

# Stable Direction Dictionaries for Strongly Positive ( $p, p$ )-Forms via Regularized Simplex Fits

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## Abstract

Let  $(X, \omega)$  be a compact Kähler manifold of complex dimension  $n$ , and let  $K_p(x)$  denote the cone of strongly positive  $(p, p)$ -covectors at  $x \in X$ . A recurring obstruction in mesh-based constructions is the absence of a stable way to *recognize* and label directions in a cone-valued form field  $\beta(x) \in K_p(x)$ : Carathéodory decompositions are highly non-unique and vary discontinuously, preventing coherent direction labeling across adjacent cells.

We introduce a dictionary-based recognizer. Fix an  $\varepsilon$ -net  $\{P_1, \dots, P_M\}$  in the calibrated (complex) Grassmannian of  $(n - p)$ -planes, and let  $\xi_i(x)$  be the associated normalized ray generators satisfying  $\langle \xi_i(x), \psi_x \rangle = 1$  where  $\psi = \omega^{n-p}/(n - p)!$  is the Kähler calibration. For each normalized target  $b(x)$  we define a recognition state  $w(x) \in \Delta_M$  by a strongly convex regularized least-squares fit on the simplex. We prove existence, uniqueness, and a Lipschitz bound

$$\|w(b) - w(b')\| \leq (\|A\|_{\text{op}}/\lambda) \|b - b'\|$$

for the weight map, where  $A : \mathbb{R}^M \rightarrow V$  is the dictionary synthesis operator  $Aw = \sum_i w_i \xi_i$  and  $\lambda > 0$  is the regularization strength. This yields stable, globally labeled coefficients that vary at the same scale as  $b(x)$ .

To connect with literal finite-resolution recognition, we consider derived finite event maps such as the winner-take-all label  $\arg \max_i w_i$  and show (via a margin lemma) that these discrete labels are robust under perturbations away from ties. Finally, we show how pointwise weights induce coherent per-cell mass budgets  $M_{Q,i}$  on a mesh, isolating direction-label stability as a quantitative choice of dictionary resolution  $\varepsilon$  and regularization  $\lambda$ , independent of later holomorphic or geometric-measure steps.

# 1 Recognition primitives

We record the recognition primitives used in this paper (state space, measurement map, quotient, and robustness modulus  $r(s)$ ). The purpose is to make explicit what structure is assumed and what structure is produced.

**Definition 1** (State space / configuration space). A *state space* is a pair  $(\mathcal{C}, d_{\mathcal{C}})$  where  $\mathcal{C}$  is a set and  $d_{\mathcal{C}}$  is a metric on  $\mathcal{C}$ . Elements  $c \in \mathcal{C}$  are called *states* (or *configurations*).

**Definition 2** (Measurement map / recognizer). Let  $(\mathcal{C}, d_{\mathcal{C}})$  be a state space and let  $\mathcal{E}$  be a set (the *event space*). A *measurement map* (or *recognizer*) is a function

$$\mathcal{R} : \mathcal{C} \rightarrow \mathcal{E}.$$

**Definition 3** (Indistinguishability and quotient). Given  $\mathcal{R} : \mathcal{C} \rightarrow \mathcal{E}$ , define an equivalence relation on  $\mathcal{C}$  by

$$c_1 \sim_{\mathcal{R}} c_2 \iff \mathcal{R}(c_1) = \mathcal{R}(c_2).$$

The *recognition quotient* is the quotient set  $\mathcal{C}_{\mathcal{R}} := \mathcal{C} / \sim_{\mathcal{R}}$ .

**Definition 4** (Robustness modulus  $r(s)$ ). Assume  $(\mathcal{E}, d_{\mathcal{E}})$  is a metric space. The *robustness modulus* of  $\mathcal{R}$  is the function  $r : [0, \infty] \rightarrow [0, \infty]$  defined by

$$r(s) := \sup \left\{ d_{\mathcal{E}}(\mathcal{R}(c), \mathcal{R}(c')) : d_{\mathcal{C}}(c, c') \leq s \right\}.$$

**Remark 1** (Lipschitz recognizers). If  $\mathcal{R} : (\mathcal{C}, d_{\mathcal{C}}) \rightarrow (\mathcal{E}, d_{\mathcal{E}})$  is  $L$ -Lipschitz, then  $r(s) \leq Ls$  for all  $s \geq 0$ .

## Assumptions and non-assumptions (no smuggled observables)

- **Given structure (assumptions).**
  - A compact Kähler manifold  $(X, \omega)$  and the induced norms on forms.
  - The calibration functional  $\alpha \mapsto \langle \alpha, \psi_x \rangle$  used to normalize  $K_p(x)$  to  $\Sigma_p(x)$ .
  - A finite dictionary  $\{\xi_i(x)\}_{i=1}^M \subset \Sigma_p(x)$  (a finite-resolution hypothesis).
  - A regularization strength  $\lambda > 0$ .
- **Not assumed (anti-smuggling constraints).**

- No canonical or continuous Carathéodory/extremal-ray decomposition is assumed.
- No globally uniform “true direction labels” are assumed; labels are produced as dictionary indices.
- No circular definition of observables: events/labels are defined by an explicit map  $\mathcal{R}$ , not by presupposing the quotient classes we intend to construct.

### Self-similarity and the golden ratio (optional scale lemma)

In zero-parameter self-similar scale updates, one often encounters the fixed-point equation  $x = 1 + 1/x$ . We record the elementary consequence for reference.

**Lemma 1** (Self-similarity fixed point forces  $\varphi$ ). *The equation  $x = 1 + \frac{1}{x}$  has exactly one positive solution, namely  $\varphi = (1 + \sqrt{5})/2$ .*

*Proof.* Rearranging gives  $x^2 = x + 1$ , i.e.  $x^2 - x - 1 = 0$ . The quadratic formula yields the two roots  $x = (1 \pm \sqrt{5})/2$ , and exactly one is positive.  $\square$

## 2 Introduction

Strongly positive  $(p,p)$ -forms on a Kähler manifold behave like “nonnegative densities with direction,” in the sense that they live in a closed convex cone whose extreme rays correspond to simple algebraic directions. In many constructions one wants to discretize a strongly positive form field  $\beta(x)$  on a mesh and propagate direction-dependent budgets from cell to cell. The immediate obstacle is not existence of decompositions but *recognition*: a pointwise decomposition of  $\beta(x)$  into extremal rays is not canonical. Even when decompositions exist, they are highly non-unique and can jump discontinuously as  $x$  varies. This makes direction labels unstable: adjacent mesh cells may use incompatible “names” for nearly the same direction.

Recognition Geometry suggests a measurement-first reframing: *without a recognizer, there is no stable labeling*. In our setting, at each  $x$  the *configuration space* is the normalized cone slice  $\mathcal{C}_x := \Sigma_p(x)$ , and we choose a finite dictionary  $\{\xi_i(x)\}_{i=1}^M \subset \Sigma_p(x)$  as an explicit finite-resolution hypothesis. We then define a *recognizer*  $\mathcal{R}_\lambda$  that maps a configuration  $b \in \mathcal{C}_x$  to a recognition state  $w = \mathcal{R}_\lambda(b) \in \Delta_M$  by a strongly convex simplex fit. Strong convexity forces uniqueness, and a monotonicity argument yields a clean Lipschitz robustness bound with an explicit constant.

To align literally with a finite-resolution event axiom, we also consider derived *finite* event maps, such as the winner-take-all label  $\arg \max_i w_i$ . These induce an indistinguishability relation  $b \sim_{\mathcal{R}} b'$  on  $\mathcal{C}_x$  and hence a recognition quotient  $\mathcal{C}_x / \sim_{\mathcal{R}}$  whose classes are “direction-label resolution cells.” The stability results in this paper quantify how large a perturbation in  $b$  is required to change the recognition state (and, away from ties, the discrete label).

The resulting mechanism has two independent stability knobs: dictionary resolution  $\varepsilon$  (approximation quality) and regularization  $\lambda$  (robustness of the recognizer).

### 3 Strong positivity and the normalized slice

Let  $(X, \omega)$  be a compact Kähler manifold of complex dimension  $n$ . Fix  $p \in \{0, 1, \dots, n\}$ . At each  $x \in X$ , let

$$V_x := \Lambda_{\mathbb{R}}^{p,p} T_x^* X$$

denote the real vector space of real  $(p, p)$ -covectors at  $x$ .

**Definition 5** (Strongly positive cone). A covector  $\alpha \in V_x$  is *strongly positive* if it is a finite sum of forms of the type

$$\left(\frac{i}{2}\right)^p \eta_1 \wedge \bar{\eta_1} \wedge \cdots \wedge \eta_p \wedge \bar{\eta_p},$$

where  $\eta_1, \dots, \eta_p \in \Lambda^{1,0} T_x^* X$ . The set of strongly positive covectors is a closed convex cone, denoted  $K_p(x) \subset V_x$ .

Define the Kähler calibration of degree  $(n-p, n-p)$  by

$$\psi := \frac{\omega^{n-p}}{(n-p)!} \in \Lambda_{\mathbb{R}}^{n-p, n-p} T_x^* X.$$

Let  $\text{vol}_{\omega} := \omega^n / n!$ .

**Definition 6** (Pairing with  $\psi$ ). For  $\alpha \in \Lambda_{\mathbb{R}}^{p,p} T_x^* X$  and  $\eta \in \Lambda_{\mathbb{R}}^{n-p, n-p} T_x^* X$ , define the scalar  $\langle \alpha, \eta \rangle$  by

$$\alpha \wedge \eta = \langle \alpha, \eta \rangle \text{vol}_{\omega}(x).$$

In particular,  $\langle \alpha, \psi_x \rangle$  is the  $\psi$ -trace of  $\alpha$ .

**Lemma 2** (Positivity of the  $\psi$ -trace). *For every  $x \in X$  and every  $\alpha \in K_p(x)$  one has  $\langle \alpha, \psi_x \rangle \geq 0$ . Moreover, if  $\alpha \in K_p(x)$  and  $\langle \alpha, \psi_x \rangle = 0$ , then  $\alpha = 0$ .*

*Proof.* In unitary coordinates at  $x$  with  $\omega = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ , each elementary strongly positive term

$$\left(\frac{i}{2}\right)^p \eta_1 \wedge \overline{\eta_1} \wedge \cdots \wedge \eta_p \wedge \overline{\eta_p}$$

has nonnegative wedge with  $\omega^{n-p}$ , hence nonnegative pairing with  $\psi_x$ . Summing preserves nonnegativity, giving  $\langle \alpha, \psi_x \rangle \geq 0$ .

If  $\langle \alpha, \psi_x \rangle = 0$  for  $\alpha \in K_p(x)$ , then every elementary term in a positive decomposition must also have zero pairing with  $\psi_x$ . In unitary coordinates, the wedge of a nonzero elementary term with  $\omega^{n-p}$  is strictly positive, so each term must be zero, hence  $\alpha = 0$ .  $\square$

**Definition 7** (Normalized slice). Define the normalized slice

$$\Sigma_p(x) := \{v \in K_p(x) : \langle v, \psi_x \rangle = 1\}.$$

The set  $\Sigma_p(x)$  is the natural compact base of the cone  $K_p(x)$  once we fix  $\psi_x$  as a strictly positive functional on  $K_p(x) \setminus \{0\}$ .

**Remark 2** (Configuration space viewpoint). For fixed  $x$ , the normalized slice  $\Sigma_p(x)$  is the space of normalized strongly positive directions. In later applications one often decomposes a form field  $\beta(x) \in K_p(x)$  as  $\beta(x) = \rho(x) b(x)$  where  $\rho(x) = \langle \beta(x), \psi_x \rangle \geq 0$  is the total density and  $b(x) \in \Sigma_p(x)$  is a normalized direction. The recognition problem addressed in this paper is to turn  $b(x)$  into stable labels and weights.

## 4 Calibrated rays and normalized ray generators

For Kähler calibrations, calibrated  $(2n - 2p)$ -planes are precisely complex  $(n-p)$ -planes. Let  $G_{n-p}^{\mathbb{C}}(T_x X)$  denote the Grassmannian of complex  $(n-p)$ -dimensional subspaces of  $T_x X$ .

**Definition 8** (Normalized ray generator associated to a complex plane). Fix  $x \in X$  and  $P \in G_{n-p}^{\mathbb{C}}(T_x X)$ . Choose a unitary frame  $e_1, \dots, e_n$  of  $T_x^{1,0} X$  such that  $P^{1,0} = \text{span}\{e_{p+1}, \dots, e_n\}$ , and let  $\zeta^1, \dots, \zeta^n$  be the dual coframe. Define

$$\xi_P(x) := \left(\frac{i}{2}\right)^p \zeta^1 \wedge \overline{\zeta^1} \wedge \cdots \wedge \zeta^p \wedge \overline{\zeta^p} \in \Lambda_{\mathbb{R}}^{p,p} T_x^* X.$$

**Lemma 3** (Well-definedness and normalization). *The covector  $\xi_P(x)$  is independent of the choice of unitary frame adapted to  $P$ . Moreover,  $\xi_P(x) \in \Sigma_p(x)$ , i.e.  $\xi_P(x)$  is strongly positive and satisfies  $\langle \xi_P(x), \psi_x \rangle = 1$ .*

*Proof.* If two unitary frames are adapted to the same splitting  $T_x^{1,0}X \cong \mathbb{C}^p \oplus \mathbb{C}^{n-p}$ , they differ by an element of  $U(p) \times U(n-p)$ . The form

$$\left(\frac{i}{2}\right)^p \zeta^1 \wedge \overline{\zeta^1} \wedge \cdots \wedge \zeta^p \wedge \overline{\zeta^p}$$

is invariant under  $U(p)$ , hence well-defined.

Strong positivity is immediate from the construction. In unitary coordinates with  $\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j$  and  $P = \text{span}\{\partial/\partial z_{p+1}, \dots, \partial/\partial z_n\}$ , one has

$$\xi_P(x) \wedge \frac{\omega^{n-p}}{(n-p)!} = \frac{\omega^n}{n!},$$

so  $\langle \xi_P(x), \psi_x \rangle = 1$ .  $\square$

Let

$$S_p(x) := \{\xi_P(x) : P \in G_{n-p}^{\mathbb{C}}(T_x X)\} \subset \Sigma_p(x).$$

The set  $S_p(x)$  is compact because the Grassmannian is compact and  $P \mapsto \xi_P(x)$  is continuous.

**Proposition 1** (Convex generation of the normalized slice). *For each  $x \in X$ , the normalized slice  $\Sigma_p(x)$  is the convex hull of  $S_p(x)$ :*

$$\Sigma_p(x) = \text{conv } S_p(x).$$

Equivalently, the cone  $K_p(x)$  is the convex cone generated by  $S_p(x)$ .

*Proof.* Every  $\xi_P(x)$  lies in  $\Sigma_p(x)$ , and  $\Sigma_p(x)$  is convex, so  $\text{conv } S_p(x) \subseteq \Sigma_p(x)$ .

For the reverse inclusion, take  $v \in \Sigma_p(x)$ . By strong positivity,  $v$  is a finite sum of elementary strongly positive forms. After scaling each summand by its  $\psi$ -trace and then renormalizing, we write

$$v = \sum_{j=1}^N a_j v_j, \quad a_j \geq 0, \quad \sum_{j=1}^N a_j = 1,$$

where each  $v_j$  is a normalized elementary strongly positive form with  $\langle v_j, \psi_x \rangle = 1$ . Any such normalized elementary form is equal to  $\xi_P(x)$  for some complex  $(n-p)$ -plane  $P$  (its nullspace in  $T_x^{1,0}X$  determines  $P$ ), hence  $v_j \in S_p(x)$ . Therefore  $v \in \text{conv } S_p(x)$ .  $\square$

## 5 Regularized simplex recognizers

This section is pointwise and finite-dimensional, and establishes existence, uniqueness, and Lipschitz stability for the dictionary recognizer. Fix a real inner-product space  $(V, \langle \cdot, \cdot \rangle_V)$ ; in applications  $V = V_x$  with the norm induced by the Kähler metric.

Fix dictionary vectors  $\xi_1, \dots, \xi_M \in V$ . Define the synthesis operator

$$A : \mathbb{R}^M \rightarrow V, \quad Aw := \sum_{i=1}^M w_i \xi_i.$$

Let  $\|\cdot\|$  denote the Euclidean norm on  $\mathbb{R}^M$  and the induced operator norm  $\|A\|_{\text{op}}$ .

Define the simplex

$$\Delta_M := \left\{ w \in \mathbb{R}^M : w_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^M w_i = 1 \right\}.$$

**Definition 9** (Regularized simplex recognizer and  $J$ -step). Fix  $\lambda > 0$ . For a target  $b \in V$ , define the objective

$$J_b(w) := \frac{1}{2} \|Aw - b\|_V^2 + \frac{\lambda}{2} \|w\|^2, \quad w \in \Delta_M,$$

and define the weight map

$$w(b) := \operatorname{argmin}_{w \in \Delta_M} J_b(w).$$

We also write  $\mathcal{R}_\lambda(b) := w(b)$  and interpret  $\mathcal{R}_\lambda : V \rightarrow \Delta_M$  as a *recognizer* whose output  $w(b)$  is an internal recognition state in the simplex. We refer to  $w(b)$  as the (unique)  $J$ -step selected by the recognition cost  $J_b$ .

**Theorem 1** (Recognition closure lemma (RCL): uniqueness of the  $J$ -step). *For every  $b \in V$  and every  $\lambda > 0$ , the minimization problem defining  $w(b)$  has a unique solution in  $\Delta_M$ .*

*Proof.* The simplex  $\Delta_M$  is nonempty, compact, and convex. The map  $w \mapsto \frac{1}{2} \|Aw - b\|_V^2$  is convex, and  $w \mapsto \frac{\lambda}{2} \|w\|^2$  is  $\lambda$ -strongly convex. Hence  $J_b$  is  $\lambda$ -strongly convex on  $\mathbb{R}^M$ , so it has at most one minimizer on any convex set. By continuity on a compact set, a minimizer exists, and by strong convexity it is unique.  $\square$

A convenient characterization is via a variational inequality.

**Lemma 4** (Variational inequality). *A point  $w_\star \in \Delta_M$  equals  $w(b)$  if and only if*

$$\langle A^*(Aw_\star - b) + \lambda w_\star, z - w_\star \rangle \geq 0 \quad \text{for all } z \in \Delta_M,$$

where  $A^* : V \rightarrow \mathbb{R}^M$  is the adjoint of  $A$  and  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product on  $\mathbb{R}^M$ .

*Proof.* This is the standard first-order optimality condition for minimizing a differentiable convex function over a closed convex set. The gradient is  $\nabla J_b(w) = A^*(Aw - b) + \lambda w$ .  $\square$

**Theorem 2** (Lipschitz stability in the target). *For every  $\lambda > 0$  and all  $b, b' \in V$ ,*

$$\|w(b) - w(b')\| \leq \frac{\|A\|_{\text{op}}}{\lambda} \|b - b'\|_V.$$

*Proof.* Let  $w := w(b)$  and  $w' := w(b')$ . Apply the variational inequality with  $z = w'$ :

$$\langle A^*(Aw - b) + \lambda w, w' - w \rangle \geq 0.$$

Apply it again with  $(b', w')$  and  $z = w$ :

$$\langle A^*(Aw' - b') + \lambda w', w - w' \rangle \geq 0,$$

which is equivalent to

$$\langle A^*(Aw' - b') + \lambda w', w' - w \rangle \leq 0.$$

Adding the two inequalities gives

$$\langle A^*A(w - w') + \lambda(w - w') - A^*(b - b'), w' - w \rangle \geq 0.$$

Let  $d := w - w'$ . Then  $w' - w = -d$ , and we obtain

$$-\|Ad\|_V^2 - \lambda\|d\|^2 + \langle A^*(b - b'), d \rangle \geq 0.$$

Dropping the nonpositive term  $-\|Ad\|_V^2$  yields

$$\lambda\|d\|^2 \leq \langle A^*(b - b'), d \rangle \leq \|A^*(b - b')\| \|d\|.$$

If  $d = 0$  there is nothing to prove. Otherwise divide by  $\lambda\|d\|$  to get

$$\|d\| \leq \frac{\|A^*(b - b')\|}{\lambda} \leq \frac{\|A^*\|_{\text{op}}}{\lambda} \|b - b'\|_V.$$

Since  $\|A^*\|_{\text{op}} = \|A\|_{\text{op}}$ , the claim follows.  $\square$

**Corollary 1** (Robustness modulus for  $\mathcal{R}_\lambda$ ). *Equip  $V$  with the metric induced by  $\|\cdot\|_V$  and equip  $\Delta_M \subset \mathbb{R}^M$  with the Euclidean norm. Then the robustness modulus  $r(s)$  of  $\mathcal{R}_\lambda : V \rightarrow \Delta_M$  satisfies*

$$r(s) \leq \frac{\|A\|_{\text{op}}}{\lambda} s \quad \text{for all } s \geq 0.$$

*Proof.* This is immediate from the Lipschitz theorem and the definition of  $r(s)$ .  $\square$

### Finite-resolution events and recognition quotients

The simplex state  $w(b) \in \Delta_M$  varies continuously with  $b$  (indeed, Lipschitz), but  $\Delta_M$  is not finite. If one wants a literal finite event space, one can post-process  $w(b)$  by a finite rule.

**Definition 10** (Winner-take-all event map). Let  $\mathcal{E} := \{1, \dots, M\}$ . Define the event map  $E_\lambda : V \rightarrow \mathcal{E}$  by

$$E_\lambda(b) := \min \left\{ i : w_i(b) = \max_{1 \leq j \leq M} w_j(b) \right\}.$$

Thus  $E_\lambda(b)$  is a deterministic tie-broken version of  $\arg \max_i w_i(b)$ .

As above,  $E_\lambda$  induces an indistinguishability relation  $b \sim_{\mathcal{R}} b'$  on any  $\mathcal{C} \subseteq V$  (e.g.  $\mathcal{C} = \Sigma_p(x)$ ) by the rule  $E_\lambda(b) = E_\lambda(b')$ , and hence a recognition quotient  $\mathcal{C}/\sim_{\mathcal{R}}$ .

**Lemma 5** (Margin implies stable discrete recognition). *Let  $b \in V$  and let  $i_\star := E_\lambda(b)$ . Suppose there exists  $\delta > 0$  such that*

$$w_{i_\star}(b) \geq w_j(b) + 2\delta \quad \text{for all } j \neq i_\star.$$

*Then for every  $b' \in V$  with  $\|b - b'\|_V \leq (\lambda \delta)/\|A\|_{\text{op}}$ , one has  $E_\lambda(b') = i_\star$ .*

*Proof.* By the Lipschitz theorem,  $\|w(b) - w(b')\| \leq (\|A\|_{\text{op}}/\lambda)\|b - b'\|_V \leq \delta$ . In particular, for each coordinate one has  $|w_j(b) - w_j(b')| \leq \|w(b) - w(b')\| \leq \delta$ . Hence for all  $j \neq i_\star$ ,

$$w_{i_\star}(b') \geq w_{i_\star}(b) - \delta \geq w_j(b) + \delta \geq w_j(b').$$

So  $i_\star$  remains the unique maximizer, and the tie-broken arg max is unchanged:  $E_\lambda(b') = i_\star$ .  $\square$

**Remark 3** (Quantized simplex events). An alternative finite-resolution bridge is to quantize the simplex output itself. For example, fix a finite set  $W \subset \Delta_M$  (a  $\tau$ -net in the Euclidean norm) and define a quantizer  $Q : \Delta_M \rightarrow W$  by selecting a nearest element of  $W$  (with deterministic tie-breaking). Then  $Q \circ \mathcal{R}_\lambda : V \rightarrow W$  is a finite event-valued recognizer, and the Lipschitz bound controls how rapidly the internal state can move relative to the quantization scale. As with winner-take-all, discrete stability is governed by separation from the quantizer’s decision boundaries.

**Remark 4** (Interpretation of the constant). The parameter  $\lambda$  is a stability knob. Larger  $\lambda$  improves Lipschitz stability but biases weights toward smaller Euclidean norm (more “spread” in many common geometries). Smaller  $\lambda$  reduces bias but amplifies sensitivity. The operator norm  $\|A\|_{\text{op}}$  measures how large a change in the synthesized form  $Aw$  can be produced by a unit change in weights.

**Remark 5** (Fieldwise versions). In geometric applications  $V$  and the dictionary vary with  $x$ . After choosing local trivializations, the pointwise Lipschitz estimate combines with standard stability estimates for compositions to yield fieldwise regularity statements on charts; we omit these routine details.

## 6 Approximation error versus dictionary resolution

The Lipschitz theorem controls *stability* of weights given a fixed dictionary. Separately, one also wants the dictionary itself to approximate the continuous set of calibrated ray generators.

We first record a basic convex-hull approximation lemma.

**Lemma 6** (Convex-hull approximation from an  $\varepsilon$ -net). *Let  $(V, \|\cdot\|_V)$  be a normed vector space and let  $S \subset V$  be compact. Suppose  $\Xi = \{\xi_1, \dots, \xi_M\} \subset S$  is an  $\varepsilon$ -net for  $S$ , meaning that for every  $s \in S$  there exists  $\xi_i \in \Xi$  with  $\|s - \xi_i\|_V \leq \varepsilon$ . Then for every  $b \in \text{conv } S$ ,*

$$\text{dist}(b, \text{conv } \Xi) \leq \varepsilon.$$

*Proof.* Write  $b = \sum_{j=1}^N a_j s_j$  with  $s_j \in S$ ,  $a_j \geq 0$ , and  $\sum_j a_j = 1$ . For each  $j$ , choose  $\xi_{i(j)} \in \Xi$  with  $\|s_j - \xi_{i(j)}\|_V \leq \varepsilon$ . Define  $b' := \sum_j a_j \xi_{i(j)} \in \text{conv } \Xi$ . Then

$$\|b - b'\|_V \leq \sum_{j=1}^N a_j \|s_j - \xi_{i(j)}\|_V \leq \sum_{j=1}^N a_j \varepsilon = \varepsilon.$$

Hence  $\text{dist}(b, \text{conv } \Xi) \leq \varepsilon$ .  $\square$

Now specialize to the Kähler setting at a point  $x$ . Recall  $S_p(x) = \{\xi_P(x)\}$  and  $\Sigma_p(x) = \text{conv } S_p(x)$ .

**Corollary 2** (Pointwise approximation on  $\Sigma_p(x)$ ). *Fix  $x \in X$ . If  $\Xi(x) = \{\xi_1(x), \dots, \xi_M(x)\} \subset S_p(x)$  is an  $\varepsilon$ -net for  $S_p(x)$  (in the norm on  $V_x$ ), then every  $b \in \Sigma_p(x)$  satisfies*

$$\text{dist}(b, \text{conv } \Xi(x)) \leq \varepsilon.$$

**Remark 6** (From plane-nets to  $\xi$ -nets). On the compact Grassmannian  $G_{n-p}^{\mathbb{C}}(T_x X)$ , the map  $P \mapsto \xi_P(x)$  is smooth, hence Lipschitz with some constant  $C = C(n, p)$  once one fixes the standard Grassmann metric induced by  $\omega_x$ . Therefore an  $\varepsilon$ -net in plane-angle distance induces a  $C\varepsilon$ -net in  $\|\cdot\|_{V_x}$ -distance among ray generators. The constant  $C$  is uniform when  $(X, \omega)$  has bounded geometry on the region of interest.

**Remark 7** (Choosing  $\varepsilon$  relative to a mesh size  $h$ ). In mesh-based constructions on cells of diameter  $h$ , one often needs dictionary approximation errors that are negligible compared to  $h$ -scale variations of the target field. A typical quantitative regime is to choose a resolution  $\varepsilon_h$  satisfying  $\varepsilon_h = o(h)$  as  $h \rightarrow 0$ . This ensures that dictionary-induced approximation errors do not dominate the geometric errors that scale linearly with the mesh diameter.

## 7 From pointwise weights to per-cell budgets on a mesh

Let  $\beta \in \Omega^{p,p}(X)$  be a continuous strongly positive form field, meaning  $\beta(x) \in K_p(x)$  for all  $x$ . Define the nonnegative density

$$\rho(x) := \langle \beta(x), \psi_x \rangle.$$

By the earlier lemma,  $\rho(x) = 0$  implies  $\beta(x) = 0$ .

On the set where  $\rho(x) > 0$ , define the normalized field

$$b(x) := \frac{\beta(x)}{\rho(x)} \in \Sigma_p(x).$$

Fix a dictionary field  $\{\xi_i(x)\}_{i=1}^M$  with  $\xi_i(x) \in \Sigma_p(x)$  and a regularization parameter  $\lambda > 0$ . Define weights

$$w(x) := w_x(b(x)) \in \Delta_M.$$

The associated dictionary reconstruction is the  $(p,p)$ -form

$$\beta_{\text{dict}}(x) := \rho(x) \sum_{i=1}^M w_i(x) \xi_i(x).$$

By construction  $\beta_{\text{dict}}(x)$  uses globally labeled directions indexed by  $i$ .

Now let  $U \subset X$  be a chart in which we place a cubical mesh of side length  $h > 0$ . For a mesh cell (cube)  $Q \subset U$ , define the per-cell mass budget assigned to label  $i$  by

$$M_{Q,i} := \int_Q w_i(x) \rho(x) dV_\omega(x),$$

where  $dV_\omega$  is the Riemannian volume measure of the Kähler metric.

**Lemma 7** (Basic identities). *For every cell  $Q$ ,*

$$\sum_{i=1}^M M_{Q,i} = \int_Q \rho(x) dV_\omega(x).$$

*Proof.* Since  $w(x) \in \Delta_M$ , one has  $\sum_i w_i(x) = 1$  pointwise, hence

$$\sum_{i=1}^M M_{Q,i} = \int_Q \left( \sum_{i=1}^M w_i(x) \right) \rho(x) dV_\omega(x) = \int_Q \rho(x) dV_\omega(x).$$

□

**Proposition 2** (Slow variation of budgets across adjacent cells). *Assume  $w_i \rho$  is Lipschitz on  $U$  with Lipschitz constant  $L_i$  (in the chart metric), and assume mesh cubes have side length  $h$ . If  $Q$  and  $Q'$  are adjacent cubes (sharing a codimension-one face), then*

$$|M_{Q,i} - M_{Q',i}| \leq C(n) L_i h \text{Vol}_\omega(Q),$$

where  $\text{Vol}_\omega(Q) = \int_Q dV_\omega$  and  $C(n)$  depends only on the real dimension  $2n$ .

*Proof.* Let  $f_i(x) := w_i(x)\rho(x)$ . For adjacent cubes  $Q, Q'$  of the same size, the difference of integrals of a Lipschitz function is controlled by the oscillation of  $f_i$  on  $Q \cup Q'$ . One convenient estimate is

$$\left| \int_Q f_i dV_\omega - \int_{Q'} f_i dV_\omega \right| \leq \int_{Q \cup Q'} |f_i(x) - f_i(x_0)| dV_\omega(x)$$

for a suitably chosen reference point  $x_0$  between the cubes. Since every point in  $Q \cup Q'$  lies within  $O(h)$  of  $x_0$ , Lipschitz continuity gives  $|f_i(x) - f_i(x_0)| \leq C(n)L_i h$  throughout  $Q \cup Q'$ , hence

$$|M_{Q,i} - M_{Q',i}| \leq C(n)L_i h \text{Vol}_\omega(Q \cup Q') \leq 2C(n)L_i h \text{Vol}_\omega(Q).$$

Absorb the factor 2 into  $C(n)$ .  $\square$

**Remark 8** (Relative form of the estimate). Since  $\text{Vol}_\omega(Q) \sim h^{2n}$ , the bound above is  $O(h^{2n+1})$  in absolute terms, which is  $O(h)$  relative to the typical cell mass scale. This is the precise sense in which budgets vary slowly across neighbors when the underlying field is Lipschitz.

## 8 Interface assumptions for later stages

This paper is designed to output only two types of information for downstream geometric constructions.

First, it outputs a fixed finite label set  $\{1, \dots, M\}$  (the dictionary indices) together with a stable recognizer output  $w(x) \in \Delta_M$  producing globally labeled coefficients. If one wants a literal finite-resolution recognizer, one may also pass forward the derived event map  $E_\lambda(x) := E_\lambda(b(x)) \in \{1, \dots, M\}$ ; the margin lemma above quantifies when this discrete output is robust.

Second, it outputs mesh-level budgets  $M_{Q,i}$  whose neighbor-to-neighbor variation can be controlled quantitatively in terms of regularity of the underlying form field and the choice of  $\lambda$  and dictionary resolution.

No holomorphic input is needed for these steps. In particular, one can treat the entire discussion above as a purely pointwise and mesh-level mechanism for turning a cone-valued field into stable labeled scalar densities.

## 9 Examples and variants

**Example 1** (The case  $p = 1$ : positive semidefinite Hermitian matrices). At a point  $x$ , a real  $(1, 1)$ -form corresponds (in unitary coordinates) to a Hermitian matrix. Strong positivity is positive semidefiniteness. The normalized slice  $\Sigma_1(x)$  is the set of positive semidefinite matrices with fixed trace against  $\omega^{n-1}$ . The calibrated rays are rank-one projectors, and a dictionary is a finite family of such projectors. The regularized simplex fit is then a stable way to express a positive semidefinite matrix as a convex combination of nearby rank-one directions with globally fixed labels.

**Remark 9** (Alternative regularizers). The quadratic regularizer  $\frac{\lambda}{2}\|w\|^2$  is chosen for two reasons: it makes the objective strongly convex and it produces a clean monotonicity argument giving the explicit Lipschitz constant. Other regularizers can be used, for example an entropic term  $\sum_i w_i \log w_i$  to enforce strictly positive weights in the simplex interior. Such choices typically yield smoother dependence on  $b$  but change the stability constant and the structure of optimality conditions.

**Remark 10** (Implementation-level stability). The optimization problem is a strongly convex quadratic program over the simplex. Strong convexity implies not only a unique solution but also good numerical conditioning as  $\lambda$  increases. In applications where  $M$  is moderate, projected gradient methods or active-set methods are standard. The theoretical Lipschitz bound is useful even if one never computes  $w$  exactly: it quantifies how much weight noise can be induced by target noise.

## Conclusion

By fixing a finite dictionary of normalized calibrated ray generators and selecting weights through a strongly convex simplex fit, we obtain a unique, stable, and Lipschitz *recognizer* for strongly positive  $(p, p)$ -forms. The recognition state  $w(b)$  provides globally labeled coefficients, and derived finite event maps (such as  $E_\lambda = \arg \max$  with deterministic tie-breaking) produce literal finite-resolution direction labels with explicit robustness margins. The induced labeled budgets on a mesh inherit slow-variation properties from the regularity of the underlying form field, with constants controlled by dictionary size/resolution and the regularization parameter  $\lambda$ . This isolates the recognition/labeling problem from subsequent geometric or holomorphic realization steps and provides a modular input for larger constructions.