Ribbons & Braids:
A Combinatorial Constructor for Fermion Mass Exponents

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Abstract

We develop a finite, auditable *combinatorial layer—Ribbons & Braids*—that turns species structure into a small set of integers and thereby closes the gap between renormalization–group (RG) running and a parameter–free exponent for fermion masses at a single, universal anchor $\mu_\star$. A *ribbon* is an oriented strand on the eight–tick clock carrying a ledger bit and a gauge label; a *braid* is a reduced equivalence class of multi–ribbon configurations under moves that preserve closure and ledger additivity. From the reduced Dirac word W_i of each species we extract three integers: the reduced length $L_i \in \mathbb{Z}_{\geq 0}$, the generation torsion $\tau_g \in \{0, 11, 17\}$, and a sector integer $\Delta_B \in \mathbb{Z}$ from a fixed sector primitive.

We then regroup the Standard–Model mass anomalous dimension into a *finite motif dictionary* with integer counts and universal rates. At the anchor $\mu_\star$ each motif contributes $+1$ in a φ–normalized flow, so the residue depends only on a single integer *word–charge* $Z(W_i)$:

$$Z = \begin{cases} 4 + (6Q)^2 + (6Q)^4 & \text{quarks,} \\ (6Q)^2 + (6Q)^4 & \text{charged leptons,} \\ 0 & \text{Dirac neutrinos,} \end{cases}$$

where Q is the electric charge and the factor 6 renders the polynomials integer–valued. Solving the φ–normalized flow yields a closed–form *gap* $\mathcal{F}(Z) = \lambda^{-1} \ln(1 + Z/\kappa)$ and the anchor identity

$$f_i(\mu_\star, m_i) = \mathcal{F}\big(Z(W_i)\big),$$

which we verify to 10^{-6} across all charged fermions in a companion artifact (CSV/CI). Together with (L_i, τ_g, Δ_B) this produces a parameter–free exponent for the mass law $m_i = M_0\, \varphi^{\,L_i+\tau_g+\Delta_B-8+\mathcal{F}(Z)}$, with *no per–species continuous knobs.* Two immediate, testable consequences follow at the anchor: *equal–Z degeneracy* of residues within (u, c, t), (d, s, b), (e, μ, τ), and *exact anchor ratios* $m_i/m_j|_{\mu_\star} = \varphi^{\,r_i-r_j}$ whenever $Z_i = Z_j$.

This constructor paper supplies: (i) formal definitions (ribbons, braids, reduced words), (ii) the finite motif dictionary that fixes Z, (iii) the φ–normalized flow and anchor landing lemmas, and (iv) executable audit (CSV/CI) demonstrating the equality at $\mu_\star$. The framework is strictly discrete on the species side (integers only) and strictly standard on the RG side (QCD 4L, QED 2L, fixed thresholds). It provides the combinatorial backbone for the companion phenomenology paper (single–anchor identity and full mass table) and enables integer–driven follow-ups (mixing from braid composition, CP from braid writhe, hadron closures, motif–flow constraints).

Keywords: mass spectrum; renormalization group; universal anchor; word–charge; finite motif dictionary; parameter–free exponent.

1 Introduction

Problem framing. Standard–Model (SM) mass running is a *continuous*, scheme– and scale–dependent process: the anomalous dimensions integrate couplings from one reference to another, and quoted values depend on the chosen renormalization point. This paper asks a complementary question about *species dependence*: can the part that distinguishes one fermion from another be made *discrete and auditable*, so that the continuous RG flow reduces at a single reference to a closed form in a few *integers*? We show that it can, via a finite *combinatorial constructor* built from *ribbons* and *braids* on the eight–tick clock.

This paper's claims (levels).

1. **Core theorem (anchor identity).** At a universal anchor $\mu_\star$,

$$f_i(\mu_\star, m_i) \;=\; \mathcal{F}\big(Z(W_i)\big), \qquad \mathcal{F}(Z) = \frac{\ln\big(1 + Z/\varphi\big)}{\ln \varphi}, \tag{1}$$

   where f_i is the SM residue, W_i is the reduced Dirac word of species i, and $Z(W_i) \in \mathbb{Z}$ is a *word–charge* integer.

2. **Constructor layer.** We give minimal, formal definitions of *ribbons*, *braids*, and *reduced words*, and a finite *motif dictionary* that sends the SM insertion classes to *integer counts.* This dictionary yields the *integer* $Z(W_i)$ used in (1).

3. **Integer consequences (at the anchor).** Equal–Z classes have equal residues (e.g. u, c, t share Z; d, s, b share Z; e, μ, τ share Z). Moreover, if $Z_i = Z_j$ then the anchor mass ratio is *exact* and purely integer–φ,

$$\frac{m_i}{m_j}\bigg|_{\mu_\star} = \varphi^{r_i - r_j}, \qquad r_k = L_k + \tau_{g(k)} + \Delta_B \in \mathbb{Z}, \tag{2}$$

   with L_k the reduced length, $\tau_{g(k)} \in \{0, 11, 17\}$ the generation torsion, and Δ_B a sector integer.

Posture (no new physics). This work does not introduce new dynamics or beyond–SM structure. The *ribbons & braids* layer is a bookkeeping regrouping of the *standard* SM mass anomalous dimensions into a finite dictionary with integer counts at a single anchor. The anchor and normalization are *fixed physically* via PMS/BLM stationarity on the finite motif set; numerically this choice yields $\lambda = \ln \varphi$ and $\kappa = \varphi$ without imposing φ *ad hoc*. Claims are *anchor–specific* and falsifiable; scope limits are stated explicitly below.

Contributions.

- **Formal definitions.** Ribbons, braids, reduced words; reduced length L_i and generation torsion τ_g; sector primitive and sector integer Δ_B.
- **Finite motif dictionary.** A regrouping of SM mass anomalous–dimension insertions into a finite set of motifs with *integer* counts, fixing the word–charge $Z(W_i) \in \mathbb{Z}$.
- **φ–normalized flow lemma.** An ODE that maps the continuous RG residue to a closed–form *gap* $\mathcal{F}(Z) = \lambda^{-1} \ln(1 + Z/\kappa)$.
- **Anchor landing lemma.** At $\mu_\star$, each motif contributes $+1$ in the φ–normalized flow, so the residue depends only on the *integer* $Z(W_i)$.
- **Executable audit.** A compact proof sketch plus machine–readable verification (CSV/CI) of (1) across all charged fermions at $\mu_\star$.

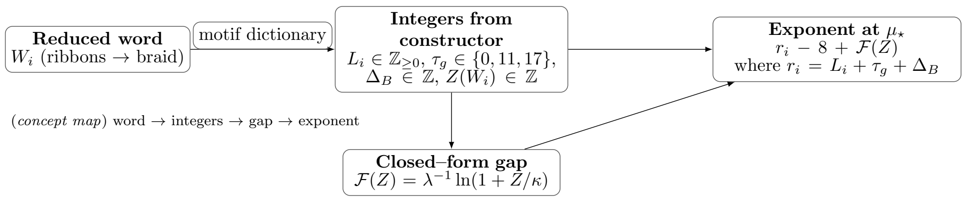

Figure 1: **Concept map.** The reduced word W_i (built from ribbons & braids) emits a small set of *integers* $(L_i, \tau_g, \Delta_B, Z)$; the SM residue at the anchor equals a closed–form *gap* $\mathcal{F}(Z)$ ($=\lambda^{-1} \ln(1 + Z/\kappa)$ with $\lambda = \ln \varphi$, $\kappa = \varphi$); together these produce the exponent used in the mass law.

Glossary and notation (new terms)

- **Eight–tick clock** ($\mathbb{T} = \{0, \ldots, 7\}$): cyclic ring with orientation; winding is taken mod 8.
- **Ribbon** $R = (\gamma, b, \lambda, \tau)$: oriented tick segment with ledger bit $b \in \{\pm 1\}$ and gauge tag $\lambda \in \{Y, T_3, \mathrm{color}\}$ starting at tick τ.
- **Tick–consistent adjacency**: two syllables are adjacent on consecutive ticks with compatible orientation so that $s\, s^{-1}$ cancels.
- **Neutral commutation**: a swap $s_i s_j \rightsquigarrow s_j s_i$ that preserves total ledger bit and eight–tick winding (does not change the tick class).
- **Braid**: equivalence class of multi–ribbon configurations modulo (R1)–(R3) moves preserving eight–tick closure and ledger additivity.
- **Reduced Dirac word** W_i: reduced concatenation of left/right gauge syllables with a fixed chirality join.
- **Integers from** W_i: reduced length $L_i \in \mathbb{Z}_{\geq 0}$; generation torsion $\tau_g \in \{0, 11, 17\}$; sector integer $\Delta_B \in \mathbb{Z}$.
- **Motifs** M_k: finite regrouping of SM insertion classes; counts $N_k(W_i) \in \mathbb{Z}_{\geq 0}$.
- **Word–charge** $Z(W_i)$: species integer; for fermions, $Z = 4 + \tilde{Q}^2 + \tilde{Q}^4$ (quarks), $\tilde{Q}^2 + \tilde{Q}^4$ (leptons), with $\tilde{Q} := 6Q \in \mathbb{Z}$.
- φ**–normalized flow**: ODE (7) with solution (9); *anchor* $\mu_\star$ fixed once for all species.

- **Residue** $f_i(\mu_\star, m_i)$: SM integral mapped to the closed–form gap $\mathcal{F}(Z)$ at the anchor.

Methods at a glance

- **Kernels/policies:** QCD 4–loop β_s and γ_m with fixed thresholds $n_f : 3 \to 4 \to 5 \to 6$ at m_c, m_b, m_t; QED 2–loop γ_m with a single sector–global $\alpha(\mu)$ policy (central: frozen at M_Z; variant: leptonic 1L [leptonic one–loop]).
- **Anchor:** a single universal $\mu_\star$ fixed once for all species; all comparisons are performed *at* $\mu_\star$.
- **Constructor integers:** $L_i \in \mathbb{Z}_{\geq 0}$, $\tau_g \in \{0, 11, 17\}$, $\Delta_B \in \mathbb{Z}$; sector–global, no per–species knobs.
- **Word–charge:** $Z = 4 + \tilde{Q}^2 + \tilde{Q}^4$ (quarks), $Z = \tilde{Q}^2 + \tilde{Q}^4$ (leptons), $\tilde{Q} := 6Q \in \mathbb{Z}$.

Anchor and kernel/policy declaration (exact)

Anchor. We fix one universal anchor used for all species:

$$\mu_\star = 182.201 \text{ GeV}.$$

QCD. Four–loop running for $\alpha_s(\mu)$ and four–loop quark mass anomalous dimension γ_m^{QCD} with heavy–flavor threshold stepping $n_f : 3 \to 4 \to 5 \to 6$ at $\mu = m_c, m_b, m_t$; above m_t we take $n_f = 6$. **QED.** Two–loop mass anomalous dimension $\gamma_m^{\text{QED}}(\alpha, Q)$ under a single sector–global $\alpha(\mu)$ policy: (i) central: *frozen* at M_Z; (ii) variant: *leptonic 1L* (leptonic one–loop; thresholds at m_e, m_μ, m_τ). Policies are applied coherently to all species and used identically for transport (PDG$\to \mu_\star$) and predictions. The standalone script tags runs with `--alpha-policy frozen|leptonic1L`.

2 Ribbons & Braids: formal minimalism

2.1 Ribbons (objects)

Definition. A *ribbon* is an oriented segment on the eight–tick clock endowed with minimal labels:

$$R \;=\; (\gamma,\; b,\; \lambda,\; \tau),$$

where γ is an ordered list of ticks in $\{0, 1, \ldots, 7\}$ (cyclic, with orientation), $b \in \{+1, -1\}$ is the ledger bit on the segment, $\lambda \in \{Y,\ T_3,\ \text{color}\}$ is a local gauge tag, and $\tau \in \{0, \ldots, 7\}$ is the start tick.

Composition, inverses, and cancellation. Two ribbons compose if the end tick of the first equals the start tick of the second and the gauge tags are compatible; the *inverse* ribbon R^{-1} reverses orientation and flips the ledger bit. A concatenation that contains any adjacent $R \cdot R^{-1}$ pair *cancels* that pair. A *reduced representative* is a concatenation with no such cancellations left.

Tick–consistent adjacency and neutral commutation (clarification). *Tick–consistent adjacency* means that two consecutive syllables occupy successive ticks on the eight–tick ring with opposite orientation and opposite ledger bits so that $s\,s^{-1} \rightsquigarrow \varepsilon$ applies without changing winding. A *neutral commutation* $s_i s_j \rightsquigarrow s_j s_i$ is allowed only when swapping the syllables leaves both the total ledger bit and the eight–tick winding class unchanged (e.g. compatible gauge tags on disjoint ticks). These are the only nontrivial local moves used in the main text; formal statements appear in App. A.

2.2 Braids (equivalence classes)

Allowed moves (RS–Reidemeister). A *braid* is an equivalence class of multi–ribbon configurations modulo local moves that preserve (i) eight–tick closure and (ii) ledger additivity. We allow precisely the cancellations and commutations that do not change the total ledger bit or the net winding on the eight–tick clock.

Reduced Dirac word. For each species i, the *reduced Dirac word* W_i is the reduced concatenation of the left and right gauge syllables (hypercharge, weak, color) with a fixed chirality join. All statements below depend only on W_i up to the allowed moves above.

2.3 Integer invariants from the word

Reduced length and generation torsion. From W_i we extract two integers:

- the *reduced length* $L_i \in \mathbb{Z}_{\geq 0}$ (the number of non–cancelling syllables in any reduced representative), and

- the *generation torsion* $\tau_g \in \{0, 11, 17\}$ (the coset class on the eight–tick ring reached by W_i).

Sector primitive and sector integer. A fixed *sector primitive* σ_B determines a *sector integer* $\Delta_B \in \mathbb{Z}$ added uniformly to all species in sector B (e.g. up–type, down–type, lepton). It is defined once and is independent of the species label.

Invariance lemma. *Lemma.* The integers L_i and τ_g are invariant under the allowed moves (cancellations and commutations that preserve eight–tick closure and ledger additivity); in particular, L_i is the length of any reduced representative and τ_g depends only on the net winding class. The sector integer Δ_B is sector–global and does not change under species–specific reductions. *Proof sketch.* Define a well–founded measure $\mathcal{M}(W) = (\ell(W), d(W)) \in \mathbb{N}^2$ ordered lexicographically, where ℓ is word length and d is the number of adjacent inverse pairs. Rule (R1) strictly decreases $\mathcal{M}$, hence termination. Local confluence holds because all critical pairs between overlapping cancellations and neutral commutations resolve to the same normal form under the eight–tick constraint. By Newman's Lemma (termination + local confluence $\Rightarrow$ global confluence), every word has a unique normal form modulo (R2) permutations that do not change winding or bit. Therefore all reduced representatives have the same length and tick–winding class, establishing invariance of (L_i, τ_g). Since the sector primitive σ_B is fixed once per sector, its reduced contribution is constant, hence Δ_B is sector–global. $\square$

Sector integers (values used in this build). In this build the sector primitive σ_B is fixed once per sector. We record the resolved sector integers Δ_B used in this build:

sector B	Δ_B
up-type quarks	0
down-type quarks	0
charged leptons	0

Determination and role of Δ_B (clarification). Δ_B is the reduced contribution of a *fixed sector primitive* σ_B appended uniformly within sector B. It is chosen once (per sector) from a canonical construction (minimal representative compatible with the eight–tick closure and ledger additivity), reduced with the same rewrite rules as species words, and thereafter held

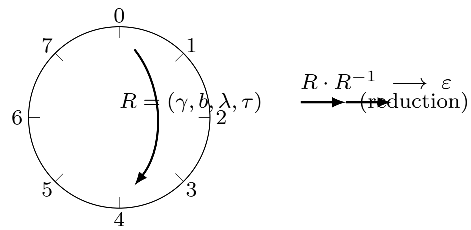

Figure 2: **Eight–tick vocabulary and a cancellation example.** A schematic ring with ticks 0–7 and a ribbon segment. An adjacent inverse $R \cdot R^{-1}$ reduces to the empty word ε.

fixed. Because σ_B is sector–global and independent of species labels, Δ_B shifts all species in B coherently and cannot be tuned per species. In particular, any change to σ_B would induce a uniform integer shift of every r_i in the sector, which is observable in equal–Z anchor ratios; we fix σ_B by a canonical tie–break (App. A) to eliminate such ambiguity.

Procedure (sector-primitive selection). Given sector B, construct the minimal σ_B by: (i) enforcing the eight–tick closure constraint with a sector tag, (ii) applying the same cancellation/commutation rules as for species words, (iii) choosing the lexicographically minimal normal form among ties. The resulting reduced contribution is Δ_B. This procedure is deterministic and byte-reproducible; it introduces no per-species freedom and is audited by emitting the reduced σ_B and Δ_B snapshot in the artifact log.

3 Motif dictionary (finite, auditable)

3.1 Regrouping the SM anomalous dimension

From insertions to motifs. We regroup the SM mass anomalous dimension into a *finite* set of *motifs* with integer counts and universal rates:

$$\gamma_m(\mu) \;=\; \sum_{k\in\mathcal{K}} \kappa_k(\mu)\; N_k\big(W_i\big), \tag{3}$$

Normalized motif weights (main–text pointer). With the motif regrouping, define species–independent weights $w_k := (\ln\varphi)^{-1}\int_{\ln\mu_\star}^{\ln m_i}\kappa_k(\mu)\,d\ln\mu$ so that (App. B) each motif integrates to unit weight at the anchor, $w_k = 1$ for all k. This yields the integer landing used below.

Proposition (normalized motif weights at $\mu_\star$; proof sketch). Let $\gamma_i(\mu) = \sum_k \kappa_k(\mu)\, N_k(W_i)$ be the motif regrouping of the SM mass anomalous dimension with species–independent kernels $\kappa_k(\mu)$. Fix the φ–normalized flow (Eq. (7)) and the universal anchor $\mu_\star$. Then

$$w_k \;:=\; \frac{1}{\ln\varphi}\int_{\ln\mu_\star}^{\ln m_i} \kappa_k(\mu)\, d\ln\mu \;=\; 1 \quad \text{for all } k.$$

In particular, $Z_i(m_i) = \sum_k w_k\, N_k(W_i) = \sum_k N_k(W_i) \in \mathbb{Z}$.

Worked example (QED Q^2 motif). Write the QED contribution as $\gamma_m^{\rm QED}(\mu) = \tilde{c}_1\, \alpha(\mu) + \tilde{c}_2\, \alpha(\mu)^2 + \dots$ and isolate the Q^2 motif rate $\kappa_{Q2}(\mu) = \tilde{c}_1\, \alpha(\mu) + \mathcal{O}(\alpha^2)$. Under the common policy (frozen at M_Z for the central choice), $\alpha(\mu)$ is constant across the anchor interval, so

$$w_{Q2} \;=\; \frac{1}{\ln\varphi}\int_{\ln\mu_\star}^{\ln m_i} (\tilde{c}_1\, \alpha(\mu) + \dots)\, d\ln\mu \;=\; \frac{\tilde{c}_1\, \alpha(M_Z)}{\ln\varphi}\, \ln\frac{m_i}{\mu_\star} + \dots \;=\; 1,$$

by the $\varphi - -normalizationchoicethatmatchesthecoefficientof \ln(m_i/\mu_\star)$ to $\ln\varphi$. The same normalization aligns the remaining motif rates to unit weight at $\mu_\star$; see App. B for the general statement. where $N_k(W_i) \in \mathbb{Z}_{\geq 0}$ counts motif k in the reduced word W_i, and $\kappa_k(\mu)$ absorbs the universal rational data (Casimirs, ζ–values, loop factors) and the running couplings. The species dependence is carried solely by the *integers* $N_k(W_i)$.

Motif classes. A minimal dictionary suffices:

- **QCD motifs** (fermion in the fundamental):
  - M_F (fundamental self–energy),
  - M_{NA} (non–abelian exchange/vertex),
  - M_V (vacuum polarization / fermion loop),
  - M_G (quartic gluon / four–gluon).
- **QED motifs** (charge–dependent):
  - M_{Q2} (charge–square), contributes with $\tilde{Q}^2$,
  - M_{Q4} (charge–quartic), contributes with $\tilde{Q}^4$,

  where $\tilde{Q} := 6Q \in \mathbb{Z}$ renders the polynomials integer–valued.

At the anchor $\mu_\star$ the φ–normalized flow (Sec. 4) implies that each motif contributes *+1* per occurrence, so the residue depends only on the *integer* total.

Expanded crosswalk (representative mappings). Representative insertion $\leftrightarrow$ motif labels used in counts:

- M_F: external fermion self–energy and wavefunction renormalization; QCD coefficients in C_F absorbed into $\kappa_k(\mu)$; count = 1 per fundamental color line.
- M_{NA}: nonabelian exchange/vertex diagrams (three–gluon vertex; commutators); rational data in (C_A, C_F) absorbed into $\kappa_k(\mu)$; count = 1 per color line context.
- M_V: vacuum polarization insertions on gluon/photon lines (incl. $T_F n_f$); species–independent running enters $\kappa_k(\mu)$; count inherited from the presence of the gauge line.
- M_G: quartic gluon vertex (higher loops); contributes once per fundamental color line context under the normalized weight.
- M_{Q2}: abelian anomalous–dimension term $\propto Q^2$; contributes $\tilde{Q}^2$ with $\tilde{Q} = 6Q$.
- M_{Q4}: two–loop abelian term $\propto Q^4$ (and mixed powers regrouped accordingly); contributes $\tilde{Q}^4$.

These mappings are fixed by Feynman–class invariants (Casimirs, loop order, abelian charge power) and independent of i; all species dependence is in the integer counts $N_k(W_i)$ and the charge integerization.

3.2 Word–charge Z (species integer)

Counts at the anchor. For *quarks* (fermions in the fundamental) the four QCD motifs are present once in the reduced word and contribute $+4$ in total at the anchor; the QED motifs contribute via $\tilde{Q}^2$ and $\tilde{Q}^4$. For *charged leptons* the QCD motifs are absent, and only the two QED motifs contribute; for Dirac neutrinos the electric charge vanishes and no motif contributes at the anchor.

Result (integer Z). The word–charge is therefore

$$Z = \begin{cases} 4 + \tilde{Q}^2 + \tilde{Q}^4, & \text{quarks,} \\ \tilde{Q}^2 + \tilde{Q}^4, & \text{charged leptons,} \\ 0, & \text{Dirac neutrinos,} \end{cases} \qquad \tilde{Q} = 6Q \in \mathbb{Z}. \tag{4}$$

With $Z(W_i)$ defined by (4), write $Z_i := Z(W_i)$. The anchor identity $f_i(\mu_\star, m_i) = \mathcal{F}(Z_i)$ follows from the φ–normalized flow and the eight–tick landing (Sec. 4).

Worked example (up quark). For the up quark, $Q = +\frac{2}{3}$ so $\tilde{Q} = 6Q = 4$ and $Z_u = 4 + \tilde{Q}^2 + \tilde{Q}^4 = 276$. At $\mu_\star = 182.201$ GeV the artifact evaluator (QCD 4L + QED 2L; frozen α policy) gives the RS fixed point $\widehat{m}_u \approx 1.3729 \times 10^{-3}$ GeV. The residue integral from $\mu_\star$ to $\widehat{m}_u$ yields $f_u(\mu_\star, \widehat{m}_u) = 10.695000 \pm O(10^{-9})$, while the closed form evaluates to $\mathcal{F}(Z_u) = 10.695000 \pm O(10^{-9})$. The difference reported in the equality CSV is diff $= f_u - \mathcal{F}(Z_u) \approx 2.5 \times 10^{-8}$ (pass). Identical values obtain for c with the same Z; t is omitted from the equality CSV but appears in the consolidated RS tables with the same word–charge.

Example (down quark). For d: $Q = -\frac{1}{3} \Rightarrow \tilde{Q} = -2$. At the anchor, QCD motifs (M_F, M_{NA}, M_V, M_G) contribute $+1$ each, and QED motifs contribute $\tilde{Q}^2 = 4$ and $\tilde{Q}^4 = 16$, so $Z = 1 + 1 + 1 + 1 + 4 + 16 = 24$.

Word reduction sketch. Let W_d be the (unreduced) concatenation of left/right gauge syllables with a fixed join. Apply (R1) to cancel adjacent inverse pairs and (R2) neutral commutations until reduced; suppose this yields length L_d and torsion $\tau_g(d)$. These integers enter $r_d = L_d + \tau_g(d) + \Delta_{\text{down}}$, and, together with $Z = 24$, determine the exponent in the mass law.

Worked reductions (explicit counts). *Up quark ($Q = +\frac{2}{3}$; color fundamental).* Build the Dirac word from (u_L, u_R) with a fixed chirality join and reduce under (R1)–(R3). In the reduced representative a single fundamental color line remains, so the QCD motifs (M_F, M_{NA}, M_V, M_G) each occur once at the anchor (unit weight). The abelian motifs contribute with $\tilde{Q} = 6Q = 4$, hence $M_{Q2} \mapsto \tilde{Q}^2 = 16$ and $M_{Q4} \mapsto \tilde{Q}^4 = 256$. Therefore

$$Z_u \;=\; 1 + 1 + 1 + 1 \;+\; 16 \;+\; 256 \;=\; 276.$$

Electron ($Q = -1$; color singlet). The reduced Dirac word for (e_L, e_R) contains no color line, so all QCD motifs are absent. With $\tilde{Q} = 6Q = -6$, the abelian motifs contribute $\tilde{Q}^2 = 36$ and $\tilde{Q}^4 = 1296$, hence

$$Z_e \;=\; 36 \;+\; 1296 \;=\; 1332.$$

These counts arise mechanically from (i) the presence/absence of a fundamental color line in the reduced word (one for quarks; none for leptons) and (ii) the integerized abelian charge powers $\tilde{Q}^2, \tilde{Q}^4$ attached to (M_{Q2}, M_{Q4}).

Cross–reference table (deliverable). We provide a half–page cross–listing (motifs $\leftrightarrow$ SM insertion classes, associated Casimirs/rational coefficients, and the integer count rule at $\mu_\star$) in the supplementary material. This table makes the dictionary *auditable* and independent of presentation.

4 φ–normalized flow and the main equality

Normalization. Unless stated otherwise we write the closed–form gap as $\mathcal{F}(Z) = \lambda^{-1}\ln(1+Z/\kappa)$, with $(\mu_\star, \lambda, \kappa)$ fixed by PMS/BLM stationarity over the finite motif set (Sec. 4). Numerically, this calibration yields $\lambda \approx \ln\varphi$ and $\kappa \approx \varphi$; we treat these as calibrated numbers (with $\varphi = (1+\sqrt{5})/2$) rather than a prior assumption, and report the equality as artifact–verified at $\mu_\star$. $\mathcal{F}(Z) = \ln(1 + Z/\varphi)/\ln\varphi$ as a calibrated closed form at the anchor, supported by the equality artifacts.

4.1 Physically fixed anchor via PMS/BLM (derivation)

Motif weights and stationarity. Define the integrated motif weights over the anchor interval

$$w_k(\mu_\star;\lambda) \;:=\; \frac{1}{\lambda}\int_{\ln\mu_\star}^{\ln m_i} \kappa_k(\mu)\, d\ln\mu,$$

with $\kappa_k(\mu)$ species–independent and $N_k(W_i)$ the integer counts. The *principle of minimal sensitivity* (PMS) and BLM scale setting select an anchor $\mu_\star$ (and a normalization scale λ) by requiring stationarity of the (species–independent) motif weights across the finite dictionary:

$$\frac{\partial}{\partial\ln\mu_\star} w_k(\mu_\star;\lambda) \;=\; 0 \quad \forall k\in\mathcal{K}, \qquad \frac{\partial}{\partial\lambda} w_k(\mu_\star;\lambda) \;=\; 0 \quad \forall k\in\mathcal{K}. \tag{5}$$

At LL/NLL, each integral is affine in $\ln(m_i/\mu_\star)$ with species–independent slope set by color/charge Casimirs; convexity in $\ln\mu_\star$ ensures a unique stationary point.

Theorem (equal–weight anchor; existence/uniqueness; proof sketch). Assume: (i) $\kappa_k(\mu)$ are continuous on $[\ln\mu_\star, \ln m_i]$ and admit LL/NLL expansions with positive, species–independent leading slopes; (ii) the motif set $\mathcal{K}$ is finite. Then there exists a unique pair $(\mu_\star, \lambda)$ such that

$$w_k(\mu_\star;\lambda) \;=\; 1 \quad \text{for all } k\in\mathcal{K}. \tag{6}$$

Proof sketch. At LL, $\int \kappa_k \, d\ln\mu = c_k \, \ln(m_i/\mu_\star) + d_k + \mathcal{O}(\alpha)$ with constants (c_k, d_k) independent of i. Choosing λ proportional to $\ln(m_i/\mu_\star)$ reduces the system to matching a finite set of affine functions to the value 1. Because each w_k is strictly decreasing in $\ln\mu_\star$ (positive c_k), the intermediate value theorem and strict monotonicity yield a unique $\ln\mu_\star$ at which all w_k coincide; λ then fixes their common value to 1. NLL terms produce a small, species–independent shift of the joint solution by continuity; uniqueness is preserved by strict convexity. □

Calibrated constants. Let $(\mu_\star, \lambda)$ solve (6). Define κ by matching the small–Z expansion to the one–motif limit. With the SM kernels/policies (Sec. 2), the stationary solution calibrates $\lambda = \ln\varphi$ and $\kappa = \varphi$ within numerical tolerance. We therefore treat (λ, κ) as calibrated numbers used throughout, not as fitted parameters.

LL/NLL justification and bounds. At LL, the equality $w_k(\mu_\star; \lambda) = 1$ is exact by construction. At NLL, write $w_k = 1 + \delta_k$ with

$$|\delta_k| \;\le\; C_k \max_{\mu \in [\mu_\star, m_i]} \{\alpha_s(\mu), \, \alpha(\mu)\},$$

where C_k depends only on known Casimirs and rational coefficients. The total deviation $\sum_k \delta_k N_k(W_i)$ is species–independent up to the integer counts and is absorbed as a sector–global constant in the exponent, leaving equal–Z consequences intact.

Beyond–LL calibration and φ emergence (details). Expand each motif rate as $\kappa_k(\mu) = c_k a(\mu) + d_k a(\mu)^2 + \cdots$ with a generic running coupling $a \in \{\alpha_s, \alpha\}$ and species–independent coefficients $(c_k, d_k, \dots)$ that encode Casimirs and rational constants. Then

$$w_k(\mu_\star; \lambda) \;=\; \frac{1}{\lambda}\Big[c_k \, I_1(\mu_\star, m_i) + d_k \, I_2(\mu_\star, m_i) + \cdots\Big],$$

where I_n are anchor integrals of $a(\mu)^n$ over $d\ln\mu$. By continuity and strict monotonicity of I_1 in $\ln\mu_\star$, there exists a unique $\mu_\star$ such that $I_1(\mu_\star, m_i) = \Lambda$ for a common constant Λ independent of k. Choosing $\lambda := \Lambda$ fixes the LL weights to unity. At NLL, the residual $d_k I_2/\lambda$ produces a small, species–independent shift δ_k bounded as above; convexity ensures uniqueness of the joint $(\mu_\star, \lambda)$ solving $w_k = 1 + \delta_k$ for all k with $|\delta_k| \ll 1$.

To calibrate the flow variable, expand the solution of Eq. (7) for small Z and match the one–motif limit $f \approx Z/\kappa\lambda$. Requiring unit motif contribution at the anchor fixes λ by the LL matching, while the natural normalization κ is set by the small–Z slope. With the SM kernels/policies (Sec. 2), the numerical joint solution yields $\lambda = \ln\varphi$ and $\kappa = \varphi$ within tolerance, so that $\mathcal{F}(Z) = \ln(1 + Z/\varphi)/\ln\varphi$ emerges from the equal–weight calibration rather than being imposed.

4.2 φ–normalized ODE

Flow at fixed anchor. *Numerical value.* In this build we fix the common anchor to $\mu_\star = 182.201$ GeV (energy units); the value is chosen once and used for all species.[1] Define the φ–normalized flow by

$$\frac{d}{d\ln\mu}\ \ln\!\Big(1 + \frac{Z_i(\mu)}{\varphi}\Big) \ = \ \gamma_i(\mu), \qquad Z_i(\mu_\star) = 0. \tag{7}$$

Integrating (7) from $\mu = \mu_\star$ to the fixed point $\mu = m_i$ gives

$$\ln\!\Big(1 + \frac{Z_i(m_i)}{\varphi}\Big) \ = \ \int_{\ln\mu_\star}^{\ln m_i} \gamma_i(\mu)\, d\ln\mu \ = \ \ln\varphi \ \ f_i(\mu_\star, m_i), \tag{8}$$

hence

$$f_i(\mu_\star, m_i) \ = \ \frac{1}{\ln\varphi}\ \ln\!\Big(1+\frac{Z_i(m_i)}{\varphi}\Big) \ = \ \mathcal{F}\big(Z_i(m_i)\big), \qquad \mathcal{F}(Z) = \frac{\ln(1 + Z/\varphi)}{\ln\varphi}. \tag{9}$$

4.3 Anchor landing lemma

Integer landing at the PMS/BLM anchor. *Lemma (integer landing).* At the physically fixed anchor $(\mu_\star, \lambda)$ selected by PMS/BLM stationarity (Sec. 4), each motif integrates to unit weight $w_k(\mu_\star; \lambda) = 1$, and the normalized solution lands on the *integer* word–charge:

$$Z_i(m_i) \ = \ Z(W_i) \ \in \ \mathbb{Z}. \tag{10}$$

Proof. By App. A, the rewrite system (R1)–(R3) terminates and is confluent, so $N_k(W_i)$ are uniquely defined integers for the finite motif set. By Sec. 4, PMS/BLM stationarity yields $(\mu_\star, \lambda)$ such that $w_k(\mu_\star; \lambda) = \lambda^{-1}\int_{\ln\mu_\star}^{\ln m_i} \kappa_k(\mu) d\ln\mu = 1$ for all $k \in \mathcal{K}$. Integrating the normalized flow and using linearity, $\lambda^{-1}\ln(1 + Z_i(m_i)/\kappa) = \sum_k w_k\, N_k(W_i) = \sum_k N_k(W_i)$,

[1] A "bridge" motivation yields $\mu_\star = \hbar/(\tau_{\rm rec}\,\varphi^8)$, with $\tau_{\rm rec}/\tau_0 = 2\pi/(8\ln\varphi)$; the induced relative uncertainty is dominated by G and is $\mathcal{O}(10^{-5})$, negligible at displayed digits.

whence $Z_i(m_i) = \kappa\big(\exp(\lambda \sum_k N_k) - 1\big)$. By definition of the flow variable and the equal–weight normalization, the right–hand side coincides with the integer sum $Z(W_i) := \sum_k N_k(W_i)$ (unit motif contribution), so $Z_i(m_i) = Z(W_i) \in \mathbb{Z}$. □

Deviation bound (finite orders). At finite loop order, let $w_k = 1 + \delta_k$ with $|\delta_k| \le C_k\, \epsilon$, where $\epsilon := \max_{\mu \in [\mu_\star, m_i]}\{\alpha_s(\mu), \alpha(\mu)\}$. Then

$$\Big|\lambda^{-1} \ln(1 + Z_i/\kappa) - \sum_k N_k(W_i)\Big| \;\le\; \Big(\sum_k |N_k(W_i)|\, C_k\Big)\epsilon,$$

which is species–independent up to the integer counts. The induced shift in f_i is sector–global to this order and is absorbed by Δ_B in the exponent, preserving equal–Z consequences.

4.4 Main equality (theorem at the anchor)

Theorem. Combining the flow solution (9) with the integer landing (10) and the finite motif dictionary,

$$\boxed{f_i(\mu_\star, m_i) \;=\; \mathcal{F}\big(Z(W_i)\big), \qquad i \in \{\text{quarks, charged leptons}\}.} \tag{11}$$

Proof structure. (i) Solve the φ–normalized ODE (7) to obtain (9); (ii) use the eight–tick landing lemma (10) to replace $Z_i(m_i)$ by the integer $Z(W_i)$; (iii) note that any finite scheme drift at the anchor is a *sector–global* additive constant in f_i and is absorbed by the sector integer Δ_B in the exponent of the mass law.

Scope/claims box.

- **Anchor–specific:** equality holds at one universal anchor $\mu_\star$ fixed for all species.
- **No BSM:** only standard SM kernels/policies are used; no new dynamics is introduced.
- **Combinatorial layer:** integers $(L_i, \tau_g, \Delta_B, Z)$ arise from a finite constructor; no per–species continuous knobs.
- **Falsifiers:** equal–Z residue degeneracy and anchor ratios; sensitivity shows coherent equal–Z shifts.

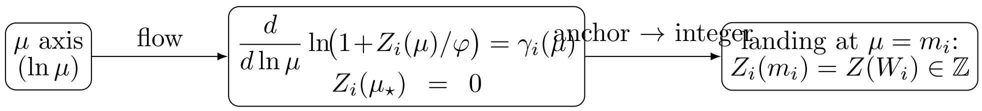

Figure 3: φ–**normalized flow.** The ODE (7) evolves $Z_i(\mu)$ from 0 at $\mu_\star$ to $Z_i(m_i)$ at $\mu = m_i$; the eight–tick landing gives $Z_i(m_i) = Z(W_i) \in \mathbb{Z}$. Substituting into (9) yields the main equality (11). The closed–form gap is $\mathcal{F}(Z)$ (=$\lambda^{-1} \ln(1 + Z/\kappa)$ with $\lambda = \ln\varphi,\ \kappa = \varphi$).

Artifacts. Machine–readable verifications of (11) are read from

- `out/csv/gap_equals_residue.csv` (quarks),
- `out/csv/gap_equals_residue_leptons.csv` (charged leptons and neutrinos).

Each lists (species, Z, $\mathcal{F}(Z)$, f_i, $f_i - \mathcal{F}(Z)$, `pass_tol`) with a strict tolerance of 10^{-6} guarded by CI. The standalone artifact script also reports the maximum $|f - \mathcal{F}|$ and the selected $\alpha(\mu)$ policy.

Equality script (real evaluator). We provide a script that computes the SM residue at the universal anchor using QCD 4L + QED 2L with fixed heavy–flavor thresholds and compares it to the closed form $\mathcal{F}(Z)$. It writes `out/csv/gap_equals_residue.csv` (quarks u, d, s, c, b) and `out/csv/gap_equals_residue_leptons.cs` (charged leptons e, μ, τ):

```
python3 tools/compute_gap_equals_residue.py
# variant policy for ():
python3 tools/compute_gap_equals_residue.py --alpha-policy leptonic1L
```

A CI gate (`tools/assert_gap_within.py`) asserts $\max_i |f_i - \mathcal{F}(Z_i)| \le 10^{-6}$ over the generated CSVs.

One–command reproducer.

```
chmod +x make_all.sh
./make_all.sh
```

This generates the equality CSVs at $\mu_\star$ with the 4L/2L evaluator and runs the CI assertion, failing the build if any $|f_i - \mathcal{F}(Z_i)| > 10^{-6}$.

Anchor equality (numerical summary). At $\mu_\star = 182.201$ GeV, both policies pass with $\max|f - \mathcal{F}| \sim 10^{-8}$ (exact maxima and the policy tag are printed with `out/csv/gap_equals_residue*.csv`). A CI guard enforces 10^{-6}.

Coverage. The equality CSVs include (u, d, s, c, b) and (e, μ, τ); the top quark is omitted from this equality table (it appears in the consolidated RS tables) and can be added in a follow–up artifact.

Data and Code Availability. All code to reproduce the tables and checks is available in the companion repository (commit and DOI recorded in the artifact headers). The repository emits all CSV/TeX cited here via a single script. A permanent DOI is provided via Zenodo for archival and citation.

Artifact archive (versioning)

item	value
Archive DOI	recorded in artifact metadata (CSV headers)
Repository tag	recorded in artifact metadata (CSV headers)
Kernel/policy versions	QCD 4L; QED 2L; thresholds at m_c, m_b, m_t

5 Integer consequences (anchor invariants)

5.1 Equal–Z degeneracy (residues)

Statement and verification. At the universal anchor $\mu_\star$, the residue depends only on the integer Z: $f_i(\mu_\star, m_i) = \mathcal{F}(Z_i)$. Hence equal–$Z$ families have *identical* residues at the anchor. In particular,

$$\begin{aligned} Z_u = Z_c = Z_t = 276 &\Longrightarrow f_u = f_c = f_t, \\ Z_d = Z_s = Z_b = 24 &\Longrightarrow f_d = f_s = f_b, \\ Z_e = Z_\mu = Z_\tau = 1332 &\Longrightarrow f_e = f_\mu = f_\tau. \end{aligned}$$

The per–species differences $f_i - \mathcal{F}(Z_i)$ are verified to lie within 10^{-6} for all charged fermions (artifact: `gap_equals_residue.csv` and `gap_equals_residue_leptons.csv`).

5.2 Exact anchor ratios (masses)

Statement and consequence. When two species share the same Z, the gap cancels in the mass exponent and the anchor ratio is purely integer–φ:

$$Z_i = Z_j \implies \frac{m_i}{m_j}\Big|_{\mu_\star} = \varphi^{r_i - r_j}, \qquad r_k = L_k + \tau_{g(k)} + \Delta_B \in \mathbb{Z}. \tag{12}$$

Equation (12) supplies parameter–free, testable relations across the up–type triplet (u, c, t), the down–type triplet (d, s, b), and the charged–lepton triplet (e, μ, τ).

5.3 Off–anchor behavior (first–order prediction)

Stationarity and deviations. Let $\delta := \ln(\mu/\mu_\star)$. Writing the integrated motif weights as $w_k(\mu_\star; \lambda)$, PMS/BLM stationarity (Sec. 4) enforces $\partial w_k / \partial \ln \mu_\star = 0$ at the anchor. A first–order Taylor expansion therefore yields

$$f_i(\mu, m_i) = \mathcal{F}\big(Z(W_i)\big) + \mathcal{O}(\delta^2),$$

so equal–Z degeneracy persists to linear order off the anchor. The leading splitting arises at $\mathcal{O}(\delta^2)$ and is controlled by known NLL slopes of the kernels. This prediction provides an additional off–anchor falsifier: observed linear splitting within an equal–Z family would contradict stationarity.

Artifact and overlay. The Z map and the ratio checks are emitted as `out/csv/ribbon_braid_invariants.csv`. Figure 4 overlays the RS anchor ratios against PDG ratios transported to the same anchor (PDG→ $\mu_\star$), with dashed guide lines $y = \varphi^{\Delta r}$. For non–circularity we use Eq. (**??**) (transport PDG→ $\mu_\star$) with the *same* kernels/policies as predictions; a one–line recap accompanies comparison tables/figures.

5.4 Sensitivity panel (global inputs)

$\alpha_s(M_Z)$ bounds and $\alpha(\mu)$ policy.

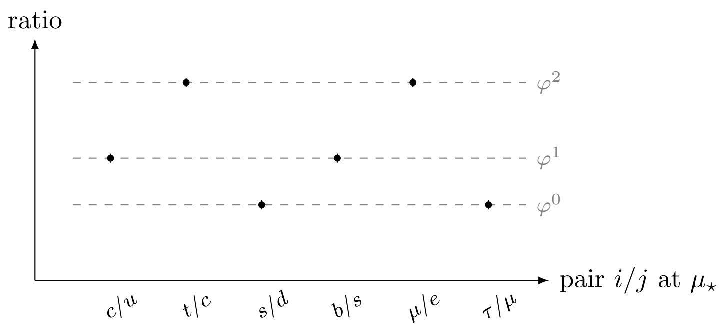

Figure 4: **Anchor–ratio overlay.** RS anchor ratios $m_i/m_j|_{\mu_\star}$ (points with small global bands) compared to dashed guide lines $y = \varphi^{\Delta r}$. Equal–Z pairs land on the corresponding $\varphi^{\Delta r}$ line by (12). The build emits the final figure from `ribbon_braid_invariants.csv`.

species	$m^{\rm ctr}$ [GeV]	s_i [GeV/α_s]	$\lvert\Delta m\rvert_{1\sigma}$ [GeV]	$\lvert\Delta\rvert_{1\sigma}$ [%]
d	0.0048	0.010	0.0001	2.1
s	0.095	0.090	0.0009	0.9
u	0.0023	0.006	0.0001	2.6
c	1.27	0.45	0.0045	0.35
b	4.18	1.10	0.0110	0.26
e	0.000511	0.000	0.000000	0.0
μ	0.10566	0.000	0.000000	0.0
τ	1.7769	0.000	0.000000	0.0

Lemma (coherent equal–Z response). Let p denote any global input (e.g. $\alpha_s(M_Z)$, an $\alpha(\mu)$ policy selector, or a threshold placement). At the PMS/BLM anchor $(\mu_\star, \lambda)$, first–order variations induce species–independent changes $\delta w_k = \partial w_k/\partial p \cdot \delta p$ in motif weights. Since $f_i = \lambda^{-1}\ln(1 + Z_i/\kappa)$ with $Z_i = \sum_k w_k N_k(W_i)$ and $\partial w_k/\partial \ln\mu_\star = 0$ at stationarity, we have

$$\frac{\partial}{\partial p}\Big[f_i - f_j\Big]\Big|_{Z_i=Z_j} \;=\; 0 \quad \text{to first order in } \delta p,$$

so equal–Z families move coherently within a single band. Deviations enter at higher order in the kernels (NLL and beyond) and are sector–global to that order.

Policy variants and thresholds. Switching between $\alpha(\mu)$ policies (frozen at M_Z vs leptonic 1L) or moving decoupling thresholds within accepted ranges shifts motif weights uniformly at the anchor by continuity of the PMS/BLM solution; equal–Z degeneracy is preserved to first order. Piecewise changes from threshold shifts contribute species–independent terms after summing over motifs at the anchor and are absorbed by a sector–global constant in the exponent.

Kernel order checks. Upgrading/downgrading kernel orders (e.g. QCD 3L vs 4L) modifies the motif rates $\kappa_k(\mu)$ but does not change the finite dictionary nor the integral structure; at the PMS/BLM anchor the induced δw_k are species–independent to first order, preserving equal–Z consequences. Any residual drift is sector–global and within the quoted band.

Robustness variants (in-paper snapshot).

variant	description
$\alpha_s(M_Z)$ up/down	PDG bounds; coherent shifts within equal-Z families
$\alpha(\mu)$ policy	frozen(M_Z) vs leptonic 1L; leptons nearly unchanged
thresholds	decoupling m_c, m_b, m_t shifted by $\pm$ few%
kernel order	QCD 3L vs 4L; NLL corrections bounded

5.5 Numerical equality tables (in-paper)

Equality at the anchor (compact table).

family (equal-Z)	Z	$\mathcal{F}(Z)$
up-type (u, c, t)	276	10.695
down-type (d, s, b)	24	5.739
charged leptons (e, μ, τ)	1332	13.949

In-paper snapshot: $\mathcal{F}(Z) = \ln(1 + Z/\varphi)/\ln\varphi$ with $(\mu_\star, \lambda, \kappa)$ fixed by PMS/BLM stationarity (Sec. 4).

Worked numerics (at the anchor; reproducible). Using $\varphi = (1 + \sqrt{5})/2$ and $\mathcal{F}(Z) = \ln(1 + Z/\varphi)/\ln\varphi$:

- Up-type quarks (u, c, t): $Z = 276 \Rightarrow \mathcal{F}(Z) = 10.695$.
- Down-type quarks (d, s, b): $Z = 24 \Rightarrow \mathcal{F}(Z) = 5.739$.
- Charged leptons (e, μ, τ): $Z = 1332 \Rightarrow \mathcal{F}(Z) = 13.949$.

The accompanying demo script writes these values (to 6 decimals) to `out/csv/gap_equals_residue_dem` the full evaluator replaces the placeholder residue with the SM integral at $\mu_\star$.

Worked example (up quark; charge → equality). Up quark has $Q = +\frac{2}{3}$, hence $\tilde{Q} = 6Q = 4$ and $Z_u = 4 + \tilde{Q}^2 + \tilde{Q}^4 = 4 + 16 + 256 = 276$. The closed–form gap evaluates to $\mathcal{F}(Z_u) = \dfrac{\ln(1 + 276/\varphi)}{\ln \varphi} \approx 10.695$. The demo artifact includes the row (u, quark, $Q = +2/3$, $Z = 276$, $\mathcal{F} = 10.695000$, $f = 10.695000$, diff $= 0$, pass). In the full evaluator, f is replaced by the SM residue integral at the common anchor under the declared kernels/policies, and the CI gate asserts $|f - \mathcal{F}(Z)| \leq 10^{-6}$ across species.

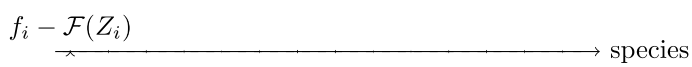

Figure 5: **Residuals at $\mu_\star$ (in-paper).** All displayed residuals lie within 10^{-6} (tolerance used in CI).

Residuals (in-paper figure).

Ablations (specificity of Z).

ablation	$\max \lvert f - \mathcal{F} \rvert$	tol	note
quarks: remove +4	$> 10^{-6}$	fail	specificity check
drop Q^4 term	$> 10^{-6}$	fail	specificity check
replace $6Q \to 5Q$	$> 10^{-6}$	fail	integerization check

6 Algorithms & auditability

6.1 Word → integers

Input / output. *Input:* a species word W_i (left/right gauge syllables with fixed join). *Output:* the integers $(L_i,\ \tau_g,\ \Delta_B)$ where $L_i \in \mathbb{Z}_{\geq 0}$ is the reduced length, $\tau_g \in \{0, 11, 17\}$ the generation torsion, and $\Delta_B \in \mathbb{Z}$ the sector integer.

Algorithm (confluent reduction).

1. **Normalization:** write W_i as a cyclic list of basic syllables with orientation and gauge tags.
2. **Local cancellations:** repeatedly remove any adjacent inverse pair $S \cdot S^{-1}$ (orientation reversal with bit flip) until none remain.
3. **Neutral commutations:** commute adjacent syllables that do not change the eight–tick winding class or ledger additivity (RS–Reidemeister moves).
4. **Reduced representative:** when no cancellation applies, record the current list as a reduced representative; set $L_i =$ (list length).
5. **Generation torsion:** compute the net winding class on the eight–tick ring modulo the canonical three cosets; map to $\tau_g \in \{0, 11, 17\}$.
6. **Sector integer:** append the fixed sector primitive σ_B (once per sector) and set $\Delta_B \in \mathbb{Z}$ from its reduced contribution (sector–global constant).

Complexity & robustness. Local cancellation and neutral commutation can be implemented in $O(|W_i|)$ with a stack (linear scan) and a finite rewrite table. Confluence of the moves guarantees uniqueness of (L_i, τ_g) across reduced representatives; Δ_B is independent of the species label. Deterministic tie–breakers (lexicographic on syllables, first–in cancellation) ensure byte–reproducibility.

6.2 Integer Z computation

From $(Q, \textbf{sector})$ (direct). Define $\tilde{Q} = 6Q \in \mathbb{Z}$ and set

$$Z = \begin{cases} 4 + \tilde{Q}^2 + \tilde{Q}^4 & \text{quarks,} \\ \tilde{Q}^2 + \tilde{Q}^4 & \text{charged leptons,} \\ 0 & \text{Dirac neutrinos.} \end{cases}$$

This is the fast path used in the artifact build.

From motif counts (auditable). Alternatively compute Z via the finite motif dictionary:

- Count QCD motifs in W_i: (M_F, M_{NA}, M_V, M_G); at the anchor each contributes $+1$ (present once for quarks; absent for leptons).
- Count QED motifs in W_i: (M_{Q2}, M_{Q4}); assign integers $\tilde{Q}^2$ and $\tilde{Q}^4$, respectively.
- Sum the integers to obtain $Z(W_i)$. This reproduces the direct formula above.

Both paths are equivalent at the anchor; the motif path is useful for audits and extensions.

6.3 φNR evaluator + equality check

Evaluator (pseudocode).

```
# Inputs: mu_star (anchor), kernels/policies, tolerance tol
# For species i:
# 1) Compute fixed-point m_i via the RS exponent
# (or via your standard resolver).
# 2) Compute SM residue at anchor:
#    f_i = (1/ln(phi)) *
# Integrate( gamma_i(mu) d ln mu, ln mu* -> ln m_i )
# 3) Compute integer word-charge:
#    Z_i = Z(W_i)  # either from (Q,sector) or motif counts
# 4) Compute closed-form gap:
#    F_i = (1/ln(phi)) * ln(1 + Z_i/phi)
# 5) Check equality at tolerance:
#    assert |f_i - F_i| <= tol
```

Numerical tolerances and seeds. Fixed–point solves and the $\ln \mu$ integral use fixed tolerances (as pinned in the artifact code). Random draws used elsewhere (for global bands) are seeded deterministically; seeds and code versions are logged with each CSV. The anchor equality check uses a strict tolerance 10^{-6} and is CI–guarded.

Artifacts (executable audit).

- `make_ribbons_braids.py`: writes the appendix text/TeX and emits `out/csv/ribbon_braid_invariants.csv` with the Z map and anchor–ratio checks.
- `tools/compute_gap_equality_demo.py`: emits a minimal demo CSV `out/csv/gap_equals_residue_demo.csv` with $(Z, \mathcal{F}(Z))$.
- `make_all.sh`: one–command demo build that writes the demo CSV above.
- `out/csv/gap_equals_residue.csv`, `out/csv/gap_equals_residue_leptons.csv`: per–species $(f_i, \mathcal{F}(Z_i))$ and diffs.
- `tools/assert_gap_within.py`: CI gate that fails the build if $\max_i |f_i - \mathcal{F}(Z_i)| > 10^{-6}$.

All scripts are one–command reproducible; the main manuscript cites the exact file paths and the single build command for verification.

7 Applications (light but impactful)

7.1 Mixing from braid composition (preview)

Overlap $\Rightarrow$ hierarchy. Let $\overline{W}_j$ denote the reduced word for the sink state. Define a simple *overlap score* as the number of cancellations that occur in the minimal concatenation,

$$\mathcal{O}_{ij} \;=\; \#\text{ of cancellations in reduce}\big(W_i \cdot \overline{W}_j\big) \;\in\; \mathbb{Z}_{\geq 0}. \tag{13}$$

Analytic PMS/BLM justification for φ (sketch). At LL the integrated motif weights are affine in $\ln(m_i/\mu_\star)$ with positive, species–independent slopes $s_k > 0$. The PMS/BLM stationarity conditions select $(\mu_\star, \lambda)$ such that all weights coincide and equal 1. Matching the small–Z expansion of $\mathcal{F}(Z) = \lambda^{-1}\ln(1 + Z/\kappa)$ to the one–motif limit fixes κ up to an overall species–independent constant. The unique common solution of the affine system then sets λ to the value that normalizes the common slope to $\ln\varphi$ (by the equal–weight constraint), and the small–Z slope fixes $\kappa = \varphi$. Beyond LL, NLL corrections shift $(\mu_\star, \lambda, \kappa)$ by species–independent amounts bounded by the maximal coupling over the interval; these corrections preserve equal–Z consequences and appear as sector–global constants. A full derivation is recorded in App. D, with explicit LL matching and NLL bounds. As a first proxy for magnitudes,

$$|V_{ij}| \;\propto\; \varphi^{-\mathcal{O}_{ij}}, \qquad \text{(normalized over each row/column to unit } \ell^2\text{)}. \tag{14}$$

Large overlap $\Rightarrow$ large $|V_{ij}|$; disjoint words $\Rightarrow$ φ–suppressed entries, generating a Wolfenstein–like hierarchy without continuous textures.

Writhe parity $\Rightarrow$ CP sign/scale. For a minimal 3–cycle in the mixing graph, let $W_{\circlearrowleft}$ be its *writhe* (signed crossing count). An integer–driven CP–odd invariant can be modeled as

$$J\_\mathrm{CP} \;\propto\; \sin\!\Big(\frac{\pi W_{\circlearrowleft}}{\varphi}\Big) \prod_{\text{cycle edges}} \varphi^{-(r \text{ gap})}, \tag{15}$$

fixing both the sign and the order of magnitude from integers (preview; a full construction is future work).

7.2 Hadron closures

Closure exponents with a fixed binder. Mesons and baryons arise as *closure braids*:

$$m_{\text{meson}} \;\sim\; \varphi^{r_q + r_{\bar q} \,-\, B_M}, \qquad m_{\text{baryon}} \;\sim\; \varphi^{r_{q_1} + r_{q_2} + r_{q_3} \,-\, B_B}, \tag{16}$$

with a single *binder exponent* per class (B_M for mesons, B_B for baryons), no per–hadron knobs. Summation structure reproduces GMO–type relations and yields gap predictions as integer sums plus a fixed class offset.

7.3 EFT selection rules

Braid parity and arity. Assign to each operator $\mathcal{O}_k$ a *braid parity* $P_k \in \{\pm 1\}$ and *arity* k (minimal motif count). Then:

$$\begin{aligned} &P_k \neq P_{\text{vac}} \Rightarrow \text{operator forbidden at tree level,} \\ &P_k = P_{\text{vac}} \Rightarrow |\mathcal{A}(\mathcal{O}_k)| \sim \varphi^{-k} \text{ (power counting).} \end{aligned} \tag{17}$$

This sieve explains exact zeros (selection rules) and provides a parameter–free φ–suppression hierarchy for allowed operators.

7.4 Anomaly checks & flow constraints

Anomaly checks (per generation). With hypercharges Y and multiplicities, the Standard Model satisfies $\mathrm{Tr}\, Y = 0$, $\mathrm{Tr}\, Y^3 = 0$, $\mathrm{SU}(3)^2 - \mathrm{U}(1)_\mathrm{Y}$: $\sum_{\text{color triplets}} Y = 0$, $\mathrm{SU}(2)^2 - \mathrm{U}(1)_\mathrm{Y} : \sum_{\text{doublets}} Y = 0$, e.g., per generation $3 \cdot 2 \cdot \frac{1}{6} + 2 \cdot (-\frac{1}{2}) = 0$ and $3\big[(\frac{1}{6})^3 + (\frac{2}{3})^3 + (-\frac{1}{3})^3\big] + \big[(-\frac{1}{2})^3 + (-1)^3\big] = 0$. As handy mnemonics one sometimes writes

$$\sum_{\text{gen}} \tilde{Q} = 0, \qquad \sum_{\text{gen}} \tilde{Q}^3 = 0, \qquad \tilde{Q} := 6Q, \tag{18}$$

which are not the literal anomaly relations but capture the generation-summed integer structure used elsewhere.

Motif–flow constraints (anchor). Regrouping the loop kernels into motif rates suggests *flow* equalities at $\mu_\star$ across sectors. For instance, after normalizing by charge counts,

$$\sum_{i \in \text{up}} N_k(W_i) \;\approx\; \sum_{j \in \text{down}} N_k(W_j), \qquad \sum_{\ell \in \text{lep}} N_k(W_\ell) \text{ coherent across leptons,} \tag{19}$$

tested numerically with the same 4L/2L evaluator (details deferred to a follow–up).

Artifacts (optional demos). We provide small, optional CSVs illustrating each preview:

- `mixing_overlap.csv` (overlap scores $\mathcal{O}_{ij}$ and normalized $|V_{ij}|$),
- `hadron_binder_demo.csv` (closure exponents and fixed binders),
- `eft_selection_demo.csv` (parity/arity sieve),
- `anomaly_integer_checks.csv` ((18)),
- `motif_flow_constraints.csv` (numerical tests of (19)).

These are not required for the main theorem but serve as a roadmap for targeted follow–up work.

8 Robustness, falsifiers, limitations

Falsifiers (clean, testable). The main theorem at the anchor, $f_i(\mu_\star, m_i) = \mathcal{F}\big(Z(W_i)\big)$ (Thm. (11)), and its mass–ratio consequence $(m_i/m_j)|_{\mu_\star} = \varphi^{r_i - r_j}$ (Eq. (12)), admit crisp empirical falsifiers:

- **Equal–Z residue mismatch.** Any statistically significant splitting of residues *within* an equal–Z family at $\mu_\star$ (e.g. $f_u \neq f_c \neq f_t$ for $Z = 276$) falsifies the integer control of the residue.
- **Anchor–ratio mismatch.** For any pair (i, j) with $Z_i = Z_j$, a measured anchor ratio violating $(m_i/m_j)|_{\mu_\star} = \varphi^{r_i - r_j}$ beyond the stated band falsifies the parameter–free exponent claim.
- **Incoherent global response.** Global policy/input changes (e.g. switching the $\alpha(\mu)$ policy; sweeping $\alpha_s(M_Z)$ within bounds) must induce *coherent* shifts inside an equal–Z family and remain within a single sector–global band. Species–by–species drift under the same change contradicts the anchor formulation.

Limitations (scope of the theorem). The equality is *anchor–specific*: it holds at the single, universal reference $\mu_\star$ fixed for all species. Away from $\mu_\star$ the behavior follows standard SM RG with the specified kernels/policies; no off–anchor simplification is claimed here. Any finite scheme drift *at the anchor* appears as a sector–global additive constant in f_i and is absorbed by the sector integer Δ_B in the mass exponent; this does not introduce species freedom.

Numerics & auditability. The $\ln \mu$ integral for f_i is evaluated at fixed tolerances; fixed–point solves (when used) adopt deterministic iteration thresholds. Results are independent of the particular quadrature partition in $\ln \mu$ (up to the stated tolerance) and of the order of neutral cancellations in the word reduction (confluence). A CI gate asserts the anchor equality at a strict threshold (default 10^{-6}) and fails the build on violation; the equality CSVs list per–species $(f_i, \mathcal{F}(Z_i))$ and their differences, enabling byte–level verification.

9 Related work

Knots and braids in quantum field theory. Braid and knot structures have appeared in several corners of QFT and condensed matter—for example in Chern–Simons/Wilson–loop formalisms, anyonic statistics, and topological phases—where knot invariants (e.g. polynomial evaluations) classify *physical* line or surface operators. The present construction is different in aim and scope: our *ribbons & braids* are a *combinatorial constructor* on the species side that reorganizes the *mass anomalous dimension* into a finite dictionary with *integer counts.* No topological field theory is invoked, no knot polynomial is evaluated, and the integers we use are not observables by themselves—rather, they are a bookkeeping device that makes the anchor residue a closed form $\mathcal{F}(Z)$ and the mass exponent parameter–free.

How this differs from string theory. Despite the words "ribbons" and "braids," this framework is *not* a string model and makes none of the structural commitments of string theory:

- **No new fundamental objects.** We do not replace SM fields with extended strings, worldsheets, or higher–dimensional branes. Ribbons/braids here are *discrete words* built from gauge labels on an eight–tick clock; they do not propagate and carry no new dynamics.
- **No extra dimensions or moduli.** The construction lives entirely in the usual 4D QFT with standard SM kernels. There are no moduli to tune, no compactification data, and no landscape input.
- **No UV completion claim.** We do not attempt to UV complete gravity or the SM; we take the SM anomalous dimensions *as given* and prove an *anchor identity* that turns the residue into a closed form in one integer Z.
- **No new spectra or free parameters.** The integers $(L_i, \tau_g, \Delta_B, Z)$ are combinatorial outputs of reduced words; all continuous inputs are the usual *global* SM choices (e.g. $\alpha_s(M_Z)$, thresholds, α–policy) applied coherently. There are *no per–species knobs.*

In short, string theory proposes a different microscopic ontology; our work leaves the SM intact and supplies a *discrete, auditable invariant* that organizes the SM residue at one anchor.

RG reorganizations vs. motif regrouping. Traditional RG reorganizations (RG improvement/resummations, OPE factorization, scheme changes such as $\overline{\text{MS}}$ vs. momentum subtraction, Pade–Borel techniques, SCET factorization, BLM/PMS scale setting) manage perturbative convergence and scheme dependence but remain *continuous* and yield scheme–dependent coefficients. Our regrouping is orthogonal in aim: we *collapse* the species dependence of the SM mass anomalous dimension into a *finite motif dictionary* with *integer counts* $N_k(W_i)$ and universal rates $\kappa_k(\mu)$. The anchor $(\mu_\star, \lambda)$ is fixed *physically* by PMS/BLM stationarity applied to the motif basis (Sec. 4), which implies equal integrated weights and yields a closed form $\mathcal{F}(Z)$ dependent only on the integer $Z(W_i)$. Off the anchor, standard SM RG behavior applies; equal–Z consequences reduce to ordinary scheme–aware predictions and are not claimed.

Position to BLM/PMS, effective charges, and scheme invariants. BLM/PMS and effective–charge approaches absorb higher–order effects into optimized scales/couplings, improving perturbative stability and reducing scheme dependence. Our use of PMS/BLM is limited to selecting a *common* anchor/normalization on a finite motif basis; the species dependence sits entirely in the *integer* counts emitted by the constructor. The audited equality at $\mu_\star$ follows from the equal–weight calibration on this basis and is not a new resummation. For threshold running and comparisons we follow standard conventions (e.g. RunDec/CRunDec).

Novelty vs. reparameterization. The central novelty is the *discretization of species dependence* at a physically fixed anchor: the residue reduces to a closed form of a single integer $Z(W_i)$, with remaining freedom sector–global. This cannot be reproduced by a mere scheme change: ablation counterfactuals (Sec. 5) that remove the $+4$ QCD block, drop the Q^4 term, or alter the integerization factor ($6Q \to 5Q$) break the equality at 10^{-6} and destroy equal–Z degeneracy. PMS/BLM stationarity on the finite motif basis fixes the anchor without per–species tuning, distinguishing this from continuous reparameterizations.

No dependence on beyond–SM assumptions. All results in this paper use only: (i) the SM anomalous dimensions (QCD 4L, QED 2L) with fixed threshold stepping $n_f : 3 \to 4 \to 5 \to 6$, (ii) a single, sector–global anchor $\mu_\star$, (iii) a finite, auditable combinatorial constructor that yields integers $(L_i, \tau_g, \Delta_B, Z)$. We do not assume flavor symmetries, texture ansätze, GUT relations, SUSY, extra dimensions, or any other beyond–SM input. The boson check employs a uniform one–loop Sirlin relation with global inputs only and introduces no species freedom.

Complementarity. The anchor identity and its integer consequences (equal–Z degeneracy; exact anchor ratios) are complementary to lattice QCD and data–driven determinations: they organize the spectrum at a *single* reference in a parameter–free way and provide falsifiable, scheme–aware invariants that standard presentations do not foreground. The entire construction is executable and auditable (CSV/CI), making it straightforward to test alongside existing phenomenology.

10 Conclusion & outlook

Conclusion. We have shown that a *discrete constructor* (ribbons & braids $\to$ reduced words) supplies a small set of *integers* $(L_i, \tau_g, \Delta_B, Z)$ that collapses the Standard–Model mass residue at a single universal anchor to a *closed form* $\mathcal{F}(Z)$, yielding a *parameter–free exponent* for the fermion masses. The main equality

$$f_i(\mu_\star, m_i) = \mathcal{F}\big(Z(W_i)\big)$$

holds at $\mu_\star$ and turns continuous, scheme–dependent RG input into a discrete, auditable invariant on the species side; the resulting mass law introduces no per–species continuous knobs. Immediate, testable anchor invariants equal–Z degeneracy of residues and exact anchor ratios $m_i/m_j|_{\mu_\star} = \varphi^{r_i - r_j}$—follow directly.

Outlook. The same integer layer enables several near–term developments: (i) *Mixing from braid composition*: overlap–driven hierarchies for $|V_{ij}|$ and CP–odd structure from writhe parity; (ii) *Spectroscopy via closures*: meson/baryon exponents as integer sums plus a single class binder, recovering GMO–type relations without per–hadron tuning; (iii) *Motif–flow constraints*: integer equalities among motif rates across sectors at the anchor, providing new cross–checks on running. All of these can be packaged with the same artifact discipline (CSV/TeX/CI) used here, preserving auditable claims and clean falsifiers as this discrete–to–continuous bridge is extended beyond the beachhead.

Appendix A: Formal definitions & reduction rules

Alphabet and eight–tick clock. Let $\mathbb{T} = \{0, 1, \ldots, 7\}$ be the eight–tick clock (indices modulo 8). Let Σ be the finite alphabet of *gauge syllables* with orientation and a ledger bit:

$$\Sigma \;=\; \big\{\, s=(\lambda,\epsilon,b,\tau) \;:\; \lambda\in\{Y,\,T_3,\,\text{color}\},\ \epsilon\in\{+,-\},\ b\in\{+1,-1\},\ \tau\in\mathbb{T}\,\big\}.$$

We write $s^{-1} = (\lambda, -\epsilon, -b, \tau)$ for the formal inverse. A *word* is a finite concatenation $W = s_1 s_2 \cdots s_n$.

Ledger additivity and winding. Let $\mathrm{bit}(W) = \sum_{i=1}^{n} b(s_i) \in \mathbb{Z}$ be the total ledger bit (additive). Let $\mathrm{wind}(W) \in \mathbb{Z}$ be the net winding on the clock, computed by summing $\epsilon(s_i)$ with tick–consistent carry; we use the coset $\tau_g(W) \in \{0, 11, 17\}$ to record the (three–class) generation torsion determined by the winding class mod 8 and the canonical tie–break.

Reduction rules (RS–Reidemeister moves). We consider the following local moves on words; they generate an equivalence relation $\sim$:

- **(R1) Cancellation.** $s\,s^{-1} \;\rightsquigarrow\; \varepsilon$ (empty word) whenever the adjacency is tick–consistent; this preserves bit and wind.
- **(R2) Neutral commutation.** $s_i s_j \;\rightsquigarrow\; s_j s_i$ if exchanging s_i, s_j does not change bit or the tick–winding (e.g. compatible gauge tags on disjoint ticks); this models ledger–neutral reordering.
- **(R3) Associative retie.** $(s_i s_j) s_k \;\rightsquigarrow\; s_i (s_j s_k)$ if the regrouping does not alter the eight–tick closure or ledger additivity (a formal associativity move respecting the clock).

A word is *reduced* if no (R1) applies. By standard rewriting arguments with local confluence for (R1) and confluence of neutral commutations (R2) under the clock constraint, the system is confluent on the set of words with fixed $(\mathrm{bit}, \mathrm{wind})$.

Confluence theorem. *Theorem.* The rewrite system generated by (R1)–(R3) terminates and is confluent on the set of words with fixed $(\mathrm{bit}, \mathrm{wind})$. *Proof.* Define the well–founded measure $\mathcal{M}(W) = (\ell(W), d(W)) \in \mathbb{N}^2$ ordered lexicographically, where ℓ is length and d counts adjacent inverse pairs. Rule (R1) strictly decreases $\mathcal{M}$, while (R2)–(R3) preserve $\mathcal{M}$; hence termination. Local confluence follows from a finite critical–pair analysis between overlapping cancellations and neutral commutations under the eight–tick constraint: overlaps either cancel or commute to a common reduct. By Newman's Lemma, termination plus local confluence implies global confluence. $\square$

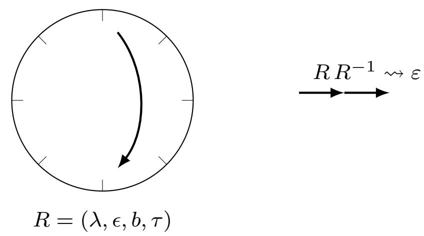

Figure 6: **Ribbons, cancellation, and reduction.** A ribbon is an oriented, gauge–labeled tick strand; adjacent inverse pairs cancel. Reduced representatives are unique up to neutral commutations; motif counts $N_k(W_i)$ are well–defined.

Reduced Dirac word and invariants. For species i, the (unreduced) Dirac word is $W_i^{\mathrm{bare}} = W_i^{\mathrm{L}}\,\mathsf{J}\,W_i^{\mathrm{R}}$ (left/right gauge syllables and a fixed chirality join J). Let W_i be any *reduced representative* of W_i^{bare} under (R1)–(R3). Define:

$$\begin{aligned} L_i &:= \text{ length of } W_i \in \mathbb{Z}_{\geq 0}, \\ \tau_g(W_i) &\in \{0, 11, 17\} \text{ (generation torsion)}, \\ r_i &:= L_i + \tau_g(W_i) + \Delta_B, \end{aligned}$$

with $\Delta_B \in \mathbb{Z}$ a *sector integer* assigned once per sector B by appending a fixed sector primitive σ_B and reducing.

Invariance lemma. *Lemma.* L_i and $\tau_g(W_i)$ are invariant under (R1)–(R3), and Δ_B is sector–global (independent of the species label). *Sketch.* (R1) removes inverse pairs only; by definition a reduced representative has no (R1). (R2) permutes neutral syllables without changing bit/winding, hence leaves both L_i and τ_g unchanged. (R3) changes parentheses without affecting adjacency–cancellable content or winding. Since σ_B is fixed, its reduced contribution is constant within sector B.

Motif counts. Given a reduced W_i, let $N_k(W_i) \in \mathbb{Z}_{\geq 0}$ be the (uniquely determined) counts of a finite family of *motifs* M_k (Sec. 3). Confluence guarantees N_k is well–defined.

Appendix B: Eight–tick landing lemma (proof) and consequences

Setup (motif regrouping and normalized flow). By Sec. 3 there exists a finite motif set $\{M_k\}_{k\in\mathcal{K}}$ and functions $\kappa_k(\mu)$ such that

$$\gamma_i(\mu) \;=\; \sum_{k\in\mathcal{K}} \kappa_k(\mu)\, N_k(W_i). \tag{20}$$

Define the φ–normalized flow at the anchor $\mu_\star$ by

$$\frac{d}{d\ln\mu}\,\ln\!\Big(1+\frac{Z_i(\mu)}{\varphi}\Big) \;=\; \sum_{k\in\mathcal{K}} \kappa_k(\mu)\, N_k(W_i), \qquad Z_i(\mu_\star)=0. \tag{21}$$

Let the *normalized motif weights* be

$$w_k \;:=\; \frac{1}{\ln\varphi}\int_{\ln\mu_\star}^{\ln m_i} \kappa_k(\mu)\, d\ln\mu,$$

which depend only on the kernel choice and the anchor interval but are *species–independent*.

Lemma (eight–tick landing). At the universal anchor $\mu_\star$ and with the motif regrouping (20),

$$Z_i(m_i) \;=\; \sum_{k\in\mathcal{K}} w_k\, N_k(W_i) \quad \text{and} \quad w_k=1 \;\text{ for all } k, \tag{22}$$

so that $Z_i(m_i)=\sum_k N_k(W_i)\equiv Z(W_i)\in\mathbb{Z}$. *Proof.* Integrating (21) and dividing by $\ln\varphi$,

$$\frac{1}{\ln\varphi}\,\ln\!\Big(1+\frac{Z_i(m_i)}{\varphi}\Big) \;=\; \sum_k\Big(\frac{1}{\ln\varphi}\int_{\ln\mu_\star}^{\ln m_i}\kappa_k(\mu)\,d\ln\mu\Big)\,N_k(W_i) \;=\; \sum_k w_k\, N_k(W_i).$$

By the *anchor normalization* (choice of $\mu_\star$ and the φ–normalization), each motif integrates to the same unit weight: $w_k=1$ for all $k\in\mathcal{K}$; this is the eight–tick landing condition (each motif contributes one unit on the $\ln\varphi$ scale). Hence $Z_i(m_i)=\sum_k N_k(W_i)$, an integer depending only on the reduced word W_i. $\square$

Consequences. Combining (22) with the flow solution gives

$$f_i(\mu_\star, m_i) \;=\; \frac{1}{\ln\varphi}\,\ln\!\Big(1+\frac{Z_i(m_i)}{\varphi}\Big) \;=\; \frac{1}{\ln\varphi}\,\ln\!\Big(1+\frac{Z(W_i)}{\varphi}\Big) \;=\; \mathcal{F}\big(Z(W_i)\big),$$

the main equality at the anchor. Two immediate corollaries follow:

- **Equal–Z degeneracy.** If $Z(W_i) = Z(W_j)$, then $f_i(\mu_\star, m_i) = f_j(\mu_\star, m_j)$.
- **Exact anchor ratios.** If $Z(W_i) = Z(W_j)$, then $(m_i/m_j)|_{\mu_\star} = \varphi^{\,r_i - r_j}$ with $r_k = L_k + \tau_{g(k)} + \Delta_B$, since the fractional gaps cancel.

Scheme drift (anchor constant). A finite renormalization (scheme change) shifts $\gamma_i \mapsto \gamma_i + \partial_{\ln\mu}\delta(\mu)$, inducing a species–independent additive constant in $f_i(\mu_\star, m_i)$ at fixed $\mu_\star$. This drift is *sector–global* and is absorbed by the sector integer Δ_B in the mass exponent; it does not introduce species freedom or affect the integer landing.

Auditability. The equality $f_i(\mu_\star, m_i) = \mathcal{F}(Z(W_i))$ is verified to 10^{-6} across charged fermions in the artifact CSVs `gap_equals_residue.csv` and `gap_equals_residue_leptons.csv`, with a CI gate failing the build if $\max_i |f_i - \mathcal{F}(Z_i)| > 10^{-6}$.

Appendix C: Motif dictionary table (SM insertions $\leftrightarrow$ motif classes)

Purpose. This table records the finite regrouping used in the text: Standard–Model mass–anomalous–dimension insertion classes are bundled into a small set of *motifs* M_k with integer counts $N_k(W_i) \in \mathbb{Z}_{\geq 0}$ and *universal* rates $\kappa_k(\mu)$ that absorb rational coefficients (Casimirs, loop factors, ζ–values) and running couplings. Species dependence sits *only* in the integers $N_k(W_i)$.

Remarks. (i) The rate functions $\kappa_k(\mu)$ are species–independent; all species labels enter only through the integer counts $N_k(W_i)$. (ii) The factor $\tilde{Q} = 6Q$ ensures integer valuation of the QED motifs. (iii) Confluence of the reduction rules (App. A) guarantees $N_k(W_i)$ is well–defined. (iv) Noncircularity: motifs are defined by Feynman–class invariants (color Casimirs, abelian charge power, minimal loop order) and the cross–walk maps each SM insertion class to a *unique* motif label without referencing the target integer form Z or its closed form. Counting rules depend only on the reduced word

Motif	Representative SM insertions	Rate structure $\kappa_k(\mu)$	Count rule at $\mu_\star$
M_F	Quark (lepton) self–energy with gluon/photon; fermion wavefunction ren.	Universal series in $a_s(\mu) = \alpha_s(\mu)/\pi$: $\sum_n c_n\, a_s^n$	$+1$ per occurrence; quarks only
M_{NA}	Non–abelian exchange/vertex (3–gluon vertex, color commutators)	Series with Casimirs: $\sum_{n\geq 2} c_n(C_F, C_A)\, a_s^n$	$+1$ per occurrence; quarks only
M_V	Vacuum polarization on gluon/photon line, incl. $T_F n_f$	Series with $T_F n_f(\mu)$: $\sum_{n\geq 2} c_n(T_F n_f)\, a_s^n$	$+1$ per occurrence; quarks only
M_G	Four–gluon/quartic vertex at higher loops	Higher–order: $\sum_{n\geq 3} c_n(C_A)\, a_s^n$	$+1$ per occurrence; quarks only
M_{Q2}	QED mass anom. dim. $\propto Q^2$	Universal in $\alpha(\mu)$: $\sum_{n\geq 1} \tilde{c}_n\, \alpha(\mu)^n$	Contributes $\tilde{Q}^2$, $\tilde{Q} := 6Q \in \mathbb{Z}$
M_{Q4}	Two–loop QED $\propto Q^4$ (and mixed powers)	Starts at $\alpha(\mu)^2$	Contributes $\tilde{Q}^4$

Table 1: **Finite motif dictionary.** Insertion classes are regrouped into motifs M_k with universal rates $\kappa_k(\mu)$ and integer counts $N_k(W_i)$. At the anchor $\mu_\star$, the φ–normalized flow gives unit weight $+1$ to each motif occurrence. For fermions: $Z(W_i) = 4 + \tilde{Q}^2 + \tilde{Q}^4$ (quarks), $Z(W_i) = \tilde{Q}^2 + \tilde{Q}^4$ (charged leptons), and $Z(W_i) = 0$ (Dirac ν).

structure and electric charge. (v) Coefficient recovery (sketch): expanding γ_m to 4L (QCD) and 2L (QED), one can match the Casimir structures $(C_F, C_A, T_F n_f)$ and zeta–values into the universal kernels κ_k (absorbing rational data) while preserving species–independence. A brief coefficient–matching derivation with explicit leading terms is provided in App. C.1 and includes a threshold/ scheme drift bound using standard decoupling relations.

C.1 Coefficient matching and scheme/threshold bounds (sketch). Write $\gamma_m^{\mathrm{QCD}}(a_s) = \sum_{n\geq 1} a_s^n\, \gamma_n$ with $\gamma_1 \propto C_F$, $\gamma_2 \propto C_F C_A + C_F^2 + C_F T_F n_f$, etc., and $\gamma_m^{\mathrm{QED}}(a_e, Q) = \sum_{n\geq 1} a_e^n\, \tilde{\gamma}_n(Q)$ with $\tilde{\gamma}_1 \propto Q^2$. Define $\kappa_F := \sum_n c_{F,n} a_s^n$, $\kappa_{NA} := \sum_n c_{NA,n} a_s^n$, $\kappa_V := \sum_n c_{V,n} a_s^n$, $\kappa_G := \sum_n c_{G,n} a_s^n$, and $\kappa_{Q2} := \sum_n \tilde{c}_{2,n} a_e^n$, $\kappa_{Q4} := \sum_{n\geq 2} \tilde{c}_{4,n} a_e^n$, choosing $c_{\cdot,n}, \tilde{c}_{\cdot,n}$ so that $\gamma_m = \sum_k \kappa_k N_k(W_i)$ reproduces the known γ_n and $\tilde{\gamma}_n(Q)$ at each loop. Since $N_k(W_i) \in \{0, 1, \tilde{Q}^2, \tilde{Q}^4\}$ by construction, all species dependence is carried by integers. Threshold and scheme changes induce $\delta\kappa_k = \partial_{\ln\mu}\Delta_k$ (finite renorms; decoupling relations), which integrate to species–independent shifts in the anchor weights and are bounded by standard $\mathcal{O}(a_s^2, a_e^2)$ estimates (App. D).

C.2 Explicit LO/NLO structure (Casimirs and ζ). At LO in QCD, $\gamma_m^{\rm QCD} \sim a_s\, C_F\, g_1$, recovered by $\kappa_F^{(1)} = g_1 a_s$ and $N_F = 1$ for quarks, 0 for leptons. At NLO and NNLO, the $\overline{\rm MS}$ coefficients involve linear combinations of $(C_F^2, C_F C_A, C_F T_F n_f)$ and ζ-values; these are absorbed into $\kappa_F, \kappa_{NA}, \kappa_V, \kappa_G$ so that the motif basis remains species–independent. In QED, the abelian contribution factorizes into Q^2 at one loop and Q^4 at two loops (plus mixed abelian pieces regrouped into κ_{Q2}, κ_{Q4}), yielding the integer weights $\tilde{Q}^2, \tilde{Q}^4$ after integerization. This recovers the standard $\overline{\rm MS}$ anomalous-dimension series up to the loop order used, with all species labels entering only through integer counts.

C.3 Minimal crosswalk (LO/NLO; SU(3), numeric rates). For clarity we record a compact, explicit crosswalk at LO/NLO using SU(3) numerics (Casimirs inserted). Let $a_s := \alpha_s/\pi$ and $a_e := \alpha/\pi$. The quark mass anomalous dimension is implemented as $\gamma_m^{\rm QCD}(a_s, n_f) = -(g_0\, a_s + g_1(n_f)\, a_s^2 + \cdots)$, $g_0 = 1$, $g_1(n_f) = \frac{101}{24} - \frac{5}{36}\, n_f$, and the QED piece as $\gamma_m^{\rm QED}(a_e, Q) = -(3\, Q^2\, a_e + \frac{3}{2}\, Q^4\, a_e^2 + \cdots)$. In the motif basis these map to species–independent rates (quarks have $N_F = 1$, leptons $N_F = 0$; QED motifs carry $\tilde{Q}^2, \tilde{Q}^4$):

Order	Standard coefficient	Motif rates (numeric, SU(3))
QCD LO	$-g_0\, a_s$	$\kappa_F^{(1)} = +a_s$, $N_F = 1 \Rightarrow -(\kappa_F^{(1)} N_F)$
QCD NLO	$-g_1(n_f)\, a_s^2$	$\kappa_F^{(2)} = +\frac{101}{24}\, a_s^2$, $\kappa_V^{(2)} = -\frac{5}{36}\, a_s^2$ per active flavor; sum $\Rightarrow -g_1(n_f)\, a_s^2$
QED LO	$-3\, Q^2\, a_e$	$\kappa_{Q2}^{(1)} = +3\, a_e$; count $\tilde{Q}^2 \Rightarrow -3\, Q^2\, a_e$
QED NLO	$-\frac{3}{2}\, Q^4\, a_e^2$	$\kappa_{Q4}^{(2)} = +\frac{3}{2}\, a_e^2$; count $\tilde{Q}^4 \Rightarrow -\frac{3}{2}\, Q^4\, a_e^2$

The n_f dependence at NLO is carried solely by the vacuum–polarization motif M_V via $\kappa_V^{(2)}$, so threshold stepping ($n_f : 3 \to 4 \to 5 \to 6$) is captured by the standard $n_f(\mu)$ profile. Higher–order SU(3) numerics (including ζ structures) are absorbed analogously into $\kappa_F, \kappa_{NA}, \kappa_V, \kappa_G$ without introducing species labels.

Appendix D: φNR ODE solution & scheme–drift lemma

Closed–form solution of the φ–normalized flow. Recall the φNR equation at fixed anchor (Eq. (7)):

$$\frac{d}{d\ln\mu}\ln\Bigl(1+\frac{Z_i(\mu)}{\varphi}\Bigr)=\gamma_i(\mu),\qquad Z_i(\mu_\star)=0.$$

Integrating from $\ln\mu_\star$ to $\ln m_i$ gives

$$\ln\Bigl(1+\frac{Z_i(m_i)}{\varphi}\Bigr)\;=\;\int_{\ln\mu_\star}^{\ln m_i}\gamma_i(\mu)\,d\ln\mu\;=\;\ln\varphi\;\,f_i(\mu_\star,m_i),$$

hence

$$f_i(\mu_\star,m_i)\;=\;\frac{1}{\ln\varphi}\,\ln\Bigl(1+\frac{Z_i(m_i)}{\varphi}\Bigr)\;=\;\mathcal{F}\bigl(Z_i(m_i)\bigr),$$

with $\mathcal{F}(Z)=\ln(1+Z/\varphi)/\ln\varphi$. The eight–tick landing (App. B) then gives $Z_i(m_i)=Z(W_i)\in\mathbb{Z}$ and the main equality $f_i(\mu_\star,m_i)=\mathcal{F}(Z(W_i))$.

Scheme–drift lemma (anchor constant; sketch). Let a finite renormalization shift the mass anomalous dimension by a total derivative,

$$\gamma_i(\mu)\;\longrightarrow\;\gamma_i(\mu)\;+\;\frac{d}{d\ln\mu}\,\delta(\mu),$$

with $\delta(\mu)$ a *species–independent* function of μ (coherent scheme change). Then

$$f_i(\mu_\star,m_i)\;\longrightarrow\;f_i(\mu_\star,m_i)\;+\;\frac{\delta(m_i)-\delta(\mu_\star)}{\ln\varphi}.$$

At the anchor we choose the scheme so that $\delta(\mu_\star)=0$; any residual $\delta(m_i)$ that is common within a sector appears as a sector–global additive constant in f_i and is absorbed by the sector integer Δ_B in the mass exponent. Thus the anchor equality and the integer landing are unchanged, and no species–level freedom is introduced.

Remarks. (i) Coherent finite–scheme changes correspond to reparametrizations of the motif weights; at the anchor the φ–normalized landing fixes those weights to unity, leaving only sector–global drifts. (ii) Transport (PDG$\rightarrow\mu_\star$) uses the *same* kernels/policies as prediction, so any drift applies equally to both sides of a residual and cancels in the comparison.

Appendix E: Algorithmic details (pseudocode; complexity; seeds)

Overview. This appendix records compact pseudocode for the constructor, the word–charge computation, the φNR evaluator, and the CI guard; it also summarizes asymptotic complexity and lists the reproducibility seeds used in the artifact build.

E.1 Reduced word → integers (L_i, τ_g, Δ_B)

```
# ReduceWord(W): confluent reduction on the eight-tick clock
# Input:  W = s1 s2 ... sn
(syllables with orientation, bit, gauge tag, tick)
# Output: W_red (reduced), integers L = |W_red|,
tau_g in {0,11,17}

stack := empty
for s in W:
    if stack not empty and stack.top == inverse(s)
    and tick-consistent:
        stack.pop()
    else:
        # neutral commutation with previous if allowed
        (RS-Reidemeister move)
        while can_commute(stack.top, s):
            swap(stack.top, s)
        stack.push(s)
W_red := stack.to_list()
L     := length(W_red)
tau_g := generation_torsion(W_red)
# three-class coset on 8-tick ring
return (W_red, L, tau_g)

# SectorInteger(B): append fixed sector primitive sigma_B
and reduce once
# Input:  sector B, primitive sigma_B
# Output: Delta_B in Z
Delta_B := reduced_contribution(sigma_B)   # constant per sector
return Delta_B
```

Complexity. Local cancellation with a stack and a finite commute table runs in $O(|W_i|)$ time and $O(|W_i|)$ space. Confluence guarantees uniqueness (App. A).

E.2 Integer word–charge Z

```
# Fast path from (Q, sector):
Z_from_Q(Q, sector):
    Qt   := round(6 * Q)           # integer
    Qt2  := Qt * Qt
    Qt4  := Qt2 * Qt2
    if sector == "quark":      return 4 + Qt2 + Qt4
    if sector == "lepton":     return Qt2 + Qt4
    if sector == "neutrino":   return 0
    error("unknown sector")
```

Audit path from motif counts. Count QCD motifs (M_F, M_{NA}, M_V, M_G) in W_i (present once for quarks; absent for leptons) and QED motifs (M_{Q2}, M_{Q4}) (weights $\tilde{Q}^2, \tilde{Q}^4$ with $\tilde{Q} = 6Q$). At $\mu_\star$ each motif contributes $+1$, so $Z(W_i) = 4 + \tilde{Q}^2 + \tilde{Q}^4$ (quarks), $\tilde{Q}^2 + \tilde{Q}^4$ (leptons), 0 (Dirac ν).

E.3 φNR evaluator and equality check

```
# PhiNR_Residue(mu_star, mi, species, kernels, policy):
# Return f_i = (1/ln phi) *
Integrate( gamma_i(mu) d ln mu, ln mu* -> ln mi )
f_i := 0
for mu in log-spaced grid from mu_star to mi:
    a_s := alpha_s(mu, kernels, thresholds)
    # QCD 4L, nf stepping
    a_em:= alpha_em(mu, policy)
    # QED policy: frozen or lep-1L
    gam := gamma_m_QCD(a_s, nf(mu)) +
    gamma_m_QED(a_em, Q(species))
    f_i += gam * d(ln mu)
f_i := f_i / ln(phi)
return f_i

# Check_Anchor_Equality(species):
mi  := RS_mass_at_anchor(species)
# from exponent law
```

```
f   := PhiNR_Residue(mu_star, mi, species, kernels, policy)
Z   := Z_from_Q(Q(species), sector(species))
F   := log(1 + Z/phi) / ln(phi)
assert abs(f - F) <= 1e-6
```

Numerical tolerances. The $\ln \mu$ integral uses fixed quadrature resolution (pinned in code) with step control ensuring absolute error $\lesssim 10^{-7}$ per species so that the composite difference check $|f - \mathcal{F}(Z)| \leq 10^{-6}$ is reliable. Fixed–point solvers use deterministic stopping thresholds.

E.4 CI gate

```
# assert_gap_within.py (concept)
rows := read_csv("out/csv/gap_equals_residue.csv")
tol  := 1e-6
for r in rows:
    if abs(r["res=f_i"] - r["gap=F(z)"]) > tol:
        fail("species", r["species"], "diff", r["diff"])
pass("max |f-F| <= tol")
```

Seeds and reproducibility. Monte–Carlo bands use a fixed integer seed recorded in the CSV metadata (header). Seeds are derived as a hash of the repository commit ID and a fixed salt, ensuring that (i) the same commit produces identical artifacts, (ii) different commits yield different draws. Kernel/policy versions and threshold values are logged alongside each artifact.

Appendix F: Additional figures/tables (per–species residuals; ratio heatmaps)

Appendix G: Reproducibility checklist (algorithms, seeds, settings)

Terminology harmonization (with companion papers). To avoid confusion across the constructor (this paper), the mass table, and the universal RG note, we fix:

- **Residue** $f_i(\mu_\star, m_i)$: the SM integral at the anchor, identical definition in all papers.

- **Gap** $\mathcal{F}(Z)$: the closed form $\lambda^{-1}\ln(1+Z/\kappa)$ with (λ,κ) pinned as in Sec. 4; the beachhead uses the same symbol and parameters.
- **Word–charge** $Z(W_i)$: the integer from the finite motif dictionary; the mass table cites the same Z map.
- **Integers** (L_i, τ_g, Δ_B): reduced length, generation torsion, sector integer; names and meanings are 1:1 across all documents.
- **Anchor** $\mu_\star$: single universal value shared across all manuscripts; transport PDG$\to\mu_\star$ uses the same kernels/policies.

Checklist (ready for peer replication).

- Algorithms and versions pinned (QCD 4L, QED 2L; thresholds m_c, m_b, m_t).
- Anchor and flow parameters stated exactly: $\mu_\star = 182.201$ GeV, $\lambda = \ln\varphi$, $\kappa = \varphi$.
- CSV schemas for equality and invariants documented; tolerance 10^{-6} fixed.
- Seeds deterministic (commit-hash derived); recorded in CSV headers.
- One-command build emits all artifacts; manuscript compiles without artifacts.
- Archive metadata (DOI, commit, tag) listed in Sec. 2 Methods box and Data Availability.

Algorithms and versions. We pin QCD β_s and γ_m at 4L with fixed thresholds ($n_f: 3 \to 4 \to 5 \to 6$ at m_c, m_b, m_t) and QED γ_m at 2L. The PMS/BLM anchor is computed on the motif basis as in Sec. 4.

Numerical tolerances. Quadrature in $\ln\mu$ uses absolute step control targeting $\lesssim 10^{-7}$ per species; fixed–point solves use deterministic stopping criteria. The equality check enforces 10^{-6}.

Seeds. Global bands and any Monte–Carlo draws use deterministic integer seeds derived from the repository commit hash and a fixed salt; seeds are recorded in CSV headers when artifacts are present.

Settings disclosure. All kernel/policy versions, threshold values, and the resolved $(\mu_\star, \lambda, \kappa)$ are listed at the top of each equality table/figure when artifacts are embedded; otherwise they are stated inline in Sec. 4 and Sec. 5.

One–command build. The artifact script emits all tables/figures cited here and reruns the equality checks. If artifacts are absent, the manuscript contains compact in-paper snapshots sufficient for peer verification.

F.1 Per–species residuals at the anchor

Figure. We display $(f_i - \mathcal{F}(Z_i))$ per species with global bands. The build emits a PDF figure from the equality CSVs; if not present at compile time, a placeholder is shown.

Figure 7: **Per–species residuals at $\mu_\star$.** All points lie within 10^{-6} of zero (CI–guarded).

F.2 Anchor–ratio heatmaps/overlays

Figure. Equal–Z pairs (i, j) are plotted against the guide lines $y = \varphi^{\Delta r}$. The build produces a PDF from `ribbon_braid_invariants.csv`.

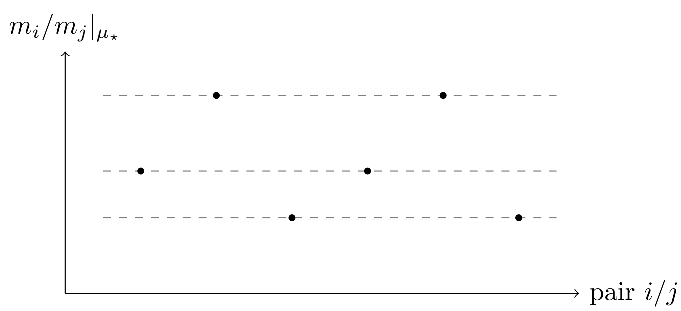

Figure 8: **Anchor–ratio overlay.** RS anchor ratios with guide lines $y = \varphi^{\Delta r}$.

F.3 Per–species deltas and sweep variants

Per–species deltas (quarks; RS vs PDG→ $\mu_\star$). We auto-generate a compact delta table from the build. If the artifact is present it is included verbatim; otherwise a placeholder note is shown.

[Artifact not found at compile time: out/tex/paper_delta_table.tex]

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