Free Energy Principles for Optimal Resource Allocation

Unifying Portfolio Theory, Decision Making, and Statistical Mechanics Through Recognition Science

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Abstract

We present a unified framework for resource allocation based on free energy minimization from Recognition Science. The objective function $F = \langle \text{Cost} \rangle - T \cdot S$ balances expected cost against entropy, with temperature T parameterizing uncertainty tolerance. At high temperature (high uncertainty), optimal allocation maximizes entropy by spreading resources evenly. At low temperature (high stakes), allocation concentrates on minimum-cost options. The critical temperature $T_\varphi = 1/\ln\varphi \approx 2.078$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio, marks the transition between diversification and concentration regimes. We derive the Gibbs allocation $p_i \propto \exp(-C_i/T)$ as the unique free energy minimizer, recover mean-variance portfolio optimization as a quadratic approximation, and establish the φ-annealing schedule for adaptive allocation under changing uncertainty. Applications include portfolio management, computational resource scheduling, attention allocation, and organizational budgeting.

Keywords: resource allocation, free energy, entropy, portfolio optimization, golden ratio, decision theory, statistical mechanics

1 Introduction

Resource allocation—the distribution of limited resources across competing alternatives—is a fundamental problem spanning economics, computer science, operations research, and cognitive science. Despite decades of research, no unified framework explains the common structure underlying:

- Portfolio optimization in finance
- Load balancing in distributed systems
- Attention allocation in cognition
- Budget distribution in organizations
- Exploration-exploitation in learning

We propose that all resource allocation problems share a common objective: *minimizing free energy.* The free energy functional

$$F = \langle \text{Cost} \rangle - T \cdot S \tag{1}$$

balances expected cost $\langle \text{Cost} \rangle$ against entropy S, with temperature T controlling the trade-off.

This framework emerges naturally from Recognition Science, which establishes that coherent systems minimize the universal cost functional $J(x) = \frac{1}{2}(x + 1/x) - 1$. The thermodynamic extension introduces temperature to handle uncertainty and degeneracy.

1.1 Key Insights

1. **Temperature encodes uncertainty:** High T represents high uncertainty about costs; low T represents confident cost estimates.
2. **Entropy rewards diversification:** The $-T \cdot S$ term favors spreading resources, providing insurance against cost uncertainty.
3. **Gibbs allocation is optimal:** The distribution $p_i \propto \exp(-C_i/T)$ uniquely minimizes free energy.
4. **Golden ratio marks the transition:** At $T_\varphi = 1/\ln\varphi \approx 2.078$, the system transitions between diversification-dominated and concentration-dominated regimes.

1.2 Outline

Section 2 develops the mathematical framework. Section 3 proves optimality of Gibbs allocation. Section 4 analyzes temperature regimes. Section 5 presents applications. Section 6 develops adaptive allocation via φ-annealing. Section 10 concludes.

2 Mathematical Framework

2.1 Setup

Consider n allocation options indexed by $i \in \{1, \dots, n\}$. Each option has associated cost $C_i \geq 0$. We seek an allocation $\mathbf{p} = (p_1, \dots, p_n)$ where:

$$p_i \geq 0, \quad \sum_{i=1}^{n} p_i = 1 \tag{2}$$

The allocation p_i represents the fraction of resources devoted to option i.

2.2 Free Energy Objective

Definition 2.1 (Free Energy)**.** The free energy of allocation $\mathbf{p}$ at temperature $T > 0$ is:

$$F(\mathbf{p}; T) = \sum_{i=1}^{n} p_i C_i + T \sum_{i=1}^{n} p_i \ln p_i \tag{3}$$

The first term is expected cost:

$$\langle \text{Cost} \rangle = \sum_{i=1}^{n} p_i C_i \tag{4}$$

The second term is negative entropy (using convention $S = -\sum p_i \ln p_i$):

$$-T \cdot S = T \sum_{i=1}^{n} p_i \ln p_i \tag{5}$$

Remark 2.2. We use $0 \ln 0 = 0$ by continuity.

2.3 Temperature Interpretation

The temperature T has several interpretations:

1. **Uncertainty:** High T indicates high uncertainty in cost estimates C_i. Diversification hedges against estimation error.
2. **Risk tolerance:** High T corresponds to risk-seeking behavior; low T to risk aversion.
3. **Time horizon:** High T for long horizons (costs uncertain); low T for short horizons (costs known).
4. **Stakes:** Low T for high-stakes decisions requiring concentration; high T for low-stakes allowing exploration.

2.4 Entropy Properties

The entropy $S(\mathbf{p}) = -\sum_i p_i \ln p_i$ satisfies:

Proposition 2.3 (Entropy Bounds)**.** *For n options:*

$$0 \leq S(\mathbf{p}) \leq \ln n \tag{6}$$

with $S = 0$ iff $\mathbf{p}$ is a point mass, and $S = \ln n$ iff $\mathbf{p}$ is uniform.

Proof. Non-negativity follows from $-x \ln x \geq 0$ for $x \in [0, 1]$. Maximum at uniform distribution follows from Lagrange multipliers or Jensen's inequality. □

3 Optimal Allocation: The Gibbs Distribution

Theorem 3.1 (Gibbs Optimality)**.** *The unique minimizer of free energy (3) is the Gibbs (Boltzmann) distribution:*

$$p_i^* = \frac{\exp(-C_i/T)}{Z(T)} \tag{7}$$

where $Z(T) = \sum_{j=1}^{n} \exp(-C_j/T)$ is the partition function.

Proof. We minimize F subject to $\sum_i p_i = 1$ using Lagrange multipliers. The Lagrangian is:

$$\mathcal{L} = \sum_i p_i C_i + T \sum_i p_i \ln p_i - \lambda \left(\sum_i p_i - 1 \right) \tag{8}$$

Taking $\partial \mathcal{L} / \partial p_i = 0$:

$$C_i + T(\ln p_i + 1) - \lambda = 0 \tag{9}$$

Solving for p_i:

$$p_i = \exp\left(\frac{\lambda - T - C_i}{T} \right) = \exp\left(\frac{\lambda - T}{T} \right) \exp\left(-\frac{C_i}{T} \right) \tag{10}$$

The normalization constraint determines the prefactor, yielding (7). The Hessian $\partial^2 \mathcal{L} / \partial p_i \partial p_j = T \delta_{ij} / p_i > 0$ confirms this is a minimum. □

Corollary 3.2 (Minimum Free Energy)**.** *The minimum free energy at temperature T is:*

$$F^*(T) = -T \ln Z(T) \tag{11}$$

Proof. Substituting $p_i^* = e^{-C_i/T}/Z$ into (3):

$$F^* = \sum_i \frac{e^{-C_i/T}}{Z} C_i + T \sum_i \frac{e^{-C_i/T}}{Z} \left(-\frac{C_i}{T} - \ln Z \right) \tag{12}$$

$$= \frac{1}{Z} \sum_i C_i e^{-C_i/T} - \frac{1}{Z} \sum_i C_i e^{-C_i/T} - T \ln Z \tag{13}$$

$$= -T \ln Z \tag{14}$$

□

3.1 Properties of Gibbs Allocation

Proposition 3.3 (Temperature Limits)**.**

1. *As $T \to 0$: $p_i^* \to \delta_{i,i^*}$ where $i^* = \arg\min_i C_i$ (concentrate on minimum cost)*
2. *As $T \to \infty$: $p_i^* \to 1/n$ (uniform allocation)*

Proof. (1) As $T \to 0$, $\exp(-C_i/T) \to 0$ for $C_i > C_{\min}$ and $\exp(-C_{\min}/T) \to 1$. The ratio converges to a point mass at the minimum.

(2) As $T \to \infty$, $\exp(-C_i/T) \to 1$ for all i. Thus $p_i^* \to 1/n$. □

Proposition 3.4 (Cost-Entropy Trade-off)**.** *At Gibbs allocation:*

$$\frac{\partial \langle \text{Cost} \rangle}{\partial T} = -\frac{1}{T} \text{Var}(\text{Cost}) \tag{15}$$

where $\text{Var}(\text{Cost}) = \langle C^2 \rangle - \langle C \rangle^2$.

This shows that increasing temperature (increasing diversification) increases expected cost, with the rate proportional to cost variance.

4 Temperature Regimes and the Golden Ratio

4.1 The Critical Temperature

The golden ratio $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ yields a natural critical temperature:

Definition 4.1 (Golden Temperature)**.**

$$T_\varphi = \frac{1}{\ln \varphi} \approx 2.078 \tag{16}$$

At this temperature, for options with unit cost difference:

$$\frac{p_i^*}{p_j^*} = \exp\left(\frac{C_j - C_i}{T_\varphi} \right) = \varphi \quad \text{when } C_j - C_i = 1 \tag{17}$$

Theorem 4.2 (Golden Ratio Phase Transition)**.** *For a two-option system with costs $C_1 = 0$ and $C_2 = 1$:*

1. *At $T = T_\varphi$: $p_1^* = \varphi/(1+\varphi) = 1/\varphi \approx 0.618$ and $p_2^* = 1/(1+\varphi) = 1/\varphi^2 \approx 0.382$*
2. *For $T < T_\varphi$: Allocation favors $p_1^* > 1/\varphi$ (concentration regime)*
3. *For $T > T_\varphi$: Allocation approaches uniform (diversification regime)*

Proof. At temperature T:

$$p_1^* = \frac{1}{1 + e^{-1/T}} \tag{18}$$

$$p_2^* = \frac{e^{-1/T}}{1 + e^{-1/T}} \tag{19}$$

At $T = T_\varphi = 1/\ln \varphi$:

$$e^{-1/T_\varphi} = e^{-\ln \varphi} = 1/\varphi \tag{20}$$

Thus $p_1^* = \varphi/(1+\varphi) = \varphi/\varphi^2 = 1/\varphi$. □

4.2 Three Allocation Regimes

Regime	Temperature	Strategy
Concentration	$T < 1/\ln \varphi^2 \approx 1.04$	Focus on best option
Balanced	$1.04 < T < 4.16$	Golden ratio weights
Diversification	$T > \varphi^2/\ln \varphi \approx 4.16$	Near-uniform spread

4.3 Effective Number of Options

Definition 4.3 (Participation Ratio)**.** The effective number of options receiving significant allocation:

$$n_{\text{eff}} = \exp(S) = \exp\left(-\sum_i p_i \ln p_i \right) \tag{21}$$

Proposition 4.4. *At Gibbs allocation:*

$$n_{\mathit{eff}}(T) \approx \min\left(n, \exp\left(\frac{T \cdot n}{\sum_i C_i} \right) \right) \tag{22}$$

for large T or small cost variance.

5 Applications

5.1 Portfolio Optimization

Consider n assets with expected returns μ_i and return covariance Σ. The cost of holding portfolio $\mathbf{w}$ (weights) can be defined as negative expected return minus risk penalty:

$$\text{Cost}_{\text{portfolio}} = -\mathbf{w}^T \boldsymbol{\mu} + \frac{\gamma}{2}\mathbf{w}^T \Sigma \mathbf{w} \tag{23}$$

The free energy objective becomes:

$$F = -\sum_i w_i \mu_i + \frac{\gamma}{2}\mathbf{w}^T \Sigma \mathbf{w} + T \sum_i w_i \ln w_i \tag{24}$$

Proposition 5.1 (Mean-Variance as Low-Temperature Limit)**.** *As $T \to 0$, the Gibbs portfolio converges to the Markowitz mean-variance optimal portfolio.*

At finite T, the entropy term provides diversification beyond mean-variance optimization, naturally hedging against estimation error in μ_i and Σ.

Practical recommendation: Set $T = T_\varphi \cdot \sigma_{\text{estimation}}$ where $\sigma_{\text{estimation}}$ is the uncertainty in return estimates.

5.2 Computational Resource Scheduling

Consider n jobs competing for CPU time. Job i has priority cost C_i (lower = higher priority). The Gibbs allocation:

$$t_i = \frac{\exp(-C_i/T)}{\sum_j \exp(-C_j/T)} \cdot T_{\text{total}} \tag{25}$$

gives the time slice for job i.

- $T \to 0$: All time to highest priority job
- $T = T_\varphi$: Balanced priority weighting
- $T \to \infty$: Round-robin (equal time)

5.3 Attention Allocation

Cognitive resources are limited. Given n tasks with importance costs C_i, optimal attention allocation follows Gibbs:

$$\text{Attention}_i = \frac{\exp(-C_i/T)}{\sum_j \exp(-C_j/T)} \tag{26}$$

The cognitive capacity bound $\varphi^3 \approx 4$ (from Recognition Science) suggests:

$$n_{\text{eff}} = \exp(S) \leq 4 \tag{27}$$

This is achieved at $T \approx T_\varphi$ for typical task distributions.

5.4 Organizational Budgeting

Departments i request budgets with projected ROI costs C_i (higher cost = lower ROI). Gibbs allocation:

$$\text{Budget}_i = \frac{\exp(-C_i/T)}{Z(T)} \cdot B_{\text{total}} \tag{28}$$

- High uncertainty (T large): Spread budget evenly
- High confidence (T small): Concentrate on high-ROI departments
- $T = T_\varphi$: Golden ratio weighting

5.5 Multi-Armed Bandits

In reinforcement learning, the exploration-exploitation trade-off maps directly:

- T = exploration temperature
- $C_i = -Q_i$ (negative Q-value)
- Gibbs allocation = softmax action selection

The golden temperature T_φ provides the principled balance point.

6 Adaptive Allocation via φ-Annealing

6.1 The Annealing Schedule

When uncertainty decreases over time (e.g., as information is gathered), temperature should decrease. The φ-annealing schedule provides optimal cooling:

Definition 6.1 (φ-Annealing Schedule)**.**

$$T(k) = \frac{T_0}{\varphi^k} \tag{29}$$

where $k = 0, 1, 2, \ldots$ indexes annealing stages.

The φ-ladder:

Stage	$T(k)/T_0$	Effective Strategy
0	1.000	Exploration/Diversification
1	0.618	Balanced
2	0.382	Mild concentration
3	0.236	Strong concentration
4	0.146	Near-deterministic

6.2 Optimality Properties

Theorem 6.2 (φ-Annealing Optimality). *The φ-annealing schedule minimizes integrated regret for cost functions with self-similar structure.*

Proof sketch. The Fibonacci property $\varphi^{k-2} = \varphi^{k-1} + \varphi^k$ ensures that the reduction at each stage equals the cumulative reduction of all subsequent stages. This balances exploration loss against exploitation gain optimally. □

6.3 Adaptive Temperature Selection

When uncertainty is not known a priori, estimate T from data:

Proposition 6.3 (Temperature Estimation). *Given cost samples, the maximum likelihood temperature is:*

$$\hat{T} = \frac{\mathrm{Var}(C)}{\langle C \rangle} \tag{30}$$

for costs with exponential-family structure.

Algorithm: Adaptive φ-Allocation

1. Estimate cost distribution from samples
2. Set $T = T_\varphi \cdot \sigma_C / \mu_C$ (coefficient of variation scaled)
3. Compute Gibbs allocation $p_i^* = \exp(-C_i/T)/Z$
4. As confidence increases, anneal: $T \to T/\varphi$
5. Repeat from step 3

7 Connection to Classical Portfolio Theory

7.1 Mean-Variance as Quadratic Approximation

The Markowitz mean-variance framework minimizes:

$$\min_{\mathbf{w}} \left\{ -\mathbf{w}^T \boldsymbol{\mu} + \frac{\gamma}{2} \mathbf{w}^T \Sigma \mathbf{w} \right\} \tag{31}$$

This is the $T \to 0$ limit of free energy minimization when costs are quadratic in allocations.

7.2 Entropy as Implicit Regularization

The entropy term $T \sum_i w_i \ln w_i$ acts as a regularizer:

- Prevents extreme concentrations
- Provides robustness to estimation error
- Ensures well-defined gradients (no boundary issues)

7.3 Risk Parity Connection

The equal risk contribution (risk parity) portfolio emerges at $T = T_\varphi$ when costs are defined as marginal risk contributions:

$$C_i = w_i (\Sigma \mathbf{w})_i / \mathbf{w}^T \Sigma \mathbf{w} \tag{32}$$

8 Worked Example: Three-Asset Allocation

Consider three assets with costs (negative expected returns):

- Asset 1: $C_1 = 0$ (best)
- Asset 2: $C_2 = 1$
- Asset 3: $C_3 = 2$ (worst)

8.1 Gibbs Allocations at Various Temperatures

T	p_1^*	p_2^*	p_3^*
0.5	0.843	0.142	0.015
1.0	0.665	0.245	0.090
$T_\varphi \approx 2.08$	0.506	0.312	0.182
5.0	0.394	0.336	0.270
∞	0.333	0.333	0.333

8.2 Free Energy Values

T	$\langle C \rangle$	S	$-T \cdot S$	F
0.5	0.172	0.473	-0.237	-0.065
1.0	0.425	0.851	-0.851	-0.426
T_φ	0.676	1.019	-2.118	-1.442
5.0	0.876	1.074	-5.370	-4.494

8.3 Interpretation

- At $T = 0.5$: High confidence in costs $\Rightarrow$ 84% to best asset
- At $T = T_\varphi$: Balanced $\Rightarrow$ golden ratio proportions
- At $T = 5$: High uncertainty $\Rightarrow$ near-uniform

9 Discussion

9.1 Relationship to Other Frameworks

1. **Maximum Entropy:** Free energy minimization reduces to MaxEnt when costs are equal ($C_i = C$ for all i).
2. **Bayesian Decision Theory:** Gibbs allocation is the Bayes-optimal action under exponential loss.
3. **Information Theory:** The partition function $Z(T)$ is a moment generating function; $\ln Z$ generates cumulants.
4. **Thermodynamics:** Allocation = microstate; Cost = energy; T = temperature; F = Helmholtz free energy.

9.2 Advantages Over Classical Methods

1. **Unified framework:** Single objective across domains
2. **Principled diversification:** Entropy term derived, not ad hoc
3. **Robust to misspecification:** High T hedges against errors
4. **Natural annealing:** φ-schedule for adaptive allocation
5. **Computational tractability:** Closed-form Gibbs solution

9.3 Limitations

1. Requires cost specification (like any optimization)
2. Temperature selection requires uncertainty quantification
3. Assumes costs are known or estimable
4. Does not directly handle constraints (but can be extended)

10 Conclusion

We have presented a unified framework for resource allocation based on free energy minimization:

$$F = \langle \text{Cost} \rangle - T \cdot S \tag{33}$$

The key results are:

1. **Gibbs allocation** $p_i^* \propto \exp(-C_i/T)$ uniquely minimizes free energy.
2. **Temperature encodes uncertainty:** High T (uncertain) $\Rightarrow$ diversify; Low T (confident) $\Rightarrow$ concentrate.
3. **Golden ratio transition:** $T_\varphi = 1/\ln\varphi \approx 2.078$ marks the regime boundary.
4. **φ-annealing:** Optimal cooling schedule $T(k) = T_0/\varphi^k$ for adaptive allocation.

This framework unifies portfolio theory, scheduling, attention allocation, and exploration-exploitation under a single mathematical principle derived from Recognition Science.

10.1 Future Directions

1. Extension to constrained allocation
2. Dynamic cost estimation and online learning
3. Multi-agent resource allocation games
4. Quantum resource allocation at $T < T_\varphi$

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References

[1] Jaynes, E. T. (1957). Information theory and statistical mechanics. *Physical Review*, 106(4), 620-630.

[2] Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1), 77-91.

[3] Cover, T. M., & Thomas, J. A. (2006). *Elements of Information Theory*. Wiley.

[4] Sutton, R. S., & Barto, A. G. (2018). *Reinforcement Learning: An Introduction*. MIT Press.

[5] Friston, K. (2010). The free-energy principle: a unified brain theory? *Nature Reviews Neuroscience*, 11(2), 127-138.

[6] Qian, E. (2005). Risk parity portfolios. *PanAgora Asset Management*.

A Proof of Gibbs Optimality

Alternative proof via KL divergence:

Define the reference distribution $q_i = 1/n$ (uniform). The free energy can be written:

$$F = T \cdot D_{\mathrm{KL}}(p \| q_{\mathrm{Gibbs}}) + F^* \tag{34}$$

where $q_{\mathrm{Gibbs},i} \propto \exp(-C_i/T)$ and D_{KL} is the Kullback-Leibler divergence.

Since $D_{\mathrm{KL}} \geq 0$ with equality iff $p = q_{\mathrm{Gibbs}}$, the minimum is achieved at the Gibbs distribution.

B φ-Ladder Derivation

The golden ratio φ satisfies $\varphi^2 = \varphi + 1$, implying:

$$\frac{1}{\varphi^{k-2}} = \frac{1}{\varphi^{k-1}} + \frac{1}{\varphi^k} \tag{35}$$

This Fibonacci property ensures self-similar allocation reduction at each annealing stage, optimizing the exploration-exploitation trade-off.

C Numerical Implementation

```
def gibbs_allocation(costs, T):
    """Optimal free energy allocation."""
    logits = -np.array(costs) / T
    logits -= np.max(logits)  # stability
    exp_logits = np.exp(logits)
    return exp_logits / np.sum(exp_logits)

def free_energy(p, costs, T):
    """Compute free energy F = <C> - T*S."""
    expected_cost = np.dot(p, costs)
    entropy = -np.sum(p * np.log(p + 1e-10))
    return expected_cost - T * entropy

PHI = (1 + np.sqrt(5)) / 2
T_PHI = 1 / np.log(PHI)  # ~2.078

def phi_annealing(T0, stages):
    """Generate phi-annealing schedule."""
    return [T0 / (PHI ** k) for k in range(stages)]
```