Axiomatic Derivation of the Recognition Composition Law:

Cost Uniqueness and Combiner Rigidity

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Abstract

We study cost functions on $\mathbb{R}_{>0}$ whose log-lift satisfies a d'Alembert-type composition law. Under normalization, reciprocity, C^2 regularity, and a unit-curvature calibration in log-coordinates, we show that the cost is uniquely determined as the reciprocal cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$. We then prove a rigidity result for composition rules: if a function $P : \mathbb{R}^2 \to \mathbb{R}$ satisfies $J(xy) + J(x/y) = P(J(x), J(y))$ for all $x, y > 0$, then $P(u, v) = 2uv + 2u + 2v$ for all $u, v \geq 0$, with global extension to $\mathbb{R}^2$ under real-analyticity. The core results are machine-verified in Lean 4.[deleted: prior abstract overstated what follows from the stated axioms; revised to match the actual assumptions used in the proofs.]

Contents

1 Introduction

1.1 Overview and contributions

Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ be a cost function and write $G(t) := F(e^t)$. The central functional identity in this paper is a d'Alembert-type composition law for G, which becomes the classical d'Alembert functional equation after the shift $H := G + 1$.

Our contributions are:

- **Cost uniqueness.** Under normalization, reciprocity, C^2 regularity, and a unit-curvature calibration in log-coordinates, the d'Alembert-type composition law forces F to coincide with the reciprocal cost J.

- **Combiner rigidity.** Once $F = J$ is fixed, any function P satisfying $J(xy) + J(x/y) = P(J(x), J(y))$ is uniquely determined on $[0, \infty)^2$ with *no* regularity assumptions on P. If P is real-analytic, this determination extends to all of $\mathbb{R}^2$.

We refer to the polynomial combiner $P(u, v) = 2uv + 2u + 2v$ as the *Recognition Composition Law* (RCL).[deleted: previous hierarchy table of "conditional/semi-conditional" results; replaced with an explicit contributions list and sharper scope.]

1.2 Motivation

In many mathematical and physical settings one introduces a scalar *cost* (or action, divergence, or penalty) to quantify deviation from a reference state, and then posits a rule for composing costs across compound comparisons. If the cost and/or the composition rule are freely chosen, the framework contains an implicit degree of freedom: changing these choices can change downstream conclusions while preserving the same qualitative narrative.

This note isolates two rigidity questions: (i) when does a natural d'Alembert-type composition law, together with a calibration at equilibrium, force the cost itself; and (ii) once the cost is fixed, whether the associated combiner can be anything other than the polynomial RCL on its natural domain.[deleted: prior motivation framed as an objection to a "semi-conditional" theorem; revised to match the current scope and terminology.]

1.3 Standing assumptions

We work with a cost function $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying:

1. **Normalization**: $F(1) = 0$

2. **Reciprocity**: $F(x) = F(x^{-1})$ for all $x > 0$

3. **Smoothness**: $F \in C^2$ (twice continuously differentiable)

4. **Calibration**: $G''(0) = 1$ where $G(t) := F(e^t)$

5. **d'Alembert-type composition law**: for all $x, y > 0$,

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y).$$

Remark. The mere existence of an arbitrary combiner P in an identity of the form $F(xy) + F(x/y) = P(F(x), F(y))$ is too weak to force a unique cost: one can construct many smooth reciprocal costs together with a tailored P. The d'Alembert-type identity above is therefore the central structural assumption.

Each assumption has a natural interpretation:

- **Normalization**: Zero deviation has zero cost (definitional).

- **Reciprocity**: Comparing x to 1 costs the same as comparing 1 to x (definitional).

- **Smoothness**: The cost function has no discontinuities or cusps (regularity).

- **Calibration**: Choice of units (convention).

- **Composition law**: A d'Alembert-type coherence condition for combining multiplicative comparisons (structural identity).

2 Part I: Derivation of the Cost Function

2.1 The d'Alembert Reduction

The key insight is that the d'Alembert-type composition law is the multiplicative avatar of a classical functional equation: after passing to log-coordinates and shifting by a constant, it becomes the d'Alembert functional equation.[deleted: prior claim that a merely existential "combiner" axiom forces the d'Alembert structure without additional hypotheses.]

Definition 2.1 (Log-coordinate cost). Given $F : \mathbb{R}_{>0} \to \mathbb{R}$, define $G : \mathbb{R} \to \mathbb{R}$ by $G(t) := F(e^t)$.

Lemma 2.2 (Composition law implies d'Alembert equation). *Assume F satisfies the d'Alembert-type composition law in the standing assumptions. Then $H(t) := G(t) + 1$ satisfies the d'Alembert functional equation:*

$$H(t + u) + H(t - u) = 2H(t)H(u) \quad \forall t, u \in \mathbb{R}$$

with $H(0) = 1$.

Proof sketch. Substitute $x = e^t$ and $y = e^u$ into the d'Alembert-type composition law to obtain
$$G(t + u) + G(t - u) = 2G(t)G(u) + 2G(t) + 2G(u).$$
Now set $H := G + 1$ and expand:

$$H(t + u) + H(t - u) = (G(t + u) + 1) + (G(t - u) + 1) = 2(G(t) + 1)(G(u) + 1) = 2H(t)H(u).$$

Finally, $H(0) = G(0) + 1 = F(1) + 1 = 1$.

Lean reference: `IndisputableMonolith.CostUniqueness.T5_uniqueness_complete` (derivation of the d'Alembert identity for H from the cosh-add identity for G). $\square$

2.2   The Aczél Classification

The d'Alembert functional equation has been completely classified.

Theorem 2.3 (Aczél 1966)**.** *The real-valued continuous solutions to $H(t+u)+H(t-u) = 2H(t)H(u)$ with $H(0) = 1$ are:*

1. $H(t) = 1$ *(constant)*

2. $H(t) = \cos(ct)$ *for some $c \in \mathbb{R}$*

3. $H(t) = \cosh(ct)$ *for some $c \in \mathbb{R}$*

   See [1, 2] for proofs and broader context.

2.3   Selection by calibration

The standing assumptions select a unique branch and fix the remaining parameter via calibration:

- The **constant solution** $H \equiv 1$ yields $H''(0) = 0$, violating the calibration $H''(0) = G''(0) = 1$.

- The cosine family $H(t) = \cos(ct)$ satisfies $H''(0) = -c^2 \leq 0$, also incompatible with $H''(0) = 1$.

- The **hyperbolic cosine** family $H(t) = \cosh(ct)$ remains, and the calibration $H''(0) = c^2 = 1$ forces $c = \pm 1$ and hence $H(t) = \cosh(t)$.

   [deleted: prior "selection" argument that appealed to non-negativity; the calibration already rules out the cosine branch for real-valued solutions.]
   Lean reference. The ODE-based uniqueness route is formalized as `IndisputableMonolith.Cost.F`

2.4   The Unique Cost Function

Theorem: Cost Uniqueness (T5)

Theorem 2.4 (Cost Uniqueness)**.** *Any cost function $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying the standing assumptions equals J on $(0, \infty)$:*

$$F(x) = J(x) = \frac{1}{2}\left(x + x^{-1}\right) - 1$$

Proof. From the d'Alembert reduction, $H(t) := G(t)+1 = \cosh(t)$, so $G(t) = \cosh(t)-1$. Therefore:

$$F(x) = G(\ln x) = \cosh(\ln x) - 1 = \frac{e^{\ln x} + e^{-\ln x}}{2} - 1 = \frac{x + x^{-1}}{2} - 1 = J(x)$$

$\square$

Lean reference: `CostUniqueness.T5_uniqueness_complete`

3 Part II: Derivation of the Combiner

Now that $F = J$ is established, we derive P directly.

3.1 The d'Alembert Identity for J

Lemma 3.1 (Composition Identity)**.** *For all $x, y > 0$:*

$$J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y)$$

Lean reference: `DAlembert.Unconditional.J_computes_P`

3.2 Surjectivity of J

Lemma 3.2 (Surjectivity)**.** *The function $J : \mathbb{R}_{>0} \to [0, \infty)$ is surjective. For any $v \geq 0$, there exists $x > 0$ with $J(x) = v$.*

Proof. For $v = 0$, take $x = 1$. For $v > 0$, solve $J(x) = v$: the equation $x^2 - (2v+2)x + 1 = 0$ has solution

$$x = v + 1 + \sqrt{v^2 + 2v} > 0$$

$\square$

Lean reference: `DAlembert.Unconditional.J_surjective_nonneg`

3.3 Determination of P on $[0, \infty)^2$

Theorem: Combiner Uniqueness

Theorem 3.3 (Combiner Uniqueness)**.** *Let $P : \mathbb{R}^2 \to \mathbb{R}$ satisfy the consistency equation with J:*

$$J(xy) + J(x/y) = P(J(x), J(y)) \quad \forall x, y > 0$$

Then for all $u, v \geq 0$:
$$P(u, v) = 2uv + 2u + 2v$$

Proof. By surjectivity, for any $u, v \geq 0$, there exist $x, y > 0$ with $J(x) = u$ and $J(y) = v$. By the consistency hypothesis:

$$P(u, v) = P(J(x), J(y)) = J(xy) + J(x/y)$$

By the composition identity:

$$J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y) = 2uv + 2u + 2v$$

$\square$

Lean reference: `DAlembert.Unconditional.rcl_unconditional`

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4 Part III: Extension to All of $\mathbb{R}^2$

The preceding theorem determines P on $[0, \infty)^2$. Can we extend to all of $\mathbb{R}^2$?

4.1 The Obstruction

The fundamental obstruction is that $J : \mathbb{R}_{>0} \to \mathbb{R}$ has range $[0, \infty)$. The consistency equation never evaluates P at negative arguments. Therefore, **no purely unconditional theorem can determine** P on $\mathbb{R}^2 \setminus [0, \infty)^2$.

4.2 The Analyticity Bridge

If we add a regularity assumption on P, the result extends.

Theorem 4.1 (Analytic Extension). *If $P : \mathbb{R}^2 \to \mathbb{R}$ is real-analytic and satisfies the consistency equation with J, then*

$$P(u, v) = 2uv + 2u + 2v \quad \forall u, v \in \mathbb{R}$$

Proof. A real-analytic function is determined by its values on any open set. Since $P(u, v) = 2uv + 2u + 2v$ on $(0, \infty)^2$ (an open set), and this polynomial is analytic, uniqueness of analytic continuation implies equality everywhere. $\qquad\square$

Corollary 4.2 (Polynomial Extension). *If P is assumed to be a polynomial, then $P(u, v) = 2uv + 2u + 2v$ on all of $\mathbb{R}^2$.*

4.3 Interpretation

The extension theorem answers the question: "Is the Recognition Composition Law the unique polynomial combiner?"

Yes. If one requires P to be polynomial (or analytic), the composition law is uniquely determined on all of $\mathbb{R}^2$, not just the first quadrant.

5 The Main Theorem

Combining all parts:

Main Theorem: Cost and combiner rigidity

Theorem 5.1 (Cost and combiner rigidity). *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfy:*

1. *$F(1) = 0$ (normalization)*

2. *$F(x) = F(x^{-1})$ for all $x > 0$ (reciprocity)*

3. *$F \in C^2$ (smoothness)*

4. *$G''(0) = 1$ where $G(t) = F(e^t)$ (calibration)*

5. *$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y)$ for all $x, y > 0$ (d'Alembert-type composition law)*

Then:

(a) *$F(x) = J(x) = \frac{1}{2}(x + x^{-1}) - 1$ for all $x > 0$*

(b) *If $P : \mathbb{R}^2 \to \mathbb{R}$ satisfies $J(xy) + J(x/y) = P(J(x), J(y))$ for all $x, y > 0$, then $P(u, v) = 2uv + 2u + 2v$ for all $u, v \geq 0$.*

(c) *If additionally P is real-analytic, then $P(u, v) = 2uv + 2u + 2v$ for all $u, v \in \mathbb{R}$.*

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6 Discussion

6.1 Uniqueness of the Metric Structure

The main theorem has two rigidity consequences: (i) within the class of costs satisfying the standing assumptions (in particular the d'Alembert-type composition law and the unit-curvature calibration), the cost is uniquely determined as J; and (ii) once the cost is fixed to J, the associated combiner is not an additional degree of freedom: any function P compatible with J agrees with the polynomial RCL on $[0, \infty)^2$ (and everywhere under real-analyticity).[deleted: prior bullet list that claimed the composition law itself is derived "by the axioms"; in the current paper the d'Alembert-type composition law is an explicit assumption, while the combiner rigidity is derived.]

6.2 Minimality of the Axioms

Each axiom is either:

- **Definitional**: What "cost of deviation" means (normalization, reciprocity)

- **Regularity**: Smoothness (no discontinuities)

- **Convention**: Choice of units (calibration)

- **Structural**: A d'Alembert-type composition identity

The proofs are purely analytic/functional-equation arguments: no empirical input is used.

6.3 Structural Determination

At the level of functional form, the theorem removes a hidden degree of freedom: within the stated class, there is no alternative choice of cost, and no alternative choice of combiner (on the natural range) once the cost is fixed.

7 Machine Verification

7.1 Verified Theorems

- `IndisputableMonolith.CostUniqueness.T5_uniqueness_complete`: cost uniqueness (Lean statement packaged with explicit hypotheses).

- `IndisputableMonolith.Foundation.DAlembert.Unconditional.rcl_unconditional`: combiner rigidity on $[0, \infty)^2$ (Theorem 3.3).

- `IndisputableMonolith.Cost.FunctionalEquation.Jcost_cosh_add_identity`: the cosh-add identity for J in log-coordinates.

- `IndisputableMonolith.Cost.FunctionalEquation.ode_cosh_uniqueness_contdiff`: ODE uniqueness for cosh.

- `IndisputableMonolith.Foundation.DAlembert.Unconditional.J_surjective_nonneg`: surjectivity of J onto $[0, \infty)$.

7.2 Explicit Hypotheses

The Lean proof uses standard regularity hypotheses from functional equation theory (made explicit as named hypotheses):

- `dAlembert_continuous_implies_smooth_hypothesis`

- `dAlembert_to_ODE_hypothesis`

- `ode_regularity_continuous_hypothesis`

- `ode_regularity_differentiable_hypothesis`

- `ode_linear_regularity_bootstrap_hypothesis`

These correspond to well-known implications used in the classical analysis of the d'Alembert functional equation (see, e.g., [1, 2]). They are stated explicitly in Lean to keep the formal proof modular.

8 Relation to companion work

The cost-uniqueness statement in Theorem 2.4 is a standard consequence of the d'Alembert functional equation and a curvature calibration at the origin (see, e.g., [1, 2]). A companion preprint by Washburn and Zlatanović (2026) [4] develops the cost-uniqueness result on $\mathbb{R}_{>0}$ with a sharper focus on minimal regularity assumptions.

The main additional contribution of the present note is the combiner rigidity theorem (Theorem 3.3) and the analytic-extension observation in Part III.[deleted: prior internal "conditional/semi-conditional" comparison table; replaced with a conventional relation-to-prior-work paragraph.]

9   Conclusion

Under the standing assumptions (in particular the d'Alembert-type composition law and the unit-curvature calibration), the cost is forced to be the reciprocal cost J. Moreover, once J is fixed, the combiner is rigid: any function P compatible with J agrees with the polynomial RCL on the natural range $[0, \infty)^2$, with global extension under real-analyticity.[deleted: prior "Given only" bullet list which overstated the generality of the assumptions by treating the d'Alembert-type composition law as a mere existential consistency condition.]

Machine Verification. The core theorems compile in Lean 4 with zero unproved assumptions (modulo explicit regularity hypotheses from functional equation theory).

Repository. `IndisputableMonolith` (Lean 4), files:

- `CostUniqueness.lean`

- `Foundation/DAlembert/Unconditional.lean`

- `Cost/FunctionalEquation.lean`

References

[1] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

[2] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, 2nd ed., Birkhäuser, 2009.

[3] F. J. Papp, The d'Alembert functional equation, *Amer. Math. Monthly* **92** (1985), 273–275.

[4] J. Washburn and M. Zlatanović, *Uniqueness of the Canonical Reciprocal Cost*, preprint, 2026.