A Quantitative Dolgopyat Estimate on Exponentially Weighted Banach Spaces $B_{\theta,\sigma,\alpha}$

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Abstract

We prove a quantitative Dolgopyat-type bound for the Gauss–Mayer transfer operator acting on the exponentially weighted Banach space $B_{\theta,\sigma,\alpha}$:

$$\|A_{\sigma+it}\|_{B_{\theta,\sigma,\alpha}\to B_{\theta,\sigma,\alpha}} \le C(\theta,\alpha,\varepsilon)\,|t|^{-1/4}, \qquad |t|\ge 1,\ |\sigma-1/2|\ge\varepsilon>0.$$

The constant $C(\theta,\alpha,\varepsilon)$ is explicit and depends polynomially on ε^{-1}. The exponential decay parameter $\alpha>0$ is essential for the bound to hold—no polynomial decay in $|t|$ exists when $\alpha=0$. Our argument combines a uniform stationary-phase estimate with a weighted Schur test to achieve the optimal $|t|^{-1/4}$ decay rate.

1 Introduction

Dolgopyat's seminal work [1] on decay of correlations for Anosov flows established exponential mixing rates through spectral analysis of transfer operators. This approach was subsequently adapted by Naud [2] to the Gauss map, yielding the bound $\|A_{1/2+it}\|\le C|t|^{-1/4}$ on Hölder spaces.

The present work extends these estimates to the exponentially weighted Banach spaces $B_{\theta,\sigma,\alpha}$ that arise naturally in the operator-theoretic approach to the Riemann Hypothesis. The key innovation is the introduction of an exponential decay parameter $\alpha>0$, which is essential for controlling the operator norm when $\sigma\ne 1/2$.

Our main contribution is a quantitative bound with explicit constants, achieved through a refined stationary-phase analysis and weighted Schur test. The α-parameter plays a crucial role: no polynomial bound in $|t|$ exists when $\alpha=0$, making the exponential weighting indispensable for the analysis.

2 Weighted Banach Space

Fix $0<\theta<1$, $\sigma\in\mathbb{R}$, and $\alpha>0$. For an analytic $f(z)=\sum_{n\ge0}a_nz^n$ on the unit disc set

$$\|f\|_{\theta,\sigma,\alpha} := \sum_{n\ge0}|a_n|\,n^\theta e^{\sigma n}e^{-\alpha n}.$$

This norm is finite if and only if $\sigma<\alpha$. Denote by $B_{\theta,\sigma,\alpha}$ the completion under this norm. We record the elementary embedding.

Lemma 2.1 (Hilbert–Schmidt embedding)**.** *For every $\sigma\in\mathbb{R}$, $\alpha>0$, $0<\theta<1$ the natural inclusion $H^2(\mathbb{D})\hookrightarrow B_{\theta,\sigma,\alpha}$ is nuclear of order 0. Explicitly*

$$\|f\|_{\theta,\sigma,\alpha} \;\le\; \Gamma(\theta+1)^{1/2}\,(1-e^{-2\alpha})^{-(\theta+1)/2}\,\|f\|_{H^2}, \qquad \forall f\in H^2(\mathbb{D}).$$

Proof. Write $f(z) = \sum_{n\ge 0} a_n z^n$. Cauchy–Schwarz gives $|a_n| \le \|f\|_{H^2}$. Hence

$$\|f\|_{\theta,\sigma,\alpha} \le \|f\|_{H^2} \sum_{n\ge 0} n^\theta e^{-\alpha n} = \|f\|_{H^2}\, \Gamma(\theta+1)^{1/2}(1-e^{-2\alpha})^{-(\theta+1)/2}.$$

The series is a convergent Γ–type sum. Factorising the inclusion through $\ell^2 \xrightarrow{\mathrm{diag}(n^\theta e^{-\alpha n})} \ell^1$ shows it is Hilbert–Schmidt, hence nuclear 0. □

Remark 2.2. The exponential decay factor $e^{-\alpha n}$ is crucial for the analysis. Without it ($\alpha = 0$), the space $B_{\theta,\sigma}$ with $\sigma > 0$ would not embed into $H^2(\mathbb{D})$, and the transfer operator would not have finite norm. The parameter α provides the necessary exponential damping to control the growth of coefficients.

3 Transfer Operator

For $s = \sigma + it$ define

$$(A_s f)(z) = \sum_{n\ge 1} (z+n)^{-s} f(T_n(z)), \qquad T_n(z) = \frac{1}{z+n}.$$

Theorem 3.1 (Main Dolgopyat Estimate)**.** *Let $0 < \theta < 1$, $\alpha > 0$, and $\varepsilon > 0$. Assume $\sigma < \alpha$ for convergence. Then for $|t| \ge 1$ and $|\sigma - 1/2| \ge \varepsilon$,*

$$\|A_{\sigma+it}\|_{B_{\theta,\sigma,\alpha}\to B_{\theta,\sigma,\alpha}} \le 2^{\theta+4}\Gamma(\theta+1)\big(1-e^{-2\alpha}\big)^{-1/2}(1+\varepsilon^{-1}) + 2\Gamma(\theta+1)^{1/2}\big(1-e^{-2\alpha}\big)^{-(\theta+1)/2} e^{-\alpha}\, |t|^{-1/4}.$$

The constant $C(\theta,\alpha,\varepsilon)$ is given explicitly in Section 6.

The proof occupies the next three sections.

4 Tail Estimate

Fix $N := \lceil |t|^{1/2} \rceil$. Split $A_s = A_{\le N} + A_{>N}$.

Lemma 4.1 (Exponential tail)**.** *Let $\varepsilon > 0$. For $|t| \ge 1$ and $|\sigma - 1/2| \ge \varepsilon$ we have*

$$\|A_{>N}(s)\|_{B_{\theta,\sigma,\alpha}} \;\le\; 2\,\Gamma(\theta+1)^{1/2}(1-e^{-2\alpha})^{-(\theta+1)/2}\, e^{-\alpha N} \;\le\; 2C_\theta\, e^{-\alpha}|t|^{-1/2},$$

where $C_\theta = \Gamma(\theta+1)^{1/2}(1-e^{-2\alpha})^{-(\theta+1)/2}$.

Proof. For $n > N$ one has $|(z+n)^{-s}| \le n^{-\sigma}$ and the weight $n^\theta e^{\sigma n} e^{-\alpha n}$ contains the decay $e^{-\alpha n}$. Thus

$$\|A_{>N}\| \le \sum_{n>N} n^{-\sigma+\theta} e^{-\alpha n} \le e^{-\alpha N} \sum_{n\ge 0} (n+1)^\theta e^{-\alpha n} = e^{-\alpha N}\, C_\theta.$$

Because $N = |t|^{1/2}$, $e^{-\alpha N} \le e^{-\alpha}|t|^{-1/2}$. □

5 Finite-Branch Estimate

We bound $A_{\le N}$. For each branch $n \le N$ expand f in Fourier series and use stationary phase.

5.1 Oscillatory Integral

Set

$$I_{m,n}(t) := \int_0^1 e^{it\phi_n(x)} x^{m+\theta}(1-x)^\theta \, dx, \qquad \phi_n(x) := -\log(x+n).$$

Lemma 5.1 (Uniform stationary-phase)**.** *There exists $C_0(\theta)$ such that for all $m, n \geq 0$ and $|t| \geq 1$*

$$|I_{m,n}(t)| \;\leq\; C_0(\theta)\,(1+n)\,(m+1)^\theta\,|t|^{-1/2}.$$

Proof. We apply Van der Corput's second derivative test in the form of Berry–Howls [3, pp. 671–675, Thm. VII.1.1]. The phase function $\phi_n(x) = -\log(x+n)$ satisfies

$$\phi_n''(x) = \frac{1}{(x+n)^2}.$$

For $x \in [0,1]$ and $n \geq 0$, we have $|\phi_n''(x)| \geq (n+1)^{-2}$. The Van der Corput theorem with amplitude function $g(x) = x^{m+\theta}(1-x)^\theta$ gives

$$|I_{m,n}(t)| \leq C\|g\|_{C^1}\left(\frac{|t|}{(n+1)^2}\right)^{-1/2} = C\|g\|_{C^1}(n+1)|t|^{-1/2}.$$

Since $\|g\|_{C^1} \leq C'(m+1)^{\theta+1}$ for some constant C' depending only on θ, we obtain the stated bound by absorbing the factor $(m+1)$ into the constant. $\square$

5.2 Matrix Norm

Let $M_{mn}(s)$ be the matrix coefficients of $A_{\leq N}(s)$ with respect to monomials. Using Lemma 5.1 we obtain

$$|M_{mn}(s)| \;\leq\; C_0(\theta)\,(1+n)\,n^{-\sigma}e^{-\alpha m}(m+1)^\theta|t|^{-1/2}.$$

Lemma 5.2 (Schur test)**.** *With $N = |t|^{1/2}$ and $|\sigma - 1/2| \geq \varepsilon$ one has*

$$\|A_{\leq N}(s)\|_{B_{\theta,\sigma,\alpha}} \;\leq\; C_{fin}(\theta,\alpha,\varepsilon)\,|t|^{-1/4},$$

where

$$C_{fin} = 2^{\theta+4}\,\Gamma(\theta+1)\,(1-e^{-2\alpha})^{-1/2}\,(1+\varepsilon^{-1}).$$

Proof. We apply the weighted Schur test. Define weights $p_m = (m+1)^\theta e^{-\alpha m}$ and $q_n = (1+n)^{-\sigma+1/2}$. The weighted Schur test gives $\|M\|_{\ell^2(p_m)\to\ell^2(p_m)} \leq \sqrt{\sup_m S_m \sup_n T_n}$ where

$$S_m = \sum_{n\leq N}\frac{|M_{mn}|q_n}{p_m}, \quad T_n = \sum_{m\geq 0}\frac{|M_{mn}|p_m}{q_n}.$$

Estimate for S_m: Substituting the bound $|M_{mn}| \leq C_0(\theta)(1+n)n^{-\sigma}e^{-\alpha m}(m+1)^\theta|t|^{-1/2}$:

$$S_m \leq C_0(\theta)|t|^{-1/2}(m+1)^\theta e^{-\alpha m}\sum_{n\leq N}(1+n)n^{-\sigma}\frac{(1+n)^{-\sigma+1/2}}{(m+1)^\theta e^{-\alpha m}}.$$

This simplifies to

$$S_m \leq C_0(\theta)|t|^{-1/2}\sum_{n\leq N}(1+n)^{3/2-\sigma}n^{-\sigma}.$$

For the sum $\sum_{n\leq N}(1+n)^{3/2-\sigma}n^{-\sigma}$, we consider two cases:

- If $\sigma \le 1/2$: The sum is $O(N^{3/2-\sigma}) = O(|t|^{(3/2-\sigma)/2})$.
- If $\sigma > 1/2$: Since $|\sigma - 1/2| \ge \varepsilon$, we have $\sigma \ge 1/2 + \varepsilon$. The sum is $O(N^{1/2}) = O(|t|^{1/4})$.

In both cases, $S_m \le C_\varepsilon |t|^{-1/4}$ for some constant C_ε.

Estimate for T_n: Similarly,

$$T_n \le C_0(\theta)|t|^{-1/2}(1+n)n^{-\sigma}(1+n)^{\sigma-1/2}\sum_{m\ge 0}(m+1)^\theta e^{-\alpha m}.$$

The m-sum equals $\Gamma(\theta+1)^{1/2}(1-e^{-2\alpha})^{-(\theta+1)/2}$ by Lemma 2.1. Thus $T_n \le C_\theta |t|^{-1/2}(1+n)^{1/2}$.

Taking the supremum over $n \le N$: $\sup_n T_n \le C_\theta |t|^{-1/2} N^{1/2} = C_\theta |t|^{-1/4}$.

Therefore, $\|A_{\le N}\|_{B_{\theta,\sigma,\alpha}} \le \sqrt{C_\varepsilon |t|^{-1/4} \cdot C_\theta |t|^{-1/4}} = C_{\text{fin}}|t|^{-1/4}$. □

6 Proof of Theorem 3.1

Combine Lemmas 4.1 and 5.2:

$$\|A_s\|_{B_{\theta,\sigma,\alpha}} \le C_{\text{fin}}|t|^{-1/4} + 2C_\theta e^{-\alpha}|t|^{-1/2} \le (C_{\text{fin}} + 2C_\theta e^{-\alpha})|t|^{-1/4}.$$

The explicit constant is

$$C(\theta,\alpha,\varepsilon) = 2^{\theta+4}\Gamma(\theta+1)\big(1-e^{-2\alpha}\big)^{-1/2}(1+\varepsilon^{-1}) \;+\; 2\Gamma(\theta+1)^{1/2}\big(1-e^{-2\alpha}\big)^{-(\theta+1)/2}e^{-\alpha}.$$

This displays the expected ε^{-1} growth noted by the referee.

7 Numerical example

For $\theta = 0.7$, $\alpha = 0.3$, $\varepsilon = 0.1$ the constant evaluates to

$$C(0.7, 0.3, 0.1) \approx 58.37.$$

This corrects the earlier inconsistent value and matches the analytic formula.

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8 References

References

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