ATTEMPTED UNCONDITIONAL CLOSURES FOR ITEMS (A)–(E) IN THE GEOMETRIC DEPLETION PROGRAM

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Abstract. This document collects the five outstanding closure items (A)–(E) that remain in the current Navier–Stokes geometric depletion manuscript (`new-version-12-11.tex`) and attempts to supply complete proofs. Where a full proof cannot be completed from classical Navier–Stokes theory (as of current knowledge), we state the sharpest provable conditional versions and explain the obstruction.

Contents

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1. Context and scope

In `new-version-12-11.tex`, the main theorem has been rewritten in a *conditional* form: global regularity is reduced to a list of scale-critical properties of an ancient tangent flow. The five key closure items are:

(A) VMO regularity of the vorticity direction $\xi^\infty = \omega^\infty/|\omega^\infty|$ for the ancient tangent flow;
(B) scale-critical local $L^{3/2}$ control of ω^∞ on shrinking parabolic cylinders;
(C) an ε-regularity theorem and Liouville rigidity for the sphere-valued drift–diffusion equation satisfied by ξ;
(D) smallness of the tangential forcing H in the critical Carleson norm at sufficiently small scales;
(E) a 2D Liouville/classification step for the reduced ancient flow after ξ^∞ is constant.

This document is written to be mechanically checkable: every step is either proved here or explicitly marked as an external classical input (with a citation placeholder), or marked as an obstruction.

Working premise and methodology. Following the "Coercive Projection Method" (CPM) viewpoint (see `CPM.tex` in this repo), we treat the blow-up analysis as a *critical-element* extraction, and we attempt to rule out any nontrivial ancient critical element by: (i) identifying a *structured set* (2D/constant-direction flows), (ii) defining a *defect* measuring distance to structure (e.g. direction oscillation/energy), and (iii) proving *coercive depletion* estimates that force the defect to vanish. This CPM framing is used only as a *proof-organizer*; every mathematical implication below is proved classically or explicitly marked as open.

2. Item (A): VMO of the tangent-flow direction field

2.1. Statement in the program.

Proposition 2.1 (Target statement (A)). *Let (u^∞, p^∞) be an ancient tangent flow obtained by CKN blow-up at a singular point, and let $\omega^\infty = \operatorname{curl} u^\infty$. Define $\xi^\infty = \omega^\infty/|\omega^\infty|$ on $\{\omega^\infty \neq 0\}$. Then ξ^∞ belongs to* VMO *in the spatial variable, locally uniformly in time.*

2.2. **What can be proved unconditionally from suitable weak compactness.** The blow-up/compactness framework for suitable weak solutions yields (on each compact cylinder)

$$u^\infty \in L^\infty_t L^2_x \cap L^2_t \dot H^1_x, \qquad p^\infty \in L^{3/2}_{t,x},$$

and hence $\nabla u^\infty \in L^2_{t,x}$ locally, so $\omega^\infty \in L^2_{t,x}$ locally. However, *this is subcritical* and does not provide scale-invariant control of ω^∞, nor any pointwise or oscillation control of ξ^∞.

Remark 2.2 (Obstruction (why (A) does not follow from the available bounds))**.** *From $\omega^\infty \in L^2_{\mathrm{loc}}$ alone, the normalized direction field $\xi^\infty = \omega^\infty/|\omega^\infty|$ need not have any quantitative oscillation control, especially near the zero set $\{\omega^\infty = 0\}$ where the normalization is singular. In particular, no implication of the form*

$$\omega^\infty \in L^2_{\mathrm{loc}} \quad \Longrightarrow \quad \xi^\infty \in \mathrm{VMO}_{\mathrm{loc}}$$

is available in classical theory.

2.3. **Provable conditional replacement.** The natural route to VMO is via *scale-invariant* control of ω^∞ in a Sobolev space and a non-degeneracy lower bound on $|\omega^\infty|$ on the region of interest. One representative statement is:

Proposition 2.3 (Conditional VMO from scale-invariant Sobolev control)**.** *Fix a cylinder $Q \subset \mathbb{R}^3 \times \mathbb{R}$. Assume*

$$\omega \in L^\infty_t W^{1,3}_x(Q) \qquad \text{and} \qquad \inf_Q |\omega| \geq c_0 > 0.$$

Then $\xi = \omega/|\omega|$ belongs to VMO *in x uniformly in t on Q.*

Proof. Under the non-degeneracy bound $|\omega| \geq c_0$, the map $v \mapsto v/|v|$ is smooth on $\{|v| \geq c_0\}$, so $\xi \in L^\infty_t W^{1,3}_x(Q)$ and

$$\|\nabla \xi\|_{L^\infty_t L^3_x(Q)} \leq C(c_0)\, \|\nabla \omega\|_{L^\infty_t L^3_x(Q)}.$$

By the classical embedding $W^{1,3}(\mathbb{R}^3) \hookrightarrow \mathrm{BMO}(\mathbb{R}^3)$ (and localization), $\xi(\cdot, t) \in \mathrm{BMO}$ uniformly in t, with a BMO seminorm controlled by $\|\nabla \xi\|_{L^3}$. To upgrade BMO to VMO, one needs the vanishing of the $W^{1,3}$ norm on small balls, which follows from absolute continuity of the integral:

$$\lim_{r \to 0} \sup_x \int_{B_r(x)} |\nabla \xi(\cdot, t)|^3 = 0,$$

uniformly in t on compact subcylinders, since $|\nabla \xi|^3$ is locally integrable and t is bounded. The John–Nirenberg characterization of VMO via vanishing mean oscillation (or the standard implication $W^{1,n} \cap L^\infty \subset \mathrm{VMO}$ in $\mathbb{R}^n$ when the $W^{1,n}$ energy is tight on small balls) then yields the claim. ☐

Remark 2.4. *Proposition 2.3 is* not *available for the tangent flow under the current blow-up bounds, since neither $W^{1,3}$ control nor a lower bound on $|\omega^\infty|$ is known.*

2.4. **What the proof-track arguments do (and do not) establish.** The proof-track files `D1_VMO_Selection_proof.txt` and `proof_phase_5_vmo.txt` argue roughly as follows: (i) by CKN partial regularity, regular points are dense; (ii) at each regular point, smoothness implies the pointwise small-scale decay $\delta_r(z_0) \to 0$ and $E(z_0, r) \to 0$ as $r \to 0$; (iii) one then attempts to upgrade this pointwise statement to a *uniform* VMO statement on compact sets.

Remark 2.5 (Why this does not yield (A) as stated)**.** *Pointwise small-scale decay at each regular point does* not *imply a uniform VMO modulus on a compact set: the good radius $r_0(z_0)$ may degenerate as z_0 approaches the (closed) singular set, and CKN only controls the singular set in parabolic Hausdorff measure, not in a way that forces uniform oscillation control of ξ^∞ near it.*

Several proof-track steps invoke additional global hypotheses (often informally referred to as "Type I" or "uniform gradient bounds") to force a uniform modulus of continuity. Those hypotheses are not supplied by the blow-up/compactness construction in `new-version-12-11.tex` *and thus cannot be used for an unconditional closure of (A).*

Proposition 2.6 (A sufficient condition for VMO: uniform Lipschitz control)**.** *Let $\xi : \mathbb{R}^3 \times I \to \mathbb{S}^2$ be such that for every compact $K \subset \mathbb{R}^3 \times I$ there exists $L_K < \infty$ with*

$$|\xi(x,t) - \xi(y,t)| \le L_K\,|x-y| \qquad \text{for all } (x,t),(y,t) \in K.$$

Then $\xi(\cdot,t) \in \mathrm{VMO}_{\mathrm{loc}}(\mathbb{R}^3)$ locally uniformly in t.

Proof. Fix a compact K and let $L = L_K$. For any ball $B_r(x)$ with $B_r(x) \times \{t\} \subset K$,

$$\frac{1}{|B_r|}\int_{B_r(x)} |\xi(y,t) - \xi(x,t)|\,dy \le \frac{1}{|B_r|}\int_{B_r(x)} L|y-x|\,dy \le C\,L\,r.$$

Taking the supremum over $(x,t) \in K$ and letting $r \to 0$ gives the VMO limit. □

3. Item (B): scale-critical $L^{3/2}$ control of ω^∞

Proposition 3.1 (Target statement (B))**.** *There exists $K_0 < \infty$ such that the ancient tangent flow satisfies*

$$\sup_{z_0 \in \mathbb{R}^3 \times (-\infty,0]}\ \sup_{0<r\le 1}\ r^{-2} \iint_{Q_r(z_0)} |\omega^\infty|^{3/2}\,dx\,dt \le K_0.$$

3.1. A possible strategic refactor: use a vorticity-max blow-up instead of CKN anchoring.

Remark 3.2 (Why this might help)**.** *The CKN-anchored blow-up used in* `new-version-12-11.tex` *is excellent for guaranteeing nontriviality of the limit, but it does not automatically provide scale-critical vorticity control needed later. By contrast, rescaling at a running vorticity maximum (Proposition 3.6) produces an ancient limit with* uniformly bounded *vorticity, which immediately implies (B) (Proposition 3.5).*

If one can (i) extract a suitable-weak ancient limit from this vorticity-max sequence and (ii) run the depletion/rigidity/2D-closure arguments for that limit, then item (B) is removed from the blocker list and the overall closure burden shifts to (C)–(E). Reconciling this with the manuscript's current CKN-based tangent-flow route is nontrivial and is tracked separately in the todo list. ***Repo status note (Dec 2025).*** *This "running-max vorticity normalization" route is now written down inside* `new-version-12-11.tex` *as an auxiliary lemma (labeled* `lem:omega32-runningmax` *there): it proves the scale-critical $L^{3/2}$ bound for any ancient limit extracted from the running-max vorticity-normalized sequence, but it does* not *yet bridge to the CKN-anchored tangent flow used for nontriviality.*

Remark 3.3 (Scaling sanity check)**.** *For 3D Navier–Stokes scaling $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, one has $\omega_\lambda = \mathrm{curl} u_\lambda = \lambda^2 \omega(\lambda x, \lambda^2 t)$ and $|Q_r| \sim r^5$. Therefore*

$$r^{-2} \iint_{Q_r} |\omega|^{3/2}$$

is scale-invariant (critical), whereas $r^{-1} \iint_{Q_r} |\omega|^{3/2}$ is not*. When importing "critical vorticity" functionals from other notes (e.g. the NS sketch in* `CPM.tex`*), this normalization must be kept consistent.*

3.2. **What compactness gives, and why it is insufficient.** From suitable weak compactness, we only obtain $\omega^\infty \in L^2_{\mathrm{loc}}$, hence

$$\iint_{Q_r(z_0)} |\omega^\infty|^{3/2} \le |Q_r|^{1/4} \left(\iint_{Q_r(z_0)} |\omega^\infty|^2 \right)^{3/4} \sim r^{5/4} \left(\iint_{Q_r(z_0)} |\omega^\infty|^2 \right)^{3/4},$$

which does not yield a uniform bound after multiplying by r^{-2}.

Remark 3.4 (Obstruction). *Item (B) is precisely a* critical *bound. As such, it cannot be derived from the subcritical local energy control alone. In the current state of Navier–Stokes theory, producing a uniform bound of this type at a singular blow-up core is not a known classical result.*

3.3. **A provable sufficient condition: L^∞ vorticity implies (B).**

Proposition 3.5 (L^∞ vorticity implies scale-critical $L^{3/2}$ control). *Let ω be a vector field on $\mathbb{R}^3 \times (-\infty, 0]$ such that*

$$\|\omega\|_{L^\infty(\mathbb{R}^3\times(-\infty,0])} \le M_\omega.$$

Then for every $z_0 \in \mathbb{R}^3 \times (-\infty, 0]$ and every $0 < r \le 1$,

$$r^{-2} \iint_{Q_r(z_0)} |\omega|^{3/2}\, dx\, dt \le C\, M_\omega^{3/2},$$

where C is a universal dimensional constant.

Proof. Since $|\omega|^{3/2} \le M_\omega^{3/2}$ and $|Q_r| = |B_r|\, r^2 \simeq r^5$,

$$r^{-2} \iint_{Q_r(z_0)} |\omega|^{3/2}\, dx\, dt \le r^{-2}\, |Q_r|\, M_\omega^{3/2} \ \lesssim\ r^3\, M_\omega^{3/2} \ \le\ C\, M_\omega^{3/2}$$

for $r \le 1$. □

3.4. **A blow-up normalization that yields a bounded-vorticity ancient limit (conditional on smooth blow-up).** The following is a standard blow-up extraction device: choose the rescaling around a *running maximum* of $\|\omega(t)\|_{L^\infty}$. It yields a sequence of smooth rescaled solutions whose vorticity is uniformly bounded by 1 for all backward times, and hence any subsequential ancient limit satisfies Proposition 3.5.

Proposition 3.6 (Ancient limit with bounded vorticity from a smooth blow-up). *Let u be a smooth solution of Navier–Stokes on $[0, T^*)$ with $T^* < \infty$ the first blow-up time. Then there exist times $t_k \uparrow T^*$ and points $x_k \in \mathbb{R}^3$ with $|\omega(x_k, t_k)| = \|\omega(t_k)\|_{L^\infty} =: A_k \to \infty$ such that the rescaled solutions*

$$u^{(k)}(y, s) := \lambda_k\, u(x_k + \lambda_k y,\ t_k + \lambda_k^2 s), \qquad \lambda_k := A_k^{-1/2},$$

are defined on time intervals $(-t_k/\lambda_k^2,\ 0]$ (which exhaust $(-\infty, 0]$ as $k \to \infty$) and satisfy the uniform bound

$$\|\omega^{(k)}\|_{L^\infty(\mathbb{R}^3\times(-t_k/\lambda_k^2,0])} \le 1,$$

where $\omega^{(k)} = \mathrm{curl} u^{(k)}$.

Proof. Since T^* is the first blow-up time, the standard continuation criterion implies $\limsup_{t\uparrow T^*} \|\omega(t)\|_{L^\infty} = \infty$. Define the nondecreasing function $M(t) := \sup_{0\le s\le t} \|\omega(s)\|_{L^\infty}$. Choose any sequence $t_k \uparrow T^*$ such that $A_k := M(t_k) \to \infty$, and pick x_k with $|\omega(x_k, t_k)| = \|\omega(t_k)\|_\infty = A_k$. Let $\lambda_k =$

$A_k^{-1/2}$ and define $u^{(k)}$ by Navier–Stokes scaling. Then $\omega^{(k)}(y,s) = \lambda_k^2\,\omega(x_k + \lambda_k y,\ t_k + \lambda_k^2 s)$, and for any $s \le 0$ with $t := t_k + \lambda_k^2 s \in [0, t_k]$,

$$\|\omega^{(k)}(s)\|_{L^\infty} = \lambda_k^2\,\|\omega(t)\|_{L^\infty} \le \lambda_k^2\,M(t_k) = \lambda_k^2\,A_k = 1,$$

since $M(t)$ is nondecreasing and $t \le t_k$. This proves the uniform bound. The rescaled time interval is $s \in (-t_k/\lambda_k^2, 0]$, and $t_k/\lambda_k^2 = t_k A_k \to \infty$ as $k \to \infty$, hence the intervals exhaust $(-\infty, 0]$. □

Remark 3.7. *Proposition 3.6 yields a bounded-vorticity ancient blow-up limit* for at least one blow-up sequence *when starting from a* smooth *finite-time blow-up. However, reconciling this particular normalization with the CKN-singular-point anchoring used to guarantee non-triviality in the suitable-weak compactness framework is subtle and must be handled carefully in the main manuscript.*

3.5. **Conditional statement obtainable from Serrin-type bounds.** If one assumes a Serrin critical velocity bound at the tangent-flow level, then (B) follows:

Proposition 3.8 (Conditional (B) from critical velocity control)**.** *Assume that on each unit cylinder $Q_1(z_0)$,*

$$\|u^\infty\|_{L_t^\infty L_x^3(Q_1(z_0))} \le M_0$$

uniformly in z_0. Then (B) holds with $K_0 = K_0(M_0)$.

Proof. On each $Q_1(z_0)$, the vorticity satisfies $\omega^\infty = \operatorname{curl} u^\infty$ in distributions. By standard local Calderón–Zygmund estimates (after localization and fixing a gauge), one can control the $L^{3/2}$ norm of ω^∞ by the L^3 norm of u^∞ and the $L^{3/2}$ norm of the pressure. Scaling then yields the stated bound on all smaller cylinders. (*A fully detailed proof would require a precise localized Biot–Savart/pressure decomposition; omitted here.*) □

Remark 3.9. *This does not close (B) unconditionally, because the required uniform $L_t^\infty L_x^3$ bound is itself a scale-critical regularity criterion and is not known to hold for general tangent flows arising from singularities.*

4. Item (C): DDE ε-regularity and Liouville rigidity

4.1. **Model equation.** Consider a sphere-valued map $\xi : \mathbb{R}^3 \times (-\infty, 0] \to \mathbb{S}^2$ solving

$$\partial_t \xi - \Delta \xi + u \cdot \nabla \xi = |\nabla \xi|^2 \xi + H, \qquad |\xi| = 1, \qquad H \cdot \xi = 0, \tag{1}$$

with a divergence-free drift u.

Remark 4.1 (Consistency note)**.** *The "proof track" file* `F_DDE_EpsReg_proof.txt` *writes a DDE of the form $\partial_t \xi - \Delta \xi + u \cdot \nabla \xi = H$ with $H \cdot \xi = 0$. Taken literally with the* full *Laplacian $\Delta\xi$, this would force $|\nabla\xi|^2 \equiv 0$ (hence ξ constant), since $\xi \cdot \Delta \xi = -|\nabla \xi|^2$. Therefore the correct geometric equation must include the curvature term $|\nabla\xi|^2\xi$ (equivalently use the projected Laplacian $P_\xi(\Delta\xi)$). In this document we work with the consistent harmonic-map form (1).*

4.2. **Drift control available in the running-max refactor.**

Lemma 4.2 (Bounded vorticity gives local Serrin drift after a Galilean gauge)**.** *Let $u(\cdot,t)$ be divergence-free on $\mathbb{R}^3$ with vorticity $\omega(\cdot,t) = \mathrm{curl} u(\cdot,t) \in L^\infty(\mathbb{R}^3)$. Fix $x_0 \in \mathbb{R}^3$, a radius $r > 0$, and $1 \le p < \infty$. Define the spatial average*

$$c_{x_0,r}(t) := \frac{1}{|B_r|} \int_{B_r(x_0)} u(x,t)\,dx.$$

Then for a.e. t,

$$\|u(\cdot,t) - c_{x_0,r}(t)\|_{L^p(B_r(x_0))} \;\le\; C_p\, r^{1+3/p}\, \|\omega(\cdot,t)\|_{L^\infty(\mathbb{R}^3)},$$

where C_p depends only on p (and dimension). In particular, if $\omega \in L^\infty(\mathbb{R}^3 \times I)$ on a time interval I, then for any $p > 3$ the gauged drift $u - c_{x_0,r}$ belongs to a local Serrin class with $q = \infty$ on $Q_r(x_0,t_0)$:

$$u - c_{x_0,r} \in L^\infty\big((t_0 - r^2, t_0); L^p(B_r(x_0))\big), \qquad \frac{2}{\infty} + \frac{3}{p} < 1.$$

Lemma 4.3 (Bounded vorticity implies local smoothness (via Serrin))**.** *Let (u,p) be a suitable weak solution of the 3D Navier–Stokes equations on $Q_{2r}(z_0)$ and assume $\omega = \mathrm{curl} u \in L^\infty(Q_{2r}(z_0))$. Then u is smooth on $Q_r(z_0)$.*

Proof sketch. Fix $p > 3$. By Lemma 4.2, after subtracting a ball average $c(t)$ (and applying the corresponding Galilean change of variables), the drift belongs to the Serrin class $L_t^\infty L_x^p$ on $Q_{2r}(z_0)$. The (local) Ladyzhenskaya–Prodi–Serrin interior regularity criterion for suitable weak solutions then yields smoothness of u on the smaller cylinder $Q_r(z_0)$. □

Remark 4.4 (Why this matters for (C))**.** *For the running-max ancient element in the main rewrite, $\omega^\infty \in L^\infty(\mathbb{R}^3 \times (-\infty, 0])$. Lemma 4.2 shows that the drift hypothesis (local Serrin control) needed in standard drift-absorption steps can be obtained* locally *after a Galilean change of coordinates (subtracting a spatially constant vector field $c(t)$). Thus, in the running-max architecture, the main remaining obstruction in (C) is not the drift integrability itself, but the fully critical ε-regularity / Campanato iteration with Carleson forcing and the verification of the needed global small-energy hypotheses for Liouville.*

4.3. **Target statement.**

Proposition 4.5 (Target statement (C))**.** *Assume that u belongs to a scale-critical Serrin class and H is small in the critical Carleson norm. Then any ancient solution ξ of (1) with finite critical energy must be constant.*

4.4. **Status.**

Remark 4.6 (Status / obstruction)**.** *The needed ε-regularity theory for (1) with* critical drift *and* critical Carleson forcing *is not a standard published black box in this exact setting, and the current proof in* `new-version-12-11.tex` *contains a scaling error in the "vanishing gradient" step.*

Assumption 4.7 (Critical-forcing ε-regularity for the DDE (missing upgrade))**.** *There exist universal constants $\varepsilon_* > 0$, $\delta_* > 0$, $\alpha \in (0,1)$, $C < \infty$, an exponent $p > 3$, and a drift threshold $\eta_* > 0$ such that the following holds. If on $Q_1(z_0)$ the direction equation (1) holds with divergence-free drift u satisfying the small Serrin bound*

$$u \in L^\infty\big((t_0 - 1, t_0); L^p(B_1(x_0))\big), \qquad \|u\|_{L_t^\infty L_x^p(Q_1(z_0))} \le \eta_*,$$

and with small initial energy and critical *forcing size*

$$E(z_0,1)\le \varepsilon_*^2, \qquad \sup_{0<r\le 1} r^{-2}\iint_{Q_r(z_0)} |H|^{3/2}\,dx\,dt \le \delta_*^{3/2},$$

then for all $\rho \le \frac{1}{2}$ *one has the quantitative decay*

$$E(z_0,\rho)\le C\rho^{2\alpha}E(z_0,1) \;+\; C\,\delta_*^2,$$

and, in particular, the scale-covariant gradient bound

$$\sup_{Q_{1/2}(z_0)} |\nabla\xi| \le C\,(\varepsilon_*+\delta_*).$$

4.5. **A correct Liouville mechanism once an ε-regularity gradient bound is available.** The proof-track file `F_DDE_Liouville_proof.txt` attempts to conclude $\nabla\xi\equiv 0$ from the small-scale limit $E(z_0,r)\to 0$ as $r\to 0$ using Lebesgue differentiation. That inference is incorrect for the scale-invariant quantity $E(z_0,r)=r^{-3}\iint_{Q_r}|\nabla\xi|^2$.

However, there is a clean (and correct) Liouville mechanism if the ε-regularity theorem provides a *scale-covariant pointwise gradient bound.*

Proposition 4.8 (Liouville from scale-invariant smallness + ε-regularity)**.** *Assume there exist universal constants $\varepsilon_*>0$ and $C<\infty$ with the following property: for any solution of* (1) *on $Q_1(z_0)$, if $E(z_0,1)\le\varepsilon_*^2$ and the drift/forcing hypotheses of the ε-regularity theorem hold, then*

$$\sup_{Q_{1/2}(z_0)} |\nabla\xi| \le C\,\varepsilon_*.$$

Assume moreover that ξ is an ancient solution of (1) *on $\mathbb{R}^3\times(-\infty,0]$ such that*

$$\sup_{z_0\in\mathbb{R}^3\times(-\infty,0]}\ \sup_{r>0}\ E(z_0,r)\ \le\ \varepsilon_*^2,$$

and that the drift/forcing hypotheses are scale-invariant and hold on every $Q_r(z_0)$ after rescaling. Then $\nabla\xi\equiv 0$ on $\mathbb{R}^3\times(-\infty,0]$, hence ξ is spatially constant.

Proof. Fix any point $z_0=(x_0,t_0)$. For each $r>0$, define the rescaled fields on $Q_1(0,0)$ by

$$\xi^{(r)}(x,t):=\xi(x_0{+}rx,\ t_0{+}r^2t),\qquad u^{(r)}(x,t):=r\,u(x_0{+}rx,\ t_0{+}r^2t),\qquad H^{(r)}(x,t):=r^2\,H(x_0{+}rx,\ t_0{+}r^2t).$$

By scale invariance of E and the hypotheses, we have $E_{\xi^{(r)}}(0,1)=E_\xi(z_0,r)\le\varepsilon_*^2$, and the drift/forcing hypotheses hold on $Q_1(0,0)$. Applying ε-regularity yields

$$|\nabla\xi^{(r)}(0,0)|\le \sup_{Q_{1/2}(0,0)}|\nabla\xi^{(r)}|\le C\,\varepsilon_*.$$

Undoing the rescaling gives

$$|\nabla\xi(z_0)|=\frac{1}{r}\,|\nabla\xi^{(r)}(0,0)|\le\frac{C\,\varepsilon_*}{r}.$$

Since this holds for every $r>0$, letting $r\to\infty$ forces $|\nabla\xi(z_0)|=0$. Because z_0 was arbitrary, $\nabla\xi\equiv 0$ everywhere. ∎

Remark 4.9. *Proposition 4.8 clarifies the correct rigidity route:* Liouville follows from a global smallness assumption plus an ε-regularity gradient estimate*, without any appeal to Lebesgue differentiation at small scales. The genuinely hard part is establishing the required ε-regularity theorem in the drift/Carleson setting, and verifying its hypotheses for the Navier–Stokes tangent flow.*

4.6. **What can be proved (conditional, classical PDE).** We record a classical-style statement that *can* be proved with standard parabolic energy methods under stronger assumptions (subcritical drift, integrable forcing). This is not sufficient for the NS application but gives a template.

Theorem 4.10 (Sketch: ε-regularity for subcritical drift)**.** *Assume $u \in L^q_t L^p_x$ with $2/q + 3/p < 1$ and $H \in L^{3/2}_{t,x}$ with sufficiently small scale-invariant norm on Q_1. Then smallness of the scale-invariant energy*

$$E(1) := \iint_{Q_1} |\nabla \xi|^2$$

implies a decay estimate $E(\theta) \le \frac{1}{2}E(1) + C\|H\|^{3/2}_{L^{3/2}(Q_1)}$ for some $\theta \in (0,1)$, and hence Hölder regularity by a Campanato iteration.

Remark 4.11. *Upgrading Theorem 4.10 to the* critical *drift/Carleson forcing regime needed for the NS program is exactly the missing content in (C). Completing that upgrade unconditionally would constitute a major new PDE result.*

5. Item (D): Carleson smallness of the tangential forcing H

Proposition 5.1 (Target statement (D))**.** *For the ancient tangent flow, the tangential forcing H in (1) satisfies*

$$\|H\|_{C^{3/2}} := \sup_{z_0} \sup_{0<r\le r_0} r^{-2} \iint_{Q_r(z_0)} |H|^{3/2}\, dx\, dt \le \delta_*^{3/2}$$

for some universal $\delta_* > 0$ at sufficiently small scales.

Remark 5.2 (Status)**.** *As currently written, proving (D) requires several hard inputs. In the original CKN-tangent-flow route, the near-field control typically uses a CRW commutator estimate with* small BMO/*VMO seminorm of ξ^∞ at small scales (this is (A)), and one must also control the "constant-direction" Calderón–Zygmund remainder quantitatively. In the running-max refactor, bounded vorticity eliminates these* near-field *issues (the commutator/oscillation term and the constant-direction remainder become Carleson-small automatically), leaving the genuinely difficult parts:*

- *a separate depletion mechanism for the far-field/tail contribution of the stretching kernel (the heuristic maximal-function tail bound in the manuscript does not yield* smallness *at small scales from (B) alone, due to borderline integrability/scaling);*
- *the log-amplitude / vorticity-zero-set issue needed to control the geometric forcing H_{geom}: for the running-max ancient element, bounded vorticity gives local smoothness so the regularized computation with $h_\varepsilon = \log(\rho + \varepsilon)$ is classical on each compact cylinder, but it is nontrivial to control the $\varepsilon \downarrow 0$ limit across $\{\rho = 0\}$ (equivalently, to obtain a scale-invariant L^2 bound on $\nabla \log \rho$). We isolate this as Assumption 5.3 below. Once such a scale-invariant L^2 bound on $\nabla \log \rho$ is available (and one assumes small direction energy), the geometric forcing is automatically Carleson-small at small scales (Lemma 5.12).*

Thus (D) is not currently closed unconditionally.

Assumption 5.3 (Log-amplitude control across the vorticity-zero set)**.** *For the ancient element (in particular, in the running-max refactor), there exists $K_h < \infty$ such that for every z_0 and every $0 < r \le 1$,*

$$\sup_{0<\varepsilon\le 1} r^{-3} \iint_{Q_r(z_0)} |\nabla \log(\rho+\varepsilon)|^2 \, dx \, dt \;\le\; K_h,$$

where $\rho := |\omega|$ is the vorticity magnitude of the ancient element.

5.1. **A clean reduction: (A)+(B) $\Rightarrow$ Carleson smallness of the *singular* forcing.** We record the precise statement that is actually used in the manuscript's "forcing depletion" step. It is classical once the hypotheses (A) and (B) are granted (the novelty is not here, but in proving (A)–(B) for tangent flows).

Definition 5.4 (Critical Carleson norm)**.** *For a measurable spacetime field F on $\mathbb{R}^3 \times (-\infty, 0]$, define*

$$\|F\|_{C^{3/2}(r_*)} := \sup_{z_0 \in \mathbb{R}^3 \times (-\infty,0]} \; \sup_{0<r\le r_*} r^{-2} \iint_{Q_r(z_0)} |F|^{3/2} \, dx \, dt.$$

When $r_* = 1$ we write $\|F\|_{C^{3/2}}$.

Definition 5.5 (Local BMO seminorm at scale r)**.** *For a locally integrable $b : \mathbb{R}^3 \to \mathbb{R}^m$ and $r > 0$, define the local seminorm*

$$\|b\|_{\mathrm{BMO}_{\le r}} \;:=\; \sup_{x\in\mathbb{R}^3} \; \sup_{0<\rho\le r} \frac{1}{|B_\rho|} \int_{B_\rho(x)} |b(y) - b_{B_\rho(x)}| \, dy,$$

where $b_{B_\rho(x)}$ denotes the spatial average of b over $B_\rho(x)$. We say $b \in \mathrm{VMO}$ if $\lim_{r\to 0} \|b\|_{\mathrm{BMO}_{\le r}} = 0$.

Proposition 5.6 (VMO $\Rightarrow$ small commutator at small scales)**.** *Let T be a Calderón–Zygmund operator on $\mathbb{R}^3$ and let $1 < p < \infty$. There exists a constant $C_{p,T}$ such that for all $b \in \mathrm{BMO}(\mathbb{R}^3)$ and all $f \in L^p(\mathbb{R}^3)$,*

$$\|[T, b]f\|_{L^p(\mathbb{R}^3)} \;\le\; C_{p,T} \, \|b\|_{\mathrm{BMO}(\mathbb{R}^3)} \, \|f\|_{L^p(\mathbb{R}^3)}.$$

Moreover, if $b \in \mathrm{VMO}(\mathbb{R}^3)$, then for every $\varepsilon > 0$ there exists $r_*(\varepsilon, b) > 0$ such that for all $0 < r \le r_*$ one has the localized smallness estimate

$$\sup_{x_0\in\mathbb{R}^3} \|[T, b]f\|_{L^p(B_r(x_0))} \;\le\; C_{p,T} \, \|b\|_{\mathrm{BMO}_{\le 2r}} \, \|f\|_{L^p(\mathbb{R}^3)} \;\le\; \varepsilon \, \|f\|_{L^p(\mathbb{R}^3)}.$$

Proof. The global bound is the Coifman–Rochberg–Weiss commutator theorem (classical input). For the localized estimate, note that for $B_r(x_0)$ and any b,

$$\|b\|_{\mathrm{BMO}(B_r(x_0))} \;\le\; \|b\|_{\mathrm{BMO}_{\le r}},$$

where $\|b\|_{\mathrm{BMO}(B_r(x_0))}$ denotes the supremum of mean oscillations over sub-balls of $B_r(x_0)$. Applying the commutator theorem to b localized to balls of radius $\le 2r$ yields the displayed inequality with $\|b\|_{\mathrm{BMO}_{\le 2r}}$. If $b \in \mathrm{VMO}$, then $\|b\|_{\mathrm{BMO}_{\le 2r}} \to 0$ as $r \to 0$, so choose r_* so that $C_{p,T}\|b\|_{\mathrm{BMO}_{\le 2r_*}} \le \varepsilon$. □

Remark 5.7 (Relevance to item (D))**.** *In the geometric depletion manuscript, the desired near-field depletion estimate for H_{sing} is obtained by rewriting H_{near} as a commutator of a Calderón–Zygmund operator with the direction field, schematically $H_{\mathrm{near}} \sim P_\xi[T, \xi]\rho$. Proposition 5.6 then explains how a* VMO modulus *for ξ yields smallness.*

Important: *deriving the precise commutator representation of H_{near} from the Biot–Savart law and the definition $H_{\mathrm{sing}} := P_\xi(S\xi)$ is a separate (nontrivial) algebraic/harmonic-analysis step. In particular, one must track the exact Biot–Savart representation of $S\xi$: after contracting S with $\xi(x)$, the resulting kernel can depend on $\xi(x)$, and CRW does not apply directly until one rewrites the expression in terms of a* fixed *Calderón–Zygmund operator plus commutator errors (or else states the commutator representation as an explicit hypothesis).*

5.2. **(New) An explicit Biot–Savart identity for the stretching term.** To move beyond purely schematic commutator notation, it is helpful to record a concrete pointwise identity for the vortex stretching term $(\omega\cdot\nabla)u = S\omega$ in terms of ω itself. This isolates which parts of H_{sing} genuinely carry a "direction difference" factor and which parts require additional cancellation.

Lemma 5.8 (Biot–Savart identity for stretching). *Let u be smooth, divergence-free on $\mathbb{R}^3$ at a fixed time t, with vorticity $\omega = \mathrm{curl}\, u$. Then for each $x \in \mathbb{R}^3$,*

$$(\omega\cdot\nabla)u(x) = \frac{1}{4\pi}\,\mathrm{p.v.}\int_{\mathbb{R}^3}\left(\frac{\omega(x)\times\omega(y)}{|x-y|^3} \;+\; 3\,\frac{(\omega(x)\cdot(x-y))\,(\omega(y)\times(x-y))}{|x-y|^5}\right)dy.$$

Proof. Write the Biot–Savart law

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)\times\omega(y)}{|x-y|^3}\,dy$$

and differentiate under the integral sign: $(\omega(x)\cdot\nabla_x)u(x) = \frac{1}{4\pi}\int(\omega(x)\cdot\nabla_x)\big((x-y)\times\omega(y)\,|x-y|^{-3}\big)\,dy$. Using $(\omega(x)\cdot\nabla_x)(x-y) = \omega(x)$ and $(\omega(x)\cdot\nabla_x)|x-y|^{-3} = -3(\omega(x)\cdot(x-y))|x-y|^{-5}$ gives the displayed formula (as a principal value integral). □

Remark 5.9 (How this impacts the commutator goal). *If $\omega = \rho\xi$ and $|\omega(x)| \neq 0$, then $S\xi(x) = (S\omega(x))/\rho(x)$ and $H_{\mathrm{sing}}(x) = P_{\xi(x)}(S\xi(x))$. Lemma 5.8 shows that one piece of $(\omega\cdot\nabla)u$ contains an explicit factor $\omega(x)\times\omega(y)$, which vanishes when directions align. However, the second piece does* not *collapse to a pure direction-difference factor without additional structure (e.g. cancellation using $\nabla\cdot\omega = 0$ or a more refined symmetric representation). This is one of the technical places where the depletion program needs a fully explicit derivation, not just a schematic "CRW commutator" label.*

Remark 5.10 (A concrete "direction-difference" bound from the Biot–Savart identity). *Writing $\omega = \rho\,\xi$ and using $|\xi(x)\times\xi(y)| \le |\xi(x)-\xi(y)|$, the first term in Lemma 5.8 yields the pointwise estimate*

$$\left|\mathrm{p.v.}\int_{\mathbb{R}^3}\frac{\omega(x)\times\omega(y)}{|x-y|^3}\,dy\right| \le C\,\rho(x)\int_{\mathbb{R}^3}\frac{\rho(y)\,|\xi(x)-\xi(y)|}{|x-y|^3}\,dy.$$

Thus, at least one component of the stretching interaction is explicitly *controlled by a singular integral of the direction oscillation. The remaining term(s) must be treated separately (either by additional cancellation identities, or by a separate "tail depletion" mechanism).*

Remark 5.11 (What this does *not* solve). *Proposition 5.6 isolates only the "VMO ⇒ commutator small" mechanism. It does* not *by itself justify the log-amplitude control needed to estimate the geometric forcing $H_{\mathrm{geom}} = 2P_\xi((\nabla\log\rho)\cdot\nabla\xi)$, but once one has a scale-invariant L^2 bound on $\nabla\log\rho$ together with small direction energy, the geometric forcing becomes Carleson-small at small scales (Lemma 5.12 below), and it does not provide the missing scale-critical estimates needed to verify (A) and (B) for tangent flows.*

5.3. **(New) Geometric forcing: Carleson smallness from log-amplitude control and small direction energy.**

Lemma 5.12 (Geometric forcing becomes Carleson-small from log-amplitude L^2 control and small direction energy)**.** *Let $\rho \geq 0$ and $\xi : \mathbb{R}^3 \times (-\infty, 0] \to \mathbb{S}^2$ be such that $h = \log \rho$ is well-defined on the region of interest and $\nabla h, \nabla \xi \in L^2_{\mathrm{loc}}$. Define*

$$H_{\mathrm{geom}} := 2P_\xi\big((\nabla h) \cdot \nabla \xi\big).$$

Assume there exists $K_h < \infty$ such that for every z_0 and every $0 < r \leq 1$,

$$r^{-3} \iint_{Q_r(z_0)} |\nabla h|^2 \, dx \, dt \leq K_h,$$

and assume there exists $\varepsilon > 0$ such that for every z_0 and every $r > 0$,

$$r^{-3} \iint_{Q_r(z_0)} |\nabla \xi|^2 \, dx \, dt \leq \varepsilon^2.$$

Then for every $0 < r_0 \leq 1$,

$$\sup_{z_0} \sup_{0<r\leq r_0} r^{-2} \iint_{Q_r(z_0)} |H_{\mathrm{geom}}|^{3/2} \, dx \, dt \leq C \, K_h^{3/4} \, \varepsilon^{3/2} \, r_0^{5/2}.$$

In particular, $\|H_{\mathrm{geom}}\|_{C^{3/2}(r_0)} \to 0$ as $r_0 \downarrow 0$ (with an explicit rate).

Proof. Using $|P_\xi v| \leq |v|$ we have $|H_{\mathrm{geom}}| \leq 2|\nabla h| \, |\nabla \xi|$, hence

$$\iint_{Q_r(z_0)} |H_{\mathrm{geom}}|^{3/2} \leq C \left(\iint_{Q_r(z_0)} |\nabla h|^2 \right)^{3/4} \left(\iint_{Q_r(z_0)} |\nabla \xi|^2 \right)^{3/4}.$$

By the hypotheses, $\iint_{Q_r(z_0)} |\nabla h|^2 \leq K_h r^3$ and $\iint_{Q_r(z_0)} |\nabla \xi|^2 \leq \varepsilon^2 r^3$, so

$$\iint_{Q_r(z_0)} |H_{\mathrm{geom}}|^{3/2} \leq C \, (K_h r^3)^{3/4} \, (\varepsilon^2 r^3)^{3/4} = C \, K_h^{3/4} \, \varepsilon^{3/2} \, r^{9/2}.$$

Multiplying by r^{-2} and using $r \leq r_0$ yields the claimed bound. □

5.4. **(New) A pointwise identity for the ξ-directional derivative and for H_{sing}.** The stretching identity above concerns $S\omega = (\omega \cdot \nabla)u$. In the direction equation, the relevant quantity is

$$H_{\mathrm{sing}} := P_\xi(S\xi).$$

Since the antisymmetric part of ∇u annihilates ξ (because $\xi \parallel \omega$), one has the simplification $S\xi = (\xi \cdot \nabla)u$. This yields an explicit principal-value formula.

Lemma 5.13 ($(\xi \cdot \nabla)u$ as a singular integral)**.** *Let u be smooth and divergence-free on $\mathbb{R}^3$ at a fixed time t, with vorticity $\omega = \mathrm{curl} u$. For any x with $\omega(x) \neq 0$, set $\xi(x) := \omega(x)/|\omega(x)|$. Then*

$$(\xi(x) \cdot \nabla)u(x) = \frac{1}{4\pi} \, \mathrm{p.v.} \int_{\mathbb{R}^3} \left(\frac{\xi(x) \times \omega(y)}{|x-y|^3} \; - \; 3 \, \frac{(\xi(x) \cdot (x-y)) \, ((x-y) \times \omega(y))}{|x-y|^5} \right) dy.$$

Proof. Differentiate the Biot–Savart formula $u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \omega(y)}{|x-y|^3} \, dy$ in the (constant) direction $\xi(x)$ at the point x. Using $(\xi(x) \cdot \nabla_x)(x-y) = \xi(x)$ and $(\xi(x) \cdot \nabla_x)|x-y|^{-3} = -3(\xi(x) \cdot (x-y))|x-y|^{-5}$ gives the displayed identity (as a principal value integral). □

Corollary 5.14 (Explicit formula for H_{sing})**.** *Under the hypotheses of Lemma 5.13, one has*

$$H_{\mathrm{sing}}(x) = P_{\xi(x)}(S\xi)(x) = P_{\xi(x)}\big((\xi(x)\cdot\nabla)u(x)\big),$$

and therefore (using $P_\xi v = \xi\times(v\times\xi)$) the integrand in Lemma 5.13 can be projected explicitly. In particular, the first term is already tangential and equals $\rho(y)\,\xi(x)\times\xi(y)/|x-y|^3$ after writing $\omega = \rho\xi$.

5.5. **A referee-checkable near-field reduction ("commutator form").** We now record a clean algebraic reduction of the near-field singular forcing to a Calderón–Zygmund operator acting on the *direction error* $\rho(\xi-\xi(x))$. This is the precise, checkable replacement for the manuscript's schematic commutator notation.

Definition 5.15 (Truncated singular integral operator)**.** *For $a\in S^2$ and $r>0$ define the truncated operator acting on vector fields $F:\mathbb{R}^3\to\mathbb{R}^3$ by*

$$(\mathcal{T}_{a,r}F)(x) := \mathrm{p.v.}\int_{B_r(x)}\left(\frac{a\times F(y)}{|x-y|^3}\ -\ 3\,\frac{(a\cdot(x-y))\,((x-y)\times F(y))}{|x-y|^5}\right)dy.$$

Lemma 5.16 (Near-field decomposition: constant-direction part + oscillation part)**.** *Let u be smooth and divergence-free at a fixed time, with $\omega=\rho\xi$ and $\xi(x)=\omega(x)/|\omega(x)|$ at points where $\omega(x)\neq 0$. Fix $r>0$ and set $a:=\xi(x)$. Define*

$$F_x(y) := \rho(y)\big(\xi(y)-a\big).$$

Then for every such x,

$$H_{\mathrm{near}}(x) = \frac{1}{4\pi}\,(\mathcal{T}_{a,r}(\rho(\cdot)\,a))(x)\ +\ P_a\Big(\frac{1}{4\pi}\,(\mathcal{T}_{a,r}F_x)(x)\Big).$$

Proof. Start from Lemma 5.13 with the integration restricted to $B_r(x)$ (this defines H_{near}), and write $\omega(y)=\rho(y)\xi(y)=\rho(y)a+F_x(y)$. By linearity of $\mathcal{T}_{a,r}$,

$$\mathcal{T}_{a,r}\omega = \mathcal{T}_{a,r}(\rho a)+\mathcal{T}_{a,r}F_x.$$

Applying P_a and noting that $\mathcal{T}_{a,r}(\rho a)$ is already tangential (since $r\times a\perp a$ and $a\times a=0$) gives the claim. □

Remark 5.17 (Turning the constant-direction part into an oscillation term)**.** *The first term in Lemma 5.16 depends only on ρ and the frozen direction $a=\xi(x)$. Using the divergence-free identity recorded in Remark 5.31, one can rewrite the* full-space *constant-direction operator $\mathcal{T}_a(\rho a)$ as a Calderón–Zygmund operator applied to $\rho(a-\xi)$ (i.e. to the direction error). To obtain a fully local near-field estimate, one performs the standard near/tail split:*

$$\mathcal{T}_{a,r}(\rho a)(x) = \mathcal{T}_a(\rho a)(x)\ -\ \int_{\mathbb{R}^3\setminus B_r(x)}\Big(-3\,\frac{(a\cdot(x-y))\,((x-y)\times(\rho(y)a))}{|x-y|^5}\Big)\,dy.$$

Thus, modulo an explicit far-field remainder (which is treated in the tail analysis), the near-field constant-direction contribution is exactly a CZ operator applied to the direction error $\rho(a-\xi)$.

Lemma 5.18 (CZ boundedness for the constant-direction remainder)**.** *Fix $a\in\mathbb{S}^2$ and define the constant-direction operator on scalars*

$$(T_af)(x) := a\times\nabla\big((a\cdot\nabla)(-\Delta)^{-1}f\big)(x).$$

Then for every $1 < p < \infty$ there exists $C_p < \infty$ such that for all $f \in L^p(\mathbb{R}^3)$,

$$\|T_a f\|_{L^p(\mathbb{R}^3)} \le C_p \|f\|_{L^p(\mathbb{R}^3)}.$$

Moreover, if $\omega = \rho\,\xi$ with $\nabla\cdot\omega = 0$, then

$$T_a\rho \;=\; a\times\nabla(-\Delta)^{-1}\nabla\cdot(\rho(a-\xi))$$

in the sense of distributions, and hence

$$\|T_a\rho\|_{L^p(\mathbb{R}^3)} \le C_p \|\rho(a-\xi)\|_{L^p(\mathbb{R}^3)}.$$

Proof. Each component of $a\times\nabla(-\Delta)^{-1}\nabla\cdot$ is a finite linear combination of Riesz transforms and is therefore a Calderón–Zygmund operator bounded on L^p for $1 < p < \infty$. The identity $T_a\rho = a\times\nabla(-\Delta)^{-1}\nabla\cdot(\rho(a-\xi))$ follows from $\nabla\cdot\omega = 0$ and the computation $a\cdot\nabla\rho = \nabla\cdot(\rho a - \omega)$. □

Remark 5.19 (What remains after the reduction)**.** *Lemma 5.16 isolates the near-field forcing into: (i) a* constant-direction *piece $\mathcal{T}_{a,r}(\rho a)$ and (ii) an* oscillation *piece $\mathcal{T}_{a,r}(\rho(\xi-a))$. Using $\nabla\cdot\omega = 0$, the* full *constant-direction operator is itself an oscillation term (Remark 5.31), so the remaining analytic problem becomes to quantify, in a scale-invariant Carleson norm, the smallness of a CZ operator applied to*

$$\rho(\xi-\xi(x)) \quad \textit{(or a comparable localized direction error such as } \rho(\xi-\xi_{B_r})\textit{).}$$

This is exactly the "VMO + scale-critical ω control $\Rightarrow$ small Carleson norm" step requested in item (D).

Lemma 5.20 (A simple closure of the constant-direction remainder under bounded vorticity)**.** *Assume $\rho = |\omega| \in L^\infty(\mathbb{R}^3\times(-\infty,0])$ with $\|\rho\|_{L^\infty} \le M$. Then for any measurable choice of unit vector $a = a(z_0,r)$ (e.g. $a = \xi(x_0,t_0)$ or a local average direction) one has*

$$\|\rho(a-\xi)\|_{C^{3/2}(r_*)} \;\le\; C\,M^{3/2}\,r_*^3 \qquad (0 < r_* \le 1),$$

and in particular $\lim_{r_*\to 0}\|\rho(a-\xi)\|_{C^{3/2}(r_*)} = 0$.

Proof. Since $|\xi| = |a| = 1$ we have $|\rho(a-\xi)| \le 2M$. Therefore for any z_0 and $0 < r \le r_*$,

$$r^{-2}\iint_{Q_r(z_0)} |\rho(a-\xi)|^{3/2}\,dx\,dt \le (2M)^{3/2}\,r^{-2}\,|Q_r| \le C\,(2M)^{3/2}\,r^3,$$

because $|Q_r| \le Cr^5$ for $r \le 1$. Taking the supremum over z_0 and $r \le r_*$ gives the claim. □

Remark 5.21 (How this interacts with the bridge problem (B))**.** *Lemma 5.20 explains why the running-max normalization is strategically attractive: if one can extract an ancient limit with* bounded *vorticity from the blow-up sequence (as in Lemma* `lem:omega32-runningmax` *of the main manuscript), then the constant-direction remainder becomes small at small scales automatically. The remaining obstruction is then to connect that running-max ancient limit to the CKN tangent-flow framework (or refactor the overall contradiction to work directly with the running-max ancient limit). This is exactly the bridge/refactor item in blocker (B).*

5.6. **Quantitative form: a uniform CRW bound implies near-field Carleson smallness under VMO+$L^{3/2}$ control.** We now package the preceding algebra into the precise *conditional* estimate that the main manuscript needs: if (A) ξ has a VMO modulus in space and (B) $\rho = |\omega|$ has a scale-critical $L^{3/2}$ Carleson bound, then the near-field singular forcing is small in the critical Carleson norm at sufficiently small scales. The proof is an application of the classical Coifman–Rochberg–Weiss commutator theorem, once the near-field terms are rewritten in commutator form.

Definition 5.22 (Fixed CZ kernels underlying $\mathcal{T}_{a,r}$)**.** *For $m, j \in \{1,2,3\}$ define the (vector-valued) kernels*

$$k_{m,j}(r) := \frac{e_m \times e_j}{|r|^3} \;-\; 3\,\frac{r_m\,((r) \times e_j)}{|r|^5}, \qquad r \in \mathbb{R}^3 \setminus \{0\},$$

and the associated truncated operators on scalars f by

$$(T_{m,j,r} f)(x) := \text{p.v.} \int_{B_r(x)} k_{m,j}(x-y)\, f(y)\, dy.$$

Then for every $a \in S^2$ and every vector field $F = (F_1, F_2, F_3)$,

$$(\mathcal{T}_{a,r} F)(x) = \sum_{m,j=1}^{3} a_m\, (T_{m,j,r} F_j)(x).$$

Lemma 5.23 (Near-field commutator term is small in the critical Carleson norm)**.** *Let $\omega = \rho\,\xi$ on $\mathbb{R}^3 \times (-\infty, 0]$ and define the* near-field commutator/oscillation term

$$H^{\text{osc}}_{\text{near}}(x,t) := P_{\xi(x,t)}\Big(\frac{1}{4\pi}\,\mathcal{T}_{\xi(x,t),r}\big(\rho(\cdot,t)(\xi(\cdot,t) - \xi(x,t))\big)(x)\Big),$$

with $\mathcal{T}_{a,r}$ as in Definition 5.15. Assume:

(A) *(*VMO in space, uniform in time*) there is a modulus $\eta : (0,1] \to [0,\infty)$ with $\eta(r) \to 0$ as $r \to 0$ such that for all $t \le 0$,*

$$\|\xi(\cdot,t)\|_{\text{BMO}_{\le r}} \le \eta(r) \qquad (0 < r \le 1);$$

(B) *(*critical $L^{3/2}$ control of ρ*) there is $M < \infty$ such that*

$$\|\rho\|_{C^{3/2}} = \sup_{z_0} \sup_{0<r\le 1} r^{-2} \iint_{Q_r(z_0)} \rho^{3/2}\, dx\, dt \le M.$$

Then there exists a universal constant $C < \infty$ such that for all $0 < r_* \le 1$,

$$\|H^{\text{osc}}_{\text{near}}\|_{C^{3/2}(r_*)} \;\le\; C\,\eta(2r_*)^{3/2}\, M.$$

In particular, $\lim_{r_* \to 0} \|H^{\text{osc}}_{\text{near}}\|_{C^{3/2}(r_*)} = 0$.

Proof (commutator reduction + CRW). Fix $z_0 = (x_0, t_0)$ and $0 < r \le r_*$. For a.e. time $t \in (t_0 - r^2, t_0)$ we work on the spatial ball $B_r(x_0)$. For $x \in B_r(x_0)$ set $a := \xi(x,t)$.

$$H^{\text{osc}}_{\text{near}}(x,t) = P_a\Big(\frac{1}{4\pi}\,\mathcal{T}_{a,r}\big(\rho(\cdot,t)(\xi(\cdot,t) - a)\big)(x)\Big).$$

Step 1: the oscillation term is a finite sum of commutators. For $x \in B_r(x_0)$ and $y \in B_r(x)$, we have $y \in B_{2r}(x_0)$, so all truncations only sample ρ, ξ inside $B_{2r}(x_0)$. Using

Definition 5.22 with $F_x(\cdot,t)=\rho(\cdot,t)(\xi(\cdot,t)-a)$ gives

$$\mathcal{T}_{a,r}F_x(x,t)=\sum_{m,j=1}^{3} a_m\, T_{m,j,r}\big(\rho(\cdot,t)(\xi_j(\cdot,t)-a_j)\big)(x).$$

Since $a=\xi(x,t)$, $(\xi_j(\cdot,t)-a_j)=\xi_j(\cdot,t)-\xi_j(x,t)$, and hence

$$T_{m,j,r}\big(\rho(\cdot,t)(\xi_j(\cdot,t)-\xi_j(x,t))\big)(x)=\big([T_{m,j,r},\xi_j(\cdot,t)](\rho(\cdot,t)\mathbf{1}_{B_{2r}(x_0)})\big)(x),$$

where $[T,b]f:=T(bf)-b\,Tf$. Using $|\xi_m|\le 1$ and that P_a is a contraction (and absorbing the harmless factor 4π from the definition of $H_{\rm near}^{\rm osc}$), we obtain on $B_r(x_0)$

$$\|H_{\rm near}^{\rm osc}(\cdot,t)\|_{L^{3/2}(B_r(x_0))}\;\lesssim\;\sum_{m,j}\big\|[T_{m,j,r},\xi_j(\cdot,t)](\rho(\cdot,t)\mathbf{1}_{B_{2r}(x_0)})\big\|_{L^{3/2}(B_r(x_0))}.$$

Applying Proposition 5.6 (uniformly in r for truncated CZ operators) and assumption (A) yields

$$\|H_{\rm near}^{\rm osc}(\cdot,t)\|_{L^{3/2}(B_r(x_0))}\;\lesssim\;\eta(2r)\,\|\rho(\cdot,t)\|_{L^{3/2}(B_{2r}(x_0))}.$$

Step 2: Carleson integration. Raising to the 3/2 power, integrating over $t\in(t_0-r^2,t_0)$ and using (B) gives

$$r^{-2}\iint_{Q_r(z_0)}|H_{\rm near}^{\rm osc}|^{3/2}\;\lesssim\;\eta(2r)^{3/2}\,r^{-2}\iint_{Q_{2r}(z_0)}\rho^{3/2}\;\lesssim\;\eta(2r)^{3/2}\,M.$$

Taking the supremum over z_0 and $r\le r_*$ proves the claim. □

Remark 5.24 (What remains for the full $H_{\rm near}$)**.** *Lemma 5.23 controls only the genuine* commutator/oscillation *part of the near-field forcing. The additional constant-direction piece $\frac{1}{4\pi}\mathcal{T}_{\xi(x,t),r}(\rho(\cdot,t)\xi(x,t))$ is known to vanish in the ideal constant-direction case by the divergence-free identity in Remark 5.31, but turning that exact cancellation into* quantitative Carleson smallness *requires (i) rewriting it as a CZ operator on the direction error* $\rho(\xi(x,t)-\xi(\cdot,t))$ plus an explicit far-field remainder*, and (ii) a separate depletion mechanism for that far-field remainder (this is the tail-control obstruction in item (D)).*

5.7. **Tail control: boundedness from critical $L^{3/2}$ control (no smallness).** Even if one grants the critical $\rho^{3/2}$ Carleson bound (B), the far-field/tail contribution is at best *bounded* in the critical Carleson norm. Smallness at small scales does *not* follow from scale-critical control alone.

Lemma 5.25 (A model tail bound via maximal truncations)**.** *Let T be a Calderón–Zygmund operator on $\mathbb{R}^3$ and let $T_{>r}$ denote its standard truncation*

$$T_{>r}f(x):=\int_{|x-y|>r}K(x-y)\,f(y)\,dy.$$

Then for every $1<p<\infty$ there exists C_p such that for all $r>0$,

$$\|T_{>r}f\|_{L^p(\mathbb{R}^3)}\le C_p\,\|f\|_{L^p(\mathbb{R}^3)}.$$

Moreover, the maximal truncation $T_*f:=\sup_{r>0}|T_{>r}f|$ satisfies $\|T_*f\|_{L^p}\le C_p\|f\|_{L^p}$.

Remark 5.26 (Consequence for Carleson norms (boundedness only)). *If ρ has the critical Carleson bound (B), then applying Lemma 5.25 at each time slice and integrating in time yields*

$$\sup_{z_0}\sup_{0<r\le 1} r^{-2}\iint_{Q_r(z_0)} |T_{>r}\rho|^{3/2} \lesssim \sup_{z_0}\sup_{0<r\le 1} r^{-2}\iint_{Q_r(z_0)} \rho^{3/2} < \infty.$$

However, smallness as $r \to 0$ does not follow *from boundedness of the right-hand side. Obtaining small tail forcing requires additional input (e.g. a vanishing-Carleson hypothesis for $\rho^{3/2}$, or a separate far-field depletion mechanism).*

Assumption 5.27 (Tail depletion (minimal form used later)). *Let $H_{\text{sing}} := P_\xi(S\xi)$ and, for each scale $r > 0$, define the near-field contribution H_{near} by restricting the Biot–Savart representation of H_{sing} to $B_r(x)$ (as in Lemma 5.16). Define the far-field/tail remainder by*

$$H_{\text{tail}} := H_{\text{sing}} - H_{\text{near}}.$$

Assume that for the relevant ancient element the tail forcing is small in the critical Carleson norm at sufficiently small scales:

$$\forall \varepsilon > 0\ \exists r_0 > 0 \text{ such that } \sup_{z_0}\sup_{0<r\le r_0} r^{-2}\iint_{Q_r(z_0)} |H_{\text{tail}}|^{3/2}\,dx\,dt \le \varepsilon.$$

Lemma 5.28 (Decomposition of H_{sing} into oscillation and "constant-direction" pieces). *Assume $\omega = \rho\xi$ with $\rho = |\omega|$ and $\xi = \omega/|\omega|$ at time t, and let x satisfy $\omega(x) \neq 0$. Then $H_{\text{sing}}(x) = P_{\xi(x)}((\xi(x)\cdot\nabla)u(x))$ admits the decomposition*

$$H_{\text{sing}}(x) = I_{\text{null}}(x) + I_{\text{const}}(x) + I_{\text{osc}}(x),$$

where (writing $r := x - y$)

$$\begin{aligned}
I_{\text{null}}(x) &:= \frac{1}{4\pi}\,\text{p.v.}\int_{\mathbb{R}^3} \frac{\rho(y)\,\xi(x)\times\xi(y)}{|r|^3}\,dy,\\
I_{\text{const}}(x) &:= -\frac{3}{4\pi}\,\text{p.v.}\int_{\mathbb{R}^3} \frac{(\xi(x)\cdot r)\,\rho(y)\,(r\times\xi(x))}{|r|^5}\,dy,\\
I_{\text{osc}}(x) &:= -\frac{3}{4\pi}\,P_{\xi(x)}\,\text{p.v.}\int_{\mathbb{R}^3} \frac{(\xi(x)\cdot r)\,\rho(y)\,(r\times(\xi(y)-\xi(x)))}{|r|^5}\,dy.
\end{aligned}$$

Moreover, I_{null} vanishes pointwise when $\xi(y) = \xi(x)$, and I_{const} is a fixed Calderón–Zygmund operator on ρ depending only on the frozen direction $\xi(x)$.

Proof. Start from Lemma 5.13 and write $\omega(y) = \rho(y)\xi(y)$. The first term becomes $\xi(x)\times\omega(y) = \rho(y)\,\xi(x)\times\xi(y)$. For the second term, write $r\times\omega(y) = \rho(y)\,r\times\xi(y) = \rho(y)\,r\times\xi(x) + \rho(y)\,r\times(\xi(y)-\xi(x))$. Substituting these into Lemma 5.13 yields the displayed decomposition. Since $r\times\xi(x) \perp \xi(x)$, the projection $P_{\xi(x)}$ does not change I_{const}. $\square$

Lemma 5.29 (Uniform Calderón–Zygmund control and Lipschitz dependence in the frozen direction). *For $a \in S^2$ define the singular integral operator on vector fields $F : \mathbb{R}^3 \to \mathbb{R}^3$ by*

$$(\mathcal{T}_a F)(x) := \text{p.v.}\int_{\mathbb{R}^3} \left(\frac{a\times F(y)}{|x-y|^3} \;-\; 3\,\frac{(a\cdot(x-y))\,((x-y)\times F(y))}{|x-y|^5} \right) dy.$$

Then for every $1 < p < \infty$ there exists $C_p < \infty$ such that uniformly for all $a \in S^2$,

$$\|\mathcal{T}_a F\|_{L^p(\mathbb{R}^3)} \le C_p\,\|F\|_{L^p(\mathbb{R}^3)}.$$

Moreover, there exists $C_p' < \infty$ such that for all $a, b \in S^2$,

$$\|(\mathcal{T}_a - \mathcal{T}_b)F\|_{L^p(\mathbb{R}^3)} \le C_p' \, |a-b| \, \|F\|_{L^p(\mathbb{R}^3)}.$$

Proof. Each $\mathcal{T}_a$ is a vector-valued Calderón–Zygmund operator: the kernel is homogeneous of degree -3, smooth on S^2, and has standard cancellation. The L^p bound follows from the classical CZ theory (uniformly because a only appears as a bounded linear coefficient). For the Lipschitz dependence, note that the kernel difference is linear in $a-b$:

$$(\mathcal{T}_a - \mathcal{T}_b)F(x) = \text{p.v.} \int_{\mathbb{R}^3} \left(\frac{(a-b) \times F(y)}{|x-y|^3} \;-\; 3\, \frac{((a-b)\cdot(x-y))\,((x-y)\times F(y))}{|x-y|^5} \right) dy,$$

so the operator norm is bounded by $C_p'|a-b|$ by the same CZ theory. □

Remark 5.30 (Why this matters for the near-field commutator step)**.** *The manuscript's schematic commutator step is obstructed by the fact that the kernel in H_{near} is "frozen" at $\xi(x)$, i.e. depends on the observation point. Lemma 5.29 is the key analytic ingredient for a CPM-style discretization: on a small ball where ξ has small mean oscillation, one may freeze the direction to a single constant a (e.g. the local average direction), replace $\mathcal{T}_{\xi(x)}$ by $\mathcal{T}_a$, and control the error by $|\xi(x) - a|$. Turning this into a full Carleson smallness proof still requires quantitative control of (i) the constant-direction remainder I_{const}, and (ii) the tail contribution.*

Remark 5.31 (Identifying the "constant-direction" remainder explicitly)**.** *Fix a point x and write $\omega = \rho\,\xi$. If one* freezes the direction *in the second term of Lemma 5.13 by replacing $\xi(y)$ with $\xi(x)$ (i.e. replacing $\omega(y)$ by $\rho(y)\,\xi(x)$), then the first term in Lemma 5.13 vanishes identically and the second term becomes an explicit Calderón–Zygmund operator acting on the scalar amplitude ρ:*

$$\frac{1}{4\pi}\,\text{p.v.} \int_{\mathbb{R}^3} \Big(-3\, \frac{(\xi(x)\cdot(x-y))\,((x-y)\times(\rho(y)\xi(x)))}{|x-y|^5} \Big)\, dy \;=\; \xi(x) \times \nabla\big((\xi(x)\cdot\nabla)(-\Delta)^{-1}\rho\big)(x).$$

This shows that the "average/constant-direction" contribution is a fixed CZ operator on ρ (with coefficients frozen at $\xi(x)$).

Key identity (using $\nabla\cdot\omega = 0$). *Let $a \in S^2$ be any constant vector and assume ω is divergence-free. Then $a\cdot\nabla\rho = \nabla\cdot(\rho a) = \nabla\cdot(\rho a - \omega)$ and hence*

$$a \times \nabla\big((a\cdot\nabla)(-\Delta)^{-1}\rho\big) \;=\; a \times \nabla(-\Delta)^{-1}\nabla\cdot(\rho a - \omega).$$

In particular, if $\omega = \rho\,\xi$ and $a = \xi(x)$, then $\rho a - \omega = \rho(\xi(x) - \xi)$, so the constant-direction term is itself a CZ operator applied to the direction error *$\rho(\xi(x)-\xi)$. Thus it is not an independent "extra forcing"; it is another oscillation-type contribution (but requires a weighted smallness mechanism to be made quantitative).*

Consistency check. *If ξ is* exactly constant *and $\omega = \rho\,\xi$ is divergence-free, then $\xi\cdot\nabla\rho = 0$ (so ρ is constant along the ξ-direction), and in Fourier variables $(\xi\cdot\nabla)(-\Delta)^{-1}\rho \equiv 0$; hence the right-hand side vanishes, as it must since $H_{\text{sing}} = P_\xi(S\xi)$ must be 0 when $\nabla\xi \equiv 0$.*

Remark 5.32 (What this buys (and what remains))**.** *Lemma 5.13 makes it clear that* at least one *part of H_{sing} is a "null form" in the directions: $\xi(x)\times\omega(y) = \rho(y)\,\xi(x)\times\xi(y)$, hence it vanishes when $\xi(y) = \xi(x)$. The second term does not exhibit such a factor directly and is a main obstruction to turning the schematic CRW commutator step into a complete proof.*

5.8. **A related classical bound: Carleson control of an extension energy (boundedness, not smallness).** Several proof-track files (notably `A3_Carleson_HalfNorm_proof.txt`) attempt to bound a Caffarelli–Silvestre-type extension energy of $|\omega|$ in a Carleson measure norm. Even when correct, such bounds yield *boundedness* of a scale-invariant quantity, not the *smallness* required in (D). We record the clean boundedness statement below because it is frequently used as an intermediate step.

Definition 5.33 (Parabolic extension energy)**.** *Let $f(x,t)$ be a scalar function on $\mathbb{R}^3 \times I$ and let $F(x,z,t)$ be its harmonic extension in the auxiliary variable $z > 0$:*

$$-(\Delta_x + \partial_{zz})F = 0 \quad (z > 0), \qquad F(x,0,t) = f(x,t).$$

For $r > 0$ define the local extension energy

$$E_r(x_0,t) := \int_{B_r(x_0)} \int_0^r z\,|\nabla_{x,z}F(x,z,t)|^2\,dz\,dx.$$

Proposition 5.34 (Time-averaged Carleson bound from an enstrophy bound)**.** *Assume $f(\cdot,t) \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ for a.e. t and that for some $K < \infty$,*

$$\sup_{x_0\in\mathbb{R}^3}\sup_{0<r\le 1} r^{-1}\iint_{Q_r(x_0,t_0)} |\nabla_x f(x,t)|^2\,dx\,dt \le K \qquad \text{for all } t_0.$$

Then there exists $C = C(3)$ such that for all x_0, t_0 and $0 < r \le 1$,

$$r^{-1}\int_{t_0-r^2}^{t_0} E_r(x_0,t)\,dt \le C\,K.$$

Proof (standard trace/extension estimate). For each fixed t, by the Caffarelli–Silvestre extension characterization of the $\dot{H}^{1/2}$ seminorm on $\mathbb{R}^3$, the quantity $E_r(x_0,t)$ is comparable (up to universal constants) to the localized Gagliardo seminorm of $f(\cdot,t)$ on $B_r(x_0)$. Localizing and using a cutoff to avoid boundary effects, one obtains an estimate of the form

$$E_r(x_0,t) \;\le\; C\int_{B_{2r}(x_0)} |\nabla_x f(x,t)|^2\,dx,$$

with C independent of $r \le 1$. Integrating this in time over $(t_0 - r^2, t_0)$ gives

$$\int_{t_0-r^2}^{t_0} E_r(x_0,t)\,dt \le C\iint_{Q_{2r}(x_0,t_0)} |\nabla_x f|^2\,dx\,dt \le C\,K\,(2r),$$

where the last step is the assumed enstrophy-type bound. Dividing by r yields the claim. □

Remark 5.35. *Taking $f = |\omega|$ (or $f = \omega$ componentwise) makes Proposition 5.34 relevant to the proof track. However, this gives only a* finite *Carleson norm, not the* smallness *demanded by (D) for the forcing H.*

6. Item (E): 2D classification / Liouville step

Proposition 6.1 (Target statement (E))**.** *If ξ^∞ is constant, then the tangent flow reduces to a 2D ancient Navier–Stokes flow and must be trivial, yielding a contradiction with non-triviality of the tangent flow.*

Remark 6.2 (Status)**.** *There are classical Liouville theorems for* bounded *ancient 2D solutions (e.g. KNSS), but the blow-up/compactness construction in* `new-version-12-11.tex` *only yields local energy and local L^3 bounds, not global boundedness. Additional hypotheses are required to apply existing 2D Liouville theorems.*

Lemma 6.3 (2D Liouville from bounded vorticity and sublinear growth)**.** *Let v be a smooth ancient solution of the 2D Navier–Stokes equations on $\mathbb{R}^2 \times (-\infty, 0]$ and let $\alpha = \partial_1 v_2 - \partial_2 v_1$ be its scalar vorticity. Assume $\alpha \in L^\infty(\mathbb{R}^2 \times (-\infty, 0])$ and that there exist constants $C > 0$ and $\beta \in [0, 1)$ such that*

$$|v(x,t)| \le C\,(1 + |x|^\beta) \qquad \text{for all } (x,t) \in \mathbb{R}^2 \times (-\infty, 0].$$

Then $\alpha \equiv 0$ and $v(x,t) = b(t)$ is spatially constant.

Proof. This is the same integral-contradiction argument as in the 2D case of [1]. Assume $M_1 := \sup \alpha > 0$. Using the maximum-principle stability lemma (Lemma 2.1 in [1]), there exist arbitrarily large cylinders $Q_R = B(\bar{x}, R) \times (\bar{t} - R^2, \bar{t})$ with $\alpha \ge M_1/2$ on Q_R, hence $\iint_{Q_R} \alpha \ge cM_1R^4$. On the other hand, Stokes' theorem gives $\int_{B(\bar{x},R)} \alpha = \int_{\partial B(\bar{x},R)} v \cdot \tau$ and the growth bound yields $\int_{B(\bar{x},R)} \alpha(\cdot, t) \le CR(1 + R^\beta)$ for each t. Integrating in time over length R^2 gives $\iint_{Q_R} \alpha \le CR^3(1 + R^\beta) = o(R^4)$ as $R \to \infty$ (since $\beta < 1$), a contradiction. Thus $\sup \alpha \le 0$. Applying the same argument to $-\alpha$ yields $\inf \alpha \ge 0$, hence $\alpha \equiv 0$. Then $\mathrm{curl} v = 0$ and $\mathrm{div} v = 0$, so $v(\cdot, t)$ is harmonic for each t. Sublinear growth forces $v(\cdot, t)$ to be constant. □

Proposition 6.4 (Running-max style E2 gate via attainment of the frozen supremum)**.** *Let ρ be a smooth bounded solution of the 2D vorticity equation*

$$\partial_t \rho + v \cdot \nabla \rho = \nu \Delta \rho \qquad \text{on } \mathbb{R}^2 \times (-\infty, 0],$$

where v is a smooth divergence-free 2D velocity field. Assume $\rho \ge 0$, $0 \le \rho \le 1$, and the supremum is frozen*:*

$$\sup_{x \in \mathbb{R}^2} \rho(x,t) = 1 \quad \text{for all } t \le 0.$$

If there exists $t_0 < 0$ such that $\rho(\cdot, t_0) \in L^p(\mathbb{R}^2)$ for some finite $p \in [1, \infty)$, then this is impossible.

Proof. Since $\rho(\cdot, t_0)$ is smooth, it is uniformly continuous. If it did not decay to 0 at infinity, one could find disjoint balls on which $\rho(\cdot, t_0) \ge c > 0$, contradicting $\rho(\cdot, t_0) \in L^p$. Hence $\rho(\cdot, t_0) \to 0$ as $|x| \to \infty$, so the supremum 1 is attained at some x_0 with $\rho(x_0, t_0) = 1$. The strong maximum principle then forces $\rho \equiv 1$ on $\mathbb{R}^2 \times [t_0, 0]$, contradicting $\rho(\cdot, t_0) \in L^p(\mathbb{R}^2)$. □

Lemma 6.5 (Running-max upgrade: bounded vorticity forces vanishing of the u_3 linear mode)**.** *Let (u, p) be a smooth ancient Navier–Stokes solution on $\mathbb{R}^3 \times (-\infty, 0]$ with constant vorticity direction $\xi \equiv e_3$, so that $\omega = (0, 0, \rho)$ with $\rho \ge 0$. Assume the global bound $\|\omega\|_{L^\infty(\mathbb{R}^3 \times (-\infty, 0])} \le M$ and nontriviality $\rho(0, 0) > 0$. If $u_3(x,t) = a(t) + b(t)x_3$ with b satisfying $\dot{b} + b^2 = 0$, then $b(0) = 0$, hence $b(t) \equiv 0$ for all $t \le 0$.*

Proof. If $b(0) > 0$, then $b(t) = \frac{b(0)}{1+b(0)t}$ blows up at a finite negative time, contradicting ancientness. Assume $b_0 := b(0) < 0$ and set $B(t) = \int_0^t b(s)\,ds = \log(1 + b_0 t)$. The scalar amplitude $\rho = \omega_3$ solves

$$\partial_t \rho - \nu \Delta \rho + u \cdot \nabla \rho = b(t)\rho.$$

Defining $\widetilde{\rho}(x,t) = e^{-B(t)}\rho(x,t)$ removes the reaction term:

$$\partial_t\widetilde{\rho} - \nu\Delta\widetilde{\rho} + u\cdot\nabla\widetilde{\rho} = 0.$$

By the parabolic maximum principle, $\|\widetilde{\rho}(\cdot,t)\|_{L^\infty}$ is non-increasing forward in time, so for any $t<0$,

$$\|\rho(\cdot,0)\|_{L^\infty} \le e^{-B(t)}\|\rho(\cdot,t)\|_{L^\infty} \le \frac{M}{1+b_0 t}.$$

Letting $t\to-\infty$ forces $\|\rho(\cdot,0)\|_{L^\infty}=0$, contradicting $\rho(0,0)>0$. Thus $b_0<0$ is impossible and $b(0)=0$. □

Remark 6.6 (Interpretation)**.** *Lemma 6.5 closes the "linear-mode" obstruction (E1)* provided one has a global vorticity bound. *This is available for the* running-max *ancient element, but it is* not *a property of a general CKN tangent flow.*

6.1. **What *can* be proved from "constant direction".** We isolate the purely geometric reduction and make explicit what extra hypothesis is needed to conclude true 2D dynamics.

Lemma 6.7 (Constant vorticity direction forces $\partial_e\omega\equiv 0$)**.** *Let $\omega:\mathbb{R}^3\to\mathbb{R}^3$ be a (distributional) vorticity field with $\nabla\cdot\omega=0$. Assume there exists a fixed unit vector $e\in\mathbb{S}^2$ such that $\omega(x)=\rho(x)\,e$ a.e. Then $\partial_e\rho=0$ in the sense of distributions. Equivalently, in coordinates with $e=e_3$, one has $\partial_3\omega_3=0$ and hence $\omega_3=\omega_3(x_1,x_2)$ is independent of x_3.*

Proof. Since $\omega=\rho e$ and e is constant, $\nabla\cdot\omega=\nabla\cdot(\rho e)=e\cdot\nabla\rho$. Thus $0=\nabla\cdot\omega=e\cdot\nabla\rho=\partial_e\rho$ in distributions. □

Lemma 6.8 (Reduction to 2D under a mild growth/decay hypothesis)**.** *Let $u:\mathbb{R}^3\to\mathbb{R}^3$ be divergence-free, and set $\omega=\mathrm{curl}u$. Assume $\omega(x)=\rho(x)\,e_3$ with $\nabla\cdot\omega=0$. Then:*

(1) *$\rho=\rho(x_1,x_2)$ is independent of x_3 (by Lemma 6.7);*
(2) *$-\Delta u=\mathrm{curl}\omega=(\partial_2\rho,\,-\partial_1\rho,\,0)$ in distributions, hence $\partial_3 u$ is harmonic:*

$$-\Delta(\partial_3 u)=\partial_3(\mathrm{curl}\omega)=0.$$

In particular, if one additionally assumes (for example) that $\partial_3 u\in L^2(\mathbb{R}^3)$, then $\partial_3 u\equiv 0$ and u is independent of x_3.

Proof. The identity $-\Delta u=\mathrm{curl}\omega$ follows from $\mathrm{curlcurl}u=\nabla(\nabla\cdot u)-\Delta u$ and $\nabla\cdot u=0$. Since ρ is independent of x_3, so is $\mathrm{curl}\omega$, which implies $-\Delta(\partial_3 u)=0$. If $\partial_3 u\in L^2(\mathbb{R}^3)$, each component is a harmonic L^2 function and must be 0. □

Remark 6.9 (Remaining gap inside (E))**.** *Lemma 6.8 shows where the extra hypothesis enters: to deduce true 2D dynamics from constant direction one needs a global control that rules out nontrivial harmonic "x_3-dependence" in $\partial_3 u$. The current blow-up construction in* `new-version-12-11.tex` *does not provide such global decay/boundedness.*

Remark 6.10 (Gauge viewpoint (Recognition Geometry analogy))**.** *At a* fixed time*, a velocity field is not uniquely determined by its vorticity unless one imposes a gauge/normalization at infinity: one may add divergence-free, curl-free (harmonic gradient) fields without changing ω. In the constant-direction regime, the potential x_3-dependence in u is exactly such a "hidden gauge" component. This matches the Recognition Geometry distinction between* indistinguishability *(same event/vorticity) and* gauge equivalence *(different configurations with the same observables).*

For the NS closure program, item (E) therefore needs an explicit gauge-fixing hypothesis (e.g. decay at spatial infinity so Biot–Savart uniquely recovers u from ω), or a separate argument showing the harmonic component is dynamically ruled out for ancient tangent flows.

7. Conclusion

Items (A)–(E) cannot currently be completed as unconditional proofs from the standard suitable weak solution framework and blow-up compactness alone. The conditional versions recorded here isolate precisely the additional scale-critical inputs required to close the geometric depletion program.

8. Appendix: CPM/RS-inspired alternate closure track (BMO^{-1} gate)

Remark 8.1 (Motivation)**.** *The main manuscript's current closure route attempts to force ξ^∞ to be constant and then reduce to a 2D ancient flow (item (E)). An alternative, suggested by the CPM Navier–Stokes instantiation in `CPM.tex`, is a* small-data gate*: if one can find a time slice t_* near the blow-up time with $\|u(\cdot,t_*)\|_{\mathrm{BMO}^{-1}}$ sufficiently small, then the Koch–Tataru theory produces a global smooth solution forward from t_*, ruling out blow-up.*

Remark 8.2 (Where this appears in CPM/RS documents)**.** *The CPM blueprint is written explicitly in `CPM.tex`, § "Navier–Stokes Instantiation (Critical Vorticity Route)": it introduces the critical vorticity window functional $\mathcal{W}(x,t;r) = r^{-2}\iint_{Q_r(x,t)}|\omega|^{3/2}$, states a "slice bridge" to a small BMO^{-1} time slice, and then applies the Koch–Tataru small-data gate. The same NS(BMO^{-1} gate) and the characteristic exponent $2/3$ are recorded in `Source-Super.txt` under `@CPM_METHOD`.*

Remark 8.3 (Lean alignment (abstract, not analytic))**.** *The abstract CPM "gate" skeleton is mirrored in Lean in `_external/reality/IndisputableMonolith/Verification/CPMBridge/Domain/NavierStokes.lean`: it packages* assumptions *(projection defect, energy control, dispersion/slice interface) into a model and derives the corresponding coercivity/aggregation inequalities via the general `LawOfExistence` theorems. This does* not *formalize the analytic Navier–Stokes estimates (e.g. the slice bridge 8.5); it only ensures that once those domain inequalities are provided, the CPM constant algebra is mechanically consistent.*

Definition 8.4 (Critical vorticity window functional)**.** *For a spacetime point (x,t) and radius $r>0$, define*

$$\mathcal{W}(x,t;r) \;:=\; r^{-2}\iint_{Q_r(x,t)} |\omega|^{3/2}\,dx\,ds.$$

Assumption 8.5 (Slice bridge to BMO^{-1})**.** *There exists an absolute constant C_B such that the following holds. If on a unit window (t_0-1,t_0) one has*

$$\sup_{(x,t)\in\mathbb{R}^3\times(t_0-1,t_0)}\ \sup_{0<r\le 1}\ \mathcal{W}(x,t;r) \;\le\; \varepsilon,$$

then there exists $t_*\in(t_0-\frac{1}{2},t_0)$ such that

$$\|u(\cdot,t_*)\|_{\mathrm{BMO}^{-1}} \;\le\; C_B\,\varepsilon^{2/3}.$$

Assumption 8.6 (Small-data gate in BMO^{-1} (Koch–Tataru))**.** *There exists $\varepsilon_{\mathrm{SD}} > 0$ such that if $\|u(\cdot, t_*)\|_{\mathrm{BMO}^{-1}} \le \varepsilon_{\mathrm{SD}}$ for a (divergence-free) time slice of a suitable weak solution, then the corresponding mild solution exists globally forward in time and is smooth for $t > t_*$.*

Proposition 8.7 (Conditional "gate" contradiction principle)**.** *Assume 8.5 and 8.6. If there exists a blow-up time T^* such that for some $t_0 \uparrow T^*$ the windowed critical vorticity functional satisfies*

$$\sup_{(x,t)\in\mathbb{R}^3\times(t_0-1,t_0)} \ \sup_{0<r\le 1} \mathcal{W}(x,t;r) \ \le \ \varepsilon \quad \text{with} \quad C_B\,\varepsilon^{2/3} \le \varepsilon_{\mathrm{SD}},$$

then T^* cannot be a singular time.

Proof. By 8.5, there exists $t_* \in (t_0 - \frac{1}{2}, t_0)$ with $\|u(\cdot, t_*)\|_{\mathrm{BMO}^{-1}} \le C_B \varepsilon^{2/3} \le \varepsilon_{\mathrm{SD}}$. Then 8.6 yields a smooth solution for all $t \ge t_*$, contradicting blow-up at $T^* > t_*$. $\square$

Remark 8.8 (Status / what remains)**.** *This appendix is* not *an unconditional closure: the hard inputs are (i) proving the slice bridge 8.5 at the level needed for suitable weak solutions, and (ii) proving that a hypothetical singularity forces the critical vorticity window defect to be small on some final window. Nevertheless, it provides a clean CPM-style "gate" target that could potentially replace item (E) if the depletion machinery can be upgraded to produce smallness of $\mathcal{W}$.*

References

[1] G. Koch, N. Nadirashvili, G. Seregin, and V. Šverák, *Liouville theorems for the Navier–Stokes equations and applications*, Acta Math. **203** (2009), no. 1, 83–105.