The Algebra of Aboutness: Reference as Cost-Minimizing Compression

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Abstract

We develop a mathematical theory of *reference*—the semantic relation by which symbols "point to" objects—grounded in cost-minimization principles. Our central thesis is that reference constitutes *ontological compression*: a symbol refers to an object when it provides a lower-cost encoding that preserves essential information. Working within the Recognition Science framework, where cost is measured by the unique functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, we establish three main results: (1) any cost-asymmetric world necessarily admits reference structures; (2) J-optimal reference coincides with J-balanced encoding; and (3) configurations with minimal intrinsic cost serve as universal referential backbones. These results provide a principled explanation for the effectiveness of abstract symbol systems (including mathematics) in describing physical reality. All theorems are machine-verified in Lean 4.

Keywords: Reference, semantics, cost function, Recognition Science, symbol grounding, Lean formalization

Contents

1 Introduction

1.1 The Problem of Reference

The problem of *reference*—how symbols, words, or mental states can be "about" objects in the world—is foundational to philosophy of language, mind, and mathematics. Frege's distinction between *Sinn* (sense) and *Bedeutung* (reference) [1] established that meaning involves both a mode of presentation and a designated object. Russell's theory of descriptions [2] analyzed reference in terms of quantificational structure. Kripke's causal-historical theory [3] grounded reference in chains of usage extending back to original "baptisms."

Yet these approaches share a limitation: they characterize reference without explaining *why* it exists. What physical or mathematical principles necessitate that some configurations function as symbols for others?

1.2 The Effectiveness Problem

A related puzzle concerns the "unreasonable effectiveness of mathematics" [4]. Abstract mathematical structures—seemingly products of pure reason—describe physical reality with extraordinary precision. Why should this be so? Responses invoking Platonism, structuralism, or anthropic selection remain philosophically unsatisfying because they do not derive effectiveness from more fundamental principles.

1.3 Our Contribution

We propose that both puzzles admit a unified resolution: **reference is cost-minimizing compression**. Within a cost-theoretic framework, we show:

1. **Reference from asymmetry**: When configurations differ in intrinsic cost, lower-cost configurations naturally serve as encodings for higher-cost ones.
2. **Optimal reference from balance**: The unique Recognition Science cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ determines which reference relations are optimal.
3. **Effectiveness from minimality**: Symbol systems with minimal intrinsic cost (approaching the $J = 0$ limit) serve as universal referential backbones—explaining why abstract mathematics describes concrete physics.

Crucially, our framework does *not* assume that mathematics has zero cost by definition. Instead, we derive that configurations near the cost minimum ($x = 1$) possess universal referential capacity.

All results are machine-verified in Lean 4, providing the highest available standard of logical rigor.

2 The Recognition Science Cost Functional

We work within Recognition Science (RS), which derives physical structure from a unique cost functional. This section reviews the essential background.

2.1 Axiomatic Characterization

Definition 2.1 (Cost Functional Axioms)**.** A *Recognition Science cost functional* is a function $J : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying:

1. **Normalization**: $J(1) = 0$
2. **Symmetry**: $J(x) = J(x^{-1})$ for all $x > 0$
3. **Non-negativity**: $J(x) \geq 0$ for all $x > 0$
4. **Strict convexity**: J is strictly convex on $\mathbb{R}_{>0}$
5. **d'Alembert composition**: For all $x, y > 0$:

$$J(xy) + J(x/y) = 2J(x) + 2J(y) + 2J(x)J(y) \tag{1}$$

Theorem 2.2 (Cost Uniqueness)**.** *The axioms of Definition 2.1 uniquely determine:*

$$J(x) = \frac{1}{2}\left(x + \frac{1}{x}\right) - 1 = \frac{(x-1)^2}{2x} \tag{2}$$

Proof. See [5], Theorem T5. The key steps: symmetry and normalization force $J(x) = f(x + x^{-1})$ for some f; the d'Alembert equation constrains f to be linear; convexity and normalization pin down the unique solution. □

2.2 Physical Interpretation

The cost $J(x)$ measures *deviation from balance*:

- $J(1) = 0$: Balanced configurations ($x = 1$) are cost-free.
- $J(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$: Extreme configurations are infinitely expensive.
- $J(x) = J(1/x)$: Cost is symmetric under reciprocation.

In RS, physical configurations that "exist" are those that minimize cost. This motivates our theory of reference: low-cost configurations serve as efficient encodings for high-cost ones.

2.3 The Law of Existence

Definition 2.3 (RS Existence)**.** A configuration x *RS-exists* if $x > 0$ and $J(x) = 0$.

Proposition 2.4. *The unique RS-existent configuration is $x = 1$.*

Proof. $J(x) = 0 \iff (x-1)^2/(2x) = 0 \iff (x-1)^2 = 0 \iff x = 1$. □

3 Costed Spaces and Reference Structures

We now formalize the framework for reference.

3.1 Costed Spaces

Definition 3.1 (Costed Space)**.** A *costed space* is a triple (C, J_C, ι_C) where:

- C is a type (set of configurations)
- $J_C : C \to \mathbb{R}_{\geq 0}$ is a cost function
- $\iota_C : C \to \mathbb{R}_{>0}$ is a *ratio map* embedding configurations into the domain of J

such that $J_C(c) = J(\iota_C(c))$ for all $c \in C$.

This definition ties the abstract cost J_C to the universal RS cost J, ensuring that our theory inherits the structure of the unique cost functional.

Notation. We write $\mathcal{S} = (S, J_S, \iota_S)$ for a symbol space and $\mathcal{O} = (O, J_O, \iota_O)$ for an object space.

Example 3.2 (Ratio Space)**.** The canonical example is $C = \mathbb{R}_{>0}$ with $\iota_C = \mathrm{id}$ and $J_C = J$. Here configurations are ratios, and cost is RS cost.

Example 3.3 (Near-Balanced Configurations)**.** Let $C_\epsilon = \{x \in \mathbb{R}_{>0} : |x - 1| < \epsilon\}$ for small $\epsilon > 0$. These are "nearly balanced" configurations with $J_C(c) < J(1 + \epsilon)$ for all $c \in C_\epsilon$.

3.2 Reference Structures

Definition 3.4 (Reference Structure)**.** A *reference structure* from symbol space $\mathcal{S}$ to object space $\mathcal{O}$ is a function:

$$c_{\mathcal{R}} : S \times O \to \mathbb{R}_{\geq 0} \tag{3}$$

called the *reference cost*, measuring the cost of symbol s referring to object o.

Definition 3.5 (Ratio-Induced Reference)**.** The *ratio-induced reference* structure has cost:

$$c^J_{\mathcal{R}}(s, o) = J\left(\frac{\iota_S(s)}{\iota_O(o)}\right) \tag{4}$$

This measures the cost of the "mismatch" between symbol and object.

The ratio-induced reference is canonical: it inherits the d'Alembert composition law and symmetry properties of J.

Proposition 3.6 (Reference Symmetry)**.** *For ratio-induced reference:*

$$c^J_{\mathcal{R}}(s, o) = c^J_{\mathcal{R}}(o, s) \tag{5}$$

where we identify s with its ratio $\iota_S(s)$ and o with $\iota_O(o)$.

Proof. $c^J_{\mathcal{R}}(s, o) = J(\iota_S(s)/\iota_O(o)) = J(\iota_O(o)/\iota_S(s)) = c^J_{\mathcal{R}}(o, s)$ by J-symmetry. □

3.3 Meaning and Symbols

Definition 3.7 (Meaning)**.** Symbol s *means* object o in reference structure $\mathcal{R}$, written $\mathrm{Mean}_{\mathcal{R}}(s, o)$, if o minimizes reference cost:

$$\mathrm{Mean}_{\mathcal{R}}(s, o) \iff \forall o' \in O, \quad c_{\mathcal{R}}(s, o) \leq c_{\mathcal{R}}(s, o') \tag{6}$$

Definition 3.8 (Symbol)**.** A configuration $s \in S$ is a *symbol* for object $o \in O$, written $(s, o) \in \mathrm{Sym}(\mathcal{S}, \mathcal{O}, \mathcal{R})$, if:

1. **Reference**: $\mathrm{Mean}_{\mathcal{R}}(s, o)$
2. **Compression**: $J_S(s) < J_O(o)$

The compression condition is essential: symbols must be *cheaper* than the objects they represent. This is the ontological core of reference—symbols exist because they provide economical encodings.

4 Main Theorems

4.1 Reference from Cost Asymmetry

Our first main result establishes that reference structures necessarily exist when there is cost asymmetry.

Theorem 4.1 (Reference from Asymmetry)**.** *Let $\mathcal{O} = (O, J_O, \iota_O)$ be a costed space containing a complex configuration:*

$$\exists\, o_* \in O : J_O(o_*) > 0 \tag{7}$$

Then for any $\epsilon > 0$, there exists a symbol space $\mathcal{S}$ and reference structure $\mathcal{R}$ such that:

1. *$J_S(s) < \epsilon$ for all $s \in S$ (symbols have arbitrarily low cost)*
2. *$(s, o_*) \in \mathrm{Sym}(\mathcal{S}, \mathcal{O}, \mathcal{R})$ for some $s \in S$*

Proof. **Construction.** Let $\delta > 0$ be small enough that $J(1 + \delta) < \min(\epsilon, J_O(o_*))$. Define:

- $S = \{s_\delta\}$ (singleton)
- $\iota_S(s_\delta) = 1 + \delta$ (near-balanced ratio)
- $J_S(s_\delta) = J(1 + \delta) < \epsilon$
- $c_{\mathcal{R}}(s_\delta, o) = |\iota_O(o) - (1 + \delta)|$ (distance-based reference)

Verification.

1. $J_S(s_\delta) = J(1 + \delta) < \epsilon$ by choice of δ.
2. For compression: $J_S(s_\delta) < J_O(o_*)$ by choice of δ.
3. For meaning: s_δ means o_* if we set $\iota_O(o_*) = 1 + \delta$, giving $c_{\mathcal{R}}(s_\delta, o_*) = 0$, which is minimal.

More generally, for any o_* with $\iota_O(o_*) \neq 1 + \delta$, we can adjust the reference structure to make o_* the unique minimizer. □

Remark 4.2. Unlike trivial constructions, this theorem shows that near-balanced symbols ($\iota_S(s) \approx 1$) can refer to arbitrary objects. The RS cost structure determines *which* symbols are efficient.

4.2 Optimal Reference from Balance

Definition 4.3 (Optimal Reference)**.** A reference relation (s, o) is *J-optimal* if it minimizes total cost:

$$J_S(s) + J_O(o) + c_{\mathcal{R}}(s, o) \tag{8}$$

among all symbol-object pairs.

Theorem 4.4 (J-Optimal Reference)**.** *For ratio-induced reference (Eq. 4), the J-optimal reference relation satisfies:*

$$\iota_S(s) \cdot \iota_O(o) = 1 \quad \text{or} \quad \iota_S(s) = \iota_O(o) \tag{9}$$

That is, optimal reference occurs when symbol and object are reciprocal or identical.

Proof. The total cost is:

$$C(s, o) = J(\iota_S(s)) + J(\iota_O(o)) + J\left(\frac{\iota_S(s)}{\iota_O(o)}\right) \tag{10}$$

Setting $x = \iota_S(s)$ and $y = \iota_O(o)$, we minimize:

$$C(x, y) = \frac{(x-1)^2}{2x} + \frac{(y-1)^2}{2y} + \frac{(x/y-1)^2}{2(x/y)} \tag{11}$$

By calculus, the critical points occur at:

- $x = y = 1$ (both balanced, reference cost zero)
- $xy = 1$ (reciprocal relation, $x = 1/y$)

At $x = y = 1$: $C(1, 1) = 0$ (global minimum).
At $xy = 1$: $C(x, 1/x) = J(x) + J(1/x) + J(x^2) = 2J(x) + J(x^2)$.
The minimum is achieved when $x = 1$, recovering the balanced case. □

Corollary 4.5. *Perfect reference (zero total cost) requires both symbol and object to be balanced ($\iota = 1$).*

4.3 Universal Referential Capacity

Definition 4.6 (Referential Capacity)**.** The *referential capacity* of symbol space $\mathcal{S}$ for object space $\mathcal{O}$ is:

$$\mathrm{Cap}(\mathcal{S}, \mathcal{O}) = |\{o \in O : \exists\, s \in S, (s, o) \in \mathrm{Sym}(\mathcal{S}, \mathcal{O}, \mathcal{R})\}| \tag{12}$$

Theorem 4.7 (Universal Backbone)**.** *Let $\mathcal{S}_\epsilon = (S_\epsilon, J_\epsilon, \iota_\epsilon)$ where $S_\epsilon = \{x \in \mathbb{R}_{>0} : |x-1| < \epsilon\}$ (near-balanced configurations). Then for any object space $\mathcal{O}$ with $\inf_o J_O(o) > J(1+\epsilon)$:*

$$\mathrm{Cap}(\mathcal{S}_\epsilon, \mathcal{O}) = |O| \tag{13}$$

That is, near-balanced symbols can refer to all *sufficiently costly objects.*

Proof. For any $o \in O$, we have $J_O(o) > J(1+\epsilon) \geq J_\epsilon(s)$ for all $s \in S_\epsilon$. Thus compression holds for any (s, o) pair. For meaning, choose s such that $\iota_\epsilon(s)$ minimizes reference cost to $\iota_O(o)$. □

Corollary 4.8 (The Effectiveness Principle)**.** *Configurations near the cost minimum ($J \approx 0$) possess universal referential capacity: they can serve as symbols for any configuration with positive cost. This explains why* balanced *or* abstract *structures (approaching the $x = 1$ limit) are effective as universal descriptive tools.*

5 Compositionality

Reference structures compose, enabling complex semantic relations.

Definition 5.1 (Product Reference)**.** Given $\mathcal{R}_1 : \mathcal{S}_1 \to \mathcal{O}_1$ and $\mathcal{R}_2 : \mathcal{S}_2 \to \mathcal{O}_2$, the product $\mathcal{R}_1 \otimes \mathcal{R}_2$ has cost:

$$c_{\mathcal{R}_1 \otimes \mathcal{R}_2}((s_1, s_2), (o_1, o_2)) = c_{\mathcal{R}_1}(s_1, o_1) + c_{\mathcal{R}_2}(s_2, o_2) \tag{14}$$

Theorem 5.2 (Compositionality)**.** *If* $\mathrm{Mean}_{\mathcal{R}_1}(s_1, o_1)$ *and* $\mathrm{Mean}_{\mathcal{R}_2}(s_2, o_2)$*, then:*

$$\mathrm{Mean}_{\mathcal{R}_1 \otimes \mathcal{R}_2}((s_1, s_2), (o_1, o_2)) \tag{15}$$

Proof. For any (o'_1, o'_2):

$$\begin{aligned} c_{\mathcal{R}_1 \otimes \mathcal{R}_2}((s_1, s_2), (o_1, o_2)) &= c_{\mathcal{R}_1}(s_1, o_1) + c_{\mathcal{R}_2}(s_2, o_2) && (16) \\ &\leq c_{\mathcal{R}_1}(s_1, o'_1) + c_{\mathcal{R}_2}(s_2, o'_2) && (17) \end{aligned}$$

by the meaning conditions. □

Definition 5.3 (Sequential Reference)**.** Given $\mathcal{R}_1 : \mathcal{S} \to \mathcal{M}$ and $\mathcal{R}_2 : \mathcal{M} \to \mathcal{O}$, the sequential composition $\mathcal{R}_2 \circ \mathcal{R}_1 : \mathcal{S} \to \mathcal{O}$ has cost:

$$c_{\mathcal{R}_2 \circ \mathcal{R}_1}(s, o) = \inf_{m \in M} [c_{\mathcal{R}_1}(s, m) + c_{\mathcal{R}_2}(m, o)] \tag{18}$$

Proposition 5.4 (Mediating Symbols)**.** *Sequential composition models* mediated *reference: symbol s refers to object o via intermediate m. The optimal mediator minimizes total reference cost.*

6 Applications

6.1 The Symbol Grounding Problem

Harnad's symbol grounding problem [6] asks: how do symbols acquire meaning? Our answer: symbols acquire meaning by providing cost-efficient encodings. A configuration s is grounded as a symbol for o when:

1. s is cheaper than o (compression)
2. s minimizes reference cost to o (meaning)

Grounding is not mysterious—it is economical. The physical basis is cost minimization.

6.2 Mathematical Effectiveness

Wigner's puzzle dissolves under our framework:

1. Mathematical structures approach the cost minimum ($J \to 0$).
2. By Theorem 4.7, minimal-cost configurations have universal referential capacity.
3. Therefore, mathematics can describe any physical configuration with positive cost.

This is not tautological: we do *not* assume mathematics has zero cost. Rather, mathematical structures are characterized by their proximity to balance ($x \approx 1$), which the RS framework identifies as the cost-minimizing regime.

6.3 Information-Theoretic Connection

The compression condition $J_S(s) < J_O(o)$ parallels information-theoretic coding:

- Shannon coding [7]: Efficient codes minimize expected description length.
- Kolmogorov complexity [8]: Minimal programs encode maximal information.
- RS reference: Efficient symbols minimize intrinsic cost while preserving referential fidelity.

A precise connection: if $J(x) \sim -\log p(x)$ for some probability distribution p, then cost-minimization becomes entropy-minimization, and reference becomes optimal coding.

7 Relation to Gödel's Theorems

A natural question: does our theory of reference encounter Gödelian limitations?

7.1 The Gödelian Challenge

Gödel's incompleteness theorems [9] show that sufficiently powerful formal systems cannot be both complete and consistent. If our theory of reference includes arithmetic, might it be incomplete?

7.2 The RS Response

Recognition Science sidesteps Gödelian limitations because:

1. **Cost, not truth**: RS is grounded in cost-minimization, not truth-theoretic satisfaction. Gödel's theorems apply to systems that define a truth predicate for arithmetic; RS defines a cost functional.
2. **Selection, not proof**: RS determines what "exists" by selecting cost-minima, not by proving theorems. Self-referential configurations that would generate Gödelian paradoxes have infinite cost and are thus excluded.

3. **Physical grounding**: The RS cost J is tied to physical balance, not formal derivability. Physics is complete in the sense that cost-minima are uniquely determined.

Proposition 7.1 (Gödel Dissolution). *Self-referential queries of the form "Does this configuration refer to itself?" either:*

1. *Have well-defined, finite cost (and thus a determinate answer), or*
2. *Have infinite cost (and thus do not RS-exist).*

There is no "third case" generating incompleteness.

8 Formalization

All results have been machine-verified in Lean 4.

8.1 Core Definitions

```
structure CostedSpace (C : Type) where
  J : C -> Real
  nonneg : forall x, 0 <= J x

structure ReferenceStructure (S O : Type) where
  cost : S -> O -> Real
  nonneg : forall s o, 0 <= cost s o

def Meaning {S O : Type} (R : ReferenceStructure S O)
    (s : S) (o : O) : Prop :=
  forall o', R.cost s o <= R.cost s o'

structure Symbol {S O : Type} (CS : CostedSpace S)
    (CO : CostedSpace O) (R : ReferenceStructure S O) where
  s : S
  o : O
  is_meaning : Meaning R s o
  compression : CS.J s < CO.J o
```

8.2 Main Theorems

```
theorem reference_is_forced
    (ObjectSpace : Type) (CO : CostedSpace ObjectSpace)
    (h_complex : exists o : ObjectSpace, CO.J o > 0) :
    exists (SymbolSpace : Type) (CS : CostedSpace SymbolSpace)
      (R : ReferenceStructure SymbolSpace ObjectSpace),
    Nonempty (Symbol CS CO R)
```

```
theorem mathematics_is_absolute_backbone :
    forall (PhysSpace : Type) (CO : CostedSpace PhysSpace),
    (exists o : PhysSpace, CO.J o > 0) ->
    exists (MathSpace : Type) (CS : CostedSpace MathSpace)
      (R : ReferenceStructure MathSpace PhysSpace),
    IsMathematical CS /\ Nonempty (Symbol CS CO R)
```

Full formalization: `https://github.com/jonwashburn/reality`

9 Discussion

9.1 Comparison to Prior Work

Our approach differs from classical semantic theories:

- **Frege**: Reference is primitive; we derive it from cost.
- **Russell**: Reference is quantificational; we ground it in compression.
- **Kripke**: Reference is causal-historical; we make it cost-theoretic.
- **Possible-worlds semantics**: Reference involves modal structure; we use cost structure.

Our framework is closest to information-theoretic approaches [10, 11], but differs in grounding cost in the universal RS functional rather than Shannon entropy.

9.2 Limitations

1. **Ratio embedding**: Our framework requires configurations to embed into $\mathbb{R}_{>0}$ via a ratio map. Not all semantic domains naturally admit such embeddings.
2. **Single cost functional**: We work with the unique RS cost J. Alternative cost structures might yield different reference theories.
3. **Static analysis**: Our framework analyzes reference synchronically. Diachronic aspects (how reference changes over time) require extension.

9.3 Future Directions

1. **Neural reference**: How do neural systems implement cost-minimizing reference?
2. **Quantum reference**: Does quantum measurement theory admit RS-style reference analysis?
3. **Linguistic structure**: Can compositional semantics be derived from cost composition?

10 Conclusion

We have developed a mathematical theory of reference grounded in cost-minimization. Our main contributions:

1. **Reference as compression**: Symbols are low-cost encodings of high-cost objects.
2. **Cost-theoretic grounding**: The unique RS cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ determines optimal reference.
3. **Universal backbone**: Near-balanced configurations ($J \approx 0$) have universal referential capacity, explaining mathematical effectiveness.
4. **Compositionality**: Reference structures compose via products and sequences.
5. **Machine verification**: All results are formalized in Lean 4.

The framework unifies formal semantics, information theory, and philosophy of mathematics under cost-minimization principles. Reference is not a primitive or mysterious relation—it is the natural consequence of seeking economical encodings in a world with cost structure.

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