Note on "Cost Uniqueness" – Comment for Sebas

This note is just based on the Manuscript, in an attempt to help imrove it. @Sebas you should cross-check with the latest files and modification of the theory done, if what I am saying is still valid. Please check everything.

1 Summary

In the current manuscript (Final version), **Theorem T5 (Cost Uniqueness)** claims that a set of five conditions (Reciprocity, Convexity, Minimality, Normalization, and Reciprocal-invariance) uniquely determines the "minimal cost function"

$$J(x) = \frac{1}{2}\left(x + x^{-1}\right) - 1, \tag{1}$$

My goal is here is to give a counter-derivation and an explicit one-parameter family of counterexamples. If addressed, it will only help paper not hurt it.

2 Problem statement

The theorem states:

There exists a *unique* function $J : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ satisfying the following five conditions:

1. **Reciprocity:** $J(x) = J(x^{-1})$ for all $x > 0$.
2. **Convexity:** J is convex on $\mathbb{R}_{>0}$.
3. **Minimality:** $J(1) = 0$ and $J(x) > 0$ for all $x \neq 1$.
4. **Normalization:** $J''(1) = 1$.
5. **Reciprocal-invariance:** J depends only on $f(x) = x + x^{-1}$, i.e. $J(x) = g(f(x))$ for some $g : [2, \infty) \to \mathbb{R}$.

Then the claim is there exist a *unique* solution:

$$J(x) = \frac{1}{2}\left(x + x^{-1}\right) - 1. \tag{2}$$

3 Derivation of conditions (1)–(5)

Define the reciprocal-invariant map

$$f(x) := x + x^{-1}, \qquad x > 0. \tag{3}$$

Then for all $x > 0$,

$$f(x^{-1}) = x^{-1} + x = f(x), \tag{4}$$

so f is automatically invariant under $x \mapsto x^{-1}$.

3.1 Condition (5) and condition (1) are redundent.

Assume condition (5): there exists g such that $J(x) = g(f(x))$. Then

$$J(x^{-1}) = g\left(f(x^{-1})\right) = g\left(f(x)\right) = J(x). \tag{5}$$

Hence reciprocity (1) holds automatically once (5) holds and it provides on additional constraint.

3.2 Normalization condition $J''(1) = 1$ into a condition on g

Assume $J(x) = g(f(x))$. By the chain rule,

$$J'(x) = g'(f(x))\, f'(x), \tag{6}$$

and

$$J''(x) = g''(f(x))\,(f'(x))^2 + g'(f(x))\, f''(x). \tag{7}$$

Compute derivatives of f:

$$f'(x) = 1 - x^{-2}, \qquad f''(x) = 2x^{-3}. \tag{8}$$

Evaluate at $x = 1$:

$$f(1) = 2, \qquad f'(1) = 0, \qquad f''(1) = 2. \tag{9}$$

Therefore,

$$J''(1) = g''(2) \cdot 0^2 + g'(2) \cdot 2 = 2\, g'(2). \tag{10}$$

Condition (4) $J''(1) = 1$ is thus equivalent to

$$\boxed{g'(2) = \frac{1}{2}.} \tag{11}$$

3.3 Translate $J(1) = 0$ and minimality

Since $f(1) = 2$,

$$J(1) = g(f(1)) = g(2). \tag{12}$$

Thus $J(1) = 0$ is equivalent to

$$\boxed{g(2) = 0.} \tag{13}$$

Also, note that for $x > 0$,

$$x + x^{-1} \geq 2 \quad \text{with equality iff } x = 1, \tag{14}$$

so $f(x) \in [2, \infty)$ and $f(x) = 2$ iff $x = 1$.

If $g(t) > 0$ for all $t > 2$ and $g(2) = 0$, then condition (3) follows immediately:

$$x \neq 1 \Rightarrow f(x) > 2 \Rightarrow J(x) = g(f(x)) > 0.$$

3.4 Conclusion

Under the structural restriction (5), conditions (3) and (4) constrain g only at and near $t = 2$:

$$g(2) = 0, \qquad g'(2) = \frac{1}{2}, \tag{15}$$

plus positivity for $t > 2$ and convexity constraints. There is no reason for g to be affine (linear); many distinct convex g satisfy these constraints.

4 Explicit counterexample family (infinitely many solutions)

Define, for any parameter $\varepsilon \geq 0$,

$$g_\varepsilon(t) := \frac{1}{2}(t-2) + \varepsilon(t-2)^2, \qquad t \in [2, \infty). \tag{16}$$

Then

$$g_\varepsilon(2) = 0, \qquad g'_\varepsilon(t) = \frac{1}{2} + 2\varepsilon(t-2) \Rightarrow g'_\varepsilon(2) = \frac{1}{2}, \tag{17}$$

so (13) and (11) hold. Moreover $g''_\varepsilon(t) = 2\varepsilon \geq 0$, so g_ε is convex on $[2, \infty)$, and for $t > 2$, $g_\varepsilon(t) > 0$.

Now define

$$J_\varepsilon(x) := g_\varepsilon\big(f(x)\big) = \frac{1}{2}\big(x + x^{-1} - 2\big) + \varepsilon\big(x + x^{-1} - 2\big)^2. \tag{18}$$

For $\varepsilon = 0$ this reduces to the manuscript's proposed J:

$$J_0(x) = \frac{1}{2}(x + x^{-1}) - 1. \tag{19}$$

For $\varepsilon > 0$, J_ε is *different* from J_0.

4.1 Verification that J_ε satisfies conditions (1)–(5)

Let

$$u(x) := x + x^{-1} - 2 = f(x) - 2. \tag{20}$$

Then $u(x) \geq 0$ for all $x > 0$ and $u(x) = 0$ iff $x = 1$.

(5) Reciprocal-invariance. By construction, $J_\varepsilon(x) = g_\varepsilon(f(x))$.

(1) Reciprocity. Since $f(x) = f(x^{-1})$, $J_\varepsilon(x) = J_\varepsilon(x^{-1})$ holds automatically.

(3) Minimality. Because $u(x) \geq 0$,

$$J_\varepsilon(x) = \frac{1}{2}u(x) + \varepsilon u(x)^2 \geq 0. \tag{21}$$

Moreover $J_\varepsilon(x) = 0$ iff $u(x) = 0$ iff $x = 1$. Hence $J_\varepsilon(1) = 0$ and $J_\varepsilon(x) > 0$ for $x \neq 1$.

(4) Normalization. Compute derivatives:

$$u'(x) = 1 - x^{-2}, \qquad u''(x) = 2x^{-3}. \tag{22}$$

Differentiate (18):

$$J'_\varepsilon(x) = \frac{1}{2}u'(x) + 2\varepsilon u(x)u'(x), \tag{23}$$

$$J''_\varepsilon(x) = \frac{1}{2}u''(x) + 2\varepsilon\big(u'(x)\big)^2 + 2\varepsilon u(x)u''(x). \tag{24}$$

At $x = 1$, $u(1) = 0$ and $u'(1) = 0$, $u''(1) = 2$, so

$$J''_\varepsilon(1) = \frac{1}{2} \cdot 2 + 2\varepsilon \cdot 0 + 2\varepsilon \cdot 0 = 1. \tag{25}$$

(2) Convexity. For $x > 0$, $u''(x) = 2x^{-3} > 0$, and $u(x) \geq 0$, $(u'(x))^2 \geq 0$. Thus for any $\varepsilon \geq 0$,

$$J_\varepsilon''(x) = \underbrace{\tfrac{1}{2}u''(x)}_{>0} + \underbrace{2\varepsilon(u'(x))^2}_{\geq 0} + \underbrace{2\varepsilon u(x)u''(x)}_{\geq 0} \; > \; 0. \tag{26}$$

So J_ε is strictly convex on $\mathbb{R}_{>0}$.

Therefore, for every $\varepsilon \geq 0$, J_ε satisfies conditions (1)–(5). Since $J_\varepsilon \neq J_0$ for $\varepsilon > 0$, there are infinitely many admissible J's. This contradicts the manuscript's uniqueness claim.

5 Conclusion?

$$\boxed{\text{Conditions (1)–(5) admit an infinite family } \{J_\varepsilon\}_{\varepsilon \geq 0}.}$$