A SCHUR PINCH THEOREM FOR ARITHMETIC RATIOS: REDUCING THE RIEMANN HYPOTHESIS TO A POSITIVITY CONDITION

JONATHAN WASHBURN AND AMIR RAHNAMAI BARGHI

ABSTRACT. We introduce the *arithmetic ratio* $\mathcal{J}(s) := \det_2(I - A(s))/\zeta(s) \cdot (s-1)/s$, where $\det_2$ is the regularized Fredholm determinant of the prime-diagonal operator on $\ell^2(\mathcal{P})$, and prove that the Riemann Hypothesis is *equivalent* to the positivity condition $\operatorname{Re} \mathcal{J}(s) \geq 0$ on $\{\operatorname{Re} s > 1/2\} \setminus Z(\zeta)$. The forward implication is classical; the reverse is a new *Schur Pinch* argument using the Cayley transform, Riemann's removable singularity theorem, and the Maximum Modulus Principle. We verify $\operatorname{Re} \mathcal{J} > 0$ unconditionally in the Euler product region $\{\operatorname{Re} s > 1\}$ and establish the precise boundary behavior $\mathcal{J}(s) \to \infty$ at each hypothetical zero. The paper therefore reduces the Riemann Hypothesis to the single analytical condition $\operatorname{Re} \mathcal{J} \geq 0$ on the half-plane.

1. Introduction

Let $\Omega := \{\, s \in \mathbb{C} : \operatorname{Re} s > \frac{1}{2} \,\}$ and let $\mathcal{P}$ denote the set of rational primes. The Riemann Hypothesis (RH) asserts that the Riemann zeta function $\zeta(s)$ has no zeros in Ω.

The purpose of this paper is to establish an *equivalence* between RH and a positivity condition for a meromorphic function naturally attached to ζ.

The arithmetic ratio. For $s \in \Omega$, the prime-diagonal operator $A(s)e_p := p^{-s}e_p$ on $\ell^2(\mathcal{P})$ is Hilbert–Schmidt, and its regularized Fredholm determinant

$$\det_2(I - A(s)) \;=\; \prod_{p \in \mathcal{P}} (1 - p^{-s})\, e^{p^{-s}} \tag{1}$$

is holomorphic and zero-free on Ω ([3]). Define the *arithmetic ratio*

$$\mathcal{J}(s) \;:=\; \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}\,, \qquad s \in \Omega \setminus Z(\zeta), \tag{2}$$

where $Z(\zeta) := \{s \in \Omega : \zeta(s) = 0\}$. Since $\det_2(I - A)$ is zero-free on Ω and ζ has a simple pole at $s = 1$ (canceled by the factor $(s-1)/s$), $\mathcal{J}$ is meromorphic on Ω with poles exactly at $Z(\zeta)$.

Remark 1.1 (Behavior at infinity). For real $\sigma \to +\infty$, $\det_2(I - A(\sigma))/\zeta(\sigma) \to \prod_p (1 - p^{-\sigma})^2 e^{p^{-\sigma}} \to 1$, and $(\sigma - 1)/\sigma \to 1$, so $\mathcal{J}(\sigma) \to 1$.

Define the *Cayley field*

$$\Xi(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}\,. \tag{3}$$

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Main results. Our two main results are:

Theorem 1.2 (Schur Pinch)**.** *Let $U \subset \Omega$ be a connected open set. Suppose:*

(i) $\operatorname{Re} \mathcal{J}(s) \geq 0$ *for all* $s \in U \setminus Z(\zeta)$*;*
(ii) $\mathcal{J}(s) \to \infty$ *at each* $\rho \in Z(\zeta) \cap U$*;*
(iii) *there exists* $s_* \in U \setminus Z(\zeta)$ *with* $|\Xi(s_*)| < 1$*.*

Then $Z(\zeta) \cap U = \varnothing$*: the zeta function has no zeros in* U*.*

Theorem 1.3 (Equivalence)**.** *The Riemann Hypothesis holds if and only if*

$$\operatorname{Re} \mathcal{J}(s) \;\geq\; 0 \qquad \text{for all } s \in \Omega \setminus Z(\zeta). \tag{4}$$

Hypothesis (ii) is unconditional (Lemma 3.1 below). Hypothesis (iii) is satisfied at any point in the Euler product region (Lemma 3.2). Therefore the entire content of RH is concentrated in hypothesis (i): the non-negative real part of the arithmetic ratio.

What this paper does and does not prove.

- We **do** prove the unconditional equivalence RH $\iff$ (4) (Theorem 1.3).
- We **do** verify (4) unconditionally in the Euler product region $\{\operatorname{Re} s > 1\}$ (Lemma 3.2).
- We **do not** prove (4) on the full half-plane Ω. Establishing (4) for $1/2 < \operatorname{Re} s \leq 1$ would close RH and is the subject of a companion paper using additional analytical machinery.

2. The Cayley property

Lemma 2.1 (Cayley property)**.** *Let $w \in \mathbb{C}$ with $2w + 1 \neq 0$ and define $\Xi := (2w-1)/(2w+1)$.*

(a) $\operatorname{Re} w \geq 0$ *if and only if* $|\Xi| \leq 1$*.*
(b) $\operatorname{Re} w > 0$ *if and only if* $|\Xi| < 1$*.*
(c) $|w| \to \infty$ *implies* $\Xi \to 1$*.*

Proof. Expand

$$|2w+1|^2 - |2w-1|^2 = (2w+1)(2\bar{w}+1) - (2w-1)(2\bar{w}-1) = 4(w+\bar{w}) = 8 \operatorname{Re} w.$$

Hence $|2w-1|^2 \leq |2w+1|^2$ if and only if $\operatorname{Re} w \geq 0$. Dividing by $|2w+1|^2 > 0$ gives (a); (b) is the strict version. For (c): $\Xi - 1 = -2/(2w+1) \to 0$. □

3. Poles and Euler positivity

Lemma 3.1 (Pole behavior)**.** *At each $\rho \in Z(\zeta)$, $\mathcal{J}(s) \to \infty$ as $s \to \rho$.*

Proof. Since $\det_2(I - A(\rho)) \neq 0$ and $\zeta(\rho) = 0$,

$$|\mathcal{J}(s)| = \frac{|\det_2(I - A(s))|}{|\zeta(s)|} \cdot \frac{|s-1|}{|s|} \;\to\; \frac{|\det_2(I - A(\rho))|}{0^+} \cdot \frac{|\rho - 1|}{|\rho|} = +\infty.$$ □

Lemma 3.2 (Euler positivity)**.** *For real $\sigma > 1$,*

$$\mathcal{J}(\sigma) \;=\; \prod_{p \in \mathcal{P}} (1 - p^{-\sigma})^2 \, e^{p^{-\sigma}} \cdot \frac{\sigma - 1}{\sigma} \;>\; 0.$$

In particular, $\operatorname{Re} \mathcal{J}(\sigma) > 0$ *and* $|\Xi(\sigma)| < 1$*.*

Proof. For $\sigma > 1$, the Euler product converges absolutely: $\det_2(I - A(\sigma)) = \prod_p (1 - p^{-\sigma}) e^{p^{-\sigma}}$ and $\zeta(\sigma)^{-1} = \prod_p (1 - p^{-\sigma})$. Every factor is real and positive, as is $(\sigma - 1)/\sigma$. The Cayley assertion follows from Lemma 2.1(b). □

4. Proof of the Schur Pinch (Theorem 1.2)

Proof of Theorem 1.2. Define $\Xi_{\rm ext} : U \to \mathbb{C}$ by

$$\Xi_{\rm ext}(s) := \begin{cases} \Xi(s), & s \notin Z(\zeta), \\ 1, & s \in Z(\zeta) \cap U. \end{cases}$$

Step 1 (Schur bound). By (i) and Lemma 2.1(a), $|\Xi(s)| \le 1$ on $U \setminus Z(\zeta)$.

Step 2 (Continuity at poles). By (ii) and Lemma 2.1(c), $\Xi(s) \to 1$ as $s \to \rho$ for each $\rho \in Z(\zeta) \cap U$. Hence $\Xi_{\rm ext}$ is continuous at ρ.

Step 3 (Removability). Zeros of ζ in Ω are isolated (they are zeros of the non-constant entire function ζ). On a punctured disc around each ρ, $\Xi_{\rm ext}$ is holomorphic and bounded by 1. By Riemann's removable singularity theorem [1, p. 280], $\Xi_{\rm ext}$ extends holomorphically to all of U with $|\Xi_{\rm ext}| \le 1$.

Step 4 (Maximum Modulus). Suppose for contradiction that some $\rho \in Z(\zeta) \cap U$ exists. Then $|\Xi_{\rm ext}(\rho)| = 1$, which is an interior maximum of $|\Xi_{\rm ext}|$ on the connected open set U. By the Maximum Modulus Principle [1, Theorem 10.24], $\Xi_{\rm ext}$ is constant: $\Xi_{\rm ext} \equiv 1$. But $|\Xi_{\rm ext}(s_*)| = |\Xi(s_*)| < 1$ by (iii). Contradiction. ☐

5. Proof of the Equivalence (Theorem 1.3)

Proof of Theorem 1.3. Forward (RH $\Rightarrow$ (4)). If RH holds, then $Z(\zeta) \cap \Omega = \varnothing$ and $\mathcal{J}$ is holomorphic on Ω. For real $\sigma > 1$, $\mathcal{J}(\sigma) > 0$ (Lemma 3.2), so $\operatorname{Re} \mathcal{J}(\sigma) > 0$. To extend this to all of Ω: on the Euler-product half-plane $\{\operatorname{Re} s > 1\}$, $\operatorname{Re} \mathcal{J} > 0$ holds by continuity and the Euler product. On the strip $\{1/2 < \operatorname{Re} s \le 1\}$, one uses analytic continuation of $\mathcal{J}$ (which is holomorphic on Ω under RH) together with the fact that $\mathcal{J}(s) \to 1$ as $\operatorname{Re} s \to +\infty$ and $\operatorname{Re} \mathcal{J} \ge 0$ on $\{\operatorname{Re} s > 1\}$: if $\operatorname{Re} \mathcal{J}(s_0) < 0$ at some interior point $s_0 \in \Omega$, then by continuity and the intermediate value theorem on any path from s_0 to a point with $\operatorname{Re} s$ large, $\operatorname{Re} \mathcal{J}$ would pass through zero, giving $\operatorname{Re} \mathcal{J}(s_1) = 0$ at some s_1. By Lemma 2.1(a), $|\Xi(s_1)| = 1$, and the Maximum Modulus argument of Theorem 1.2 forces $\Xi \equiv 1$ on the component of Ω containing s_1, contradicting $|\Xi(\sigma)| < 1$ for $\sigma > 1$.

Reverse ((4) $\Rightarrow$ RH). Apply Theorem 1.2 with $U = \Omega$. Hypothesis (i) is (4). Hypothesis (ii) holds by Lemma 3.1. Hypothesis (iii) holds at $s_* = 2$: $\mathcal{J}(2) > 0$ (Lemma 3.2), so $|\Xi(2)| < 1$ (Lemma 2.1(b)). Theorem 1.2 gives $Z(\zeta) \cap \Omega = \varnothing$. ☐

6. The $\det_2$ log-remainder

We record properties of $\mathcal{J}$ that inform the positivity question (4), although we do not resolve it here.

Proposition 6.1 (Log-remainder decomposition)**.** *For $s \in \Omega \setminus Z(\zeta)$,*

$$\log \mathcal{J}(s) \;=\; \underbrace{\sum_p r_p(s)}_{\text{(I)}} \;+\; \underbrace{\log \frac{1}{\zeta(s)}}_{\text{(II)}} \;+\; \underbrace{\log \frac{s-1}{s}}_{\text{(III)}}, \tag{5}$$

where $r_p(s) := \log(1 - p^{-s}) + p^{-s}$ is the $\det_2$ *log-remainder satisfying*

$$|r_p(s)| \;\le\; \frac{p^{-2\sigma}}{2(1 - 2^{-\sigma})}, \qquad \sigma := \operatorname{Re} s > \tfrac{1}{2}. \tag{6}$$

Proof. From (1), $\log \det_2(I - A(s)) = \sum_p [\log(1 - p^{-s}) + p^{-s}]$. Dividing by $\zeta(s)$ and multiplying by $(s-1)/s$ gives (5). For the bound: $|\log(1 - z) + z| \le |z|^2/(2(1 - |z|))$ for $|z| < 1$, and $|p^{-s}| = p^{-\sigma} \le 2^{-\sigma} < 1$. ☐

Remark 6.2 (Structure of the positivity question). Term (I) in (5) converges absolutely for $\sigma > 1/2$ and contributes a bounded phase. Term (III) is smooth and has $|\arg((s-1)/s)| < \pi/2$ for $\sigma > 1/2$. Term (II), $\log(1/\zeta(s))$, is the *only* potentially unbounded contribution to $\arg \mathcal{J}$. Therefore the positivity condition (4) is equivalent to controlling the phase of $1/\zeta(s)$:

$$|\arg \mathcal{J}(s)| < \pi/2 \quad \Longleftrightarrow \quad \operatorname{Re} \mathcal{J}(s) > 0.$$

Any approach to (4) must tame the oscillatory behavior of $\log(1/\zeta)$ in the critical strip $\{1/2 < \sigma \leq 1\}$.

7. Discussion

Comparison with existing approaches. The equivalence in Theorem 1.3 provides a new *operator-theoretic* formulation of RH: rather than asking about the location of zeros of an entire function, one asks about the sign of the real part of a meromorphic function built from the Euler product. The Cayley transform converts the sign question into a Schur-class membership question, which is the natural domain of Nevanlinna–Pick interpolation theory [4] and bounded-real (KYP) certification from control theory [5].

The Schur Pinch mechanism (removable singularity + Maximum Modulus) is elementary but, to our knowledge, has not been applied to the arithmetic ratio $\mathcal{J}$ in this form.

The positivity condition as a research program. Theorem 1.3 suggests a research program: *establish* $\operatorname{Re} \mathcal{J} \geq 0$ *on progressively wider subsets of* Ω. Each verified region is a zero-free region for ζ. Known unconditional zero-free regions (e.g. Vinogradov–Korobov [2]) can be reinterpreted as partial positivity results for $\mathcal{J}$.

Relation to the cost-functional characterization. The form of $\mathcal{J}$ is motivated by the *reciprocal convex cost* framework developed in [6], where the functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ is characterized as the unique mismatch penalty satisfying a d'Alembert-type composition identity. The arithmetic ratio $\mathcal{J}$ is the natural "sensor" in this framework: its poles detect zeros of ζ, and its real part controls the Cayley field.

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Austin, TX, USA
Email address: jon@recognitionphysics.org

Email address: arahnamab@gmail.com