A boundary product–certificate reduction of the Riemann Hypothesis

Jonathan Washburn
Recognition Science, Recognition Physics Institute
Austin, Texas, USA
jon@recognitionphysics.org

September 2025

Abstract

We reduce the Riemann Hypothesis to a boundary product–certificate route on $\{\Re s > \frac{1}{2}\}$. The globalization/pinch mechanism is unconditional once an almost-everywhere boundary wedge (P+) is available for a normalized ratio. The remaining analytic bottleneck is a scale-free Whitney box-energy input for the ξ-field (Hypothesis 26); equivalently, a uniform prohibition on extreme zero–clustering at the microscopic scale $H \asymp 1/\log\langle T\rangle$, which can be formulated as a uniform weighted short-interval variation bound for the classical $S(T)$ term (Lemma 28). Unconditionally we only prove the weaker $O(|I|\log\langle T\rangle)$ Whitney-box energy bound (Lemma 24). Under the scale-free box-energy hypothesis, the manuscript's certificate machinery yields (P+) and hence RH by the Schur–Herglotz pinch.

Keywords. Riemann zeta function; Hardy/Smirnov spaces; Herglotz/Schur functions; Carleson measures; Hilbert–Schmidt determinants.

MSC 2020. 11M26, 30D15, 30C85; secondary 47A12, 47B10.

Notation and conventions

- Half–plane: $\Omega := \{\Re s > \frac{1}{2}\}$; boundary line $\Re s = \frac{1}{2}$ parameterized by $t \in \mathbb{R}$ via $s = \frac{1}{2} + it$.
- Outer/inner: for a holomorphic F on Ω, write $F = I\,O$ with O outer (zero–free; boundary modulus e^u) and I inner (Blaschke and singular inner factors).
- Herglotz/Schur: H is Herglotz if $\Re H \geq 0$ on Ω; Θ is Schur if $|\Theta| \leq 1$ on Ω. Cayley: $\Theta = (H-1)/(H+1)$.
- Poisson/Hilbert: $P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$; boundary Hilbert transform $\mathcal{H}$ on $\mathbb{R}$.
- Off-critical zeros: the (half-plane) *defect measure* is

$$\nu \;:=\; \sum_{\substack{\rho=\beta+i\gamma \\ \beta>1/2}} 2(\beta - \tfrac{1}{2})\,\delta_\rho \qquad \text{on } \Omega,$$

and the associated *boundary balayage* is the absolutely continuous measure μ on $\mathbb{R}$ with density

$$\frac{d\mu}{dt}(t) \;=\; \sum_{\substack{\rho=\beta+i\gamma \\ \beta>1/2}} 2(\beta - \tfrac{1}{2})\,P_{\beta-1/2}(t-\gamma).$$

- Windows: fix an even C^∞ flat-top window $\psi : \mathbb{R} \to [0,1]$ with $\psi \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi \subset [-2,2]$ (see *Printed window*). For $L > 0$ and $t_0 \in \mathbb{R}$ set

$$\psi_{L,t_0}(t) := \psi\Big(\frac{t-t_0}{L}\Big), \qquad \varphi_{L,t_0}(t) := \frac{1}{L}\,\psi\Big(\frac{t-t_0}{L}\Big), \qquad m_\psi := \int_{\mathbb{R}} \psi.$$

  Then $\int_{\mathbb{R}} \varphi_{L,t_0} = m_\psi$ and $\operatorname{supp}\varphi_{L,t_0} \subset [t_0-2L, t_0+2L]$, while $\varphi_{L,t_0} \equiv L^{-1}$ on $[t_0-L, t_0+L]$.

- Carleson boxes: $Q(\alpha I) = I \times (0, \alpha|I|]$; C_{box} uses the area measure $\lambda := |\nabla U|^2\,\sigma\,dt\,d\sigma$.

- Constants/macros: $c_0(\psi) = 0.17620819$, $C_\psi^{(H^1)} = 0.2400$, $C_H(\psi) = 2/\pi$, K_ξ, $C_{\text{box}}^{(\zeta)} = K_0 + K_\xi$, $M_\psi = (4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}$, $\Upsilon = (2/\pi)\,(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}/0.17620819$.

- Scope convention: throughout, $C_{\text{box}}^{(\zeta)}$ denotes the supremum over all boxes $Q(\alpha I)$ with $I \subset \mathbb{R}$ (fixed $\alpha \in [1,2]$).

- Terminology (used once and consistently): PSC = product certificate route (active); AAB = adaptive analytic bandlimit (archival, not used in the main chain); KYP = Kalman–Yakubovich–Popov (appears only in archived material; not used in proofs).

Standing properties (used in the pinch)

The globalization/pinch step (Section 2) uses two auxiliary facts: a right-edge normalization (N1) for the chosen normalized ratio (ensuring a fixed limiting value for the Cayley transform as $\sigma \to +\infty$), and a non-cancellation statement (N2) (ensuring no cancellation at off-critical ζ-zeros). Both are recorded explicitly in Section 2; (N2) is verified in the paragraph "Proof of (N2)" using the diagonal determinant and the fact that outers are zero-free.

Reader's guide

- **What is unconditional.** The globalization/pinch mechanism is unconditional once an a.e. boundary wedge (P+) is available (Section 2 together with (N1)–(N2)).

- **What remains open.** The certificate step *as a proof of RH* requires a *scale-free* Whitney box-energy input for the ξ-field (Hypothesis 26); unconditionally we only prove the weaker $O(|I|\log\langle T\rangle)$ Whitney-box bound (Lemma 24).

- **Equivalent formulation of the bottleneck.** By Lemma 28, the scale-free hypothesis is implied by a uniform bound on a weighted microscopic short-interval variation of $S(T) = \frac{1}{\pi}\operatorname{Arg}\zeta(\frac{1}{2}+iT)$ at the Whitney scale $L = c/\log\langle T\rangle$; informally, it forbids arbitrarily large zero clusters in windows of length $\asymp 1/\log T$.

- Structural innovations: outer cancellation with energy bookkeeping, outer-phase $\mathcal{H}[u']$ identity, and phase–velocity calculus with smoothed $\to$ boundary passage.

- Where numerics enter: diagnostic tables (gated by `\shownumerics`) record outward-rounded enclosures for K_0, the hypothesized scale-free constant K_ξ (Hypothesis 26), and derived thresholds. These do not constitute proofs; they are recorded only for reproducibility/diagnostics.

- How (P+) would be proved under the hypothesis: the phase–velocity identity paired with CR–Green bounds yields a quantitative windowed phase control (Theorem 60). The final conversion "certificate $\Rightarrow$ boundary wedge" is handled by an explicit wedge lemma (Lemma 8).

- How RH follows *once (P+) holds*: assuming additionally the Poisson representation (H), one has $\Re F_\Lambda \geq 0$ on $\Omega \setminus Z(\zeta)$ and hence Θ_Λ is Schur there; removability and the (N1)–(N2) pinch then rule out any ζ–zero in Ω, yielding RH (Section 2, Theorem 57, Corollary 53).

Dependency map (load-bearing chain)

All proofs not explicitly listed below are either auxiliary or marked *diagnostic/archival* in the text.

1. **Phase–velocity identity + boundary passage.** Theorem 6, based on Lemmas 1, 14, 2, 5, and 10, yields the phase–velocity identity with a positive distribution $-w'$ and explicit Poisson/atomic terms.

2. **Box-energy hypothesis (bottleneck).** Hypothesis 26 supplies a scale-free Whitney box-energy constant K_ξ for the ξ-field. Unconditionally we only prove $O(\log\langle T\rangle)$ growth on Whitney boxes (Lemma 24).

3. **Certificate $\Rightarrow$ (P+) under the hypothesis.** Under Hypothesis 26, the CR–Green windowed phase bound (Lemma 31 with Lemma 3) feeds into Theorem 60, and the explicit wedge lemma (Lemma 8) yields (P+).

4. **Globalization and RH once (P+) holds.** Assuming (H), Poisson transport and Cayley yield F_Λ Herglotz and Θ_Λ Schur on $\Omega \setminus Z(\zeta)$ (Corollaries 54 and 56). Removability across punctures uses Lemma 51. The Schur pinch then eliminates $Z(\zeta) \cap \Omega$ (Theorem 57), and Corollary 53 yields RH.

1 Introduction

Conceptual motivation. The Euler product for ζ separates the $k = 1$ prime layer from all higher prime powers. On the right half–plane $\{\Re s > \frac{1}{2}\}$ the diagonal prime operator $A(s)e_p := p^{-s}e_p$ has finite Hilbert–Schmidt norm ($\sum_p p^{-2\sigma} < \infty$), so the $k \geq 2$ tail is naturally encoded by the 2–modified determinant $\det_2(I - A)$. After dividing by an outer normalizer one arrives at a ratio $\mathcal{J}$ that shares its zero/pole geometry with ξ but whose boundary modulus is unimodular. This puts the problem into the bounded–real/Schur/Herglotz framework: boundary positivity for $2\mathcal{J}$ transports to the interior by Poisson, and Cayley converts positivity into a Schur contractive bound for $\Theta = (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$. The central analytic mechanism is local: a Cauchy–Riemann/Green pairing against a Poisson test on a Whitney box controls windowed boundary phase by the Dirichlet energy of $U = \Re \log \mathcal{J}$. The globalization/pinch step is unconditional once (P+) holds; the remaining bottleneck is the scale-free Whitney ξ-energy input isolated as Hypothesis 26. **Main result and one-route proof outline (conditional bottleneck made explicit).** The route is a boundary product–certificate mechanism in the ζ–normalized gauge (no C_P term). The steps are:

- Phase–velocity identity with outer normalization and boundary passage (Lemma 17).

- Derivative envelope and the H^1–BMO link yielding M_ψ (Lemmas 19,45).

- Box–energy control for the paired field (Lemma 34), using the arithmetic tail bound (Lemma 22) and the ξ-energy input (Hypothesis 26; unconditionally only Lemma 24).

- Boundary wedge from the certificate (Theorem 60).

- Globalization/pinch across $Z(\zeta)$ and conclusion (Section 2).

We retain two compatible RHS bounds (CR–Green + box energy, and the Hilbert envelope). Any printed numerics are diagnostic only and do not substitute for the missing scale-free ξ-energy input.

Non-circularity (active certificate).

- Active RHS uses only three inputs: $c_0(\psi)$ (plateau), the CR–Green box constant $C(\psi)$, and the box-energy constant $C_{\text{box}}^{(\zeta)}$.
- Closure of (P+) uses the explicit wedge lemma (Lemma 8), avoiding the earlier informal "wedge criterion" phrasing.
- The envelope constants $C_H(\psi)$ and M_ψ are auxiliary and do not enter the load-bearing inequality for (P+).

One-route outline (what actually happens). Section 6 establishes the phase–velocity identity with outer normalization and boundary passage. Under the scale-free Whitney ξ-energy hypothesis (Hypothesis 26), the window/CR–Green machinery converts box-energy control into windowed phase control (Theorem 60); combined with a smallness condition $\Upsilon < \frac{1}{2}$, the wedge lemma yields (P+) (Lemma 8). The globalization/pinch step (Section 2) is unconditional once (P+) holds. Unconditionally, Lemma 24 provides only $O(\log\langle T\rangle)$ growth on Whitney boxes and does not supply the scale-free constant needed for the quantitative closure. The Riemann Hypothesis (RH) admits several analytic formulations. In this paper we pursue a bounded-real (BRF) route on the right half-plane

$$\Omega \;:=\; \{\, s \in \mathbb{C} : \; \Re s > \tfrac{1}{2} \,\},$$

which is naturally expressed in terms of Herglotz/Schur functions and passive systems. Let $\mathcal{P}$ be the primes, and define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p \;:=\; p^{-s} e_p.$$

For $\sigma := \Re s > \frac{1}{2}$ we have $\|A(s)\|_{\mathcal{S}_2}^2 = \sum_{p\in\mathcal{P}} p^{-2\sigma} < \infty$ and $\|A(s)\| \le 2^{-\sigma} < 1$. With the completed zeta function

$$\xi(s) \;:=\; \tfrac{1}{2} s(1-s)\, \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s)$$

and the Hilbert–Schmidt regularized determinant $\det_2$, we study the analytic function

$$\mathcal{J}(s) \;:=\; \frac{\det_2(I - A(s))}{\mathcal{O}(s)\, \xi(s)}, \qquad \Theta(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}.$$

The BRF assertion is that the Cayley transform associated to the normalized ratio is Schur on zero-free rectangles (equivalently, the ratio is Herglotz there). In the globalization step (Section 2) we apply this in a Λ–normalized gauge (avoiding the trivial zero of ξ at $s = 1$) and pinch across $Z(\zeta) \cap \Omega$.

Our method combines four ingredients:

- **Schur–determinant splitting.** For a block operator $T(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$ with finite auxiliary part, one has

  $$\log\det_2(I - T) \;=\; \log\det_2(I - A) \;+\; \log\det(I - S), \qquad S \;:=\; D - C(I - A)^{-1}B,$$

  which separates the Hilbert–Schmidt ($k \ge 2$) terms from the finite block.

- **HS continuity for** $\det_2$**.** Prime truncations $A_N \to A$ in the HS topology, uniformly on compacts in Ω, imply local-uniform convergence of $\det_2(I - A_N)$ (Section 9). Division by ξ is justified only on compacts avoiding its zeros; throughout we explicitly state such hypotheses when needed.

Unsmoothing $\det_2$: routed through smoothed testing (A1)

Lemma 1 (Smoothed distributional bound for $\partial_\sigma \Re \log \det_2$)**.** *Let $I \Subset \mathbb{R}$ be a compact interval and fix $\varepsilon_0 \in (0, \frac{1}{2}]$. There exists a finite constant*

$$C_* \; := \; \sum_p \sum_{k\geq 2} \frac{p^{-k/2}}{k^2 \, \log p} \; < \; \infty$$

such that for all $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$ and every $\varphi \in C_c^2(I)$,

$$\Big| \int_{\mathbb{R}} \varphi(t) \, \partial_\sigma \Re \log \det_2 \left(I - A(\sigma + it)\right) dt \Big| \; \leq \; C_* \, \|\varphi''\|_{L^1(I)}.$$

In particular, testing against smooth, compactly supported windows yields bounds uniform in σ.

Proof. For $\sigma > \frac{1}{2}$ one has $\sum_p |p^{-(\sigma+it)}|^2 = \sum_p p^{-2\sigma} < \infty$, so the diagonal product formula for $\det_2$ gives

$$\log \det_2(I - A(s)) \; = \; \sum_p \left(\log(1 - p^{-s}) + p^{-s} \right) \; = \; -\sum_p \sum_{k\geq 2} \frac{p^{-ks}}{k},$$

with absolute convergence (uniform on compact subsets of $\{\Re s > \frac{1}{2}\}$). Differentiating termwise in $\sigma = \Re s$ yields the absolutely convergent expansion

$$\partial_\sigma \Re \log \det_2 \left(I - A(\sigma + it)\right) \; = \; \sum_p \sum_{k\geq 2} (\log p) \, p^{-k\sigma} \cos(kt \log p).$$

For each frequency $\omega = k \log p \geq 2 \log 2$, two integrations by parts give

$$\Big| \int_{\mathbb{R}} \varphi(t) \cos(\omega t) \, dt \Big| \; \leq \; \frac{\|\varphi''\|_{L^1(I)}}{\omega^2}.$$

Since $\sum_{p,k\geq 2} (\log p) \, p^{-k\sigma}/(k \log p)^2 \leq C_*$ uniformly in $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$, Tonelli/Fubini allows summing after testing against φ. Summing the resulting majorant yields

$$\Big| \int \varphi \, \partial_\sigma \Re \log \det_2 \; dt \Big| \; \leq \; \|\varphi''\|_{L^1} \sum_p \sum_{k\geq 2} \frac{(\log p) \, p^{-k\sigma}}{(k \log p)^2} \; \leq \; \|\varphi''\|_{L^1} \sum_p \sum_{k\geq 2} \frac{p^{-k/2}}{k^2 \, \log p},$$

uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$, since the rightmost double series converges. This proves the claim. $\square$

Note. The single-interval density route is archived; the small-L scaling $c_0 L \leq C \, L^{1/2}$ does not contradict the RHS bound.

Lemma 2 (De-smoothing to L^1 control)**.** *Fix a compact interval $I \Subset \mathbb{R}$. Suppose a family $g_\varepsilon \in \mathcal{D}'(I)$ satisfies*

$$|\langle g_\varepsilon, \phi'' \rangle| \; \leq \; C_I \, \|\phi''\|_{L^1(I)} \qquad \forall \, \phi \in C_c^\infty(I), \; \forall \, \varepsilon \in (0, \varepsilon_0].$$

Then g_ε is uniformly bounded in $W^{-2,\infty}(I)$ and there exist primitives $u_\varepsilon \in BV(I)$ with $u_\varepsilon' = g_\varepsilon$ in $\mathcal{D}'(I)$ such that, along a subsequence, $u_\varepsilon \to u$ in $L^1(I)$. In particular, applied to $g_\varepsilon = \partial_\sigma \Re \log \det_2(\frac{1}{2} + \varepsilon + it)$ together with the tested L^1 bound for $\partial_\sigma \Re \log \xi$, this yields the L^1 Cauchy property used in Proposition 21.

Proof. 1) Uniform $W^{-2,\infty}$ bound. Define the linear functionals $\Lambda_\varepsilon(\psi) := \langle g_\varepsilon, \psi\rangle$ for $\psi \in C_c^\infty(I)$. For any $\psi \in C_c^\infty(I)$ let $\Phi \in C_c^\infty(I)$ solve $\Phi'' = \psi$ with zero boundary data on I (obtainable by two integrations). Then $\|\Phi''\|_{L^1} = \|\psi\|_{L^1}$ and by hypothesis

$$|\Lambda_\varepsilon(\psi)| \;=\; |\langle g_\varepsilon, \Phi''\rangle| \;\le\; C_I\,\|\Phi''\|_{L^1} \;=\; C_I\,\|\psi\|_{L^1}.$$

Thus $\|g_\varepsilon\|_{W^{-2,\infty}(I)} \le C_I$ uniformly in ε.

2) Construction of primitives and BV bound. Fix any $x_0 \in I$. Let G be the Green operator for ∂_t^2 on I with homogeneous boundary data. Define $u_\varepsilon := G[g_\varepsilon] + c_\varepsilon$, where c_ε is the constant making $\int_I u_\varepsilon = 0$. Then $u_\varepsilon \in W^{1,\infty}(I)^*$ and $u_\varepsilon' = g_\varepsilon$ in distributions. For $\varphi \in C_c^\infty(I)$ with $\|\varphi\|_{L^1} \le 1$,

$$|\langle u_\varepsilon', \varphi\rangle| \;=\; |\langle g_\varepsilon, \varphi\rangle| \;\le\; C_I,$$

so the total variation $\mathrm{Var}_I(u_\varepsilon) \le C_I$. Together with the zero-mean choice, this yields a uniform $BV(I)$ bound on u_ε.

3) Compactness and L^1 convergence. By the compactness theorem for BV (Helly selection: bounded sets in $BV(I)$ are relatively compact in $L^1(I)$; see, e.g., Ambrosio–Fusco–Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Thm. 3.23), there exists a subsequence (not relabeled) such that $u_\varepsilon \to u$ in $L^1(I)$ and pointwise a.e. on I. This proves the claim. $\square$

Lemma 3 (Neutralization bookkeeping for CR–Green on a Whitney box)**.** *Let $I = [t_0 - L, t_0 + L]$ and $Q(\alpha' I)$ be as above. Let B_I be the product of half–plane Blaschke factors for the zeros/poles of J in $Q(\alpha' I)$ and set $\widetilde{U} := \Re \log(J/B_I)$ on $Q(\alpha' I)$. Then with the same cutoff χ_{L,t_0} and Poisson test V_{ψ,L,t_0},*

$$\iint_{Q(\alpha' I)} \nabla \widetilde{U} \cdot \nabla(\chi V)\, dt\, d\sigma = \int_{\mathbb{R}} \psi_{L,t_0}(t) \;-\; w'(t)\, dt \;+\; \mathcal{E}_{\mathrm{side}} \;+\; \mathcal{E}_{\mathrm{top}},$$

where the error terms obey the uniform bound

$$|\mathcal{E}_{\mathrm{side}}| + |\mathcal{E}_{\mathrm{top}}| \;\le\; C_{\mathrm{neu}}(\alpha, \psi)\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\, \sigma\Big)^{1/2}.$$

In particular,

$$\int_{\mathbb{R}} \psi_{L,t_0}(-w') \;\le\; C(\psi)\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\, \sigma\Big)^{1/2},$$

with constants independent of t_0 and L.

Proof. Let $\mathcal{Z}_I$ denote the (finite) set of zeros/poles of J in the bounded box $Q(\alpha' I)$ (counted with multiplicity). For $\rho = \beta + i\gamma \in \mathcal{Z}_I$ write $\rho^* := 1 - \overline{\rho}$ for reflection across the boundary line $\Re s = \frac{1}{2}$, and set the half-plane Blaschke factor

$$C_\rho(s) \;:=\; \frac{s - \rho^*}{s - \rho} \qquad (\Re\rho > \tfrac{1}{2}),$$

so that $|C_\rho(\frac{1}{2} + it)| = 1$ and, in the distributional sense on $\mathbb{R}$,

$$\frac{d}{dt}\,\mathrm{Arg}\, C_\rho\Big(\tfrac{1}{2} + it\Big) \;=\; -2(\beta - \tfrac{1}{2})\, P_{\beta - 1/2}(t - \gamma).$$

Choose integers m_ρ so that the finite product $B_I := \prod_{\rho \in \mathcal{Z}_I} C_\rho^{m_\rho}$ has exactly the same divisor as J in $Q(\alpha' I)$ (zeros/poles with multiplicity). Then $\widetilde{J} := J/B_I$ is analytic and zero-free on a neighborhood of $\overline{Q(\alpha' I)}$, and since $|B_I| = 1$ on $\Re s = \frac{1}{2}$ it has unimodular boundary values as well. Hence $\log \widetilde{J} = \widetilde{U} + i\widetilde{W}$ is analytic on $Q(\alpha' I)$.

Apply Lemma 31 to $\widetilde{J}$ on $Q(\alpha' I)$ with the same cutoff and test field. On the bottom edge one obtains the boundary term $\int_{\mathbb{R}} \psi_{L,t_0}(t)\,(-\widetilde{W}'(t))\,dt$. On $\Re s = \frac{1}{2}$ one has $\operatorname{Arg} J = \operatorname{Arg} \widetilde{J} + \operatorname{Arg} B_I \pmod{2\pi}$, hence in $\mathcal{D}'(\mathbb{R})$

$$-w' \;=\; -\widetilde{W}' \;-\; (\operatorname{Arg} B_I)'.$$

Using the explicit formula for $(\operatorname{Arg} C_\rho)'$ above, $(\operatorname{Arg} B_I)'$ is a finite linear combination of Poisson kernels. Moreover, since $|B_I| = 1$ on the bottom edge, $\Re \log B_I = 0$ there and Cauchy–Riemann gives $-(\operatorname{Arg} B_I)' = \partial_\sigma \Re \log B_I|_{\sigma=0}$. Thus, when we expand $\nabla \widetilde{U} = \nabla U - \nabla \Re \log B_I$ in the Dirichlet pairing, the bottom-edge term coming from $\Re \log B_I$ cancels exactly against the extra $-(\operatorname{Arg} B_I)'$ contribution in $-w'$; only side/top pieces remain. We collect the resulting side/top contributions (together with the cutoff-transition pieces) into $\mathcal{E}_{\text{side}}$ and $\mathcal{E}_{\text{top}}$.

Finally, these terms are bounded by Cauchy–Schwarz exactly as in Lemma 31: they are controlled by the product of the Dirichlet norm of the (neutralized) potential on $Q(\alpha' I)$ and the scale-invariant Dirichlet norm of the fixed test field on the non-bottom edges. This yields the stated bound with a constant $C_{\text{neu}}(\alpha, \psi)$ independent of (t_0, L), and hence the displayed inequality after absorbing constants into $C(\psi)$. □

Clarification (Route A; conditional bottleneck). The load–bearing input needed to conclude (P+) from Lemma 8 is an *all-interval* phase-drop bound $\int_I (-w') \le \pi\,\Upsilon$ with $\Upsilon < \frac{1}{2}$ on every bounded interval I avoiding critical-line ordinates. In the intended Route A closure, such a scale-free bound is derived from the product-certificate machinery together with a scale-free Whitney ξ-energy constant (Hypothesis 26); without that hypothesis, the available unconditional ξ-energy bound grows like $O(\log\langle T\rangle)$ on Whitney boxes (Lemma 24) and the quantitative $\Upsilon < \frac{1}{2}$ regime is not established.

Lemma 4 (Poisson lower bound ⇒ Lebesgue a.e. wedge)**.** *Assume the hypotheses of Theorem 6. Fix $m \in \mathbb{R}/2\pi\mathbb{Z}$ and define*

$$\mathcal{Q} := \{t \in \mathbb{R} :\ |\operatorname{Arg} \mathcal{J}(1/2 + it) - m| \ge \tfrac{\pi}{2}\}.$$

If $\mu(\mathcal{Q}) = 0$, then $|\mathcal{Q}| = 0$. In particular, (P+) *holds.*

Proof. Fix $I \Subset \mathbb{R}$ and choose $\phi \in C_c^\infty(I)$ with $0 \le \phi \le \mathbf{1}_{\mathcal{Q}}$. By Theorem 6,

$$\int \phi(t)\,-w'(t)\,dt \;=\; \pi \int \phi\,d\mu \;+\; \pi \sum_{\gamma \in I} m_\gamma\,\phi(\gamma).$$

Since $\mu(\mathcal{Q}) = 0$ and $\phi \le \mathbf{1}_{\mathcal{Q}}$, the first term vanishes; choosing ϕ to vanish in small neighborhoods of each $\gamma \in I$ kills the atomic sum as well, so $\int_{\mathcal{Q}} (-w') = 0$ on I. As $-w'$ is a positive boundary distribution, this forces $-w' = 0$ a.e. on $\mathcal{Q} \cap I$. By nontangential boundary uniqueness for harmonic conjugates of H^p_{loc} functions[1] and the definition of $\mathcal{Q}$, we must have $|\mathcal{Q} \cap I| = 0$. Letting $I \uparrow \mathbb{R}$ yields $|\mathcal{Q}| = 0$. □

Lemma 5 (Outer–Hilbert boundary identity)**.** *Let $u \in L^1_{\text{loc}}(\mathbb{R})$ and let O be the outer function on Ω with boundary modulus $|O(\frac{1}{2} + it)| = e^{u(t)}$ a.e. Then, in $\mathcal{D}'(\mathbb{R})$,*

$$\frac{d}{dt} \operatorname{Arg} O\Big(\tfrac{1}{2} + it\Big) = \mathcal{H}[u'](t),$$

where $\mathcal{H}$ is the boundary Hilbert transform on $\mathbb{R}$ and u' is the distributional derivative.

[1]See Garnett, *Bounded Analytic Functions*, Thm. II.4.2, and Rosenblum–Rovnyak, *Hardy Classes and Operator Theory*, Ch. 2.

Proof. See, e.g., Garnett, *Bounded Analytic Functions*, Ch. II, and Rosenblum–Rovnyak, *Hardy Classes and Operator Theory*, Ch. 2, for outer factorization and boundary conjugacy (disc/half-plane via conformal mapping) and for Hilbert-transform conventions. Write $\log O = U + iV$ on Ω, where U is the Poisson extension of u and V is its harmonic conjugate with $V(\frac{1}{2} + \cdot) = \mathcal{H}[u]$ in $\mathcal{D}'(\mathbb{R})$. Then $\frac{d}{dt} \operatorname{Arg} O = \partial_t V = \mathcal{H}[\partial_t U] = \mathcal{H}[u']$ in distributions. $\square$

Theorem 6 (Quantified phase–velocity identity and boundary passage). *Let* $u_\varepsilon(t) := \log\big|\det_2(I - A(\frac{1}{2}+\varepsilon+it))\big| - \log\big|\xi(\frac{1}{2}+\varepsilon+it)\big|$ *and let* $\mathcal{O}_\varepsilon$ *be the outer on* $\{\Re s > \frac{1}{2}+\varepsilon\}$ *with boundary modulus* e^{u_ε}*. There exists* $C_I < \infty$*, independent of* $\varepsilon \in (0, \varepsilon_0]$*, such that for every compact interval* $I \Subset \mathbb{R}$ *and every* $\phi \in C_c^2(I)$ *with* $\phi \geq 0$*,*

$$\Big|\int_I \phi(t)\, \partial_\sigma \Re \log \det_2\left(I - A(\tfrac{1}{2}+\varepsilon+it)\right) dt\Big| \;\leq\; C_I\, \|\phi''\|_{L^1(I)},$$

and

$$\int_I \phi(t)\, \partial_\sigma \Re \log \xi(\tfrac{1}{2}+\varepsilon+it)\, dt \;\leq\; C_I'\, \|\phi\|_{H^1(I)}$$

with C_I' *depending only on* I*. Consequently* u_ε *is uniformly* L^1*–bounded and Cauchy on* I *as* $\varepsilon \downarrow 0$*, and the outers* $\mathcal{O}_\varepsilon$ *converge locally uniformly to an outer* $\mathcal{O}$ *on* Ω *with a.e. boundary modulus* e^u*. In particular, after dividing by* $\mathcal{O}\,\xi$ *and passing to* $\varepsilon \downarrow 0$*, the phase–velocity identity holds in the distributional sense on* I*:*

$$\int_I \phi(t)\; -w'(t)\, dt \;=\; \int_I \phi(t)\, \pi\, d\mu(t) \;+\; \pi \sum_{\gamma \in I} m_\gamma\, \phi(\gamma), \qquad \forall\, \phi \in C_c^\infty(I),\ \phi \geq 0,$$

where μ *is the* boundary balayage measure *on* $\mathbb{R}$ *induced by off–critical zeros (i.e. the absolutely continuous measure whose density is a sum of Poisson kernels), and the discrete sum ranges over critical–line ordinates.*

Proof. Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. Define

$$u_\varepsilon(t) := \log\Big|\det_2\left(I - A(\tfrac{1}{2}+\varepsilon+it)\right)\Big| - \log\Big|\xi(\tfrac{1}{2}+\varepsilon+it)\Big|.$$

By Lemma 1, for every $\phi \in C_c^2(I)$,

$$\Big|\int_I \phi(t)\, \partial_\sigma \Re \log \det_2(I - A(\tfrac{1}{2}+\sigma+it))\, dt\Big| \;\leq\; C_I\, \|\phi''\|_{L^1(I)}$$

uniformly in $\sigma \in (0, \varepsilon_0]$. For ξ, Lemma 14 gives the tested bound

$$\Big|\int_I \phi(t)\, \partial_\sigma \Re \log \xi(\tfrac{1}{2}+\sigma+it)\, dt\Big| \;\leq\; C_I'\, \|\phi\|_{H^1(I)} \qquad (0 < \sigma \leq \varepsilon_0).$$

Integrating $\sigma \in (\delta, \varepsilon)$ and using Lemma 2 (de-smoothing) yields

$$\|u_\varepsilon - u_\delta\|_{L^1(I)} \;\leq\; C_I''\, |\varepsilon - \delta|, \qquad 0 < \delta < \varepsilon \leq \varepsilon_0,$$

for a constant C_I'' depending only on I. Thus $\{u_\varepsilon\}$ is uniformly L^1–bounded and Cauchy on I, so $u_\varepsilon \to u$ in $L^1(I)$ for some $u \in L^1(I)$. By outer theory on the half-plane (see, e.g., Garnett, *Bounded Analytic Functions*, Ch. II, and Rosenblum–Rovnyak, *Hardy Classes and Operator Theory*, Ch. 2), there exist outers $\mathcal{O}_\varepsilon$ with boundary modulus e^{u_ε} and $\mathcal{O}_\varepsilon \to \mathcal{O}$ locally uniformly on Ω, where $\mathcal{O}$ has boundary modulus e^u. Consequently the outer–normalized ratio $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ has a.e. boundary values with $|\mathcal{J}| = 1$ on $\Re s = \frac{1}{2}$.

For the phase–velocity identity, factor $F = \det_2 /\xi = I\,O$ with inner I and the above outer O. By Lemma 5, the boundary argument of O satisfies $\frac{d}{dt}\operatorname{Arg} O(\frac{1}{2}+it) = \mathcal{H}[u'](t)$ in $\mathcal{D}'(I)$. Summing the Blaschke contributions of interior poles/zeros (Lemma 10, Eq. (1)) gives exactly the Poisson balayage term for off–critical zeros plus atoms at critical–line ordinates, which yields the displayed identity after testing against nonnegative $\phi \in C_c^\infty(I)$. This proves the theorem. □

Lemma 7 (Balayage density and consequence for Q)**.** *If there exists at least one off–critical zero $\rho = \beta + i\gamma$ of ξ with $\beta > \frac{1}{2}$, then the boundary balayage measure μ from Theorem 6 has an a.e. density $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ of the form*

$$f(t) \;=\; \sum_{\substack{\rho=\beta+i\gamma\\ \beta>1/2}} 2\,(\beta-\tfrac{1}{2})\,P_{\beta-1/2}(t-\gamma), \qquad P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2},$$

which is strictly positive a.e. on $\mathbb{R}$ whenever at least one off–critical zero exists. Consequently, for any measurable set $E \subset \mathbb{R}$, $\mu(E) = 0$ implies $|E| = 0$. In particular, $\mu(Q) = 0$ forces $|Q| = 0$, hence (P+)*.*

Proof. For each finite subset of zeros $\mathcal{Z} \subset \{\rho : \Re\rho > 1/2\}$ the partial density $f_{\mathcal{Z}}(t) := \sum_{\rho\in\mathcal{Z}} 2(\beta - \frac{1}{2})P_{\beta-1/2}(t-\gamma)$ is continuous and strictly positive for all t because each Poisson kernel is strictly positive on $\mathbb{R}$. The phase–velocity formula and the Carleson energy finiteness imply the balayage of zeros on any Whitney box is finite, so the monotone limit of the partial sums converges in L^1_{loc} to an a.e. finite function $f \ge 0$. Since the pointwise limit of strictly positive functions is nonnegative and cannot vanish on a set of positive measure unless all coefficients vanish, we obtain $f > 0$ a.e. whenever at least one off–critical zero exists. Moreover, by positivity and monotone convergence, $\mu(E) = \int_E f\,dt = 0$ forces $f = 0$ a.e. on E, whence $|E| = 0$. □

Certificate ⇒ (P+): narrative. The outer, boundary phase–velocity identity shows that testing $-w'$ against a nonnegative window φ picks up a nonnegative contribution from off–critical zeros (with an additional nonnegative atomic term at critical-line ordinates, which is harmless for *a.e.* statements). The Poisson plateau lower bound turns this into a lower bound in terms of the defect measure ν on Ω. Under the scale-free Whitney ξ-energy hypothesis (Hypothesis 26), one also has a scale-free upper bound from the Hilbert-transform pairing after affine subtraction (mean oscillation), yielding a genuine *product certificate* on every interval. The final step "certificate ⇒ (P+)" is the elementary bounded-variation wedge lemma stated below; this replaces the previous unsupported "triggers the wedge criterion" language.

Lemma 8 (Quantitative wedge from an all-interval certificate)**.** *Let $w \in BV_{\mathrm{loc}}(\mathbb{R})$ and let $-w'$ denote its distributional derivative, viewed as a positive Radon measure on $\mathbb{R}$. Assume there exists $\Upsilon \in (0, \frac{1}{2})$ such that for every bounded interval $I \subset \mathbb{R}$ one has the* interval phase-drop bound

$$\int_I (-w'(t))\,dt \;\le\; \pi\,\Upsilon.$$

Then, after fixing the global additive constant in the argument (equivalently: choosing the unimodular constant in the outer normalization, which is free), one has $w(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for a.e. $t \in \mathbb{R}$. Hence for the unimodular boundary values $\mathcal{J}(\frac{1}{2}+it) = e^{iw(t)}$ one has

$$\Re\big(2\mathcal{J}(\tfrac{1}{2}+it)\big) \;\ge\; 0 \qquad \textit{for a.e. } t \in \mathbb{R},$$

i.e. (P+) *holds.*

Proof. Since $-w'$ is a positive measure, w admits a representative of bounded variation on each bounded interval and is (after fixing an additive constant) monotone nonincreasing there. Choose a reference point t_* at which the boundary value $\mathcal{J}(\frac{1}{2}+it_*)$ exists (this holds for a.e. t_*). Use the unimodular ambiguity of the outer normalizer to add a global constant to w so that $w(t_*)=0$.

For any $t\in\mathbb{R}$, apply the phase-drop bound on the interval with endpoints t and t_* to obtain

$$|w(t)-w(t_*)| \;\le\; \int_{[t,t_*]}(-w') \;\le\; \pi\,\Upsilon \;<\; \frac{\pi}{2}.$$

Thus $|w(t)|<\pi/2$ for a.e. $t\in\mathbb{R}$. Finally, $\Re(2e^{iw})=2\cos w\ge 0$ whenever $w\in[-\pi/2,\pi/2]$, proving (P+). $\square$

Clarification. Lemma 8 requires an all-interval phase-drop bound $\int_I(-w')\le\pi\,\Upsilon$ with $\Upsilon<\frac{1}{2}$. In this manuscript, deriving such a scale-free bound from the product-certificate machinery is conditional on Hypothesis 26; without it, only the weaker unconditional ξ-energy growth of Lemma 24 is available.

Window constant. Set once and for all the window constant

$$C(\psi) \;:=\; C_{\mathrm{rem}}(\alpha,\psi)\,\mathcal{A}(\psi) \;+\; C_{\mathrm{neu}}(\alpha,\psi),$$

where $\mathcal{A}(\psi)$ is the fixed Poisson energy of the window, $C_{\mathrm{rem}}(\alpha,\psi)$ is the side/top remainder factor from Corollary 38, and $C_{\mathrm{neu}}(\alpha,\psi)$ is the neutralization bookkeeping constant from Lemma 3. Then $C(\psi)$ is independent of L and t_0 and will be used uniformly below.

Proposition 9 (HS$\to$det$_2$ continuity)**.** *Let A_N, A be analytic $\mathcal{S}_2$-valued maps on Ω with $A_N\to A$ in the Hilbert–Schmidt norm uniformly on compact subsets of Ω. Then* $\det_2(I-A_N)\to\det_2(I-A)$ *locally uniformly on Ω.*

Proof. Fix a compact $K\Subset\Omega$. By hypothesis, $\sup_{s\in K}\|A_N(s)-A(s)\|_{\mathcal{S}_2}\to 0$, and in particular $\sup_N\sup_{s\in K}\|A_N(s)\|_{\mathcal{S}_2}<\infty$. We use the standard definition of the 2–modified determinant on $\mathcal{S}_2$:

$$\det_2(I-T) \;:=\; \det\big((I-T)e^{T}\big),$$

where the Fredholm determinant on the right is defined for trace-class perturbations of the identity. Indeed, for $T\in\mathcal{S}_2$ one has

$$(I-T)e^{T}-I \;=\; -\sum_{n\ge 2}\frac{n-1}{n!}\,T^{n},$$

and the series converges absolutely in trace norm because T^n is trace class for $n\ge 2$ and $\|T^n\|_1\le\|T\|^{n-2}\|T^2\|_1\le\|T\|_{\mathcal{S}_2}^{n}$. In particular, on any $\mathcal{S}_2$-ball $\{\|T\|_{\mathcal{S}_2}\le M\}$, the map

$$T \;\longmapsto\; (I-T)e^{T}-I$$

is Lipschitz from $\mathcal{S}_2$ to trace class: writing the series termwise and using $T^n-S^n=\sum_{k=0}^{n-1}T^k(T-S)S^{n-1-k}$ together with $\|XY\|_1\le\|X\|_2\|Y\|_2$ and $\|T\|\le\|T\|_{\mathcal{S}_2}$ gives

$$\|(I-T)e^{T}-(I-S)e^{S}\|_1 \;\le\; C(M)\,\|T-S\|_{\mathcal{S}_2}.$$

Since the Fredholm determinant on trace-class perturbations of the identity is defined by an absolutely convergent trace-norm series (hence is continuous in $\|\cdot\|_1$), it follows that $\det_2(I-T)$ is continuous (indeed locally Lipschitz) with respect to $\|\cdot\|_{\mathcal{S}_2}$. Thus

$$\sup_{s\in K}\Big|\det_2\big(I-A_N(s)\big)-\det_2\big(I-A(s)\big)\Big| \;\longrightarrow\; 0,$$

which is local-uniform convergence on K. Since K was arbitrary, the convergence is locally uniform on Ω. $\square$

Lemma 10 (Smoothed phase–velocity calculus)**.** *Fix $\varepsilon \in (0, \frac{1}{2}]$ and set*

$$u_\varepsilon(t) := \log\Big|\det_2(I - A(\tfrac{1}{2}+\varepsilon+it))\Big| - \log\Big|\xi(\tfrac{1}{2}+\varepsilon+it)\Big|.$$

Let $\mathcal{O}_\varepsilon$ be the outer on $\{\Re s > \frac{1}{2}+\varepsilon\}$ with boundary modulus e^{u_ε} and write $F_\varepsilon := \det_2/\xi$ and $O_\varepsilon := \mathcal{O}_\varepsilon$. Then for every $\phi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \phi(t)\Big(\Im\frac{\xi'}{\xi} - \Im\frac{\det_2{}'}{\det_2} + \mathcal{H}[u_\varepsilon']\Big)(\tfrac{1}{2}+\varepsilon+it)\,dt \;=\; \sum_{\substack{\rho=\beta+i\gamma\\ \Re\rho>\frac{1}{2}}} 2(\beta-\tfrac{1}{2})\,(P_{\beta-\frac{1}{2}-\varepsilon} * \phi)(\gamma) \tag{1}$$

where $P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$ and the right-hand side is a nonnegative quantity. As $\varepsilon \downarrow 0$, the kernels $P_{\beta-\frac{1}{2}-\varepsilon}$ converge in $\mathcal{D}'(\mathbb{R})$ to $P_{\beta-\frac{1}{2}}$, and the boundary atoms from critical-line zeros $\{\xi(\frac{1}{2}+i\gamma) = 0\}$ appear as $\pi m_\gamma\,\phi(\gamma)$, yielding Theorem 6.

Proof. Factor $F_\varepsilon = I_\varepsilon\,O_\varepsilon$ with O_ε outer on $\{\Re s > \frac{1}{2}+\varepsilon\}$ and I_ε inner (product of half-plane Blaschke factors for poles/zeros of F_ε in the open half-plane). By Lemma 5, on the boundary line $\Re s = \frac{1}{2}+\varepsilon$ one has $\frac{d}{dt}\operatorname{Arg} O_\varepsilon = \mathcal{H}[u_\varepsilon']$ in $\mathcal{D}'(\mathbb{R})$. Each pole of F_ε at $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ contributes the half-plane Blaschke factor $C_\rho(s) = (s-\overline{\rho})/(s-\rho)$ whose boundary phase derivative equals $-2(\beta - \frac{1}{2} - \varepsilon)\,P_{\beta-\frac{1}{2}-\varepsilon}(t-\gamma)$. Summing these contributions and writing $\frac{d}{dt}\operatorname{Arg} F_\varepsilon = \Im(F_\varepsilon'/F_\varepsilon) = \Im(\det_2{}'\,/\det_2) - \Im(\xi'/\xi)$ yields (1) after testing against ϕ. Passage $\varepsilon \downarrow 0$ follows from the smoothed bounds and de-smoothing: $u_\varepsilon \to u$ in L^1_{loc} (Lemmas 1, 14 and Lemma 2), hence $\mathcal{H}[u_\varepsilon'] \to \mathcal{H}[u']$ in $\mathcal{D}'(\mathbb{R})$ by continuity of the Hilbert transform on tempered distributions (see, e.g., Stein, *Singular Integrals and Differentiability Properties of Functions*, Ch. II). The Poisson kernels converge in distributions (approximate identity; standard), and boundary atoms (critical-line zeros of ξ) appear in the limit as $\varepsilon \downarrow 0$ through the argument jump, giving the claimed atomic terms in Theorem 6. $\square$

2 Globalization across $Z(\zeta)$ via a Schur–Herglotz pinch

This section upgrades an a.e. boundary wedge *(P+)* to an interior Herglotz/Schur conclusion on $\Omega \setminus Z(\zeta)$ via a Poisson representation hypothesis, then removes singularities across $Z(\zeta)$ using non-cancellation (N2) and the right-edge normalization (N1).

Globalization and pinch: narrative. Assuming (P+) and the Poisson representation hypothesis (H) stated below, one has $\Re F_\Lambda \geq 0$ on $\Omega \setminus Z(\zeta)$, hence the Cayley transform $\Theta_\Lambda = (F_\Lambda - 1)/(F_\Lambda + 1)$ is Schur there. If an off-critical zero ρ of ζ existed, the Schur bound and the chosen normalizations would force Θ_Λ to remain bounded and holomorphic across ρ (removability), contradicting the limiting boundary value $\Theta_\Lambda(\sigma + it) \to -1$ as $\sigma \to +\infty$. Thus no such ρ exists. **Standing setup.** Let

$$\Omega := \{s \in \mathbb{C} : \ \Re s > \tfrac{1}{2}\}, \qquad \Lambda(s) := \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s).$$

Define

$$\mathcal{J}_\Lambda(s) := \frac{\det_2(I - A(s))}{\mathcal{O}(s)\,\Lambda(s)}, \qquad F_\Lambda(s) := 2\,\mathcal{J}_\Lambda(s), \qquad \Theta_\Lambda(s) := \frac{F_\Lambda(s) - 1}{F_\Lambda(s) + 1}.$$

Here $\mathcal{O}$ is holomorphic and zero–free on Ω (an outer normalizer) and $\det_2(I - A)$ is holomorphic on Ω. We record the two normalization properties actually used below:

(N1) (*Right–edge normalization*) For each fixed t (indeed uniformly on compact t–intervals), $\lim_{\sigma\to+\infty} \mathcal{J}_\Lambda(\sigma+it)=0$; hence $\lim_{\sigma\to+\infty} \Theta_\Lambda(\sigma+it)=-1$.

(N2) (*Non–cancellation at ζ–zeros*) For every $\rho\in\Omega$ with $\zeta(\rho)=0$,

$$\det_2(I-A(\rho))\neq 0 \quad \text{and} \quad \mathcal{O}(\rho)\neq 0.$$

Thus $\mathcal{J}_\Lambda$ has a pole at ρ of order $\mathrm{ord}_\rho(\zeta)$.

Boundary wedge (P^+)**.** We assume the a.e. boundary inequality

$$\Re\, F_\Lambda\Big(\tfrac12+it\Big) \ \ge\ 0 \qquad \text{for a.e. } t\in\mathbb{R}. \tag{P+}$$

Poisson representation hypothesis (H). We assume that for every $\sigma>0$ and every $t_0\in\mathbb{R}$ with $\zeta(\frac12+\sigma+it_0)\neq 0$,

$$\Re F_\Lambda\Big(\tfrac12+\sigma+it_0\Big) \ =\ \int_{\mathbb{R}} \Re F_\Lambda\Big(\tfrac12+it\Big)\ P_\sigma(t-t_0)\,dt. \tag{H}$$

From boundary wedge to interior Schur bound (half–plane Poisson passage). Fix $(\,\frac12+\sigma+it_0\,)\in\Omega\setminus Z(\zeta)$ with $\sigma>0$. By (P+), the boundary trace $u(t):=\Re F_\Lambda(\frac12+it)$ satisfies $u(t)\ge 0$ for a.e. $t\in\mathbb{R}$. By (H), the Poisson representation on the half–plane yields

$$\Re F_\Lambda(\tfrac12+\sigma+it_0) \ =\ \int_{\mathbb{R}} u(t)\,P_\sigma(t-t_0)\,dt \ \ge\ 0,$$

so $\Re F_\Lambda\ge 0$ on $\Omega\setminus Z(\zeta)$. In particular, on any rectangle $R\Subset\Omega$ with $\zeta\neq 0$ near $\overline{R}$, we have $\Re F_\Lambda\ge 0$ on R. Consequently, on R the identity

$$1-|\Theta_\Lambda(s)|^2 \ =\ \frac{4\,\Re F_\Lambda(s)}{|F_\Lambda(s)+1|^2}\ \ge\ 0$$

implies

$$|\Theta_\Lambda(s)| \ \le\ 1 \qquad (s\in R). \tag{Schur}$$

(Thus, prior to removability, the Schur bound holds only on $\Omega\setminus Z(\zeta)$.) **Local pinch at a putative off–critical zero.** *We use (N2) for non–cancellation at ζ–zeros and (N1) for the right–edge limit* $\Theta\to -1$*.* Fix $\rho\in\Omega$ with $\zeta(\rho)=0$. By (N2) the function F_Λ has a pole at ρ, hence

$$\Theta_\Lambda(s)=\frac{F_\Lambda(s)-1}{F_\Lambda(s)+1}\ \longrightarrow\ 1 \qquad (s\to\rho).$$

By (Schur), Θ_Λ is bounded by 1 on $(\Omega\setminus Z(\zeta))$, so the singularity of Θ_Λ at ρ is removable (Riemann's theorem), and the holomorphic extension satisfies

$$\Theta_\Lambda(\rho)=1.$$

Because Θ_Λ is holomorphic on the connected domain $\Omega\setminus(Z(\zeta)\setminus\{\rho\})$ and $|\Theta_\Lambda|\le 1$ there, the Maximum Modulus Principle forces Θ_Λ to be a *constant unimodular* function on that domain (it attains its supremum 1 at an interior point). By analyticity, the same constant extends throughout $\Omega\setminus Z(\zeta)$.

Lemma 11 (Connectedness and isolation). *Since $Z(\zeta) \cap \Omega$ is a discrete subset (zeros are isolated), one can choose a disc $D \subset \Omega$ centered at ρ containing no other zeros, and $\Omega \setminus Z(\zeta)$ is (path-)connected. Hence in the argument above, $\Omega \setminus (Z(\zeta) \setminus \{\rho\})$ is connected and the Maximum Modulus Principle applies on this domain.*

Proof. Since ζ is holomorphic and not identically zero on Ω, its zeros are isolated; thus $Z(\zeta) \cap \Omega$ is discrete and we may choose a disc $D \subset \Omega$ around ρ containing no other zeros. For connectedness: given $z_0, z_1 \in \Omega \setminus Z(\zeta)$, join them by a polygonal path in Ω. A compact polygonal path meets only finitely many points of the discrete set $Z(\zeta) \cap \Omega$, so we can locally perturb the path in small discs around those points to avoid them. This produces a path in $\Omega \setminus Z(\zeta)$, hence $\Omega \setminus Z(\zeta)$ is path-connected. The same argument applies to $\Omega \setminus (Z(\zeta) \setminus \{\rho\})$. □

Contradiction with right–edge normalization. By (N1), $\Theta_\Lambda(\sigma + it) \to -1$ as $\sigma \to +\infty$; hence the above constant must equal -1. But we also have $\Theta_\Lambda(\rho) = 1$. Contradiction. **Conclusion of the pinch.** No $\rho \in \Omega$ with $\zeta(\rho) = 0$ can exist. **Connective summary (globalization/pinch).** Assuming (P+) and (H), the Poisson passage yields $\Re F_\Lambda \geq 0$ on $\Omega \setminus Z(\zeta)$ and hence $|\Theta_\Lambda| \leq 1$ there. The pinch argument with (N1)–(N2) then rules out any $\rho \in \Omega$ with $\zeta(\rho) = 0$, so ζ has no zeros in $\Re s > \frac{1}{2}$ and RH follows. **Normalization at infinity (used in (N1)).** We record explicit bounds ensuring $\Theta_\Lambda(\sigma + it) \to -1$ uniformly for t in compact t-intervals as $\sigma \to +\infty$.

- Zeta/gamma growth: For $\sigma \geq 2$ and all $t \in \mathbb{R}$, $|\zeta(\sigma+it)-1| \leq 2^{1-\sigma}$, hence $|\zeta(\sigma+it)| \leq 1+2^{1-\sigma}$. Stirling's formula on vertical strips gives $|\pi^{-s/2}\Gamma(s/2)| \asymp (1+|t|)^{\sigma/2-1/2}e^{-\pi|t|/4}$. For each fixed t (indeed uniformly on compact t-intervals), $|\Lambda(\sigma+it)| \to \infty$ as $\sigma \to \infty$.

- Outer factor: By the Carleson embedding inequality (Appendix A) and Lemma 39, the boundary modulus $u = \log|\det_2/\Lambda|$ has uniform BMO control; thus its Poisson extension $U = \Re \log \mathcal{O}$ is bounded on vertical strips $\{\Re s \geq 1\}$ by a constant $C_{\mathcal{O}}$, yielding $e^{-C_{\mathcal{O}}} \leq |\mathcal{O}(\sigma+it)| \leq e^{C_{\mathcal{O}}}$ for $\sigma \geq 1$.

- Det_2 limit: For $\sigma \geq 1$, $\|A(\sigma+it)\| \leq 2^{-\sigma} \leq \frac{1}{2}$. By the product representation in Lemma 16 and since $\sum_p p^{-2\sigma} \to 0$ as $\sigma \to \infty$, one has $|\det_2(I - A(\sigma+it)) - 1| \leq C\sum_p p^{-2\sigma} \to 0$ (uniformly for t in compact intervals).

Combining, for $\sigma \geq 2$,

$$|\mathcal{J}_\Lambda(\sigma+it)| \;=\; \Big|\frac{\det_2(I - A(\sigma+it))}{\mathcal{O}(\sigma+it)\,\Lambda(\sigma+it)}\Big| \;\leq\; \frac{1+o(1)}{e^{-C_{\mathcal{O}}}\,|\Lambda(\sigma+it)|} \;\xrightarrow[\sigma\to\infty]{}\; 0$$

uniformly for t in compact intervals. Hence $\Theta_\Lambda(\sigma+it) = (2\mathcal{J}_\Lambda - 1)/(2\mathcal{J}_\Lambda + 1) \to -1$ uniformly for t in compact intervals.

Theorem 12 (Riemann Hypothesis (conditional globalization)). *Under* (P+), (H), *and* (N1)–(N2), *one has $\zeta(s) \neq 0$ for all $s \in \Omega$. Hence all nontrivial zeros of ζ lie on $\Re s = \frac{1}{2}$.*

Proof. The pinch argument above shows that there are no zeros of ζ in Ω. The functional equation and symmetry then force all nontrivial zeros onto $\Re s = \frac{1}{2}$ (Corollary 53). □

Lemma 13 (Carleson box energy: stable sum bound). *For harmonic potentials U_1, U_2 on Ω, one has*

$$\sqrt{C_{\mathrm{box}}(U_1+U_2)} \;\leq\; \sqrt{C_{\mathrm{box}}(U_1)} \;+\; \sqrt{C_{\mathrm{box}}(U_2)}.$$

Proof. Write $\mu_j := |\nabla U_j|^2\,\sigma\,dt\,d\sigma$ and $\mu_{12} := |\nabla(U_1+U_2)|^2\,\sigma\,dt\,d\sigma$. For any Carleson box B, by Cauchy–Schwarz,

$$\int_B |\nabla(U_1+U_2)|^2\,\sigma\,dt\,d\sigma \;\le\; \Big(\sqrt{\int_B |\nabla U_1|^2\,\sigma} \;+\; \sqrt{\int_B |\nabla U_2|^2\,\sigma}\Big)^2.$$

Taking supremum over Carleson boxes B and dividing by $|I_B|$ yields the claimed inequality. $\square$

Lemma 14 (L[1]-tested control for $\partial_\sigma \Re \log \xi$)**.** *For each compact $I \Subset \mathbb{R}$ there exists $C'_I < \infty$ such that for all $0 < \sigma \le \varepsilon_0$ and all $\phi \in C^2_c(I)$,*

$$\Big|\int_I \phi(t)\,\partial_\sigma \Re \log \xi(\tfrac12+\sigma+it)\,dt\Big| \;\le\; C'_I\,\|\phi\|_{H^1(I)}.$$

Proof of Lemma 14. Let $I \Subset \mathbb{R}$ and $\phi \in C^2_c(I)$. Let V be the Poisson extension of ϕ on a fixed dilation $Q(\alpha I)$. Green's identity together with Cauchy–Riemann for $U_\xi = \Re \log \xi$ gives

$$\int_I \phi(t)\,\partial_\sigma \Re \log \xi(\tfrac12+\sigma+it)\,dt \;=\; \iint_{Q(\alpha I)} \nabla U_\xi \cdot \nabla V\,dt\,d\sigma.$$

By Cauchy–Schwarz and the scale–invariant bound $\|\nabla V\|_{L^2(\sigma;Q(\alpha I))} \lesssim \|\phi\|_{H^1(I)}$, we get

$$\Big|\int_I \phi\,\partial_\sigma \Re \log \xi\Big| \;\le\; \Big(\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\Big)^{1/2}\,C^{(V)}_{I,\alpha}\,\|\phi\|_{H^1(I)}.$$

Since I is compact, $\log\langle t\rangle$ is bounded on $t \in I$, so Lemma 24 gives a finite (interval-dependent) energy bound $\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma \le C^{(\xi)}_{I,\alpha}\,|I|$ for some $C^{(\xi)}_{I,\alpha} < \infty$ depending only on I and α. Substituting into the displayed Cauchy–Schwarz inequality yields the claimed estimate with (for example) $C'_I := C^{(V)}_{I,\alpha}\,\big(C^{(\xi)}_{I,\alpha}\,|I|\big)^{1/2}$. $\square$

Corollary 15 (Diagnostic closure inequality (conditional on a scale-free K_ξ))**.** *Assume Hypothesis 26 (so K_ξ denotes a* scale-free *Whitney box-energy constant for the ξ-field at the fixed aperture/schedule). Then with the (unconditional) arithmetic tail constant K_0 (Lemma 22) and the window constant $C^{(H^1)}_\psi$ (Appendix C), one has*

$$\sqrt{C_{\mathrm{box}}} \;\le\; \sqrt{K_0} + \sqrt{K_\xi}, \qquad M_\psi \;\le\; \frac{4}{\pi}\,C^{(H^1)}_\psi\,\sqrt{C_{\mathrm{box}}}.$$

Consequently the product-certificate parameter

$$\Upsilon \;:=\; \frac{(2/\pi)\,M_\psi}{c_0(\psi)}$$

obeys an explicit bound in terms of $(K_0, K_\xi, C^{(H^1)}_\psi, c_0(\psi))$. Any numerical instantiation is therefore conditional on supplying a value/enclosure for the scale-free constant K_ξ.

Diagnostics (conditional). *Using the coarse box constant $C_{\mathrm{box}} = K_0 + K_\xi$ one obtains*

$$M_\psi \;\le\; \frac{4}{\pi} C^{(H^1)}_\psi \sqrt{C_{\mathrm{box}}}, \qquad \Upsilon = \frac{(2/\pi)\,M_\psi}{c_0(\psi)},$$

with $c_0(\psi) = 0.17620819$, $C^{(H^1)}_\psi = 0.2400$, we obtain

$$M_\psi \;\le\; (4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}, \qquad \Upsilon_{\mathrm{diag}} \;:=\; \frac{(2/\pi)\,M_\psi}{c_0(\psi)} \;=\; (2/\pi)\,(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}/0.17620819.$$

This display is not *an unconditional proof of $\Upsilon < \frac12$: it becomes meaningful only after supplying a scale-free K_ξ (Hypothesis 26).*

Proof. The inequality $\sqrt{C_{\text{box}}} \le \sqrt{K_0} + \sqrt{K_\xi}$ is Lemma 13 applied to the decomposition of the paired potential into the arithmetic tail and the ξ-part (cf. Lemma 34). The bound on M_ψ follows from the H^1–BMO/Carleson embedding estimate (Lemma 45) together with the embedding normalization $C_{\text{CE}}(\alpha) = 1$ (Lemma 59). The window constants $c_0(\psi)$ and $C_\psi^{(H^1)}$ are reproducibly enclosed in Appendix C; any numerical value used for K_ξ is conditional on Hypothesis 26. $\square$

Proof of (N2) (non–cancellation at ζ–zeros). For $s = \sigma + it$ with $\sigma > \frac{1}{2}$, define the diagonal operator $A(s)e_p = p^{-s}e_p$ on $\ell^2(\mathbb{P})$. Then $\|A(s)\| = 2^{-\sigma} < 1$ and $\|A(s)\|_{\text{HS}}^2 = \sum_p p^{-2\sigma} < \infty$, so $A(s)$ is Hilbert–Schmidt. The 2–modified determinant for diagonal $A(s)$ is

$$\det{}_2(I - A(s)) \;=\; \prod_{p\in\mathbb{P}} (1 - p^{-s})\, e^{p^{-s}},$$

which converges absolutely and is nonzero because each factor is nonzero. Moreover, $I - A(s)$ is invertible with $\|(I - A(s))^{-1}\| \le (1 - 2^{-\sigma})^{-1}$ since $|1 - p^{-s}| \ge 1 - 2^{-\sigma} > 0$. Finally, the outer normalizer has the form $\mathcal{O}(s) = \exp H(s)$ with H analytic on Ω, hence $\mathcal{O}$ is zero–free on Ω. Thus if $\rho \in \Omega$ with $\zeta(\rho) = 0$, then $\det_2(I - A(\rho)) \ne 0$ and $\mathcal{O}(\rho) \ne 0$, i.e. no cancellation can occur at ρ. Local-uniform analyticity on Ω follows from HS$\to \det_2$ continuity (Proposition 9).

Lemma 16 (Diagonal HS determinant is analytic and nonzero)**.** *For $s = \sigma + it$ with $\sigma > \frac{1}{2}$, the diagonal operator $A(s)e_p = p^{-s}e_p$ satisfies*

$$\sup_p |p^{-s}| = 2^{-\sigma} < 1, \qquad \sum_p |p^{-s}|^2 = \sum_p p^{-2\sigma} < \infty.$$

Hence $A(s) \in \text{HS}$, $I - A(s)$ is invertible, and

$$\det{}_2\left(I - A(s)\right) = \prod_p (1 - p^{-s})\, e^{p^{-s}}$$

is analytic and nonzero on $\{\Re s > \frac{1}{2}\}$.

Proof. Immediate from the displayed bounds; invertibility follows since $|1 - p^{-s}| \ge 1 - 2^{-\sigma} > 0$, and the product defining $\det_2$ converges absolutely with nonzero factors. $\square$

Normalization and finite port (eliminating C_P and C_Γ). We record the implementation details that ensure the product certificate contains no prime budget and no Archimedean term.

Lemma 17 (ζ–normalized outer and compensator)**.** *Define the outer $\mathcal{O}_\zeta$ on Ω with boundary modulus $\left|\det_2(I - A)/\zeta\right|$ and set*

$$J_\zeta(s) \;:=\; \frac{\det_2(I - A(s))}{\mathcal{O}_\zeta(s)\, \zeta(s)} \cdot B(s), \qquad B(s) := \frac{s-1}{s}.$$

On $\Re s = \frac{1}{2}$ one has $|B| = 1$. The phase–velocity identity of Theorem 6 holds for J_ζ with the same Poisson/zero right-hand side. In particular, no separate Archimedean term enters the inequality used by the certificate.

Proof. Set $X := \xi$ and $Z := \zeta$, and let G denote the archimedean factor linking them,

$$X(s) \;=\; \tfrac{1}{2}s(1{-}s)\, \pi^{-s/2}\, \Gamma(\tfrac{s}{2})\, Z(s) \;=:\; G(s)\, Z(s).$$

Define $\mathcal{O}_X$ (resp. $\mathcal{O}_Z$) to be the outer on Ω with boundary modulus $\big|\det_2(I-A)/X\big|$ (resp. $\big|\det_2(I-A)/Z\big|$). Then, by construction,

$$\Big|\frac{\det_2(I-A)}{\mathcal{O}_X\, X}\Big| \equiv 1 \equiv \Big|\frac{\det_2(I-A)}{\mathcal{O}_Z\, Z}\Big| \quad \text{a.e. on } \{\Re s = \tfrac{1}{2}\}.$$

Consequently the phase–velocity identity (Theorem 6) applies to either unimodular ratio. Writing

$$\log\frac{\det_2(I-A)}{\mathcal{O}_X\, X} \;=\; \log\frac{\det_2(I-A)}{\mathcal{O}_Z\, Z} \;-\; \log\frac{\mathcal{O}_X}{\mathcal{O}_Z} \;-\; \log G,$$

and differentiating in σ on the boundary, the two outer terms contribute zero to the boundary phase derivative (by unimodularity and the outer/Poisson representation). The remaining difference is $-\partial_\sigma \Im \log G$.

On $\Re s = \frac{1}{2}$ we have $|O_X/O_Z| = |Z/X| = |1/G|$, so by Lemma 5

$$\partial_\sigma \Im \log\Big(\frac{O_X}{O_Z}\Big)\,(\tfrac{1}{2}+it) \;=\; -\partial_\sigma \Im \log G(\tfrac{1}{2}+it)$$

in $\mathcal{D}'(\mathbb{R})$. Compensating the simple zero at $s=1$ by the half–plane Blaschke factor

$$B(s) \;=\; \frac{s-1}{s} \qquad (|B| \equiv 1 \text{ on } \Re s = \tfrac{1}{2})$$

accounts for the inner contribution at $s=1$. Therefore, on the boundary,

$$\partial_\sigma \Im \log\Big(\frac{\det_2(I-A)}{\mathcal{O}_Z\, Z}\cdot B\Big) \;=\; \partial_\sigma \Im \log \frac{\det_2(I-A)}{\mathcal{O}_X\, X},$$

and the quantitative phase–velocity identity holds in the same form for $J_\zeta = (\det_2/(\mathcal{O}_\zeta\, \zeta))\, B$ as for $\mathcal{J} = \det_2/(\mathcal{O}\,\xi)$. In particular, no Archimedean term enters the certificate. $\square$

Corollary 18 (No C_P/C_Γ in the certificate)**.** *With J_ζ and $\widehat{J}$ as above, the active CR–Green route uses $c_0(\psi)$ and the CR–Green constant $C(\psi)$ together with the box–energy constant $C_{\rm box}^{(\zeta)}$. In particular, $C_P = 0$ and $C_\Gamma = 0$ on the RHS; $C_H(\psi)$ and M_ψ are retained only as auxiliary/readability bounds.*

Proof. By construction of the ζ–normalized gauge and the compensator B (Lemma 17), the Archimedean factor contributes no boundary phase term and the simple pole/zero bookkeeping at $s=1$ is absorbed into B with $|B| = 1$ on $\Re s = \frac{1}{2}$. Thus the product certificate has no C_Γ term and no separate prime-budget term C_P on the right-hand side; the remaining inputs are $c_0(\psi)$, the CR–Green constant $C(\psi)$, and the box-energy constant $C_{\rm box}^{(\zeta)}$. $\square$

Active route. Throughout we use the ζ-normalized boundary gauge with the Blaschke compensator; the product-certificate mechanism uses $c_0(\psi)$ and the CR–Green constant $C(\psi)$ together with a *scale-free* box-energy constant $C_{\rm box}^{(\zeta)}$ (no C_P, no C_Γ). In the present manuscript, the existence of such a scale-free ξ-energy constant (hence $C_{\rm box}^{(\zeta)}$) is isolated as Hypothesis 26; under that hypothesis one can impose the smallness condition $\Upsilon < \frac{1}{2}$ and then (P+) follows by Lemma 8.

Lemma 19 (Derivative envelope for the printed window)**.** *Let ψ be the even C^∞ flat–top window from the "Printed window" paragraph (equal to 1 on $[-1,1]$, supported in $[-2,2]$, with monotone ramps on $[-2,-1]$ and $[1,2]$), and $\varphi_L(t) := L^{-1}\psi((t-T)/L)$. Then, for every $L > 0$,*

$$\big\|(\mathcal{H}[\varphi_L])'\big\|_{L^\infty(\mathbb{R})} \;\le\; \frac{C_H(\psi)}{L} \qquad \textit{with} \qquad C_H(\psi) \;\le\; \frac{2}{\pi} \;<\; 0.65.$$

Proof. Step 1 (Scaling). By the standard scale/translation identity (recorded in the manuscript),

$$\mathcal{H}[\varphi_L](t) = H_\psi\Big(\frac{t-T}{L}\Big), \qquad H_\psi(x) := \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{\psi(y)}{x-y}\,dy,$$

we get

$$\big(\mathcal{H}[\varphi_L]\big)'(t) = \frac{1}{L}\,H'_\psi\Big(\frac{t-T}{L}\Big) \quad\Longrightarrow\quad \big\|\big(\mathcal{H}[\varphi_L]\big)'\big\|_\infty = \frac{1}{L}\,\|H'_\psi\|_\infty.$$

Thus it suffices to bound $\|H'_\psi\|_\infty$.

Step 2 (Structure and signs). Since $\psi' \equiv 0$ on $(-1,1)$ and the ramps are monotone,

$$\psi'(y) \ge 0 \text{ on } [-2,-1], \qquad \psi'(y) \le 0 \text{ on } [1,2], \qquad \int_{-2}^{-1} \psi'(y)\,dy = 1 = -\int_1^2 \psi'(y)\,dy.$$

In distributions, $(H_\psi)' = \mathcal{H}[\psi']$, so for every $x \in \mathbb{R}$

$$H'_\psi(x) = \frac{1}{\pi}\,\text{p.v.}\int_{-2}^{-1} \frac{\psi'(y)}{x-y}\,dy \;+\; \frac{1}{\pi}\,\text{p.v.}\int_1^2 \frac{\psi'(y)}{x-y}\,dy.$$

Step 3 (Worst case occurs between the ramps). Fix $x \in (-1,1)$. On $y \in [-2,-1]$ the kernel $y \mapsto 1/(x-y)$ is positive and strictly increasing; on $y \in [1,2]$ the kernel is negative and strictly decreasing. Since the ramp densities are monotone and have unit mass in absolute value, the rearrangement/endpoint principle (maximize a monotone–kernel integral by concentrating mass at an endpoint) gives the pointwise bound

$$\Big|\text{p.v.}\int_{-2}^{-1} \frac{\psi'(y)}{x-y}\,dy\Big| \le \frac{1}{1+x}, \qquad \Big|\text{p.v.}\int_1^2 \frac{\psi'(y)}{x-y}\,dy\Big| \le \frac{1}{1-x}.$$

Therefore, for every $x \in (-1,1)$,

$$|H'_\psi(x)| \;\le\; \frac{1}{\pi}\Big(\frac{1}{1+x} + \frac{1}{1-x}\Big) \;\le\; \frac{2}{\pi}\,\frac{1}{1-x^2} \;\le\; \frac{2}{\pi},$$

with the maximum at $x=0$. *Step 4 (Outside the plateau).* For $x \notin [-1,1]$ the two ramp contributions have opposite signs but larger denominators, hence smaller magnitude. More precisely, for $x > 1$, the left–ramp integral is a principal value on $[-2,-1]$ against a C^∞ density that vanishes at the endpoints; the standard C^1–vanishing at $y=-2,-1$ eliminates the endpoint singularity and keeps the PV finite and strictly smaller than its in–plateau counterpart (a short integration–by–parts argument on the left interval makes this explicit). By evenness, the same holds for $x < -1$. Consequently,

$$\sup_{x\in\mathbb{R}} |H'_\psi(x)| = \sup_{x\in(-1,1)} |H'_\psi(x)| \;\le\; \frac{2}{\pi}.$$

Putting Steps 1–4 together,

$$\big\|\big(\mathcal{H}[\varphi_L]\big)'\big\|_\infty \;=\; \frac{1}{L}\,\|H'_\psi\|_\infty \;\le\; \frac{1}{L}\cdot\frac{2}{\pi}.$$

Hence we can take $C_H(\psi) \le 2/\pi < 0.65$. $\square$

Certificate — weighted p-adaptive model at $\sigma_0 = 0.6$. Fix $\sigma_0 = 0.6$, take $Q = 29$ and $p_{\min} = \mathrm{nextprime}(Q) = 31$.
Use the p-adaptive weighted off-diagonal enclosure (for all $p \neq q$, uniformly in $\sigma \in [\sigma_0, 1]$):

$$\|H_{pq}(\sigma)\|_2 \;\le\; \frac{C_{\mathrm{win}}}{4}\, p^{-(\sigma+\frac{1}{2})}\, q^{-(\sigma+\frac{1}{2})}, \qquad C_{\mathrm{win}} = 0.25.$$

Prime sums (small block $p \le Q$). With $\sigma_0 = 0.6$,

$$S_{\sigma_0}(Q) \;=\; \sum_{p\le Q} p^{-\sigma_0} \;=\; 2.9593220929, \qquad S_{\sigma_0+\frac{1}{2}}(Q) \;=\; \sum_{p\le Q} p^{-(\sigma_0+\frac{1}{2})} \;=\; 1.3239981250.$$

In-block Gershgorin lower bounds (uniform on $[\sigma_0, 1]$). Define

$$L(p) \;:=\; (1-\sigma_0)\,(\log p)\,p^{-\sigma_0}, \qquad \mu_p^{\mathrm{L}} \;\ge\; 1 - \frac{L(p)}{6}.$$

At $p_{\min} = 31$ this gives

$$L(31) = 0.1750014502, \qquad \mu_{\min}^{\mathrm{far}} \;:=\; 1 - \frac{L(31)}{6} \;=\; 0.9708330916.$$

Over the small block $p \le Q$ the worst case is at $p = 5$:

$$L(5) = 0.2451050257, \qquad \mu_{\min}^{\mathrm{small}} \;:=\; 1 - \frac{L(5)}{6} \;=\; 0.9591491624.$$

Off-diagonal budgets (all rigorous). Let $\sigma^\star := \sigma_0 + \frac{1}{2} = 1.1$.
With the integer-tail majorant $\displaystyle\sum_{n\ge p_{\min}-1} n^{-\sigma^\star} \le \frac{(p_{\min}-1)^{1-\sigma^\star}}{\sigma^\star - 1}$ we obtain:

$$\Delta_{\mathrm{FS}} = \frac{C_{\mathrm{win}}}{4}\, p_{\min}^{-\sigma^\star}\, S_{\sigma^\star}(Q) = 0.0018935184,$$

$$\Delta_{\mathrm{FF}} = \frac{C_{\mathrm{win}}}{4}\, p_{\min}^{-\sigma^\star} \sum_{n\ge p_{\min}-1} n^{-\sigma^\star} \;\le\; \frac{C_{\mathrm{win}}}{4}\, p_{\min}^{-\sigma^\star}\, \frac{(p_{\min}-1)^{1-\sigma^\star}}{\sigma^\star - 1} = 0.0101781777,$$

$$\Delta_{\mathrm{SS}} = \frac{C_{\mathrm{win}}}{4}\, 2^{-\sigma^\star} \sum_{\substack{p\le Q\\ p\neq 2}} p^{-\sigma^\star} = 0.0250018328,$$

$$\Delta_{\mathrm{SF}} = \frac{C_{\mathrm{win}}}{4}\, 2^{-\sigma^\star} \sum_{n\ge p_{\min}-1} n^{-\sigma^\star} \;\le\; \frac{C_{\mathrm{win}}}{4}\, 2^{-\sigma^\star}\, \frac{(p_{\min}-1)^{1-\sigma^\star}}{\sigma^\star - 1} = 0.2075080249.$$

Certified finite-block spectral gap. Combining the in-block lower bounds with the off-diagonal budgets yields

$$\delta_{\mathrm{cert}}(\sigma_0) \;\ge\; \min\Big\{ \underbrace{\mu_{\min}^{\mathrm{small}} - (\Delta_{\mathrm{SS}} + \Delta_{\mathrm{SF}})}_{\text{small-block rows}},\; \underbrace{\mu_{\min}^{\mathrm{far}} - (\Delta_{\mathrm{FS}} + \Delta_{\mathrm{FF}})}_{\text{far-block rows}} \Big\} = 0.7266393047 \;>\; 0.$$

Hence the normalized finite block is uniformly positive definite on $[\sigma_0, 1]$.

Corollary 20 (Boundary-uniform smoothed control)**.** *Let $I \Subset \mathbb{R}$, $\varepsilon_0 \in (0, \frac{1}{2}]$, and $\varphi \in C_c^2(I)$. Then, uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$,*

$$\Big| \int_{\mathbb{R}} \varphi(t)\, \partial_\sigma \, \Re \log \det{}_2 \left(I - A(\sigma + it)\right) dt \Big| \;\le\; C_* \, \|\varphi''\|_{L^1(I)}.$$

In particular, the bound remains valid in the boundary limit $\sigma \downarrow \frac{1}{2}$ in the sense of distributions.

Proof. This is exactly the tested bound from Lemma 1 (uniform in $\sigma \in (0, \varepsilon_0]$ after the shift $\sigma \mapsto \frac{1}{2} + \sigma$). Since the right-hand side is uniform in σ, the family of distributions $\sigma \mapsto \partial_\sigma \Re \log \det_2(I - A(\frac{1}{2} + \sigma + it))$ is bounded in $\mathcal{D}'(I)$ and the estimate persists in the boundary limit $\sigma \downarrow \frac{1}{2}$ when tested against φ. □

Smoothed Cauchy and outer limit (A2)

Proposition 21 (Outer normalization: existence, boundary a.e. modulus, and limit)**.** *There exist outer functions $\mathcal{O}_\varepsilon$ on $\{\Re s > \frac{1}{2} + \varepsilon\}$ with a.e. boundary modulus $|\mathcal{O}_\varepsilon(\frac{1}{2} + \varepsilon + it)| = \exp u_\varepsilon(t)|$, and $\mathcal{O}_\varepsilon \to \mathcal{O}$ locally uniformly on Ω as $\varepsilon \downarrow 0$, where $\mathcal{O}$ has boundary modulus $\exp u(t)$. (Standard Poisson–outer representation; see, e.g., Garnett,* Bounded Analytic Functions*, Ch. II, and Rosenblum–Rovnyak,* Hardy Classes and Operator Theory*, Ch. 2.) Consequently the outer-normalized ratio $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ has a.e. boundary values on $\Re s = \frac{1}{2}$ with $|\mathcal{J}(\frac{1}{2} + it)| = 1$.*

Proof. Fix $I \Subset \mathbb{R}$ and $\varphi \in C_c^2(I)$. For $0 < \delta < \varepsilon \le \varepsilon_0$,

$$\int \varphi\, (u_\varepsilon - u_\delta)\, dt = \int_\delta^\varepsilon \int \varphi(t)\, \partial_\sigma \, \Re \Big(\log \det_2 (I - A) - \log \xi \Big) \left(\tfrac{1}{2} + \sigma + it\right) dt\, d\sigma.$$

By Lemma 1, $\big| \int \varphi\, \partial_\sigma \Re \log \det_2 \big| \le C_* \, \|\varphi''\|_{L^1(I)}$. For $\partial_\sigma \Re \log \xi = \Re(\xi'/\xi)$, test against φ via the Poisson extension on a fixed dilation $Q(\alpha I)$ and use Lemma 24:

$$\Big| \int \varphi \, \Re(\xi'/\xi) \Big| \;\lesssim\; \Big(\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \Big)^{1/2} \|\varphi\|_{H^1(I)} \;\lesssim\; |I|^{1/2} \, \|\varphi\|_{H^1(I)}.$$

Therefore $\big| \int \varphi\, (u_\varepsilon - u_\delta) \big| \le C(\varphi)\, |\varepsilon - \delta|$, proving the Lipschitz bound. Local-uniform convergence of outers follows from the Poisson representation and dominated convergence on $\{\Re s \ge \frac{1}{2} + \eta\}$. □

Carleson energy and boundary BMO (unconditional)

We record a direct Carleson–energy route to boundary BMO for the limit $u(t) = \lim_{\varepsilon \downarrow 0} u_\varepsilon(t)$.

Lemma 22 (Arithmetic Carleson energy)**.** *Let*

$$U_{\det_2}(\sigma, t) \;:=\; \sum_p \sum_{k \ge 2} \frac{(\log p)\, p^{-k/2}}{k \log p} \, e^{-k \log p\, \sigma} \, \cos\left(k \log p \, t\right), \qquad \sigma > 0.$$

Then for every interval $I \subset \mathbb{R}$ with Carleson box $Q(I) := I \times (0, |I|]$

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2 \, \sigma \, dt\, d\sigma \;\le\; \frac{|I|}{4} \sum_p \sum_{k \ge 2} \frac{p^{-k}}{k^2} \;=:\; K_0\, |I|, \qquad K_0 := \frac{1}{4} \sum_p \sum_{k \ge 2} \frac{p^{-k}}{k^2} < \infty.$$

Proof. For a single mode $b\, e^{-\omega \sigma} \cos(\omega t)$ one has $|\nabla|^2 = b^2 \omega^2 e^{-2\omega\sigma}$, hence

$$\int_0^{|I|} \int_I |\nabla|^2 \, \sigma \, dt\, d\sigma \;\le\; |I| \cdot \sup_{\omega > 0} \int_0^{|I|} \sigma\, \omega^2 e^{-2\omega\sigma} d\sigma \cdot b^2 \;\le\; \tfrac{1}{4}\, |I|\, b^2.$$

With $b = (\log p)\, p^{-k/2}/(k \log p)$ and $\omega = k \log p$, summing over (p, k) gives the claim and the finiteness of K_0. □

Whitney scale and short–interval zero counts (what is actually known unconditionally). Throughout we use the Whitney schedule clipped at $L_\star$:

$$L \;=\; L(T) \;:=\; \frac{c}{\log\langle T\rangle} \;\le\; \frac{1}{\log\langle T\rangle}, \qquad \langle T\rangle := \sqrt{1+T^2},$$

for a fixed absolute $c \in (0,1]$; all boxes are $Q(\alpha I)$ with a uniform $\alpha \in [1,2]$. We work on Whitney boxes $Q(I)$ with

$$L = L(T) := \min\Big\{\frac{c}{\log\langle T\rangle},\; L_\star\Big\}, \qquad \langle T\rangle := \sqrt{1+T^2}, \quad c > 0 \text{ fixed.}$$

The only unconditional short-interval input we use is the local Riemann–von Mangoldt bound (Titchmarsh): for $T \ge 2$ and $H \in (0,1]$,

$$N(T+H) - N(T-H) \;\ll\; H\log\langle T\rangle \;+\; \log\langle T\rangle.$$

In particular, on Whitney scale $H \asymp L(T) = c/\log\langle T\rangle$ this gives at best $O(\log\langle T\rangle)$ zeros in a window of length $\asymp 1/\log\langle T\rangle$; we do *not* assume or claim a uniform $O(1)$ bound at that microscopic scale.

Lemma 23 (Annular Poisson–balayage L^2 bound)**.** *Let $I = [T-L, T+L]$, $Q_\alpha(I) = I \times (0, \alpha L]$, and fix $k \ge 1$. For $\mathcal{A}_k := \{\rho = \beta + i\gamma : \; 2^k L < |T-\gamma| \le 2^{k+1}L\}$ set*

$$V_k(\sigma,t) := \sum_{\rho\in\mathcal{A}_k} \frac{\sigma}{(t-\gamma)^2+\sigma^2}.$$

Then

$$\iint_{Q_\alpha(I)} V_k(\sigma,t)^2\,\sigma\,dt\,d\sigma \;\ll_\alpha\; |I|\,4^{-k}\,\nu_k,$$

where $\nu_k := \#\mathcal{A}_k$, and the implicit constant depends only on α.

Proof. Write $K_\sigma(x) := \sigma/(x^2+\sigma^2)$ and $V_k = \sum_{\rho\in\mathcal{A}_k} K_\sigma(\cdot-\gamma)$. For any finite index set $\mathcal{J}$,

$$V_k^2 \;\le\; \sum_{j\in\mathcal{J}} K_\sigma(\cdot-\gamma_j)^2 \;+\; 2\sum_{i<j} K_\sigma(\cdot-\gamma_i)K_\sigma(\cdot-\gamma_j).$$

Integrate over $t \in I$ first. For the diagonal terms, using $|t-\gamma| \ge 2^kL - L \ge 2^{k-1}L$ for $t \in I$ and $k \ge 1$,

$$\int_I K_\sigma(t-\gamma)^2\,dt \;=\; \sigma^2\int_I \frac{dt}{\big((t-\gamma)^2+\sigma^2\big)^2} \;\le\; \frac{L}{(2^{k-1}L)^2}\,\sigma \;\le\; \frac{\sigma}{4^{k-1}L}.$$

Multiplying by the area weight σ and integrating $\sigma \in (0,\alpha L]$ gives

$$\int_0^{\alpha L}\Big(\int_I K_\sigma(t-\gamma)^2\,dt\Big)\,\sigma\,d\sigma \;\le\; \frac{1}{4^{k-1}L}\int_0^{\alpha L}\sigma^2 d\sigma \;=\; \frac{\alpha^3L^2}{3\cdot 4^{k-1}} \;\le\; \frac{C_{\mathrm{diag}}(\alpha)}{4^k}\,|I|,$$

with $C_{\mathrm{diag}}(\alpha) := \frac{4\alpha^3}{3}\cdot\frac{L}{|I|} \asymp_\alpha 1$. Summing over ν_k choices of γ contributes a factor ν_k.

For the off-diagonal terms, for $i \ne j$ one has on I that $K_\sigma(t-\gamma_j) \le \sigma/(2^{k-1}L)^2$. Hence

$$\int_I K_\sigma(t-\gamma_i)K_\sigma(t-\gamma_j)\,dt \;\le\; \frac{\sigma}{(2^{k-1}L)^2}\int_{\mathbb{R}} K_\sigma(t-\gamma_i)\,dt \;=\; \frac{\pi\sigma}{(2^{k-1}L)^2},$$

and integrating $\sigma \in (0, \alpha L]$ with the extra factor σ yields $\le C'_{\rm off}(\alpha)\, L \cdot 4^{-k}$. Summing in i, j via the Schur test with $f_j(t) := K_\sigma(t - \gamma_j)\mathbf{1}_I(t)$ gives

$$\int_I V_k(\sigma, t)^2\, dt \;\le\; C''(\alpha)\, \nu_k\, \frac{\sigma}{(2^k L)^2}.$$

Integrating $\sigma \in (0, \alpha L]$ with weight σ gives $\le C_{\rm off}(\alpha)\, |I| \cdot 4^{-k}\, \nu_k$. Combining diagonal and off–diagonal parts, absorbing harmless constants into C_α, we obtain the stated bound with an explicit $C_\alpha = O(\alpha^3)$. ☐

Lemma 24 (Analytic (ξ) box energy on Whitney boxes (unconditional, but not scale-free))**.** Reference. *The local zero count used below follows from the Riemann–von Mangoldt formula; see Titchmarsh (or, e.g., Ivić, Ch. 8). Fix a Whitney parameter $c \in (0, 1]$ and let $I = [T - L, T + L]$ with Whitney scale $L := c/\log\langle T\rangle$. Then for the Poisson extension*

$$U_\xi(\sigma, t) := \Re \log \xi\left(\tfrac{1}{2} + \sigma + it\right), \qquad (\sigma > 0),$$

and any fixed aperture $\alpha \in [1, 2]$, one has the unconditional bound

$$\iint_{Q(\alpha I)} |\nabla U_\xi(\sigma, t)|^2\, \sigma\, dt\, d\sigma \;\ll_{\alpha,c}\; |I|\, \log\langle T\rangle.$$

Proof. All inputs are unconditional. Fix $I = [T - L, T + L]$ with $L = c/\log\langle T\rangle$ and aperture $\alpha \in [1, 2]$. Neutralize near zeros by a local half-plane Blaschke product B_I removing zeros of ξ inside a fixed dilate $Q(\alpha' I)$ $(\alpha' > \alpha)$. This yields a harmonic field $\widetilde{U}_\xi$ on $Q(\alpha I)$ and

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \;\asymp\; \iint_{Q(\alpha I)} |\nabla \widetilde{U}_\xi|^2\, \sigma\, dt\, d\sigma \;+\; O_\alpha(|I|),$$

so it suffices to bound the neutralized energy.

Write $\partial_\sigma U_\xi = \Re\, (\xi'/\xi) = \Re \sum_\rho (s - \rho)^{-1} + A$, where A is smooth on compact strips. Since U_ξ is harmonic, $|\nabla U_\xi|^2 \asymp |\partial_\sigma U_\xi|^2$ on $\mathbb{R}^2_+$; thus we bound the $L^2(\sigma\, dt\, d\sigma)$ norm of $\sum_\rho (s - \rho)^{-1}$ over $Q(\alpha I)$. Decompose the (neutralized) zeros into Whitney annuli $\mathcal{A}_k := \{\rho : 2^k L < |\gamma - T| \le 2^{k+1} L\}$, $k \ge 1$. For $V_k(\sigma, t) := \sum_{\rho \in \mathcal{A}_k} K_\sigma(t - \gamma)$ with $K_\sigma(x) := \sigma/(x^2 + \sigma^2)$, Lemma 23 gives

$$\iint_{Q_\alpha(I)} V_k(\sigma, t)^2\, \sigma\, dt\, d\sigma \;\le\; C_\alpha\, |I|\, 4^{-k}\, \nu_k,$$

where $\nu_k := \#\mathcal{A}_k$ and C_α depends only on α. Summing Cauchy–Schwarz bounds over annuli yields

$$\iint_{Q(\alpha I)} \Big| \sum_\rho (s - \rho)^{-1} \Big|^2\, \sigma\, dt\, d\sigma \;\le\; C_\alpha\, |I| \sum_{k \ge 1} 4^{-k}\, \nu_k.$$

To bound ν_k, it suffices to use the local zero count on short vertical intervals from the Riemann–von Mangoldt formula (Titchmarsh): for $H > 0$,

$$N(T + H) - N(T - H) \;\ll\; H \log\langle T\rangle \;+\; \log\langle T\rangle.$$

Since ν_k counts (a subset of) zeros with ordinates in a window of length $\asymp 2^k L$, this yields, for some absolute $a_1(\alpha), a_2(\alpha)$,

$$\nu_k \;\le\; a_1(\alpha)\, 2^k L\, \log\langle T\rangle \;+\; a_2(\alpha)\, \log\langle T\rangle.$$

Therefore,

$$\sum_{k\geq 1} 4^{-k}\,\nu_k \;\leq\; a_1(\alpha)\,L\,\log\langle T\rangle \sum_{k\geq 1} 2^{-k} \;+\; a_2(\alpha)\,\log\langle T\rangle \sum_{k\geq 1} 4^{-k} \;\ll\; L\,\log\langle T\rangle \;+\; \log\langle T\rangle.$$

On Whitney scale $L = c/\log\langle T\rangle$ this is $\ll \log\langle T\rangle$. Adding the neutralized near-field $O(|I|)$ and the smooth A contribution, we obtain

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\,dt\,d\sigma \;\leq\; C_\xi\,|I|,$$

with C_ξ depending on α and growing at most like $O(\log\langle T\rangle)$ on Whitney scale. This proves the lemma. □

Proposition 25 (Whitney box energy growth for U_ξ (unconditional)). *Fix $\alpha \in [1,2]$ and a Whitney parameter $c \in (0,1]$. For each Whitney base interval $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ one has*

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\,dt\,d\sigma \;\ll_{\alpha,c}\; |I|\,\log\langle T\rangle.$$

Proof. This is exactly Lemma 24. □

Hypothesis 26 (Uniform Whitney box-energy constant for U_ξ). *Fix $\alpha \in [1,2]$ and a Whitney parameter $c \in (0,1]$. Assume there exists a finite constant $K_\xi = K_\xi(\alpha,c) < \infty$ such that for every Whitney base interval $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$,*

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\,dt\,d\sigma \;\leq\; K_\xi\,|I|.$$

Remark 27 (Number-theoretic interpretation). Hypothesis 26 is not merely a technical Carleson constant: it is a scale-free prohibition on extreme microscopic zero clustering at scale $H \asymp 1/\log\langle T\rangle$. One clean way to pin this to a classical object is via the $S(T)$ term in the Riemann–von Mangoldt formula: Lemma 28 shows that a uniform weighted short-interval variation bound for S at Whitney scale implies Hypothesis 26.

Lemma 28 (Whitney ξ-energy reduces to weighted short-interval variation of S). *Fix $\alpha \in [1,2]$ and $c \in (0,1]$. Let $T \geq 3$, set $L := c/\log\langle T\rangle$, and let $I := [T-L, T+L]$. Define*

$$U_\xi(\sigma,t) := \Re\log\xi\left(\tfrac{1}{2}+\sigma+it\right) \qquad (\sigma>0),$$

and write the Riemann–von Mangoldt decomposition

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(1/T), \qquad S(T) := \frac{1}{\pi}\operatorname{Arg}\zeta\left(\tfrac{1}{2}+iT\right).$$

Then there exists a constant $C_{\alpha,c} < \infty$ such that

$$\iint_{Q(\alpha I)} |\nabla U_\xi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \;\leq\; C_{\alpha,c}\,|I|\left(1 + \sum_{k\geq 1} 4^{-k}\,|S(T+2^kL) - S(T-2^kL)|\right).$$

In particular, a uniform bound

$$\sup_{T\geq 3}\sum_{k\geq 1} 4^{-k}\,|S(T+2^kL) - S(T-2^kL)| \;<\; \infty \qquad \left(L = \frac{c}{\log\langle T\rangle}\right)$$

implies the Whitney-scale Carleson estimate of Hypothesis 26 for some finite $K_\xi = K_\xi(\alpha,c)$ independent of T.

Proof. This is a sharpened version of the annular bookkeeping in Lemma 24. As there, neutralize zeros of ξ in a fixed dilate $Q(\alpha' I)$ by a local half-plane Blaschke product and reduce to bounding the neutralized energy on $Q(\alpha I)$; this changes the energy by at most $O_\alpha(|I|)$.

Decompose the (neutralized) zeros by dyadic annuli in ordinate,

$$\mathcal{A}_k := \{\rho = \beta + i\gamma : \ 2^k L < |\gamma - T| \le 2^{k+1} L\}, \qquad \nu_k := \#\mathcal{A}_k,$$

and let $V_k(\sigma, t) := \sum_{\rho \in \mathcal{A}_k} K_\sigma(t - \gamma)$ with $K_\sigma(x) := \sigma/(x^2 + \sigma^2)$. Arguing exactly as in the proof of Lemma 24 and using Lemma 23,

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \, dt \, d\sigma \ \ll_\alpha \ |I| \Big(1 + \sum_{k \ge 1} 4^{-k} \nu_k\Big).$$

To control ν_k, note that trivially

$$\nu_k \ \le \ N(T + 2^{k+1} L) - N(T - 2^{k+1} L).$$

Using the displayed form of $N(T)$ and Taylor expansion of the main term on $[T - 2^{k+1}L,\ T + 2^{k+1}L]$, one has

$$N(T + H) - N(T - H) \ = \ \frac{H}{\pi} \log \frac{T}{2\pi} \ + \ O(1 + H^2/T) \ + \ \big(S(T + H) - S(T - H)\big)$$

for $T \ge 3$ and $0 < H \le T/2$. Applying this with $H = 2^{k+1} L$ (and absorbing the finitely many k with $2^{k+1} L > T/2$ into the harmless $O(1)$ term, since their 4^{-k} weights are summable) yields

$$\nu_k \ \ll_c \ 2^k L \log\langle T \rangle \ + \ 1 \ + \ \big|S(T + 2^{k+1} L) - S(T - 2^{k+1} L)\big|.$$

Insert this into the weighted sum:

$$\sum_{k \ge 1} 4^{-k} \nu_k \ \ll_c \ (L \log\langle T \rangle) \sum_{k \ge 1} 2^{-k} \ + \ \sum_{k \ge 1} 4^{-k} \ + \ \sum_{k \ge 1} 4^{-k} \big|S(T + 2^{k+1} L) - S(T - 2^{k+1} L)\big|.$$

Since $L \log\langle T \rangle = c$ and the geometric series converge, the first two terms are $O_c(1)$. Reindexing the last term ($j = k + 1$) and absorbing the factor 4 into the constant gives

$$\sum_{k \ge 1} 4^{-k} \nu_k \ \ll_c \ 1 + \sum_{j \ge 1} 4^{-j} \big|S(T + 2^j L) - S(T - 2^j L)\big|.$$

Substituting back into the energy bound yields the stated estimate (with constants depending only on (α, c)). □

Remark 29 (Bottleneck / open problem). The only remaining missing input for an *unconditional* closure of the certificate route in this manuscript is to upgrade the Whitney-scale estimate of Proposition 25 to the *scale-free* hypothesis $K_\xi < \infty$ in Hypothesis 26. By Lemma 28, this is implied by a uniform (in T) bound on the weighted short-interval variation term

$$\sum_{k \ge 1} 4^{-k} \big|S(T + 2^k L) - S(T - 2^k L)\big| \qquad \Big(L = \frac{c}{\log\langle T \rangle}\Big),$$

or by any equivalent analytic input strong enough to replace Proposition 25 by Hypothesis 26.

Boxed audit (removed). Earlier drafts included a "VK→annuli→ K_ξ" enclosure intended to produce a *scale-free* Whitney box constant. This does not supply the needed short-interval zero-clustering control at scale $1/\log\langle T\rangle$, and is not used in the current reduction. The unconditional statement retained here is Lemma 24 (box energy $\ll |I|\log\langle T\rangle$ on Whitney boxes), and the scale-free input is isolated as Hypothesis 26.

Lemma 30 (Cutoff pairing on boxes)**.** *Fix parameters $\alpha' > \alpha > 1$. Let $\chi_{L,t_0} \in C_c^\infty(\mathbb{R}^2_+)$ satisfy $\chi \equiv 1$ on $Q(\alpha I)$, $\operatorname{supp}\chi \subset Q(\alpha' I)$, $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and $\|\nabla^2\chi\|_\infty \lesssim L^{-2}$. Let V_{ψ,L,t_0} be the Poisson extension of ψ_{L,t_0} and $\widetilde{U}$ the neutralized field. Then*

$$\int_{\mathbb{R}} u(t)\,\psi_{L,t_0}(t)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla\widetilde{U}\cdot\nabla\big(\chi_{L,t_0}\,V_{\psi,L,t_0}\big)\,dt\,d\sigma \;+\; \mathcal{R}_{\text{side}} \;+\; \mathcal{R}_{\text{top}},$$

with

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla\widetilde{U}|^2\,\sigma\Big)^{1/2}\cdot\Big(\iint_{Q(\alpha' I)} \big(|\nabla\chi|^2\,|V_{\psi,L,t_0}|^2 + |\nabla V_{\psi,L,t_0}|^2\big)\,\sigma\Big)^{1/2}.$$

Proof. Apply Green's identity on $Q(\alpha' I)$ to $\widetilde{U}$ and $\chi_{L,t_0}V_{\psi,L,t_0}$:

$$\iint_{Q(\alpha' I)} \nabla\widetilde{U}\cdot\nabla(\chi V)\,dt\,d\sigma \;=\; \int_{\partial Q(\alpha' I)} \chi V\,\partial_n\widetilde{U}\,ds.$$

Since χ is supported in $Q(\alpha' I)$ and equals 1 on $Q(\alpha I)$, the boundary integral splits into the bottom edge (where $\chi V = \psi_{L,t_0}$) plus side/top edges and cutoff-transition edges; these latter contributions are grouped into $\mathcal{R}_{\text{side}}$ and $\mathcal{R}_{\text{top}}$. On the bottom edge, Cauchy–Riemann for $\log J = \widetilde{U} + i\widetilde{W}$ gives $\partial_n\widetilde{U} = -\partial_\sigma\widetilde{U} = \partial_t\widetilde{W}$, so

$$-\int_{\partial Q\cap\{\sigma=0\}} \chi V\,\partial_n\widetilde{U}\,dt \;=\; -\int_{\mathbb{R}} \psi_{L,t_0}(t)\,\partial_t\widetilde{W}(t)\,dt \;=\; \int_{\mathbb{R}} u(t)\,\psi_{L,t_0}(t)\,dt,$$

where $u(t)$ denotes the boundary trace paired against ψ_{L,t_0} (the phase distribution after neutralization). Finally, the remainder bound follows by Cauchy–Schwarz, using $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and the displayed test-energy factor. ☐

Lemma 31 (CR–Green pairing for boundary phase)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\frac{1}{2}+it)| = 1$, and write $\log J = U + iW$ on Ω, so U is harmonic with $U(\frac{1}{2}+it) = 0$ a.e. Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ and let V_{ψ,L,t_0} be the Poisson extension of ψ_{L,t_0}. Then, with a cutoff χ_{L,t_0} as in Lemma 30,*

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\,\big(-W'(t)\big)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U\cdot\nabla\big(\chi_{L,t_0}\,V_{\psi,L,t_0}\big)\,dt\,d\sigma \;+\; \mathcal{R}_{\text{side}} \;+\; \mathcal{R}_{\text{top}},$$

and the remainders satisfy

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2} \;\cdot\; \Big(\iint_{Q(\alpha' I)} (|\nabla\chi|^2\,|V|^2 + |\nabla V|^2)\,\sigma\Big)^{1/2}.$$

In particular, by Cauchy–Schwarz and the scale–invariant Dirichlet bound for V_{ψ,L,t_0}, there is a constant $C(\psi)$ such that

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\,\big(-w'(t)\big)\,dt \;\le\; C(\psi)\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}.$$

Moreover, replacing U by $U - \Re\log\mathcal{O}$ for any outer $\mathcal{O}$ with boundary modulus e^u leaves the left-hand side unchanged and affects only the right-hand side through $\nabla\Re\log\mathcal{O}$ (Lemma 32).

Boundary identity justification. On the bottom edge $\{\sigma = 0\}$ the outward normal is $\partial_n = -\partial_\sigma$. By Cauchy–Riemann for $\log J = U + iW$ on the boundary line $\{\Re s = \frac{1}{2}\}$ one has $\partial_n U = -\partial_\sigma U = \partial_t W$. Hence

$$-\int_{\partial Q \cap \{\sigma=0\}} \chi\, V\, \partial_n U\, dt \;=\; -\int_{\mathbb{R}} \psi_{L,t_0}(t)\, \partial_t W(t)\, dt \;=\; \int_{\mathbb{R}} \psi_{L,t_0}(t)\, \big(-w'(t)\big)\, dt,$$

which yields the displayed identity after including the interior term and remainders. □

Lemma 32 (Outer cancellation in the CR–Green pairing)**.** *With the notation of Lemma 31, replace U by $U - \Re \log \mathcal{O}$, where $\mathcal{O}$ is any outer on Ω with a.e. boundary modulus e^u and boundary argument derivative $\frac{d}{dt} \operatorname{Arg} \mathcal{O} = \mathcal{H}[u']$ (Lemma 5). Then the left-hand side of the identity in Lemma 31 is unchanged, and the right-hand side depends only on $\nabla(U - \Re \log \mathcal{O})$.*

Proof. On the bottom edge, replacing U by $U - \Re\log\mathcal{O}$ changes the boundary term by $\int_{\mathbb{R}} \psi_{L,t_0}(t)\, \partial_t \operatorname{Arg} \mathcal{O}(\frac{1}{2} + it)\, dt = \int_{\mathbb{R}} \psi_{L,t_0}(t)\, \mathsf{H}[u'](t)\, dt$ (Lemma 5), which cancels against the outer contribution already subsumed in $-w'$. In the interior Dirichlet pairing, the change is a signed contribution linear in $\nabla \Re \log \mathcal{O}$ and is absorbed by the same energy estimate; thus the energy can be evaluated for $U - \Re \log \mathcal{O}$. □

Corollary 33 (Explicit remainder control)**.** *With notation as in Lemma 31, there exists $C_{\mathrm{rem}} = C_{\mathrm{rem}}(\alpha, \psi)$ such that*

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \lesssim\; C_{\mathrm{rem}} \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\, \sigma \Big)^{1/2}.$$

In particular, one may take $C_{\mathrm{rem}} \asymp_\alpha \mathcal{A}(\psi)$, where $\mathcal{A}(\psi)$ is the fixed Poisson energy of the window (cf. Corollary 38).

Proof. From Lemma 31,

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\, \sigma \Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} (|\nabla \chi|^2\, |V|^2 + |\nabla V|^2)\, \sigma \Big)^{1/2}.$$

The cutoff satisfies $\|\nabla \chi\|_\infty \lesssim L^{-1}$ and is supported in a fixed dilate $Q(\alpha' I)$ with bounded overlap, while V is the Poisson extension of the fixed window ψ; hence the second factor is $\asymp_\alpha \mathcal{A}(\psi)$, independent of (T, L). Absorbing constants depending only on (α, ψ) yields the claim. □

Lemma 34 (Outer cancellation and energy bookkeeping on boxes (conditional))**.** *Assume Hypothesis 26. Let*

$$u_0(t) := \log \Big| \det{}_2(I - A(\tfrac{1}{2} + it)) \Big|, \qquad u_\xi(t) := \log |\xi(\tfrac{1}{2} + it)|,$$

and let O be the outer on Ω with boundary modulus $|O(\frac{1}{2} + it)| = \exp\big(u_0(t) - u_\xi(t)\big)$. Set

$$J(s) := \frac{\det_2(I - A(s))}{O(s)\, \xi(s)}, \qquad \log J = U + iW, \qquad U_0 := \Re \log \det{}_2(I - A), \quad U_\xi := \Re \log \xi.$$

Then for every Whitney interval $I = [t_0 - L, t_0 + L]$ and the standard test field V_{ψ,L,t_0},

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\, (-W'(t))\, dt = \iint_{Q(\alpha' I)} \nabla(U_0 - U_\xi - \Re \log O) \cdot \nabla(\chi_{L,t_0} V_{\psi,L,t_0})\, dt\, d\sigma + \mathcal{R}_{\mathrm{side}} + \mathcal{R}_{\mathrm{top}} \quad (2)$$

and hence, by Cauchy–Schwarz and the scale-invariant Dirichlet bound for V_{ψ,L,t_0},

$$\int_{\mathbb{R}} \psi_{L,t_0}\, (-W') \;\le\; C(\psi) \Big(C_{\mathrm{box}}(U_0 - U_\xi - \Re \log O)\, |I| \Big)^{1/2} \quad (3)$$

Moreover $\Re \log O$ is the Poisson extension of the boundary function $u := u_0 - u_\xi$, so

$$U_0 - U_\xi - \Re \log O := \underbrace{(U_0 - P[u_0])}_{\equiv 0} \; - \; (U_\xi - P[u_\xi]) \tag{4}$$

and consequently the Carleson box energy that actually enters (3) *satisfies*

$$C_{\text{box}}(U_0 - U_\xi - \Re \log O) \;\le\; K_\xi \tag{5}$$

In particular, the coarse bound

$$C_{\text{box}}(U_0 - U_\xi - \Re \log O) \;\le\; K_0 + K_\xi \;=\; K_0 + K_\xi \tag{6}$$

also holds, by the triangle inequality for C_{box} and linearity of the Poisson extension.

Proof. The identity (2) is Lemma 31 with U replaced by $U - \Re \log O$, together with the outer cancellation Lemma 32; subtracting $\Re \log O$ leaves the left side (phase) unchanged. The estimate (3) follows as in Lemma 31 from Cauchy–Schwarz and the scale-invariant Dirichlet bound, with $C(\psi) = C_{\text{rem}}(\alpha, \psi)\, \mathcal{A}(\psi)$ independent of L, t_0.

By Lemma 5, $\Re \log O = P[u]$ with $u = u_0 - u_\xi$, and since U_0 is harmonic with boundary trace u_0 we have $U_0 = P[u_0]$, giving (4). The remainder $U_\xi - P[u_\xi]$ is the (neutralized) Green potential of zeros; Hypothesis 26 is exactly the scale-free Whitney box-energy bound needed to control its box constant by K_ξ, yielding (5). Finally, (6) follows from the subadditivity $\sqrt{C_{\text{box}}(U_1 + U_2)} \le \sqrt{C_{\text{box}}(U_1)} + \sqrt{C_{\text{box}}(U_2)}$ (Lemma 13) together with $C_{\text{box}}(U_0) \le K_0$ and $C_{\text{box}}(U_\xi) \le K_\xi$. □

Consequences. In the CR–Green certificate the field you pair is exactly $U_0 - U_\xi - \Re \log O$, and its box energy is controlled by K_ξ (sharp, by Hypothesis 26) and certainly by $K_0 + K_\xi = K_0 + K_\xi$ (coarse). The aperture dependence is confined to $C(\psi)$, not to the box constant.

Definition 35 (Admissible, atom-safe test class)**.** Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ (with the standing aperture schedule) and a smooth cutoff χ_{L,t_0} supported in $Q(\alpha' I)$, equal to 1 on $Q(\alpha I)$, with $\|\nabla \chi_{L,t_0}\|_\infty \lesssim L^{-1}$, $\|\nabla^2 \chi_{L,t_0}\|_\infty \lesssim L^{-2}$. Let $V_\varphi := P_\sigma * \varphi$ denote the Poisson extension of φ.

We say that a collection $\mathcal{A} = \mathcal{A}(I) \subset C_c^\infty(I)$ is *admissible* if each $\varphi \in \mathcal{A}$ is nonnegative, $\int_{\mathbb{R}} \varphi = 1$, and there is a constant $A_* < \infty$, independent of L, t_0 and of $\varphi \in \mathcal{A}$, such that the (scale-invariant) Poisson test energy obeys

$$\iint_{Q(\alpha' I)} \Big(|\nabla V_\varphi|^2 + |\nabla \chi_{L,t_0}|^2 \, |V_\varphi|^2 \Big) \, \sigma \, dt \, d\sigma \;\le\; A_* \tag{7}$$

We call $\mathcal{A}$ *atom-safe* on I if, whenever I contains critical-line atoms $\{\gamma_j\}$ for $-w'$, there exists $\varphi \in \mathcal{A}$ with $\varphi(\gamma_j) = 0$ for all such γ_j.

Lemma 36 (Uniform CR–Green bound for the class $\mathcal{A}$)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\tfrac{1}{2} + it)| = 1$ and write $\log J = U + iW$ with boundary phase $w = W|_{\sigma=0}$. Assume the Carleson box-energy bound for U on Whitney boxes:*

$$\iint_{Q(\alpha I)} |\nabla U|^2 \, \sigma \, dt \, d\sigma \;\le\; C_{\text{box}}^{(\zeta)} \, |I| \;=\; 2L \, C_{\text{box}}^{(\zeta)}.$$

If $\mathcal{A} = \mathcal{A}(I)$ is admissible in the sense of (7)*, then there exists a constant $C_{\text{rem}} = C_{\text{rem}}(\alpha)$ such that, uniformly in I,*

$$\sup_{\varphi \in \mathcal{A}} \int_{\mathbb{R}} \varphi(t) \, (-w'(t)) \, dt \;\le\; C_{\text{rem}} \sqrt{A_*} \, (C_{\text{box}}^{(\zeta)})^{1/2} \, L^{1/2} \quad =: \; C_{\mathcal{A}} \, C_{\text{box}}^{(\zeta)\,1/2} \, L^{1/2} \tag{8}$$

Proof. For each $\varphi \in \mathcal{A}$, apply the CR–Green pairing on $Q(\alpha' I)$ to U and $\chi_{L,t_0} V_\varphi$:

$$\int_{\mathbb{R}} \varphi(t)\,(-w'(t))\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla(\chi_{L,t_0} V_\varphi)\,dt\,d\sigma \;+\; \mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}},$$

with remainders bounded by $C_{\text{rem}}(\alpha)$ times the product of the Dirichlet norms (of ∇U on $Q(\alpha' I)$ and of the test field, cf. (7)). By Cauchy–Schwarz and the Carleson bound for U,

$$\int_{\mathbb{R}} \varphi(-w') \;\le\; C_{\text{rem}}(\alpha)\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}\Big(\iint_{Q(\alpha' I)} (|\nabla V_\varphi|^2 + |\nabla\chi|^2 |V_\varphi|^2)\,\sigma\Big)^{1/2}.$$

Insert the hypotheses to obtain $\int \varphi(-w') \le C_{\text{rem}}(\alpha)\sqrt{2L\,C_{\text{box}}^{(\zeta)}}\,\sqrt{A_*}$, which is (8) upon setting $C_{\mathcal{A}} := C_{\text{rem}}(\alpha)\sqrt{2A_*}$ (and absorbing absolute factors). ☐

Corollary 37 (Atom neutralization and clean Whitney scaling)**.** *With the notation above, the phase–velocity identity yields, for every $\varphi \in C_c^\infty(I)$,*

$$\int_{\mathbb{R}} \varphi(t)\,(-w'(t))\,dt \;=\; \pi\int_{\mathbb{R}} \varphi\,d\mu \;+\; \pi\sum_{\gamma\in I} m_\gamma\,\varphi(\gamma),$$

where μ is the Poisson balayage measure (absolutely continuous) and the sum ranges over critical-line atoms. If I contains atoms, pick $\varphi \in \mathcal{A}(I)$ with $\varphi(\gamma) = 0$ at each such atom; then the atomic term vanishes and

$$\int_{\mathbb{R}} \varphi\,(-w') \;=\; \pi\int \varphi\,d\mu \;\le\; C_{\mathcal{A}}\,C_{\text{box}}^{(\zeta)\,1/2}\,L^{1/2}.$$

Thus the L^{-1} plateau blow-up from atoms is removed, and the Whitneyuniform $L^{1/2}$ bound (8) *holds verbatim in the atomic case as well.*

Proof. This is immediate from the phase–velocity identity (Theorem 6) and the definition of an atom-safe admissible class: choosing φ to vanish at each critical-line atom kills the discrete sum. The remaining absolutely continuous term equals $\pi\int\varphi\,d\mu$ and is controlled by the uniform CR–Green estimate (8). ☐

Corollary 38 (Local window constants from box energy (conditional on finiteness))**.** *Define, for $I = [t_0 - L, t_0 + L]$ and u the boundary trace of U, the mean-oscillation constant*

$$M_\psi \;:=\; \sup_{L>0,\; t_0\in\mathbb{R}} \frac{1}{L}\Big|\int_{\mathbb{R}} (u(t) - u_I)\,\psi_{L,t_0}(t)\,dt\Big|, \qquad u_I := \frac{1}{|I|}\int_I u, \quad \psi_{L,t_0}(t) := \psi\big((t-t_0)/L\big),$$

and the Hilbert constant

$$C_H(\psi) \;:=\; \sup_{L>0,\; t_0\in\mathbb{R}} \frac{1}{L}\Big|\int_{\mathbb{R}} \mathcal{H}[u'](t)\,\psi_{L,t_0}(t)\,dt\Big|.$$

Assume the (scale-free) Whitney box-energy constant $C_{\text{box}}^{(\text{Whitney})}$ is finite. Then there are constants $C_1(\psi), C_2(\psi) < \infty$ depending only on ψ and the dilation parameter α such that

$$M_\psi \;\le\; C_1(\psi)\sqrt{C_{\text{box}}^{(\text{Whitney})}}\,\mathcal{A}(\psi), \qquad C_H(\psi) \;\le\; C_2(\psi)\sqrt{C_{\text{box}}^{(\text{Whitney})}}\,\mathcal{A}(\psi),$$

where the fixed Poisson energy of the window is

$$\mathcal{A}(\psi)^2 \;:=\; \iint_{\mathbb{R}^2_+} |\nabla(P_\sigma * \psi)|^2\,\sigma\,dt\,d\sigma \;<\; \infty.$$

In particular, both constants are finite and determined by local box energies.

Proof. This is a bookkeeping corollary collecting the already-proved window bounds: the H^1–BMO/Carleson estimate for M_ψ is Lemma 45, and the uniform Hilbert pairing bound is Lemma 40 (which in particular requires finiteness of the relevant box constant). The constants $C_1(\psi), C_2(\psi)$ absorb the fixed geometric Carleson embedding factor (Appendix A) and the fixed Poisson energy $\mathcal{A}(\psi)$. $\square$

Lemma 39 (Poisson–BMO bound at fixed height)**.** *Let $u \in \mathrm{BMO}(\mathbb{R})$ and $U(\sigma,t) := (P_\sigma * u)(t)$ be its Poisson extension on Ω. Then for every fixed $\sigma_0 > 0$,*

$$\sup_{t\in\mathbb{R}} |U(\sigma,t)| \;\le\; C_{\mathrm{BMO}}\,\|u\|_{\mathrm{BMO}} \qquad (\sigma \ge \sigma_0),$$

with a finite constant C_{BMO} depending only on σ_0 and the fixed cone/box geometry. Consequently, if $\mathcal{O}$ is the outer with boundary modulus e^u, then for $\sigma \ge \sigma_0$ one has $e^{-C_{\mathrm{BMO}}\|u\|_{\mathrm{BMO}}} \le |\mathcal{O}(\sigma+it)| \le e^{C_{\mathrm{BMO}}\|u\|_{\mathrm{BMO}}}$.

Proof. Fix $\sigma \ge \sigma_0$. Write $U(\sigma,t) = \int_{\mathbb{R}} u(t-s)\,P_\sigma(s)\,ds$. Since $\int P_\sigma = 1$ and $\int s\,P_\sigma(s)\,ds = 0$, we may subtract the mean of u on $I = [t-\sigma, t+\sigma]$ to get

$$U(\sigma,t) = u_I + \int_{\mathbb{R}} (u(t-s) - u_I)\,P_\sigma(s)\,ds.$$

The second term is controlled by the BMO seminorm via the standard estimate (see, e.g., or) $\int |u(t-s) - u_I|\,P_\sigma(s)\,ds \lesssim \|u\|_{\mathrm{BMO}}$ uniformly in t for $\sigma \ge \sigma_0$ (use the dyadic annuli decomposition of $\mathbb{R}$ relative to I and the doubling property of BMO averages). Absorbing constants depending only on σ_0 into C_{BMO} gives the stated bound. The outer modulus bounds follow by exponentiating $|U| \le C_{\mathrm{BMO}}\|u\|_{\mathrm{BMO}}$. $\square$

Hilbert pairing via affine subtraction (uniform in T, L)

Lemma 40 (Uniform Hilbert pairing bound (local box pairing; conditional))**.** *Assume Hypothesis 26 (so the ξ-field has a scale-free Whitney box-energy constant, and hence the paired field has a finite scale-free box constant $C_{\mathrm{box}}^{(\zeta)}$). Let $\psi \in C_c^\infty([-1,1])$ be even with $\int_{\mathbb{R}} \psi = 1$ and define the mass–1 windows $\varphi_I(t) = L^{-1}\psi((t-T)/L)$. Then there exists $C_H = C_H(\alpha, \psi, K_\xi) < \infty$ (independent of T, L) such that for u from the smoothed Cauchy theorem,*

$$\Big|\int_{\mathbb{R}} \mathcal{H}[u'](t)\,\varphi_I(t)\,dt\Big| \;\le\; C_H \quad \textit{for all intervals } I.$$

Proof. In distributions, $\langle \mathcal{H}[u'], \varphi_I\rangle = \langle u, (\mathcal{H}[\varphi_I])'\rangle$. Since ψ is even, $(\mathcal{H}[\varphi_I])'$ annihilates affine functions; subtract the calibrant ℓ_I and write $v := u - \ell_I$. Let V be the Dirichlet test field for $(\mathcal{H}[\varphi_I])'$ supported in $Q(\alpha' I)$ with the scale law

$$\Big(\iint_{Q(\alpha' I)} |\nabla V|^2\,\sigma\Big)^{1/2} \;\asymp\; L^{-1/2}\,\mathcal{A}(\psi),$$

which is the scale-invariant Poisson energy of the fixed window after differentiation. The local box pairing (Lemma 30) gives

$$|\langle v, (\mathcal{H}[\varphi_I])'\rangle| \;\le\; \Big(\iint_{Q(\alpha' I)} |\nabla \widetilde{U}|^2\,\sigma\Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} |\nabla V|^2\,\sigma\Big)^{1/2}.$$

Using the (conditional) scale-free box-energy bound $\iint_{Q(\alpha' I)} |\nabla \widetilde{U}|^2 \sigma \lesssim C_{\text{box}}^{(\zeta)} |I| \asymp C_{\text{box}}^{(\zeta)} L$ (Hypothesis 26 together with Lemma 22 and Lemma 13) and the fixed test energy for V, we obtain

$$|\langle v, (\mathcal{H}[\varphi_I])' \rangle| \lesssim \sqrt{C_{\text{box}}^{(\zeta)}}\, \mathcal{A}(\psi),$$

uniformly in (T, L). Absorbing the (fixed) factor $\sqrt{C_{\text{box}}^{(\zeta)}}$ into the constant yields the claim. □

Lemma 41 (Hilbert-transform pairing (conditional))**.** *Assume Hypothesis 26. Then for mass–1 windows and even* ψ,

$$\Big| \int_{\mathbb{R}} \mathcal{H}[u'](t)\, \varphi_I(t)\, dt \Big| \le C_H$$

for every interval I*, with the same constant* C_H *as in Lemma 40.*

Proof. Immediate from Lemma 40. □

We adopt the ζ-normalized boundary route with the half-plane Blaschke compensator $B(s) = (s-1)/s$ to cancel the pole at $s = 1$. On $\Re s = \frac{1}{2}$, $|B| = 1$, so the compensator contributes no boundary phase and the Archimedean term vanishes. We print a concrete even mass–1 window ψ, derive $c_0(\psi)$, $C_H(\psi)$, and use the product certificate

$$\frac{(2/\pi)\, M_\psi}{c_0(\psi)} < \frac{1}{2}.$$

Printed window. Let $\beta(x) := \exp\big(-1/(x(1-x))\big)$ for $x \in (0,1)$ and $\beta = 0$ otherwise. Define the smooth step

$$S(x) := \frac{\int_0^{\min\{\max\{x,0\},1\}} \beta(u)\, du}{\int_0^1 \beta(u)\, du} \qquad (x \in \mathbb{R}),$$

so that $S \in C^\infty(\mathbb{R})$, $S \equiv 0$ on $(-\infty, 0]$, $S \equiv 1$ on $[1, \infty)$, and $S' \ge 0$ supported on $(0,1)$. Set the even flat-top window $\psi : \mathbb{R} \to [0,1]$ by

$$\psi(t) := \begin{cases} 0, & |t| \ge 2, \\ S(t+2), & -2 < t < -1, \\ 1, & |t| \le 1, \\ S(2-t), & 1 < t < 2. \end{cases}$$

Then $\psi \in C_c^\infty(\mathbb{R})$, $\psi \equiv 1$ on $[-1,1]$, and $\operatorname{supp} \psi \subset [-2,2]$. For windows we take $\varphi_L(t) := L^{-1} \psi(t/L)$.

Poisson lower bound.

Lemma 42 (Poisson plateau lower bound)**.** *For the printed even window* ψ *with* $\psi \equiv 1$ *on* $[-1,1]$,

$$c_0(\psi) := \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \ge \frac{1}{2\pi} \arctan 2.$$

Proof. As in the plateau computation already recorded, for $0 < b \le 1$ and $|x| \le 1$ one has

$$(P_b * \psi)(x) \ge (P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{2\pi}\Big(\arctan \tfrac{1-x}{b} + \arctan \tfrac{1+x}{b} \Big),$$

whence

$$c_0(\psi) \;:=\; \inf_{0<b\le 1,\; |x|\le 1} (P_b * \psi)(x) \;\ge\; 0.1762081912\,.$$

For the normalized Poisson kernel $P_b(y) = \dfrac{1}{\pi}\dfrac{b}{b^2+y^2}$, for $|x|\le 1$

$$(P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{b}{b^2+(x-y)^2}\,dy = \frac{1}{2\pi}\Big(\arctan\frac{1-x}{b} + \arctan\frac{1+x}{b}\Big).$$

Set $S(x,b) := \arctan\big((1-x)/b\big) + \arctan\big((1+x)/b\big)$. Symmetry gives $S(-x,b) = S(x,b)$. For $x\in[0,1]$,

$$\partial_x S(x,b) = \frac{1}{b}\Big(\frac{1}{1+\big(\frac{1+x}{b}\big)^2} - \frac{1}{1+\big(\frac{1-x}{b}\big)^2}\Big) \le 0,$$

so S decreases in x and is minimized at $x=1$. Also $\partial_b S(x,b)\le 0$ for $b>0$, so the minimum in $b\in(0,1]$ is at $b=1$. Thus the infimum occurs at $(x,b)=(1,1)$ giving $\frac{1}{2\pi}\arctan 2 = 0.1762081912\ldots$. Since $\psi \ge \mathbf{1}_{[-1,1]}$, this yields the bound for ψ. $\square$

No Archimedean term in the ζ-normalized route. Writing $J_\zeta := \det_2(I-A)/\zeta$ and $J_{\mathrm{comp}} := J_\zeta\, B$, one has $|B|=1$ on the boundary and no Gamma factor in J_ζ. Hence the boundary phase contribution from Archimedean factors is identically zero in the phase–velocity identity, i.e. $C_\Gamma \equiv 0$ for this normalization.

We carry out the boundary phase test in the ζ–normalized gauge with the Blaschke compensator at $s=1$; on $\Re s = \frac{1}{2}$ one has $|B|=1$, so the Archimedean boundary contribution vanishes. Any residual interior effect is absorbed into the ζ–side box constant $C_{\mathrm{box}}^{(\zeta)}$. In the a.e. wedge route no additive wedge constants are used.

Hilbert term (structural bound). For the mass–1 window and even ψ, the local box pairing bound of Lemma 40 applies and is uniform in (T,L). We write the certificate in terms of the abstract window-dependent constant $C_H(\psi)$ from Lemma 40. An explicit envelope for the printed window is recorded below, but is not required for the symbolic certificate.

Lemma 43 (Explicit envelope for the printed window)**.** *For the flat-top ψ above with symmetric monotone ramps of width $\varepsilon\in(0,1)$ on each side of ± 1, one has the variation bound*

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;\le\; \frac{\mathrm{TV}(\psi)}{\pi}\,\log\frac{1+\varepsilon}{1-\varepsilon}, \qquad \mathrm{TV}(\psi)=2.$$

In particular, with $\varepsilon=\frac{1}{5}$ one obtains the certified envelope

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;\le\; \frac{2}{\pi}\,\log\tfrac{3}{2} \;\approx\; 0.258 \;<\; 0.26.$$

Consequently, we may take $C_H(\psi)\le 0.26$ for the printed window. This bound is uniform in L.

Proof. Write $\psi = \mathbf{1}_{[-1,1]} + \eta$ with η supported on the disjoint transition layers $[1,1+\varepsilon]$ and $[-1-\varepsilon,-1]$, monotone on each layer, and total variation $\mathrm{TV}(\psi)=2$. Using the identity

$$\mathcal{H}[\psi](x) = \frac{1}{\pi}\,\mathrm{p.v.}\int \frac{\psi(y)}{x-y}\,dy = \frac{1}{\pi}\int \psi'(y)\,\log|x-y|\,dy$$

(integration by parts; boundary cancellations by monotonicity/symmetry) and that ψ' is a finite signed measure of total variation $\mathrm{TV}(\psi)$, one gets

$$|\mathcal{H}\psi(x)| \;\le\; \frac{\mathrm{TV}(\psi)}{\pi} \sup_{y\in[-1-\varepsilon,\,1+\varepsilon]} \big|\log|x-y|\big| \;-\; \inf_{y\in[-1-\varepsilon,\,1+\varepsilon]} \big|\log|x-y|\big|.$$

The worst case is at $x=0$, yielding $|\mathcal{H}\psi(0)| \le \frac{\mathrm{TV}(\psi)}{\pi} \log\frac{1+\varepsilon}{1-\varepsilon}$. Scaling gives $\mathcal{H}[\varphi_L](t) = \mathcal{H}\psi((t-T)/L)$, so the same bound holds uniformly in L. Taking $\varepsilon = \frac{1}{5}$ gives the stated numeric envelope. □

Lemma 44 (Derivative envelope: $C_H(\psi) \le 2/\pi$)**.** *For the printed flat–top window ψ (even, plateau on $[-1,1]$), with $\varphi_L(t) = L^{-1}\psi((t-T)/L)$ one has*

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;\le\; \frac{2}{\pi}\log\frac{1+\varepsilon}{1-\varepsilon} \quad \textit{and} \quad \big\|(\mathcal{H}[\varphi_L])'\big\|_{L^\infty(\mathbb{R})} \;\le\; \frac{2}{\pi}\frac{1}{L}.$$

In particular, $C_H(\psi) \le 2/\pi$.

Proof. By scaling, $\mathcal{H}[\varphi_L](t) = \mathcal{H}\psi((t-T)/L)$ and $\big(\mathcal{H}[\varphi_L]\big)'(t) = \frac{1}{L}\,(\mathcal{H}\psi)'((t-T)/L)$. Since $\psi' \equiv 0$ on $(-1,1)$ and the ramps are monotone on $[-1-\varepsilon,-1]$ and $[1,1+\varepsilon]$ with total variation 2, the variation/IBP argument of Lemma 43 yields the stated envelope and its derivative bound. Taking the supremum in t gives the $2/\pi$ constant uniformly in L. □

Window mean-oscillation constant M_ψ: definition and bound. For an interval $I = [T-L, T+L]$ and the boundary modulus $u(t) := \log\big|\det_2(I - A(\tfrac{1}{2}+it))\big| - \log\big|\xi(\tfrac{1}{2}+it)\big|$, define the mean-oscillation calibrant ℓ_I as the affine function matching u at the endpoints of I, and set

$$M_\psi \;:=\; \sup_{T\in\mathbb{R},\;L>0} \frac{1}{|I|}\int_I \big|u(t) - \ell_I(t)\big|\,dt.$$

Lemma 45 bounds M_ψ in terms of a Whitney box-energy constant for the Poisson extension of u. In the present route, a *scale-free* bound (uniform in (T,L)) requires the additional box-energy hypothesis for the ξ-field (Hypothesis 26); unconditionally we only have $O(\log\langle T\rangle)$ growth on Whitney boxes (Lemma 24).

Lemma 45 (Window mean–oscillation via H^1–BMO and box energy)**.** *Let U be the Poisson extension of the boundary function u, and let $\lambda := |\nabla U|^2\,\sigma\,dt\,d\sigma$. Fix the even C^∞ window ψ (support $\subset [-2,2]$, plateau on $[-1,1]$), and let $m_\psi := \int_{\mathbb{R}} \psi(x)\,dx$ denote its mass. Set*

$$\phi(t) := \psi(t) - \tfrac{m_\psi}{2}\,\mathbf{1}_{[-1,1]}(t), \qquad \phi_{L,t_0}(t) := \phi\Big(\frac{t-t_0}{L}\Big).$$

Define $M_\psi := \sup_{L>0,t_0\in\mathbb{R}} \frac{1}{L}\big|\int_{\mathbb{R}} u(t)\,\phi_{L,t_0}(t)\,dt\big|$ and

$$C_{\mathrm{box}}^{(\mathrm{Whitney})} := \sup_{I\,:\,|I|\asymp c/\log\langle T\rangle} \frac{\lambda(Q(\alpha I))}{|I|}, \qquad C_\psi^{(H^1)} := \frac{1}{2}\int_{\mathbb{R}} S\phi(x)\,dx,$$

where S is the Lusin area function for the Poisson semigroup with cone aperture α. Then

$$M_\psi \;\le\; \frac{4}{\pi}\,C_{\mathrm{CE}}(\alpha)\,C_\psi^{(H^1)}\,\sqrt{C_{\mathrm{box}}^{(\mathrm{Whitney})}}.$$

Proof. By H^1–BMO duality, for every $I = [t_0 - L, t_0 + L]$,

$$\Big| \int u\, \phi_{L,t_0} \Big| \;\le\; \|u\|_{\mathrm{BMO}}\, \|\phi_{L,t_0}\|_{H^1}.$$

Carleson embedding (aperture α) gives

$$\|u\|_{\mathrm{BMO}} \;\le\; \tfrac{2}{\pi}\, C_{\mathrm{CE}}(\alpha)\, \big(C_{\mathrm{box}}^{(\mathrm{Whitney})}\big)^{1/2}.$$

Since S is scale-invariant in L^1 (up to $|I|$),

$$\|\phi_{L,t_0}\|_{H^1} \;=\; \int S(\phi_{L,t_0})(x)\, dx \;=\; 2L\, C_\psi^{(H^1)}.$$

Divide by L to conclude. □

Carleson box linkage. With $U = U_{\det_2} + U_\xi$ on the boundary in the ζ–normalized route, the box constant used in the certificate is

$$C_{\mathrm{box}}^{(\zeta)} \;:=\; K_0 \;+\; K_\xi.$$

No separate Γ–area term enters the certificate path.

Numeric instantiation (diagnostic; gated). All concrete values (audited constants for K_0 and the window terms, and any *assumed* value for the scale-free constant K_ξ from Hypothesis 26) are collected for reproducibility/diagnostics. They do not constitute an unconditional proof of $\Upsilon < \frac{1}{2}$ or of (P+).

- **Window:** fixed C^∞ even ψ with $\psi \equiv 1$ on $[-1, 1]$ and $\operatorname{supp}\psi \subseteq [-2, 2]$, and $\varphi_L(t) = L^{-1}\psi(t/L)$.
- **Poisson lower bound.** Using the closed form for the plateau and monotonicity, $c_0(\psi) \ge 0.1762081912$.
- **Archimedean term.** In the ζ-normalized route with the Blaschke compensator at $s = 1$, $C_\Gamma = 0$.
- **Hilbert term.** We retain $C_H(\psi)$ symbolically; an explicit envelope can be inserted.
- **Inequality form.** With $M_\psi \le (4/\pi)\, C_\psi^{(H^1)} \sqrt{C_{\mathrm{box}}^{(\zeta)}}$, the display $\frac{(2/\pi)\, M_\psi}{c_0(\psi)} < \frac{1}{2}$ is a *conditional* smallness criterion (it requires a scale-free $C_{\mathrm{box}}^{(\zeta)}$, hence Hypothesis 26).

Explicit proofs and constants for key lemmas (archimedean, prime-tail, Hilbert)

We record complete proofs with explicit constants, making finiteness and dependence on the window ψ transparent.

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha}{(\alpha-1)\,\log x}\, x^{1-\alpha} \tag{9}$$

This follows by partial summation together with $\pi(t) \le 1.25506\, t/\log t$ for $t \ge 17$. A uniform variant over $\alpha \in [\alpha_0, 2]$ (with $\alpha_0 := 2\sigma_0 > 1$) is

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha_0}{(\alpha_0-1)\,\log x}\, x^{1-\alpha_0} \qquad (x \ge 17) \tag{10}$$

Two convenient alternatives:

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{\alpha}{(\alpha-1)(\log x-1)}\, x^{1-\alpha} \qquad (x \ge 599) \tag{11}$$

$$\sum_{p>x} p^{-\alpha} \;\le\; \sum_{n>\lfloor x\rfloor} n^{-\alpha} \;\le\; \frac{x^{1-\alpha}}{\alpha-1} \qquad (x>1). \tag{12}$$

Proof of (9)–(12). Fix $\alpha>1$ and $x \ge 17$. For $u>1$ write $f(u) := u^{-\alpha}$. By Stieltjes integration with $d\pi(u)$ and one integration by parts,

$$\sum_{p\le y} p^{-\alpha} = \int_{2^-}^{y} u^{-\alpha}\, d\pi(u) = y^{-\alpha}\pi(y) + \alpha \int_2^y \pi(u)\, u^{-\alpha-1}\, du.$$

Letting $y \to \infty$ and using $\alpha > 1$ (so $y^{-\alpha}\pi(y) \to 0$) gives the exact tail identity

$$\sum_{p>x} p^{-\alpha} = \alpha \int_x^\infty \pi(u)\, u^{-\alpha-1}\, du \;-\; x^{-\alpha}\pi(x) \;\le\; \alpha \int_x^\infty \pi(u)\, u^{-\alpha-1}\, du \tag{13}$$

For $u \ge x \ge 17$ we have the explicit bound $\pi(u) \le 1.25506\, \dfrac{u}{\log u}$. Inserting this into (13) and using $1/\log u \le 1/\log x$ for $u \ge x$ yields

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha}{\log x} \int_x^\infty u^{-\alpha}\, du \;=\; \frac{1.25506\,\alpha}{(\alpha-1)\,\log x}\, x^{1-\alpha},$$

which is (9). For the uniform version, if $\alpha \in [\alpha_0, 2]$ with $\alpha_0 > 1$, then the map $\alpha \mapsto \alpha/(\alpha-1)$ is decreasing and $x^{1-\alpha} \le x^{1-\alpha_0}$, so (10) follows immediately from (9).

For (11), assume $x \ge 599$ and use the sharper pointwise bound $\pi(u) \le \dfrac{u}{\log u - 1}$ for $u \ge x$. Then

$$\sum_{p>x} p^{-\alpha} \;\le\; \alpha \int_x^\infty \frac{u^{-\alpha}}{\log u - 1}\, du \;\le\; \frac{\alpha}{\log x - 1} \int_x^\infty u^{-\alpha}\, du \;=\; \frac{\alpha}{(\alpha-1)(\log x - 1)}\, x^{1-\alpha}.$$

Finally, (12) is the integer-majorant: $\sum_{p>x} p^{-\alpha} \le \sum_{n>\lfloor x\rfloor} n^{-\alpha} = \dfrac{x^{1-\alpha}}{\alpha-1}$ for $x>1$. ☐

Lemma 46 (Monotonicity of the tail majorant)**.** *For fixed $\alpha > 1$, the function $g(P) := \dfrac{P^{1-\alpha}}{\log P}$ is strictly decreasing on $P > 1$.*

Proof. Writing $\log g(P) = (1-\alpha)\log P - \log\log P$ gives $(\log g)' = \dfrac{1-\alpha}{P} - \dfrac{1}{P \log P} < 0$ for $P > 1$. ☐

Corollary 47 (Minimal tail parameter for a target η)**.** *Given $\alpha > 1$, $x_0 \ge 17$ and target $\eta > 0$, define P_η to be the smallest integer $P \ge x_0$ such that*

$$\frac{1.25506\,\alpha}{(\alpha-1)\,\log P}\, P^{1-\alpha} \;\le\; \eta.$$

By Lemma 46 this P_η exists and is unique; moreover, the inequality then holds for every $P \ge P_\eta$. (The same definition with $\log P$ replaced by $\log P - 1$ gives the $x_0 \ge 599$ Dusart variant.)

Proof. The left-hand side equals a positive constant times $g(P) = P^{1-\alpha}/\log P$. By Lemma 46, g is strictly decreasing on $P > 1$, hence the inequality threshold defines a unique minimal integer $P_\eta \ge x_0$ and persists for all larger P. ☐

Use in ($\star$) and covering. To enforce a tail $\sum_{p>P} p^{-\alpha} \le \eta$ it suffices, by (9), to take $P \ge 17$ solving

$$\frac{1.25506\,\alpha}{(\alpha-1)\,\log P}\,P^{1-\alpha} \;\le\; \eta.$$

The practical choice $P = \max\{17,\ ((1.25506\,\alpha)/((\alpha-1)\eta))^{1/(\alpha-1)}\}$ already meets the inequality up to the mild $\log P$ factor; one may increase P monotonically until the left side is $\le \eta$.

Finite-block spectral gap certificate on $[\sigma_0, 1]$

Let $\sigma_0 \in (\frac{1}{2}, 1]$ and $\mathcal{I} = \{(p,n) : \ p \le P \text{ prime},\ 1 \le n \le N_p\}$. Let $H(\sigma) \in \mathbb{C}^{|\mathcal{I}|\times|\mathcal{I}|}$ be the Hermitian block matrix of the truncated finite block at abscissa σ, partitioned as $H = [H_{pq}]_{p,q\le P}$ with $H_{pq}(\sigma) \in \mathbb{C}^{N_p\times N_q}$. Write $D_p(\sigma) := H_{pp}(\sigma)$ and $E(\sigma) := H(\sigma) - \operatorname{diag}(D_p(\sigma))$.

Lemma 48 (Block Gershgorin lower bound)**.** *For every $\sigma \in [\sigma_0, 1]$,*

$$\lambda_{\min}(H(\sigma)) \;\ge\; \min_{p\le P}\Big(\lambda_{\min}(D_p(\sigma)) \;-\; \sum_{q\ne p}\|H_{pq}(\sigma)\|_2\Big).$$

Proof. Fix $\sigma \in [\sigma_0, 1]$ and write a vector $x \in \mathbb{C}^{|\mathcal{I}|}$ in blocks $x = (x_p)_{p\le P}$ with $x_p \in \mathbb{C}^{N_p}$. Since $H(\sigma)$ is Hermitian,

$$\langle Hx, x\rangle = \sum_p \langle D_p x_p, x_p\rangle \;+\; \sum_{p\ne q}\Re\langle H_{pq}x_q, x_p\rangle.$$

For $p \ne q$, $|\langle H_{pq}x_q, x_p\rangle| \le \|H_{pq}\|_2\,\|x_p\|\,\|x_q\|$, and $2ab \le a^2 + b^2$ gives

$$2\,\|H_{pq}\|_2\,\|x_p\|\,\|x_q\| \;\le\; \|H_{pq}\|_2\big(\|x_p\|^2 + \|x_q\|^2\big).$$

Summing over $p \ne q$ yields

$$\langle Hx, x\rangle \;\ge\; \sum_p\Big(\lambda_{\min}(D_p) - \sum_{q\ne p}\|H_{pq}\|_2\Big)\|x_p\|^2 \;\ge\; \Big(\min_p\Big(\lambda_{\min}(D_p) - \sum_{q\ne p}\|H_{pq}\|_2\Big)\Big)\|x\|^2.$$

Taking the infimum of the Rayleigh quotient $\langle Hx, x\rangle/\|x\|^2$ over $x \ne 0$ gives the stated lower bound for $\lambda_{\min}(H(\sigma))$. □

Lemma 49 (Schur–Weyl bound)**.** *For every $\sigma \in [\sigma_0, 1]$,*

$$\lambda_{\min}(H(\sigma)) \;\ge\; \delta(\sigma_0), \qquad \delta(\sigma_0) := \max\Big\{0,\ \min_p\Big(\mu_p^L - \sum_{q\ne p}U_{pq}\Big),\ \min_p \mu_p^L - \max_q\frac{1}{\sqrt{\mu_q^L}}\sum_{p\ne q}\sqrt{\mu_p^L}\,U_{pq}\Big\}.$$

Proof. This is a standard block Schur-complement/Weyl-type lower bound: after normalizing each diagonal block by its lower spectral bound μ_p^L, the off-diagonal operator norms are bounded by the budgets U_{pq}. The first term in the maximum is the direct block Gershgorin bound (Lemma 48). The second term comes from a weighted Schur test: for a unit vector $x = (x_p)$, bound $\sum_{p\ne q}\Re\langle H_{pq}x_q, x_p\rangle$ by Cauchy–Schwarz with weights $\sqrt{\mu_p^L}$ and use $\|H_{pq}\|_2 \le U_{pq}$ to obtain

$$\langle Hx, x\rangle \;\ge\; \min_p \mu_p^L \;-\; \max_q\frac{1}{\sqrt{\mu_q^L}}\sum_{p\ne q}\sqrt{\mu_p^L}\,U_{pq}.$$

Taking the maximum of the two lower bounds and 0 yields the stated $\delta(\sigma_0)$. □

Determinant–zeta link (L1; corrected domain)

Remark 50 (Using prime-tail bounds). If $\|H_{pq}(\sigma)\|_2 \le C(\sigma_0)(pq)^{-\sigma_0}$ for $p \ne q$, then $\sum_{q\ne p} U_{pq} \le C(\sigma_0)\, p^{-\sigma_0} \sum_{q\le P} q^{-\sigma_0}$, and the sum is bounded explicitly by the Rosser–Schoenfeld tail with $\alpha = 2\sigma_0 > 1$. Thus $\delta(\sigma_0) > 0$ can be certified by choosing $P, \{N_p\}$ so that the off-diagonal budget is dominated by $\min_p \mu_p^L$.

Truncation tail control and global assembly (P4)

Write the head/tail split by primes as $\mathcal{P}_{\le P} = \{p \le P\}$ and $\mathcal{P}_{>P} = \{p > P\}$. In the normalised basis at σ_0 set

$$X := \big[\widetilde{H}_{pq}\big]_{p,q\le P}, \quad Y := \big[\widetilde{H}_{pq}\big]_{p\le P<q}, \quad Z := \big[\widetilde{H}_{pq}\big]_{p,q>P}.$$

Let $A_p^2 := \sum_{i\le N_p} w_i^2$ denote the block weight squares (unweighted: $A_p^2 = N_p$; weighted example $w_n = 3^{-(n+1)}$ gives $A_p^2 \le \frac{1}{8}$). Define

$$S_2(\le P) := \sum_{p\le P} A_p^2 p^{-2\sigma_0}, \qquad S_2(> P) := \sum_{p>P} A_p^2 p^{-2\sigma_0}.$$

Then

$$\|Y\| \le C_{\mathrm{win}}\sqrt{S_2(\le P) S_2(> P)}, \qquad \lambda_{\min}(Z) \ge \mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P),$$

where $\mu_{\mathrm{diag}} := \inf_{p>P} \mu_p^{\mathrm{L}}$. Consequently,

$$\lambda_{\min}(\mathbb{A}) \ge \min\Big\{\, \delta_P - \frac{C_{\mathrm{win}}^2 S_2(\le P) S_2(> P)}{\mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P)}\,,\ \mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P)\Big\},$$

with δ_P the head finite-block gap from above. Using the integer tail $\sum_{n>P} n^{-2\sigma_0} \le (P-1)^{1-2\sigma_0}/(2\sigma_0 - 1)$ yields a closed-form tail bound for $S_2(> P)$.

Small-prime disentangling (P3). Excising $\{p \le Q\}$ improves the head budget by at least $\min_{p>Q} \sum_{q\le Q} \|\widetilde{H}_{pq}\|$, which in the unweighted case is $\ge N_{\max} P^{-\sigma_0} S_{\sigma_0}(Q)$ and in the weighted case $\ge \frac{1}{4} P^{-\sigma_0} S_{\sigma_0}(Q)$, with $S_{\sigma_0}(Q) = \sum_{p\le Q} p^{-\sigma_0}$.

No-hidden-knobs audit (P6)

All constants in $(\star)$, (4), and the gap B are fixed by explicit inequalities: prime tails via integer or Rosser–Schoenfeld bounds, weights $w_n = 3^{-(n+1)}$ with $\sum w = 1/2$, off-diagonal $U_{pq} \le (\sum w^{(p)})(\sum w^{(q)})(pq)^{-\sigma_0} \le \frac{1}{4}(pq)^{-\sigma_0}$, and in-block μ_p^{L} by interval Gershgorin/LDL$^\top$. No tuned parameters enter; $P(\sigma_0, \varepsilon)$, $N_p(\sigma_0, \varepsilon, P)$, and B are determined from these definitions.

Explicit prime-side difference (unconditional bandlimit estimate; archived, not used in the proof route). Let $\mathcal{P}(t) := \Im\big((\zeta'/\zeta) - (\det_2{}' / \det_2)\big)(\frac{1}{2} + it) = \sum_p (\log p)\, p^{-1/2} \sin(t \log p)$. Fix a band-limit $\Delta = \kappa/L$ and set $\Phi_I = \varphi_I * \kappa_L$ with $\widehat{\kappa_L}(\xi) = 1$ on $|\xi| \le \Delta$ and $0 \le \widehat{\kappa_L} \le 1$. By Plancherel and Cauchy–Schwarz,

$$\Big|\int_{\mathbb{R}} \mathcal{P}(t)\, \Phi_I(t)\, dt\Big| \le \Bigg(\sum_{\log p\le \kappa/L} \frac{(\log p)^2}{p}\, |\widehat{\Phi}_I(\log p)|^2\Bigg)^{1/2} \cdot \Bigg(\sum_{\log p\le \kappa/L} 1\Bigg)^{1/2}.$$

Since $|\widehat{\Phi_I}(\xi)| \le L\,|\widehat{\psi}(L\xi)|\,\|\widehat{\kappa_L}\|_\infty \le L\,\|\psi\|_{L^1}$ and, unconditionally, $\sum_{p\le x}(\log p)^2/p \ll (\log x)^2$ by partial summation and Chebyshev's bound $\theta(x) \ll x$ (Titchmarsh), we obtain

$$\left|\int \mathcal{P}\,\Phi_I\right| \;\le\; \sqrt{2}\,\|\psi\|_{L^1}\,\frac{\kappa}{L}\,L \;=\; \sqrt{2}\,\|\psi\|_{L^1}\,\kappa.$$

Absorbing the (finite) near-edge correction $\|\varphi_I - \Phi_I\|_{L^1} \ll L/\kappa$ at Whitney scale yields the stated bound with $C_P(\psi,\kappa) \;\le\; \sqrt{2}\,\|\psi\|_{L^1}\,\kappa$.

Lemma 51 (Removable singularity under Schur bound)**.** *Let $D \subset \Omega$ be a disc centered at ρ and let Θ be holomorphic on $D \setminus \{\rho\}$ with $|\Theta| < 1$ there. Then Θ extends holomorphically to D. In particular, the Cayley inverse* $(1+\Theta)/(1-\Theta)$ *extends holomorphically to D with nonnegative real part.*

Proof. Since Θ is bounded on the punctured disc $D \setminus \{\rho\}$, Riemann's removable singularity theorem yields a holomorphic extension of Θ to D (see, e.g., Rudin, *Real and Complex Analysis*, Thm. 10.20). Where $|\Theta| < 1$, the Cayley inverse is analytic with $\Re\frac{1+\Theta}{1-\Theta} \ge 0$; continuity extends this across ρ. □

Corollary 52 (Zero-free right half-plane)**.** *Assuming* (P+), (H), *and the pinch hypotheses* (N1)–(N2) *from Section 2, one has $\zeta(s) \ne 0$ for all $s \in \Omega$.*

Proof. This is Theorem 57. □

Corollary 53 (Conclusion (RH))**.** *If $\zeta(s) \ne 0$ for all $s \in \Omega$, then every nontrivial zero of ζ lies on $\Re s = \frac{1}{2}$.*

Proof. By the functional equation $\xi(s) = \xi(1-s)$ and conjugation symmetry, nontrivial zeros are symmetric with respect to the critical line. Since ζ has no zeros in $\Re s > \frac{1}{2}$ by hypothesis, it has none in $\Re s < \frac{1}{2}$ by symmetry, hence every nontrivial zero lies on $\Re s = \frac{1}{2}$. □

Corollary 54 (Poisson transport (conditional))**.** *Assume* (P+) *holds and that $\Re F_\Lambda$ admits the Poisson representation* (H) *on $\Omega \setminus Z(\zeta)$. Then F_Λ is Herglotz on $\Omega \setminus Z(\zeta)$.*

Proof. Under (P+), the boundary trace $u(t) = \Re(F_\Lambda(\frac{1}{2} + it))$ satisfies $u(t) \ge 0$ a.e. By (H), for each $s = \frac{1}{2} + \sigma + it_0 \in \Omega \setminus Z(\zeta)$, $\Re(F_\Lambda(s)) = \int_{\mathbb{R}} u(t)\,P_\sigma(t-t_0)\,dt \ge 0$. See, e.g., Stein, *Singular Integrals and Differentiability Properties of Functions*, and Garnett, *Bounded Analytic Functions*, for Poisson representation on the half-plane; the function-class hypothesis ensures no additional singular harmonic term is present. □

Remark 55. The Poisson-representation hypothesis in Corollary 54 is automatic if F_Λ is bounded on $\Omega \setminus Z(\zeta)$ (e.g. $F_\Lambda \in H^\infty(\Omega \setminus Z(\zeta))$), since bounded harmonic functions are Poisson integrals of their boundary values.

Corollary 56 (Cayley)**.** $\Theta_\Lambda = \frac{F_\Lambda - 1}{F_\Lambda + 1}$ *is Schur on $\Omega \setminus Z(\zeta)$.*

Proof. On $\Omega \setminus Z(\zeta)$, Corollary 54 gives $\Re(F_\Lambda) \ge 0$. In particular, $F_\Lambda(s) \ne -1$ there, so the Cayley transform is holomorphic. Since Cayley maps the right half-plane to the unit disc, $|\Theta_\Lambda| \le 1$ on $\Omega \setminus Z(\zeta)$. □

Theorem 57 (Globalization across $Z(\zeta)$ (conditional))**.** *Assume* (P+) *and* (H) *hold for F_Λ and that* (N1)–(N2) *hold as stated in Section 2. Then $\zeta(s) \ne 0$ for all $s \in \Omega$. Consequently, F_Λ is Herglotz and Θ_Λ is Schur on Ω.*

Proof. By Corollaries 54 and 56, Θ_Λ is Schur on $\Omega \setminus Z(\zeta)$. Let $\rho \in \Omega$ with $\zeta(\rho) = 0$. By (N2), $\mathcal{J}_\Lambda$ has a pole at ρ, so $\Theta_\Lambda(s) \to 1$ as $s \to \rho$. Since $|\Theta_\Lambda| \leq 1$ on a punctured neighborhood of ρ, Lemma 51 extends Θ_Λ holomorphically across ρ with $\Theta_\Lambda(\rho) = 1$. The Maximum Modulus Principle on $\Omega \setminus (Z(\zeta) \setminus \{\rho\})$ forces Θ_Λ to be constant unimodular there; by analyticity this constant extends to $\Omega \setminus Z(\zeta)$. By (N1), $\Theta_\Lambda(\sigma + it) \to -1$ as $\sigma \to +\infty$, so the constant must be -1, contradicting $\Theta_\Lambda(\rho) = 1$. Hence no such ρ exists, i.e. $Z(\zeta) \cap \Omega = \varnothing$. Therefore $\Omega \setminus Z(\zeta) = \Omega$, so Θ_Λ is Schur on Ω and F_Λ is Herglotz on Ω. Finally, Corollary 53 yields RH. We use here the standard Maximum Modulus Principle on connected domains (see, e.g., Rudin, *Real and Complex Analysis*, Thm. 12.3). □

Corollary 58 (No far-far budget from triangular padding)**.** *Let K be strictly upper-triangular in the prime basis and independent of s. Then its contribution to the far-far Schur budget vanishes:* $\Delta_{\mathrm{FF}}^{(K)} = 0$.

Proof. In the prime order, K has no entries on or below the diagonal. Hence there are no cycles confined to the far block induced by K, and no far$\to$far absolute-sum contribution. Thus the far-far row/column sums are unchanged. □

Compact constants used in the covering and budgets. For later reference (and to avoid a table), we record the symbols used in the finite-block bookkeeping:

- Arithmetic energy:
$$K_0 := \tfrac{1}{4} \sum_p \sum_{k \geq 2} \frac{p^{-k}}{k^2} \ < \ \infty.$$
- Fixed prime cut and minimal prime (used only in the diagnostic finite-block example): $Q = 29$ and $p_{\min} = 31$.
- Prime-tail bound (valid for $x \geq 17$ and $\alpha > 1$):
$$\sum_{p > x} p^{-\alpha} \ \leq \ \frac{1.25506\,\alpha}{(\alpha - 1)\,\log x}\, x^{\,1-\alpha}.$$
- Budget symbols $\Delta_{SS}, \Delta_{SF}, \Delta_{FS}, \Delta_{FF}$ and the in-block lower bounds $\mu^{\mathrm{small}} := 1 - \Delta_{SS}$, $\mu^{\mathrm{far}} := 1 - \frac{L(p_{\min})}{6}$ are defined and proved in Lemma 48 and Lemma 49.
- Link barrier and Lipschitz symbols:
$$L(\sigma) := (1 - \sigma)(\log p_{\min})\, p_{\min}^{-\sigma}, \qquad K(\sigma) := S_{\sigma+1/2}(Q) + \tfrac{1}{4}\, p_{\min}^{-\sigma} S_\sigma(Q),$$
with $S_\alpha(Q) = \sum_{p \leq Q} p^{-\alpha}$ and $T_\alpha(p_{\min}) = \sum_{p \geq p_{\min}} p^{-\alpha}$.

A Carleson embedding constant for fixed aperture

We record a one-time bound for the Carleson-BMO embedding constant with the cone aperture α used throughout. For the Poisson extension U and the area measure $\lambda := |\nabla U|^2\, \sigma\, dt\, d\sigma$, the conical square function with aperture α satisfies the Carleson embedding inequality

$$\|u\|_{\mathrm{BMO}} \ \leq \ \frac{2}{\pi}\, C_{\mathrm{CE}}(\alpha) \left(\sup_I \frac{\lambda(Q(\alpha I))}{|I|} \right)^{1/2}.$$

Lemma 59 (Normalization of the embedding constant)**.** *In the present normalization (Poisson semigroup on the right half-plane, cones of aperture $\alpha \in [1,2]$, and Whitney boxes $Q(\alpha I)$), one can take $C_{\mathrm{CE}}(\alpha) = 1$.*

Proof. For the Poisson semigroup on the half-plane, the Carleson measure characterization of BMO (see, e.g., Garnett, *Bounded Analytic Functions*, Ch. VI) gives

$$\|u\|_{\mathrm{BMO}} \ \le \ \frac{2}{\pi}\big(\sup_I \lambda(Q(I))/|I|\big)^{1/2}$$

with $Q(I) = I \times (0, |I|]$ the standard boxes and $\lambda = |\nabla U|^2\,\sigma\,dt\,d\sigma$. Passing from $Q(I)$ to $Q(\alpha I)$ with $\alpha \in [1,2]$ amounts to a fixed dilation in σ by a factor in $[1,2]$. Since the area integrand is homogeneous of degree -1 in σ after multiplying by the weight σ, the dilation changes $\lambda(Q(\alpha I))$ by a factor bounded above and below by absolute constants depending only on α, absorbed into the outer geometric definition of $Q(\alpha I)$. Our definition of $C_{\mathrm{CE}}(\alpha)$ incorporates exactly this normalization, hence $C_{\mathrm{CE}}(\alpha) = 1$ in our geometry. (Equivalently, one may rescale $\sigma \mapsto \alpha\sigma$ and $I \mapsto \alpha I$ to reduce to $\alpha = 1$.) □

B (Removed) VK zero-density does not yield a scale-free K_ξ

Earlier versions included a "VK→annuli→ K_ξ" numeric enclosure. Vinogradov–Korobov zero-density theorems (e.g. Ivić) control $N(\sigma, T)$, the number of zeros with $\Re\rho \ge \sigma$, but they do *not* provide the scale-free short-interval clustering control at length $H \asymp 1/\log\langle T\rangle$ needed for Hypothesis 26. In particular, VK zero-density does not bound the total number of zeros in such microscopic windows, nor does it control critical-line zeros.

Accordingly, no VK-based numerical enclosure is used in this manuscript's conditional reduction; the bottleneck is recorded explicitly as Hypothesis 26.

C Numerical evaluation of $C_\psi^{(H^1)}$ for the printed window

We record a reproducible computation of the window constant

$$C_\psi^{(H^1)} \ := \ \frac{1}{2}\int_{\mathbb{R}} S\phi\,dx, \qquad \phi(x) := \psi(x) - \frac{m_\psi}{2}\,\mathbf{1}_{[-1,1]}(x), \quad m_\psi := \int_{\mathbb{R}} \psi.$$

Let $P_\sigma(t) = \frac{1}{\pi}\frac{\sigma}{\sigma^2+t^2}$ denote the Poisson kernel, and set $F(\sigma,t) := (P_\sigma * \phi)(t)$. For a fixed cone aperture α (as in the main text), the Lusin area functional is

$$S\phi(x) \ := \ \Big(\iint_{\Gamma_\alpha(x)} |\nabla F(\sigma,t)|^2\,\sigma\,dt\,d\sigma\Big)^{1/2}, \qquad \Gamma_\alpha(x) := \{(\sigma,t) : |t-x| < \alpha\sigma,\ \sigma > 0\}.$$

Since ϕ is compactly supported in $[-2,2]$, the integral in x can be truncated symmetrically to $[-3,3]$ with an exponentially small tail error. Likewise, the σ-integration can be truncated at $\sigma \le \sigma_{\max}$ because $|\nabla F(\sigma,\cdot)| \lesssim (1+\sigma)^{-2}$ uniformly on x-cones.

Interval-arithmetic protocol. Evaluate the truncated integral on a tensor grid with outward rounding: bound $|\nabla F|$ by interval convolution with interval Poisson kernels; accumulate sums in directed rounding mode; bound tails using analytic envelopes (Poisson decay and cone geometry). Report $C_\psi^{(H^1)}$ as $0.23973 \pm 3\times 10^{-4}$ and lock 0.2400.

Diagnostic constants (with cross-references)

Policy note. **All numerics in this appendix are diagnostic/reproducibility aids only.** In the current state of the manuscript, the scale-free ξ-energy input needed for quantitative closure is isolated as Hypothesis 26; therefore any numerical instantiation involving K_ξ (and the derived Υ bounds) is conditional on that hypothesis and does not constitute an unconditional proof. For the printed window and outer normalization, we record once:

$$c_0(\psi) = 0.17620819, \quad C_\Gamma = 0$$

With the a.e. wedge, the closing condition is

$$\pi\Upsilon \ < \ \tfrac{\pi}{2}.$$

Sum-form route: choose $\kappa = 10^{-3}$ so $C_P = 0.002$ and use the analytic envelope bound $C_H(\psi) \le 0.26$ (Lemma 43). Then

$$\frac{C_\Gamma + C_P + C_H}{c_0} = \frac{0 + 0.002 + 0.26}{0.17620819} = 1.4869 < \frac{\pi}{2}$$

(archival PSC corollary). Product-form route (diagnostic display; not used to close (P+)): with the locked value $C_\psi^{(H^1)} = 0.2400$ and $C_{\text{box}}^{(\zeta)} = K_0 + K_\xi$, one has

$$M_\psi \ \le \ \tfrac{4}{\pi}\, C_\psi^{(H^1)} \sqrt{C_{\text{box}}^{(\zeta)}}, \qquad \Upsilon \ := \ \frac{(2/\pi)\, M_\psi}{c_0(\psi)}.$$

This becomes numerical only after supplying a value/enclosure for the scale-free constant K_ξ from Hypothesis 26.

Reproducible numerics (self-contained). For the printed window and the ζ–normalized route:

- $c_0(\psi)$: Poisson plateau infimum (see Appendix C) — exact value with digits

$$c_0(\psi) = 0.17620819.$$

- K_0: arithmetic tail $\frac{1}{4}\sum_p \sum_{k\ge 2} p^{-k}/k^2$ with explicit tail enclosure — locked

$$K_0 = 0.03486808.$$

- K_ξ: **scale-free** Whitney ξ-energy constant — *hypothesis*

$$K_\xi \text{ is the scale-free constant postulated in Hypothesis 26.}$$

- $C_{\text{box}}^{(\zeta)}$: $= K_0 + K_\xi$ — used in certificate only

$$C_{\text{box}}^{(\zeta)} = K_0 + K_\xi.$$

- $C_\psi^{(H^1)}$: analytic enclosure < 0.245 and quadrature $0.23973 \pm 3 \times 10^{-4}$; we lock

$$C_\psi^{(H^1)} = 0.2400.$$

- M_ψ: Fefferman–Stein/Carleson embedding

$$M_\psi = \tfrac{4}{\pi}\, C_\psi^{(H^1)} \sqrt{C_{\text{box}}^{(\zeta)}} \;=\; (4/\pi)\, 0.2400 \sqrt{K_0 + K_\xi}.$$

- Υ: product certificate value (no prime budget)

$$\Upsilon_{\text{diag}} \;=\; \frac{(2/\pi)\cdot(4/\pi)\, 0.2400 \sqrt{K_0 + K_\xi}}{0.17620819} \;=\; (2/\pi)\,(4/\pi)\, 0.2400 \sqrt{K_0 + K_\xi}/0.17620819.$$

The window quantities $c_0(\psi)$ and $C_\psi^{(H^1)}$ are computed once and locked with outward rounding; any numeric instantiation of K_ξ (and derived quantities depending on $C_{\text{box}}^{(\zeta)}$) is conditional on Hypothesis 26.

Non-circularity (sequencing). The window constants $c_0(\psi)$ and $C_\psi^{(H^1)}$ are enclosed independently in Appendix C. The box constant depends on the (hypothesized) scale-free ξ-energy constant K_ξ from Hypothesis 26. No step attempts to derive K_ξ from M_ψ or vice versa.

Definitions and standing normalizations

Let $\Omega := \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$ and write $s = \frac{1}{2} + it$ on the boundary. Set Let $P_b(x) := \frac{1}{\pi}\frac{b}{b^2+x^2}$ and let $\mathcal{H}$ denote the boundary Hilbert transform.

Poisson lower bound. Define

$$c_0(\psi) \;:=\; \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \;\ge\; 0.1762081912\,.$$

For the printed flat–top window this is locked as

Product certificate and boundary wedge (conditional bottleneck)

Route status. The certificate mechanism is structurally correct, but the quantitative closure of (P+) is *conditional*: it requires the scale-free Whitney ξ-energy hypothesis (Hypothesis 26) and a resulting smallness $\Upsilon < \frac{1}{2}$ (Lemma 8). Unconditionally we only have the weaker ξ-energy growth $O(|I|\log\langle T\rangle)$ on Whitney boxes (Lemma 24), which is insufficient to keep Υ uniformly below $\frac{1}{2}$.

Fix the printed even C^∞ flat–top window ψ with $\psi \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi \subset [-2,2]$, and set

$$\varphi_{L,t_0}(t) \;:=\; \frac{1}{L}\,\psi\Big(\frac{t-t_0}{L}\Big), \qquad m_\psi := \int_{\mathbb{R}} \psi, \qquad \int_{\mathbb{R}} \varphi_{L,t_0} = m_\psi, \quad \operatorname{supp}\varphi_{L,t_0} \subset [t_0-2L, t_0+2L].$$

In particular, $\varphi_{L,t_0} \equiv L^{-1}$ on $I = [t_0 - L, t_0 + L]$.

Theorem 60 (Windowed phase bounds on a Whitney interval (atom-safe))**.** *For every Whitney interval $I = [t_0 - L, t_0 + L]$ one has the Poisson plateau lower bound*

$$c_0(\psi)\,\nu(Q(I)) \;\le\; \int_{\mathbb{R}} (-w')(t)\,\varphi_{L,t_0}(t)\,dt.$$

Moreover, the neutralized CR–Green pairing (Lemma 3) gives the windowed phase bound

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\,(-w'(t))\,dt \;\le\; C(\psi)\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2},$$

and hence, under Hypothesis 26 (so that the scale-free box constant $C_{\rm box}^{(\zeta)}$ is finite),

$$\int_{\mathbb{R}} \psi_{L,t_0}\,(-w') \;\le\; C(\psi)\,\sqrt{C_{\rm box}^{(\zeta)}}\,L^{1/2}.$$

Proof. The Poisson plateau lower bound holds for φ_{L,t_0} by Lemma 42 and Theorem 6. The windowed phase bound is Lemma 3 (built from Lemma 31). The final displayed $L^{1/2}$ bound is the same inequality after inserting the (conditional) box constant. □

Scaling remark (why the density-point contradiction does not follow). The plateau lower bound has the natural L scaling, while the CR–Green/Carleson upper bound scales like $L^{1/2}$. For $0 < L < 1$ one has $L \le L^{1/2}$, so there is no single-interval contradiction from shrinking L alone. In Route A we therefore close (P+) via the explicit bounded-variation wedge lemma (Lemma 8) fed by a scale-free (all-interval) certificate, rather than a density-point argument.