A Cost-Minimization Theory of Reference: Aboutness from Balance and Compression

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Abstract

We develop a mathematical theory of *reference*: the relation by which a symbol (a word, code, or internal configuration) "points to" an object. The framework equips symbol and object spaces with positive scales ι, and defines a reference cost

$$c_{\mathcal{R}}(s, o) = J\left(\frac{\iota_S(s)}{\iota_O(o)}\right),$$

where J is the Recognition Science (RS) cost functional. *Meaning* is defined by optimization: a symbol means those objects minimizing $c_{\mathcal{R}}(s, \cdot)$. We prove (i) existence of meanings under mild closure assumptions, (ii) explicit decision geometry for finite object dictionaries (geometric-mean boundaries and stability), (iii) exact compositionality for product symbols/objects, and (iv) an optimal mediation principle showing how intermediate representations can reduce reference cost. We close with concrete scenarios and open problems for richer costed spaces and learned scales ι.

Keywords: Reference, semantics, cost function, Recognition Science, symbol grounding

Contents

1 Introduction

This manuscript proposes a mathematically explicit account of *aboutness* by treating reference as an *optimization problem.* Given a space of potential symbols and a space of potential objects, we ask:

> When does a configuration function as a symbol for another configuration, and what determines what that symbol is *about*?

The guiding hypothesis is that reference is not a primitive relation but an *optimality phenomenon*: a symbol is about whatever it can refer to with *least cost*, relative to an intrinsic notion of cost/complexity assigned to configurations. The goal of the paper is to formalize this hypothesis as a checkable variational theory and to derive structural consequences (existence, stability, compositionality, and capacity-type bounds) from explicit axioms.

1.1 Reference, grounding, and the need for a generative constraint

Analyses of reference in the philosophy of language emphasize logical form and truth conditions (e.g. Frege and Russell) [1, 2], but they do not by themselves provide a *generative* mechanism that selects, from the many possible symbol–object pairings, which pairings are admissible and which admissible pairing(s) are preferred. This gap is closely connected to the symbol-grounding problem: purely formal symbol systems do not acquire semantics from additional formal relations alone; some further constraint is required that ties symbols to what they are about [4].

In this work, the additional constraint is a *cost principle.* Rather than postulating a primitive semantics, we specify a reference cost and define meaning as the set of cost-minimizing targets.

1.2 Costed spaces and a ratio-induced reference cost

We work with two *costed spaces*: a symbol space $\mathcal{S}$ and an object space $\mathcal{O}$, each equipped with a positive *scale* map ι (Section 3). Intuitively, $\iota_S(s)$ and $\iota_O(o)$ quantify the "size" or "complexity" of s and o in a common currency.

A *reference structure* assigns a nonnegative cost $c_{\mathcal{R}}(s, o)$ to the act of using s to refer to o (Definition 3.4). The central class studied here is *ratio-induced* reference:

$$c_{\mathcal{R}}(s, o) = J\left(\frac{\iota_S(s)}{\iota_O(o)}\right), \tag{1}$$

where $J : (0, \infty) \to [0, \infty)$ is the Recognition Science (RS) cost functional (Section 2; Definition 2.3). Aboutness is then defined by an optimization rule:

$$\mathrm{Mean}_{\mathcal{R}}(s, o) \quad \Longleftrightarrow \quad c_{\mathcal{R}}(s, o) \leq c_{\mathcal{R}}(s, o') \ \text{ for all } o' \in O,$$

i.e. the meaning(s) of a symbol are the cost-minimizing objects it can target (Definition 3.8).

The RS cost J (Definition 2.3) penalizes *multiplicative mismatch.* In particular, it is symmetric under $x \mapsto x^{-1}$ and uniquely minimized at $x = 1$. Consequently, low reference cost forces a near-balance condition $\iota_S(s) \approx \iota_O(o)$ in a quantitative way: *aboutness is driven by scale matching.*

To avoid degeneracy, we restrict to *admissible* reference structures (Definition 3.6), which encode feasibility constraints on reference patterns imposed by the underlying spaces. This admissibility layer is where nontrivial existence and stability questions enter.

1.3 Balance, compression, and the effectiveness motif

A nonnegative cost is naturally interpretable as a proxy for resource usage or description length, connecting semantics to compression-oriented viewpoints in information theory and minimum description length [5, 6, 7]. Separately, Wigner famously asked why abstract mathematics can describe concrete physical phenomena with striking accuracy [3]. This manuscript does not claim an empirical resolution of that question. Instead, it provides a precise mathematical template in which “effective description” and “aboutness” are realized by the same operation: selecting minimizers of a cost functional.

1.4 Main results (informal overview)

Fix the RS cost functional J and restrict to admissible ratio-induced reference. The paper proves:

- **Existence of meaning under mild closure/attainment.** Under an attainment hypothesis on the feasible object ratio-set, each symbol admits at least one meaning (Theorem 4.6).
- **Finite-dictionary geometry and stability.** For finite object sets, decision boundaries between competing meanings are described by geometric means of object scales, and meanings are stable away from these boundaries (Theorem 7.3 and Corollary 7.4).
- **Compositionality for product symbols.** In product settings, meaning factorizes componentwise under the model’s separability hypotheses (Theorem 5.2).
- **Mediation and efficiency gains.** Introducing optimal intermediate ratios reduces reference cost and can strictly improve efficiency (Theorem 5.6 and Corollary 5.7).
- **A near-balance window and a capacity-type constraint.** Low-cost reference forces scale ratios into an explicit bounded window, yielding a quantitative constraint on what low-cost symbols can be about (Theorem 4.11 and Corollary 8.3).

Further sections develop multidimensional and robustness extensions (Theorems 6.10–6.12) and worked examples illustrating how the minimization rule produces interpretable semantic partitions.

1.5 Scope and status of the axioms

RS is used here as an *axiomatic* modeling framework: we do not rely on any external RS publication for foundational claims. All primitives and assumptions needed for the results are stated explicitly in the body of the paper, and the logical form of each theorem is “If the axioms hold, then the stated semantic properties follow.” Any stronger claim—for example, that the RS cost is the *correct* cost for a particular cognitive, linguistic, or physical system—is outside the logical scope of this manuscript and would require separate validation.

1.6 Organization

Section 2 introduces the RS cost functional and its basic analytic properties. Section 3 defines costed spaces, reference structures, admissibility, and the meaning relation. Section 4 proves the main results, followed by compositionality (Section 5), extensions and examples, and a discussion of positioning and potential applications.

2 The Recognition Science Cost Functional

We work within Recognition Science (RS), which derives physical structure from a normalized RS cost functional. This section reviews the essential background.

2.1 Axiomatic Characterization

Definition 2.1 (Cost Functional Axioms)**.** A *Recognition Science cost functional* is a function $J : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying:

1. **Normalization**: $J(1) = 0$
2. **Symmetry**: $J(x) = J(x^{-1})$ for all $x > 0$
3. **Non-negativity**: $J(x) \geq 0$ for all $x > 0$
4. **Strict convexity**: J is strictly convex on $\mathbb{R}_{>0}$
5. **d'Alembert composition**: For all $x, y > 0$:

$$J(xy) + J(x/y) = 2J(x) + 2J(y) + 2J(x)J(y) \tag{2}$$

Lemma 2.2 (Uniqueness of the zero-cost point)**.** *If J satisfies Definition 2.1, then $J(x) = 0$ implies $x = 1$.*

Proof. By non-negativity and normalization, $x = 1$ is a global minimizer of J. By strict convexity, a global minimizer is unique. Hence $J(x) = 0 = J(1)$ forces $x = 1$. □

Definition 2.3 (RS cost functional)**.** In the remainder of this paper we work with the explicit Recognition Science (RS) cost functional

$$J(x) = \tfrac{1}{2}(x + x^{-1}) - 1 = \tfrac{(x-1)^2}{2x} \qquad (x > 0). \tag{3}$$

This choice is the simplest symmetric, strictly convex, nonnegative "deviation from balance" functional compatible with the multiplicative d'Alembert identity used below.

Proposition 2.4 (The RS cost satisfies the axioms)**.** *The function* (3) *satisfies Definition 2.1.*

Proof. Normalization and symmetry are immediate from (3). Non-negativity follows from the representation $J(x) = \frac{(x-1)^2}{2x} \geq 0$. For strict convexity, differentiate $J(x) = \frac{1}{2}(x+x^{-1}) - 1$ to get

$$J'(x) = \tfrac{1}{2} - \tfrac{1}{2x^2}, \qquad J''(x) = \tfrac{1}{x^3} > 0 \ (x > 0),$$

so J is strictly convex on $\mathbb{R}_{>0}$. Finally, set $C(x) = 1 + J(x) = \frac{1}{2}(x + x^{-1})$. Then

$$C(xy) + C(x/y) = \tfrac{1}{2}\Big(xy + \tfrac{1}{xy} + \tfrac{x}{y} + \tfrac{y}{x}\Big) = \tfrac{1}{2}\big(x + \tfrac{1}{x}\big)\big(y + \tfrac{1}{y}\big) = 2C(x)C(y),$$

which is equivalent to the d'Alembert composition axiom (2) after substituting $C = 1 + J$ and expanding. □

Remark 2.5. One may study the axioms in Definition 2.1 abstractly; in this manuscript we fix the explicit RS functional (3) and develop the resulting algebra of reference.

2.2 Physical Interpretation

The cost $J(x)$ measures *deviation from balance*:

- $J(1) = 0$: Balanced configurations ($x = 1$) are cost-free.
- $J(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$: Extreme configurations are infinitely expensive.
- $J(x) = J(1/x)$: Cost is symmetric under reciprocation.

In RS, physical configurations that "exist" are those that minimize cost. This motivates our theory of reference: low-cost configurations serve as efficient encodings for high-cost ones.

2.3 The Law of Existence

Definition 2.6 (RS Existence)**.** A configuration x *RS-exists* if $x > 0$ and $J(x) = 0$.

Proposition 2.7. *The unique RS-existent configuration is* $x = 1$*.*

Proof. $J(x) = 0 \iff (x-1)^2/(2x) = 0 \iff (x-1)^2 = 0 \iff x = 1$. □

3 Costed Spaces and Reference Structures

We now formalize the framework for reference.

3.1 Costed Spaces

Definition 3.1 (Costed Space)**.** A *costed space* is a triple (C, J_C, ι_C) where:

- C is a type (set of configurations)
- $J_C : C \to \mathbb{R}_{\geq 0}$ is a cost function
- $\iota_C : C \to \mathbb{R}_{>0}$ is a *ratio map* embedding configurations into the domain of J

such that $J_C(c) = J(\iota_C(c))$ for all $c \in C$.

This definition ties the abstract cost J_C to the universal RS cost J, ensuring that our theory inherits the structure of the normalized RS cost functional.

Notation. We write $\mathcal{S} = (S, J_S, \iota_S)$ for a symbol space and $\mathcal{O} = (O, J_O, \iota_O)$ for an object space.

Example 3.2 (Ratio Space)**.** The canonical example is $C = \mathbb{R}_{>0}$ with $\iota_C = \mathrm{id}$ and $J_C = J$. Here configurations are ratios, and cost is RS cost.

Example 3.3 (Near-Balanced Configurations)**.** Let $C_\epsilon = \{x \in \mathbb{R}_{>0} : |x-1| < \epsilon\}$ for small $\epsilon > 0$. These are "nearly balanced" configurations with $J_C(c) < J(1+\epsilon)$ for all $c \in C_\epsilon$.

3.2 Reference Structures

Definition 3.4 (Reference Structure)**.** A *reference structure* from symbol space $\mathcal{S}$ to object space $\mathcal{O}$ is a function:

$$c_{\mathcal{R}} : S \times O \to \mathbb{R}_{\geq 0} \tag{4}$$

called the *reference cost*, measuring the cost of symbol s referring to object o.

Definition 3.5 (Ratio-Induced Reference)**.** The *ratio-induced reference* structure has cost:

$$c^J_{\mathcal{R}}(s, o) = J\left(\frac{\iota_S(s)}{\iota_O(o)}\right) \tag{5}$$

This measures the cost of the "mismatch" between symbol and object.

Definition 3.6 (Admissible Reference Structure)**.** A reference structure $\mathcal{R}$ from $\mathcal{S}$ to $\mathcal{O}$ is called *admissible* (with respect to the cost functional J and ratio maps ι_S, ι_O) if it is ratio-induced, i.e.

$$c_{\mathcal{R}}(s, o) = J\Big(\frac{\iota_S(s)}{\iota_O(o)}\Big) \qquad \forall (s, o) \in S \times O. \tag{6}$$

When we invoke symmetry or d'Alembert composition inherited from J, we will explicitly assume $\mathcal{R}$ is admissible.

The ratio-induced reference is canonical: it inherits the d'Alembert composition law and symmetry properties of J.

Proposition 3.7 (Reference Symmetry)**.** *For ratio-induced reference:*

$$c^J_{\mathcal{R}}(s, o) = c^J_{\mathcal{R}}(o, s) \tag{7}$$

where we identify s with its ratio $\iota_S(s)$ and o with $\iota_O(o)$.

Proof. $c^J_{\mathcal{R}}(s, o) = J(\iota_S(s)/\iota_O(o)) = J(\iota_O(o)/\iota_S(s)) = c^J_{\mathcal{R}}(o, s)$ by J-symmetry. □

3.3 Meaning and Symbols

Definition 3.8 (Meaning)**.** Symbol s *means* object o in reference structure $\mathcal{R}$, written $\mathrm{Mean}_{\mathcal{R}}(s, o)$, if o minimizes reference cost:

$$\mathrm{Mean}_{\mathcal{R}}(s, o) \iff \forall o' \in O, \quad c_{\mathcal{R}}(s, o) \leq c_{\mathcal{R}}(s, o') \tag{8}$$

Definition 3.9 (Symbol)**.** A configuration $s \in S$ is a *symbol* for object $o \in O$, written $(s, o) \in \mathrm{Sym}(\mathcal{S}, \mathcal{O}, \mathcal{R})$, if:

1. **Reference**: $\mathrm{Mean}_{\mathcal{R}}(s, o)$
2. **Compression**: $J_S(s) < J_O(o)$

The compression condition is essential: symbols must be *cheaper* than the objects they represent. This is the ontological core of reference—symbols exist because they provide economical encodings.

4 Main Theorems

Central claim. The core formal contribution is that *aboutness* can be defined and studied as an optimization principle driven by a universal cost functional J (Definition 2.1). In this draft, the most explicit mathematical instantiation of that principle is the balanced-reference theorem (Theorem 4.8), while the remaining results develop existence/robustness, capacity, and compositionality.

4.1 Reference from Cost Asymmetry

The previous version of this subsection used an *unconstrained* reference cost $c_{\mathcal{R}}$; under no structural restrictions, one can always *design* $c_{\mathcal{R}}$ so that a prescribed object becomes a minimizer for a prescribed symbol. In the remainder of this paper we therefore work with *admissible* reference structures (Definition 3.6), for which reference cost is induced from the ratio maps and the universal cost J. This makes existence and capacity statements genuinely nontrivial.

Lemma 4.1 (Sublevel intervals for the RS cost)**.** *Assume the explicit RS cost* (3)*. For each $\epsilon > 0$, the sublevel set*

$$L_\epsilon := \{x \in \mathbb{R}_{>0} : \ J(x) \le \epsilon\}$$

is the closed interval $[a_\epsilon, b_\epsilon]$ with $0 < a_\epsilon \le 1 \le b_\epsilon < \infty$, where

$$b_\epsilon := (1+\epsilon) + \sqrt{\epsilon(2+\epsilon)}, \qquad a_\epsilon := (1+\epsilon) - \sqrt{\epsilon(2+\epsilon)} = \frac{1}{b_\epsilon}.$$

Proof. Using $J(x) = \frac{(x-1)^2}{2x}$, the inequality $J(x) \le \epsilon$ is equivalent (after multiplying by $2x > 0$) to

$$(x-1)^2 \le 2\epsilon x \quad \Longleftrightarrow \quad x^2 - 2(1+\epsilon)x + 1 \le 0.$$

The quadratic polynomial $q(x) := x^2 - 2(1+\epsilon)x + 1$ has discriminant

$$\Delta = 4\big((1+\epsilon)^2 - 1\big) = 4\epsilon(2+\epsilon),$$

and hence two positive roots

$$x_\pm = (1+\epsilon) \pm \sqrt{\epsilon(2+\epsilon)}.$$

Since q opens upward, $q(x) \le 0$ holds exactly for $x \in [x_-, x_+]$. Set $a_\epsilon := x_-$ and $b_\epsilon := x_+$. Then $a_\epsilon b_\epsilon = (1+\epsilon)^2 - \epsilon(2+\epsilon) = 1$, so $a_\epsilon = 1/b_\epsilon$, and clearly $0 < a_\epsilon \le 1 \le b_\epsilon < \infty$. □

Theorem 4.2 (Low-cost symbols cannot freely name high-cost objects)**.** *Let $\mathcal{O} = (O, J_O, \iota_O)$ be a costed space that contains at least one* balanced *configuration $o_0 \in O$ with $\iota_O(o_0) = 1$ (equivalently $J_O(o_0) = 0$). Let $\mathcal{S} = (S, J_S, \iota_S)$ be any symbol space and let $\mathcal{R}$ be an admissible reference structure, i.e.*

$$c_{\mathcal{R}}(s, o) = J\Big(\frac{\iota_S(s)}{\iota_O(o)}\Big).$$

If s means o in the sense of Definition 3.8, then

$$c_{\mathcal{R}}(s, o) \ \le \ c_{\mathcal{R}}(s, o_0) \ = \ J(\iota_S(s)) \ = \ J_S(s).$$

In particular, for every $\epsilon > 0$, if $J_S(s) \le \epsilon$ then $\iota_S(s)/\iota_O(o) \in [a_\epsilon, b_\epsilon]$ (Lemma 4.1), hence

$$\frac{\iota_S(s)}{b_\epsilon} \ \le \ \iota_O(o) \ \le \ \frac{\iota_S(s)}{a_\epsilon}.$$

Proof. Since o is a minimizer of $o' \mapsto c_{\mathcal{R}}(s, o')$, we have $c_{\mathcal{R}}(s, o) \le c_{\mathcal{R}}(s, o_0)$. The equality $c_{\mathcal{R}}(s, o_0) = J(\iota_S(s)) = J_S(s)$ follows from admissibility and $\iota_O(o_0) = 1$. The final bounds follow by applying Lemma 4.1 to the inequalities $c_{\mathcal{R}}(s, o) \le \epsilon$ and $J_S(s) \le \epsilon$. □

Corollary 4.3 (Near-balanced symbols force near-balanced meanings)**.** *Under the hypotheses of Theorem 4.2, if $J_S(s) \le \epsilon$ and s means o, then $\iota_O(o) \in [a_\epsilon/b_\epsilon,\ b_\epsilon/a_\epsilon] = [1/b_\epsilon^2,\ b_\epsilon^2]$. In particular, as $\epsilon \downarrow 0$, any meaning of an ϵ-cheap symbol must have $\iota_O(o) \to 1$.*

Proof. Combine $\iota_S(s) \in [a_\epsilon, b_\epsilon]$ (since $J_S(s) = J(\iota_S(s)) \le \epsilon$) with the bounds in Theorem 4.2, and use $a_\epsilon = 1/b_\epsilon$. □

4.2 Optimal Reference from Balance

Definition 4.4 (Optimal Reference)**.** A reference relation (s, o) is J*-optimal* if it minimizes total cost:

$$J_S(s) + J_O(o) + c_{\mathcal{R}}(s, o) \tag{9}$$

among all symbol-object pairs.

(Technical note: the phrase "minimizes total cost among all symbol–object pairs" is an *existence* claim that depends on the spaces. Theorem 4.8 first identifies the *ratio-level* global minimizer of the total cost, and then gives a sufficient condition for realizing it inside (S, O).)

Lemma 4.5 (Coercivity of the RS cost)**.** *Assume J is given by Eq. 3. Then $J(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$. In particular, for each $M \geq 0$ the sublevel set $\{x \in \mathbb{R}_{>0} : \ J(x) \leq M\}$ is compact in $\mathbb{R}$.*

Proof. Using Eq. 3, $J(x) = \frac{1}{2}(x + x^{-1}) - 1$. As $x \to \infty$ the term $\frac{1}{2}x$ dominates, so $J(x) \to \infty$. As $x \to 0^+$ the term $\frac{1}{2}x^{-1}$ dominates, so $J(x) \to \infty$. If $J(x) \leq M$ then $\frac{1}{2}(x + x^{-1}) \leq M + 1$, hence both x and x^{-1} are bounded; thus the sublevel set is closed and bounded away from 0 and ∞, hence compact. □

Theorem 4.6 (Existence of meanings for ratio-induced reference)**.** *Assume $\mathcal{R}$ is admissible in the sense of Definition 3.6 and that J is given by Eq. 3. Let $Y := \iota_O(O) \subset \mathbb{R}_{>0}$ be nonempty and closed. Then for every $s \in S$ there exists $o \in O$ such that $\mathrm{Mean}_{\mathcal{R}}(s, o)$. Moreover, if $x := \iota_S(s) \in Y$, then any $o \in O$ with $\iota_O(o) = x$ is a meaning and satisfies $c_{\mathcal{R}}(s, o) = 0$.*

Proof. Fix s and set $x := \iota_S(s)$. Consider the function $f : Y \to \mathbb{R}_{\geq 0}$ defined by

$$f(y) := J(x/y).$$

This is continuous in y. By Lemma 4.5, $f(y) \to \infty$ as $y \to 0^+$ or $y \to \infty$. Therefore the infimum of f over the closed set Y is attained at some $y_* \in Y$. Choose $o \in O$ with $\iota_O(o) = y_*$ (possible since $y_* \in \iota_O(O)$). Then for any $o' \in O$ with $y' = \iota_O(o') \in Y$ we have

$$c_{\mathcal{R}}(s, o) = J(x/y_*) \leq J(x/y') = c_{\mathcal{R}}(s, o'),$$

which is exactly $\mathrm{Mean}_{\mathcal{R}}(s, o)$. If $x \in Y$, then $y = x$ is admissible and $J(x/x) = J(1) = 0$, hence $c_{\mathcal{R}}(s, o) = 0$ for any o with $\iota_O(o) = x$. □

Remark 4.7. If $Y = \iota_O(O)$ is not closed, the minimum need not be attained; in that case $\mathrm{Mean}_{\mathcal{R}}(s, o)$ may fail to hold for every $o \in O$ even though $\inf_{o \in O} c_{\mathcal{R}}(s, o)$ exists.

Theorem 4.8 (Balanced Reference Principle)**.** *Assume ratio-induced reference (Eq. 5) and intrinsic costs $J_S(s) = J(\iota_S(s))$, $J_O(o) = J(\iota_O(o))$. Define the total cost*

$$C(s, o) \ := \ J_S(s) + J_O(o) + c_{\mathcal{R}}(s, o) \ = \ J(\iota_S(s)) + J(\iota_O(o)) + J\left(\frac{\iota_S(s)}{\iota_O(o)}\right).$$

Then $C(s, o) \geq 0$ for all $s \in S$, $o \in O$. Moreover,

$$C(s, o) = 0 \quad \Longleftrightarrow \quad \iota_S(s) = 1 \text{ and } \iota_O(o) = 1.$$

In particular, if there exist $s_0 \in S$ and $o_0 \in O$ with $\iota_S(s_0) = \iota_O(o_0) = 1$, then (s_0, o_0) is a global minimizer of C among all pairs in $S \times O$.

Proof. By Definition 2.1(3), each term in the displayed expression for $C(s,o)$ is nonnegative, hence $C(s,o) \geq 0$. If $C(s,o) = 0$, then all three terms vanish; by Lemma 2.2 this forces $\iota_S(s) = 1$, $\iota_O(o) = 1$, and $\iota_S(s)/\iota_O(o) = 1$ (which is then automatic). Conversely, if $\iota_S(s) = \iota_O(o) = 1$, then each term equals $J(1) = 0$ and hence $C(s,o) = 0$. $\square$

Corollary 4.9. *Perfect reference (zero total cost) requires both symbol and object to be balanced ($\iota = 1$).*

4.3 Universal Referential Capacity

Definition 4.10 (Referential Capacity)**.** The *referential capacity* of symbol space $\mathcal{S}$ for object space $\mathcal{O}$ is:

$$\mathrm{Cap}(\mathcal{S}, \mathcal{O}; \mathcal{R}) = |\{o \in O : \exists\, s \in S, (s,o) \in \mathrm{Sym}(\mathcal{S}, \mathcal{O}, \mathcal{R})\}| \tag{10}$$

Theorem 4.11 (Backbone window for near-balanced symbols (corrected))**.** *Let $\mathcal{S}_\delta = (S_\delta, J_\delta, \iota_\delta)$ be the near-balanced ratio space*

$$S_\delta := \{x \in \mathbb{R}_{>0} : \ |x - 1| < \delta\}, \qquad \iota_\delta = \mathrm{id}, \qquad J_\delta = J|_{S_\delta}.$$

Let $\mathcal{O} = (O, J_O, \iota_O)$ be a costed space such that $Y := \iota_O(O) \subset \mathbb{R}_{>0}$ is nonempty, closed, and contains 1*. Assume $\mathcal{R}$ is admissible (Definition 3.6) and J is given by Eq. 3.*

Set $\epsilon_\delta := J(1+\delta)$ and let $[a_{\epsilon_\delta}, b_{\epsilon_\delta}]$ be the sublevel interval from Lemma 4.1. Then:

1. *For every $s \in S_\delta$ there exists at least one meaning $o \in O$ (i.e.* $\mathrm{Mean}_{\mathcal{R}}(s,o)$ *holds).*
2. *If $s \in S_\delta$ means o, then the ratio of o lies in the bounded "backbone window"*

$$\iota_O(o) \in I_\delta := \left[\frac{1-\delta}{b_{\epsilon_\delta}},\ \frac{1+\delta}{a_{\epsilon_\delta}}\right].$$

Equivalently, if $\iota_O(o) \notin I_\delta$, then no $s \in S_\delta$ can mean o under admissible reference.

In particular, the referential capacity defined in Definition 4.10 satisfies the bound

$$\mathrm{Cap}(\mathcal{S}_\delta, \mathcal{O}; \mathcal{R}) \ \leq\ \big|\{o \in O : \ \iota_O(o) \in I_\delta\}\big|.$$

Proof. Fix $s \in S_\delta$ and set $x := \iota_\delta(s) \in (1-\delta, 1+\delta)$. Since Y is nonempty, closed, and J is coercive (Lemma 4.5), Theorem 4.6 yields existence of a meaning $o \in O$; this proves (1).

For (2), let o be any meaning of s. Because $1 \in Y$, choose $o_0 \in O$ with $\iota_O(o_0) = 1$. Then Theorem 4.2 gives

$$c_{\mathcal{R}}(s,o) \leq c_{\mathcal{R}}(s,o_0) = J(\iota_\delta(s)) = J(x) \leq J(1+\delta) = \epsilon_\delta.$$

Hence $J(x/\iota_O(o)) \leq \epsilon_\delta$, so by Lemma 4.1 we have

$$\frac{x}{\iota_O(o)} \in [a_{\epsilon_\delta}, b_{\epsilon_\delta}] \quad \Longrightarrow \quad \frac{x}{b_{\epsilon_\delta}} \leq \iota_O(o) \leq \frac{x}{a_{\epsilon_\delta}}.$$

Using $x \in [1-\delta, 1+\delta]$ gives $\iota_O(o) \in I_\delta$ as claimed. The contrapositive form is immediate.

Finally, if (s,o) is counted toward $\mathrm{Cap}(\mathcal{S}_\delta, \mathcal{O}; \mathcal{R})$, then in particular s means o, hence $\iota_O(o) \in I_\delta$; the displayed bound follows. $\square$

Corollary 4.12 (Local effectiveness; compositionality for global reach)**.** *Near-balanced symbol systems have a mathematically provable* local *backbone property: under admissible (ratio-induced) reference, they can only directly mean objects whose ratios lie in a bounded window I_δ around balance (Theorem 4.11). To extend "effectiveness" to high-cost objects, one must use additional structure beyond one-shot admissible reference—for example, multi-part descriptions via product reference or multi-hop mediated reference (Section 5 discusses compositional mechanisms).*

5 Compositionality

Reference structures compose, enabling complex semantic relations.

Definition 5.1 (Product Reference)**.** Given $\mathcal{R}_1 : \mathcal{S}_1 \to \mathcal{O}_1$ and $\mathcal{R}_2 : \mathcal{S}_2 \to \mathcal{O}_2$, the product $\mathcal{R}_1 \otimes \mathcal{R}_2$ has cost:

$$c_{\mathcal{R}_1 \otimes \mathcal{R}_2}((s_1, s_2), (o_1, o_2)) = c_{\mathcal{R}_1}(s_1, o_1) + c_{\mathcal{R}_2}(s_2, o_2) \tag{11}$$

Theorem 5.2 (Compositionality of Product Meaning)**.** *Let $\mathcal{R}_1 : \mathcal{S}_1 \to \mathcal{O}_1$ and $\mathcal{R}_2 : \mathcal{S}_2 \to \mathcal{O}_2$ be reference structures, and let $\mathcal{R}_1 \otimes \mathcal{R}_2$ be the product reference from Definition 5.1. Fix $(s_1, s_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ and assume the minimizers defining meaning exist for s_1 and s_2, i.e.*

$$\arg\min_{o_1 \in \mathcal{O}_1} c_{\mathcal{R}_1}(s_1, o_1) \neq \emptyset, \qquad \arg\min_{o_2 \in \mathcal{O}_2} c_{\mathcal{R}_2}(s_2, o_2) \neq \emptyset.$$

Then for any $(o_1, o_2) \in \mathcal{O}_1 \times \mathcal{O}_2$,

$$\mathrm{Mean}_{\mathcal{R}_1 \otimes \mathcal{R}_2}((s_1, s_2), (o_1, o_2)) \iff \mathrm{Mean}_{\mathcal{R}_1}(s_1, o_1) \text{ and } \mathrm{Mean}_{\mathcal{R}_2}(s_2, o_2).$$

In particular, if meanings exist componentwise then a meaning exists for the product symbol (s_1, s_2).

Proof. Define $f_1(o_1) := c_{\mathcal{R}_1}(s_1, o_1)$ on $\mathcal{O}_1$ and $f_2(o_2) := c_{\mathcal{R}_2}(s_2, o_2)$ on $\mathcal{O}_2$. By Definition 5.1,

$$c_{\mathcal{R}_1 \otimes \mathcal{R}_2}((s_1, s_2), (o_1, o_2)) = f_1(o_1) + f_2(o_2).$$

If $\mathrm{Mean}_{\mathcal{R}_1}(s_1, o_1)$ and $\mathrm{Mean}_{\mathcal{R}_2}(s_2, o_2)$, then for all (o_1', o_2') we have $f_1(o_1) \leq f_1(o_1')$ and $f_2(o_2) \leq f_2(o_2')$, hence $f_1(o_1) + f_2(o_2) \leq f_1(o_1') + f_2(o_2')$, proving product meaning. Conversely, if (o_1, o_2) minimizes $f_1 + f_2$, then for any $o_1' \in \mathcal{O}_1$,

$$f_1(o_1) + f_2(o_2) \leq f_1(o_1') + f_2(o_2),$$

so $f_1(o_1) \leq f_1(o_1')$ and $\mathrm{Mean}_{\mathcal{R}_1}(s_1, o_1)$ holds; similarly $\mathrm{Mean}_{\mathcal{R}_2}(s_2, o_2)$.

□

Lemma 5.3 (Closedness under products)**.** *If $A \subset X$ and $B \subset Y$ are closed subsets of topological spaces, then $A \times B$ is closed in $X \times Y$ (with the product topology).*

Proof. Since A and B are closed, their complements $X \setminus A$ and $Y \setminus B$ are open. One checks that

$$(X \times Y) \setminus (A \times B) = ((X \setminus A) \times Y) \;\cup\; (X \times (Y \setminus B)),$$

a union of open sets in $X \times Y$. Hence $A \times B$ is closed. □

Corollary 5.4 (Existence of product meanings under admissible reference)**.** *Assume $\mathcal{R}_i : \mathcal{S}_i \to \mathcal{O}_i$ are admissible reference structures (Definition 3.6) and their object ratio-sets $Y_{O_i} := \iota_{O_i}(\mathcal{O}_i) \subset \mathbb{R}_{>0}$ are closed. Then for every symbol $(s_1, s_2) \in \mathcal{S}_1 \times \mathcal{S}_2$ the product reference $\mathcal{R}_1 \otimes \mathcal{R}_2$ admits at least one meaning, and Theorem 5.2 applies.*

Definition 5.5 (Sequential Reference)**.** Given $\mathcal{R}_1 : \mathcal{S} \to \mathcal{M}$ and $\mathcal{R}_2 : \mathcal{M} \to \mathcal{O}$, the sequential composition $\mathcal{R}_2 \circ \mathcal{R}_1 : \mathcal{S} \to \mathcal{O}$ has cost

$$c_{\mathcal{R}_2 \circ \mathcal{R}_1}(s, o) = \inf_{m \in \mathcal{M}} \left[c_{\mathcal{R}_1}(s, m) + c_{\mathcal{R}_2}(m, o) \right]. \tag{12}$$

A *mediator* m is *optimal* for (s, o) if it attains the infimum in (12).

Theorem 5.6 (Optimal mediator for ratio-induced sequential reference). *Assume $\mathcal{R}_1$ and $\mathcal{R}_2$ are* admissible *(ratio-induced) reference structures built from the same cost J (Definition 3.6), with ratio maps $\iota_S, \iota_M, \iota_O$ for $\mathcal{S}, \mathcal{M}, \mathcal{O}$ respectively. Fix $s \in \mathcal{S}$ and $o \in \mathcal{O}$ and set $a := \iota_S(s)$ and $c := \iota_O(o)$. Let $Y_M := \iota_M(\mathcal{M}) \subset \mathbb{R}_{>0}$ and assume Y_M is closed and contains $b_* := \sqrt{ac}$. Then the infimum in* (12) *is attained by an optimal mediator $m_* \in \mathcal{M}$ with $\iota_M(m_*) = b_*$, and*

$$c_{\mathcal{R}_2 \circ \mathcal{R}_1}(s, o) = J\left(\frac{a}{b_*}\right) + J\left(\frac{b_*}{c}\right) = 2\,J\left(\sqrt{\frac{a}{c}}\right).$$

Moreover, at the level of ratios $b \in (0, \infty)$ this minimizer is unique.

Proof. Under admissibility, the objective depends on m only through $b := \iota_M(m) \in Y_M$, namely

$$F(b) := J\left(\frac{a}{b}\right) + J\left(\frac{b}{c}\right).$$

Using the explicit RS form (3), $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, hence

$$F(b) = \tfrac{1}{2}\left(\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b}\right) - 2.$$

A direct computation gives

$$F'(b) = \tfrac{1}{2}\left(-\frac{a+c}{b^2} + \frac{1}{a} + \frac{1}{c}\right), \qquad F''(b) = \frac{a+c}{b^3} > 0,$$

so F is strictly convex on $(0, \infty)$ and has a unique critical point, necessarily the global minimizer, satisfying $b^2 = ac$, i.e. $b = b_*$. By the hypothesis $b_* \in Y_M$ and closedness, the infimum over Y_M is attained at b_*, realized by some $m_* \in \mathcal{M}$ with $\iota_M(m_*) = b_*$. Substituting $b_* = \sqrt{ac}$ yields $J(a/b_*) = J(\sqrt{a/c}) = J(b_*/c)$ and the stated formula. □

Corollary 5.7 (Mediation can reduce reference cost). *For every $x > 0$ one has*

$$2\,J(\sqrt{x}) \le J(x),$$

with equality if and only if $x = 1$. Consequently, whenever the hypotheses of Theorem 5.6 hold and a direct admissible reference cost is available between $\mathcal{S}$ and $\mathcal{O}$, one has

$$c_{\mathcal{R}_2 \circ \mathcal{R}_1}(s, o) \le c_{\mathcal{R}}(s, o),$$

with equality if and only if $\iota_S(s) = \iota_O(o)$.

Proof. Let $t := \sqrt{x} > 0$. Using (3), a direct calculation gives

$$J(t^2) - 2\,J(t) = \frac{1}{2}\Big((t-1)^2 + (t^{-1} - 1)^2\Big) \ge 0,$$

with equality if and only if $t = 1$, i.e. $x = 1$. The final inequality follows by substituting $x = \iota_S(s)/\iota_O(o)$ for the direct admissible cost. □

6 Extensions: Multi-dimensional Aboutness and Robustness

The core development above uses a single positive "balance" coordinate $\iota(\cdot) \in \mathbb{R}_{>0}$. In practice, symbols and objects carry multiple attributes (size, tone, topic, context, etc.). This section shows how the same cost-minimization semantics extends to a multi-dimensional ratio map, how the RS cost becomes a smooth geometry in log-space, and how to quantify robustness of meanings under small perturbations.

6.1 Multi-dimensional costed spaces

Definition 6.1 (Multi-dimensional ratio map and induced cost)**.** Let $d \in \mathbb{N}$. A *d-dimensional costed space* is a triple (C, J_C, ι_C) where $\iota_C : C \to (\mathbb{R}_{>0})^d$ and the intrinsic cost is the separable sum

$$J_C(c) \;:=\; \sum_{i=1}^{d} J(\iota_C(c)_i).$$

Definition 6.2 (Multi-dimensional admissible reference)**.** Let (S, J_S, ι_S) and (O, J_O, ι_O) be d-dimensional costed spaces. The reference structure $\mathcal{R}$ is called *(multi-dimensional) admissible* if

$$c_{\mathcal{R}}(s, o) \;=\; \sum_{i=1}^{d} J\!\left(\frac{\iota_S(s)_i}{\iota_O(o)_i}\right) \qquad (s \in S,\ o \in O).$$

We then write $\mathcal{R} \in \mathrm{AdmRef}(S, O)$.

Theorem 6.3 (Coordinatewise meaning for product dictionaries)**.** *Assume $O = \prod_{i=1}^{d} O_i$ with $\iota_O(o)_i = \iota_{O_i}(o_i)$ and $S = \prod_{i=1}^{d} S_i$ with $\iota_S(s)_i = \iota_{S_i}(s_i)$. Let $\mathcal{R} \in \mathrm{AdmRef}(S, O)$ be multi-dimensional admissible. Then $(o_1, \ldots, o_d)$ is a meaning of $(s_1, \ldots, s_d)$ iff each o_i is a meaning of s_i in the 1D admissible sense:*

$$\mathrm{Mean}_{\mathcal{R}}\big((s_1, \ldots, s_d), (o_1, \ldots, o_d)\big) \iff \forall i,\ \mathrm{Mean}_{\mathcal{R}_i}(s_i, o_i),$$

where $\mathcal{R}_i$ is the induced 1D admissible reference on (S_i, O_i).

Proof. By Definition 6.2, the total reference cost is a sum of d nonnegative terms, each depending only on (s_i, o_i). Minimizing a separable sum over a product set is equivalent to minimizing each summand over its coordinate. Formally, for fixed s, write $F(o) = \sum_{i=1}^{d} f_i(o_i)$ with $f_i(o_i) = J(\iota_{S_i}(s_i)/\iota_{O_i}(o_i))$. Then o is a minimizer of F iff each o_i is a minimizer of f_i, proving the equivalence. □

6.2 Log-space geometry and the quadratic regime

Lemma 6.4 (RS cost in log-coordinates)**.** *For the RS functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, one has for all $t \in \mathbb{R}$:*

$$J(e^t) \;=\; \cosh(t) - 1.$$

Proof. Compute $J(e^t) = \frac{1}{2}(e^t + e^{-t}) - 1 = \cosh(t) - 1$. □

Proposition 6.5 (Quadratic approximation with an explicit remainder bound)**.** *For all $t \in \mathbb{R}$,*

$$0 \;\le\; J(e^t) - \frac{t^2}{2} \;\le\; \frac{t^4}{24}\cosh(|t|).$$

In particular, for $|t| \le 1$,

$$\frac{t^2}{2} \;\le\; J(e^t) \;\le\; \frac{t^2}{2} + \frac{\cosh(1)}{24}\, t^4.$$

Proof. By Lemma 6.4, it suffices to bound $\cosh(t) - 1 - \frac{1}{2}t^2$. Using Taylor's theorem at 0 with remainder, $\cosh(t) = 1 + \frac{1}{2}t^2 + \frac{1}{24}t^4\cosh(\xi)$ for some ξ between 0 and t. Since cosh is even and increasing on $\mathbb{R}_{\ge 0}$, $\cosh(\xi) \le \cosh(|t|)$, giving the stated upper bound; moreover the same Taylor remainder shows nonnegativity since $\cosh(\xi) > 0$. □

Corollary 6.6 (Local Euclidean geometry in log-space). *Let* $\mathcal{R} \in \mathrm{AdmRef}(S, O)$ *be admissible and set* $x = \iota_S(s)$, $y = \iota_O(o)$. *When* $|\log(x/y)|$ *is small, the reference cost satisfies*

$$c_{\mathcal{R}}(s, o) \;=\; \frac{1}{2}\big(\log(x/y)\big)^2 + O\big((\log(x/y))^4\big),$$

so meanings behave like nearest neighbors in log-ratio *at small scales.*

6.3 Margin stability under perturbations

Definition 6.7 (Decision margin). Fix $s \in S$ and a finite object dictionary $O = \{o_1, \ldots, o_N\}$. Let $C_k := c_{\mathcal{R}}(s, o_k)$ and let k^* be any index achieving $\min_k C_k$. The *decision margin* at s is

$$\Delta(s) \;:=\; \min_{j \neq k^*} (C_j - C_{k^*}) \;\in [0, \infty).$$

Proposition 6.8 (Robustness of meaning under bounded cost perturbations). *In the finite-dictionary setting of Definition 6.7, suppose the reference costs are perturbed to* $\widetilde{C}_k$ *with*

$$\max_{1 \le k \le N} |\,\widetilde{C}_k - C_k\,| \;\le\; \eta.$$

If $\Delta(s) > 2\eta$, *then the set of minimizers is unchanged: the indices minimizing* C_k *are exactly the indices minimizing* $\widetilde{C}_k$.

Proof. Let k^* minimize C_k. For any $j \neq k^*$, we have $C_j \ge C_{k^*} + \Delta(s)$. Then $\widetilde{C}_j \ge C_j - \eta \ge C_{k^*} + \Delta(s) - \eta$ while $\widetilde{C}_{k^*} \le C_{k^*} + \eta$. If $\Delta(s) > 2\eta$ then $\widetilde{C}_j > \widetilde{C}_{k^*}$ for all $j \neq k^*$, so k^* remains the unique minimizer. The same argument applies when there are multiple original minimizers (replace k^* by any minimizer). □

Remark 6.9. (Interpretation) In real systems, costs are estimated from data and are noisy. Proposition 6.8 says that meanings are stable whenever the best match is separated from the runner-up by a gap (margin) exceeding the noise level.

6.4 Existence, uniqueness, and continuity of meanings in d dimensions

Theorem 6.10 (Existence of meanings for multi-dimensional admissible reference). *Let* $d \in \mathbb{N}$ *and let* (S, J_S, ι_S) *and* (O, J_O, ι_O) *be* d*-dimensional costed spaces in the sense of Definition 6.1. Assume* $\mathcal{R} \in \mathrm{AdmRef}(S, O)$ *is multi-dimensional admissible (Definition 6.2). Let* $Y := \iota_O(O) \subset (\mathbb{R}_{>0})^d$ *be nonempty and closed (in the Euclidean topology). Then for every* $s \in S$ *there exists* $o \in O$ *such that* $\mathrm{Mean}_{\mathcal{R}}(s, o)$. *Moreover, if* $x := \iota_S(s) \in Y$, *then any* $o \in O$ *with* $\iota_O(o) = x$ *is a meaning and satisfies* $c_{\mathcal{R}}(s, o) = 0$.

Proof. Fix $s \in S$ and write $x = \iota_S(s) = (x_1, \ldots, x_d) \in (\mathbb{R}_{>0})^d$. Define the continuous objective on Y,

$$F_x(y) \;:=\; \sum_{i=1}^{d} J\!\left(\frac{x_i}{y_i}\right), \qquad y = (y_1, \ldots, y_d) \in Y.$$

By Lemma 4.5, for each $M \ge 0$ the 1D sublevel set $K_M := \{z > 0 :\ J(z) \le M\}$ is nonempty and compact, hence admits $a_M := \min K_M > 0$ and $b_M := \max K_M < \infty$ with $a_M \le 1 \le b_M$. If $F_x(y) \le M$, then each summand is $\le M$, hence $J(x_i/y_i) \le M$ and thus $x_i/y_i \in K_M$. Equivalently,

$$\frac{x_i}{b_M} \;\le\; y_i \;\le\; \frac{x_i}{a_M} \qquad (i = 1, \ldots, d).$$

Therefore the sublevel set $\{y \in Y : F_x(y) \le M\}$ is contained in the compact box $\prod_{i=1}^d [x_i/b_M,\ x_i/a_M]$ and is closed (as Y is closed and F_x is continuous), hence compact. Consequently, F_x attains its minimum on Y at some $y_* \in Y$. Choose $o \in O$ with $\iota_O(o) = y_*$; then o minimizes $c_{\mathcal{R}}(s, \cdot)$ by Definition 6.2, i.e. $\mathrm{Mean}_{\mathcal{R}}(s, o)$.

If $x \in Y$, then $y = x$ is admissible and gives $F_x(x) = \sum_{i=1}^d J(1) = 0$. Since each term is nonnegative, 0 is the global minimum, so any o with $\iota_O(o) = x$ is a meaning and has zero reference cost. □

Definition 6.11 (Log-dictionary and log-convexity)**.** For a set $Y \subset (\mathbb{R}_{>0})^d$, define its *log-image* by

$$\log Y \ := \ \{(\log y_1, \dots, \log y_d) : \ y \in Y\} \ \subset \ \mathbb{R}^d.$$

We say Y is *log-convex* if $\log Y$ is convex.

Theorem 6.12 (Uniqueness and continuity under log-convex dictionaries)**.** *Assume the setting of Theorem 6.10, and in addition assume $U := \log Y$ is closed and convex in $\mathbb{R}^d$. For each $x \in (\mathbb{R}_{>0})^d$, the minimizer $y_*(x) \in Y$ of F_x is unique at the ratio level. Equivalently, the set of meanings of a symbol s is exactly the fiber*

$$\arg\min_{o \in O} c_{\mathcal{R}}(s, o) \ = \ \{o \in O : \ \iota_O(o) = y_*(\iota_S(s))\}.$$

Moreover, the map $t \mapsto u_*(t)$ is continuous, where $t = \log x$ and $u_*(t) := \log y_*(e^t) \in U$.

Proof. Write $t = \log x \in \mathbb{R}^d$ and $u = \log y \in U$. By Lemma 6.4, the objective becomes

$$F_x(y) = \sum_{i=1}^{d} \big(\cosh(t_i - u_i) - 1\big) \ =: \ G_t(u).$$

Since cosh is strictly convex on $\mathbb{R}$, each term $u_i \mapsto \cosh(t_i - u_i) - 1$ is strictly convex, hence G_t is strictly convex on $\mathbb{R}^d$. Restricting a strictly convex function to a convex set preserves strict convexity, so G_t has at most one minimizer on the convex set U. Existence of a minimizer follows from Theorem 6.10 (equivalently, from coercivity of G_t and closedness of U). Hence the ratio-level minimizer $u_*(t) \in U$ is unique, which implies uniqueness of $y_*(x) = e^{u_*(\log x)}$.

For continuity, let $t_n \to t$ and write $u_n := u_*(t_n) \in U$. Fix any $u_0 \in U$. Since u_n minimizes G_{t_n} on U, we have $G_{t_n}(u_n) \le G_{t_n}(u_0)$ for all n. The right-hand side is bounded because $t_n \to t$ and $G_t(u_0)$ is finite and continuous in t. To bound $\{u_n\}$ explicitly, note that $G_{t_n}(u_n) \le B$ for some $B < \infty$. Since each summand is nonnegative, we have $\cosh(t_{n,i} - u_{n,i}) - 1 \le B$ for each coordinate i, hence $|t_{n,i} - u_{n,i}| \le \operatorname{arcosh}(B + 1)$. Because $t_n \to t$, the sequence $\{t_n\}$ is bounded, so each coordinate $u_{n,i} = t_{n,i} - (t_{n,i} - u_{n,i})$ is bounded as well. Therefore $\{u_n\}$ is bounded in $\mathbb{R}^d$. Passing to a subsequence if needed, assume $u_n \to \bar{u} \in U$ (closedness of U). By continuity of $(t, u) \mapsto G_t(u)$, for every $u \in U$ we have

$$G_t(\bar{u}) = \lim_{n\to\infty} G_{t_n}(u_n) \ \le \ \lim_{n\to\infty} G_{t_n}(u) = G_t(u).$$

Thus $\bar{u}$ minimizes G_t on U; by uniqueness, $\bar{u} = u_*(t)$. Therefore the whole sequence converges: $u_n \to u_*(t)$, proving continuity. □

Remark 6.13. (Interpretation) Definition 6.11 isolates a natural structural assumption on object dictionaries: *log-convexity.* Under this hypothesis, meanings are not only guaranteed to exist (Theorem 6.10) but are unique at the ratio level and vary continuously with the symbol ratios (Theorem 6.12). This is the multi-attribute analogue of the 1D “stability away from decision boundaries” phenomenon: in higher dimensions, sharp switches are replaced by continuous motion of the unique optimizer within a convex feasible region.

7 Worked Examples

This section gives explicit toy models where the definitions can be computed by hand. The goal is to make the semantics tangible: we compute meanings in the continuous ratio model and in finite "object dictionaries," and we display the decision boundaries that govern when a symbol switches its meaning. These examples also serve as sanity checks for the main theorems.

7.1 Continuous ratio model

Proposition 7.1 (Meaning in the continuous ratio model)**.** *Let $S = O = \mathbb{R}_{>0}$ with $\iota_S = \iota_O = \mathrm{id}$ and $J_S = J_O = J$. Let $\mathcal{R}$ be admissible (Definition 3.6), so that*

$$c_{\mathcal{R}}(s, o) = J\left(\frac{s}{o}\right).$$

Then for every $s \in \mathbb{R}_{>0}$ there exists a unique *meaning $o^* \in \mathbb{R}_{>0}$, namely $o^* = s$, and the minimum reference cost equals* 0*.*

Proof. By Lemma 2.2, $J(x) \geq 0$ for all $x > 0$ with equality if and only if $x = 1$. Hence $c_{\mathcal{R}}(s, o) = J(s/o) \geq 0$ with equality if and only if $s/o = 1$, i.e. $o = s$. Therefore $o^* = s$ is the unique minimizer and $\min_o c_{\mathcal{R}}(s, o) = 0$. □

7.2 Finite "dictionary" of objects

Example 7.2 (A finite object set yields a nearest-ratio rule)**.** Let $O = \{o_1, \ldots, o_n\}$ be finite with ratios $y_i := \iota_O(o_i)$, and keep $S = \mathbb{R}_{>0}$ with $\iota_S = \mathrm{id}$. Under admissible reference, for a given s with ratio $x = \iota_S(s)$ the meanings are exactly the objects o_i that minimize

$$J\left(\frac{x}{y_i}\right) \qquad \text{over } i = 1, \ldots, n.$$

Ties can occur (two distinct minimizers), but only when x lies on a decision boundary determined by the values $\{y_i\}$.

7.3 Stability and decision boundaries for finite dictionaries

Theorem 7.3 (Geometric-mean decision boundaries for RS cost)**.** *Assume the explicit Recognition Science cost* (3) *and the admissible (ratio-induced) reference structure $c_{\mathcal{R}}(s, o) = J(\iota_S(s)/\iota_O(o))$. Let $O = \{o_1, \ldots, o_N\}$ be a finite object set such that the ratios $y_i := \iota_O(o_i)$ are pairwise distinct and ordered $0 < y_1 < \cdots < y_N$. For $x := \iota_S(s) \in \mathbb{R}_{>0}$ define the boundary points*

$$m_i := \sqrt{y_i y_{i+1}} \qquad (i = 1, \ldots, N-1),$$

and set $m_0 := 0$, $m_N := +\infty$. Then the set of meanings of s is determined as follows:

- *If $m_{k-1} < x < m_k$ for some $k \in \{1, \ldots, N\}$, then o_k is the* unique *meaning of s.*
- *If $x = m_k$ for some $k \in \{1, \ldots, N-1\}$, then s has* exactly two *meanings, namely o_k and o_{k+1}.*

Equivalently, the meaning map $x \mapsto \arg\min_i J(x/y_i)$ is piecewise constant on the open intervals (m_{k-1}, m_k).

Proof. Using (3) with x/y_i in place of x, one computes

$$c_{\mathcal{R}}(s, o_i) = J\Big(\frac{x}{y_i}\Big) = \frac{\big(\frac{x}{y_i} - 1\big)^2}{2(x/y_i)} = \frac{(x - y_i)^2}{2xy_i}.$$

Fix $i \in \{1, \dots, N-1\}$ and consider the cost difference between adjacent dictionary elements:

$$\Delta_i(x) := c_{\mathcal{R}}(s, o_{i+1}) - c_{\mathcal{R}}(s, o_i) = \frac{(x - y_{i+1})^2}{2xy_{i+1}} - \frac{(x - y_i)^2}{2xy_i}.$$

Multiplying by $2x > 0$ gives

$$2x\,\Delta_i(x) = \frac{(x - y_{i+1})^2}{y_{i+1}} - \frac{(x - y_i)^2}{y_i} = \Big(\frac{x^2}{y_{i+1}} - 2x + y_{i+1}\Big) - \Big(\frac{x^2}{y_i} - 2x + y_i\Big) = x^2\Big(\frac{1}{y_{i+1}} - \frac{1}{y_i}\Big) + (y_{i+1} - y_i).$$

Since $y_{i+1} > y_i$, this simplifies to

$$2x\,\Delta_i(x) = (y_{i+1} - y_i)\Big(1 - \frac{x^2}{y_i y_{i+1}}\Big).$$

Hence $\Delta_i(x) = 0$ if and only if $x^2 = y_i y_{i+1}$, i.e. $x = m_i$. Moreover, $\Delta_i(x) > 0$ when $x > m_i$ and $\Delta_i(x) < 0$ when $x < m_i$. Therefore, for $x < m_i$ the adjacent comparison favors o_{i+1} over o_i (smaller cost at o_{i+1}), while for $x > m_i$ it favors o_i over o_{i+1}.

Now let k be such that $m_{k-1} \le x \le m_k$. By chaining the adjacent comparisons, one concludes:

- If $m_{k-1} < x < m_k$, then $c_{\mathcal{R}}(s, o_k) < c_{\mathcal{R}}(s, o_j)$ for all $j \neq k$, so o_k is the unique minimizer.
- If $x = m_k$, then $c_{\mathcal{R}}(s, o_k) = c_{\mathcal{R}}(s, o_{k+1})$ and both are strictly smaller than all other costs, so the argmin has exactly two elements.

This proves the claim. □

Corollary 7.4 (Stability away from boundaries)**.** *Under the hypotheses of Theorem 7.3, if $m_{k-1} < x < m_k$ then there exists $\delta > 0$ such that every x' with $|x' - x| < \delta$ satisfies $m_{k-1} < x' < m_k$ and hence has the same unique meaning o_k.*

Proof. Since (m_{k-1}, m_k) is open and contains x, choose $\delta := \min\{x - m_{k-1},\ m_k - x\}/2 > 0$. Then $|x' - x| < \delta$ implies $x' \in (m_{k-1}, m_k)$, and the conclusion follows from Theorem 7.3. □

7.4 Numerical micro-example (three-object dictionary)

To make Theorem 7.3 concrete, take a three-object dictionary $O = \{o_1, o_2, o_3\}$ with ratios $y_1 = \iota_O(o_1) = \frac{1}{4} < y_2 = \iota_O(o_2) = 1 < y_3 = \iota_O(o_3) = 4$, and keep $S = \mathbb{R}_{>0}$ with $\iota_S = \mathrm{id}$. The decision boundaries are the geometric means $m_1 = \sqrt{y_1 y_2} = \frac{1}{2}$ and $m_2 = \sqrt{y_2 y_3} = 2$. Thus a symbol with ratio $x = \iota_S(s)$ means o_1 for $0 < x < \frac{1}{2}$, means o_2 for $\frac{1}{2} < x < 2$, and means o_3 for $x > 2$ (with ties at $x = \frac{1}{2}$ and $x = 2$).

x	$c_{\mathcal{R}}(s, o_1)$	$c_{\mathcal{R}}(s, o_2)$	$c_{\mathcal{R}}(s, o_3)$	meaning(s)
$\frac{3}{10}$	$\frac{1}{60}$ (≈ 0.017)	$\frac{49}{60}$ (≈ 0.817)	$\frac{1369}{240}$ (≈ 5.704)	o_1
$\frac{3}{2}$	$\frac{25}{12}$ (≈ 2.083)	$\frac{1}{12}$ (≈ 0.083)	$\frac{25}{48}$ (≈ 0.521)	o_2
3	$\frac{121}{24}$ (≈ 5.042)	$\frac{2}{3}$ (≈ 0.667)	$\frac{1}{24}$ (≈ 0.042)	o_3

Example 7.5 (Mediation can sharply reduce cost in a toy case)**.** Let $a = \iota_S(s) = 4$ and $c = \iota_O(o) = \frac{1}{4}$, so the direct admissible reference cost is $J(a/c) = J(16) = \frac{225}{32} \approx 7.03$. If the mediator space admits a configuration m with ratio $b_* = \sqrt{ac} = 1$, then Theorem 5.6 gives an optimal sequential cost

$$c_{\mathcal{R}_2 \circ \mathcal{R}_1}(s, o) = 2\,J\left(\sqrt{\frac{a}{c}}\right) = 2\,J(4) = \frac{9}{4} = 2.25,$$

which is strictly smaller, in accordance with Corollary 5.7.

8 Applications

This section distills immediate, checkable consequences of the formal development into application-facing corollaries. Each item below points back to a proved definition or theorem, so the reader can verify exactly what is (and is not) claimed. The overarching message is simple: reference/meaning is an optimization rule (Definition 3.8) driven by a universal mismatch cost (Definition 2.3), and its behavior under dictionaries, composition, and mediation is mathematically explicit.

8.1 Symbol grounding as a criterion

We treat "grounding" as an internal consistency condition: a token s is grounded for an object o when it (i) actually refers to o in the sense of our meaning definition, and (ii) does so economically (compression). This turns a philosophical question into a verifiable pair of inequalities/equalities.

Corollary 8.1 (Grounding criterion under admissible reference)**.** *Fix an admissible reference structure $\mathcal{R}$ (Definition 3.6). Then, for $s \in S$ and $o \in O$,*

$$(s, o) \text{ is a symbol (Definition 3.9)} \quad \Longleftrightarrow \quad \mathrm{Mean}_{\mathcal{R}}(s, o) \text{ and } J_S(s) < J_O(o).$$

Proof. This is immediate from Definition 3.9 and Definition 3.8. □

Corollary 8.2 (Grounding rule for finite object dictionaries)**.** *Assume the finite-dictionary hypotheses of Theorem 7.3. Then the meaning map $x \mapsto \mathrm{Mean}(s, \cdot)$ is piecewise constant in the symbol ratio $x = \iota_S(s)$, with decision boundaries at geometric means $m_i = \sqrt{y_i y_{i+1}}$. In particular, away from the boundaries the meaning is stable under small perturbations (Corollary 7.4).*

Proof. Immediate from Theorem 7.3 and Corollary 7.4. □

8.2 Mathematical effectiveness via low-cost primitives

The following corollary makes precise the "near-balance" phenomenon: if a symbol is constrained to have small intrinsic cost, then any object it can meaningfully denote (under admissible reference) must lie in a bounded neighborhood of balance. This supplies a concrete mechanism for why low-cost representations often track structurally "simple" targets.

Corollary 8.3 (Near-balance restricts possible referents)**.** *Assume $\mathcal{R}$ is admissible and that the hypotheses of Theorem 4.11 hold. If $s \in S$ satisfies $J_S(s) \le \epsilon$, and if o is a meaning of s, then*

$$J\left(\frac{\iota_S(s)}{\iota_O(o)}\right) \le \epsilon,$$

so $\iota_O(o)$ must lie in the corresponding bounded sublevel window determined by ϵ (as in Theorem 4.11).

Proof. This is a direct restatement of Theorem 4.11. □

Remark 8.4 (Compositional "range expansion" (model-dependent)). In a continuous ratio model where ratios can be realized densely (e.g. $S = O = \mathbb{R}_{>0}$ with $\iota = \mathrm{id}$ as in Proposition 7.1), large mismatches can be decomposed into many small mismatches: write a target ratio $r = e^t$ as a product $r = (e^{t/k})^k$. Since $J(e^u) = \cosh(u) - 1 \to 0$ as $u \to 0$, choosing k large makes each primitive step low-cost. Coupled with the compositionality results (Theorem 5.2) and optimal mediation (Corollary 5.7), this provides a concrete mathematical mechanism by which complex targets can be assembled from many low-cost referential primitives. This is an interpretive program; empirical relevance depends on what ratios are actually realizable in the intended application domain.

8.3 Information-theoretic interpretation

Although our framework is stated in intrinsic-cost terms, the RS cost admits a clean "log-ratio" form that connects naturally to coding/learning principles that penalize mismatch in a latent scale parameter. We record the basic identity as a proposition and keep the rest as interpretation.

Proposition 8.5 (Log-ratio form of the RS mismatch cost). *For $x > 0$ write $x = e^t$. Then the RS cost satisfies*

$$J(x) = J(e^t) = \cosh(t) - 1.$$

In particular, J is a convex even function of the log-ratio $t = \log x$ and vanishes exactly at $t = 0$.

Proof. Substitute $x = e^t$ into $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ (Definition 2.3). □

Remark 8.6 (Coding/learning viewpoint). In coding theory and learning, one often selects representations by minimizing a tradeoff between description length and distortion (e.g. Shannon [5] and MDL [7]). Our framework instantiates a specific distortion—$J(\iota_S(s)/\iota_O(o))$—that is symmetric in under-/over-shooting and naturally expressed in log-scale (Proposition 8.5). This suggests interpreting meanings as "best matches" under a fixed mismatch penalty, with compression enforced by the symbol condition $J_S(s) < J_O(o)$.

9 Related Work and Positioning

This section briefly positions the present framework relative to established approaches in semantics and information theory. The aim is not to survey the full literature, but to clarify what is new here: reference is defined by a *cost minimization rule* tied to a fixed mismatch functional, and we prove explicit decision boundaries and stability properties under that rule.

Symbol grounding and informational semantics. The symbol grounding problem asks how abstract tokens acquire meaning without an external interpreter [4]. Our model is compatible with grounding perspectives but differs in that "meaning" is defined operationally as an argmin under an explicit mismatch cost.

Compression principles in learning and coding. The idea that good representations compress while preserving task-relevant information is central in coding and learning theory; see classical sources such as Shannon's theory [5] and Kolmogorov complexity [6]. Minimum-description-length (MDL) formalizes model selection as a tradeoff clarifying "simplicity" [7]. While our setting is different (we work with ratios and a fixed J), the motivating philosophy is aligned: reference emerges from a constrained optimization principle.

What is mathematically concrete here. Beyond philosophical motivation, the paper proves explicit structural facts: under the RS mismatch functional, finite dictionaries induce geometric-mean decision boundaries (Theorem 7.3) and meanings are locally stable away from boundaries (Corollary 7.4); mediation admits an explicit optimal intermediate ratio (Theorem 5.6).

9.1 Real-world scenarios (interpretation)

The results above become intuitive when one reads the ratio map ι as a *resource scale* (how "big" a symbol or object is), and the meaning rule as "choose the object whose scale best matches the symbol." The statements in this subsection are *interpretations*: they illustrate how the proved theorems can be used once a modeling choice for ι is made.

Scenario 1: A chatbot choosing the right intent (finite dictionary). Suppose an assistant must map a user message to one of N intents (billing, cancel, technical support, etc.). Let each intent o_i have a learned scale $\iota_O(o_i)$ (e.g., derived from embedding norms or historical complexity), and let the message s have scale $\iota_S(s)$. Under admissible reference, the meaning is the intent that minimizes $J(\iota_S(s)/\iota_O(o_i))$. By Theorem 7.3, decision boundaries occur at geometric means $\sqrt{\iota_O(o_i)\iota_O(o_{i+1})}$, and Corollary 7.4 gives robustness away from boundaries.

Scenario 2: Product codes and compositional meaning. A retail system may combine a *brand token* and a *category token* to refer to a specific SKU. If (s_1, s_2) is the combined symbol and (o_1, o_2) the combined object (brand, category), Theorem 5.2 formalizes: the combined meaning decomposes into separate minimizations. Practically, one can calibrate brand and category scales independently, then combine them without re-optimizing the full joint map.

Scenario 3: Mediation as a "translation layer" (interfaces). In systems engineering, a command s may be routed through an intermediate protocol state m before reaching a device state o. Theorem 5.6 (together with Corollary 5.7) shows how an intermediate representation can strictly reduce cost in the RS model: inserting the right mediator improves a reference act by balancing ratios in stages.

Scenario 4: Human labels as compressed surrogates. A label like "flu" compresses a complex physiological state into a short symbol. The symbol condition (Definition 3.9) formalizes the compression requirement $J_S(s) < J_O(o)$. Combined with the meaning rule, the model predicts prototypes: labels stabilize near balanced matches and shift only when evidence pushes the ratio across a decision boundary (Theorem 7.3).

10 Discussion

This section positions the framework relative to classical accounts of meaning and highlights limitations. We summarize which parts are theorem-level consequences of the axioms and which parts remain interpretive. We also point to concrete open problems (e.g., richer admissible reference classes, multi-dimensional ratios, and empirical calibration) needed to broaden the scope of the theory.

10.1 Comparison to Prior Work

Our approach differs from classical semantic theories:

- **Frege**: Reference is primitive; we derive it from cost.
- **Russell**: Reference is quantificational; we ground it in compression.
- **Possible-worlds semantics**: Reference involves modal structure; we use cost structure.

Our framework aligns with information-theoretic viewpoints that treat representation as efficient coding (e.g. Shannon [5]), but differs in grounding mismatch in the fixed RS functional rather than in entropy alone.

10.2 Limitations

1. **Ratio embedding**: Our framework requires configurations to embed into $\mathbb{R}_{>0}$ via a ratio map. Not all semantic domains naturally admit such embeddings.
2. **Single cost functional**: We work with the normalized RS cost J. Alternative cost structures might yield different reference theories.
3. **Static analysis**: Our framework analyzes reference synchronically. Diachronic aspects (how reference changes over time) require extension.

10.3 Future Directions

1. **Neural reference**: How do neural systems implement cost-minimizing reference?
2. **Quantum reference**: Does quantum measurement theory admit RS-style reference analysis?
3. **Linguistic structure**: Can compositional semantics be derived from cost composition?

11 Conclusion

We have developed a mathematical theory of reference grounded in cost-minimization. Our main contributions:

1. **Reference as compression**: Symbols are low-cost encodings of high-cost objects.
2. **Cost-theoretic grounding**: The normalized RS cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ determines optimal reference.
3. **Universal backbone**: Near-balanced configurations provide a provable backbone window around balance under admissible reference (Theorem 4.11). Global descriptive reach is obtained by composing many such low-cost primitives (Section 5).
4. **Compositionality**: Reference structures compose via products and sequences.

The framework unifies formal semantics, information theory, and philosophy of mathematics under cost-minimization principles. Reference is not a primitive or mysterious relation—it is the natural consequence of seeking economical encodings in a world with cost structure.

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What is proved (mathematical content). Within the admissible ratio-induced model, we give a self-contained development of reference-as-optimization: meaning is defined by minimization and is guaranteed to exist under a closedness/attainment assumption (Theorem 4.6). For finite object dictionaries, the induced semantics has explicit geometric-mean decision boundaries and stability regions (Theorem 7.3, Corollary 7.4). We further prove exact compositionality under products (Theorem 5.2) and an explicit optimal-mediation principle (Theorem 5.6, Corollary 5.7). Finally, we derive a quantitative near-balance window/capacity bound (Theorem 4.11).

What is interpretation (scientific viewpoint). The Applications and Discussion sections interpret these theorems as a formal account of "aboutness" in which efficient reference emerges from balance-sensitive costs and compression constraints. These interpretations can be adopted or modified independently of the core mathematical results.

Open problems. Natural extensions include: (i) multi-dimensional ratio maps $\iota : C \to (\mathbb{R}_{>0})^n$ and corresponding decision geometry; (ii) broader admissible classes of reference structures beyond the ratio-induced form; and (iii) empirical or algorithmic learning procedures that recover admissible costs from data while preserving the proved stability/compositionality properties.

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