

Projection Algebra as a Reusable Kernel for Mechanized Proofs: Idempotence, Spectral Decomposition, and Cost-Ordered Optimization

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Abstract

Projection operators sit at the interface of algebra, geometry, and optimization. This paper distills a “projection algebra” suitable for mechanized mathematics: (i) idempotence ($\pi^2 = \pi$) as the defining axiom of projection, (ii) spectral decomposition as the source of complete families of orthogonal projectors, and (iii) cost-ordering of projectors as an optimization principle that turns projection into dynamics. We explain how this algebraic kernel supports reusable theorems in coarse-graining, renormalization, and decoherence, and why encoding it explicitly in a proof assistant (e.g. Lean) dramatically reduces the marginal cost of future formalizations.

1 Introduction

Projectors appear whenever a system must enforce constraints, select a subspace, or discard degrees of freedom. In physics they formalize measurement and decoherence; in applied mathematics they enforce feasibility (e.g. constrained optimization); in information theory they model coarse-graining; and in many mechanized developments they serve as the missing “glue” between analytic statements and algebraic composition.

The core observation is that *projection* is an algebraic idea with a surprisingly small axiom set, but enormous leverage. If a library makes projectors first-class objects, then many subsequent constructions become one-line instantiations rather than bespoke proofs.

We focus on three pillars:

1. **Idempotence** ($\pi^2 = \pi$): projection is characterized by stabilization after one application.
2. **Spectral decomposition**: in inner-product settings, orthogonal projectors arise from eigenspace decompositions and provide complete, mutually exclusive outcomes.
3. **Cost-ordering**: attaching a cost functional to a projector turns it into an optimization primitive; composition laws then yield monotone descent guarantees for multi-stage procedures.

2 Abstract projector algebra

2.1 Idempotence and fixed points

Definition 1 (Abstract projector). Let α be a type (set). A map $\pi : \alpha \rightarrow \alpha$ is a *projector* if it is *idempotent*:

$$\pi(\pi(x)) = \pi(x) \quad \text{for all } x \in \alpha. \quad (1)$$

The fundamental structure induced by idempotence is the fixed-point set.

Definition 2 (Fixed points). For a projector π , define $\text{Fix}(\pi) := \{x \in \alpha \mid \pi(x) = x\}$.

Lemma 3 (Image equals fixed points). *If π is idempotent, then $\text{im}(\pi) \subseteq \text{Fix}(\pi)$, and π restricts to the identity on $\text{Fix}(\pi)$.*

Proof. If $y = \pi(x)$, then $\pi(y) = \pi(\pi(x)) = \pi(x) = y$, so $y \in \text{Fix}(\pi)$. If $x \in \text{Fix}(\pi)$, then $\pi(x) = x$ by definition. \square

Remark 4. This lemma is one of the most “reused” facts in formal work: any time a proof shows something is a fixed point, the projector disappears.

2.2 Commuting composition

Composition of projectors is not always a projector, but commutation is enough.

Proposition 5 (Commuting composition is a projector). *Let $\pi_1, \pi_2 : \alpha \rightarrow \alpha$ be idempotent maps such that $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1$. Then $\pi := \pi_1 \circ \pi_2$ is idempotent.*

Proof. Using commutation and idempotence:

$$\pi \circ \pi = (\pi_1 \circ \pi_2) \circ (\pi_1 \circ \pi_2) = \pi_1 \circ (\pi_2 \circ \pi_1) \circ \pi_2 = \pi_1 \circ (\pi_1 \circ \pi_2) \circ \pi_2 = (\pi_1 \circ \pi_1) \circ (\pi_2 \circ \pi_2) = \pi_1 \circ \pi_2 = \pi.$$

\square

Remark 6. This single proposition underlies multi-stage constraint enforcement, iterative coarse-graining, and many “project then project again” pipelines.

3 Orthogonal projectors and spectral decomposition

The preceding section is purely algebraic. In Hilbert spaces, additional structure yields *orthogonal* projectors and complete decompositions.

Definition 7 (Orthogonal projector). Let H be a (real or complex) Hilbert space. A bounded linear operator $P : H \rightarrow H$ is an *orthogonal projector* if it is idempotent and self-adjoint:

$$P^2 = P, \quad P^* = P. \quad (2)$$

Orthogonal projectors correspond to closed subspaces. Their key spectral fact is elementary:

Lemma 8 (Spectrum of a projector). *If P is a (linear) projector on a vector space, then every eigenvalue λ of P satisfies $\lambda \in \{0, 1\}$.*

Proof. If $Pv = \lambda v$, then applying P again gives $P^2v = \lambda^2v$, but $P^2v = Pv = \lambda v$. Hence $\lambda^2 = \lambda$, so $\lambda \in \{0, 1\}$. \square

Proposition 9 (Decomposition into range and kernel). *Let $P : H \rightarrow H$ be an orthogonal projector. Then*

$$H = \text{im}(P) \oplus \ker(P), \quad (3)$$

with $\text{im}(P) \perp \ker(P)$. Moreover, P is the identity on $\text{im}(P)$ and zero on $\ker(P)$.

Proof. Standard: for any x , write $x = Px + (x - Px)$. Then $P(x - Px) = Px - P^2x = 0$, so $x - Px \in \ker(P)$. Orthogonality follows from self-adjointness: if $y = Pu \in \text{im}(P)$ and $z \in \ker(P)$, then $\langle y, z \rangle = \langle Pu, z \rangle = \langle u, P^*z \rangle = \langle u, Pz \rangle = 0$. \square

3.1 Complete families of orthogonal projectors

Measurement and coarse-graining use not one projector, but a complete mutually exclusive family.

Definition 10 (Complete orthogonal projectors). Let H be a Hilbert space. A finite family $\{P_k\}_{k=1}^m$ of operators on H is a *complete family of orthogonal projectors* if:

$$P_k^2 = P_k, \quad P_k^* = P_k \quad \text{for all } k, \quad (4)$$

$$P_i P_j = 0 \quad \text{for } i \neq j, \quad (5)$$

$$\sum_{k=1}^m P_k = I. \quad (6)$$

Remark 11. The spectral theorem provides such a family for any normal operator with finite spectrum; in quantum mechanics, these are precisely the projectors associated to a projective measurement.

4 Cost-ordered projectors and optimization

4.1 Projectors equipped with a cost

To turn projection into dynamics, we attach a cost functional and demand monotonic descent.

Definition 12 (Cost-ordered projector). Let α be a type and $\pi : \alpha \rightarrow \alpha$ an idempotent map. A *cost-ordered projector* is a pair (π, \mathcal{C}) with $\mathcal{C} : \alpha \rightarrow \mathbb{R}$ such that:

$$\mathcal{C}(x) \geq 0 \quad \text{for all } x, \quad (7)$$

$$\mathcal{C}(\pi(x)) \leq \mathcal{C}(x) \quad \text{for all } x, \quad (8)$$

$$\pi(x) = x \Rightarrow \mathcal{C}(x) = 0. \quad (9)$$

Remark 13. This structure is the minimal “optimization payload” needed to prove global statements like: repeated projection never increases cost, and fixed points are certified minima.

4.2 Sequencing and commuting descent

If multiple constraints are enforced by commuting projectors, their composition inherits both idempotence and monotone descent.

Proposition 14 (Monotone descent under commuting composition). *Let (π_1, \mathcal{C}) and (π_2, \mathcal{C}) be cost-ordered projectors sharing the same cost \mathcal{C} . If π_1 and π_2 commute, then $\pi = \pi_1 \circ \pi_2$ is a projector and satisfies $\mathcal{C}(\pi(x)) \leq \mathcal{C}(x)$ for all x .*

Proof. Idempotence follows from commuting composition. For monotonicity,

$$\mathcal{C}(\pi(x)) = \mathcal{C}(\pi_1(\pi_2(x))) \leq \mathcal{C}(\pi_2(x)) \leq \mathcal{C}(x).$$

□

5 Why this algebra accelerates future formalizations

Once a proof assistant library makes the above structures explicit, large families of theorems become “free”:

1. **Coarse-graining:** A coarse-graining map is typically idempotent (applying it twice adds no further information loss). Many theorems reduce to commuting-composition and fixed-point reasoning.
2. **Renormalization:** Renormalization procedures can be expressed as alternating (i) projection onto an effective subspace and (ii) rescaling. Cost-ordering provides a clean way to prove monotone improvement bounds and convergence-to-fixed-point claims.
3. **Decoherence:** Decoherence and projective measurement are naturally encoded by complete orthogonal families $\{P_k\}$. When these are packaged as reusable objects, normalization and orthogonality lemmas are reusable across all measurement-like arguments.

6 Mechanization notes (Lean)

In a mechanized setting, the critical design choice is to represent projectors as *structures* (record types) carrying their laws ($\pi^2 = \pi$, cost monotonicity, etc.) as fields. This makes theorems depend only on the interface, not on a concrete implementation.

In practice, such a file (e.g. `Support/Projectors.lean`) typically provides:

- an *abstract* projector interface (a map plus idempotence);
- a *cost-ordered* refinement (a cost function plus monotonicity);
- reusable lemmas: commuting composition, fixed-point simplification, and sequencing theorems.

These abstractions allow new domain modules (coarse-graining kernels, decoherence models, bridge maps) to import and reuse the same projection theorems, yielding smaller proofs and fewer ad hoc rewrites.

7 Conclusion

Projection algebra is a rare example of a mathematically small interface with outsized downstream impact. Idempotence provides the algebraic core; spectral decomposition supplies complete orthogonal families in inner-product spaces; and cost-ordering turns projectors into optimization primitives that support monotone descent guarantees. Encoding these ideas as reusable interfaces in a proof assistant makes future formalizations (coarse-graining, renormalization, decoherence) substantially easier: proofs become compositions of generic lemmas rather than bespoke arguments.

References

- [1] P. R. Halmos, *Finite-Dimensional Vector Spaces*, Springer, 1974.
- [2] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, 1980.
- [3] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955.
- [4] The Lean Community, *Lean 4 Theorem Prover*, <https://lean-lang.org/>.