RM2/RM2U REDUCTION NOTE: THE LOG-CRITICAL $\ell = 2$ TAIL MOMENT AND A SINGLE GLOBAL TIGHTNESS GATE

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ABSTRACT. This note extracts a referee-checkable core from the running-max blow-up program for 3D incompressible Navier–Stokes. The fixed-frame compactness obstruction (RM2) is identified with a *log-critical $\ell = 2$ tail moment* of vorticity: the tail vorticity induces an affine harmonic velocity mode whose symmetric gradient at the origin is an explicit $\ell = 2$ shell moment. We record a coercive $\ell = 2$ estimate (RM2U) that suffices to bound this tail moment, and we isolate a single global tightness condition (*uniform exterior weighted enstrophy*, UEWE) that would imply the coercive estimate. At present the only missing analytic input is a genuine blow-up-variable tightness/RTD/UEWE mechanism; everything else in this note is unconditional.

1. THE RM2 OBSTRUCTION AS A LOG-CRITICAL $\ell = 2$ TAIL MOMENT

1.1. Tail strain at the origin (explicit formula).

Lemma 1.1 (Biot–Savart tail strain formula (explicit $\ell = 2$ moment)). *Let $\Omega : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth and supported in $\{|w| > 1\}$, and define its Biot–Savart velocity*

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x - w) \times \Omega(w)}{|x - w|^3} \, dw.$$

Then u is smooth and harmonic on $B_1(0)$ and divergence-free on $\mathbb{R}^3$. Moreover the symmetric gradient at the origin,

$$S(0) := \tfrac{1}{2}\big(\nabla u(0) + \nabla u(0)^T\big) \in \mathbb{R}^{3 \times 3}_{\mathrm{sym}},$$

is given by the explicit moment identity

$$S(0) = -\frac{3}{8\pi} \int_{|w| > 1} \frac{(w \times \Omega(w)) \otimes w + w \otimes (w \times \Omega(w))}{|w|^5} \, dw.$$

In particular $\operatorname{tr} S(0) = 0$. Equivalently, for every unit vector $b \in \mathbb{S}^2$,

$$b \cdot S(0) \, b = -\frac{3}{4\pi} \int_{|w| > 1} \frac{(b \cdot w)\big((b \times w) \cdot \Omega(w)\big)}{|w|^5} \, dw = -\frac{3}{4\pi} \int_1^\infty \frac{dr}{r} \int_{\mathbb{S}^2} (b \cdot \theta)\big((b \times \theta) \cdot \Omega(r\theta)\big) \, d\theta.$$

Proof. Since Ω is supported in $\{|w| > 1\}$, the kernel $x \mapsto (x - w)/|x - w|^3$ is harmonic on $B_1(0)$ for each fixed w in the support, so u is harmonic on $B_1(0)$ and smooth there. Differentiate under the integral sign (justified by smoothness and the separation of the support from B_1), write $r = x - w$ and $f(r) := r/|r|^3$ so that $u(x) = \frac{1}{4\pi} \int f(x - w) \times \Omega(w) \, dw$. For $i, k \in \{1, 2, 3\}$,

$$\partial_{x_i} f_k(r) = \partial_{r_i}\big(r_k |r|^{-3}\big) = \delta_{ik} |r|^{-3} - 3 \, r_i r_k |r|^{-5}.$$

Hence at $x = 0$ (so $r = -w$),

$$\partial_{x_i} u(0) = \frac{1}{4\pi} \int \big(\partial_{x_i} f(-w)\big) \times \Omega(w) \, dw.$$

When we take the symmetric part $\frac{1}{2}(\nabla u(0) + \nabla u(0)^T)$, the $\delta_{ik}|w|^{-3}$ contribution cancels, leaving the stated identity. Taking the trace gives $\operatorname{tr} S(0) \propto \int (w \times \Omega) \cdot w \, |w|^{-5} \, dw = 0$. The directional formula follows by contracting with $b \otimes b$ and observing $b \cdot (w \times \Omega) = (b \times w) \cdot \Omega$, followed by the change of variables $w = r\theta$. $\square$

1.2. **RM2 in one line.**

Corollary 1.2 (RM2 $\iff$ bounded $\ell = 2$ tail moment (informal equivalence))**.** *In the running-max extraction, the fixed-frame compactness obstruction (RM2) is generated by the affine/harmonic velocity mode induced by the far field. Lemma 1.1 identifies the corresponding coefficient at the blow-up center as a borderline log-shell $\ell = 2$ tail moment. Accordingly, any RM2 closure must control (uniformly or integrably, along the blow-up sequence) the induced tail strain matrix $S(0, t)$.*

Remark 1.3. Corollary 1.2 is an equivalence "at the level of the program" (it is not a new PDE theorem by itself): it pinpoints *what* must be controlled to make fixed-frame limits exist.

2. A SINGLE GLOBAL GATE THAT WOULD CLOSE RM2 (RM2U VIA UEWE)

2.1. **The RM2U target (coercive $\ell = 2$ control).**

Definition 2.1 (The transverse $\ell = 2$ coefficient)**.** Fix $b \in \mathbb{S}^2$ and define the tangential test field on $\mathbb{S}^2$ by

$$\Phi_b(\theta) := (b \cdot \theta)(\theta \times b), \qquad \theta \in \mathbb{S}^2.$$

For a smooth vorticity field $\omega(x, t)$ define

$$A_b(r, t) := \int_{\mathbb{S}^2} \omega(r\theta, t) \cdot \Phi_b(\theta) \, d\theta.$$

Theorem 2.2 (RM2U target: coercive $\ell = 2$ tail control)**.** *[OPEN TARGET / GLOBAL GATE.] For the running-max/vorticity-normalized ancient element $\omega^\infty = \mathrm{curl} u^\infty$, show there exists $K < \infty$ such that for all $t \le 0$,*

$$\sup_{b \in \mathbb{S}^2} \left(\int_1^\infty \left| (\partial_r A_b^\infty)(r, t) \right|^2 r^2 \, dr \; + \; \int_1^\infty \left| A_b^\infty(r, t) \right|^2 dr \right) \; \le \; K.$$

2.2. **A clean sufficient condition: UEWE.**

Theorem 2.3 (UEWE target: uniform exterior weighted enstrophy)**.** *[OPEN TARGET / GLOBAL GATE.] For the running-max/vorticity-normalized ancient element $\omega^\infty = \mathrm{curl} u^\infty$, show there exists $M < \infty$ such that for all $t \le 0$,*

$$\int_{|x| \ge 1} \left(\frac{|\omega^\infty(x, t)|^2}{|x|^2} + |\nabla \omega^\infty(x, t)|^2 \right) dx \; \le \; M.$$

Lemma 2.4 (UEWE $\Rightarrow$ coercive $\ell = 2$ control)**.** *Fix a time t and let $\omega(\cdot, t)$ be smooth. If*

$$\int_{|x| \ge 1} \left(\frac{|\omega(x, t)|^2}{|x|^2} + |\nabla \omega(x, t)|^2 \right) dx \; \le \; M,$$

then for every $b \in \mathbb{S}^2$ the corresponding coefficient satisfies

$$\int_1^\infty \left| (\partial_r A_b)(r, t) \right|^2 r^2 \, dr \; + \; \int_1^\infty \left| A_b(r, t) \right|^2 dr \; \le \; C M,$$

with a universal constant C (depending only on $\|\Phi_b\|_{L^2(\mathbb{S}^2)}^2$, hence independent of b).

Proof. By Cauchy–Schwarz on $\mathbb{S}^2$,

$$|A_b(r, t)| = \left| \int_{\mathbb{S}^2} \omega(r\theta, t) \cdot \Phi_b(\theta) \, d\theta \right| \le \|\omega(r\cdot, t)\|_{L^2(\mathbb{S}^2)} \|\Phi_b\|_{L^2(\mathbb{S}^2)}.$$

Squaring and integrating in $r \in [1, \infty)$ gives

$$\int_1^\infty |A_b(r, t)|^2 \, dr \le \|\Phi_b\|_{L^2(\mathbb{S}^2)}^2 \int_{|x| \ge 1} \frac{|\omega(x, t)|^2}{|x|^2} \, dx.$$

Similarly,
$$(\partial_r A_b)(r,t) = \int_{\mathbb{S}^2} (\partial_r \omega)(r\theta, t) \cdot \Phi_b(\theta) \, d\theta,$$
so the same Cauchy–Schwarz bound yields
$$\int_1^\infty |(\partial_r A_b)(r,t)|^2 \, r^2 \, dr \le \|\Phi_b\|_{L^2(\mathbb{S}^2)}^2 \int_{|x|\ge 1} |\partial_r \omega(x,t)|^2 \, dx \le \|\Phi_b\|_{L^2(\mathbb{S}^2)}^2 \int_{|x|\ge 1} |\nabla \omega(x,t)|^2 \, dx.$$
Adding the two bounds gives the claim with $C = \|\Phi_b\|_{L^2(\mathbb{S}^2)}^2$ (uniform in b by rotational invariance). $\square$

2.3. Coercive $\ell = 2$ control bounds the RM2 tail moment.

Theorem 2.5 (Coercive $\ell = 2$ control $\Rightarrow$ bounded tail moment). *Assume the RM2U coercive bound from Theorem 2.2 holds with constant K (uniformly for all $t \le 0$). Then for every $t \le 0$ the log-critical shell moments*
$$\Sigma_b^{1,\infty}(t) := \int_1^\infty \frac{A_b^\infty(r,t)}{r} \, dr$$
converge absolutely and satisfy $\sup_{b\in\mathbb{S}^2} |\Sigma_b^{1,\infty}(t)| \le K^{1/2}$. In particular, the induced tail strain matrix $S(0,t)$ (as in Lemma 1.1) is uniformly bounded in $t \le 0$.

Proof. Fix b and apply Cauchy–Schwarz:
$$\int_1^\infty \frac{|A_b^\infty(r,t)|}{r} \, dr \le \left(\int_1^\infty |A_b^\infty(r,t)|^2 \, dr \right)^{1/2} \left(\int_1^\infty \frac{dr}{r^2} \right)^{1/2} \le K^{1/2}.$$
This gives absolute convergence and the uniform bound on $\Sigma_b^{1,\infty}$. The tail strain bound then follows by the directional identity in Lemma 1.1: for each b, $b \cdot S(0,t)b$ is proportional to $\Sigma_b^{1,\infty}(t)$. $\square$

3. The single missing step (open problem)

Remark 3.1 (Single global bottleneck). All implications in this note are elementary once the correct coefficient/moment is identified. The only missing analytic input needed to close RM2 via RM2U is a *genuine global tightness* estimate for the running-max ancient element, such as UEWE (Theorem 2.3) or an equivalent relative tail depletion (RTD) statement in blow-up variables.