ZEROS OF THE RIEMANN ZETA FUNCTION VIA INNER FUNCTIONS AND BLASCHKE PRODUCTS

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Abstract. Starting from the Euler product and the regularized determinant $\det_2(I - A(s))$ over primes, we construct an inner function $\mathcal{I}$ on $\{\Re s > \frac{1}{2}\}$ whose zero set coincides with that of ζ, and prove that $\mathcal{I}$ is a *pure Blaschke product* (the singular inner factor is trivial). The Riemann Hypothesis is equivalent to the statement that this Blaschke product has no zeros. The construction proceeds via the arithmetic ratio $\mathcal{J} := \det_2(I - A(s))/\zeta(s) \cdot (s-1)/s$, whose poles coincide with the zeros of ζ; outer normalization produces a function with unit boundary modulus, and the inner reciprocal $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$ converts those poles into zeros of an inner function. The proof that the singular inner factor vanishes ($S \equiv 1$) uses only the convexity bound for ζ, the convergence of $\sum(1+\gamma^2)^{-1}$, and the explicit Fourier structure of $\det_2(I - A)$.

1. Introduction

The Riemann zeta function

$$\zeta(s) \;=\; \sum_{n\geq 1} \frac{1}{n^s}, \qquad \Re s > 1,$$

extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ and satisfies a functional equation after completion. Its nontrivial zeros govern the distribution of prime numbers, and the Riemann Hypothesis asserts that all such zeros lie on the critical line $\Re s = \frac{1}{2}$; see [2, 4, 6, 12] for background.

Theorem 1 (Inner-function encoding of the zeros of ζ)**.** *Let $\Omega := \{\, s \in \mathbb{C} : \Re s > \frac{1}{2} \,\}$. There exists a function $\mathcal{I}$, constructed explicitly from ζ, the regularized determinant $\det_2(I - A(s))$, and an outer normalizer $\mathcal{O}_\zeta$ (§§2–3, Lemma 16), with the following properties:*

(a) *$\mathcal{I}$ is holomorphic on Ω with $|\mathcal{I}(s)| \leq 1$ for all $s \in \Omega$.*

(b) *$|\mathcal{I}(\frac{1}{2} + it)| = 1$ for Lebesgue-a.e. $t \in \mathbb{R}$.*

(c) *The zeros of $\mathcal{I}$ in Ω are exactly the nontrivial zeros of ζ in Ω, with the same multiplicities.*

(d) *$\mathcal{I}$ is a pure Blaschke product: the singular inner factor is trivial, $S \equiv 1$.*

Corollary 2 (Equivalence with the Riemann Hypothesis)**.** *The Riemann Hypothesis is equivalent to the statement $\mathcal{I} \equiv e^{i\theta}$ for some $\theta \in \mathbb{R}$, i.e., the Blaschke product is empty.*

Proof. If RH holds, $\mathcal{I}$ has no zeros and is inner, hence a unimodular constant. Conversely, if $\mathcal{I} \equiv e^{i\theta}$, part (c) of Theorem 1 implies ζ has no zeros in Ω. □

Theorem 1 and Corollary 2 are proved in §§2–3 and Appendix A.

Notation. Throughout we use the following conventions.

- $\Omega := \{\, s \in \mathbb{C} : \Re s > \frac{1}{2} \,\}$ denotes the open half-plane to the right of the critical line, with boundary $\partial\Omega = \{\frac{1}{2} + it : t \in \mathbb{R}\}$.
- $\sigma := \Re s - \frac{1}{2}$ is the distance from the critical line.

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- $\langle T\rangle := (1+T^2)^{1/2}$ is the Japanese bracket.
- For a compact interval $I \subset \mathbb{R}$, $|I|$ denotes its length and
$$Q_\alpha(I) \ := \ \{\, \tfrac{1}{2}+\sigma+it : 0<\sigma\le \alpha\,|I|,\ t\in I\,\}$$
is the Whitney box with aperture $\alpha>0$.
- "A.e." refers to Lebesgue measure on $\mathbb{R}$ unless stated otherwise.

Strategy. On Ω we construct an *inner reciprocal* $\mathcal{I} := B^2/\mathcal{J}_{\mathrm{out}}$, where $B(s)=(s-1)/s$, from the Riemann zeta function, the regularized determinant $\det_2(I-A(s))$ over primes, and an outer normalizer $\mathcal{O}_\zeta$; the construction is carried out in §2–§3. Lemma 16 shows that $\mathcal{I}$ is holomorphic on Ω with $|\mathcal{I}|\le 1$ (via the Phragmén–Lindelöf principle) and boundary modulus 1 a.e. Crucially, zeros of ζ in Ω become *zeros* (not poles) of $\mathcal{I}$. The proof that $S\equiv 1$ (Proposition 17) then identifies $\mathcal{I}$ as a pure Blaschke product, yielding Theorem 1.

2. Definitions and main objects

This section introduces the principal objects of the proof: the prime-diagonal operator $A(s)$ and its regularized determinant $\det_2(I-A(s))$, and the arithmetic ratio $\mathcal{J}$ formed from $\det_2$ and ζ.

The completed zeta function. Let $\zeta(s)$ denote the Riemann zeta function. We write $\xi(s)$ for the completed zeta function
$$\xi(s) \ := \ \tfrac{1}{2}\, s(s-1)\,\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s),$$
which is entire and satisfies the functional equation $\xi(s)=\xi(1-s)$; see [12]. Throughout, by a *zero* we mean a zero of ζ (equivalently of ξ, away from the canceled singularities at $s=0,1$) lying in the half-plane Ω.

The prime-diagonal operator and the regularized determinant. Let $\mathcal{P}$ denote the set of primes and write $\ell^2(\mathcal{P})$ for the Hilbert space with orthonormal basis $\{e_p\}_{p\in\mathcal{P}}$. For $s\in\mathbb{C}$ define the prime-diagonal operator
$$A(s):\ell^2(\mathcal{P})\to\ell^2(\mathcal{P}), \qquad A(s)e_p := p^{-s}e_p.$$
For $\Re s>1/2$,
$$\|A(s)\|_{\mathrm{HS}}^2=\sum_{p\in\mathcal{P}}|p^{-s}|^2=\sum_{p\in\mathcal{P}}p^{-2\Re s}\le\sum_{n\ge 2}n^{-2\Re s}<\infty,$$
so $A(s)$ is Hilbert–Schmidt on Ω. In particular, the regularized determinant $\det_2(I-A(s))$ is well-defined and holomorphic on Ω (see [9, Ch. III] and [10, Ch. 9]).

Lemma 3 (Diagonal product formula for $\det_2$)**.** *Let T be a diagonal Hilbert–Schmidt operator on ℓ^2 with eigenvalues $\{\lambda_n\}$ satisfying $\sum_n|\lambda_n|^2<\infty$. Then*
$$\det_2(I-T) \ = \ \prod_n(1-\lambda_n)\,e^{\lambda_n},$$
where the product converges absolutely. In particular, $\det_2(I-T)=0$ *iff* $\lambda_n=1$ *for some* n.

Proof. This holds for the $\mathcal{S}_2$-regularized determinant; see [9, Ch. III] or [10, Ch. 9]. (We only use the diagonal case and the zero criterion $\lambda_n=1$.) □

Applying Lemma 3 to $T=A(s)$ on Ω gives the explicit product
$$\det_2(I-A(s)) \ = \ \prod_{p\in\mathcal{P}}(1-p^{-s})\,e^{p^{-s}}. \tag{2.1}$$

Since $\Re s>1/2$ implies $|p^{-s}|<1$ for every prime p, each factor in (2.1) is nonzero. Hence $\det_2(I-A(s))$ is holomorphic and zero-free on Ω.

The arithmetic ratio $\mathcal{J}$. Fix a domain $D \subset \Omega$. To allow numerically stable bounds later, we permit a holomorphic nonvanishing *normalizer* (or *gauge*) $\mathcal{O}$ on D, and define

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s} \cdot \frac{1}{\mathcal{O}(s)}, \qquad s \in D. \tag{2.2}$$

The factor $(s-1)$ cancels the simple pole of ζ at $s = 1$; the factor $1/s$ plays no role on $D \subset \Omega$ (but is convenient in later normalization). Since $\Omega \subset \{\Re s > 1/2\}$ lies away from $s = 0$, the compensator $1/s$ introduces no pole on the working domain. Unless explicitly stated otherwise, we work in the *raw ζ-gauge* $\mathcal{O} \equiv 1$ and denote the resulting objects by $\mathcal{J}_{\mathrm{raw}}$; for readability we usually drop the subscript in this default gauge.

Remark 4 (Gauge invariance of the pole set). Since $\mathcal{O}$ is holomorphic and nonvanishing on D, the pole set of $\mathcal{J}$ on D is independent of the choice of gauge. In the default gauge $\mathcal{O} \equiv 1$ one has $\mathcal{J}(s) \to 1$ as $\Re s \to +\infty$.

Lemma 5 (Zeros of ζ produce poles of $\mathcal{J}$)**.** *Let $D \subset \Omega$ be a domain and assume the chosen gauge $\mathcal{O}$ is holomorphic and nonvanishing on D. If $\rho \in D$ is a zero of $\zeta(s)$, then ρ is a pole of $\mathcal{J}(s)$ defined in* (2.2)*.*

Proof. By (2.2), the only possible singularities of $\mathcal{J}$ on D arise from zeros of ζ and from zeros of $\mathcal{O}$. The latter do not occur by assumption. The factor $(s-1)/s$ is holomorphic and nonzero on $D \subset \Omega$. Finally, $\det_2(I - A(s))$ is holomorphic and nonzero on Ω by (2.1). Hence a zero of ζ at ρ forces a pole of $\mathcal{J}$ at ρ. □

3. Outer normalization

The arithmetic ratio $\mathcal{J}$ from §2 has poles at the zeros of ζ, but its boundary modulus need not equal 1. We now divide by an outer function to impose unit boundary modulus, producing the outer-normalized ratio $\mathcal{J}_{\mathrm{out}}$ that serves as the principal object in the construction of the inner reciprocal $\mathcal{I}$. The construction proceeds in three stages: first we verify that the ratio F (i.e., (2.2) with $\mathcal{O} \equiv 1$) has well-behaved boundary values (Lemmas 6–11), then we extract the outer factor $\mathcal{O}_\zeta$ (Lemma 12), and finally we form $\mathcal{J}_{\mathrm{out}} = F/\mathcal{O}_\zeta$.

The ratio F and its boundary regularity. Define

$$F(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}, \qquad \Re s > \tfrac{1}{2},$$

and extend F to $\Omega \setminus Z(\zeta)$ by analytic continuation, where $Z(\zeta)$ denotes the zero set of ζ in Ω.

Lemma 6 (Boundary admissibility and Smirnov class for F)**.** *Let F be as above. Then on each connected component of $\Omega \setminus Z(\zeta)$:*

(1) F belongs to the Smirnov class N^+ (see, e.g., [3, Ch. 10]) and therefore admits nontangential boundary values $F^*(t) = \mathrm{n.t.}\lim_{\sigma \downarrow \frac{1}{2}} F(\sigma + it)$ for Lebesgue-a.e. $t \in \mathbb{R}$.

(2) The boundary log-modulus $u(t) := \log|F^*(t)|$ lies in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Moreover, if $|u(t)| \le C \log(2 + |t|)$ for $|t| \ge 1$, then $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$.

Proof. Fix a connected component U of $\Omega \setminus Z(\zeta)$. By Lemma 7, for every compact interval $I \Subset \mathbb{R}$ with $Q_\alpha(I) \Subset U$ the restriction of F to $Q_\alpha(I)$ is of bounded type. Since U is covered by such Whitney regions and bounded type is local on simply connected subdomains, it follows that F is of bounded type on U.

Next, on each such $Q_\alpha(I) \Subset U$, the boundary log-modulus of $\det_2(I - A)$ lies in $L^1(I)$ by Lemma 9, and $\log|\zeta(\frac{1}{2} + it)| \in L^1(I)$ with L^1-convergence from the interior by Lemma 10. Unwinding the definition of F (as a holomorphic combination of $\det_2(I - A)$ and ζ on U), this gives $\log|F^*| \in L^1_{\mathrm{loc}}$

on $\partial U \cap \{\Re s = \frac{1}{2}\}$. Applying Lemma 8 on each Whitney region yields $F \in N^+(U)$, hence F admits nontangential boundary values a.e. and $u(t) = \log |F^*(t)| \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Finally, if $|u(t)| \le C \log(2 + |t|)$ for $|t| \ge 1$, then

$$\int_{\mathbb{R}} \frac{|u(t)|}{1+t^2}\, dt \;\le\; C \int_{\mathbb{R}} \frac{\log(2+|t|)}{1+t^2}\, dt \;<\; \infty,$$

so $u \in L^1(\mathbb{R}, (1+t^2)^{-1} dt)$. □

The following two lemmas supply the inputs to Lemma 6: a local bounded-type criterion, and the Smirnov upgrade.

Lemma 7 (Local bounded-type control for F)**.** *Fix a compact interval $I \Subset \mathbb{R}$ and a Whitney region $Q_\alpha(I) \Subset \Omega$. Assume that the arithmetic Carleson energy bound of Lemma 14 holds on $Q_\alpha(I)$, so that* $\log |\det_2(I - A)|$ *has a BMO boundary trace on I (Lemma 9). Then F is of bounded type on $Q_\alpha(I)$.*

Proof. The outer normalizer construction (Lemma 13) provides a holomorphic, zero-free function $\mathcal{O}$ on $Q_\alpha(I)$. Define $\mathcal{J} := \det_2(I - A)/(\mathcal{O}\,\xi)$ on $Q_\alpha(I)$; since $\mathcal{O}$ is outer and ξ is holomorphic and nonvanishing on $Q_\alpha(I) \subset \Omega \setminus Z(\zeta)$, this ratio is of bounded type. By the definition of F, it is obtained from $\mathcal{J}$ by composing with holomorphic operations that preserve bounded type (products and quotients by nonvanishing bounded-type functions). Therefore F is of bounded type on $Q_\alpha(I)$. □

Lemma 8 (Smirnov upgrade from bounded type and boundary log-modulus)**.** *Let $U \subset \Omega$ be a simply connected domain with rectifiable boundary segment on $\Re s = \frac{1}{2}$ (e.g. a Whitney region $Q_\alpha(I)$ as in §A.1 of Appendix A). Let g be holomorphic on U and of bounded type (Nevanlinna class) on U. Assume g admits nontangential boundary values $g^*(t)$ for Lebesgue-a.e. t along $\partial U \cap \{\Re s = \frac{1}{2}\}$ and that* $\log |g^*(t)| \in L^1_{\mathrm{loc}}(dt)$ *on that boundary segment. Then $g \in N^+(U)$, and in particular g has nontangential boundary limits a.e. on $\partial U \cap \{\Re s = \frac{1}{2}\}$.*

Proof. By conformal mapping, it suffices to treat the case of the unit disk $\mathbb{D}$ (or upper half-plane) with boundary arc corresponding to the given rectifiable boundary segment. Since g is of bounded type on U, it belongs to the Nevanlinna class on U; equivalently, $g = h/k$ with $h, k \in H^\infty(U)$ and $k \not\equiv 0$. The hypothesis $\log |g^*| \in L^1_{\mathrm{loc}}$ on the boundary segment implies that the boundary values of $\log |k^*|$ are locally integrable there as well (because h is bounded), so the outer-function construction on U produces an outer function k_{out} with $|k^*_{\mathrm{out}}| = |k^*|$ a.e. on that segment. Replacing k by k_{out} and h by $h\,k/k_{\mathrm{out}}$ (which remains bounded and holomorphic) yields a representation $g = \tilde{h}/k_{\mathrm{out}}$ with $\tilde{h} \in H^\infty(U)$ and k_{out} outer. This is precisely $g \in N^+(U)$. In particular, functions in $N^+(U)$ admit nontangential boundary limits a.e. on the corresponding boundary segment. □

We next record the boundary regularity of the individual factors $\det_2(I - A)$ and ζ, which together control $\log |F^*|$.

Lemma 9 (From Carleson energy to L^1 boundary control for $\log |\det_2|$)**.** *Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. Let*

$$U_{\det_2}(\sigma, t) \;:=\; \log \Big| \det_2\Big(I - A(\tfrac{1}{2} + \sigma + it)\Big)\Big|, \qquad (\sigma, t) \in (0, \varepsilon_0] \times I,$$

where $\log |\det_2(I - A)|$ *is the real part of any analytic branch of* $\mathrm{Log}(\det_2(I - A))$*; it is subharmonic on Ω and harmonic away from the discrete zero set. Assume the Carleson energy bound of Lemma 14 for $\nabla U_{\det_2}$ on $Q(I)$, uniformly up to height ε_0. Then the boundary trace $u_{\det_2}(t) := \lim_{\sigma \downarrow 0} U_{\det_2}(\sigma, t)$ exists in* $\mathrm{BMO}(I)$ *(hence in $L^1(I)$), and in particular*

$$\sup_{0 < \sigma \le \varepsilon_0} \|U_{\det_2}(\sigma, \cdot)\|_{L^1(I)} \;<\; \infty.$$

Proof. On $\Omega \setminus Z(\det_2(I-A))$ the function $U_{\det_2} = \log|\det_2(I-A)|$ is harmonic. The Carleson energy hypothesis (Lemma 14) provides a Carleson-measure bound for $|\nabla U_{\det_2}|^2\,\sigma\,d\sigma\,dt$ on the box above I. By the Carleson-measure characterization of BMO boundary traces [11, Ch. IV],[5, Ch. VI], the nontangential boundary trace $u_{\det_2}(t) := \lim_{\sigma\downarrow 0} U_{\det_2}(\sigma,t)$ exists in $\mathrm{BMO}(I) \subset L^1(I)$, and $U_{\det_2}(\sigma,\cdot) \to u_{\det_2}$ in $L^1(I)$ as $\sigma \downarrow 0$. The discrete zero set is polar and does not affect boundary trace statements. ☐

Lemma 10 (Boundary log-modulus control for ζ on components)**.** *Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. Let U be a connected component of $\Omega \setminus Z(\zeta)$ intersecting $Q_{\varepsilon_0}(I)$. Then ζ is holomorphic and nonvanishing on U, hence $u(s) = \log|\zeta(s)|$ is harmonic on U. Moreover, the boundary trace $t \mapsto \log|\zeta(\frac{1}{2}+it)|$ lies in $L^1(I)$ and*

$$\log|\zeta(\tfrac{1}{2}+\varepsilon+it)| \to \log|\zeta(\tfrac{1}{2}+it)| \quad \textit{in } L^1(I) \textit{ as } \varepsilon \downarrow 0.$$

Proof. Let U be a connected component of $\Omega \setminus Z(\zeta)$ intersecting $Q_{\varepsilon_0}(I)$. Then ζ is holomorphic and nonvanishing on U, hence $u(s) = \log|\zeta(s)|$ is harmonic on U. On the compact strip segment $\{\sigma+it : \sigma \in [\frac{1}{2}, \frac{1}{2}+\varepsilon_0],\ t \in I\}$, ζ has only finitely many zeros (counted with multiplicity). For each zero s_k in this compact set, write $\zeta(s) = (s-s_k)^{m_k} g_k(s)$ with g_k holomorphic and nonvanishing in a neighborhood of s_k. Covering the compact strip by finitely many such neighborhoods and a zero-free remainder shows that on the strip

$$\log|\zeta(s)| = \sum_k m_k \log|s-s_k| + O(1),$$

with the $O(1)$ bounded on the strip. For each fixed s_k, the functions $t \mapsto \log|(\frac{1}{2}+\varepsilon+it)-s_k|$ are uniformly $L^1(I)$-bounded for $\varepsilon \in (0,\varepsilon_0]$ and converge in $L^1(I)$ as $\varepsilon \downarrow 0$. Therefore dominated convergence yields the stated $L^1(I)$ convergence $\log|\zeta(\frac{1}{2}+\varepsilon+it)| \to \log|\zeta(\frac{1}{2}+it)|$ as $\varepsilon \downarrow 0$. ☐

Combining the two preceding lemmas yields the local L^1 control of the full ratio F.

Lemma 11 (Local L^1 control of $\log|F^*|$ on boundary intervals)**.** *Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$, and set*

$$Q_{\varepsilon_0}(I) \;:=\; \{\, \tfrac{1}{2}+\sigma+it :\ 0 < \sigma \le \varepsilon_0,\ t \in I \,\} \Subset \Omega.$$

Let

$$F(s) \;:=\; \det{}_2(I-A(s))\,\frac{s-1}{s\,\zeta(s)}, \qquad s \in \Omega \setminus Z(\zeta).$$

Assume:

(i) $\log|\det_2(I-A(\frac{1}{2}+\varepsilon+it))| \in L^1(I)$ *uniformly for $\varepsilon \in (0,\varepsilon_0]$, and the nontangential boundary limit* $\log|\det_2(I-A(\frac{1}{2}+it))|$ *exists in $L^1(I)$;*

(ii) for each connected component U of $\Omega \setminus Z(\zeta)$ intersecting $Q_{\varepsilon_0}(I)$, the function $\log|\zeta(\frac{1}{2}+\varepsilon+it)|$ *has an $L^1(I)$-limit as $\varepsilon \downarrow 0$ when restricted to U.*

Then on each such component U, the nontangential boundary values $F^*(t)$ exist for Lebesgue-a.e. $t \in I$, and $\log|F^*(t)| \in L^1_{\mathrm{loc}}(I)$ on U.

Proof. Fix a component U as in the statement. For $s = \frac{1}{2}+\varepsilon+it$ with $0 < \varepsilon \le \varepsilon_0$ and $t \in I$, we have

$$\log|F(s)| = \log|\det{}_2(I-A(s))| + \log|s-1| - \log|s| - \log|\zeta(s)|.$$

Since I is compact and $\varepsilon \in (0,\varepsilon_0]$, the functions $t \mapsto \log|\frac{1}{2}+\varepsilon+it|$ and $t \mapsto \log|-\frac{1}{2}+\varepsilon+it|$ are bounded on I, uniformly in ε; hence $\log|s|$ and $\log|s-1|$ contribute uniformly bounded $L^1(I)$ terms. Assumptions (i)–(ii) therefore imply that $\log|F(\frac{1}{2}+\varepsilon+it)|$ is uniformly in $L^1(I)$ and has an $L^1(I)$ limit as $\varepsilon \downarrow 0$ along U. In particular, after passing to a subsequence if needed, $F(\frac{1}{2}+\varepsilon+it)$ has a nontangential boundary limit for a.e. $t \in I$, and the limiting boundary modulus satisfies $\log|F^*(t)| \in L^1_{\mathrm{loc}}(I)$ on U. ☐

Extracting the outer factor. The boundary regularity established above permits the construction of the outer normalizer $\mathcal{O}_\zeta$.

Lemma 12 (Outer factor from boundary modulus on Ω)**.** *Under the hypotheses of Lemma 6, assume in addition that $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$. Then there exists a holomorphic function $\mathcal{O}_\zeta$ on Ω, unique up to a unimodular constant, with no zeros on Ω, such that the nontangential boundary values satisfy*

$$|\mathcal{O}_\zeta(\tfrac{1}{2}+it)| = |F^*(t)| \qquad \text{for Lebesgue-a.e. } t \in \mathbb{R}.$$

Moreover, $\log|\mathcal{O}_\zeta(s)|$ is the Poisson extension of $u(t)$ from the boundary line $\Re s = \frac{1}{2}$.

Proof. Translate Ω to the right half-plane $\{\,\Re w > 0\,\}$ via $w = s - \frac{1}{2}$. Since $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$, its Poisson extension $U = \mathcal{P}[u]$ is a harmonic function on Ω with nontangential boundary trace u a.e. Choose a harmonic conjugate V of U on Ω and set $\mathcal{O}_\zeta := \exp(U + iV)$. Then $\mathcal{O}_\zeta$ is holomorphic and zero-free on Ω, and by Fatou theory its boundary modulus is $e^{u(t)}$ for a.e. t. Uniqueness up to a unimodular constant follows because the ratio of two such outer functions has boundary modulus 1 a.e. and hence is an inner constant; see Garnett [5, Ch. II]. $\square$

The outer-normalized ratio. Define

$$\mathcal{J}_{\text{out}}(s) \;:=\; \frac{F(s)}{\mathcal{O}_\zeta(s)} \;=\; \frac{\det_2(I - A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)} \cdot \frac{s-1}{s}. \tag{3.1}$$

By construction, $|\mathcal{J}_{\text{out}}(\frac{1}{2}+it)| = 1$ for Lebesgue-a.e. t.

Concluding remarks

Summary of results. Theorem 1 establishes that the zeros of ζ in Ω are encoded as a pure Blaschke product $\mathcal{I}$ on $\{\Re s > \frac{1}{2}\}$, with the singular inner factor provably trivial ($S \equiv 1$, Proposition 17). The Riemann Hypothesis is equivalent to the triviality of this Blaschke product (Corollary 2).

Principal ingredients. The construction at the heart of the paper—converting the arithmetic ratio $\mathcal{J}$ into an inner function via outer normalization—rests on the inner–outer factorization theory of Hardy spaces, a central tool in complex and harmonic analysis since the work of Beurling [1]; see [3, 5] for comprehensive treatments. The principal ingredients are:

- the explicit product formula for $\det_2(I - A)$ and the resulting Carleson energy bound (Lemma 14),
- the Smirnov-class regularity of the ratio F (Lemma 6),
- the Phragmén–Lindelöf bound $|\mathcal{I}| \le 1$ (Lemma 16), and
- the proof that $S \equiv 1$ (Proposition 17), which uses only the convexity bound for ζ, the convergence of $\sum(1+\gamma^2)^{-1}$, and the explicit Fourier structure of $\det_2(I - A)$.

From equivalence to proof of RH. Corollary 2 reduces RH to showing that the Blaschke product $\mathcal{I}$ has no zeros, i.e. that $\mathcal{I}$ is a unimodular constant. Two natural avenues toward this goal are:

(i) *Direct spectral gap.* Prove, using the explicit product structure of $\det_2(I - A)$ and the convexity bound for ζ, that the Cayley transform $\Xi := (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ satisfies a Schur bound $|\Xi| \le 1$ on all of Ω. This would exclude all poles of $\mathcal{J}$ and hence all zeros of ζ.

(ii) *Energy-theoretic approach.* Establish that the nonnegative potential $W = -\log|\mathcal{I}|$ vanishes identically, by showing that the Dirichlet energy of W on Whitney boxes decays to zero on the correct scale.

Extensions. The framework applies naturally to any L-function with an Euler product: the arithmetic ratio, outer normalization, and inner-function encoding generalize immediately. The key input is always the explicit product formula for the regularized determinant and the Smirnov-class regularity of the resulting ratio. For Dirichlet L-functions $L(s,\chi)$, the same construction produces a pure Blaschke product whose triviality is equivalent to GRH for χ.

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Appendix A. Analytic prerequisites

This appendix collects the analytic lemmas supporting Theorem 1: the outer normalizer construction (§A.1), the arithmetic Carleson energy bound and Riemann–von Mangoldt zero count (§A.2), and the inner reciprocal with its Phragmén–Lindelöf bound and the proof that $S \equiv 1$ (§A.3).

A.1. **Outer functions and standing notation.**

The conventions of §1 remain in force throughout.

Lemma 13 (Outer normalizer from boundary log-modulus)**.** *Let $u \in L^1(\mathbb{R},(1+t^2)^{-1}dt)$ be real-valued. Then there exists an outer function O on Ω (zero-free and holomorphic on Ω) whose nontangential boundary values satisfy*

$$|O(\tfrac{1}{2}+it)| = e^{u(t)} \quad \textit{for a.e. } t \in \mathbb{R}.$$

Moreover O is unique up to a unimodular constant.

Proof. Define the Poisson extension U of u to Ω by

$$U(\tfrac{1}{2}+\sigma+it) \;:=\; \frac{1}{\pi}\int_{\mathbb{R}} u(\tau)\,\frac{\sigma}{\sigma^2+(t-\tau)^2}\,d\tau, \qquad \sigma>0.$$

The weighted integrability $u \in L^1(\mathbb{R},(1+t^2)^{-1}dt)$ ensures the integral converges and that U is harmonic on Ω. Let V be a harmonic conjugate of U on Ω (defined up to an additive constant), and set

$$O(s) \;:=\; \exp\big(U(s)+iV(s)\big).$$

Then O is holomorphic and zero-free on Ω. By the nontangential boundary limit theorem for Poisson extensions of L^1_{loc} boundary data, one has $U(\frac{1}{2}+\varepsilon+it) \to u(t)$ for a.e. t as $\varepsilon \downarrow 0$; hence the nontangential boundary values satisfy $|O(\frac{1}{2}+it)| = e^{u(t)}$ for a.e. t; see Duren [3, Ch. II] or Garnett [5, Ch. II]. Uniqueness up to unimodular constant follows because the ratio of two such outer functions has a.e. boundary modulus 1 and hence is an inner constant. □

A.2. **Arithmetic Carleson energy and zero density.**

Lemma 14 (Arithmetic Carleson energy)**.** *Let*

$$U_{\det_2}(\sigma,t) \;:=\; \Re\log\det{}_2\Big(I-A(\tfrac{1}{2}+\sigma+it)\Big) \;=\; -\sum_p\sum_{k\ge 2}\frac{p^{-k/2}}{k}\,e^{-k\log p\,\sigma}\,\cos(k\log p\,t), \qquad \sigma>0,$$

where the series converges absolutely for every $\sigma>0$. Then for every interval $I \subset \mathbb{R}$ with Carleson box $Q(I) := I \times (0,|I|]$,

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2\,\sigma\,dt\,d\sigma \;\le\; \frac{|I|}{4}\sum_p\sum_{k\ge 2}\frac{p^{-k}}{k^2} \;=:\; K_0\,|I|, \qquad K_0 := \frac{1}{4}\sum_p\sum_{k\ge 2}\frac{p^{-k}}{k^2} < \infty.$$

Proof. For a single mode $b\,e^{-\omega\sigma}\cos(\omega t)$ one has $|\nabla|^2 = b^2\omega^2 e^{-2\omega\sigma}$, hence

$$\int_0^{|I|}\int_I |\nabla|^2\,\sigma\,dt\,d\sigma \;\le\; |I|\cdot\sup_{\omega>0}\int_0^{|I|}\sigma\,\omega^2 e^{-2\omega\sigma}d\sigma\cdot b^2 \;\le\; \tfrac{1}{4}\,|I|\,b^2.$$

With $b = p^{-k/2}/k$ and $\omega = k\log p$, summing over (p,k) gives the claim and the finiteness of K_0. □

Whitney scale and zero counts. Throughout, Whitney boxes are based at height T with

$$L = L(T) := \min\Big\{\frac{c}{\log\langle T\rangle},\ L_\star\Big\}, \qquad c \in (0,1] \text{ fixed.}$$

The only input about the number of zeros is the classical Riemann–von Mangoldt bound:

$$N(T;H) \ := \ \#\{\rho = \beta + i\gamma : \gamma \in [T, T+H]\} \ \le \ C_{\mathrm{RvM}}\,(1+H)\,\log\langle T\rangle, \tag{A.1}$$

for all $T \geq 2$ and $H > 0$, with C_{RvM} an absolute constant; see [12]. On Whitney scale $H = 2L$ this gives $N(T; 2L) = O(\log\langle T\rangle)$.

Lemma 15 (Local L^1 control for $\log|\xi|$ along vertical approach)**.** *Fix a compact interval $I \Subset \mathbb{R}$. Then the family $t \mapsto \log|\xi(\frac{1}{2}+\varepsilon+it)|$ is bounded in $L^1(I)$ uniformly for $\varepsilon \in (0,1]$. Moreover, for $\varepsilon, \varepsilon' \downarrow 0$ the difference $\log|\xi(\frac{1}{2}+\varepsilon+it)| - \log|\xi(\frac{1}{2}+\varepsilon'+it)|$ tends to 0 in $L^1(I)$.*

Proof. Write ξ in Hadamard form $\xi(s) = e^{a+bs}\prod_\rho(1-\frac{s}{\rho})e^{s/\rho}$, where the product runs over nontrivial zeros ρ of ζ. Fix $I = [T_0, T_1] \Subset \mathbb{R}$ and $\varepsilon \in (0,1]$. Split the zeros into a finite set $\mathcal{Z}_R := \{\rho : \ |\Im\rho| \le R\}$ and the complement, with $R \geq 2 + \max(|T_0|, |T_1|)$. For $\rho \in \mathcal{Z}_R$, the map $t \mapsto \log|(\frac{1}{2}+\varepsilon+it) - \rho|$ lies in $L^1(I)$, with an $L^1(I)$ bound depending only on I and $\mathcal{Z}_R$ (local integrability of $\log|t-\gamma|$ near $\gamma = \Im\rho$). For $\rho \notin \mathcal{Z}_R$ and $t \in I$, one has $|(\frac{1}{2}+\varepsilon+it)/\rho| \ll_I 1/|\rho|$, so

$$\log\Big|\Big(1 - \frac{\frac{1}{2}+\varepsilon+it}{\rho}\Big)e^{(\frac{1}{2}+\varepsilon+it)/\rho}\Big| = O_I(|\rho|^{-2}),$$

uniformly in $t \in I$ and $\varepsilon \in (0,1]$. Since $\sum_\rho |\rho|^{-2} < \infty$ (order 1 entire function), the tail contributes an absolutely convergent $L^\infty(I)$ error uniformly in ε. Combining these bounds gives $\sup_{\varepsilon\in(0,1]} \|\log|\xi(\frac{1}{2}+\varepsilon+i\cdot)|\|_{L^1(I)} < \infty$.

For the Cauchy property, write the difference as a sum over the same factorization. The finite set $\mathcal{Z}_R$ contributes a term that tends to 0 in $L^1(I)$ as $\varepsilon, \varepsilon' \downarrow 0$ by dominated convergence away from the finitely many points $t = \Im\rho$ and the local integrability of $\log|t - \Im\rho|$. The tail is uniformly $O_I\Big(\sum_{\rho\notin\mathcal{Z}_R}|\rho|^{-2}\Big)$ and hence uniformly small; letting $R \to \infty$ yields the $L^1(I)$-Cauchy claim. ☐

A.3. **Inner reciprocal and triviality of S.** The key device is the *inner reciprocal* $\mathcal{I} := B^2/\mathcal{J}_{\mathrm{out}}$, which converts poles of $\mathcal{J}_{\mathrm{out}}$ (at ζ-zeros) into zeros, yielding an inner function on Ω whose zero set coincides with that of ζ in Ω.

Lemma 16 (Inner reciprocal and nonnegative potential)**.** *Let $\mathcal{J}_{\mathrm{out}}$ be as in* (3.1) *and $B(s) := (s-1)/s$. Define*

$$\mathcal{I}(s) \ := \ \frac{B(s)^2}{\mathcal{J}_{\mathrm{out}}(s)} \ = \ \frac{B(s)\,\mathcal{O}_\zeta(s)\,\zeta(s)}{\det_2(I - A(s))}\,.$$

Then:

(1) $\mathcal{I}$ is holomorphic on Ω. (The simple pole of ζ at $s = 1$ is canceled by B; zeros of ζ become zeros of $\mathcal{I}$; the denominator $\det_2(I-A)$ is nonvanishing on Ω.)

(2) $|\mathcal{I}(\frac{1}{2}+it)| = 1$ for Lebesgue-a.e. t. (On $\partial\Omega$: $|B| = 1$ and $|\mathcal{J}_{\mathrm{out}}| = 1$ a.e.)

(3) $|\mathcal{I}(s)| \le 1$ for all $s \in \Omega$. (Phragmén–Lindelöf: $\log|\mathcal{I}|$ is subharmonic on Ω with boundary trace 0 a.e. and at most polynomial growth; see below.)

In particular, the function

$$W(s) \ := \ -\log|\mathcal{I}(s)| \ \ge \ 0 \qquad (s \in \Omega)$$

is nonnegative, and one has the identity

$$U(s) \ := \ \log|\mathcal{J}_{\mathrm{out}}(s)| \ = \ 2\log|B(s)| \ + \ W(s) \qquad (s \in \Omega \setminus Z(\zeta)).$$

Proof. Part (1). Write $\mathcal{I} = B\,\mathcal{O}_\zeta\,\zeta/\det_2(I-A)$. The factor $B\zeta = (s-1)\zeta(s)/s$ is holomorphic on Ω (the simple pole of ζ at $s=1$ is canceled by the zero of $s-1$, and $s=0 \notin \Omega$). The remaining factors $\mathcal{O}_\zeta$ (outer, zero-free) and $1/\det_2(I-A)$ (nonvanishing by (2.1)) are holomorphic on Ω. Hence $\mathcal{I}$ is holomorphic on Ω, with zeros exactly at the nontrivial zeros of ζ in Ω (same multiplicities).

Part (2). On $\partial\Omega$: $|B(\frac{1}{2}+it)|^2 = |(-\frac{1}{2}+it)/(\frac{1}{2}+it)|^2 = (\frac{1}{4}+t^2)/(\frac{1}{4}+t^2) = 1$, and $|\mathcal{J}_{\mathrm{out}}(\frac{1}{2}+it)| = 1$ a.e. by construction. Hence $|\mathcal{I}(\frac{1}{2}+it)| = |B|^2/|\mathcal{J}_{\mathrm{out}}| = 1$ a.e.

Part (3): $|\mathcal{I}| \le 1$ *via Phragmén–Lindelöf.* Since $\mathcal{I}$ is holomorphic on Ω, $u := \log|\mathcal{I}|$ is subharmonic on Ω.

Boundary trace. For $\varepsilon > 0$ set $s_\varepsilon := \frac{1}{2} + \varepsilon + it$. Each factor of $\mathcal{I} = B\,\mathcal{O}_\zeta\,\zeta/\det_2(I-A)$ has L^1_{loc}-convergent log-modulus as $\varepsilon \downarrow 0$:

- $\log|B(s_\varepsilon)| \to 0$ uniformly (B is continuous and $|B^*| = 1$);
- $\log|\mathcal{O}_\zeta(s_\varepsilon)| \to u(t)$ in L^1_{loc} ($\mathcal{O}_\zeta$ is the Poisson extension of $u := \log|F^*|$);
- $\log|\zeta(s_\varepsilon)| \to \log|\zeta^*(t)|$ in L^1_{loc} (Lemma 10 or 15);
- $\log|\det_2(s_\varepsilon)| \to \log|\det_2{}^*(t)|$ in L^1_{loc} (BMO boundary trace from the arithmetic Carleson energy, Lemma 14).

Since $u = \log|\det_2{}^*| - \log|\zeta^*|$ by construction of $\mathcal{O}_\zeta$, the sum of boundary traces is $0 + u + \log|\zeta^*| - \log|\det_2{}^*| = 0$. Hence $u^*(\frac{1}{2}+it) = \log|\mathcal{I}^*(t)| = 0$ for a.e. t. No Smirnov or Hardy class membership is invoked; only the L^1_{loc} convergence of each factor's log-modulus is needed.

Growth. $|\mathcal{I}(s)| \le C(1+|t|)^N$ for some N and all $s = \frac{1}{2}+\sigma+it$ with $\sigma \in (0,1]$ (this follows from the convexity bound for ζ, the absolutely convergent product for $\det_2$, and the Poisson-controlled modulus of $\mathcal{O}_\zeta$). Hence $u(s) = O(\log(2+|s|)) = o(|s|)$ as $|s| \to \infty$ in Ω.

Conclusion. By the Phragmén–Lindelöf principle for subharmonic functions on the half-plane (e.g. [7, Ch. III] or [8, Thm. 5.3.4]): a subharmonic function on Ω with nontangential boundary trace ≤ 0 a.e. and growth $o(|s|)$ satisfies $u \le 0$ on Ω. Hence $|\mathcal{I}| \le 1$ and $W = -\log|\mathcal{I}| \ge 0$. ☐

With Lemma 16 in hand, it remains to show that the singular inner factor is trivial.

Proposition 17 (Triviality of the singular inner factor)**.** *The inner function $\mathcal{I}$ from Lemma* 16 *has trivial singular inner factor: $S \equiv 1$. Hence $\mathcal{I}$ is a pure Blaschke product, and Theorem* 1(d) *is proved.*

Proof. The singular inner factor satisfies $S \equiv 1$ if and only if

$$\lim_{\sigma\to 0^+} \int_{\mathbb{R}} \frac{W(\frac{1}{2}+\sigma+it)}{1+t^2}\,dt \;=\; 0$$

(see Garnett [5, Ch. II]). We prove this by showing that each factor of $\mathcal{I} = B\,\mathcal{O}_\zeta\,\zeta/\det_2$ has log-modulus converging in $L^1(\mathbb{R}, dt/(1+t^2))$ as $\sigma \to 0$, and that the boundary traces sum to 0.

Term $\log|B|$. $B = (s-1)/s$ is continuous with $|B^*| = 1$; convergence is uniform.

Term $\log|\mathcal{O}_\zeta|$. $\mathcal{O}_\zeta$ is the outer function with boundary modulus $\exp(u)$, so $\log|\mathcal{O}_\zeta(\sigma)| = P_\sigma[u] \to u$ in $L^1(dt/(1+t^2))$ by Poisson convergence.

Term $\log|\det_2|$. By explicit Fourier computation,

$$\int_{\mathbb{R}} \frac{\log|\det_2(\sigma,t)|}{1+t^2}\,dt \;=\; -\pi \sum_p \sum_{k\ge 2} \frac{p^{-k(\frac{3}{2}+\sigma)}}{k}\,,$$

which converges absolutely to $-\pi\sum_p\sum_{k\ge2} p^{-3k/2}/k$ as $\sigma \to 0$.

Term $\log|\zeta|$ *(the key term).* We must show $\int \log|\zeta(\frac{1}{2}+\sigma+it)|/(1+t^2)\,dt \to \int \log|\zeta^*(t)|/(1+t^2)\,dt$ as $\sigma \to 0$.

(a) The $\log^+$ *part.* $\log^+|\zeta(\frac{1}{2}+\sigma+it)| \le A\log(2+|t|)$ uniformly for $\sigma \in (0,1]$ (convexity bound; Titchmarsh [12, Ch. V]). Since $A\log(2+|t|)/(1+t^2) \in L^1$, dominated convergence applies.

(b) The $\log^-$ *part.* Cover $\mathbb{R}$ by unit intervals $I_n = [n, n+1]$. On each I_n, Jensen's inequality for the subharmonic function $\log|\zeta(\frac{1}{2}+\sigma+i\cdot)|$ on a disc of radius 2 centered at $n+\frac{1}{2}+i\sigma$ gives

$$\int_{I_n} \log^- |\zeta(\tfrac{1}{2}+\sigma+it)|\, dt \ \le \ \pi\cdot 4\cdot \big(A\log(3+|n|)+C\big) \ + \ \pi\cdot 4\cdot N_n\cdot \log 4,$$

where N_n is the number of ζ-zeros with $|\gamma-(n+\frac{1}{2})|\le 4$ and the right side comes from the standard Jensen bound ($\int \log^-|f| \le$ mean of $\log^+|f|$ on a larger circle + zero count $\cdot \log(\text{ratio})$). By (A.1): $N_n \le C_1(1+4)\log\langle n\rangle = O(\log\langle n\rangle)$. Hence

$$\int_{I_n} \log^- |\zeta(\sigma,t)|\, dt \ \le \ C_2\log(2+|n|) \qquad \text{uniformly for } \sigma\in(0,1].$$

Dividing by $1+t^2 \ge 1+n^2$ and summing: $\int_{\mathbb{R}} \log^-|\zeta(\sigma)|/(1+t^2) \le \sum_n C_2\log(2+|n|)/(1+n^2) < \infty$. This bound is uniform in σ.

(c) Convergence. L^1_{loc} convergence $\log|\zeta(\sigma)| \to \log|\zeta^*|$ holds by Lemma 15. Combined with the σ-uniform $L^1(dt/(1+t^2))$ bound from (a) and (b), Vitali's convergence theorem gives $\int \log|\zeta(\sigma)|/(1+t^2) \to \int \log|\zeta^*|/(1+t^2)$.

Assembly. By the construction of $\mathcal{O}_\zeta$: $u = \log|\det_2{}^*| - \log|\zeta^*|$, so the boundary traces satisfy $0 + u + \log|\zeta^*| - \log|\det_2{}^*| = 0$. Hence

$$\lim_{\sigma\to 0}\int \frac{W(\sigma,t)}{1+t^2}\, dt \ = \ 0-(-u)-(-\log|\zeta^*|)+(-\log|\det_2{}^*|) \ = \ 0.$$

Therefore $S \equiv 1$. (This argument uses only: the convexity bound for ζ, the convergence of $\sum 1/(1+\gamma^2)$, the outer construction of $\mathcal{O}_\zeta$, and the explicit Fourier series for $\det_2$. No zero-free hypothesis is used.) ☐

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