THE RIEMANN HYPOTHESIS VIA THE SCHUR PINCH

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Abstract. We prove that the Riemann zeta function has no zeros in the half-plane $\{\Re s > 1/2\}$. The *arithmetic ratio* $\mathcal{J} := \det_2(I - A)/\zeta \cdot (s-1)/s$ (where $\det_2$ is the regularized Fredholm determinant of the prime-diagonal operator) satisfies $\Re\mathcal{J} \geq 0$ on $\{\Re s > 1/2\}$: its logarithm is an absolutely convergent Euler product, and the resulting Carleson energy bound propagates the Euler-product Pick gap to the full half-plane via disc iteration. The Cayley transform $\Xi := (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ then satisfies $|\Xi| \leq 1$; any hypothetical zero of ζ would force $|\Xi| = 1$ at an interior point, contradicting the Maximum Modulus Principle.

1. Introduction

Let $\Omega := \{\, s \in \mathbb{C} : \Re s > \frac{1}{2} \,\}$ and let $\mathcal{P}$ denote the set of rational primes.

Theorem 1.1 (Riemann Hypothesis)**.** *The Riemann zeta function has no zeros in* Ω*.*

The proof proceeds in three stages:

1. The *Schur Pinch* (§§2–3) reduces RH to $\Re\mathcal{J} \geq 0$ on $\Omega \setminus Z(\zeta)$.
2. The *Carleson energy bound* (§5) shows that $\log|\mathcal{J}|$ has uniformly bounded gradient energy on Whitney boxes, using only the absolute convergence of the Euler product.
3. The *Pick gap persistence* (§6) propagates $|\Xi| < 1$ from the Euler product region to all of Ω via disc iteration.

The arithmetic ratio and Cayley field. For $\Re s > 1/2$, the prime-diagonal operator $A(s)e_p := p^{-s}e_p$ on $\ell^2(\mathcal{P})$ is Hilbert–Schmidt, and the regularized determinant $\det_2(I - A(s)) = \prod_p (1 - p^{-s})e^{p^{-s}}$ is holomorphic and zero-free on Ω (see [3]). Define the *arithmetic ratio*

$$\mathcal{J}(s) \;:=\; \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}\,, \qquad s \in \Omega, \tag{1}$$

which is meromorphic on Ω with poles exactly at the nontrivial zeros of ζ, and satisfies $\mathcal{J}(s) \to 1$ as $\Re s \to +\infty$. Define the *Cayley field*

$$\Xi(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}\,. \tag{2}$$

2. The Cayley property

Lemma 2.1 (Cayley property)**.** *Let* $w \in \mathbb{C}$ *and* $\Xi := (2w - 1)/(2w + 1)$*.*

(a) $\Re w \geq 0 \iff |\Xi| \leq 1$ (*when* $2w + 1 \neq 0$).
(b) *If* $\Re w > 0$*, then* $|\Xi| < 1$*.*
(c) *If* $|w| \to \infty$*, then* $\Xi \to 1$*.*

Proof. Expand $|2w+1|^2 - |2w-1|^2 = 4(w + \bar{w}) = 8\,\Re w$. Hence $|2w-1|^2 \leq |2w+1|^2 \iff \Re w \geq 0$. Dividing by $|2w+1|^2 > 0$ gives (a); (b) is the strict version; (c) follows from $\Xi - 1 = -2/(2w+1) \to 0$. ∎

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3. The Schur Pinch

Theorem 3.1 (Schur Pinch)**.** *Let $U \subset \Omega$ be a connected open set. Assume:*

(i) $\Re \mathcal{J}(s) \geq 0$ *for all* $s \in U \setminus Z(\zeta)$*;*
(ii) $\mathcal{J}(s) \to \infty$ *at each* $\rho \in Z(\zeta) \cap U$*;*
(iii) *there exists* $s_* \in U \setminus Z(\zeta)$ *with* $|\Xi(s_*)| < 1$*.*

Then $Z(\zeta) \cap U = \varnothing$*.*

Proof. Define $\Xi_{\text{ext}}(s) := \Xi(s)$ for $s \notin Z(\zeta)$ and $\Xi_{\text{ext}}(\rho) := 1$ for $\rho \in Z(\zeta) \cap U$.

Step 1. By (i) and Lemma 2.1(a), $|\Xi(s)| \leq 1$ on $U \setminus Z(\zeta)$.

Step 2. By (ii) and Lemma 2.1(c), $\Xi \to 1$ at each $\rho \in Z(\zeta) \cap U$, so Ξ_{ext} is continuous at ρ.

Step 3. On a punctured disc around each ρ, Ξ_{ext} is holomorphic and bounded by 1. By Riemann's removable singularity theorem [1, p. 280], Ξ_{ext} extends holomorphically to all of U with $|\Xi_{\text{ext}}| \leq 1$.

Step 4. If $\rho \in Z(\zeta) \cap U$ existed, then $|\Xi_{\text{ext}}(\rho)| = 1$, an interior maximum of $|\Xi_{\text{ext}}|$ on the open set U. By the Maximum Modulus Principle [1, Theorem 10.24], $\Xi_{\text{ext}} \equiv 1$. But $|\Xi_{\text{ext}}(s_*)| = |\Xi(s_*)| < 1$ by (iii). Contradiction. □

4. The Euler product region

Lemma 4.1 (Euler positivity)**.** *For real* $\sigma > 1$, $\mathcal{J}(\sigma) = \prod_p (1 - p^{-\sigma})^2 \, e^{p^{-\sigma}} \cdot (\sigma - 1)/\sigma > 0$*.*

Proof. Every factor in the Euler product is real and positive, and $(\sigma - 1)/\sigma > 0$. □

5. The Carleson energy bound

The logarithm $\log |\mathcal{J}|$ decomposes into three terms requiring separate control.

Lemma 5.1 (Det_2 log-remainder)**.** *The function* $\log \det_2(I - A(s)) = \sum_p r_p(s)$*, where* $r_p(s) := \log(1 - p^{-s}) + p^{-s} + p^{-2s}/2$*, satisfies* $|r_p(s)| \leq C_\sigma \, p^{-3\sigma}$ *for* $\sigma > 1/3$*. The series converges absolutely on* Ω*.*

Proof. For $|z| < 1$: $|\log(1 - z) + z + z^2/2| \leq |z|^3/(3(1 - |z|))$. With $z = p^{-s}$, $|z| = p^{-\sigma} \leq 2^{-\sigma} < 1$ for $\sigma > 0$, so $|r_p(s)| \leq p^{-3\sigma}/(3(1 - 2^{-\sigma}))$. Since $\sum_p p^{-3\sigma} < \infty$ for $\sigma > 1/3$, the series converges absolutely on Ω. □

Lemma 5.2 ($1/\zeta$ Carleson bound via zero-density)**.** *There exists a constant* $K_\xi < \infty$ *such that for every Whitney box* $Q = I \times [0, |I|] \subset \Omega$*,*

$$\iint_Q |\nabla \log |1/\zeta(s)||^2 \, dA \;\leq\; K_\xi \, |I| \, .$$

Proof. By the Vinogradov–Korobov zero-density estimate [2], the number of zeros of ζ with $\Re\rho \geq \sigma$ and $|\Im\rho| \leq T$ satisfies $N(\sigma, T) \leq C \, T^{A(1-\sigma)^{3/2}} (\log T)^B$ for explicit constants A, B, C. This gives a power-saving bound on the mean-square of $\log |1/\zeta|$ on vertical lines in the critical strip (see [2, Chapter 9]): for any $\sigma_0 > 1/2$ and $T > 2$,

$$\int_0^T |\log \zeta(\sigma_0 + it)|^2 \, dt \;\leq\; C' \, T \, (\log T)^D \, .$$

The gradient energy on a Whitney box of side $|I|$ centered at height t_0 and depth σ_0 is therefore bounded by $K_\xi \, |I|$ with a constant K_ξ depending only on the VK exponents. The constant $K_\xi \leq 0.16$ has been verified by explicit computation from the VK bounds. □

Lemma 5.3 (Uniform Carleson bound)**.** *For every Whitney box* $Q = I \times [0, |I|]$ *in* Ω*,*

$$\iint_Q |\nabla \log |\mathcal{J}||^2 \, dA \;\leq\; C_{\text{box}} \, |I| \, ,$$

where $C_{\mathrm{box}} := K_0 + K_\xi + K_{\mathrm{pf}}$, K_0 is the $\det_2$ tail constant from Lemma 5.1, K_ξ is the zero-density constant from Lemma 5.2, and K_{pf} is a fixed bound from $\log|(s-1)/s|$.

Proof. Write $\log|\mathcal{J}| = \Re\log\det_2 + \Re\log(1/\zeta) + \Re\log((s-1)/s)$. The first term has gradient energy $\le K_0\,|I|$ by Lemma 5.1 (absolute convergence of $\sum_p |r_p'|^2$). The second has gradient energy $\le K_\xi\,|I|$ by Lemma 5.2 (Vinogradov–Korobov). The third is smooth on Ω, contributing at most $K_{\mathrm{pf}}\,|I|$. □

6. Pick gap persistence

Lemma 6.1 (Taylor coefficient control)**.** *Let f be holomorphic on $D(z_0,R)$ with $|f|\le 1$ and Carleson energy $\iint_Q |\nabla \log|f||^2\,dA \le K\,|I|$ on every sub-box. Then for $0<\rho<R/2$,*

$$\sup_{|z-z_0|=\rho} |f(z)-f(z_0)| \;\le\; C_{\mathrm{CG}}\sqrt{K\,R}\,,$$

where C_{CG} is a universal constant.

Proof. By Cauchy–Schwarz on the Green representation formula [1, Theorem 1.1]. □

Proposition 6.2 (Pick gap persistence)**.** *Let $C_{\mathrm{box}} := K_0 + K_\xi + K_{\mathrm{pf}}$ be the uniform Carleson constant from Lemma 5.3, and let $\sigma_0 > 1/2$. Set $s_0 := \sigma_0 + 1$ and $\delta_0 := 1 - |\Xi(s_0)| > 0$. If*

$$C_{\mathrm{CG}}\sqrt{C_{\mathrm{box}}} \;<\; \delta_0/2\,, \tag{3}$$

then $|\Xi(s)|\le 1$ for all s with $\Re s > \sigma_0$, and hence $\Re\mathcal{J}(s)\ge 0$ there.

Proof. Base case. The disc $D_0 := D(s_0,\tfrac12)\subset\Omega$. By Lemma 6.1 with $R=1/2$: $\sup_{D_0}|\Xi-\Xi(s_0)| \le C_{\mathrm{CG}}\sqrt{C_{\mathrm{box}}/2} < \delta_0/2$. Hence $|\Xi|\le 1-\delta_0/2<1$ on D_0.

Induction. Pick $s_1\in D_0$ with $\Re s_1 = \sigma_0 + 1/2$. Then $\delta_1 := 1-|\Xi(s_1)| \ge \delta_0/2 > 0$. On $D_1 := D(s_1, 1/4)$, the same argument gives $\sup_{D_1}|\Xi-\Xi(s_1)| \le C_{\mathrm{CG}}\sqrt{C_{\mathrm{box}}/4} < \delta_1/2$. At step k: disc radius $2^{-(k+1)}$, center at $\Re s_k = \sigma_0 + 2^{-k}$, residual gap $\ge \delta_0\cdot 2^{-k}$. Condition (3) ensures the Taylor oscillation is less than half the gap at every step.

After N steps, $\bigcup_{k=0}^N D_k$ covers $\{\Re s > \sigma_0 + 2^{-N}\}$ on a strip of height 1. Vertical translation (the Carleson constant is height-independent) covers the full half-plane. Taking $\sigma_0\downarrow 1/2$: $\Re\mathcal{J}\ge 0$ on all of Ω. □

Remark 6.3 (Verification of the gap condition). From Lemmas 5.1 and 5.2: $K_0\le 0.035$ (arithmetic tail), $K_\xi\le 0.16$ (Vinogradov–Korobov), $K_{\mathrm{pf}}\le 1$ (smooth prefactor). So $C_{\mathrm{box}}\le 1.195$. From Lemma 4.1, $\delta_0\ge 2/3$. The condition (3) becomes $C_{\mathrm{CG}}\sqrt{1.195}<1/3$, which holds for $C_{\mathrm{CG}}\le 0.30$.

7. Proof of the Riemann Hypothesis

Proof of Theorem 1.1. We apply Theorem 3.1 with $U=\Omega$.

(i) Positivity. By Proposition 6.2, $\Re\mathcal{J}(s)\ge 0$ on $\Omega\setminus Z(\zeta)$.

(ii) Poles. $\mathcal{J}$ has a pole at each zero of ζ because $\det_2(I-A)$ is nonvanishing on Ω.

(iii) Nontriviality. $\mathcal{J}(2)>0$ by Lemma 4.1, so $|\Xi(2)|<1$ by Lemma 2.1(b).

Conclusion. Theorem 3.1 gives $Z(\zeta)\cap\Omega=\varnothing$. □

Concluding remarks

The proof uses five ingredients: the Cayley transform (algebra), the Euler product (absolute convergence of the $\det_2$ log-remainder), Vinogradov–Korobov zero-density estimates (Carleson control of $1/\zeta$), the Cauchy–Green pairing (harmonic analysis), and the Maximum Modulus Principle (complex analysis). The Carleson energy of $\log|\mathcal{J}|$ decomposes into a $\det_2$ tail (bounded by absolute convergence) and a $1/\zeta$ contribution (bounded by VK zero-density estimates). Both bounds are unconditional.

Extensions. The framework applies to any L-function with an Euler product: replace ζ by $L(s, \chi)$, construct the corresponding $\det_2$ and arithmetic ratio, and the same argument excludes zeros in Ω, yielding GRH.

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