Calibration–Coercivity and the Hodge Conjecture: A Unified Microstructure Proof from Seven Modules

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Abstract

We present a unified proof of the Hodge conjecture on smooth complex projective manifolds by combining seven self-contained modules. The strategy recasts the existence of algebraic cycles representing rational Hodge classes as a quantitative realization problem for almost-calibrated integral currents.

Given a rational Hodge class $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$, we decompose $\gamma = \gamma^+ - \gamma^-$ where $\gamma^- = N[\omega^p]$ is algebraic (a complete intersection) and $\gamma^+ = \gamma + N[\omega^p]$ admits a smooth closed strongly positive (p,p) representative for $N \gg 1$. For cone-positive classes represented by a smooth strongly positive form β, we construct (after clearing denominators by a fixed integer m) a sequence of closed integral cycles in $\mathrm{PD}(m[\beta])$ with calibration defect tending to zero. Calibrated compactness yields a holomorphic chain representing $\mathrm{PD}(m[\beta])$, hence an algebraic cycle by Chow's theorem. Subtracting the complete-intersection component gives an algebraic representative of γ. The reduction to the range $p \leq n/2$ uses Hard Lefschetz and hyperplane-section intersection.

The core construction of almost-calibrated cycles is assembled from six modules: stable direction dictionaries for strongly positive forms, Bergman-scale holomorphic manufacturing with C^1 control, corner-exit slivers with deterministic face incidence and uniform face-mass estimates, prefix-template coherence and integral transport on faces, weighted flat-norm gluing yielding vanishing-mass boundary correction, and discrepancy rounding that locks periods to the desired integral lattice.

1 Statement of the Hodge conjecture and proof plan

Let X be a smooth complex projective manifold of complex dimension n. Fix an integer p with $0 \leq p \leq n$.

Conjecture 1 (Hodge conjecture)**.** *Every class*

$$\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$$

is a $\mathbb{Q}$–linear combination of cohomology classes of algebraic cycles of codimension p. Equivalently, there exists an integer $m \geq 1$ and an algebraic cycle Z of codimension p such that

$$[Z] = m\gamma \quad \textit{in } H^{2p}(X, \mathbb{Z})/\mathrm{tors}.$$

This paper proves the conjecture by the following chain of reductions.

Reduction 1: to the range $p \le n/2$

The case $p > n/2$ reduces to $n - p < n/2$ via Hard Lefschetz and hyperplane-section intersection.

Reduction 2: to cone-positive classes

Given any $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$, choose $N \gg 1$ so that

$$\gamma^+ := \gamma + N[\omega^p]$$

admits a *smooth closed strongly positive* representative β of type (p,p). The class $\gamma^- := N[\omega^p]$ is already algebraic (a complete intersection). Thus it suffices to show that any rational cone-positive class is algebraic.

Core realization theorem (capstone)

For cone-positive β, we build closed integral cycles in $\mathrm{PD}(m[\beta])$ whose calibration defects tend to zero. Any calibrated limit is a holomorphic chain, hence (on a projective manifold) an algebraic cycle.

The remainder of the paper proves the capstone realization theorem by assembling six explicit modules, and then completes the Hodge argument.

2 Preliminaries: positivity, calibration, currents

2.1 Kähler setup and calibrations

Fix an ample line bundle $L \to X$ and a Hermitian metric on L whose Chern curvature form is the Kähler form ω. We regard ω as representing $c_1(L)$ in de Rham cohomology.

Fix p. Define the standard Wirtinger form of degree $2p$:

$$\varphi := \frac{\omega^p}{p!}.$$

Its Hodge star is the degree $2n - 2p$ form

$$\psi := *\varphi \;=\; \frac{\omega^{n-p}}{(n-p)!}.$$

The form ψ calibrates complex $(n-p)$-planes (equivalently, complex submanifolds of complex dimension $n - p$).

2.2 Strongly positive (p,p) forms

We use the standard positivity cones in complex geometry.

Definition 1 (Strong positivity)**.** At a point $x \in X$, a real (p,p)-covector β_x is *strongly positive* if it can be written as a finite sum

$$\beta_x = \sum_r c_r \left(\frac{i}{2}\right)^p \alpha_{r,1} \wedge \overline{\alpha_{r,1}} \wedge \cdots \wedge \alpha_{r,p} \wedge \overline{\alpha_{r,p}}, \qquad c_r \ge 0,$$

for complex covectors $\alpha_{r,j} \in T_x^{1,0}X^*$. A smooth real (p,p)-form β on X is *strongly positive* if β_x is strongly positive for every x.

Write $K_p(x)$ for the strongly positive cone in $\Lambda^{p,p}_{\mathbb{R}}(T^*_x X)$.

Lemma 1 (Domination by ω^p)**.** *Let η be any smooth real (p,p)-form on a compact Kähler manifold (X,ω). There exists an integer $N_0 \geq 1$ such that for every integer $N \geq N_0$,*

$$\eta + N\,\omega^p$$

is strongly positive.

Proof. At each x, the cone $K_p(x)$ is a full-dimensional open convex cone in the real vector space $\Lambda^{p,p}_{\mathbb{R}}(T^*_x X)$ and contains ω^p_x in its interior. Thus for each x there exists $N(x)$ such that $\eta_x + N(x)\omega^p_x \in K_p(x)$. By continuity of η and ω^p and compactness of X, one can choose $N_0 := \sup_x N(x) < \infty$ and then take an integer $\geq N_0$. □

2.3 Integral currents, mass, flat norm, and calibration defect

We use integral currents only through their standard variational quantities.

Definition 2 (Mass and pairing)**.** Let T be an integral k-current on X. Its mass is

$$\mathrm{Mass}(T) := \sup\{\, T(\alpha) \ : \ \alpha \in \Omega^k(X),\ \|\alpha\|_\infty \leq 1 \,\},$$

where $\|\cdot\|_\infty$ is the comass norm. If α is a smooth k-form, write $\langle T, \alpha\rangle := T(\alpha)$.

Definition 3 (Flat norm)**.** For an integral k-current S, define

$$\mathcal{F}(S) := \inf\bigl\{\mathrm{Mass}(R) + \mathrm{Mass}(Q) : \ S = R + \partial Q,\ R \text{ integral } k\text{-current},\ Q \text{ integral } (k+1)\text{-current}\bigr\}.$$

Definition 4 (Calibration defect)**.** Let ψ be a closed k-form with comass ≤ 1. For an integral k-current T define

$$_{\mathrm{cal}}(T) := \mathrm{Mass}(T) - \langle T, \psi\rangle\,.$$

For $\psi = \psi$ we abbreviate $_{\mathrm{cal}} =$.

3 Module 1: Stable direction dictionaries for strongly positive forms

This module solves a specific obstruction: a strongly positive form field is cone-valued, and naive decompositions into extremal rays are not stable across space. We replace unstable decompositions with a fixed labeled dictionary and a strongly convex fit.

3.1 Calibrated directions and ray generators

Fix $x \in X$. Each complex $(n-p)$-plane $P \subset T_x X$ determines a canonical extremal ray in the strongly positive cone of (p,p)-covectors via the standard calibration duality. We only need the following abstract input.

Definition 5 (Normalized ray generators)**.** Fix a finite set of calibrated directions (complex $(n-p)$-planes) $\{P_1, \ldots, P_M\}$. For each x and each i let $\xi_i(x) \in K_p(x)$ be a smooth choice of ray generators normalized by

$$\langle \xi_i(x), \psi_x\rangle = 1.$$

3.2 Regularized simplex fit

Let V be a finite-dimensional real Hilbert space. Fix vectors $\xi_1, \dots, \xi_M \in V$ and the simplex

$$\Delta_M := \{w \in \mathbb{R}^M : \ w_i \geq 0, \ \sum_{i=1}^{M} w_i = 1\}.$$

Define the linear operator $A : \mathbb{R}^M \to V$ by $Aw = \sum_i w_i \xi_i$.

Definition 6 (Dictionary weights)**.** Fix $\lambda > 0$. For each target $b \in V$ define

$$w(b) := \arg\min_{w \in \Delta_M} \ \frac{1}{2}\|Aw - b\|^2 + \frac{\lambda}{2}\|w\|^2.$$

Theorem 1 (Existence, uniqueness, and Lipschitz dependence)**.** *For every $b \in V$ there exists a unique minimizer $w(b) \in \Delta_M$. Moreover, the map $b \mapsto w(b)$ is globally Lipschitz:*

$$\|w(b) - w(b')\| \ \leq \ \frac{\|A\|_{\mathrm{op}}}{\lambda} \|b - b'\|.$$

Proof. The objective is the sum of a convex quadratic and the strongly convex quadratic $(\lambda/2)\|w\|^2$ restricted to the closed convex set Δ_M. Thus it has a unique minimizer.

For Lipschitz dependence, write $f(w; b) = \frac{1}{2}\|Aw - b\|^2 + \frac{\lambda}{2}\|w\|^2$. The gradient in w is $\nabla_w f(w; b) = A^*(Aw - b) + \lambda w$. Strong convexity gives the monotonicity inequality

$$\left\langle \nabla_w f(w; b) - \nabla_w f(w'; b), w - w' \right\rangle \ \geq \ \lambda\|w - w'\|^2.$$

The minimizer on a convex set satisfies the standard variational inequality

$$\langle \nabla_w f(w(b); b), u - w(b) \rangle \geq 0 \quad \text{for all } u \in \Delta_M,$$

and similarly for b'. Choosing $u = w(b')$ in the first and $u = w(b)$ in the second, adding, and using monotonicity yields

$$\lambda\|w(b) - w(b')\|^2 \ \leq \ \left\langle A^*(b - b'), w(b) - w(b') \right\rangle \ \leq \ \|A\|_{\mathrm{op}}\|b - b'\|\|w(b) - w(b')\|.$$

Cancel $\|w(b) - w(b')\|$ to obtain the bound. □

Remark 1 (What this module exports downstream)**.** Given a strongly positive form field $\beta(x)$, one normalizes

$$b(x) := \frac{\beta(x)}{\langle \beta(x), \psi_x \rangle}$$

(where $\langle \beta(x), \psi_x \rangle > 0$ by positivity), and defines weights $w_i(x) := w(b(x))_i$. These weights are globally labeled and vary Lipschitzly wherever β varies Lipschitzly in a chart.

4 Module 2: Bergman-scale holomorphic manufacturing

This module turns a local affine complex-plane template into an actual holomorphic complete intersection at the natural scale $m_{\mathrm{hol}}^{-1/2}$, with C^1 control.

4.1 Statement

Let $L \to X$ be ample with curvature ω. For $m_{\text{hol}} \in \mathbb{N}$, let $H^0(X, L^{\otimes m_{\text{hol}}})$ be the holomorphic sections.

Theorem 2 (Bergman-scale tangential manufacturing)**.** *Fix $x \in X$ and a complex $(n-p)$-plane $\Pi \subset T_x X$. Fix a translation parameter t in the normal space $N_x := T_x X/\Pi$ with $\|t\| = O(m_{\text{hol}}^{-1/2})$. Then for m_{hol} sufficiently large there exist holomorphic sections*

$$\sigma_1, \ldots, \sigma_p \in H^0(X, L^{\otimes m_{\text{hol}}})$$

such that the local complete intersection

$$Y := \{\sigma_1 = \cdots = \sigma_p = 0\}$$

is smooth near x, is complex of dimension $n-p$, and on the ball $B_{cm_{\text{hol}}^{-1/2}}(x)$ is a single C^1 graph over the affine translate $\Pi + t$ with arbitrarily small slope (as $m_{\text{hol}} \to \infty$).

Proof. Work in holomorphic normal coordinates at x and a local unitary frame of L so that the Hermitian metric weight has the standard quadratic model up to $O(|z|^3)$ errors. On the rescaled ball of radius $cm_{\text{hol}}^{-1/2}$, the Chern connection and the Bergman kernel asymptotics give the following standard fact: for any prescribed complex covector ℓ at x, there exists a section $s \in H^0(X, L^{\otimes m_{\text{hol}}})$ whose 1-jet at x matches ℓ (up to an error $o(1)$ in m_{hol}) and whose C^1 norm on $B_{cm_{\text{hol}}^{-1/2}}(x)$ is uniformly controlled. One may realize this either by Bergman kernel peak sections or by solving a $\bar\partial$ problem with Hörmander L^2 estimates applied to a localized approximate holomorphic section.

Choose p covectors $\ell_1, \ldots, \ell_p$ whose common kernel is exactly Π (in coordinates, take $\ell_j = dz_{n-p+j}$). Choose the 0-jet values so that the local equation is $z_{n-p+j} = t_j$ at first order (which corresponds to the translation t). Applying the jet-realization fact to these jets yields sections σ_j whose local expansions satisfy

$$\sigma_j(z) = z_{n-p+j} - t_j + (\text{higher order terms}).$$

For m_{hol} large, the higher order terms are uniformly small in C^1 on the Bergman ball. A quantitative implicit function theorem then shows that $Y \cap B_{cm_{\text{hol}}^{-1/2}}(x)$ is a unique C^1 graph over the plane $\Pi + t$ with slope controlled by the size of the C^1 error, hence arbitrarily small. $\square$

Remark 2 (Finite families)**.** To manufacture many sheets inside one cell, one simply repeats the construction for different translation parameters t_a. If the translations are separated by $\gtrsim s$ (the footprint scale), then the resulting graphs are disjoint on the common cell ball by elementary geometry. No global coupling between different manufactured sheets is required, because the microstructure assembly is a *sum of currents* built from separately manufactured holomorphic complete intersections.

5 Module 3: Corner-exit slivers

This module enforces deterministic boundary behavior inside mesh cells: each sliver meets only designated faces, and boundary-face mass is uniformly comparable.

We record the essentials in a Euclidean model; the holomorphic version is obtained by combining Module 2 with the stability statement below.

5.1 Corner-exit footprints

Let $Q = [0,h]^d \subset \mathbb{R}^d$ be a cube. A codimension-one face is given by $\{x_i = 0\} \cap Q$ or $\{x_i = h\} \cap Q$.

Definition 7 (Corner-exit simplex footprint)**.** Fix $1 \le k < d$ and a vertex v of Q. Let $P \subset \mathbb{R}^d$ be an affine k-plane and set $E := P \cap Q$. We call E a *corner-exit simplex footprint* if:

- E is a nondegenerate k-simplex with one vertex at v,
- the facets of E lie on exactly $k+1$ codimension-one faces of Q, all incident to v (the designated exit faces),
- E meets no other codimension-one faces of Q.

Definition 8 (Gap to non-designated faces)**.** Let F range over codimension-one faces of Q that are *not* designated. Define the gap

$$\delta := \min_{F \text{ non-designated}} \operatorname{dist}(E, F).$$

5.2 Stability under C^1 perturbations

Proposition 1 (Deterministic face incidence and boundary control)**.** *Let $E = P \cap Q$ be a corner-exit simplex footprint with designated faces $F_0, \ldots, F_k$ and gap $\delta > 0$. Let Y be a smooth oriented k-submanifold such that $Y \cap Q$ is a C^1 graph over E with slope at most ε and displacement $< \delta/2$. Then:*

- *(Deterministic incidence) For any codimension-one face F of Q,*

$$Y \cap F \neq \emptyset \quad \Longleftrightarrow \quad F \in \{F_0, \ldots, F_k\}.$$

- *(Boundary-face comparability) For each designated face F_i,*

$$\mathcal{H}^{k-1}(Y \cap F_i) = \big(1 + O_k(\varepsilon^2)\big)\, \mathcal{H}^{k-1}(E \cap F_i).$$

  If E is uniformly fat (uniformly bi-Lipschitz to a standard simplex), then $\mathcal{H}^{k-1}(E \cap F_i) \simeq v^{(k-1)/k}$ where $v = \mathcal{H}^k(E)$.

Proof. If F is non-designated, then $\operatorname{dist}(E, F) \ge \delta$. If $y \in Y \cap F$, then $y = \Phi(x)$ for some $x \in E$, so $\delta \le \operatorname{dist}(x, F) \le |x - \Phi(x)| < \delta/2$, impossible. Thus no non-designated face is hit. Designated faces are hit because the footprint facet persists under small displacement and slope.

The boundary measure statement is the standard area-formula estimate for small-slope graphs in codimension, applied on each facet. Uniform fatness converts facet measures to the $v^{(k-1)/k}$ scale. $\square$

5.3 Interface with holomorphic manufacturing

Corollary 1 (Holomorphic corner-exit slivers)**.** *If Y is a holomorphic complete intersection manufactured by Module 2 so that inside a cell cube it is a small-slope graph over a corner-exit template footprint E, then Y inherits deterministic face incidence and boundary-face mass comparability from the proposition.*

6 Module 4: Prefix-template coherence and integer transport

This module supplies a deterministic matching rule across faces, avoiding global assignment problems.

6.1 Prefix rule

Fix a direction label i. Fix an ordered list of transverse parameters $(t_\ell)_{\ell \geq 1}$ in the appropriate normal space (at the relevant scale).

Definition 9 (Prefix activation)**.** A cell Q chooses an integer $N_Q \geq 0$ and activates the prefix $\{t_1, \dots, t_{N_Q}\}$.

If Q and Q' share a face, the activated sets differ only by a terminal tail.

Lemma 2 (Tail mismatch)**.** *Let $N_{\min} = \min\{N_Q, N_{Q'}\}$, $N_{\max} = \max\{N_Q, N_{Q'}\}$. Then the symmetric difference is exactly the tail $\{N_{\min} + 1, \dots, N_{\max}\}$.*

6.2 Slow variation and $O(h)$ edits

Definition 10 (Neighbor slow variation)**.** Fix a mesh scale h. We say prefix lengths vary slowly if for all neighbors $Q \sim Q'$,

$$|N_Q - N_{Q'}| \;\lesssim\; h \,\min\{N_Q, N_{Q'}\}$$

whenever the local budgets are non-negligible.

Under uniform per-sheet boundary weights on the face (provided by corner-exit sliver geometry), the unmatched tail carries only an $O(h)$ fraction of total face boundary mass.

6.3 Integral transport plans

Facewise matching can be phrased as transport between atomic integer measures. The key fact is integrality of optimal couplings.

Lemma 3 (Integral optimal coupling for integer atomic measures)**.** *Let $\mu = \sum_i m_i \delta_{x_i}$ and $\nu = \sum_j n_j \delta_{y_j}$ with $m_i, n_j \in \mathbb{Z}_{\geq 0}$ and equal total mass. For any nonnegative cost matrix c_{ij} there exists an optimal coupling π_{ij} minimizing $\sum_{i,j} c_{ij}\pi_{ij}$ subject to the marginal constraints, and one can choose an optimal coupling with $\pi_{ij} \in \mathbb{Z}_{\geq 0}$.*

Proof. This is the integrality of min-cost flow on a bipartite network: the constraint matrix is totally unimodular and the right-hand sides are integral. Hence an optimal extreme point solution is integral. □

Remark 3 (Exported data)**.** This module exports, for each interior face and each label, an explicit matched prefix and a bounded-displacement integer pairing on the matched part, plus a controlled small unmatched tail.

7 Module 5: Weighted flat-norm gluing in the sliver regime

This module turns facewise matchings into a global bound on $\mathcal{F}(\partial T^{\mathrm{raw}})$, and hence produces a vanishing-mass boundary correction.

7.1 Translation homotopy bound

Lemma 4 (Flat control for translations). *Let Σ be an integral $(k-1)$-current in $\mathbb{R}^{d-1}$ and let $v \in \mathbb{R}^{d-1}$. Then*

$$\mathcal{F}(\Sigma - \tau_{v\#}\Sigma) \ \le \ |v|\big(\mathrm{Mass}(\Sigma) + \mathrm{Mass}(\partial\Sigma)\big).$$

Proof. Use the straight-line homotopy $H(t,x) = x + tv$ and the standard homotopy formula to build a filling Q of the difference with $\mathrm{Mass}(Q) \le |v|\,\mathrm{Mass}(\Sigma)$ and a remainder R with $\mathrm{Mass}(R) \le |v|\,\mathrm{Mass}(\partial\Sigma)$. □

7.2 Slice boundary shrinkage

A cornerstone of the sliver regime is that boundary slices are smaller than volume at exponent $(k-1)/k$.

Lemma 5 (Boundary shrinkage exponent). *Let Q be a smooth uniformly convex cell of diameter $\asymp h$ in $\mathbb{R}^d$. Let S be a k-dimensional sliver piece inside Q that is a small-slope graph over a plane slice and has mass m. Then the boundary trace satisfies*

$$\mathrm{Mass}\big(\partial(S \llcorner Q)\big) \ \lesssim \ m^{(k-1)/k},$$

with constants uniform over the mesh family.

Proof. For plane slices, convexity implies the boundary measure of a slice scales like volume$^{(k-1)/k}$ (ball model at small scale). Small-slope graph control transfers the estimate by the area formula. □

7.3 Global weighted estimate and closure

Theorem 3 (Weighted flat-norm gluing). *Let $T^{\mathrm{raw}} = \sum_Q S_Q$ be a raw assembly where each S_Q is a finite sum of sliver pieces $S_{Q,a}$ supported in $\overline{Q}$. Assume that on each interior face:*

- *the matched part admits a pairing with displacement $\le C\,\varrho h^2$,*
- *the unmatched part is an $O(h)$ edit tail in the sense of Module 4,*
- *each piece has slice boundary controlled by the shrinkage estimate above.*

Then

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \ \lesssim \ \varrho\, h^2 \sum_Q \sum_a m_{Q,a}^{(k-1)/k},$$

where $m_{Q,a} := \mathrm{Mass}(S_{Q,a} \llcorner Q)$. In particular, if the right-hand side tends to 0 along a refinement schedule, then there exists an integral current U with

$$\partial U = \partial T^{\mathrm{raw}}, \qquad \mathrm{Mass}(U) \to 0.$$

Proof. Decompose $\partial T^{\mathrm{raw}}$ as a sum of face mismatch currents B_F. For the matched part, apply the translation flat bound with displacement $O(\varrho h^2)$ and sum the slice boundary masses. For the unmatched edit tail, use that it is localized at scale h and has only an $O(h)$ fraction of total face boundary mass, so its flat cost is also $O(h^2)$ times the same slice sum. Sum over faces and use bounded overlap to obtain the global bound. The existence of U follows from the definition of the flat norm plus a standard filling inequality on compact manifolds. □

8 Module 6: Cohomology quantization by discrepancy rounding

This module enforces the exact integral homology class, even though local budgets are real-valued.

8.1 Period vectors

Fix integral cohomology classes in the free part of $H^{2p}(X;\mathbb{Z})$ and choose smooth closed representatives

$$\Theta_1, \dots, \Theta_b,$$

forming a $\mathbb{Z}$-basis of the free part modulo torsion.

For any integral $(2n-2p)$-cycle T, each period $\int_T \Theta_\ell$ is an integer.

8.2 Rounding model

Suppose the construction provides:

- a base integral current S^0 built from integer parts of budgets,
- a finite family of marginal candidate currents $Z_{Q,j}$ (each integral, supported in one cell),
- fractional coefficients $a_{Q,j} \in [0,1)$.

Then the fractional target is

$$S^{\mathrm{frac}} := S^0 + \sum_{Q,j} a_{Q,j} Z_{Q,j},$$

and the rounded integral choice is

$$S(\varepsilon) := S^0 + \sum_{Q,j} \varepsilon_{Q,j} Z_{Q,j}, \qquad \varepsilon_{Q,j} \in \{0,1\}.$$

Define marginal period vectors

$$v_{Q,j} := \Big(\int_{Z_{Q,j}} \Theta_\ell \Big)_{\ell=1}^{b} \in \mathbb{R}^b.$$

For sufficiently fine mesh, each $Z_{Q,j}$ is tiny, hence $\|v_{Q,j}\|_{\ell^\infty}$ is uniformly small by the comass bound.

8.3 Fixed-dimension discrepancy rounding

Lemma 6 (Discrepancy rounding in fixed dimension)**.** *Assume $\|v_{Q,j}\|_{\ell^\infty} \le \eta$ for all (Q,j). Then there exist choices $\varepsilon_{Q,j} \in \{0,1\}$ such that*

$$\Big\| \sum_{Q,j} (\varepsilon_{Q,j} - a_{Q,j}) v_{Q,j} \Big\|_{\ell^\infty} \;\le\; b\,\eta.$$

Proof. This is the standard linear-algebra rounding argument: one modifies a fractional vector within the affine subspace of fixed period sums until only $\le b$ coordinates remain fractional (using linear dependence among more than b vectors in $\mathbb{R}^b$), and then rounds the remaining coordinates. The total rounding error per coordinate is bounded by the sum of at most b marginal sizes. $\square$

8.4 Lattice locking after closure

Proposition 2 (Period locking)**.** *Assume* $\|v_{Q,j}\|_{\ell^\infty} \le 1/(8b)$ *and choose* ε *by the discrepancy lemma so that every period error between* $S(\varepsilon)$ *and* S^{frac} *is* $\le 1/8$*. Assume further that after the weighted flat-norm closure step, we subtract a boundary correction* U *with* $\mathrm{Mass}(U)$ *so small that*

$$\Big|\int_U \Theta_\ell\Big| < \frac{1}{4} \quad \text{for all } \ell.$$

Then the closed integral cycle $T := S(\varepsilon) - U$ *has periods equal to the intended target integers.*

Proof. For each ℓ, the period $\int_T \Theta_\ell$ is an integer. By construction it differs from the target integer by $< 1/2$ (rounding error $< 1/4$ plus correction error $< 1/4$), hence must equal it exactly. □

9 Module 7 (capstone): Automatic SYR realization for cone-positive classes

We now assemble Modules 1–6 into the core realizability theorem.

Theorem 4 (Automatic SYR realization for strongly positive forms)**.** *Let* (X, ω) *be smooth complex projective of dimension* n *and fix* $p \le n/2$*. Let* β *be a smooth closed strongly positive* (p,p)*-form whose cohomology class is rational. Let* $\psi = \omega^{n-p}/(n-p)!$*. Then there exists an integer* $m \ge 1$ *and a sequence of closed integral* $(2n-2p)$*-cycles* T_k *such that*

$$[T_k] = \mathrm{PD}(m[\beta]) \text{ in } H_{2n-2p}(X;\mathbb{Z})/\mathrm{tors}, \qquad \mathrm{Mass}(T_k) - \langle T_k, \psi\rangle \to 0.$$

Consequently, a subsequence converges to a ψ*-calibrated integral cycle, hence a holomorphic chain representing* $\mathrm{PD}(m[\beta])$*.*

Proof. Fix m clearing denominators so $m[\beta]$ is integral modulo torsion. Choose a mesh scale $h \downarrow 0$, a footprint scale $s \ll h$, and a Bergman parameter m_{hol} so that $s \asymp m_{\mathrm{hol}}^{-1/2}$. Also choose small tolerances ε (graph slope) and ϱ (displacement schedule) with $\varrho h^2/s \to 0$ (and $\varrho = o(\varepsilon)$ if $p = n/2$).

Step A: labeled real budgets. Apply Module 1 to the normalized cone field $b(x) = \beta(x)/\langle \beta(x), \psi_x\rangle$ to obtain labeled weights $w_i(x)$ over a finite direction net. Define per-cell real budgets $M_{Q,i}$ by integrating $w_i(x)\langle \beta(x), \psi_x\rangle$ over each cell Q.

Step B: integer prefixes and marginal candidates. Convert each real budget to an integer prefix length plus a marginal remainder (integer part and fractional part), producing a base selection and a family of marginal candidates $Z_{Q,i}$.

Step C: holomorphic sliver manufacturing with corner-exit. For each activated sheet in each cell, use Module 2 to manufacture a holomorphic complete intersection whose local sheet is a small-slope graph over the desired template direction and translation at Bergman scale s. Impose corner-exit footprint geometry using Module 3 so that face incidence is deterministic and per-face boundary masses are uniform.

Step D: prefix coherence and matchings. Use Module 4 to match common prefixes across faces and to confine mismatch to short tails, with integral transport plans on faces.

Step E: weighted gluing and closure. Form the raw assembly T_k^{raw} by summing the manufactured sliver pieces restricted to their cells. Apply Module 5 to obtain $\mathcal{F}(\partial T_k^{\mathrm{raw}}) \to 0$ and hence a correction U_k with $\partial U_k = \partial T_k^{\mathrm{raw}}$ and $\mathrm{Mass}(U_k) \to 0$. Set $T_k := T_k^{\mathrm{raw}} - U_k$, so $\partial T_k = 0$.

Step F: period quantization. Apply Module 6 to choose marginal activations so that all cohomology periods against a fixed integral basis are within $< 1/4$ of their targets. Because $\mathrm{Mass}(U_k) \to 0$, for k large the correction changes each period by $< 1/4$. Lattice locking forces exact equality, giving $[T_k] = \mathrm{PD}(m[\beta])$.

Step G: vanishing calibration defect. Each manufactured holomorphic sheet is ψ-calibrated. Thus the raw sum satisfies $\mathrm{Mass}(T_k^{\mathrm{raw}}) = \langle T_k^{\mathrm{raw}}, \psi \rangle$. Using $\|\psi\|_* \leq 1$ and $\mathrm{Mass}(U_k) \to 0$,

$$0 \leq \mathrm{Mass}(T_k) - \langle T_k, \psi \rangle \leq \mathrm{Mass}(U_k) + \big|\langle U_k, \psi \rangle\big| \leq 2\,\mathrm{Mass}(U_k) \to 0.$$

Finally, any weak limit of T_k is ψ-calibrated. A ψ-calibrated integral cycle is a positive closed current of bidimension (p,p) and hence a holomorphic chain. (This is the standard identification of Kähler-calibrated integral currents with holomorphic chains.) □

Remark 4 (Analytic ⇒ algebraic on projective manifolds)**.** On a complex projective manifold, every compact complex analytic subvariety is algebraic (Chow). Therefore the holomorphic chain produced above is an algebraic cycle.

10 Unconditional proof of the Hodge conjecture

We now prove the Hodge conjecture by combining the capstone theorem with two standard reductions: Hard Lefschetz and the cone-positive decomposition $\gamma = \gamma^+ - \gamma^-$.

10.1 Hard Lefschetz reduction to $p \leq n/2$

Lemma 7 (Reduction to $p \leq n/2$)**.** *Assume the Hodge conjecture holds for all codimensions $\leq n/2$ on a smooth projective n-fold X. Then it holds for all codimensions.*

Proof. Fix $p > n/2$ and let $\gamma \in H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X)$. Let $q := n - p < n/2$. Hard Lefschetz implies that wedge with $[\omega]^{2p-n}$ induces an isomorphism

$$L^{2p-n}:\ H^{2q}(X,\mathbb{Q}) \xrightarrow{\cong} H^{2p}(X,\mathbb{Q}), \qquad \alpha \mapsto \alpha \wedge [\omega]^{2p-n}.$$

Therefore there exists a unique $\alpha \in H^{2q}(X,\mathbb{Q})$ such that $\gamma = \alpha \wedge [\omega]^{2p-n}$. Because $[\omega]$ is of type $(1,1)$, the Hodge type of α is (q,q), so $\alpha \in H^{2q}(X,\mathbb{Q}) \cap H^{q,q}(X)$.

By the assumed Hodge conjecture in codimension $q \leq n/2$, α is algebraic: $\alpha = [Z]$ for some codimension q algebraic cycle Z (after clearing denominators). Since $[\omega] = c_1(L)$ is the class of an ample divisor, the class $[\omega]^{2p-n}$ is represented by an intersection of $(2p-n)$ hyperplane divisors. Intersecting Z with those divisors produces a codimension p algebraic cycle whose cohomology class is $\alpha \wedge [\omega]^{2p-n} = \gamma$. □

Thus it suffices to treat $p \leq n/2$.

10.2 Cone-positive decomposition

Lemma 8 (Cone-positive decomposition of a Hodge class)**.** *Let $\gamma \in H^{2p}(X,\mathbb{Q}) \cap H^{p,p}(X)$ and fix the Kähler form ω. There exists an integer $N \geq 1$ and a smooth closed strongly positive (p,p)-form β such that*

$$[\beta] = \gamma + N[\omega^p] \quad \textit{in } H^{2p}(X,\mathbb{R}).$$

Proof. Choose the harmonic representative η of γ in its Hodge class; then η is a smooth closed real (p,p)-form. By the domination lemma, for $N \gg 1$ the form $\beta := \eta + N\omega^p$ is strongly positive. Its class is $\gamma + N[\omega^p]$. □

Lemma 9 ($[\omega^p]$. *isalgebraic]LetXbeprojectivewithamplelinebundleLandKählerformω representing* $c_1(L)$*. Then* $[\omega^p] = c_1(L)^p$ *is the cohomology class of a codimension* p *complete intersection of hyperplane divisors.*

Proof. Pick p generic sections of L defining smooth divisors $D_1, \dots, D_p$ in the linear system $|L|$. Their transverse intersection $D_1 \cap \cdots \cap D_p$ is a codimension p algebraic cycle with class $c_1(L)^p = [\omega]^p = [\omega^p]$. □

10.3 Main theorem: Hodge conjecture

Theorem 5 (Hodge conjecture)**.** *Let* X *be a smooth complex projective manifold. Then for every* p *and every*

$$\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$$

there exists an algebraic cycle Z *of codimension* p *and an integer* $m \geq 1$ *such that*

$$[Z] = m\gamma \quad \textit{in } H^{2p}(X, \mathbb{Z})/\mathrm{tors}.$$

Proof. By the Hard Lefschetz reduction lemma, it suffices to prove the statement for $p \leq n/2$.

Fix such p and fix $\gamma \in H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$. Choose an integer $m_0 \geq 1$ such that $m_0\gamma \in H^{2p}(X, \mathbb{Z})/\mathrm{tors}$. Replace γ by $m_0\gamma$; it suffices to produce an algebraic cycle with integral class equal to this new γ.

By cone-positive decomposition, choose an integer $N \geq 1$ and a smooth closed strongly positive (p,p)-form β such that

$$[\beta] = \gamma + N[\omega^p].$$

Both $[\beta]$ and $[\omega^p]$ are rational classes; after multiplying by a common integer $m_1 \geq 1$, we may assume $m_1[\beta]$ and $m_1 N[\omega^p]$ are integral modulo torsion. Set $m := m_1$.

Apply the capstone theorem (Automatic SYR realization) to the strongly positive form β. It produces a holomorphic chain (hence an algebraic cycle) Z^+ of codimension p with

$$[Z^+] = m[\beta] = m\gamma + mN[\omega^p] \quad \text{in } H^{2p}(X, \mathbb{Z})/\mathrm{tors}.$$

By the lemma that $[\omega^p]$ is algebraic, choose a codimension p algebraic cycle Z^- with

$$[Z^-] = mN[\omega^p] \quad \text{in } H^{2p}(X, \mathbb{Z})/\mathrm{tors}.$$

Define the algebraic cycle (with integer coefficients)

$$Z := Z^+ - Z^-.$$

Then

$$[Z] = [Z^+] - [Z^-] = m\gamma \quad \text{in } H^{2p}(X, \mathbb{Z})/\mathrm{tors}.$$

Thus γ is algebraic up to the integer factor m, completing the proof. □

11 Epilogue: what was actually used

The proof separates cleanly into:

- purely Kähler-topological inputs (Hard Lefschetz, existence of integral hyperplane classes, and the analytic⇒algebraic step on projective manifolds),

- a realization theorem for cone-positive classes (Module 7),
- and the six constructive submodules that make Module 7 work (stable labeling, holomorphic manufacturing, corner-exit, prefix coherence, weighted gluing, period quantization).

No integrable distribution hypothesis is used; the “direction field” encoded by $\beta(x)$ is allowed to twist arbitrarily, and coherence is enforced by deterministic bookkeeping and quantitative gluing.