Response on NS draft and Lemma 2.6 (Ancient Tangent Flow)

December 11, 2025

To: Milan Zlatanović
Cc: Prof. Elshad Allahyarov
Re: "Geometric Depletion Mechanisms in the 3D Incompressible Navier–Stokes Equations" (Overleaf project: link)

Dear Milan,

Thank you for the careful pass and for restructuring Lemma 2.6. I agree with your assessment: the *idea* of Lemma 2.6 is right, but the original presentation was too compressed for a referee-level check. Your decomposition into four lemmas *(singular point existence; normalization; domain exhaustion; ancient limit)* is exactly the right way to make the logic verifiable.

Layperson subtext (what your comments are really asking): the "tangent flow" step is the bridge from the classical smooth solution (where we can do pointwise/vorticity arguments) to the weak/limit object (where we can only pass properties that are stable under compactness). So the question is not whether one can *define* a rescaling, but whether one can (i) choose centers/scales that do not drift to infinity, (ii) obtain enough uniform bounds to extract a subsequence, and (iii) guarantee the limit is *nontrivial* in a way that survives convergence. These are exactly the three places where blow-up arguments can silently fail if not spelled out.

My answers / how the model closes the proof:

1. **Your Lemma "Existence of a singular point" is the correct anchor.** Once we know there exists at least one CKN-singular point (x^*, T^*), we should *anchor the blow-up sequence to that point.* This is what prevents the "escape to infinity" issue: we choose all centers x_k inside a fixed ball around x^* and choose $t_k \uparrow T^*$.

2. **Normalization: prefer a normalization that passes to the limit robustly.** There are two common normalizations:

   - *Vorticity normalization* $|\omega^{(k)}(0,0)| = 1$ is intuitive, but to pass it to the limit one needs enough compactness to control $\omega^{(k)}$ (or an indirect argument that prevents $u^\infty \equiv 0$).
   - *CKN-functional normalization* is more robust: because (x^*, T^*) is singular, there exists a sequence $r_k \downarrow 0$ such that

   $$r_k^{-2} \iint_{Q_{r_k}(x^*,T^*)} \left(|u|^3 + |p|^{3/2}\right) dx\, dt \geq \varepsilon_{\mathrm{CKN}}.$$

   If we set $\lambda_k := r_k$ and center at (x^*, T^*), then the rescaled sequence satisfies the *scale-1 lower bound* on Q_1 automatically, and this lower bound survives to the limit by lower semicontinuity. This gives nontriviality of u^∞ in a way that is hard to lose.

   Practically: your current vorticity-based normalization lemma is fine, but I recommend (for referee-proof clarity) adding one short remark that we *can also* normalize via the CKN functional at the singular point, and that this is the cleanest route to nontriviality of the limit.

3. **Domain exhaustion under rescaling: your Lemma is correct.** The key estimate is $\lambda_k^{-2} t_k \to \infty$, so $(-R^2, 0] \subset (-\lambda_k^{-2} t_k, 0]$ for large k.

4. **Compactness / existence of the ancient limit: what needs to be made explicit.** To close the final lemma rigorously, we should state the precise compactness input (a standard theorem for suitable weak solutions): from the local energy inequality we get uniform bounds on each cylinder Q_R:

$$u^{(k)} \text{ bounded in } L_t^\infty L_x^2(Q_R) \cap L_t^2 H_x^1(Q_R), \qquad p^{(k)} \text{ bounded in } L^{3/2}(Q_R),$$

   and then Aubin–Lions gives strong compactness in L^q_{loc} for $q < 3$ (and strong L^3_{loc} away from $s = 0$ is standard in blow-up literature, as you wrote). This yields a subsequence converging to an ancient suitable weak solution (u^∞, p^∞) on $\mathbb{R}^3 \times (-\infty, 0]$.

5. **Nontriviality of the limit: the missing closure step.** The cleanest closure is:

   (a) Choose the rescaling so that a scale-invariant quantity is *uniformly bounded below* (e.g. the CKN functional on Q_1) along the sequence.

   (b) Pass to the limit using lower semicontinuity/strong L^3_{loc} convergence to obtain a positive lower bound for u^∞ on some Q_r, which implies $u^\infty \not\equiv 0$.

   This is exactly what your statement "$\int_{Q_r} |u^\infty|^3 \geq c$" encodes; the only remaining work is to tie it explicitly to the singular-point lower bound before rescaling.

6. **One technical simplification: remove any global L^∞ claim for u^∞.** I agree with your earlier concern in the old draft: asserting $u^\infty \in L^\infty$ is not justified from local energy bounds alone and is not needed for the geometric depletion mechanism. The right inherited quantities are local suitable-weak bounds and scale-invariant $L^3/L^{3/2}$ controls.

Physical-motivation paragraph (suggested insertion): I agree it would help the exposition to add a short physics paragraph near the start of the blow-up section: the blow-up procedure is a "microscope"/renormalization idea—we zoom into regions where vorticity becomes large and ask whether the rescaled flow has a nontrivial limiting profile. The geometric depletion method then shows that any such limiting profile must have essentially constant vorticity direction (hence becomes 2D), and 2D dynamics cannot sustain a 3D blow-up mechanism. This connects the analysis to the physical intuition that vortex stretching requires *misalignment* of vorticity direction, and that strong alignment kills stretching.

Bottom line: your four-lemma decomposition is the right structure. To "close" Lemma 2.6 cleanly, I suggest anchoring the blow-up at a CKN singular point and (optionally) normalizing via the CKN functional to make nontriviality of the limit immediate by semicontinuity. With those minor clarifications, the lemma becomes fully referee-checkable.

Thank you again—please proceed with preparing the Recognition Geometry manuscript while we finalize this lemma chain; I'm happy for you to keep polishing the blue-verified parts in parallel.

Sincerely,
Jonathan Washburn