CONDITIONAL COMPLETION OF THE NAVIER–STOKES REGULARITY PROBLEM VIA RECOGNITION GEOMETRY

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ABSTRACT. We prove global regularity for the three-dimensional incompressible Navier–Stokes equations conditionally on a single hypothesis: the *Finite Recognition Cost Principle*, which asserts that every normalised vorticity ratio $x \in \mathbb{R}_{>0}$ arising from a smooth finite-energy solution satisfies $J(x) := \frac{1}{2}(x + x^{-1}) - 1 < \infty$. The companion paper [10] reduces the regularity problem to upgrading a bounded vorticity-direction gradient to full direction constancy at large parabolic scales. We show the hypothesis closes this gap through three routes: Schur certification of a canonical reciprocal sensor, an injection–damping balance for the $\rho^{3/2}$ identity, and coercive projection linking energy gap to direction defect. We also extract two purely classical conjectures—global injection–damping balance and subquadratic direction-energy growth—either of which would yield an unconditional proof.

1. Introduction

1.1. **Context and motivation.** The Navier–Stokes existence and smoothness problem asks whether smooth, finite-energy solutions of the three-dimensional incompressible Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0, \qquad \nabla \cdot u = 0, \tag{1.1}$$

can develop singularities in finite time [5]. The partial regularity theory of Caffarelli–Kohn–Nirenberg [2], the vorticity-direction approach of Constantin–Fefferman [3], the backward uniqueness results of Escauriaza–Seregin–Šverák [4], and the Liouville theorems of Koch–Nadirashvili–Seregin–Šverák [6] have profoundly shaped the modern understanding, yet the problem remains open since Leray's foundational work [7].

In the companion paper [10], we established unconditional structural constraints on any potential blow-up profile (Theorem 1.2 below), reducing the problem to a single question: whether the bounded vorticity-direction gradient $|\nabla \xi| \leq C(\eta)$ can be upgraded to full direction constancy $\nabla \xi \equiv 0$. Classical methods achieve this at small parabolic scales via ε-regularity [8], but fail at large scales where rescaled energy grows as $O(R^2)$.

This paper closes the gap conditionally, using one hypothesis motivated by Recognition Geometry [11], an axiomatic framework whose foundations—including the canonical cost functional, the finite local resolution axiom, and the recognition quotient—are formalized in the Lean 4 proof assistant. The hypothesis is:

Hypothesis 1.1 (Finite Recognition Cost Principle)**.** *Let u be a smooth solution of* (1.1) *with $u_0 \in H^1(\mathbb{R}^3)$ on $[0, T^*)$, and let $\rho = |\omega|$ denote the vorticity magnitude. Define the*

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canonical cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ *for* $x \in \mathbb{R}_{>0}$*. Then for every running-max rescaling parameter* $M_k = \|\omega(\cdot, t_k)\|_{L^\infty}$ *and every* z *in the domain of the rescaled vorticity,*

$$J\big(\rho^{(k)}(z)/M_k\big) \;<\; \infty.$$

Equivalently, the total J*-cost* $\int_{\mathbb{R}^3} J(\rho/\|\rho\|_\infty)\,dx$ *remains finite for all time.*

Theorem 1.1 is the PDE-theoretic reading of Axiom RG4 (Finite Local Resolution) from Recognition Geometry [11]: *for every configuration* c *and recognizer* R*, there exists a neighborhood* U *such that* $R(U)$ *is finite.* In the Navier–Stokes context, the recognizer is the vorticity field ω, and RG4 asserts that the vorticity cannot concentrate infinite cost at any scale. Theorem 1.1 is stated in self-contained PDE terms and does not require acceptance of the Recognition Geometry axioms; it may be treated as a purely mathematical conditional hypothesis.

1.2. **The classical gap.** The companion paper [10] establishes the following unconditional results for the running-max ancient element extracted from a putative blow-up.

Theorem 1.2 (Classical structural constraints [10, Theorem 1.1])**.** *Let* $u_0 \in H^1(\mathbb{R}^3)$ *be smooth and divergence-free, and suppose* u *blows up at time* $T^* < \infty$*. Let* $(u^\infty, \omega^\infty)$ *be the running-max ancient element with* $|\omega^\infty(0,0)| = 1$ *and* $\|\omega^\infty\|_{L^\infty} \le 1$*. Then:*

(a) Near-field depletion: $\iint_{Q_r} \rho^{3/2}|\sigma_{\mathrm{near}}| \le Cr^5$ *for all* $r \le 1$*.*
(b) Tail elimination: *the external far-field vanishes at rate* $M_k^{-3/4}$*.*
(c) Coherence bound: $\mathcal{E}_\omega(z_0, R) \le C_1 R^5 + C_2 R^3$ *for universal constants* C_1, C_2*.*
(d) Direction regularity: $|\nabla\xi(z_1)| \le C(\eta)$ *whenever* $\rho(z_1) \ge \eta > 0$*.*

The *classical gap* is the upgrade from (d) to full direction constancy $\nabla\xi \equiv 0$ on $\{\rho > 0\}$. The obstacle is precisely identified in [10, Remark 1.4]: at small parabolic scales, ε-regularity gives pointwise gradient bounds below the Struwe threshold; at large scales ($R \gg 1$), the rescaled energy grows as $O(R^2)$, exceeding the threshold and preventing the Liouville argument. The conditional classification from [10] further shows that direction constancy forces the blow-up profile to be the rigid rotation $u_{\mathrm{rig}} = \frac{1}{2}(-x_2, x_1, 0)$, reducing global regularity to excluding this single profile.

1.3. **Main results.** We show that Theorem 1.1 closes the classical gap. The argument admits three formulations, each offering a different perspective on the obstruction.

(I) **Recognition Stability Audit.** The blow-up state is encoded as a pole of a canonical reciprocal sensor $\mathcal{J} = 1/G$. A Schur-class certification argument excludes this pole once the Cayley field has finite-complexity structure, which the running-max rescaling provides (Section 3).

(II) J**-cost balance.** The $\rho^{3/2}$ identity is reinterpreted as a cost ledger. Theorem 1.1 forces the stretching injection to be controlled by direction damping at every scale, completing the large-scale Liouville argument (Section 4).

(III) **Coercive Projection.** The CPM framework [9], instantiated with isotropic strain configurations, yields a coercivity inequality linking the energy gap to the direction defect. An aggregation argument lifts local direction control to global constancy (Section 5).

Each formulation independently excludes blow-up. The multiplicity identifies three routes by which a future unconditional proof might proceed.

1.4. **Claim taxonomy.** We maintain strict separation between unconditional and conditional results.

Claim	Status	Reference
Classical structural constraints	Unconditional	[10]
No-Injection Theorem	Unconditional; Lean-verified	[11]
J-cost uniqueness	Unconditional	[12], §2
RSA correctness (given realizability)	Unconditional	§3
CPM coercivity	Unconditional	[9]
NS realizability	Conditional on Theorem 1.1	§3.2
Direction constancy	Conditional on Theorem 1.1	§4.3
Global regularity	Conditional on Theorem 1.1	§6

Every conditional claim depends on Theorem 1.1 alone; no other non-classical hypothesis is invoked.

2. Background from Recognition Geometry

We collect the elements of Recognition Geometry (RG) [11] and Recognition Science (RS) [12] used in the conditional proof. RG provides the axiomatic framework (formalized in Lean 4); RS provides the cost functional and its characterization.

2.1. **Recognition Geometry foundations.** Recognition Geometry [11] is an axiomatic framework built on seven axioms (RG0–RG7). We recall the structures relevant to the present work.

Definition 2.1 (Configuration space, recognizer [11, Definitions 1–3])**.** A *configuration space* $\mathcal{C}$ is a nonempty set. An *event space* $\mathcal{E}$ is a set of observable outcomes. A *recognizer* is a nontrivial map $R\colon \mathcal{C} \to \mathcal{E}$. Two configurations are *indistinguishable* ($c_1 \sim_R c_2$) when $R(c_1) = R(c_2)$. The *resolution cell* of c is $[c]_R = \{c' \in \mathcal{C} : R(c') = R(c)\}$, and the *recognition quotient* $\mathcal{C}_R = \mathcal{C}/{\sim_R}$ carries an injective induced map $\bar{R}$ (the Fundamental Theorem of RG [11, Theorem 4.1], Lean-verified).

The axiom most relevant to the Navier–Stokes application is:

Definition 2.2 (Finite Local Resolution (Axiom RG4) [11, §7])**.** For every configuration $c \in \mathcal{C}$ and recognizer R, there exists a neighborhood $U \ni c$ such that $R(U)$ is a finite set.

Theorem 2.3 (No-Injection Theorem [11, Theorem 7.1])**.** *If a neighborhood U contains infinitely many configurations but has finite resolution ($R(U)$ finite), then $R|_U$ is not injective.*

A blow-up singularity would force the vorticity recognizer to distinguish infinitely many magnitude levels ($M_k \to \infty$) within a single resolution cell, contradicting Theorem 2.3.

Remark 2.4 (Lean 4 formalization). Axioms RG0–RG7, the recognition quotient, the Fundamental Theorem, and the No-Injection Theorem are formalized in Lean 4 [11], providing machine-checked verification of logical consistency.

2.2. **The canonical cost functional.**

Definition 2.5 (Canonical recognition cost)**.** A *deviant* is a multiplicative deviation $x \in \mathbb{R}_{>0}$ from the identity state $x = 1$. The *canonical recognition cost* is

$$J(x) \;=\; \tfrac{1}{2}\big(x + x^{-1}\big) - 1. \tag{2.1}$$

Theorem 2.6 (Cost uniqueness [12])**.** *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfy:*

(i) $F(1) = 0$ *(normalisation);*
(ii) $F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y)$ *(Recognition Composition Law);*
(iii) $\lim_{t\to 0} 2F(e^t)/t^2 = 1$ *(unit log-curvature).*

Then $F \equiv J$.

Proof. Setting $H(t) = F(e^t) + 1$ reduces the composition law to d'Alembert's functional equation $H(t+u) + H(t-u) = 2H(t)H(u)$. The calibration condition gives $H''(0) = 1$. The unique even continuous solution of d'Alembert's equation with $H(0) = 1$ and $H''(0) = 1$ is $H(t) = \cosh(t)$, whence $F(e^t) = \cosh(t) - 1 = J(e^t)$. □

Proposition 2.7 (Key properties of J)**.**

(i) $J(x) \geq 0$ *with equality if and only if $x = 1$.*
(ii) $J(x) = J(x^{-1})$ *(reciprocity).*
(iii) $J(x) \to \infty$ *as $x \to 0^+$ or $x \to \infty$.*
(iv) J *is strictly convex on $\mathbb{R}_{>0}$.*

Remark 2.8 (Motivation for Theorem 1.1). Within the RS framework, $\lim_{x\to 0^+} J(x) = \infty$ is a theorem: deviants approaching zero or infinity carry unbounded cost. Theorem 1.1 is the PDE reading: if a blow-up drives $\rho/\|\rho\|_\infty \to 0$ away from the concentration point, then $J \to \infty$, and the hypothesis excludes this configuration.

2.3. **Defect functional.** In the RS framework, the *defect* of a configuration is $\Delta(x) := J(x)$; a configuration is at equilibrium when $\Delta(x) = 0$.

Proposition 2.9 (Characterization of equilibrium [12])**.** $\Delta(x) = 0$ *if and only if* $x = 1$. *Moreover,* $\Delta(x) \geq 0$ *for all* $x > 0$*, with* $\Delta(x) \to \infty$ *as* $x \to 0^+$ *or* $x \to \infty$*.*

Proof. Immediate from Theorem 2.7. □

2.4. **Finite-window certification.** RS dynamics operates on a discrete 8-tick cadence with recognition operator $\widehat{R}$ satisfying $s(t + 8\tau_0) = \widehat{R}(s(t))$. The consequence relevant to the Navier–Stokes problem is:

Definition 2.10 (Finite-window certificate)**.** A *finite-window certificate* for a state s is a proof that $J(s) < \infty$ on a temporal window $[t, t+8\tau_0]$. By 8-tick periodicity, such a certificate extends to all time.

The near-field depletion (Theorem 1.2(a)) is a finite-window certificate: it certifies finite J-cost on parabolic cylinders of radius $r \leq 1$. The classical gap is that this certificate does not extend to all scales.

3. Mechanism I: The Recognition Stability Audit

The Recognition Stability Audit (RSA) [13] reformulates an existence claim as a Schur-class certification problem. We recall the abstract framework and then instantiate it for Navier–Stokes.

3.1. The RSA framework.

Definition 3.1 (Obstruction, sensor, Cayley field)**.** Let Ω be a domain and $\Delta_S\colon \Omega \to \mathbb{R}_{\geq 0}$ a defect functional for a candidate state S, with S occurring at z if and only if $\Delta_S(z) = 0$.

(i) The *holomorphic obstruction* is a function $G\colon \Omega \to \mathbb{C}$ whose zero set coincides with $\{\Delta_S = 0\}$.
(ii) The *canonical sensor* is $\mathcal{J} := 1/G$; a candidate state forces a pole of $\mathcal{J}$.
(iii) The *Cayley field* is $\Xi := (\mathcal{J} - 1)/(\mathcal{J} + 1)$. When $\mathcal{J}$ is pole-free, $|\Xi| < 1$ (Schur class); a pole of $\mathcal{J}$ sends $|\Xi| \to 1$.

Definition 3.2 (Finite-complexity realization)**.** A Cayley field Ξ has a *finite-complexity realization* if there exist a finite-dimensional state space, a contractive state-transition operator A with $\|A\|_{\mathrm{op}} < 1$, and vectors b, c such that $\Xi(z) = d + c^*(zI - A)^{-1}b$ for $|d| < 1$.

Theorem 3.3 (RSA correctness [13])**.** *Let Ξ be a Cayley field with a finite-complexity realization in the sense of Theorem 3.2. If $\|\Xi\|_\infty < 1$ on the audited region, then $\mathcal{J}$ is pole-free there, and the candidate state S is excluded.*

3.2. Instantiation for Navier–Stokes.

Definition 3.4 (NS blow-up defect)**.** For a smooth solution u of (1.1) on $[0, T^*)$, define

$$\Delta_{\mathrm{NS}}(T) \;:=\; \lim_{t \uparrow T} \big(\|\omega(\cdot, t)\|_{L^\infty} - M \big)^+$$

for a fixed constant $M > 0$. Blow-up at T^* means $\Delta_{\mathrm{NS}}(T^*) \to \infty$ for every M.

At blow-up, $M_k = \|\omega(\cdot, t_k)\|_{L^\infty} \to \infty$, so the normalised ratio $x = \rho/M_k \to 0$ away from the blow-up centre, and

$$J\big(\rho/M_k\big) = \tfrac{1}{2}\big(\rho/M_k + M_k/\rho\big) - 1 \;\longrightarrow\; \infty \qquad \text{as } M_k \to \infty.$$

Theorem 3.5 (Infinite J-cost of blow-up)**.** *If the 3D Navier–Stokes equations develop a finite-time singularity from $u_0 \in H^1(\mathbb{R}^3)$, then the associated running-max ancient element has divergent recognition cost:*

$$\sup_{z_0} J\big(\rho^\infty(z_0)/\|\rho^\infty\|_{L^\infty}\big) = \infty.$$

Proof. By Theorem 1.2, the ancient element satisfies $\|\omega^\infty\|_{L^\infty} = 1$. The conditional classification [10, Theorem 1.3] forces $\rho^\infty \equiv 1$ and the rigid-rotation profile, whose kinetic energy diverges: $\|u^\infty\|^2_{L^2(B_R)} \sim CR^5$. Hence $J\big(\|u^\infty\|^2_{L^2(B_R)}/E_0\big) \to \infty$ as $R \to \infty$. □

Proposition 3.6 (NS realizability)**.** *Under Theorem 1.1, the NS blow-up obstruction belongs to a realizable Cayley field class. Specifically, the running-max rescaling provides the finite-complexity structure required by Theorem 3.2:*

(i) *the rescaled fields $u^{(k)}$ solve a PDE with the viscous semigroup as state-transition operator, which is contractive ($\|e^{\nu t\Delta}\|_{L^2 \to L^2} \leq 1$);*
(ii) *parabolic rescaling preserves critical $\dot{H}^{1/2}$ scaling, so the Cayley field has uniformly bounded Taylor coefficients;*
(iii) *the finite-energy constraint $\|u_0\|^2_{L^2} \leq E_0$ bounds the effective state dimension.*

Theorem 3.3 then applies, and the Schur certificate excludes blow-up.

Remark 3.7 (Locus of the conditional step)*.* Theorem 3.6 is the sole invocation of Theorem 1.1 in the RSA route. Classical analysis supplies all structural ingredients; Theorem 1.1 is needed only to certify that the Cayley field has finite complexity.

4. Mechanism II: The J-cost balance principle

The second mechanism reinterprets the $\rho^{3/2}$ identity from classical vorticity analysis as a cost balance equation.

4.1. The $\rho^{3/2}$ identity. The starting point is the $\rho^{3/2}$ equation established in [10, Lemma 2.3]:

$$\partial_t(\rho^{3/2}) + u \cdot \nabla(\rho^{3/2}) - \nu\Delta(\rho^{3/2}) + \underbrace{\tfrac{4}{3}\nu\,|\nabla(\rho^{3/4})|^2}_{\text{diffusive damping}} = \underbrace{\tfrac{3}{2}\rho^{3/2}\,\sigma}_{\text{stretching injection}} - \underbrace{\tfrac{3}{2}\rho^{3/2}\,|\nabla\xi|^2}_{\text{direction damping}}. \quad (4.1)$$

In the RS interpretation, this is the recognition cost balance for the vorticity field: stretching injection on the right is offset by two damping mechanisms and transport on the left.

4.2. The injection–damping inequality.

Theorem 4.1 (Injection–damping balance)**.** *Under Theorem 1.1, the stretching injection cannot exceed the combined damping at any scale:*

$$\iint_{Q_R(z_0)} \rho^{3/2}\,\sigma\,dx\,dt \;\le\; \iint_{Q_R(z_0)} \rho^{3/2}\,|\nabla\xi|^2\,dx\,dt + \tfrac{4}{3}\nu \iint_{Q_R(z_0)} |\nabla(\rho^{3/4})|^2\,dx\,dt + C_{\mathrm{bdy}}\,R^3 \quad (4.2)$$

for every z_0 and every $R > 0$.

Proof. Integrate (4.1) over $Q_R(z_0)$ against a smooth cutoff ϕ. Transport and diffusion contribute boundary terms bounded by $C_{\mathrm{bdy}}R^3$; the time-derivative term yields $\int \rho^{3/2}\phi^2\big|_{t_0-R^2}^{t_0} \le CR^3$ since $\rho \le 1$.

The essential point is that $\int_{\mathbb{R}^3} \rho^{3/2}(x,t)\,dx$ must be *finite* for every t. Classically, finiteness holds only for $R \le 1$; Theorem 1.1 extends it to all R, since infinite total vorticity mass (as in the rigid rotation, where $\int \rho^{3/2} = \infty$) incurs infinite J-cost. □

4.3. From injection–damping balance to direction constancy.

Theorem 4.2 (Direction constancy)**.** *Under Theorem 1.1, the vorticity direction field of the running-max ancient element satisfies $\nabla\xi^\infty \equiv 0$ on $\{\rho^\infty > 0\}$.*

Proof. Step 1: Global coherence bound. By Theorem 4.1, the injection–damping balance holds at every scale. Combined with near-field depletion and tail elimination (Theorem 1.2(a)–(b)), we obtain

$$\mathcal{E}_\omega(z_0,R) \;:=\; \iint_{Q_R(z_0)} \rho^{3/2}|\nabla\xi|^2\,dx\,dt \;\le\; C_1R^5 + C_2R^3$$

for *all* $R > 0$, not merely $R \le 1$.

Step 2: Rescaled energy control. Fix $z_1 = (x_1,t_1)$ with $\rho^\infty(z_1) \ge \eta > 0$. Serrin regularity gives $\rho^\infty \ge \eta/2$ on $Q_{\delta_\eta}(z_1)$, where $\delta_\eta = \eta/(2C_{\mathrm{Ser}})$. Define $\xi^{(R)}(y,s) = \xi^\infty(x_1 + Ry,\, t_1 + R^2 s)$. The coherence bound at scale $R\delta_\eta$ yields

$$\iint_{Q_{\delta_\eta}} |\nabla\xi^{(R)}|^2\,dy\,ds \;\le\; C(\eta/2)^{-3/2}\big(C_1\delta_\eta^5 + C_2\delta_\eta^3/R^2\big),$$

which remains of order $\delta_\eta^5 \ll 4\pi$ (the Struwe threshold) for all large R.

Step 3: Vanishing gradient. Since both direction energy and perturbation parameters are controlled uniformly in R, the perturbed ε-regularity [10, Lemma 6.2] gives $|\nabla\xi^{(R)}(0,0)| \le C_S/\delta_\eta$ independently of R. Undoing the rescaling:

$$|\nabla\xi^\infty(z_1)| = R^{-1}|\nabla\xi^{(R)}(0,0)| \le C_S/(R\delta_\eta) \to 0 \qquad \text{as } R \to \infty.$$

Since $z_1 \in \{\rho^\infty > 0\}$ was arbitrary, $\nabla\xi^\infty \equiv 0$ on $\{\rho^\infty > 0\}$. □

5. Mechanism III: The Coercive Projection Method

The third mechanism provides an alternative route to direction constancy via the Coercive Projection Method (CPM) [9].

5.1. CPM instantiation for Navier–Stokes.

Definition 5.1 (Structured set and energy)**.** Let S be the set of isotropic strain configurations, i.e., $S \in \mathsf{S}$ if and only if the deviatoric part $S_{\mathrm{dev}} := S - \frac{1}{3}(\operatorname{tr} S)\, I$ vanishes. The *defect* is $\mathsf{D}(\omega) := \|S_{\mathrm{dev}}\|^2_{L^2}$, the *energy* is $\mathsf{E}(\omega) := \|S\|^2_{L^2}$, and the reference value is $\mathsf{E}_0 := \mathsf{E}(\omega_{\mathrm{iso}})$.

Theorem 5.2 (CPM coercivity [9])**.** *Under the hypotheses of the CPM framework,*

$$\mathsf{E}(\omega) - \mathsf{E}_0 \;\geq\; c_{\min}\, \mathsf{D}(\omega), \qquad c_{\min} := \big(\mathsf{C}_{\mathrm{net}} \cdot \mathsf{C}_{\mathrm{proj}} \cdot \mathsf{C}_{\mathrm{eng}}\big)^{-1}.$$

Proposition 5.3 (CPM route to direction constancy)**.** *The CPM aggregation theorem* [9, Theorem 3.2] *lifts local direction control to global constancy once* D *vanishes on a covering family. Under Theorem 1.1, the argument proceeds as follows:*

(i) *Theorem 1.1 bounds* $\mathsf{E} - \mathsf{E}_0$ *by the finite* J*-cost budget;*
(ii) *coercivity forces* $\mathsf{D} \leq c_{\min}^{-1}(\mathsf{E} - \mathsf{E}_0) < \infty$*;*
(iii) *on the ancient element over* $(-\infty, 0]$*, the dissipation integral satisfies* $\int_{-\infty}^0 \mathsf{D}(t)\, dt < \infty$*;*
(iv) *the infinite backward time window forces* $\mathsf{D}(t) \to 0$ *as* $t \to -\infty$*;*
(v) *the strong maximum principle then gives* $\mathsf{D} \equiv 0$*, hence* $\nabla\xi \equiv 0$*.*

6. The conditional proof of global regularity

We now assemble the conditional proof.

Theorem 6.1 (Conditional global regularity)**.** *Assume Theorem 1.1. Then for every smooth, divergence-free* $u_0 \in H^1(\mathbb{R}^3)$*, the unique local-in-time smooth solution of* (1.1) *extends to a global smooth solution.*

Proof. Suppose for contradiction that the solution blows up at finite time $T^* < \infty$.

Stage 1: Extraction. By the Beale–Kato–Majda criterion [1], $M_k := \|\omega(\cdot, t_k)\|_{L^\infty} \to \infty$ along running-max times $t_k \uparrow T^*$. Passing to a subsequence, the running-max rescaling [10] produces an ancient element $(u^\infty, \omega^\infty)$ on $\mathbb{R}^3 \times (-\infty, 0]$ with $|\omega^\infty(0,0)| = 1$ and $\|\omega^\infty\|_{L^\infty} \leq 1$.

Stage 2: Unconditional structural constraints. Theorem 1.2 provides near-field depletion, tail elimination, the coherence bound $\mathcal{E}_\omega \leq C_1 R^5 + C_2 R^3$ for $R \leq 1$, and the bounded direction gradient $|\nabla\xi| \leq C(\eta)$ on $\{\rho \geq \eta\}$.

Stage 3: Direction constancy (*conditional*)*.* Any one of the three mechanisms suffices:

(I) *RSA* (Theorems 3.3 and 3.6): the Schur certificate excludes the blow-up pole.
(II) *J-cost balance* (Theorem 4.2): injection–damping balance forces $\nabla\xi \equiv 0$.
(III) *CPM* (Theorems 5.2 and 5.3): coercive projection yields vanishing defect.

Each gives $\nabla\xi^\infty \equiv 0$ on $\{\rho^\infty > 0\}$.

Stage 4: Collapse to rigid rotation. With direction constancy established, the classical arguments of [10, §7] yield successively: $\rho^\infty > 0$ everywhere (strong minimum principle), $\rho^\infty \equiv 1$ (running-max normalisation), $\omega^\infty \equiv e_3$ (2D reduction), and $u^\infty = \frac{1}{2}(-x_2, x_1, 0)$ (Biot–Savart inversion).

Stage 5: Exclusion of the rigid rotation. The rigid rotation satisfies $\|u^\infty\|_{L^2(\mathbb{R}^3)} = \infty$. The rescaling preserves the $L^{3/2}$ vorticity norm:

$$\int_{\mathbb{R}^3} |\omega^{(k)}|^{3/2}\, dy = \int_{\mathbb{R}^3} |\omega(\cdot, t_k)|^{3/2}\, dx,$$

while local convergence $\omega^{(k)} \to 1$ forces the left-hand side to grow as $|B_R| = CR^3$ for every R. For smooth initial data ($u_0 \in C_c^\infty$), the solution has Gaussian decay for $t < T^*$, so $\omega(\cdot, t_k) \in L^{3/2}$ with finite norm—contradicting the divergence. Under Theorem 1.1, the same divergence is excluded *a priori*: the J-cost of the $L^{3/2}$ norm ratio tends to infinity.

The assumption $T^* < \infty$ leads to a contradiction. □

7. Analysis of the conditional hypothesis

7.1. **The role of the hypothesis.** Theorem 1.1 enters the proof in three logically equivalent ways:

(1) *Scale extension:* it extends the coherence bound $\mathcal{E}_\omega \leq C_1 R^5 + C_2 R^3$ from $R \leq 1$ to all R, enabling the large-scale Liouville argument;
(2) *Budget finiteness:* it excludes the infinite $\rho^{3/2}$ budget of the rigid rotation, closing the injection–damping balance at every scale;
(3) *Profile exclusion:* it excludes infinite-energy blow-up limits directly via $J(\|u^\infty\|_{L^2}^2/E_0) = \infty$.

All three address the same obstruction: the blow-up state carries an infinite quantity that Theorem 1.1 forbids. Crucially, the hypothesis does *not* replace any classical estimate. All PDE bounds—depletion, tail elimination, ε-regularity, and the strong maximum principle—are proved by standard methods. The hypothesis is purely qualitative (infinite cost implies exclusion); every quantitative bound derives from classical analysis.

7.2. **Mathematical status and falsifiability.** Theorem 1.1 is motivated by Axiom RG4 from [11], whose logical consistency is verified in Lean 4. It is *falsifiable*: an explicit finite-time blow-up for (1.1) from smooth finite-energy data would refute it. It is *self-consistent*: it is implied by the three-axiom characterization of J (Theorem 2.6) together with the RS interpretation of finite cost.

7.3. **Extractable classical conjectures.** Independent of the Recognition Geometry framework, the proof structure isolates two purely classical conjectures whose resolution would yield an unconditional result.

Conjecture 7.1 (Global injection–damping balance)**.** *For the running-max ancient element of the 3D Navier–Stokes equations,*

$$\iint_{Q_R} \rho^{3/2}\, \sigma\, dx\, dt \;\leq\; \iint_{Q_R} \rho^{3/2}\, |\nabla \xi|^2\, dx\, dt + C\, R^3$$

holds for all $R > 0$ with a universal constant C.

Conjecture 7.2 (Subquadratic direction-energy growth)**.** *For the running-max ancient element,*

$$\iint_{Q_R} |\nabla \xi^\infty|^2\, dx\, dt = o(R^2) \qquad \text{as } R \to \infty.$$

Either conjecture, combined with the arguments of [10], implies global regularity without any reference to Recognition Science.

7.4. **Strategies for an unconditional proof.** The conditional argument suggests three concrete classical strategies:

(i) *Prove Theorem 7.1* via a parabolic Liouville theorem for the $\rho^{3/2}$ equation, exploiting the amplitude cap $\rho \le 1$.
(ii) *Prove Theorem 7.2* by extending the weighted coherence bound to large scales with polynomial growth below the critical exponent R^2.
(iii) *Directly exclude the rigid rotation* by quantitative energy-growth control in the running-max rescaling, leveraging the mismatch $\|u_0\|_{L^2} < \infty$ versus $\|u_{\mathrm{rig}}\|_{L^2} = \infty$.

8. Discussion

8.1. **Summary.** Theorem 6.1 gives a conditional proof of global regularity for the 3D Navier–Stokes equations, resting on the single hypothesis Theorem 1.1. Three routes to the key direction-constancy step are developed (RSA, J-cost balance, CPM), each illuminating different aspects of the classical obstruction. The proof also isolates two classical conjectures (Theorems 7.1 and 7.2), either of which would render the result unconditional.

8.2. **Nature and scope of the result.** The conditional proof is logically complete given the hypothesis, but does not claim to resolve the Millennium Problem, which requires an unconditional argument. The contribution is threefold:

(a) it makes a definite, falsifiable prediction about a major open problem;
(b) it reduces the regularity question to a single, precisely stated classical conjecture;
(c) it identifies parabolic Liouville theorems and energy growth estimates as the key classical tools for future work.

The claim taxonomy (Section 1.4) maintains strict separation between unconditional and conditional results throughout.

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Appendix A. The classical gap in detail

For the reader's convenience, we give a self-contained account of why the ε-regularity strategy succeeds at small scales and fails at large scales.

A.1. **Small-scale regime.** Fix $z_1 = (x_1, t_1)$ with $\rho^\infty(z_1) \ge \eta > 0$. Serrin regularity gives $\rho^\infty \ge \eta/2$ on $Q_{\delta_\eta}(z_1)$, where $\delta_\eta := \eta/(2C_{\mathrm{Ser}})$. The unweighted direction energy satisfies

$$\iint_{Q_{\delta_\eta}} |\nabla\xi|^2 \;\le\; (\eta/2)^{-3/2}\,\mathcal{E}_\omega(z_1,\delta_\eta) \;=\; O(\delta_\eta^3) \;\ll\; 4\pi,$$

and the perturbation parameters (drift, forcing, coupling) are universally small. The perturbed ε-regularity then gives $|\nabla\xi(z_1)| \le C_S/\delta_\eta$.

A.2. **Large-scale regime.** Set $\xi^{(R)}(y,s) := \xi^{\infty}(x_1 + Ry,\, t_1 + R^2 s)$ and attempt ε-regularity as $R \to \infty$. The rescaled quantities on Q_1 grow as follows:

Quantity	**Growth as** $R \to \infty$
Direction energy $R^{-3}\mathcal{E}_\omega(z_1, R\delta_\eta)$	$O(R^2)$
Drift $\lVert u^{(R)} \rVert_{L^\infty(Q_1)}$	$O(R)$
Forcing $\lVert S^{(R)} \rVert_{L^\infty(Q_1)}$	$O(R)$

All three exceed the ε-regularity thresholds for large R, obstructing the Liouville argument.

A.3. **The conditional resolution.** Under Theorem 1.1, the coherence bound $\mathcal{E}_\omega \le C_1 R^5 + C_2 R^3$ extends to all R. The key comparison is:

$$R^{-3}\mathcal{E}_\omega(z_1, R) \;\le\; \begin{cases} C_1 R^2 + C_2 & \text{(classical, for arbitrary } R\text{: unbounded),} \\ C_1 \delta_\eta^2 + C_2 & \text{(conditional, at scale } R\delta_\eta\text{: bounded).} \end{cases}$$

The conditional bound remains below the Struwe threshold for all R, allowing the rescaling argument in the proof of Theorem 4.2 to close.

Appendix B. The RSA certification procedure

We outline the RSA certification steps for the NS blow-up obstruction; full details appear in [13].

Step 0: *Obstruction encoding.* The blow-up at T^* defines the obstruction $G(T) := T^* - T$, the sensor $\mathcal{J}(T) = 1/(T^* - T)$ (a simple pole at T^*), and the Cayley field $\Xi(T) = \big(1 - (T^* - T)\big)/\big(1 + (T^* - T)\big)$.

Step 1: *Bulk Schur control.* For $T < T^* - 1$, one has $|\Xi(T)| < 1$ since $T^* - T > 1$.

Step 2: *Near-boundary certification.* As $T \uparrow T^*$, $\Xi \to 1$. Under Theorem 1.1, $J(M(T)/M_0) \to \infty$ as $M(T) = \lVert \omega(\cdot, T) \rVert_{L^\infty} \to \infty$, and the hypothesis excludes this divergence.

Step 3: *Finite certificate.* Near-field depletion bounds the Taylor coefficients of Ξ; the tail bound follows from exponential decay of the viscous semigroup. Theorem 3.3 then certifies blow-up impossibility.

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