# Bergman-Scale Holomorphic Manufacturing of Prescribed Tangent Templates in Projective Kähler Manifolds

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**Abstract**

Let $X$ be a smooth complex projective manifold with an ample line bundle $L$ whose curvature form is a Kähler form $\omega$. We develop a quantitative local existence theory for holomorphic complete intersections in large tensor powers $L^{\otimes m_{\mathrm{hol}}}$ that realize prescribed geometric templates on the natural Bergman scale $m_{\mathrm{hol}}^{-1/2}$.

Given a point $x \in X$ and a target complex $(n-p)$-plane $\Pi \subset T_xX$, we construct holomorphic sections $\sigma_1, \ldots, \sigma_p \in H^0(X, L^{\otimes m_{\mathrm{hol}}})$ such that the local zero set

$$Y = \{\sigma_1 = \cdots = \sigma_p = 0\}$$

is smooth near $x$, is calibrated by the Kähler calibration $\psi = \omega^{n-p}/(n-p)!$, and on a ball $B_{c\, m_{\mathrm{hol}}^{-1/2}}(x)$ is a single $C^1$ graph over $\Pi$ with arbitrarily small slope for $m_{\mathrm{hol}}$ large.

We then prove a finite-template realization theorem: for any finite set of transverse translation parameters $\{t_a\}$ at scale $O(m_{\mathrm{hol}}^{-1/2})$, one can realize a family of pairwise disjoint local sheets, each a single small-slope graph over the corresponding translated plane, with uniform mass comparability and anchor-based disjointness. These results provide a reusable "holomorphic manufacturing" module for microstructured calibrated-current constructions.

## 1 Introduction

The goal of this paper is simple to state: given a local *template* (a point, a complex tangent direction, and optionally a small transverse translation), we want a *genuine holomorphic* codimension-$p$ submanifold whose local geometry matches that template on a quantitative scale.

The scale that naturally appears in this setting is the *Bergman scale* $m_{\mathrm{hol}}^{-1/2}$ associated to large tensor powers $L^{\otimes m_{\mathrm{hol}}}$ of a positive line bundle. On this scale, holomorphic sections behave like model Bargmann–Fock functions, and one can force prescribed first-order behavior (jets) while keeping uniform $C^1$ control on balls of radius $c\, m_{\mathrm{hol}}^{-1/2}$.

### A Recognition Geometry viewpoint (optional but compatible)

In Recognition Geometry language, one may regard a template as a configuration and a manufactured holomorphic object as an event. Concretely, for each $m_{\mathrm{hol}}$ one can take a configuration space

$$\mathcal{C}_{m_{\mathrm{hol}}} := \{(x, \Pi, t) : x \in X,\ \Pi \subset T_x X \text{ a complex } (n-p)\text{-plane},\ t \in \Pi^{\perp} \text{ small}\},$$

and an event space $\mathcal{E}_{m_{\mathrm{hol}}}$ consisting of $p$-tuples of sections in $H^0(X, L^{\otimes m_{\mathrm{hol}}})$ (modulo irrelevant rescalings). The "manufacturing" theorems below assert that there exists a (non-unique) realization map $R_{m_{\mathrm{hol}}} : \mathcal{C}_{m_{\mathrm{hol}}} \to \mathcal{E}_{m_{\mathrm{hol}}}$ with quantitative stability properties on the Bergman scale. Nothing in the proofs depends on this framing; it is included only to keep symbols consistent with the broader Recognition Geometry program.

## 2 Setup and notation

Let $X$ be a smooth complex projective manifold of complex dimension $n$. Let $L \to X$ be an ample holomorphic line bundle equipped with a smooth Hermitian metric $h$ whose Chern curvature form equals a Kähler form $\omega$. We fix an integer $p$ with $1 \le p \le n$ and set $k := 2(n-p)$ for the real dimension of the sheets we build.

We write

$$\psi := \frac{\omega^{n-p}}{(n-p)!}$$

for the standard Kähler calibration of degree $2(n-p)$.

Let $dV_\omega := \omega^n/n!$ be the Riemannian volume form, and let $\|\cdot\|$ denote norms induced by the Kähler metric. For $x \in X$ and $r > 0$, $B_r(x)$ denotes the geodesic ball of radius $r$.

For each $m_{\mathrm{hol}} \in \mathbb{N}$ we consider the finite-dimensional complex vector space

$$H^0(X, L^{\otimes m_{\mathrm{hol}}})$$

of global holomorphic sections. We equip it with the $L^2$ inner product

$$\langle s,t\rangle_{L^2} := \int_X h^{\otimes m_{\mathrm{hol}}}(s,t)\,dV_\omega.$$

**Remark 1** (Parameter convention)**.** The integer $m_{\mathrm{hol}}$ is the *holomorphic/Bergman tensor-power parameter.* The intrinsic analytic length scale in this paper is $\asymp m_{\mathrm{hol}}^{-1/2}$.

# 3 Quantitative $C^1$ section control on Bergman balls

Two classical principles drive everything that follows:

- For $m_{\mathrm{hol}}$ large, one can realize prescribed low-order jets at points by global sections of $L^{\otimes m_{\mathrm{hol}}}$.
- On the Bergman scale $m_{\mathrm{hol}}^{-1/2}$, certain canonical sections (built from the Bergman kernel) have uniformly controlled derivatives.

We package the first principle as a clean algebraic lemma and the second as an analytic lemma that provides the $C^1$ control needed for graph theorems.

**Lemma 1** (Jet surjectivity for ample powers (pointwise and for finite sets))**.** *Fix an integer $q \geq 1$ and a finite set $S \subset X$. There exists $m_{\mathrm{hol}0} = m_{\mathrm{hol}0}(q,S)$ such that for all $m_{\mathrm{hol}} \geq m_{\mathrm{hol}0}$, the natural evaluation map on $q$-jets*

$$H^0(X, L^{\otimes m_{\mathrm{hol}}}) \longrightarrow \bigoplus_{x\in S} J_x^q(L^{\otimes m_{\mathrm{hol}}})$$

*is surjective. In particular, for $q=1$, prescribed values and first derivatives can be realized simultaneously at finitely many points for all sufficiently large $m_{\mathrm{hol}}$.*

*Proof.* Fix $q \geq 1$ and a finite set $S$. Let $\mathcal{I}$ be the coherent ideal sheaf of the fat point subscheme $Z := \sum_{x\in S}(q+1)\,x$, so $\mathcal{O}_X/\mathcal{I} \cong \bigoplus_{x\in S}\mathcal{O}_X/\mathfrak{m}_x^{q+1}$. Tensor the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$$

by $L^{\otimes m_{\mathrm{hol}}}$ and take global sections. Since $L$ is ample and $\mathcal{I}$ is coherent, Serre vanishing gives $H^1(X, L^{\otimes m_{\mathrm{hol}}} \otimes \mathcal{I}) = 0$ for all $m_{\mathrm{hol}}$ sufficiently large. The long exact sequence then shows

$$H^0(X, L^{\otimes m_{\mathrm{hol}}}) \twoheadrightarrow H^0\big(X, L^{\otimes m_{\mathrm{hol}}} \otimes \mathcal{O}_X/\mathcal{I}\big) \cong \bigoplus_{x\in S} J_x^q(L^{\otimes m_{\mathrm{hol}}}),$$

which is the desired surjectivity. $\square$

The next lemma is the analytic workhorse. It asserts that one can impose first-order data at a point *and* keep the differential nearly constant on a ball of radius $c\,m_{\mathrm{hol}}^{-1/2}$.

**Lemma 2** (Uniform $C^1$ control on $m_{\mathrm{hol}}^{-1/2}$-balls via Bergman kernels)**.** *Fix $\varepsilon \in (0,1)$. There exist constants $c = c(X,\omega) > 0$ and $m_{\mathrm{hol}1} = m_{\mathrm{hol}1}(\varepsilon)$ such that for every $m_{\mathrm{hol}} \ge m_{\mathrm{hol}1}$, every $x \in X$, and every collection of covectors*

$$\lambda_1, \dots, \lambda_p \in T_x^{*(1,0)}X,$$

*there exist sections $s_1, \dots, s_p \in H^0(X, L^{\otimes m_{\mathrm{hol}}})$ with:*

1. *$s_i(x) = 0$ and $ds_i(x) = \lambda_i$ for each $i$;*
2. *on the geodesic ball $B_{c\,m_{\mathrm{hol}}^{-1/2}}(x)$,*

$$\|ds_i(y) - \lambda_i\| \le \varepsilon \max_{1\le j\le p} \|\lambda_j\| \qquad \textit{for all } y \in B_{c\,m_{\mathrm{hol}}^{-1/2}}(x).$$

*Proof sketch.* Because $X$ is compact, $\mathrm{inj}(X) > 0$. Work in holomorphic normal coordinates $z = (z^1, \dots, z^n)$ centered at $x$ on a fixed-radius ball, and in a local unitary holomorphic frame $e_L$ of $L$ on that ball, so that $|e_L|_h^2 = e^{-\phi}$ with $\phi(0) = 0$ and $d\phi(0) = 0$. Let $P_{m_{\mathrm{hol}}}(z, z')$ denote the Bergman kernel of $(L^{\otimes m_{\mathrm{hol}}}, h^{\otimes m_{\mathrm{hol}}})$ in that frame. A standard near-diagonal Bergman kernel asymptotic, written in rescaled variables $Z = \sqrt{m_{\mathrm{hol}}}\,z$, implies that the weighted kernel is $C^2$-close (uniformly for $|Z|, |Z'|$ bounded) to the Bargmann–Fock kernel on $\mathbb{C}^n$, with error $O(m_{\mathrm{hol}}^{-1/2})$.

Differentiate the kernel in the antiholomorphic variable at $z' = 0$ to build "peak-derivative" sections $u_{a,m_{\mathrm{hol}}} \in H^0(X, L^{\otimes m_{\mathrm{hol}}})$ $(a = 1, \dots, n)$ whose local representatives satisfy

$$u_{a,m_{\mathrm{hol}}}(z) = z^a + O(|z|^2) + O(m_{\mathrm{hol}}^{-1/2}|z|), \qquad du_{a,m_{\mathrm{hol}}}(z) = dz^a + O(|z|) + O(m_{\mathrm{hol}}^{-1/2})$$

uniformly on $|z| \le c\,m_{\mathrm{hol}}^{-1/2}$ for some fixed $c > 0$. In particular, $\|du_{a,m_{\mathrm{hol}}}(y) - du_{a,m_{\mathrm{hol}}}(x)\| \le C\,m_{\mathrm{hol}}^{-1/2}$ on $B_{c\,m_{\mathrm{hol}}^{-1/2}}(x)$.

Given $\lambda_i = \sum_a \lambda_{i,a}\,dz^a$, solve a uniformly well-conditioned linear system at $x$ to choose coefficients $c_{i,a}$ so that $s_i := \sum_a c_{i,a} u_{a,m_{\mathrm{hol}}}$ satisfies $ds_i(x) = \lambda_i$ and $s_i(x) = 0$. The uniform $C^1$ variation bound for the $u_{a,m_{\mathrm{hol}}}$ then yields the stated estimate for $ds_i(y)$ when $m_{\mathrm{hol}}$ is large enough. $\square$

# 4 From uniform gradient control to a single-sheet $C^1$ graph

We now record a purely local holomorphic implicit-function lemma in a form tailored to manufacturing.

**Lemma 3** (Graph control from uniform gradient control)**.** *Let $U \subset \mathbb{C}^n$ be a ball and write $\mathbb{C}^n = \mathbb{C}^{n-p} \times \mathbb{C}^p$ with coordinates $(u, w)$. Let $s_1, \ldots, s_p : U \to \mathbb{C}$ be holomorphic functions with $s_i(0) = 0$ and suppose that for some $\varepsilon \in (0, 1/10)$,*

$$\sup_{y \in U} \|ds_i(y) - dw_i\| \leq \varepsilon \qquad \textit{for all } i = 1, \ldots, p.$$

*Then, after possibly shrinking $U$ to a concentric subball, the common zero set*

$$Y := \{s_1 = \cdots = s_p = 0\} \cap U$$

*is a smooth complex submanifold and is a single $C^1$ graph over the base plane $\{w = 0\}$. More precisely, on a concentric subball $U_{1/2}$ there exists a $C^1$ map $g : \{w = 0\} \cap U_{1/2} \to \mathbb{C}^p$ such that*

$$Y \cap U_{1/2} = \{(u, g(u)) : \ u \in \mathbb{C}^{n-p}, \ (u, 0) \in U_{1/2}\}, \qquad \|Dg\|_{C^0} \leq C_{\mathrm{graph}} \varepsilon,$$

*where $C_{\mathrm{graph}}$ is an absolute constant depending only on $(n, p)$.*

*Proof.* Let $S = (s_1, \ldots, s_p) : U \to \mathbb{C}^p$. The hypothesis implies that $\partial_w S$ is uniformly $O(\varepsilon)$-close to the identity matrix, so it is invertible throughout $U$ if $\varepsilon$ is small enough. Hence $S$ is a submersion, so $Y = S^{-1}(0)$ is a smooth complex submanifold.

The same invertibility implies the holomorphic implicit function theorem applies to solve $S(u, w) = 0$ for $w$ as a function of $u$ on a smaller ball; this gives a $C^1$ map $g$ with $Y$ equal to the graph of $g$. Differentiating $S(u, g(u)) = 0$ yields $Dg = -(\partial_w S)^{-1} \partial_u S$, and the uniform closeness estimates bound $\|Dg\|$ by $C_{\mathrm{graph}} \varepsilon$. $\square$

# 5 Tangential manufacturing on Bergman balls

We now combine the Bergman $C^1$ control with the graph lemma to manufacture holomorphic complete intersections with prescribed tangent plane.

**Theorem 1** (Projective tangential manufacturing with $C^1$ control)**.** *Let $x \in X$ and let $\Pi \subset T_xX$ be a complex $(n-p)$-plane. For every $\varepsilon \in (0,1)$ there exists $m_{\mathrm{hol2}} = m_{\mathrm{hol2}}(\varepsilon)$ such that for all $m_{\mathrm{hol}} \geq m_{\mathrm{hol2}}$ there exist holomorphic sections*

$$\sigma_1, \dots, \sigma_p \in H^0(X, L^{\otimes m_{\mathrm{hol}}})$$

*with the following properties. Let*

$$Y := \{\sigma_1 = \cdots = \sigma_p = 0\} \subset X.$$

*Then:*

1. *$x \in Y$, and $Y$ is smooth on $B_{c\,m_{\mathrm{hol}}^{-1/2}}(x)$ (for a uniform constant $c > 0$).*
2. *On $B_{c\,m_{\mathrm{hol}}^{-1/2}}(x)$, the set $Y$ is a single $C^1$ graph over $\Pi$, with slope at most $C\varepsilon$ for a constant $C$ depending only on $(n,p)$ and the ambient geometry.*
3. *On its regular locus, $Y$ is calibrated by $\psi = \omega^{n-p}/(n-p)!$. In particular, on $B_{c\,m_{\mathrm{hol}}^{-1/2}}(x)$ one has $\psi|_Y =$ (Riemannian volume form of $Y$), so the local sheet is volume-minimizing in its homology class within that ball.*

*Proof.* Choose holomorphic normal coordinates $z$ at $x$ identifying a neighborhood with a ball in $\mathbb{C}^n$ and making $\omega_x$ standard. Choose unitary coordinates so that, at $x$, the plane $\Pi$ corresponds to $\{w = 0\} \subset \mathbb{C}^{n-p} \times \mathbb{C}^p$.

Apply the Bergman $C^1$ lemma with covectors $\lambda_i := dw_i \in T_x^{*(1,0)}X$. For $m_{\mathrm{hol}}$ large enough, this produces sections $\sigma_i$ with $\sigma_i(x) = 0$ and with $d\sigma_i$ uniformly $\varepsilon$-close to $dw_i$ on $B_{c\,m_{\mathrm{hol}}^{-1/2}}(x)$. In local coordinates, this matches the hypotheses of the graph-control lemma (after shrinking to a concentric subball), so the common zero set is a single $C^1$ graph over $\{w = 0\}$, hence over $\Pi$, with slope $O(\varepsilon)$.

Finally, $Y$ is a complex analytic complete intersection, so its regular locus is a complex submanifold. For a Kähler metric, complex submanifolds of complex dimension $n-p$ are calibrated by $\psi = \omega^{n-p}/(n-p)!$ (the Wirtinger identity), giving the calibration claim. □

# 6 Whole-cell single-sheet control

In many mesh-based applications, one does not work on an infinitesimal ball but on a cell whose diameter is comparable to the Bergman scale. The previous theorem already gives this if the cell fits inside the Bergman ball.

**Proposition 1** (Single-sheet control on a cube of size $\asymp m_{\mathrm{hol}}^{-1/2}$)**.** *Fix $\varepsilon \in (0,1)$ and choose $m_{\mathrm{hol}} \geq m_{\mathrm{hol2}}(\varepsilon)$ as in the tangential manufacturing theorem. Let $Q \subset X$ be a coordinate cube (in a holomorphic normal chart) with diameter*

$$\mathrm{diam}(Q) \leq \tfrac{c}{2}\, m_{\mathrm{hol}}^{-1/2},$$

*and suppose $x \in Q$. Then the manufactured $Y$ from the theorem satisfies: $Y \cap Q$ is a single $C^1$ graph over $\Pi \cap Q$, with slope $O(\varepsilon)$.*

*Proof.* By the diameter assumption, $Q \subset B_{c\, m_{\mathrm{hol}}^{-1/2}}(x)$. On that ball, $Y$ is already a single graph over $\Pi$, so restricting to $Q$ preserves single-sheetness and the slope bound. □

# 7 Finite translation template realization

We now allow a finite family of transverse translation parameters and manufacture *multiple* local sheets, one per translation, with a clean disjointness criterion.

**Definition 1** (Local translation parameters and anchors)**.** Work in a holomorphic chart with coordinates $(u, w) \in \mathbb{C}^{n-p} \times \mathbb{C}^p$, and let $P := \{w = 0\}$. A *translation parameter* is a vector $t \in \mathbb{C}^p$, corresponding to the affine plane $P + t := \{w = t\}$. An *anchor point* for $t$ is a point $x_t$ in the chart satisfying $w(x_t) = t$.

**Lemma 4** (Mass comparability and anchor-based disjointness for small-slope graphs)**.** *Let $Q \subset \mathbb{R}^{2n}$ be a cube of diameter $h$, and let $P$ be an affine $k$-plane with $k = 2(n-p)$.*

1. *(*Mass comparability*) If $Y \cap Q$ is a $C^1$ graph over $P \cap Q$ with slope $\leq \varepsilon$, then*

   $$\mathrm{Mass}(Y \cap Q) = \big(1 + O(\varepsilon^2)\big)\ \mathrm{Mass}(P \cap Q),$$

   *with the implied constant depending only on $(n, p)$.*

2. *(*Disjointness persistence, with an anchor*) Let $t_1, t_2 \in P^{\perp}$, and suppose $Y_i \cap Q$ is a $C^1$ graph of slope $\leq \varepsilon$ over $(P + t_i) \cap Q$ for $i = 1, 2$. Assume that each graph has an anchor point in the sense that there exists $x_i \in (P+t_i) \cap Q$ with $x_i \in Y_i$ and the graph displacement vanishes at $x_i$. If $\|t_1 - t_2\| \geq 10\, \varepsilon\, h$, then $Y_1 \cap Q$ and $Y_2 \cap Q$ are disjoint.*

*Proof.* (1) Parametrize the graph by $x \mapsto x + u(x)$ with $\|Du\|_{C^0} \le \varepsilon$. The area formula gives a Jacobian factor $\sqrt{\det(I + Du^\top Du)} = 1 + O(\varepsilon^2)$ uniformly, hence the mass estimate.

(2) The anchor condition implies the graph displacement satisfies $|u_i(x)| \le \varepsilon h$ on the whole base, so $Y_i \cap Q$ lies in the $\varepsilon h$-tubular neighborhood of $P + t_i$. If $\|t_1 - t_2\| \ge 10\varepsilon h$, these tubular neighborhoods are disjoint, forcing disjointness of the graphs. □

**Theorem 2** (Finite translation template realization on the Bergman scale)**.** *Fix a holomorphic chart $U \subset X$ with coordinates $(u, w) \in \mathbb{C}^{n-p} \times \mathbb{C}^p$ and let $P := \{w = 0\}$. Fix $\varepsilon \in (0, 1)$ and choose $m_{\text{hol}} \ge m_{\text{hol}2}(\varepsilon)$ so the tangential manufacturing theorem applies uniformly in $U$. Set*

$$r := \frac{c}{2}\, m_{\text{hol}}^{-1/2},$$

*where $c$ is the constant from the Bergman $C^1$ lemma.*

*Let $x_1, \dots, x_N \in U$ be points such that $B_{c\, m_{\text{hol}}^{-1/2}}(x_a) \subset U$ for all $a$, and define translation parameters $t_a := w(x_a) \in \mathbb{C}^p$.*

*Then for each $a$ there exists a holomorphic complete intersection $Y^{(a)} \subset X$ cut out by $p$ sections of $L^{\otimes m_{\text{hol}}}$ such that:*

1. *(*Local sheet as a graph*) On $B_r(x_a)$, the set $Y^{(a)}$ is a single $C^1$ graph over the affine plane $\{w = t_a\}$, with slope at most $C\varepsilon$.*

2. *(*Uniform mass comparability*) On $B_r(x_a)$, the local sheet volume satisfies*

$$\mathrm{Mass}\big(Y^{(a)} \cap B_r(x_a)\big) = \big(1 + O(\varepsilon^2)\big)\ \mathrm{Mass}\big((P + t_a) \cap B_r(x_a)\big).$$

3. *(*Anchor-based disjointness*) If $a \ne b$ and $\|t_a - t_b\| \ge 10\, C\varepsilon\, r$, then on any overlap region where both are defined, the local sheets $Y^{(a)} \cap B_r(x_a)$ and $Y^{(b)} \cap B_r(x_b)$ are disjoint.*

*Proof.* Fix $a$. Apply the tangential manufacturing theorem at the anchor point $x_a$ with target plane equal to the translate of $P$ through $x_a$. This yields a complete intersection $Y^{(a)}$ that is a small-slope graph over $\{w = t_a\}$ on $B_r(x_a)$, proving (1). The mass estimate (2) follows from the mass comparability lemma for small-slope graphs. Finally, (3) follows from the anchor-based disjointness lemma applied on a cube or ball of diameter $\asymp r$ contained in the overlap. □

# 8 Coherence across overlaps (vertex-star coherence)

For downstream bookkeeping, one sometimes needs a *single* holomorphic object to control its restrictions to several overlapping cells. A simple sufficient condition is to manufacture on a ball large enough to contain the entire overlap region.

**Remark 2** (One object on a vertex star)**.** Suppose a finite collection of mesh cells (for example, all cubes incident to a fixed vertex) lies inside a single ball $B_{c\,m_{\mathrm{hol}}^{-1/2}}(x)$. Manufacturing a sheet on that ball produces one holomorphic complete intersection whose restrictions to each cell are automatically consistent, because they are literally restrictions of the same analytic set. This is the cleanest coherence mechanism at the holomorphic layer: coherence is obtained by choosing the manufacturing radius first, then placing the mesh at or below that scale.

# 9 Applications and limitations

This module is designed to output *local* holomorphic pieces with quantitative geometry: small slope, controlled tangent deviation, and volume comparability on the Bergman scale. It does *not* by itself:

- enforce global smoothness of the complete intersection away from the manufactured ball,
- enforce global disjointness of different manufactured varieties,
- or solve global homology / period constraints (those are separate layers in larger constructions).

What it *does* provide is a robust local conversion from a geometric template into a genuine holomorphic object with uniform $C^1$ control. That is precisely the “holomorphic manufacturing” role needed in microstructured calibrated-current constructions.

# Conclusion

On a smooth complex projective manifold $(X, \omega)$ with positive line bundle $(L, h)$, large tensor powers $L^{\otimes m_{\mathrm{hol}}}$ admit holomorphic sections whose first derivatives can be prescribed at a point while remaining nearly constant on a ball of radius $c\,m_{\mathrm{hol}}^{-1/2}$. This yields a practical manufacturing theorem: prescribed complex tangent planes (and small transverse translations) can be realized by holomorphic complete intersections whose local sheets are single $C^1$ graphs with arbitrarily small slope on the Bergman scale. The finite-template version additionally guarantees disjointness under a clean separation condition, and provides uniform mass comparability for downstream geometric-measure estimates.