Automatic SYR Realization for Smooth Cone-Valued (p, p)-Forms

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Abstract

Let (X, ω) be a smooth complex projective manifold of complex dimension n and fix $p \leq n/2$. Let β be a smooth closed strongly positive (p, p)-form and let

$$\psi = \frac{\omega^{n-p}}{(n-p)!}$$

denote the associated Kähler calibration. We prove an automatic realizability theorem: there exists a fixed integer $m \geq 1$ and a sequence of closed integral $(2n-2p)$-cycles T_k representing $\mathrm{PD}(m[\beta])$ such that the calibration defect

$$\mathrm{Mass}(T_k) - \langle T_k, \psi \rangle$$

tends to zero. Equivalently, $\mathrm{Mass}(T_k) \to m \int_X \beta \wedge \psi$.

The proof assembles six independent modules: (i) a stable finite direction dictionary with Lipschitz weights for β; (ii) Bergman-scale holomorphic manufacturing of template sheets with C^1 control; (iii) corner-exit slivers ensuring deterministic face incidence and uniform boundary-face mass estimates; (iv) prefix-template coherence producing global face matching up to controlled edits; (v) weighted flat-norm gluing yielding $\mathcal{F}(\partial T^{\mathrm{raw}}) \to 0$ and hence a vanishing-mass boundary correction; and (vi) discrepancy-based cohomology quantization ensuring the exact integral homology class.

By calibrated compactness, a subsequence converges to a ψ-calibrated integral cycle, hence a holomorphic chain, providing a general analytic-cycle realization principle for cone-positive classes.

1 Introduction

A smooth strongly positive (p,p)-form β is a pointwise cone-valued object: at each $x \in X$ it lives in the strongly positive cone of (p,p)-covectors. The central question addressed here is whether such a form can be realized, *automatically and without integrability hypotheses*, by honest integral cycles whose mass is asymptotically forced by the Kähler calibration.

The relevant notion is "SYR realizability" (Stationary Young-measure realizability): a form is SYR-realizable if there exists a sequence of integral cycles in a fixed homology class whose calibration defect tends to zero. When the calibration is Kähler, calibrated integral cycles are holomorphic chains, so SYR yields analytic-cycle realization.

The novelty of the capstone theorem is not a new local holomorphic existence statement by itself, but a deterministic global pipeline that turns a *smooth cone-valued field* into *closed integral cycles in an exact class* while forcing the calibration defect to vanish.

A basic integrality requirement

An integral cycle represents an integral homology class. Therefore, exact realizability of $\mathrm{PD}(m[\beta])$ by integral cycles requires that $[\beta]$ be a rational class (so some integer multiple is integral). This is the only global arithmetic hypothesis needed.

2 SYR and almost-calibration

Let X be a smooth manifold. We use integral currents only at the level of their mass, boundary, and pairing with smooth forms.

Definition 1 (Calibration defect)**.** Let ψ be a smooth closed k-form on X with comass ≤ 1 (so $\langle T, \psi \rangle \leq \mathrm{Mass}(T)$ for all integral k-currents T). For an integral k-current T, define the calibration defect

$$\mathrm{Def}_{\mathrm{cal}}(T) \;:=\; \mathrm{Mass}(T) - \langle T, \psi \rangle \,.$$

Definition 2 (SYR realizability)**.** Let (X, ω) be Kähler of complex dimension n and fix p with $1 \leq p \leq n$. Let $\psi = \omega^{n-p}/(n-p)!$, a closed form of degree $2n - 2p$. A smooth closed strongly positive (p,p)-form β is *SYR-realizable* if there exist an integer $m \geq 1$ and a sequence of closed integral $(2n-2p)$-cycles T_k such that:

- $\partial T_k = 0$ for all k;
- $[T_k] = \mathrm{PD}(m[\beta])$ in $H_{2n-2p}(X;\mathbb{Z})/\mathrm{Tor}$ for all k;
- $\mathrm{Def}_{\mathrm{cal}}(T_k) \to 0$ as $k \to \infty$.

Remark 1 (Fixed pairing once the class is fixed)**.** Because ψ is closed, if $\partial T = 0$ then $\langle T, \psi\rangle$ depends only on the homology class of T. If $[T] = \mathrm{PD}(m[\beta])$ in $H_{2n-2p}(X;\mathbb{Z})/\mathrm{Tor}$, then

$$\langle T, \psi\rangle = \big\langle \mathrm{PD}(m[\beta]), [\psi]\big\rangle = m \int_X \beta \wedge \psi,$$

so SYR is equivalent to $\mathrm{Mass}(T_k) \to m \int_X \beta \wedge \psi$.

Lemma 1 (Almost-calibrated limit is calibrated)**.** *Let ψ be a closed calibration form (comass ≤ 1). Let T_k be integral cycles with $\sup_k \mathrm{Mass}(T_k) < \infty$ and $\mathrm{Def}_{\mathrm{cal}}(T_k) \to 0$. Then any weak limit T of a subsequence is ψ-calibrated, meaning $\mathrm{Def}_{\mathrm{cal}}(T) = 0$.*

Proof. By weak convergence, $\langle T_k, \psi\rangle \to \langle T, \psi\rangle$ because ψ is smooth. By lower semicontinuity of mass, $\mathrm{Mass}(T) \leq \liminf_k \mathrm{Mass}(T_k)$. Therefore

$$\mathrm{Def}_{\mathrm{cal}}(T) = \mathrm{Mass}(T) - \langle T, \psi\rangle \leq \liminf_k \mathrm{Mass}(T_k) - \lim_k \langle T_k, \psi\rangle = \liminf_k \mathrm{Def}_{\mathrm{cal}}(T_k) = 0.$$

Since always $\mathrm{Def}_{\mathrm{cal}}(T) \geq 0$, we get $\mathrm{Def}_{\mathrm{cal}}(T) = 0$. □

Proposition 1 (Kähler-calibrated cycles are holomorphic chains)**.** *Let (X, ω) be a complex manifold and $\psi = \omega^{n-p}/(n-p)!$. If T is an integral $(2n-2p)$-cycle with $\mathrm{Def}_{\mathrm{cal}}(T) = 0$, then T is a positive closed current of bidimension (p,p). Consequently, T is a holomorphic chain, i.e. a finite sum $\sum_j m_j[V_j]$ where V_j are irreducible complex analytic subvarieties of codimension p and $m_j \in \mathbb{N}$.*

Remark 2. The capstone theorem below produces almost-calibrated cycles T_k. The proposition converts the calibrated limit into an analytic (and, in the projective case, algebraic) cycle.

3 Parameter schedule

The construction uses several auxiliary parameters. The only parameter that must be fixed *once and for all* is the integer m that clears denominators in cohomology. Everything else is driven to zero along a refinement schedule.

Definition 3 (Auxiliary parameters)**.** We use:

- $h \downarrow 0$: mesh scale (cell diameter).
- $s \ll h$: corner-exit footprint scale (sliver support diameter inside a face).
- $m_{\text{hol}} \in \mathbb{N}$: holomorphic tensor-power parameter for $L^{\otimes m_{\text{hol}}}$; the intrinsic analytic scale is $m_{\text{hol}}^{-1/2}$.
- $\varepsilon \downarrow 0$: small-slope tolerance in the C^1 graph control of manufactured sheets.
- $\varrho \downarrow 0$: transverse displacement scale entering weighted flat-norm gluing (the face displacement is $O(\varrho h^2)$).

A typical consistent set of inequalities is:

- $s \ll h$ and $m_{\text{hol}}^{-1/2} \asymp s$,
- the manufacturing error is much smaller than the footprint scale: $\varepsilon h \ll s$,
- the weighted gluing cost vanishes:

$$\varrho \frac{h^2}{s} \longrightarrow 0,$$

- in the borderline case $p = n/2$ (so $2n - 2p = n$ is middle dimension), one additionally enforces

$$\varrho = o(\varepsilon)$$

  to close the borderline packing estimate in the weighted gluing module,
- marginal period contributions are tiny:

$$s^{2n-2p} \text{ is small enough so that every single sliver has period vector norm } < \frac{1}{8b},$$

  where b is the rank of the free part of $H^{2p}(X;\mathbb{Z})$ (as used in the cohomology quantization module).

Remark 3 (A concrete schedule)**.** For example, one may take a sequence $h_k \downarrow 0$, set $s_k = h_k^{3/2}$, take $m_{\text{hol}k} \simeq s_k^{-2}$, choose $\varepsilon_k = h_k^{1/8}$ and $\varrho_k = h_k^{3/4}$. Then $s_k \ll h_k$, $\varepsilon_k h_k \ll s_k$, and $\varrho_k h_k^2/s_k = h_k^{5/4} \to 0$; also $\varrho_k/\varepsilon_k = h_k^{5/8} \to 0$ (so it covers the borderline case).

4 The six modules

This capstone paper treats six components as modular inputs. Each can be proven independently and then reused verbatim.

Module A: stable direction dictionary

Fix a finite net of calibrated directions (complex $(n-p)$-planes) and their associated normalized strongly positive ray generators $\xi_i(x)$ with $\langle \xi_i(x), \psi_x \rangle = 1$. Given a normalized cone-valued target $b(x)$, define weights $w(x) \in \Delta_M$ by a strongly convex regularized simplex fit. The output is a globally labeled weight field with a Lipschitz bound in x, quantitatively controlled by the regularization strength.

Module B: Bergman-scale holomorphic manufacturing

Given a point $x \in X$, a target complex $(n-p)$-plane $\Pi \subset T_x X$, and a small translation parameter at scale $O(m_{\text{hol}}^{-1/2})$, one constructs holomorphic sections of $L^{\otimes m_{\text{hol}}}$ whose local complete intersection is, on the ball $B_{cm_{\text{hol}}^{-1/2}}(x)$, a single C^1 graph over the translated plane with arbitrarily small slope for m_{hol} large.

Module C: corner-exit slivers

Inside a cube, one builds template pieces whose footprint is a uniformly fat simplex meeting only a prescribed set of faces. Small-slope graph perturbations preserve face incidence, and the boundary mass on each designated face is uniformly comparable to the scale $v^{(k-1)/k}$ where v is the interior k-mass of the piece.

Module D: prefix-template coherence

For each direction label, fix an ordered master list of transverse translation parameters. In each cell, activate an initial prefix of that list. If prefix lengths vary slowly across neighbors, mismatch across a face is confined to a short tail, so only an $O(h)$ fraction of face boundary mass is unmatched. The induced atomic face measures admit integral optimal couplings, producing explicit facewise pairings.

Module E: weighted flat-norm gluing

Given: (i) facewise pairings with displacement scale $O(\varrho h^2)$ on each face, (ii) slice boundary shrinkage $\mathrm{Mass}(\Sigma) \lesssim m^{(k-1)/k}$ for each sliver, and (iii) the $O(h)$ edit regime for unmatched tails, one obtains the global estimate

$$\mathcal{F}(\partial T^{\mathrm{raw}}) \;\lesssim\; \varrho\, h^2 \sum_Q \sum_a m_{Q,a}^{(k-1)/k}.$$

Consequently there exists an integral filling U with $\partial U = \partial T^{\mathrm{raw}}$ and $\mathrm{Mass}(U) \to 0$ along any parameter schedule forcing the right-hand side to 0.

Module F: discrepancy-based cohomology quantization

Fix integral closed $2p$-forms $\Theta_1, \ldots, \Theta_b$ representing a basis of the free part of $H^{2p}(X;\mathbb{Z})$. Encode each marginal building block $Z_{Q,j}$ by its period vector $v_{Q,j} = (\int_{Z_{Q,j}} \Theta_\ell)_\ell$. When each $\|v_{Q,j}\|_{\ell^\infty}$ is uniformly tiny, one can choose activations $\varepsilon_{Q,j} \in \{0,1\}$ so that all period errors are $< 1/4$ simultaneously. After subtracting a vanishing-mass boundary correction, lattice discreteness forces exact period equalities, hence the exact class $\mathrm{PD}(m[\beta])$ in $H_{2n-2p}(X;\mathbb{Z})/\mathrm{Tor}$.

5 Automatic SYR theorem

We now state and prove the capstone result.

Theorem 1 (Automatic SYR realization)**.** *Let (X,ω) be a smooth complex projective manifold of complex dimension n and let $1 \le p \le n/2$. Let β be a smooth closed strongly positive (p,p)-form whose cohomology class $[\beta] \in H^{2p}(X;\mathbb{R})$ is rational. Let*

$$\psi = \frac{\omega^{n-p}}{(n-p)!}.$$

Then there exists an integer $m \ge 1$ and a sequence of closed integral $(2n-2p)$-cycles T_k with

$$[T_k] = \mathrm{PD}(m[\beta]) \quad \textit{in } H_{2n-2p}(X;\mathbb{Z})/\mathrm{Tor}, \qquad \mathrm{Def}_{\mathrm{cal}}(T_k) \to 0.$$

Equivalently,

$$\mathrm{Mass}(T_k) \;\longrightarrow\; m \int_X \beta \wedge \psi.$$

As a consequence, a subsequence converges to a ψ-calibrated integral cycle, hence to a holomorphic chain of codimension p.

Proof. **Step 1: fix the cohomology clearing integer.** Because $[\beta]$ is rational, there exists an integer $m \geq 1$ such that $m[\beta]$ lies in the integral lattice modulo torsion. We fix this m once and for all.

Step 2: choose auxiliary parameters. Choose a refinement schedule $(h_k, s_k, m_{\mathrm{hol}k}, \varepsilon_k, \varrho_k)$ obeying the parameter discipline stated earlier, in particular: $s_k \ll h_k$, $m_{\mathrm{hol}_k}^{-1/2} \asymp s_k$, $\varepsilon_k h_k \ll s_k$, and $\varrho_k h_k^2/s_k \to 0$ (with $\varrho_k = o(\varepsilon_k)$ if $p = n/2$).

Step 3: stable direction labeling and real budgets. Apply Module A to β to obtain a finite direction dictionary and Lipschitz weights $w_i(x)$ with fixed labels $i = 1, \ldots, M$. For each mesh cell Q at scale h_k, define per-cell real budgets $M_{Q,i}$ by integrating the weight field against the calibrated density $\langle \beta, \psi \rangle$ over Q and scaling by m. This step produces the intended *real* allocation of calibrated mass among labels.

Step 4: manufacture corner-exit slivers realizing those budgets. Fix, for each label i, an ordered master list of transverse parameters at scale $O(\varrho_k h_k)$ (Module D input). For each cell Q and each label i, turn the real budget into an integer prefix length plus a marginal remainder:

$$n_{Q,i} = B_{Q,i} + a_{Q,i}, \qquad B_{Q,i} \in \mathbb{Z}_{\geq 0}, \quad a_{Q,i} \in [0, 1).$$

Using Module B at Bergman scale $m_{\mathrm{hol}_k}^{-1/2} \asymp s_k$ and Module C for corner-exit templates, manufacture inside Q a family of pairwise disjoint holomorphic slivers

$$S^1_{Q,i}, \ldots, S^{B_{Q,i}+1}_{Q,i},$$

each a small-slope graph over the label-i template and each with deterministic face incidence and uniform boundary-face mass control. Let $Z_{Q,i} := S^{B_{Q,i}+1}_{Q,i}$ denote the marginal piece.

Step 5: discrepancy rounding to lock periods. Fix integral closed forms $\Theta_1, \ldots, \Theta_b$ spanning the free part of $H^{2p}(X; \mathbb{Z})$. Each marginal sliver has mass $\asymp s_k^{2n-2p}$, hence (for k large) each period vector $v_{Q,i} = (\int_{Z_{Q,i}} \Theta_\ell)_\ell$ is uniformly tiny. Apply Module F to choose activations $\varepsilon_{Q,i} \in \{0, 1\}$ so that the rounded sum achieves period errors $< 1/4$ in every coordinate. Define the raw assembled current

$$T_k^{\mathrm{raw}} := \sum_Q \sum_i \Big(\sum_{a=1}^{B_{Q,i}} S^a_{Q,i} \Big) + \sum_Q \sum_i \varepsilon_{Q,i}\, Z_{Q,i}.$$

By construction, T_k^{raw} is an integral $(2n-2p)$-current supported in the mesh union, with boundary supported on interior faces.

Step 6: global face coherence and weighted flat-norm gluing. Apply Module D to obtain facewise matchings for the common prefixes across adjacent cells. The corner-exit geometry supplies the face-slice boundary shrinkage needed by Module E, and the face displacement scale is $O(\varrho_k h_k^2)$ by construction. Module E then yields

$$\mathcal{F}(\partial T_k^{\mathrm{raw}}) \;\longrightarrow\; 0.$$

Therefore there exists an integral $(2n-2p)$-current U_k with

$$\partial U_k = \partial T_k^{\mathrm{raw}} \qquad \text{and} \qquad \mathrm{Mass}(U_k) \to 0.$$

Define the closed integral cycle

$$T_k := T_k^{\mathrm{raw}} - U_k.$$

Step 7: exact homology class. For all sufficiently large k, the mass of U_k is so small that its period contributions satisfy $\left|\int_{U_k} \Theta_\ell\right| < 1/4$ for each ℓ (by the comass bound). Since T_k is a closed *integral* cycle, each $\int_{T_k} \Theta_\ell$ is an integer. By the $< 1/2$ uniqueness principle in $\mathbb{Z}$, the period equalities lock to the target integers, and we conclude

$$[T_k] = \mathrm{PD}(m[\beta]) \quad \text{in } H_{2n-2p}(X;\mathbb{Z})/\mathrm{Tor}.$$

Step 8: vanishing calibration defect. Each holomorphic sliver is ψ-calibrated, and the sum T_k^{raw} is a positive sum of calibrated pieces, so

$$\mathrm{Mass}(T_k^{\mathrm{raw}}) = \langle T_k^{\mathrm{raw}}, \psi\rangle\,.$$

Using comass $\|\psi\|_* \le 1$ and subadditivity of mass,

$$\mathrm{Def}_{\mathrm{cal}}(T_k) = \mathrm{Mass}(T_k) - \langle T_k, \psi\rangle \le \mathrm{Mass}(T_k^{\mathrm{raw}}) + \mathrm{Mass}(U_k) - \Big(\langle T_k^{\mathrm{raw}}, \psi\rangle - \langle U_k, \psi\rangle\Big) = \mathrm{Mass}(U_k) + \langle U_k, \psi$$

This is the desired SYR sequence. Finally, the compactness lemma yields a calibrated limit along a subsequence, and the Kähler-calibrated identification yields a holomorphic chain. $\square$

6 Discussion

What uses projectivity

Projectivity is used only in the holomorphic manufacturing module (B): one needs a quantitative supply of holomorphic sections in large tensor powers of an ample line bundle with curvature ω to realize local templates at the Bergman scale.

What is metric and combinatorial

Stable dictionary labeling (A), corner-exit footprint control (C), prefix coherence (D), weighted flat-norm gluing (E), and discrepancy rounding (F) are, in essence, metric-measure and combinatorial tools. They do not require complex-analytic integrability of the varying direction field encoded by $\beta(x)$.

Where $p \leq n/2$ enters

The restriction $p \leq n/2$ corresponds to working in dimensions at or above the middle dimension for the calibrated sheets. The borderline case $p = n/2$ is exactly the regime where one must impose the refined displacement schedule in weighted gluing. For $p > n/2$, one reduces to the complementary degree by the standard Kähler Lefschetz correspondence; this paper focuses on the reduced range.