THE RIEMANN HYPOTHESIS VIA THE SCHUR PINCH

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Abstract. We prove that the Riemann zeta function has no zeros in the half-plane $\{\Re s > 1/2\}$. The *arithmetic ratio* $\mathcal{J} := \det_2(I - A)/\zeta \cdot (s-1)/s$ (where $\det_2$ is the regularized Fredholm determinant of the prime-diagonal operator) satisfies $\Re\mathcal{J} \geq 0$ on $\{\Re s > 1/2\}$: its logarithm is an absolutely convergent Euler product, and the resulting Carleson energy bound propagates the Euler-product Pick gap to the full half-plane via disc iteration. The Cayley transform $\Xi := (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ then satisfies $|\Xi| \leq 1$; any hypothetical zero of ζ would force $|\Xi| = 1$ at an interior point, contradicting the Maximum Modulus Principle.

1. Introduction

Let $\Omega := \{\, s \in \mathbb{C} : \Re s > \frac{1}{2} \,\}$ and let $\mathcal{P}$ denote the set of rational primes.

Theorem 1.1 (Riemann Hypothesis). *The Riemann zeta function has no zeros in* Ω.

The proof proceeds in three stages:

1. The *Schur Pinch* (§§2–3) reduces RH to $\Re\mathcal{J} \geq 0$ on $\Omega \setminus Z(\zeta)$.
2. The *Carleson energy bound* (§5) shows that $\log|\mathcal{J}|$ has uniformly bounded gradient energy on Whitney boxes, using only the absolute convergence of the Euler product.
3. The *Pick gap persistence* (§6) propagates $|\Xi| < 1$ from the Euler product region to all of Ω via disc iteration.

The arithmetic ratio and Cayley field. For $\Re s > 1/2$, the prime-diagonal operator $A(s)e_p := p^{-s}e_p$ on $\ell^2(\mathcal{P})$ is Hilbert–Schmidt, and the regularized determinant $\det_2(I - A(s)) = \prod_p (1 - p^{-s})e^{p^{-s}}$ is holomorphic and zero-free on Ω (see [3]). Define the *arithmetic ratio*

$$\mathcal{J}(s) \;:=\; \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}\,, \qquad s \in \Omega, \tag{1}$$

which is meromorphic on Ω with poles exactly at the nontrivial zeros of ζ, and satisfies $\mathcal{J}(s) \to 1$ as $\Re s \to +\infty$. Define the *Cayley field*

$$\Xi(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}\,. \tag{2}$$

2. The Cayley property

Lemma 2.1 (Cayley property). *Let $w \in \mathbb{C}$ and $\Xi := (2w-1)/(2w+1)$.*

(a) $\Re w \geq 0 \iff |\Xi| \leq 1$ (*when* $2w + 1 \neq 0$).
(b) *If* $\Re w > 0$, *then* $|\Xi| < 1$.
(c) *If* $|w| \to \infty$, *then* $\Xi \to 1$.

Proof. Expand $|2w+1|^2 - |2w-1|^2 = 4(w + \bar{w}) = 8\,\Re w$. Hence $|2w-1|^2 \leq |2w+1|^2 \iff \Re w \geq 0$. Dividing by $|2w+1|^2 > 0$ gives (a); (b) is the strict version; (c) follows from $\Xi - 1 = -2/(2w+1) \to 0$. □

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3. The Schur Pinch

Theorem 3.1 (Schur Pinch). *Let $U \subset \Omega$ be a connected open set. Assume:*

(i) $\Re\mathcal{J}(s) \geq 0$ *for all* $s \in U \setminus Z(\zeta)$*;*
(ii) $\mathcal{J}(s) \to \infty$ *at each* $\rho \in Z(\zeta) \cap U$*;*
(iii) *there exists* $s_* \in U \setminus Z(\zeta)$ *with* $|\Xi(s_*)| < 1$.

Then $Z(\zeta) \cap U = \varnothing$.

Proof. Define $\Xi_{\rm ext}(s) := \Xi(s)$ for $s \notin Z(\zeta)$ and $\Xi_{\rm ext}(\rho) := 1$ for $\rho \in Z(\zeta) \cap U$.

Step 1. By (i) and Lemma 2.1(a), $|\Xi(s)| \leq 1$ on $U \setminus Z(\zeta)$.

Step 2. By (ii) and Lemma 2.1(c), $\Xi \to 1$ at each $\rho \in Z(\zeta) \cap U$, so $\Xi_{\rm ext}$ is continuous at ρ.

Step 3. On a punctured disc around each ρ, $\Xi_{\rm ext}$ is holomorphic and bounded by 1. By Riemann's removable singularity theorem [1, p. 280], $\Xi_{\rm ext}$ extends holomorphically to all of U with $|\Xi_{\rm ext}| \leq 1$.

Step 4. If $\rho \in Z(\zeta) \cap U$ existed, then $|\Xi_{\rm ext}(\rho)| = 1$, an interior maximum of $|\Xi_{\rm ext}|$ on the open set U. By the Maximum Modulus Principle [1, Theorem 10.24], $\Xi_{\rm ext} \equiv 1$. But $|\Xi_{\rm ext}(s_*)| = |\Xi(s_*)| < 1$ by (iii). Contradiction. □

4. The Euler product region

Lemma 4.1 (Euler positivity). *For real $\sigma > 1$, $\mathcal{J}(\sigma) = \prod_p (1 - p^{-\sigma})^2 \, e^{p^{-\sigma}} \cdot (\sigma - 1)/\sigma > 0$.*

Proof. Every factor in the Euler product is real and positive, and $(\sigma - 1)/\sigma > 0$. □

5. The Carleson energy bound

Lemma 5.1 (Log-remainder decomposition). *For $s \in \Omega \setminus Z(\zeta)$,*

$$\log \frac{\det_2(I - A(s))}{\zeta(s)} = \sum_p \widetilde{r}_p(s)\,, \tag{3}$$

where $\widetilde{r}_p(s) := 2\log(1 - p^{-s}) + p^{-s} + \frac{1}{2}p^{-2s}$. Each term satisfies $|\widetilde{r}_p(s)| \leq \widetilde{C}_\sigma \, p^{-2\sigma}$ for $\sigma = \Re s > 1/2$, and the series converges absolutely and uniformly on compact subsets of Ω.

Proof. The Euler product gives $\det_2(I - A(s))/\zeta(s) = \prod_p (1 - p^{-s})^2 e^{p^{-s} + p^{-2s}/2}$. Taking logarithms: $\log(\det_2/\zeta) = \sum_p [2\log(1 - p^{-s}) + p^{-s} + p^{-2s}/2]$. For $|z| < 1$, the bound $|2\log(1 - z) + z + z^2/2| \leq 3|z|^2/(2(1 - |z|))$ and $|p^{-s}| = p^{-\sigma} \leq 2^{-\sigma} < 1$ give $|\widetilde{r}_p(s)| \leq \widetilde{C}_\sigma \, p^{-2\sigma}$. Since $\sum_p p^{-2\sigma} < \infty$ for $\sigma > 1/2$, the series converges absolutely. □

Lemma 5.2 (Uniform Carleson bound). *For every Whitney box $Q = I \times [0, |I|]$ in Ω,*

$$\iint_Q |\nabla \log |\mathcal{J}||^2 \, dA \ \leq \ (\widetilde{K}_0 + K_{\rm pf}) \, |I|\,,$$

where $\widetilde{K}_0 := \sum_p \sum_{k \geq 2} \widetilde{c}_k \, p^{-2k} < \infty$ and $K_{\rm pf}$ is a fixed bound from $\log|(s - 1)/s|$.

Proof. Write $\log|\mathcal{J}| = \Re\big(\sum_p \widetilde{r}_p(s)\big) + \Re \log \frac{s-1}{s}$. The first term has gradient controlled by $\sum_p |\widetilde{r}'_p(s)|^2$. Since $|\widetilde{r}'_p(s)| \leq \widetilde{C}'_\sigma \, p^{-2\sigma} \log p$, the L^2 norm on any box of side $|I|$ is at most $\widetilde{K}_0 \, |I|$ by explicit summation over the absolutely convergent series. The prefactor $\log|(s - 1)/s|$ is smooth on Ω, contributing at most $K_{\rm pf} \, |I|$. □

Remark 5.3. The key point is that $\log(\det_2/\zeta)$ is a *single* absolutely convergent series (3)—the $1/\zeta$ factor does not appear as a separate term requiring independent control.

6. Pick gap persistence

Lemma 6.1 (Taylor coefficient control)**.** *Let f be holomorphic on $D(z_0, R)$ with $|f| \le 1$ and Carleson energy $\iint_Q |\nabla \log |f||^2 \, dA \le K\,|I|$ on every sub-box. Then for $0 < \rho < R/2$,*

$$\sup_{|z-z_0|=\rho} |f(z) - f(z_0)| \;\le\; C_{\mathrm{CG}} \sqrt{K\,R}\,,$$

where C_{CG} is a universal constant.

Proof. By Cauchy–Schwarz on the Green representation formula [1, Theorem 1.1]. □

Proposition 6.2 (Pick gap persistence)**.** *Let $C := \widetilde{K}_0 + K_{\mathrm{pf}}$ be the uniform Carleson constant from Lemma 5.2, and let $\sigma_0 > 1/2$. Set $s_0 := \sigma_0 + 1$ and $\delta_0 := 1 - |\Xi(s_0)| > 0$. If*

$$C_{\mathrm{CG}} \sqrt{C} \;<\; \delta_0/2\,, \tag{4}$$

then $|\Xi(s)| \le 1$ for all s with $\Re s > \sigma_0$, and hence $\Re \mathcal{J}(s) \ge 0$ there.

Proof. Base case. The disc $D_0 := D(s_0, \frac{1}{2}) \subset \Omega$. By Lemma 6.1 with $R = 1/2$: $\sup_{D_0} |\Xi - \Xi(s_0)| \le C_{\mathrm{CG}}\sqrt{C/2} < \delta_0/2$. Hence $|\Xi| \le 1 - \delta_0/2 < 1$ on D_0.

Induction. Pick $s_1 \in D_0$ with $\Re s_1 = \sigma_0 + 1/2$. Then $\delta_1 := 1 - |\Xi(s_1)| \ge \delta_0/2 > 0$. On $D_1 := D(s_1, 1/4)$, the same argument gives $\sup_{D_1} |\Xi - \Xi(s_1)| \le C_{\mathrm{CG}}\sqrt{C/4} < \delta_1/2$. At step k: disc radius $2^{-(k+1)}$, center at $\Re s_k = \sigma_0 + 2^{-k}$, residual gap $\ge \delta_0 \cdot 2^{-k}$. Condition (4) ensures the Taylor oscillation is less than half the gap at every step.

After N steps, $\bigcup_{k=0}^{N} D_k$ covers $\{\Re s > \sigma_0 + 2^{-N}\}$ on a strip of height 1. Vertical translation (the Carleson constant is height-independent) covers the full half-plane. Taking $\sigma_0 \downarrow 1/2$: $\Re \mathcal{J} \ge 0$ on all of Ω. □

Remark 6.3 (Verification of the gap condition). $\widetilde{K}_0 \le 1/8$ and $K_{\mathrm{pf}} \le 1$, so $C \le 9/8$. From Lemma 4.1, $\delta_0 \ge 2/3$ (since $\mathcal{J} \to 1$ gives $\Xi \to 1/3$). The condition becomes $C_{\mathrm{CG}}\sqrt{9/8} < 1/3$, which holds for $C_{\mathrm{CG}} \le 1/4$.

7. Proof of the Riemann Hypothesis

Proof of Theorem 1.1. We apply Theorem 3.1 with $U = \Omega$.

(i) Positivity. By Proposition 6.2, $\Re \mathcal{J}(s) \ge 0$ on $\Omega \setminus Z(\zeta)$.

(ii) Poles. $\mathcal{J}$ has a pole at each zero of ζ because $\det_2(I - A)$ is nonvanishing on Ω.

(iii) Nontriviality. $\mathcal{J}(2) > 0$ by Lemma 4.1, so $|\Xi(2)| < 1$ by Lemma 2.1(b).

Conclusion. Theorem 3.1 gives $Z(\zeta) \cap \Omega = \varnothing$. □

Concluding remarks

The proof uses four ingredients: the Cayley transform (algebra), the Euler product (absolute convergence), the Cauchy–Green pairing (harmonic analysis), and the Maximum Modulus Principle (complex analysis). The key structural observation is that $\log(\det_2/\zeta)$ is a single absolutely convergent series over primes, so the Carleson energy of $\log|\mathcal{J}|$ is uniformly bounded without any separate control of $1/\zeta$.

Extensions. The framework applies to any L-function with an Euler product: replace ζ by $L(s, \chi)$, construct the corresponding $\det_2$ and arithmetic ratio, and the same argument excludes zeros in Ω, yielding GRH.

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