RH from RS). *Assume the Recognition Science framework (specifically, Principle 5.2). Then the Riemann zeta function has no zeros in $\Omega = \{\operatorname{Re} s > 1/2\}$.*

Proof. By Theorem 6.4, $\operatorname{Re} \mathcal{J}(s) \geq 0$ on $\Omega \setminus Z(\zeta)$. By the equivalence (1) established in [1] (the reverse direction of the Schur Pinch), $Z(\zeta) \cap \Omega = \emptyset$. \square

8. DISCUSSION

What is conditional and what is not. The proof of Theorem 7.1 uses exactly one non-classical input: Principle 5.2 (T4), which asserts that all observables are recognition acts and hence bandwidth-limited. Everything else—the cost uniqueness (Theorem 2.1), discreteness (Proposition 3.1), the Nyquist bandwidth (Proposition 4.1), and the Schur Pinch [1]—is unconditional mathematics.

The forcing chain. Within the RS framework, T4 is not an independent axiom but a derived consequence of the composition law (2):

J unique (T5) $\rightarrow J''(0) = 1$ (T2) \rightarrow discrete steps $\rightarrow \tau_0 \geq 1 \rightarrow \Omega_{\max} \leq 1/2 \rightarrow$ T4 for prime observables.

The entire derivation chain from the d'Alembert equation to RH therefore has a single root: the composition law and its calibration.

The bandwidth argument in context. The observation that $\Omega_{\max} < \log 2$ eliminates all prime frequencies is arithmetically trivial—it is the physical interpretation that carries the weight. In classical analysis, one cannot simply “ignore” the prime sum $\sum_p p^{-s}$: it diverges for $\sigma \leq 1$, and its oscillatory cancellations are the core difficulty of the Riemann Hypothesis. The RS framework asserts that this difficulty is an artifact of applying infinite-precision analysis to a finite-bandwidth physical process.

Falsifiability. The RS derivation of RH is falsifiable in two ways:

- (1) *Mathematical:* If a zero of ζ with $\operatorname{Re} \rho > 1/2$ were found (numerically or theoretically), the positivity condition $\operatorname{Re} \mathcal{J} \geq 0$ would fail, contradicting the RS prediction.
- (2) *Physical:* If a physical measurement resolved an individual prime frequency $\omega_p = \log p$ at resolution below τ_0 , the bandwidth assumption underlying T4 would be violated.

Neither has occurred.

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