TWO-CHART EGOROV THEOREMS (INTERIOR AND AIRY-GLANCING) FOR THE KAPUSTIN–JOUKOWSKY TRANSFORM AND GLOBAL LOCALITY FOR THE DE BRANGES TRANSPORTED PROLATE OPERATOR

[ANONYMOUS]

Abstract. We develop a global microlocal calculus for the Kapustin–Joukowsky transform linking the Connes–Consani–Moscovici semi-local trace formula with Kapustin's de Branges canonical system. Building on the semiclassical Fourier-integral-operator (FIO) structure and Airy glancing normal form established by Cipollina–Washburne, we prove a two-chart Egorov theorem in a precise λ-dependent symbol class: an interior pseudodifferential chart away from the hard wall $t = 2\pi$ and an Airy-glancing chart at $t = 2\pi$ obtained via Melrose–Taylor boundary FIO theory. We glue the two charts by an explicit microlocal partition of unity on $T^*(2\pi, \infty)$ and deduce global polynomial off-diagonal decay for the Schwartz kernel $Q(t, t')$ of the de Branges transported prolate operator W_π^{dB} (thus proving the "Archimedean locality" conjecture of [3]). As consequences we obtain uniform weighted resolvent estimates and eigenfunction localization, which feed into the regularized determinant controls appearing in the zeta spectral triple framework of Connes–Consani–Moscovici. We also record (as an appendix) the rank–2 IIKS integrable-kernel representation for the Connes–Consani–van Suijlekom (CCvS) truncation matrices, isolating the arithmetic deformation layer as an explicit long-term conjecture.

Contents

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1. Introduction and statement of results

1.1. **Motivation.** The Connes–Consani–Moscovici (CCM) program proposes a family of finite-dimensional, self-adjoint operators $D_{\log}^{(\lambda,N)}$ whose spectra numerically approximate the lowest non-trivial zeros of $\zeta(\frac{1}{2}+is)$ with high accuracy, and whose normalized regularized determinants are expected to converge to the Riemann Ξ-function [1]. A rigorous proof of that convergence would imply the Riemann Hypothesis.

On the Archimedean side, CCM's semi-local trace formula involves the prolate wave operator W_π acting on the Sonin space with gaps. Cipollina–Washburne [3] introduced a unitary "bridge" transform T_B from Kapustin's canonical variable space $L^2([2\pi,\infty))$ to the Sonin space and transported W_π to a canonically-defined self-adjoint operator $W_\pi^{\mathrm{dB}} := T_B^{-1} W_\pi T_B$ on $L^2([2\pi,\infty))$. They conjectured a strong global locality statement for the Schwartz kernel $Q(t,t')$ of W_π^{dB}.

Cipollina–Washburne [4] then established two decisive microlocal inputs for this Archimedean operator: (i) in the interior (away from $t=2\pi$) the semiclassically rescaled Kapustin–Joukowsky transform T_h is an elliptic semiclassical FIO associated with an explicit canonical transformation χ; (ii) at the hard wall $t=2\pi$, T_h admits a Melrose–Taylor Airy normal form.

The present paper completes the missing analytic step in this circle of ideas: we remove microlocal cutoffs, control the glancing regime in a quantifiable calculus, and deduce *global* kernel decay and resolvent stability, uniform in the large parameter (equivalently, uniform in λ).

1.2. **Main objects.** We briefly recall the bridge kernel and the quadratic form kernel. Let $K(x,t)$ be the distributional kernel of T_B in [3, Prop. 4.2], defined via a regularized inverse Mellin representation. The key point for this paper is that the quadratic form of W_π^{dB} can be written as

$$Q(t,t') = Q_1(t,t') + Q_2(t,t'), \qquad Q_1(t,t') := \int_{|x|\geq\pi} (\pi^2 - x^2)\, \partial_x K(x,t)\, \partial_x K(x,t')\, dx, \tag{1.1}$$

$$Q_2(t,t') := 4\pi^4 \int_{|x|\geq\pi} x^2\, K(x,t)\, K(x,t')\, dx. \tag{1.2}$$

This formula identifies Q as the Schwartz kernel of the boundary pseudodifferential representative of W_π^{dB} (precisely formulated in §2).

1.3. **A precise λ-dependent symbol class.** To quantify uniformity in the CCM infrared parameter $\lambda = e^{L/2}$ we work semiclassically with

$$h := \frac{2\pi}{L} = \frac{\pi}{\log \lambda} \in (0, h_0]. \tag{1.3}$$

All constants in the remainder estimates below are polynomial in h^{-1}, equivalently polynomial in $\log \lambda$. We package this into a λ-dependent symbol class S^m_λ; see Definition 3.2.

1.4. **Results.** Our first main theorem is a two-chart Egorov theorem (interior and glancing) for conjugation by T_h. It is stated in the t-notation used in [3] and with the conjugation convention $T_h^{-1} A_h T_h$ as in [4, Eq. (16)].

Theorem 1.1 (Two-chart Egorov in S^m_λ)**.** *Let $Y = (2\pi, \infty)$ and let $T_h : L^2(Y) \to L^2(\mathbb{R})$ be the semiclassically rescaled Kapustin–Joukowsky transform as in* [4]*. Fix $m \in \mathbb{R}$ and suppose $A_h = \mathrm{Op}_h^{\mathrm{W}}(a_h)$ is a semiclassical Weyl pseudodifferential operator on $\mathbb{R}$ with symbol $a_h \in S^m_\lambda(T^*\mathbb{R})$.*

There exist a conic neighbourhood $\Gamma_{\mathrm{gla}} \subset T^*Y$ of the glancing set at $t = 2\pi$, an interior region $\Gamma_{\mathrm{int}} \Subset T^*Y$ with $\overline{\Gamma_{\mathrm{int}}} \cap \overline{\Gamma_{\mathrm{gla}}} = \emptyset$, and a microlocal partition of unity $\psi_{\mathrm{int}} + \psi_{\mathrm{gla}} = 1$ on a neighbourhood of $\mathrm{WF}(T_h)$ such that:

(i) ***Interior chart.*** *Microlocally on Γ_{int},*

$$\mathrm{Op}_h(\psi_{\mathrm{int}})\, T_h^{-1} A_h T_h\, \mathrm{Op}_h(\psi_{\mathrm{int}}) = \mathrm{Op}_h^{\mathrm{W}}\big(a_h \circ \chi\big) + h\, \mathrm{Op}_h^{\mathrm{W}}(r_{1,h}) + \cdots + h^N \mathrm{Op}_h^{\mathrm{W}}(r_{N,h}) + R_N(h),$$

where $\chi : T^*Y \to T^*\mathbb{R}$ is the canonical transformation of [4, Thm. A]*, each $r_{j,h} \in S^{m-j}_\lambda(T^*Y)$, and $\|R_N(h)\|_{L^2 \to L^2} = \mathcal{O}(h^N)$ with implied constants polynomial in h^{-1}.*

(ii) ***Glancing Airy chart.*** *Microlocally on Γ_{gla}, there exist microlocally unitary semiclassical FIOs U_h on $L^2(Y)$ and V_h on $L^2(\mathbb{R})$ and an elliptic symbol c_h such that*

$$V_h T_h U_h^* = \mathrm{Op}_h^{\mathrm{W}}(c_h) \circ \mathcal{A}_h + \mathcal{O}(h^\infty),$$

where $\mathcal{A}_h$ is the standard Airy transform written in the blown-up variables $Y = (t - 2\pi)/h^{2/3}$, and conjugation by $\mathcal{A}_h$ maps S^m_λ to S^m_λ with a full asymptotic expansion, again with operator-norm remainders $\mathcal{O}(h^N)$ and seminorm control polynomial in h^{-1}.

Our second main theorem proves global locality of the transported prolate operator, i.e. Conjecture 5.7 of [3].

Theorem 1.2 (Global locality for W_π^{dB})**.** *Let W_π be the prolate wave operator on the Sonin space and let $W_\pi^{\mathrm{dB}} = T_B^{-1} W_\pi T_B$ be its de Branges transport to $L^2(Y)$, with quadratic form kernel $Q(t, t')$ given by* (1.1)–(1.2)*. Then for every $N \in \mathbb{N}$ there exist constants $C_N > 0$ and $M_N \geq 0$ such that for all $t, t' \geq 2\pi$,*

$$\big|Q(t, t')\big| \leq C_N \, \langle t + t' \rangle^{M_N} \, \langle t - t' \rangle^{-N}. \tag{1.4}$$

In particular, the integral kernel of W_π^{dB} is smooth off the diagonal $\{t = t'\}$ and rapidly decaying in the off-diagonal variable $t - t'$.

Our third main theorem derives the resolvent/eigenfunction stability estimates used as analytic input for regularized determinant control in CCM.

Theorem 1.3 (Uniform weighted resolvent bounds and eigenfunction localization)**.** *Let W_π^{dB} be as in Theorem 1.2. For each $k \in \mathbb{N}$ and each compact set $\Omega \Subset \mathbb{C} \setminus \mathbb{R}$ there exists $C_{k,\Omega} > 0$ such that the resolvent satisfies*

$$\Big\| \langle t \rangle^k \, (W_\pi^{\mathrm{dB}} - z)^{-1} \, \langle t \rangle^k \Big\|_{L^2 \to L^2} \leq C_{k,\Omega}, \qquad z \in \Omega, \tag{1.5}$$

and each normalized eigenfunction φ_n of W_π^{dB} belongs to $\mathcal{S}([2\pi,\infty))$ and satisfies for each k

$$\sup_{t\geq 2\pi} \langle t\rangle^k \, |\varphi_n(t)| \leq C_k \, (1+\mu_n)^{k/2}, \tag{1.6}$$

where μ_n is the corresponding eigenvalue and C_k depends only on k.

Finally, we record a clean formulation of how these estimates feed into the CCM regularized determinant. CCM show that, for fixed (λ, N), the regularized determinant is controlled by the Fourier transform of a minimal eigenvector of a truncated Weil quadratic form [1, §5.6]. We isolate the analytic input needed for uniformity.

Corollary 1.4 (Determinant control from resolvent/eigenfunction stability)**.** *Assume the uniform weighted resolvent bounds* (1.5) *for the Archimedean transported operator, uniformly along the semiclassical scaling* (1.3)*. Then the family of normalized regularized determinants*

$$F_{\lambda,N}(z) := \lambda^{iz} \det_{\mathrm{reg}} \big(D_{\log}^{(\lambda,N)} - z\big)$$

is locally bounded and hence forms a normal family of entire functions of z. In particular, any subsequential limit is an entire function with real zeros (by the finite-N Carathéodory–Fejér mechanism and Hurwitz).

Remark 1.5 (Arithmetic deformation remains conjectural)**.** A proof of the Riemann Hypothesis in the CCM framework requires identifying the unique subsequential limit of $F_{\lambda,N}$ with the Riemann Ξ-function. This is where the arithmetic deformation problem enters. We state a precise uniform convergence conjecture in §9 and discuss evidence-building strategies.

2. The bridge kernel and identification of the quadratic form kernel

We work throughout with the t-variable of [3], i.e. $t \in Y = (2\pi, \infty)$. The bridge transform T_B is defined (distributionally) by its kernel $K(x,t)$ via

$$(T_B u)(x) = \int_{2\pi}^{\infty} K(x,t)\, u(t)\, dt.$$

In [3, Prop. 4.2], $K(x,t)$ is given by an inverse Mellin representation built from the modified Bessel functions $K_s(t)$ and a weight $W(s)$, and is *a priori* only defined as a distribution in x for each fixed t. We adopt exactly the regularization of [3, Rem. 4.3].

Definition 2.1 (Regularized bridge kernel)**.** Let $L = \{s \in \mathbb{C} : \Re(s) = \frac{1}{2}\}$. For each smooth cutoff $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0, define for $\epsilon > 0$

$$K_\epsilon(x,t) := \frac{C'}{2\pi} \int_L \big(K_s(t) - K_{s-1}(t)\big)\, (x^{s-1} - x^{-s})\, W(s)\, \chi(\epsilon\, \Im s)\, \frac{ds}{i}. \tag{2.1}$$

Then K_ϵ is smooth in (x,t) and the limit $K := \lim_{\epsilon\downarrow 0} K_\epsilon$ exists in $\mathcal{S}'(\mathbb{R}_x)$ for each t.

Proposition 2.2 (Quadratic form kernel equals the Schwartz kernel of W_π^{dB})**.** *Let W_π be the prolate operator on the Sonin space and define $W_\pi^{\mathrm{dB}} = T_B^{-1} W_\pi T_B$ on $L^2(Y)$. For $u, v \in C_c^\infty(Y)$ one has*

$$\Big\langle W_\pi^{\mathrm{dB}} u, v \Big\rangle_{L^2(Y)} = \int_{2\pi}^{\infty} \int_{2\pi}^{\infty} Q(t,t')\, u(t)\, \overline{v(t')}\, dt\, dt',$$

where Q is defined by (1.1)–(1.2) *with K replaced by the regularized kernel K_ϵ and then $\epsilon \downarrow 0$.*

Proof. This is precisely the content of [3, Prop. 4.2], together with the regularization prescription [3, Rem. 4.3]. We reproduce the key point for completeness: on the Sonin side, the quadratic form of W_π is local and given by an integral of $(\pi^2 - x^2)\,|f'(x)|^2 + 4\pi^4 x^2\,|f(x)|^2$ over the gap region $|x| \ge \pi$; transporting by T_B and using $(T_B u)(x) = \int K(x,t)u(t)dt$ yields (1.1)–(1.2) after polarization and Fubini, with the regularization ensuring the interchange is justified. □

2.1. **An explicit elliptic parametrix in S^m_λ.** We next record a fully explicit parametrix construction in the λ–dependent semiclassical Weyl calculus. This is the basic analytic input behind the weighted resolvent bounds of Theorem 1.3. We keep the argument entirely in one dimension (the only case used in this paper).

Lemma 2.3 (Elliptic parametrix with explicit recursion)**.** *Let $Y = (2\pi, \infty)$ and let $a_h \in S^m_\lambda(T^*Y)$ be* real-valued. *Fix a compact set $\Omega \Subset \mathbb{C} \setminus \mathbb{R}$ and set*

$$\eta := \operatorname{dist}(\Omega, \mathbb{R}) > 0. \tag{2.2}$$

For $z \in \Omega$ set $a_{h,z} := a_h - z$. Then for each $N \in \mathbb{N}$ there exists a symbol

$$b_{N,h}(z) \in S^{-m}_\lambda(T^*Y) \tag{2.3}$$

such that the corresponding Weyl operator $B_{N,h}(z) := \mathrm{Op}_h^{\mathrm{W}}(b_{N,h}(z))$ satisfies

$$\mathrm{Op}_h^{\mathrm{W}}(a_{h,z})\, B_{N,h}(z) = \mathbf{1} + h^N\, R_{N,h}(z), \tag{2.4}$$

$$B_{N,h}(z)\, \mathrm{Op}_h^{\mathrm{W}}(a_{h,z}) = \mathbf{1} + h^N\, R'_{N,h}(z), \tag{2.5}$$

where $R_{N,h}(z), R'_{N,h}(z)$ are semiclassical pseudodifferential operators with symbols in $S^{-N}_\lambda(T^*Y)$. In particular, for each N there exist constants $C_N > 0$ and exponents $M_N \ge 0$ such that

$$\sup_{z\in\Omega} \Big(\|R_{N,h}(z)\|_{L^2\to L^2} + \big\|R'_{N,h}(z)\big\|_{L^2\to L^2} \Big) \le C_N\, h^{-M_N}, \qquad h \in (0, h_0]. \tag{2.6}$$

Moreover, $b_{N,h}(z)$ is given by an explicit finite expansion

$$b_{N,h}(z) = \sum_{j=0}^{N-1} h^j\, b_{j,h}(z), \tag{2.7}$$

with $b_{j,h}(z) \in S^{-m-j}_\lambda$ defined recursively as follows. Let $b_{0,h}(z) := a_{h,z}^{-1}$ and, for $j \ge 1$,

$$b_{j,h}(z) := -b_{0,h}(z) \sum_{n=1}^{j} \mathcal{P}_n\big(a_{h,z}, b_{j-n,h}(z)\big), \tag{2.8}$$

where the bidifferential operator $\mathcal{P}_n$ is the n–th Weyl–product coefficient

$$\mathcal{P}_n(c,d) := \frac{1}{n!}\left(\frac{i}{2}\right)^n \sum_{\alpha+\beta=n} (-1)^\beta \binom{n}{\alpha} \big(\partial_\tau^\alpha \partial_t^\beta c\big)\, \big(\partial_t^\alpha \partial_\tau^\beta d\big). \tag{2.9}$$

The seminorms of $b_{j,h}(z)$ and of the remainders $R_{N,h}(z), R'_{N,h}(z)$ are controlled (uniformly for $z \in \Omega$) by finitely many S^m_λ–seminorms of a_h.

Proof. We work with the one-dimensional Weyl product $\#$. For symbols c, d one has the standard Moyal formula

$$c\#d = \sum_{n=0}^{N-1} h^n\, \mathcal{P}_n(c,d) \;+\; h^N\, \mathcal{R}_{N,h}(c,d), \tag{2.10}$$

where $\mathcal{P}_n$ is given by (2.9) and $\mathcal{R}_{N,h}(c,d)$ is a finite linear combination of terms involving N derivatives in (t,τ) of c and d (see e.g. [7, Ch. 18]).

Step 1: symbol-level inversion. Since a_h is real-valued and $z \in \Omega \subset \mathbb{C} \setminus \mathbb{R}$, we have the uniform pointwise bound

$$|a_{h,z}(t,\tau)| \geq |\Im z| \geq \eta, \qquad (t,\tau) \in T^*Y. \tag{2.11}$$

Thus $b_{0,h}(z) = a_{h,z}^{-1}$ is well-defined and belongs to S_λ^{-m} by the chain rule applied to (2.11) together with the defining seminorm bounds of S_λ^m.

Define $b_{j,h}(z)$ recursively by (2.8), and set $b_{N,h}(z)$ by (2.7). By induction on j, using that $\mathcal{P}_n$ differentiates n times in each variable and therefore lowers order by n, one obtains $b_{j,h}(z) \in S_\lambda^{-m-j}$ with seminorms bounded by a finite collection of seminorms of a_h (and powers of η^{-1}), uniformly for $z \in \Omega$.

Step 2: composition and remainder. Apply (2.10) with $c = a_{h,z}$ and $d = b_{N,h}(z)$. The recursion (2.8) exactly cancels the coefficients of h^j for $1 \leq j \leq N-1$, so that

$$a_{h,z} \# b_{N,h}(z) = 1 + h^N r_{N,h}(z), \tag{2.12}$$

with $r_{N,h}(z) \in S_\lambda^{-N}$. Quantizing (2.12) yields (2.4); the right parametrix (2.5) is obtained similarly from $b_{N,h}(z) \# a_{h,z} = 1 + h^N r'_{N,h}(z)$.

Finally, since $r_{N,h}(z) \in S_\lambda^{-N}$, the Calderón–Vaillancourt theorem in the semiclassical calculus gives (2.6) with constants polynomial in h^{-1} (i.e. compatible with Definition 3.2). □

3. Semiclassical and λ-dependent symbol classes

3.1. Standard semiclassical symbols.

We use the one-dimensional semiclassical Weyl quantization

$$\mathrm{Op}_h^{\mathrm{W}}(a)u(t) := (2\pi h)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\frac{i}{h}(t-t')\tau}\, a\Big(\frac{t+t'}{2}, \tau; h\Big)\, u(t')\, dt'\, d\tau.$$

Definition 3.1 (Semiclassical symbol class S^m)**.** A family $a(\cdot,\cdot;h) \in C^\infty(T^*Y)$ belongs to $S^m(T^*Y)$ if for all $\alpha, \beta \in \mathbb{N}$ there exists $C_{\alpha\beta}$ such that

$$\sup_{(t,\tau)\in T^*Y} \langle\tau\rangle^{-m+\beta} \left|\partial_t^\alpha \partial_\tau^\beta a(t,\tau;h)\right| \leq C_{\alpha\beta}, \qquad h \in (0,h_0].$$

3.2. λ-dependent symbols.

We now incorporate polynomial growth in $\log\lambda \sim h^{-1}$ into seminorms.

Definition 3.2 (λ-dependent symbol class S_λ^m)**.** Fix $m \in \mathbb{R}$. A family $a(\cdot,\cdot;h)$ belongs to $S_\lambda^m(T^*Y)$ if for all $\alpha, \beta \in \mathbb{N}$ there exist constants $C_{\alpha\beta}$ and exponents $M_{\alpha\beta} \geq 0$ such that

$$\sup_{(t,\tau)\in T^*Y} \langle\tau\rangle^{-m+\beta} \left|\partial_t^\alpha \partial_\tau^\beta a(t,\tau;h)\right| \leq C_{\alpha\beta}\, h^{-M_{\alpha\beta}}, \qquad h \in (0,h_0].$$

Equivalently, the same bound holds with h^{-1} replaced by a fixed polynomial in $\log\lambda$ via (1.3).

Remark 3.3 (Tempered remainder bookkeeping)**.** All symbol classes used below are stable under multiplication and under composition in the standard semiclassical calculus. We record explicit seminorm bookkeeping in Lemma 4.1 and in the Airy-chart Lemma 5.2.

In the λ-dependent setting, we interpret remainder statements such as

$$R_N(h) = \mathcal{O}(h^N)$$

in the following *tempered* sense: for each fixed $N \in \mathbb{N}$ there exist constants $C_N > 0$ and an exponent $M_N \geq 0$ such that

$$\|R_N(h)\|_{L^2 \to L^2} \leq C_N\, h^N\, h^{-M_N}, \qquad h \in (0, h_0]. \tag{3.1}$$

Equivalently, using $h \sim (\log \lambda)^{-1}$, the bound is $\|R_N(h)\| \leq C_N\, h^N\, (1 + \log \lambda)^{M_N}$.

For all constructions in this paper, the coefficient at order h^j depends only on finitely many derivatives of the input symbols of order $\leq Cj$ (with a fixed constant C determined by the calculus), hence one may take

$$M_N \;=\; \max\{M_{\alpha\beta} : \;\alpha + \beta \leq CN\}$$

in (3.1) for some fixed C. Whenever we require smallness of a remainder, we fix N and then choose N large enough that $N - M_N$ is positive, which is precisely the regime used in the Neumann-series steps of §**??**.

4. Interior pseudodifferential chart: kernel decay from symbol bounds

The following lemma is the interior mechanism converting symbol seminorm bounds into rapid off-diagonal kernel decay.

Lemma 4.1 (Kernel decay for semiclassical pseudodifferential operators)**.** *Let $a_h \in S^m_\lambda(T^*Y)$ and let $A_h = \mathrm{Op}_h^{\mathrm{W}}(a_h)$. Then its Schwartz kernel $K_{A_h}(t,t')$ satisfies: for every $N \in \mathbb{N}$ there exist constants $C_N > 0$ and integers $M_N \geq 0$ such that for all $t, t' \in Y$ and $h \in (0, h_0]$,*

$$\left|K_{A_h}(t,t')\right| \leq C_N\, h^{-1-M_N}\, \langle t + t'\rangle^{M_N} \left\langle \frac{t - t'}{h} \right\rangle^{-N}. \tag{4.1}$$

In particular, for fixed h the kernel is rapidly decaying in $t - t'$.

Proof. By definition,

$$K_{A_h}(t,t') = (2\pi h)^{-1} \int_{\mathbb{R}} e^{\frac{i}{h}(t-t')\tau}\, a_h\Big(\frac{t+t'}{2}, \tau; h\Big)\, d\tau.$$

Fix $N \in \mathbb{N}$. Define the integration-by-parts operator

$$L := \frac{h}{i(t-t')}\,\partial_\tau, \qquad \text{so that} \qquad L\big(e^{\frac{i}{h}(t-t')\tau}\big) = e^{\frac{i}{h}(t-t')\tau} \quad \text{for } t \neq t'.$$

Integrating by parts N times gives, for $t \neq t'$,

$$K_{A_h}(t,t') = (2\pi h)^{-1} \int_{\mathbb{R}} e^{\frac{i}{h}(t-t')\tau}\, L^N\Big[a_h\Big(\frac{t+t'}{2}, \tau; h\Big)\Big]\; d\tau,$$

and hence

$$\left|K_{A_h}(t,t')\right| \leq (2\pi h)^{-1} \Big(\frac{h}{|t-t'|}\Big)^N \int_{\mathbb{R}} \left|\partial_\tau^N a_h\big(\frac{t+t'}{2}, \tau; h\big)\right| d\tau.$$

Choose $M_N := M_{0,N}$ from Definition 3.2. Since $\partial_\tau^N a_h \in S^{m-N}_\lambda$,

$$\left|\partial_\tau^N a_h\big(\frac{t+t'}{2}, \tau; h\big)\right| \leq C\, h^{-M_N}\, \langle \tau\rangle^{m-N}.$$

If $N > m + 2$ the τ-integral is finite and bounded by an absolute constant; if not, one splits $\mathbb{R}$ into $|\tau| \leq 1$ and $|\tau| \geq 1$ and uses the same inequality with an additional polynomial factor in τ absorbed into $\langle t + t'\rangle^{M_N}$ (using the fact that for symbols supported in a fixed conic set one has $|\tau| \lesssim \langle t + t'\rangle$ after microlocal cutoff, as in §6). This yields (4.1). The case $t = t'$ follows from the crude bound $|K_{A_h}(t,t)| \leq (2\pi h)^{-1} \int |a_h((t+t')/2, \tau; h)|\, d\tau \lesssim h^{-1-M_0} \langle t\rangle^{M_0}$. ☐

5. Glancing Airy chart: an explicit Airy conjugation lemma

We now write the key kernel bound in the glancing Airy chart in the same t-notation as [3]. Set the normal coordinate

$$y := t - 2\pi \geq 0, \qquad Y := h^{-2/3} y.$$

The Melrose–Taylor theory (in the form used by Cipollina–Washburne) provides, microlocally near the glancing set, a factorization of T_h into elliptic FIOs and a model Airy transform; see [4, Thm. 5.2].

Definition 5.1 (Model Airy transform in blown-up variables)**.** Let $\mathcal{F}_h$ denote the semiclassical Fourier transform on $\mathbb{R}_Y$, $(\mathcal{F}_h u)(\eta) = (2\pi h)^{-1/2} \int e^{-iY\eta/h} u(Y)\, dY$. Define $\mathcal{A}_h$ on $L^2(\mathbb{R}_Y)$ by

$$\mathcal{A}_h := \mathcal{F}_h^{-1} \, M_{e^{i\eta^3/(3h)}} \, \mathcal{F}_h,$$

i.e. $(\mathcal{A}_h u)\widehat{\ }(\eta) = e^{i\eta^3/(3h)}\, \hat{u}(\eta)$. Then $\mathcal{A}_h$ is unitary on $L^2(\mathbb{R}_Y)$ and is a semiclassical FIO associated with the canonical map $\kappa(Y,\eta) = (Y + \eta^2, \eta)$.

We need a version of Egorov's theorem and kernel decay inside this Airy calculus.

Lemma 5.2 (Airy conjugation kernel lemma: explicit transport and kernel decay)**.** *Let $a_h \in S^m_\lambda(T^*\mathbb{R}_Y)$ and let $A_h = \mathrm{Op}_h^{\mathrm{W}}(a_h)$ act on $\mathbb{R}_Y$ by semiclassical Weyl quantization,*

$$(A_h u)(Y) = (2\pi h)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\frac{i}{h}(Y-Y')\eta} \, a_h\left(\frac{Y+Y'}{2}, \eta; h\right) u(Y')\, dY'\, d\eta. \tag{5.1}$$

Let $\mathcal{A}_h$ be the Airy transform

$$(\mathcal{A}_h u)(Y) = (2\pi h)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\frac{i}{h}\left((Y-Y')\eta + \frac{\eta^3}{3}\right)} u(Y')\, dY'\, d\eta, \tag{5.2}$$

and define $B_h := \mathcal{A}_h^{-1} A_h \mathcal{A}_h$. Then the following hold.

(i) Symbol transport and explicit transport equation. *There exists $b_h \in S^m_\lambda(T^*\mathbb{R}_Y)$ such that*

$$B_h = \mathrm{Op}_h^{\mathrm{W}}(b_h).$$

Moreover, if $b(s; Y, \eta; h)$ denotes the Weyl symbol of

$$B_h(s) := \mathcal{A}_h^{-s} A_h \mathcal{A}_h^{s} \qquad (s \in [0,1]),$$

then b satisfies the exact *transport equation*

$$\partial_s b(s) = \eta^2 \, \partial_Y b(s) + \frac{h^2}{12} \, \partial_Y^3 b(s), \qquad b(0) = a_h. \tag{5.3}$$

In particular, setting $\kappa(Y,\eta) = (Y+\eta^2, \eta)$, the principal symbol is the pullback by κ,

$$b_h(Y,\eta) = b(1; Y, \eta) = a_h(\kappa(Y,\eta)) + O_{S^m_\lambda}(h^2), \tag{5.4}$$

and for each $N \in \mathbb{N}$ one has the explicit finite expansion

$$b_h(Y,\eta) = \sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{h^2}{12}\right)^k \left(\partial_Y^{3k} a_h\right)(Y + \eta^2, \eta) + h^{2N} r_{N,h}(Y,\eta), \tag{5.5}$$

where $r_{N,h} \in S^m_\lambda(T^*\mathbb{R}_Y)$ and the S^m_λ–seminorms of $r_{N,h}$ are controlled by finitely many S^m_λ–seminorms of a_h (up to $3N$ additional Y–derivatives).

(ii) Oscillatory kernel representation and off–diagonal decay. *The Schwartz kernel of B_h admits the standard Weyl oscillatory integral representation*

$$K_{B_h}(Y,Y') = (2\pi h)^{-1} \int_{\mathbb{R}} e^{\frac{i}{h}(Y-Y')\eta}\, b_h\!\left(\frac{Y+Y'}{2}, \eta; h\right) d\eta. \tag{5.6}$$

For every $N \in \mathbb{N}$ there exist constants $C_N > 0$ and $M_N \geq 0$, depending only on finitely many seminorms of a_h in S^m_λ, such that

$$\left|K_{B_h}(Y,Y')\right| \leq C_N\, h^{-1}\, \langle Y+Y'\rangle^{M_N} \left\langle \frac{Y-Y'}{h} \right\rangle^{-N}. \tag{5.7}$$

Proof. **Step 1: the generator and the Moyal commutator.** Let F_h denote the semiclassical Fourier transform on $\mathbb{R}_Y$,

$$(F_h u)(\eta) = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{-\frac{i}{h}Y\eta}\, u(Y)\, dY,$$

so that $F_h(hD_Y)F_h^{-1}$ is multiplication by η. From (5.2) one checks that

$$\mathcal{A}_h = F_h^{-1}\, e^{\frac{i}{3h}\eta^3}\, F_h = \exp\!\left(\frac{i}{h}\, \mathrm{Op}_h^{\mathrm{W}}\!\left(\frac{\eta^3}{3}\right)\right),$$

where $\mathrm{Op}_h^{\mathrm{W}}(\eta^3/3)$ is the semiclassical Weyl quantization of $H(\eta) := \eta^3/3$ (a polynomial symbol independent of Y).

Let $C_h = \mathrm{Op}_h^{\mathrm{W}}(c_h)$ and $D_h = \mathrm{Op}_h^{\mathrm{W}}(d_h)$ be semiclassical Weyl operators. The Weyl product is governed by the Moyal formula

$$c_h \# d_h = \exp\!\left(\frac{ih}{2}\left(\partial_Y\partial_{\eta'} - \partial_\eta\partial_{Y'}\right)\right) c_h(Y,\eta)\, d_h(Y',\eta')\Big|_{(Y',\eta')=(Y,\eta)}.$$

Hence the commutator corresponds to the Moyal bracket:

$$\frac{i}{h}\left[C_h, D_h\right] = \mathrm{Op}_h^{\mathrm{W}}(\{c_h, d_h\}_{\mathrm{M}}), \qquad \{c_h, d_h\}_{\mathrm{M}} := \frac{2}{h}\, c_h \sin\!\left(\frac{h}{2}\left(\partial_Y\partial_{\eta'} - \partial_\eta\partial_{Y'}\right)\right) d_h\Big|_{(Y',\eta')=(Y,\eta)}.$$

When $c_h = H(\eta)$ depends only on η, the differential operator $\partial_Y\partial_{\eta'} - \partial_\eta\partial_{Y'}$ reduces to $-\partial_\eta\partial_{Y'}$, and the sine series truncates because $H^{(5)} \equiv 0$. A direct expansion yields the *exact* identity

$$\{H, d_h\}_{\mathrm{M}} = H'(\eta)\, \partial_Y d_h + \frac{h^2}{24}\, H^{(3)}(\eta)\, \partial_Y^3 d_h = \eta^2\, \partial_Y d_h + \frac{h^2}{12}\, \partial_Y^3 d_h. \tag{5.8}$$

Step 2: Heisenberg evolution and transport equation. Define $B_h(s) := \mathcal{A}_h^{-s} A_h \mathcal{A}_h^{s}$. Differentiating in s gives the Heisenberg equation

$$\partial_s B_h(s) = \frac{i}{h}\left[B_h(s), \mathrm{Op}_h^{\mathrm{W}}(H)\right], \qquad B_h(0) = A_h.$$

Writing $B_h(s) = \mathrm{Op}_h^{\mathrm{W}}(b(s))$ and using (5.8) yields (5.3).

Step 3: principal transport and the explicit finite expansion. Since $\eta^2\partial_Y$ commutes with ∂_Y^3, equation (5.3) can be solved by the commuting semigroup formula

$$b(1) = \exp\!\left(\frac{h^2}{12}\partial_Y^3\right)\, \exp(\eta^2\partial_Y)\, a_h.$$

The operator $\exp(\eta^2\partial_Y)$ is the translation $f(Y) \mapsto f(Y+\eta^2)$, hence

$$b(1; Y, \eta) = \exp\!\left(\frac{h^2}{12}\partial_Y^3\right)\, a_h(Y+\eta^2, \eta).$$

Expanding the exponential to order $N-1$ gives (5.5), with the remainder

$$r_{N,h}(Y,\eta) = \frac{1}{(N-1)!}\int_0^1 (1-\theta)^{N-1}\left(\frac{1}{12}\right)^N \partial_Y^{3N} a_h\big(Y+\eta^2,\eta;h\big)\,d\theta,$$

which is visibly in S^m_λ with seminorm bounds controlled by finitely many seminorms of a_h (up to $3N$ additional Y–derivatives). This proves (i).

Step 4: oscillatory kernel and integration by parts. The kernel formula (5.6) is the defining oscillatory representation for $\mathrm{Op}_h^{\mathrm{W}}(b_h)$. Fix $N \in \mathbb{N}$. On the region $\{|Y-Y'| \le h\}$ one estimates (5.6) trivially:

$$|K_{B_h}(Y,Y')| \le (2\pi h)^{-1}\int_{\mathbb{R}} \Big|b_h\Big(\frac{Y+Y'}{2},\eta;h\Big)\Big|\,d\eta \lesssim h^{-1}\langle Y+Y'\rangle^{M_N},$$

using the order-m growth in η and the S^m_λ–bounds. On the complementary region $\{|Y-Y'| > h\}$ we integrate by parts in η using the operator

$$L := \frac{h}{i(Y-Y')}\,\partial_\eta, \qquad L\,e^{\frac{i}{h}(Y-Y')\eta} = e^{\frac{i}{h}(Y-Y')\eta}. \tag{5.9}$$

Applying L^N to the integrand in (5.6) and integrating by parts N times gives

$$K_{B_h}(Y,Y') = (2\pi h)^{-1}\int_{\mathbb{R}} e^{\frac{i}{h}(Y-Y')\eta}\,(L^*)^N\left[b_h\Big(\frac{Y+Y'}{2},\eta;h\Big)\right]\,d\eta,$$

with

$$(L^*)^N = \left(-\frac{h}{i(Y-Y')}\right)^N \partial_\eta^N.$$

Therefore,

$$|K_{B_h}(Y,Y')| \le (2\pi h)^{-1}\left(\frac{h}{|Y-Y'|}\right)^N \int_{\mathbb{R}} \left|\partial_\eta^N b_h\Big(\frac{Y+Y'}{2},\eta;h\Big)\right|\,d\eta.$$

The η–derivatives are controlled by S^m_λ–seminorms of b_h, which in turn are controlled by finitely many seminorms of a_h via (5.5). Combining the two regions yields (5.7). □

Lemma 5.3 (Reflected kernel and reflected trace decay)**.** *Let* $b_h \in S^m_\lambda(T^*\mathbb{R}_Y)$ *and let* $B_h = \mathrm{Op}_h^{\mathrm{W}}(b_h)$ *be the semiclassical Weyl operator on* $\mathbb{R}_Y$ *defined by* (5.1)*. Let* R *denote the reflection operator*

$$(\mathsf{R}u)(Y) := u(-Y), \qquad u \in \mathcal{S}(\mathbb{R}_Y).$$

Then the composition $B_h\mathsf{R}$ *has Schwartz kernel*

$$K_{B_h\mathsf{R}}(Y,Y') = (2\pi h)^{-1}\int_{\mathbb{R}} e^{\frac{i}{h}(Y+Y')\eta}\,b_h\Big(\frac{Y-Y'}{2},\eta;h\Big)\,d\eta, \qquad Y,Y' \in \mathbb{R}. \tag{5.10}$$

Moreover, for every $N \in \mathbb{N}$ *there exist constants* $C_N > 0$ *and integers* $M_N \ge 0$ *(depending only on finitely many* S^m_λ*–seminorms of* b_h*) such that for all* $Y,Y' \in \mathbb{R}$ *and all* $h \in (0,h_0]$*,*

$$\big|K_{B_h\mathsf{R}}(Y,Y')\big| \le C_N\,h^{-1-M_N}\,\langle Y+Y'\rangle^{M_N}\left\langle\frac{Y+Y'}{h}\right\rangle^{-N}. \tag{5.11}$$

In particular, if $\chi \in C_c^\infty(\mathbb{R})$ *satisfies* $\mathrm{supp}\chi \subset \{Y \ge \delta\}$ *for some* $\delta > 0$*, then for every* $N \in \mathbb{N}$*,*

$$\Big|\mathrm{Tr}\,\big(\chi\,B_h\mathsf{R}\,\chi\big)\Big| = \Big|\int_{\mathbb{R}} \chi(Y)^2\,K_{B_h\mathsf{R}}(Y,Y)\,dY\Big| \le C_{\chi,N}\,h^N, \tag{5.12}$$

and hence $\mathrm{Tr}(\chi B_h\mathsf{R}\chi) = O(h^\infty)$ *as* $h \to 0$*.*

Proof. **Step 1: the explicit kernel** (5.10)**.** By definition,

$$(B_h \mathsf{R} u)(Y) = (2\pi h)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\frac{i}{h}(Y-Y'')\eta}\, b_h\Big(\frac{Y+Y''}{2}, \eta; h\Big)\; u(-Y'')\, dY''\, d\eta\,.$$

Set $Y' = -Y''$ to obtain (5.10).

Step 2: an integration–by–parts operator in the phase variable. Fix $N \in \mathbb{N}$ and define, for $Y + Y' \neq 0$, the IBP operator

$$L_{\mathrm{ref}} := \frac{h}{i(Y+Y')}\,\partial_\eta, \qquad \text{so that} \qquad L_{\mathrm{ref}}\Big(e^{\frac{i}{h}(Y+Y')\eta}\Big) = e^{\frac{i}{h}(Y+Y')\eta}\,. \tag{5.13}$$

Integrating by parts N times in (5.10) yields, for $Y + Y' \neq 0$,

$$K_{B_h\mathsf{R}}(Y,Y') = (2\pi h)^{-1} \int_{\mathbb{R}} e^{\frac{i}{h}(Y+Y')\eta}\, L_{\mathrm{ref}}^N \Big[b_h\Big(\frac{Y-Y'}{2}, \eta; h\Big)\Big]\; d\eta\,. \tag{5.14}$$

Since $L_{\mathrm{ref}}^N = (h/(i(Y+Y')))^N \partial_\eta^N$, we obtain

$$\big|K_{B_h\mathsf{R}}(Y,Y')\big| \le (2\pi h)^{-1} \Big(\frac{h}{|Y+Y'|}\Big)^N \int_{\mathbb{R}} \Big|\partial_\eta^N b_h\Big(\frac{Y-Y'}{2}, \eta; h\Big)\Big|\, d\eta\,. \tag{5.15}$$

Step 3: S^m_λ seminorm bookkeeping. Let $M_N := M_{0,N}$ be the exponent from Definition 3.2 applied to the seminorm $\sup\langle\eta\rangle^{-m+N}|\partial_\eta^N b_h|$. Then $\partial_\eta^N b_h \in S_\lambda^{m-N}$ and

$$\Big|\partial_\eta^N b_h\Big(\frac{Y-Y'}{2}, \eta; h\Big)\Big| \le C\, h^{-M_N}\, \langle\eta\rangle^{m-N}\,.$$

If $N > m+2$ the η–integral in (5.15) is finite and bounded by an absolute constant; if $N \le m+2$, we simply replace N by $N_0 := \lceil m \rceil + 3$ in the IBP argument and use that $\langle (Y+Y')/h\rangle^{-N_0} \le \langle (Y+Y')/h\rangle^{-N}$. In either case we obtain, for $Y + Y' \neq 0$,

$$\big|K_{B_h\mathsf{R}}(Y,Y')\big| \le C_N\, h^{-1-M_N} \Big(\frac{h}{|Y+Y'|}\Big)^N\,.$$

On the complementary region $\{|Y+Y'| \le h\}$ we use the crude bound from (5.10),

$$|K_{B_h\mathsf{R}}(Y,Y')| \le (2\pi h)^{-1} \int_{\mathbb{R}} \Big| b_h\Big(\frac{Y-Y'}{2}, \eta; h\Big)\Big|\, d\eta \lesssim h^{-1-M_0}\, \langle Y+Y'\rangle^{M_0}\,,$$

with $M_0 := M_{0,0}$, and note that $\langle (Y+Y')/h\rangle^{-N} \simeq 1$ on this region. Combining the two regions yields (5.11).

Step 4: reflected trace away from the boundary. Let χ be supported in $\{Y \ge \delta\}$. Then on the diagonal $Y' = Y$ we have $|Y+Y'| = 2Y \ge 2\delta$, and applying (5.11) with $Y' = Y$ gives, for every N,

$$|K_{B_h\mathsf{R}}(Y,Y)| \le C_N\, h^{-1-M_N}\, \langle 2Y\rangle^{M_N} \Big\langle \frac{2Y}{h}\Big\rangle^{-N} \le C'_{N,\chi}\, h^N$$

after choosing N sufficiently large and absorbing the fixed δ–dependence into the constant. Integrating against $\chi(Y)^2$ proves (5.12). □

5.1. **Closed Airy evaluation of the model reflection phase.** The Melrose–Taylor glancing normal form expresses the microlocal boundary reflection in terms of a scalar *Airy reflection coefficient* (a unit-modulus function of the glancing parameter). In later trace and density expansions, the boundary contribution is governed by the *derivative* of the corresponding phase. For completeness we record here a closed evaluation of this derivative in the pure Airy model; the proof is a one-line Wronskian computation.

Lemma 5.4 (Airy phase derivative)**.** *Let* Ai *and* Bi *denote the standard Airy functions, and set*

$$F(\sigma) := \mathrm{Ai}(-\sigma) + i\ \mathrm{Bi}(-\sigma), \qquad \sigma \in \mathbb{R}.$$

Then $F(\sigma) \neq 0$ for all $\sigma \in \mathbb{R}$, so the phase function

$$\delta(\sigma) := \arg F(\sigma)$$

is well-defined and $C^\infty(\mathbb{R})$. Moreover one has the exact identity

$$\delta'(\sigma) = \Im\Big(\frac{F'(\sigma)}{F(\sigma)}\Big) = -\frac{1}{\pi\big(\mathrm{Ai}(-\sigma)^2 + \mathrm{Bi}(-\sigma)^2\big)}. \tag{5.16}$$

Equivalently, for the unit-modulus reflection coefficient

$$\mathcal{R}(\sigma) := \frac{F(\sigma)}{\overline{F(\sigma)}} = e^{2i\delta(\sigma)}$$

one has

$$\frac{1}{2\pi i}\,\partial_\sigma \log \mathcal{R}(\sigma) = \frac{1}{\pi}\,\delta'(\sigma). \tag{5.17}$$

Proof. Since $\mathrm{Ai}(-\sigma), \mathrm{Bi}(-\sigma) \in \mathbb{R}$ for $\sigma \in \mathbb{R}$, we have $|F(\sigma)|^2 = \mathrm{Ai}(-\sigma)^2 + \mathrm{Bi}(-\sigma)^2 > 0$, so δ is C^∞ and $\delta'(\sigma) = \Im(F'(\sigma)/F(\sigma))$. Writing $A(\sigma) := \mathrm{Ai}(-\sigma)$, $B(\sigma) := \mathrm{Bi}(-\sigma)$, we have $F(\sigma) = A(\sigma) + iB(\sigma)$ and $F'(\sigma) = A'(\sigma) + iB'(\sigma)$. Hence

$$\Im\Big(\frac{F'(\sigma)}{F(\sigma)}\Big) = \frac{A(\sigma)B'(\sigma) - A'(\sigma)B(\sigma)}{A(\sigma)^2 + B(\sigma)^2}.$$

Since $A'(\sigma) = -\mathrm{Ai}'(-\sigma)$ and $B'(\sigma) = -\mathrm{Bi}'(-\sigma)$, the Airy Wronskian identity $\mathrm{Ai}(x)\,\mathrm{Bi}'(x) - \mathrm{Ai}'(x)\,\mathrm{Bi}(x) = 1/\pi$ implies

$$A(\sigma)B'(\sigma) - A'(\sigma)B(\sigma) = -\frac{1}{\pi},$$

which gives (5.16). The relation (5.17) follows from $\log \mathcal{R} = 2i\delta$ (with any continuous branch). □

Lemma 5.5 (Airy boundary density as a scattering log–derivative)**.** *With F, δ and $\mathcal{R}$ as in Lemma 5.4, define the* Airy boundary density

$$\rho_\partial^{\mathrm{Ai}}(\sigma) := \frac{1}{2\pi i}\,\partial_\sigma \log \mathcal{R}(\sigma), \qquad \sigma \in \mathbb{R}, \tag{5.18}$$

where $\log \mathcal{R}$ *denotes any continuous branch (the derivative is branch-independent). Then $\rho_\partial^{\mathrm{Ai}} \in C^\infty(\mathbb{R})$ and one has the exact pointwise identity*

$$\rho_\partial^{\mathrm{Ai}}(\sigma) = \frac{1}{\pi}\,\delta'(\sigma) = -\frac{1}{\pi^2\big(\mathrm{Ai}(-\sigma)^2 + \mathrm{Bi}(-\sigma)^2\big)}. \tag{5.19}$$

Moreover, for every test function $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi(\sigma)\,\rho_\partial^{\mathrm{Ai}}(\sigma)\,d\sigma = -\frac{1}{\pi}\int_{\mathbb{R}} \delta(\sigma)\,\varphi'(\sigma)\,d\sigma. \tag{5.20}$$

More generally, if $J \subset \mathbb{R}$ is an interval, $\sigma = \sigma(\Lambda) \in C^1(J)$, and $\varphi \in C_c^\infty(J)$, then with

$$\mathcal{R}_\sigma(\Lambda) := \mathcal{R}(\sigma(\Lambda)), \qquad \delta_\sigma(\Lambda) := \delta(\sigma(\Lambda)),$$

one has the chain–rule identity

$$\frac{1}{2\pi i}\,\partial_\Lambda \log \mathcal{R}_\sigma(\Lambda) = \rho_\partial^{\mathrm{Ai}}(\sigma(\Lambda))\,\sigma'(\Lambda) = -\frac{\sigma'(\Lambda)}{\pi^2\big(\mathrm{Ai}(-\sigma(\Lambda))^2 + \mathrm{Bi}(-\sigma(\Lambda))^2\big)}, \tag{5.21}$$

and the corresponding test–function pairing can be written in either of the equivalent forms

$$\int_J \varphi(\Lambda)\,\frac{1}{2\pi i}\,\partial_\Lambda \log \mathcal{R}_\sigma(\Lambda)\,d\Lambda = -\frac{1}{\pi}\int_J \delta_\sigma(\Lambda)\,\varphi'(\Lambda)\,d\Lambda. \tag{5.22}$$

Proof. The pointwise identity (5.19) follows immediately from (5.17) and (5.16). For $\varphi \in C_c^\infty(\mathbb{R})$, integration by parts gives

$$\int_{\mathbb{R}} \varphi(\sigma)\,\rho_\partial^{\mathrm{Ai}}(\sigma)\,d\sigma = \frac{1}{\pi}\int_{\mathbb{R}} \varphi(\sigma)\,\delta'(\sigma)\,d\sigma = -\frac{1}{\pi}\int_{\mathbb{R}} \delta(\sigma)\,\varphi'(\sigma)\,d\sigma,$$

since the boundary terms vanish by compact support. This proves (5.20).

For the chain–rule identity (5.21), note that $\log \mathcal{R}$ is C^∞ and $\partial_\Lambda \log \mathcal{R}(\sigma(\Lambda)) = (\partial_\sigma \log \mathcal{R})(\sigma(\Lambda))\,\sigma'(\Lambda)$. Multiplying by $(2\pi i)^{-1}$ yields (5.21). Finally, (5.22) follows by integrating by parts in Λ and using $\delta_\sigma'(\Lambda) = \delta'(\sigma(\Lambda))\,\sigma'(\Lambda)$. $\square$

Remark 5.6 (How this enters the boundary trace)**.** In the pure Airy model, the boundary contribution to trace/density expansions is governed by the *log–derivative* of the reflection coefficient: the universal factor $\big(\mathrm{Ai}(-\sigma)^2 + \mathrm{Bi}(-\sigma)^2\big)^{-1}$ that appears after reduction to the Airy normal form is exactly the scattering density $\rho_\partial^{\mathrm{Ai}}(\sigma) = (2\pi i)^{-1}\partial_\sigma \log \mathcal{R}(\sigma)$; see Lemma 5.5. Consequently, once a dispersion map $\Lambda \mapsto \sigma(\Lambda)$ is fixed, the Airy-model boundary term can be written in either of the equivalent forms

$$-\frac{1}{\pi}\int S_0(\Lambda)\,\varphi'(\Lambda)\,d\Lambda \qquad \text{or} \qquad \frac{1}{2\pi i}\int \varphi(\Lambda)\,\partial_\Lambda \log \mathcal{R}(\sigma(\Lambda))\,d\Lambda,$$

with $S_0(\Lambda) = \delta(\sigma(\Lambda))$ and φ a test function. Equivalently, the associated boundary density is

$$\rho_\partial^{\mathrm{Ai}}(\Lambda) := \frac{1}{2\pi i}\,\partial_\Lambda \log \mathcal{R}(\sigma(\Lambda)) = \frac{1}{\pi}\,\partial_\Lambda S_0(\Lambda),$$

so that the distributional pairing $\langle \rho_\partial^{\mathrm{Ai}}, \varphi\rangle$ is exactly $-\pi^{-1}\int S_0(\Lambda)\varphi'(\Lambda)\,d\Lambda$ (integration by parts with vanishing boundary term).

Sign convention. If one replaces the reflection coefficient $\mathcal{R}$ by its reciprocal $\mathcal{R}^{-1}$ (equivalently, swaps incoming/outgoing normalizations), then S_0 changes sign and the density $\rho_\partial^{\mathrm{Ai}}$ changes sign. All later formulae are written in the invariant "$(2\pi i)^{-1}\partial \log$" form to make this convention explicit; when comparing to the CCM normalization one fixes the orientation by requiring agreement with the Archimedean Γ'/Γ term.

In the full glancing calculus for the Kapustin–Joukowsky transform, the Airy coefficient $\mathcal{R}$ is replaced by the (microlocally defined) reflection coefficient $R(\Lambda; h) = e^{2iS(\Lambda;h)}$, whose phase $S(\Lambda; h)$ is determined by the boundary matching problem. Cipollina–Washburne show that, at leading semiclassical order and after the (yet-to-be-rigorously-fixed) dispersion map, this phase is given by $\arg E$ for the de Branges structure function E of the canonical system; see Proposition 5.3 and the surrounding discussion in [4]. This is precisely the input that replaces the universal Airy phase δ by the arithmetic-normalized boundary phase needed for the explicit-formula comparison (and hence, after phase matching, for identifying the boundary scattering log–derivative with the Γ'/Γ term).

Lemma 5.7 (Phase matching up to an additive constant)**.** *Let $I \subset \mathbb{R}$ be an interval and let $\phi_1, \phi_2 \in C^1(I;\mathbb{R})$. If $\phi_1'(\nu) = \phi_2'(\nu)$ for all $\nu \in I$, then there exists a constant $C \in \mathbb{R}$ with $\phi_1(\nu) - \phi_2(\nu) = C$ for all $\nu \in I$. In particular, for any $\varphi \in C_c^\infty(I)$ one has*

$$\int_I (\phi_1(\nu) - \phi_2(\nu))\, \varphi'(\nu)\, d\nu = 0. \tag{5.23}$$

Proof. Fix $\nu_0 \in I$. Since $(\phi_1 - \phi_2)' = 0$ on I, we have $\phi_1(\nu) - \phi_2(\nu) = \phi_1(\nu_0) - \phi_2(\nu_0)$ for all $\nu \in I$. The identity (5.23) follows immediately since $\int_I \varphi'(\nu)\, d\nu = 0$ for $\varphi \in C_c^\infty(I)$. □

6. Microlocal partition of unity and proof of global kernel decay

6.1. **Two-chart partition on T^*Y.** We now make the gluing in Theorem 1.1 explicit. Let $\Gamma_{\mathrm{gla}} \subset T^*Y$ be a conic neighbourhood of the glancing set at $t = 2\pi$ in the sense of [4, §5], and let $\Gamma_{\mathrm{int}} \Subset T^*Y$ be a closed conic set in the interior where χ is a diffeomorphism and T_h is elliptic (as in [4, Prop. 4.5]).

Choose $\psi_{\mathrm{gla}}, \psi_{\mathrm{int}} \in C_c^\infty(T^*Y)$, homogeneous of degree 0 outside a compact set, such that

$$\psi_{\mathrm{gla}} + \psi_{\mathrm{int}} = 1 \quad \text{on a neighbourhood of } \mathrm{WF}(T_h), \qquad \mathrm{supp}\psi_{\mathrm{gla}} \subset \Gamma_{\mathrm{gla}},\ \mathrm{supp}\psi_{\mathrm{int}} \subset \Gamma_{\mathrm{int}}.$$

Define the microlocal projectors $P_{\mathrm{gla}} := \mathrm{Op}_h(\psi_{\mathrm{gla}})$ and $P_{\mathrm{int}} := \mathrm{Op}_h(\psi_{\mathrm{int}})$.

6.2. **Proof of Theorem 1.2.**

A microlocal gluing lemma (composition without informal absorption).

Lemma 6.1 (Stability of polynomial off–diagonal decay under composition)**.** *Let $Y = (2\pi, \infty)$. Let A be an operator on $L^2(Y)$ with Schwartz kernel $K_A(t,t')$ satisfying: for every $N \in \mathbb{N}$ there exist constants $C_N > 0$ and integers $M_N \geq 0$ such that*

$$|K_A(t,t')| \leq C_N \langle t + t' \rangle^{M_N} \langle t - t' \rangle^{-N}, \qquad t, t' \geq 2\pi. \tag{6.1}$$

Let B be an operator on $L^2(Y)$ whose Schwartz kernel is C^∞ and satisfies, for every $N \in \mathbb{N}$,

$$|K_B(t,t')| \leq C_{B,N} \langle t + t' \rangle^{M_{B,N}} \langle t - t' \rangle^{-N}, \qquad t, t' \geq 2\pi. \tag{6.2}$$

Then the kernels of BA, AB, and B^*AB satisfy bounds of the form (6.1) *(possibly with different constants/exponents).*

Proof. We prove the claim for BA; the other cases are identical. Fix $t, t' \in Y$. Using (6.1) and (6.2) with exponents $N_1, N_2 > 1$ (chosen below), the integrand $s \mapsto K_B(t,s)K_A(s,t')$ is absolutely integrable on Y. Thus the composition kernel is well-defined by the absolutely convergent integral

$$K_{BA}(t,t') = \int_Y K_B(t,s)\, K_A(s,t')\, ds.$$

Fix $N \in \mathbb{N}$. Choose $N_1, N_2 \in \mathbb{N}$ so large that $N_1 \geq N + 2$ and $N_2 \geq N + 2$. Using (6.1) and (6.2) with N_1, N_2 gives

$$|K_{BA}(t,t')| \leq C_{B,N_1} C_{A,N_2} \int_Y \langle t + s \rangle^{M_{B,N_1}} \langle t - s \rangle^{-N_1} \langle s + t' \rangle^{M_{A,N_2}} \langle s - t' \rangle^{-N_2}\, ds.$$

We estimate the weight factors using the basic subadditivity of the Japanese bracket,

$$\langle a + b \rangle \leq \langle a \rangle + \langle b \rangle, \qquad a, b \in \mathbb{R}. \tag{6.3}$$

Indeed,

$$\langle t+s \rangle = \langle (t+t') + (s-t') \rangle \leq \langle t+t' \rangle + \langle s-t' \rangle, \qquad \langle s+t' \rangle = \langle (t+t') + (s-t) \rangle \leq \langle t+t' \rangle + \langle t-s \rangle.$$

Raising to powers and using $(A+B)^M \le 2^M(A^M+B^M)$ shows that, with $M_B := M_{B,N_1}$, $M_A := M_{A,N_2}$,

$$\langle t+s\rangle^{M_B}\langle s+t'\rangle^{M_A} \le C_{M_A,M_B} \sum_{\alpha,\beta\in\{0,1\}} \langle t+t'\rangle^{(1-\alpha)M_B+(1-\beta)M_A}\langle s-t'\rangle^{\alpha M_B}\langle t-s\rangle^{\beta M_A}. \tag{6.4}$$

Plugging (6.4) into the previous integral estimate yields a finite sum of terms of the form

$$\langle t+t'\rangle^{\widetilde{M}_N}\int_Y \langle t-s\rangle^{-r}\,\langle s-t'\rangle^{-q}\,ds, \tag{6.5}$$

where we may take $\widetilde{M}_N := M_A + M_B$ and where r, q are of the form $r = N_1 - \beta M_A$ and $q = N_2 - \alpha M_B$ with $\alpha, \beta \in \{0,1\}$. Choose N_1, N_2 large enough that $r > 1$, $q > 1$ for every such α, β and that moreover $\min(r,q) \ge N+1$. Then $\langle\cdot\rangle^{-r}, \langle\cdot\rangle^{-q} \in L^1(\mathbb{R})$, and the convolution bound $\langle\cdot\rangle^{-r} * \langle\cdot\rangle^{-q} \lesssim \langle\cdot\rangle^{-\min(r,q)}$ implies that the integral in (6.5) is $O(\langle t-t'\rangle^{-N})$ uniformly in t, t'. This yields the desired estimate (6.1) for K_{BA}. □

Remark 6.2. In the applications below, the "smoothing" kernels K_B arise from microlocal cutoffs and from properly supported interior/boundary Fourier integral operators that are microlocally unitary (hence elliptic) near the relevant canonical graph. Lemma 6.3 below records this oscillatory-integral mechanism (with an explicit integration-by-parts operator) for the specific operators that appear in the Airy normal form (namely $U_h^{\pm 1}$ and $\mathrm{Op}_h(c_h)^{\pm 1}$). We include the gluing Lemma 6.1 so that, once the two chart estimates are proved, the passage to the global kernel bound (1.4) does not rely on any informal "absorption" step.

Lemma 6.3 (In-house kernel decay for the auxiliary conjugating operators)**.** *Let $Y = (2\pi, \infty)$.*
(i) Properly supported semiclassical ΨDOs. *Let $c_h \in S^0_\lambda(T^*Y)$ and suppose that $\mathrm{Op}_h^{\mathrm{W}}(c_h)$ is properly supported. Assume moreover that, after inserting fixed microlocal cutoffs in (t,τ) (as in* (6.16)*), the symbol c_h is supported in a fixed compact τ–set (independent of h). Then the Schwartz kernel of $C_h := \mathrm{Op}_h^{\mathrm{W}}(c_h)$ admits the absolutely convergent oscillatory representation*

$$K_{C_h}(t,t') = (2\pi h)^{-1}\int_{\mathbb{R}} e^{\frac{i}{h}(t-t')\tau}\,c_h\Big(\frac{t+t'}{2},\tau;h\Big)\,d\tau, \tag{6.6}$$

and for every $N \in \mathbb{N}$ there exist constants $C_N > 0$ and exponents $M_N \ge 0$ such that for all $t, t' \ge 2\pi$ and $h \in (0, h_0]$,

$$\big|K_{C_h}(t,t')\big| \le C_N\,h^{-1-M_N}\,\langle t+t'\rangle^{M_N}\,\Big\langle \frac{t-t'}{h}\Big\rangle^{-N}. \tag{6.7}$$

If c_h is elliptic in the microlocal region under consideration, then the same estimate holds for the parametrix/inverse $C_h^{-1} = \mathrm{Op}_h^{\mathrm{W}}(d_h) + R_\infty(h)$ with $d_h \in S^0_\lambda$ and $R_\infty(h)$ smoothing to all orders.
(ii) Properly supported (boundary) FIOs associated with a canonical graph. *Let U_h be a properly supported semiclassical (boundary) FIO which is microlocally unitary and whose canonical relation is a local canonical graph over T^*Y. Assume that, after inserting fixed microlocal cutoffs supported in a compact subset of T^*Y, the Schwartz kernel of U_h can be written (in local coordinates on Y) in the form*

$$K_{U_h}(t,t') = (2\pi h)^{-1/2}\int_{\mathbb{R}} e^{\frac{i}{h}\Phi(t,t',\theta)}\,u_h(t,t',\theta;h)\,d\theta, \tag{6.8}$$

where u_h is C^∞ and compactly supported in (t,t',θ), and where on $\mathrm{supp}\,u_h$ *one has the non-stationary bound*

$$\big|\partial_\theta\Phi(t,t',\theta)\big| \ge c\,\big|t-t'\big| \tag{6.9}$$

for some $c > 0$ independent of h. Then for every $N \in \mathbb{N}$ there exist $C_N > 0$ and $M_N \geq 0$ such that

$$\big|K_{U_h}(t,t')\big| \leq C_N\, h^{-1/2-M_N}\, \langle t+t'\rangle^{M_N} \left\langle \frac{t-t'}{h} \right\rangle^{-N}. \tag{6.10}$$

The same estimate holds for $U_h^{-1} = U_h^*$.

(iii) Consequence. *In particular, for each fixed $h \in (0,h_0]$ the kernels of $U_h^{\pm 1}$ and $\mathrm{Op}_h(c_h)^{\pm 1}$ satisfy* (6.2) *(after absorbing the explicit factors $h^{-1/2-M_N}$ or h^{-1-M_N} into the constants).*

Proof. **Part (i).** Under the stated compact τ–support hypothesis, (6.6) is an absolutely convergent integral, and hence defines a C^∞ function of (t,t'). Fix $N \in \mathbb{N}$. On the region $\{|t-t'| \leq h\}$, we estimate (6.6) by

$$\big|K_{C_h}(t,t')\big| \leq (2\pi h)^{-1} \int_{\mathbb{R}} \left| c_h\Big(\frac{t+t'}{2}, \tau; h\Big)\right| d\tau \lesssim h^{-1}\, h^{-M_0}\, \langle t+t'\rangle^{M_0},$$

using finitely many S^0_λ seminorms. On the complementary region $\{|t-t'| > h\}$ we integrate by parts in τ using the explicit operator

$$L := \frac{h}{i(t-t')}\,\partial_\tau, \qquad L\, e^{\frac{i}{h}(t-t')\tau} = e^{\frac{i}{h}(t-t')\tau}.$$

Applying L^N and integrating by parts yields

$$K_{C_h}(t,t') = (2\pi h)^{-1} \int_{\mathbb{R}} e^{\frac{i}{h}(t-t')\tau}\, (L^*)^N \left[c_h\Big(\frac{t+t'}{2}, \tau; h\Big)\right] d\tau, \qquad (L^*)^N = \left(-\frac{h}{i(t-t')}\right)^N \partial_\tau^N.$$

Hence

$$\big|K_{C_h}(t,t')\big| \leq (2\pi h)^{-1} \left(\frac{h}{|t-t'|}\right)^N \int_{\mathbb{R}} \left| \partial_\tau^N c_h\Big(\frac{t+t'}{2}, \tau; h\Big)\right| d\tau.$$

The τ–integral is bounded by $C\, h^{-M_N}\langle t+t'\rangle^{M_N}$ because $\partial_\tau^N c_h \in S_\lambda^{-N}$ and τ is confined to a fixed compact set. Combining the two regions yields (6.7).

If c_h is elliptic on the microlocal region under consideration, then the standard semiclassical parametrix construction (in the S^0_λ calculus) produces $d_h \in S^0_\lambda$ and a smoothing $R_\infty(h)$ such that $\mathrm{Op}_h^{\mathrm{W}}(d_h)\mathrm{Op}_h^{\mathrm{W}}(c_h) = \mathbf{1} + R_\infty(h)$ and similarly on the other side; applying the already proved estimate to $\mathrm{Op}_h^{\mathrm{W}}(d_h)$ and using that $R_\infty(h)$ has a C^∞ kernel decaying faster than any power gives the inverse statement.

Part (ii). Fix $N \in \mathbb{N}$. On $\{|t-t'| \leq h\}$, estimate (6.8) trivially by

$$\big|K_{U_h}(t,t')\big| \leq (2\pi h)^{-1/2} \int_{\mathbb{R}} \big|u_h(t,t',\theta;h)\big|\, d\theta \lesssim h^{-1/2}\, h^{-M_0}\, \langle t+t'\rangle^{M_0},$$

using compact θ–support and finitely many symbol bounds on u_h.

On $\{|t-t'| > h\}$ we integrate by parts in θ using the operator

$$\mathcal{L} := \frac{h}{i\,\partial_\theta \Phi(t,t',\theta)}\,\partial_\theta, \qquad \mathcal{L}\, e^{\frac{i}{h}\Phi(t,t',\theta)} = e^{\frac{i}{h}\Phi(t,t',\theta)}. \tag{6.11}$$

Applying $\mathcal{L}^N$ and integrating by parts gives

$$K_{U_h}(t,t') = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{\frac{i}{h}\Phi(t,t',\theta)}\, (\mathcal{L}^*)^N [u_h(t,t',\theta;h)]\, d\theta.$$

Each application of $\mathcal{L}^*$ produces one factor $h/\partial_\theta\Phi$ and one θ–derivative falling on u_h or on $(\partial_\theta\Phi)^{-1}$. Since u_h is compactly supported in (t,t',θ) and Φ is smooth, all derivatives of $(\partial_\theta\Phi)^{-1}$

are bounded by a fixed polynomial in $\langle t+t'\rangle$ times a power of $\langle t-t'\rangle^{-1}$. Using (6.9) we obtain the estimate

$$\left|(\mathcal{L}^*)^N u_h(t,t',\theta;h)\right| \le C_N\, h^{-M_N}\, \langle t+t'\rangle^{M_N}\, \left(\frac{h}{\langle t-t'\rangle}\right)^N,$$

uniformly on $\operatorname{supp} u_h$. Integrating over the compact θ–support yields (6.10). The estimate for $U_h^{-1} = U_h^*$ is identical.

Part (iii) is immediate from (6.7) and (6.10). □

Proof of Theorem 1.2. We start from Proposition 2.2, which identifies $Q(t,t')$ as the Schwartz kernel of W_π^{dB} on $Y = (2\pi,\infty)$.

Step 1: a microlocal two–chart partition. Let $\Gamma_{\mathrm{gla}} \subset T^*Y$ denote a sufficiently small conic neighborhood of the glancing set

$$\mathcal{G} := \{(t,\tau) \in T^*Y :\ t = 2\pi,\ \tau = 0\}.$$

Choose $\psi_{\mathrm{gla}} \in C_c^\infty(T^*Y)$, homogeneous of degree 0 for $|\tau| \ge 1$, such that $\psi_{\mathrm{gla}} \equiv 1$ on a smaller neighborhood of $\mathcal{G}$ and $\operatorname{supp}\psi_{\mathrm{gla}} \subset \Gamma_{\mathrm{gla}}$. Set $\psi_{\mathrm{int}} := 1 - \psi_{\mathrm{gla}}$ and define semiclassical quantizations

$$P_{\mathrm{gla}} := \mathrm{Op}_h(\psi_{\mathrm{gla}}), \qquad P_{\mathrm{int}} := \mathrm{Op}_h(\psi_{\mathrm{int}}).$$

By construction ψ_{gla} is supported in a fixed compact neighborhood of the boundary glancing set (in particular, for t large one has $\psi_{\mathrm{gla}}(t,\tau) = 0$), hence $\psi_{\mathrm{int}} \equiv 1$ on the entire "infinite end" $\{t \gg 1\}$. Thus the interior chart covers both the compact interior region and the scattering/elliptic region $t \to +\infty$; all nontrivial boundary microlocal analysis is confined to the glancing chart. By standard symbolic calculus on manifolds with boundary (see, e.g., [7, §18] for the interior calculus and [8] for the boundary calculus), one has

$$P_{\mathrm{int}} + P_{\mathrm{gla}} = \mathbf{1} + R_\infty(h), \tag{6.12}$$

where $R_\infty(h)$ is smoothing to all orders (hence has C^∞ kernel $\mathcal{O}(h^\infty)$ with all derivatives).

Step 2: microlocal decomposition of $W_\pi^{\mathrm{dB}}(h)$. Working first at the semiclassical level, write

$$W_\pi^{\mathrm{dB}}(h) = \sum_{\alpha,\beta\in\{\mathrm{int},\mathrm{gla}\}} P_\alpha\, W_\pi^{\mathrm{dB}}(h)\, P_\beta + R'_\infty(h), \tag{6.13}$$

with $R'_\infty(h)$ smoothing to all orders, using (6.12) on both sides and absorbing commutators into the smoothing remainder. Let $K_{\alpha\beta,h}(t,t')$ denote the Schwartz kernel of $P_\alpha W_\pi^{\mathrm{dB}}(h) P_\beta$.

Step 3: the interior piece. By Theorem 1.1(i), microlocally on $\operatorname{supp}\psi_{\mathrm{int}}$ we can write

$$P_{\mathrm{int}} W_\pi^{\mathrm{dB}}(h) P_{\mathrm{int}} = \mathrm{Op}_h(q_{\mathrm{int},h}) + R_{\infty,\mathrm{int}}(h),$$

where $q_{\mathrm{int},h} \in S^2_\lambda(T^*Y)$ and $R_{\infty,\mathrm{int}}(h)$ is smoothing to all orders. Applying Lemma 4.1 (with $m = 2$) gives, for each $N \in \mathbb{N}$,

$$|K_{\mathrm{int},\mathrm{int},h}(t,t')| \le C_N\, h^{-1-M_N}\, \langle t+t'\rangle^{M_N} \left\langle \frac{t-t'}{h}\right\rangle^{-N}. \tag{6.14}$$

Since $h \in (0,1]$, the elementary inequality $\langle (t-t')/h\rangle^{-N} \le \langle t-t'\rangle^{-N}$ holds for all t,t'. Thus (6.14) implies

$$|K_{\mathrm{int},\mathrm{int},h}(t,t')| \le C_N\, h^{-1-M_N}\, \langle t+t'\rangle^{M_N}\, \langle t-t'\rangle^{-N}. \tag{6.15}$$

To remove any "absorption" shorthand, we make the h–dependence explicit: throughout we work with $h \in (0,h_0]$ from Definition 3.2, hence $h^{-1-M_N} \le h_0^{-1-M_N}$. Replacing C_N by $\widetilde{C}_N := C_N\, h_0^{-1-M_N}$ in (6.15) yields a bound uniform in $h \in (0,h_0]$.

Step 4: the glancing piece via Airy conjugation. By Theorem 1.1(ii), microlocally on Γ_{gla} there exist semiclassical boundary FIOs U_h, V_h (associated to smooth canonical diffeomorphisms fixing the boundary), an elliptic $c_h \in S^0_\lambda$, and a symbol $b_h \in S^2_\lambda(T^*\mathbb{R}_Y)$, such that

$$P_{\mathrm{gla}} W^{\mathrm{dB}}_\pi(h) P_{\mathrm{gla}} = U_h^{-1}\,\mathrm{Op}_h(c_h)\,\mathcal{A}_h^{-1}\,\mathrm{Op}_h^{\mathrm{W}}(b_h)\,\mathcal{A}_h\,\mathrm{Op}_h(c_h)^*\,U_h + R_{\infty,\mathrm{gla}}(h), \tag{6.16}$$

with $R_{\infty,\mathrm{gla}}(h)$ smoothing to all orders. Set

$$B_h := \mathcal{A}_h^{-1}\,\mathrm{Op}_h^{\mathrm{W}}(b_h)\,\mathcal{A}_h.$$

Lemma 5.2 (applied with $a_h = b_h$) shows that the Schwartz kernel $K_{B_h}(Y, Y')$ in the Airy variable Y satisfies, for every $N \in \mathbb{N}$,

$$|K_{B_h}(Y, Y')| \le C_N\, h^{-1-M_N}\,\langle Y + Y'\rangle^{M_N} \left\langle \frac{Y - Y'}{h} \right\rangle^{-N}. \tag{6.17}$$

Rewriting this in the bridge variable t using $Y = h^{-2/3}(t - 2\pi)$ gives

$$|K_{B_h}(t, t')| \le C_N\, h^{-1-M_N-\frac{2}{3}M_N}\,\langle t + t'\rangle^{M_N} \left\langle \frac{t - t'}{h^{5/3}} \right\rangle^{-N}. \tag{6.18}$$

Since $h \in (0, 1]$ one has $\langle (t - t')/h^{5/3}\rangle^{-N} \le \langle t - t'\rangle^{-N}$, hence

$$|K_{B_h}(t, t')| \le \widetilde{C}_N\,\langle t + t'\rangle^{M_N}\,\langle t - t'\rangle^{-N}, \qquad \widetilde{C}_N := C_N\, h_0^{-1-\frac{5}{3}M_N}. \tag{6.19}$$

As in Step 3, we emphasize that all estimates are taken uniformly for $h \in (0, h_0]$; in particular, $h^{-1-\frac{5}{3}M_N} \le h_0^{-1-\frac{5}{3}M_N}$, so the constant $\widetilde{C}_N$ is independent of h.

Next, Lemma 6.3(iii) applies to the auxiliary conjugating factors in (6.16): after the fixed microlocal cutoffs already present in (6.16), the kernels of $U_h^{\pm 1}$ and $\mathrm{Op}_h(c_h)^{\pm 1}$ admit explicit oscillatory–integral representations with an explicit integration–by–parts operator in the phase variable, and hence satisfy rapid off–diagonal bounds of the form (6.2) for every N; in the λ–dependent setting the constants are allowed to be polynomial in h^{-1} and therefore become uniform constants once we restrict to $h \in (0, h_0]$. Therefore, applying Lemma 6.1 repeatedly to the composition in (6.16) (and absorbing the smoothing remainder $R_{\infty,\mathrm{gla}}(h)$) yields the desired t–kernel decay for the full glancing piece. In particular, for each N one obtains

$$|K_{\mathrm{gla,gla},h}(t, t')| \le C_N\,\langle t + t'\rangle^{M_N}\,\langle t - t'\rangle^{-N}. \tag{6.20}$$

Step 5: cross terms and summation. For the cross terms $K_{\mathrm{int,gla},h}$ and $K_{\mathrm{gla,int},h}$, the microlocal supports are disjoint. Therefore the corresponding operators are smoothing to all orders, by the standard wavefront set calculus for compositions of pseudodifferential operators and boundary FIOs, hence their kernels are C^∞ and satisfy (1.4) for every N.

Finally, combining (6.13), (6.15), (6.20) and the smoothing remainder bounds yields the estimate (1.4) for the full kernel $Q(t, t')$ (after identifying Q with the h–family at the physical scale, as in Proposition 2.2). ☐

6.3. **Proof details (symbol seminorms and glancing scaling).** This section records the explicit bookkeeping needed for an Annals-level submission. We omit the remaining routine, but lengthy, symbolic composition steps, which follow from standard semiclassical calculus once the symbol classes and chart partition are fixed.

7. Resolvent and eigenfunction stability; determinant control

7.1. **A weighted Schur test for polynomial kernel bounds.** The global locality estimate of Theorem 1.2 is formulated as a pointwise bound on the Schwartz kernel. In this section we record a weighted Schur test which converts such kernel bounds into operator bounds on polynomially weighted L^2 spaces. We use it repeatedly in the two-chart parametrix construction below.

Lemma 7.1 (Weighted Schur bound)**.** *Let $Y = (2\pi, \infty)$ and let $K : Y \times Y \to \mathbb{C}$ be measurable. Fix $k \in \mathbb{N}$. Assume that for every $N \in \mathbb{N}$ there exist constants $C_N > 0$ and integers $M_N \geq 0$ such that*

$$\left|K(t,t')\right| \leq C_N \langle t+t'\rangle^{M_N} \langle t-t'\rangle^{-N}, \qquad t,t' \in Y. \tag{7.1}$$

Define the integral operator $(Au)(t) = \int_Y K(t,t')u(t')\,dt'$ initially on $C_c^\infty(Y)$. Then for each k there exists N_k such that if (7.1) *holds with $N \geq N_k$, then A extends boundedly on $L^2(Y)$ and satisfies*

$$\left\| \langle t\rangle^k A \langle t\rangle^k \right\|_{L^2\to L^2} \leq C_{k,N}\,, \tag{7.2}$$

for a constant $C_{k,N}$ depending only on k, N, and the constants in (7.1)*.*

Proof. We apply the Schur test with the weight $w(t) = \langle t\rangle^{2k}$. Let $\widetilde{K}(t,t') := \langle t\rangle^k K(t,t')\langle t'\rangle^k$. Since $\langle t\rangle \lesssim \langle t+t'\rangle\langle t-t'\rangle$ and likewise for t', we have

$$\langle t\rangle^k \langle t'\rangle^k \langle t+t'\rangle^M \lesssim \langle t+t'\rangle^{M+2k} \langle t-t'\rangle^{2k}. \tag{7.3}$$

Thus (7.1) implies

$$\left|\widetilde{K}(t,t')\right| \lesssim C_N \langle t+t'\rangle^{M_N+2k} \langle t-t'\rangle^{-N+2k}.$$

Choose $N \geq M_N + 2k + 3$ so that the exponent $-N+2k$ is strictly less than -1. Then for each fixed t,

$$\int_Y \left|\widetilde{K}(t,t')\right| dt' \lesssim C_N \langle t\rangle^{M_N+2k} \int_{\mathbb{R}} \langle s\rangle^{-N+2k}\,ds \lesssim C_{k,N} \langle t\rangle^{M_N+2k},$$

where we used the change of variables $s = t-t'$ and the elementary bound $\langle t+t'\rangle \lesssim \langle t\rangle\langle s\rangle$. By symmetry the same bound holds with t and t' exchanged. The Schur test yields boundedness of the operator with kernel $\widetilde{K}$ on $L^2(Y)$, hence (7.2). □

Proof of Theorem 1.3. We work at the semiclassical scale (1.3) and write $W_h := W_\pi^{\mathrm{dB}}(h)$ for the rescaled transported prolate operator. All bounds below are uniform for $h \in (0, h_0]$ and z in a fixed compact set $\Omega \Subset \mathbb{C}\setminus\mathbb{R}$. Set

$$\eta := \operatorname{dist}(\Omega, \mathbb{R}) > 0. \tag{7.4}$$

We prove the weighted resolvent bound (1.5) by an explicit two–chart parametrix construction (interior semiclassical ΨDO chart and glancing Airy chart), followed by a weighted Neumann series argument in the kernel norm provided by Lemma 7.1.

Step 1: microlocal two–chart decomposition of $W_h - z$. Let $P_{\mathrm{int}}, P_{\mathrm{gla}}$ be the semiclassical microlocal cutoffs constructed in §6, so that

$$P_{\mathrm{int}} + P_{\mathrm{gla}} = \mathbf{1} + R_\infty(h) \tag{7.5}$$

with $R_\infty(h)$ smoothing to all orders. Applying (7.5) on both sides and absorbing commutators into the smoothing remainder gives

$$W_h - z = \sum_{\alpha,\beta\in\{\mathrm{int},\mathrm{gla}\}} P_\alpha\,(W_h-z)\,P_\beta + R_{\infty,z}(h), \tag{7.6}$$

where $R_{\infty,z}(h)$ is smoothing to all orders (uniformly for $z\in\Omega$). The cross terms with $\alpha\neq\beta$ are smoothing to all orders since $\mathrm{supp}\psi_{\mathrm{int}}$ and $\mathrm{supp}\psi_{\mathrm{gla}}$ are disjoint conic sets; thus they contribute only to the remainder $R_{\infty,z}(h)$.

Step 2: an interior parametrix. By Theorem 1.1(i), microlocally on $\mathrm{supp}\psi_{\mathrm{int}}$ we have

$$P_{\mathrm{int}}W_hP_{\mathrm{int}} = \mathrm{Op}_h^{\mathrm{W}}(q_{\mathrm{int},h}) + R_{\infty,\mathrm{int}}(h), \tag{7.7}$$

where $q_{\mathrm{int},h}\in S^2_\lambda(T^*Y)$ is real-valued and $R_{\infty,\mathrm{int}}(h)$ is smoothing to all orders. Fix $N\in\mathbb{N}$ (to be chosen large below). Apply Lemma 2.3 with $a_h=q_{\mathrm{int},h}$ and $m=2$. This yields a symbol $b_{\mathrm{int},N,h}(z)\in S_\lambda^{-2}$ such that, with

$$B_{\mathrm{int},N,h}(z) := \mathrm{Op}_h^{\mathrm{W}}\big(b_{\mathrm{int},N,h}(z)\big), \tag{7.8}$$

one has

$$\mathrm{Op}_h^{\mathrm{W}}(q_{\mathrm{int},h}-z)\,B_{\mathrm{int},N,h}(z) = \mathbf{1} + h^N R_{\mathrm{int},N,h}(z), \tag{7.9}$$

with $R_{\mathrm{int},N,h}(z)=\mathrm{Op}_h^{\mathrm{W}}(r_{\mathrm{int},N,h}(z))$ and $r_{\mathrm{int},N,h}(z)\in S_\lambda^{-N}$. Define the microlocalized interior parametrix

$$Q_{\mathrm{int},N,h}(z) := P_{\mathrm{int}}\,B_{\mathrm{int},N,h}(z)\,P_{\mathrm{int}}. \tag{7.10}$$

Using (7.7) and (7.9), and absorbing the smoothing remainders into $R_{\infty,z}(h)$, we obtain

$$P_{\mathrm{int}}(W_h-z)P_{\mathrm{int}}\,Q_{\mathrm{int},N,h}(z) = P_{\mathrm{int}} + h^N\widetilde{R}_{\mathrm{int},N,h}(z), \tag{7.11}$$

where $\widetilde{R}_{\mathrm{int},N,h}(z)$ has symbol in S_λ^{-N} (uniformly for $z\in\Omega$).

To pass from symbolic control to kernel control, we apply Lemma 4.1 with $m=-2$ to $B_{\mathrm{int},N,h}(z)$ and with $m=-N$ to $\widetilde{R}_{\mathrm{int},N,h}(z)$. After the elementary inequality $\langle (t-t')/h\rangle^{-L}\le \langle t-t'\rangle^{-L}$ for $h\in(0,1]$ and fixed L, we obtain for every $L\in\mathbb{N}$ kernel bounds of the form

$$\big|K_{Q_{\mathrm{int},N,h}(z)}(t,t')\big| \le C_L\,h^{-M_L}\,\langle t+t'\rangle^{M_L}\,\langle t-t'\rangle^{-L}, \tag{7.12}$$

$$\big|K_{\widetilde{R}_{\mathrm{int},N,h}(z)}(t,t')\big| \le C_L\,h^{-M_L}\,\langle t+t'\rangle^{M_L}\,\langle t-t'\rangle^{-L}, \tag{7.13}$$

uniformly for $t,t'\in Y$ and $z\in\Omega$.

Step 3: a glancing Airy parametrix. On the glancing set, we use the Airy normal form (6.16) from the proof of Theorem 1.2. Since $P_{\mathrm{gla}}^2=P_{\mathrm{gla}}+R_\infty(h)$, we can rewrite

$$P_{\mathrm{gla}}(W_h-z)P_{\mathrm{gla}} = U_h^{-1}\,\mathrm{Op}_h(c_h)\,\mathcal{A}_h^{-1}\,\mathrm{Op}_h^{\mathrm{W}}(b_h-z)\,\mathcal{A}_h\,\mathrm{Op}_h(c_h)^*\,U_h + R_{\infty,\mathrm{gla},z}(h), \tag{7.14}$$

with $R_{\infty,\mathrm{gla},z}(h)$ smoothing to all orders uniformly for $z\in\Omega$. We now invert the middle Weyl operator $\mathrm{Op}_h^{\mathrm{W}}(b_h-z)$ on $\mathbb{R}_Y$.

Fix the same integer N as above and apply Lemma 2.3 (on $\mathbb{R}_Y$ in place of Y) with $a_h=b_h$ and $m=2$. This yields $d_{\mathrm{gla},N,h}(z)\in S_\lambda^{-2}(T^*\mathbb{R}_Y)$ such that, with

$$D_{\mathrm{gla},N,h}(z) := \mathrm{Op}_h^{\mathrm{W}}\big(d_{\mathrm{gla},N,h}(z)\big), \tag{7.15}$$

one has

$$\mathrm{Op}_h^{\mathrm{W}}(b_h-z)\,D_{\mathrm{gla},N,h}(z) = \mathbf{1} + h^N R_{\mathrm{gla},N,h}(z), \tag{7.16}$$

with $R_{\mathrm{gla},N,h}(z)$ a Weyl operator with symbol in S_λ^{-N}. Conjugate by $\mathcal{A}_h$ and set

$$\widetilde{D}_{\mathrm{gla},N,h}(z) := \mathcal{A}_h^{-1} D_{\mathrm{gla},N,h}(z)\, \mathcal{A}_h. \tag{7.17}$$

Then (7.16) becomes

$$\mathcal{A}_h^{-1}\mathrm{Op}_h^{\mathrm{W}}(b_h - z)\mathcal{A}_h\, \widetilde{D}_{\mathrm{gla},N,h}(z) = \mathbf{1} + h^N \widetilde{R}_{\mathrm{gla},N,h}(z), \tag{7.18}$$

where $\widetilde{R}_{\mathrm{gla},N,h}(z) := \mathcal{A}_h^{-1} R_{\mathrm{gla},N,h}(z)\mathcal{A}_h$. Define the microlocalized glancing parametrix

$$Q_{\mathrm{gla},N,h}(z) := P_{\mathrm{gla}}\, U_h^{-1}\, \mathrm{Op}_h(c_h)\, \widetilde{D}_{\mathrm{gla},N,h}(z)\, \mathrm{Op}_h(c_h)^*\, U_h\, P_{\mathrm{gla}}. \tag{7.19}$$

Combining (7.14) and (7.18) (and absorbing smoothing remainders) yields

$$P_{\mathrm{gla}}(W_h - z)P_{\mathrm{gla}}\, Q_{\mathrm{gla},N,h}(z) = P_{\mathrm{gla}} + h^N \widetilde{R}^{\sharp}_{\mathrm{gla},N,h}(z), \tag{7.20}$$

where $\widetilde{R}^{\sharp}_{\mathrm{gla},N,h}(z)$ is smoothing to all orders and (more precisely) has kernel satisfying the same class of polynomial off–diagonal bounds as in Theorem 1.2.

To make this explicit, note first that Lemma 5.2 applied with $a_h = d_{\mathrm{gla},N,h}(z)$ (of order -2) gives a kernel bound in the Y–variable for $\widetilde{D}_{\mathrm{gla},N,h}(z)$, and the change of variables $Y = h^{-2/3}(t - 2\pi)$ converts this into a t–kernel bound of the form

$$\left|K_{\widetilde{D}_{\mathrm{gla},N,h}(z)}(t,t')\right| \le C_L\, h^{-M_L}\, \langle t + t'\rangle^{M_L}\, \langle t - t'\rangle^{-L} \qquad (\forall L \in \mathbb{N}), \tag{7.21}$$

uniformly for $z \in \Omega$. Second, Lemma 6.3(iii) gives the same style of kernel bound for the fixed microlocal cutoffs of $U_h^{\pm 1}$ and $\mathrm{Op}_h(c_h)^{\pm 1}$. Finally, Lemma 6.1 is stable under composition. Applying it repeatedly to the composition defining $Q_{\mathrm{gla},N,h}(z)$ yields: for each $L \in \mathbb{N}$ there exist C_L, M_L such that

$$\left|K_{Q_{\mathrm{gla},N,h}(z)}(t,t')\right| \le C_L\, h^{-M_L}\, \langle t + t'\rangle^{M_L}\, \langle t - t'\rangle^{-L}, \tag{7.22}$$

and likewise for the remainder kernel $K_{\widetilde{R}^{\sharp}_{\mathrm{gla},N,h}(z)}$.

Step 4: global parametrix, Neumann series, and weighted bounds. Define the global two–chart parametrix

$$Q_{N,h}(z) := Q_{\mathrm{int},N,h}(z) + Q_{\mathrm{gla},N,h}(z). \tag{7.23}$$

Using (7.5) and the microlocal identities (7.11) and (7.20), we obtain

$$(W_h - z)\, Q_{N,h}(z) = \mathbf{1} + h^N R_{N,h}(z), \tag{7.24}$$

where $R_{N,h}(z)$ is smoothing to all orders and satisfies, for every $L \in \mathbb{N}$, a kernel bound of the form

$$\left|K_{R_{N,h}(z)}(t,t')\right| \le C_L\, h^{-M_L}\, \langle t + t'\rangle^{M_L}\, \langle t - t'\rangle^{-L}, \tag{7.25}$$

uniformly for $z \in \Omega$.

Fix $k \in \mathbb{N}$. Choose L in Lemma 7.1 large enough so that the weighted Schur bound applies to the kernels (7.12), (7.22), and (7.25) after multiplying by $\langle t\rangle^k \langle t'\rangle^k$. Then Lemma 7.1 gives

$$\sup_{z\in\Omega} \left\| \langle t\rangle^k Q_{N,h}(z) \langle t\rangle^k \right\|_{L^2\to L^2} \le C_{k,\Omega}\, h^{-M}, \tag{7.26}$$

for some exponent M depending only on finitely many seminorms in the two charts. Similarly, by (7.25) and Lemma 7.1,

$$\sup_{z\in\Omega} \left\| \langle t\rangle^k R_{N,h}(z) \langle t\rangle^k \right\|_{L^2\to L^2} \le C_{k,\Omega}\, h^{-M}. \tag{7.27}$$

Therefore

$$\sup_{z\in\Omega} \left\| \langle t\rangle^k \, h^N R_{N,h}(z) \, \langle t\rangle^k \right\|_{L^2\to L^2} \le C_{k,\Omega} \, h^{N-M}. \tag{7.28}$$

Choose N so large that $N > M+1$ and then fix $h_0 > 0$ small enough so that the right-hand side of (7.28) is $\le 1/2$ for all $h \in (0, h_0]$. Then $(\mathbf{1} + h^N R_{N,h}(z))$ is invertible on $L^2(Y)$ and on the weighted space $\langle t\rangle^{-k} L^2(Y)$ by a convergent Neumann series, with

$$\sup_{h\in(0,h_0]} \sup_{z\in\Omega} \left\| (\mathbf{1} + h^N R_{N,h}(z))^{-1} \right\|_{\langle t\rangle^{-k} L^2 \to \langle t\rangle^{-k} L^2} \le 2. \tag{7.29}$$

Multiplying (7.24) on the right by $(\mathbf{1} + h^N R_{N,h}(z))^{-1}$ gives the exact resolvent identity

$$(W_h - z)^{-1} = Q_{N,h}(z) \, (\mathbf{1} + h^N R_{N,h}(z))^{-1}. \tag{7.30}$$

Combining (7.26) and (7.29) yields

$$\sup_{h\in(0,h_0]} \sup_{z\in\Omega} \left\| \langle t\rangle^k \, (W_h - z)^{-1} \, \langle t\rangle^k \right\|_{L^2\to L^2} \le C_{k,\Omega} \, h^{-M}, \tag{7.31}$$

which is the claimed weighted resolvent bound (1.5) (with the stated convention that all constants are polynomial in $h^{-1} \sim \log\lambda$).

Step 5: eigenfunction localization. Let φ_n be an L^2–normalized eigenfunction with eigenvalue $\mu_n \in \mathbb{R}$. Applying (7.31) with $z = \mu_n + i$ and using $(W_h - (\mu_n + i))\varphi_n = -i\varphi_n$ gives

$$\left\| \langle t\rangle^k \varphi_n \right\|_{L^2} \le C_k \, h^{-M_k}. \tag{7.32}$$

To upgrade to pointwise bounds and full Schwartz regularity, we use elliptic regularity in the two-chart calculus. Fix $j \in \mathbb{N}$ and apply the interior and glancing parametrices to the equation $(W_h - \mu_n)\varphi_n = 0$ to obtain

$$\|\varphi_n\|_{H^j(Y)} + \left\| \langle t\rangle^j \varphi_n \right\|_{L^2} \le C_j \, (1 + \mu_n)^{j/2} \, h^{-M_j}, \tag{7.33}$$

with constants depending only on finitely many symbol seminorms. (Concretely, one applies the parametrix construction above to $(W_h - \mu_n \pm i)$, observes that $(W_h - \mu_n \pm i)\varphi_n = \mp i\varphi_n$, and iterates in j.) Finally, by Sobolev embedding in one dimension, $H^j(Y) \hookrightarrow L^\infty(Y)$ for $j \ge 1$, so taking $j \ge k+1$ in (7.33) yields (1.6) (after adjusting constants). This also shows $\varphi_n \in \mathcal{S}([2\pi, \infty))$. □

Proof of Corollary 1.4. CCM compute the regularized determinant explicitly in terms of the Fourier transform $\widehat{\xi}$ of a minimal eigenvector ξ of a truncated Weil quadratic form; in particular [1, Eq. (5.27)] yields

$$\det_{\mathrm{reg}}(D_{\log}^{(\lambda,N)} - z) = -i \, e^{-izL/2} \, \widehat{\xi}(z) = -i \, \lambda^{-iz} \, \widehat{\xi}(z).$$

The resolvent/eigenfunction bounds of Theorem 1.3 provide uniform control of $\widehat{\xi}$ on compact subsets of $\mathbb{C}$ via standard Paley–Wiener and Sobolev embedding arguments (the eigenfunctions are compactly supported on the physical side in the CCM normalization). Hence $F_{\lambda,N}(z) = \lambda^{iz} \det_{\mathrm{reg}}(D_{\log}^{(\lambda,N)} - z)$ is locally bounded in z uniformly in (λ, N), and Montel's theorem implies normality. Real-zero localization for finite (λ, N) follows from CCM's self-adjointness argument; passing to limits then uses Hurwitz. □

8. Two-chart trace asymptotics and the Archimedean boundary functional

The kernel locality and resolvent parametrices established in §??–§?? allow one to pass from microlocal structure to spectral asymptotics. The goal of this section is to make the semiclassical trace formula used in the CCM bridge completely explicit and self-contained: we implement the Helffer–Sjöstrand functional calculus, insert the two-chart resolvent parametrix, and extract the diagonal contributions in the interior and Airy charts.

Throughout we fix a real-valued test function $\varphi \in \mathcal{S}(\mathbb{R})$. (The heat family $\varphi_\varepsilon(\Lambda) = e^{-\varepsilon\Lambda}$ is included.) We write $z = x + iy$ and $dL(z) = dx\,dy$.

8.1. **Almost-analytic extensions and Helffer–Sjöstrand.** Fix a cutoff $\chi \in C_c^\infty((-2,2))$ with $\chi \equiv 1$ on $[-1,1]$. For an integer $J \geq 3$ define the standard almost-analytic extension

$$\widetilde{\varphi}_J(x+iy) := \chi(y) \sum_{k=0}^{J-1} \frac{(iy)^k}{k!}\, \varphi^{(k)}(x). \tag{8.1}$$

Lemma 8.1 (Almost-analytic estimates). *For each $J \geq 3$ there is $C_J < \infty$ such that*

$$\left|\partial_{\bar z}\widetilde{\varphi}_J(x+iy)\right| \leq C_J\, |y|^{J-1} \sum_{k=0}^{J} \|\varphi^{(k)}\|_{L^\infty(\mathbb{R})} \qquad (x \in \mathbb{R},\ y \in \mathbb{R}). \tag{8.2}$$

Moreover $\operatorname{supp}(\partial_{\bar z}\widetilde{\varphi}_J) \subset \{|y| \leq 2\}$.

Proof. Differentiate (8.1) and use cancellation of the Cauchy–Riemann operator on the Taylor polynomial; the only non-cancelling term appears at order J and is controlled by $|y|^{J-1}$. The cutoff χ localizes in y. □

We use the sign convention compatible with the resolvent parametrices of §?? (which are built for $W_h - z$).

Proposition 8.2 (Helffer–Sjöstrand). *Let W_h be the self-adjoint semiclassical operator of §??. For each $J \geq 3$ and each $\varphi \in \mathcal{S}(\mathbb{R})$,*

$$\varphi(W_h) = -\frac{1}{\pi} \int_{\mathbb{C}} \partial_{\bar z}\widetilde{\varphi}_J(z)\, (W_h - z)^{-1}\, dL(z) \tag{8.3}$$

with convergence in operator norm.

Proof sketch. This is standard; see e.g. Helffer–Sjöstrand. The identity follows from the corresponding scalar formula and the holomorphic functional calculus. The decay of φ and (8.2) ensure absolute integrability against the resolvent bound $\|(W_h - z)^{-1}\| \leq |\Im z|^{-1}$. □

8.2. **Trace-class bounds for the HS integral remainder.** The HS formula (8.3) becomes a trace identity once the integrand is trace class. In our setting this is achieved by inserting the two-chart resolvent parametrix from §?? and using the compact energy microlocalization encoded by the interior cutoff P_{int} and the glancing cutoff P_{gla}. The point of this subsection is to make the $y \to 0$ integrability in the HS integral completely transparent.

Lemma 8.3 (Trace-class from rapid kernel decay). *Let $A_h : L^2(Y) \to L^2(Y)$ have Schwartz kernel $K_h(t,t')$ satisfying, for some $N > 2$ and all $t, t' \in Y$,*

$$|K_h(t,t')| \leq C_N\, h^{-M}\, \langle t\rangle^{-N}\, \langle t'\rangle^{-N}\, \langle t-t'\rangle^{-N}. \tag{8.4}$$

Then A_h is trace class and

$$\|A_h\|_{\mathfrak{S}_1} \leq \iint_{Y\times Y} |K_h(t,t')|\, dt\, dt' \leq C'_N\, h^{-M}. \tag{8.5}$$

Proof. The first inequality in (8.5) is the elementary L^1-kernel criterion. The integrability of (8.4) on $Y \times Y$ follows from $N > 2$. $\square$

Lemma 8.4 (HS remainder integrability)**.** *Fix a compact interval* $I \Subset \mathbb{R}$ *containing* $\operatorname{supp}\varphi$*. Assume* $R_h(z)$ *is trace class for* $z \in I + i(-2,2)$ *and satisfies*

$$\|R_h(z)\|_{\mathfrak{S}_1} \le C\, h^{N_0}\, |\Im z|^{-m} \qquad (z \in I + i(-2,2),\ \Im z \neq 0) \tag{8.6}$$

for some $m \ge 1$*. Choose* $J \ge m+2$ *in* (8.1)*. Then*

$$\Big\| \int_{\mathbb{C}} \partial_{\bar z}\widetilde{\varphi}_J(z)\, R_h(z)\, dL(z) \Big\|_{\mathfrak{S}_1} \le C_{\varphi,J}\, h^{N_0}. \tag{8.7}$$

Proof. By (8.2) and (8.6), for $|y| \le 2$,

$$\big\|\partial_{\bar z}\widetilde{\varphi}_J(x+iy)\, R_h(x+iy)\big\|_{\mathfrak{S}_1} \lesssim |y|^{J-1}\, |y|^{-m} = |y|^{J-1-m}.$$

Since $J \ge m+2$, one has $J-1-m \ge 1$ and therefore $|y|^{J-1-m}$ is integrable near $y = 0$. The x-integral is over a fixed compact set (depending on $\operatorname{supp}\varphi$), so (8.7) follows. $\square$

8.3. **Interior diagonal extraction to order h^0.** We now refine the bulk trace computation so that all $O(1)$ terms are explicit. This addresses the $O(1)$–precision issue raised in the referee reports: the boundary scattering functional lives at order h^0, and so does the interior subprincipal contribution.

Let $Q_{\mathrm{int},N,h}(z) = \mathrm{Op}_h^{\mathrm{W}}(d_{\mathrm{int},N,h}(z))$ be the interior parametrix from §**??**. Write the full interior Weyl symbol of W_h as

$$q_{\mathrm{int},h}(t,\tau) \sim q_0(t,\tau) + h\, q_1(t,\tau) + h^2 q_2(t,\tau) + \cdots \qquad \text{in } S^2_\lambda. \tag{8.8}$$

Here q_0 is the quadratic principal symbol and q_1 is the (Weyl) subprincipal symbol in the interior chart.

Proposition 8.5 (Bulk trace term, including the subprincipal contribution)**.** *Fix* $\varphi \in \mathcal{S}(\mathbb{R})$ *real-valued. Let* $a_{\mathrm{int}} \in S^0_\lambda$ *be the interior cutoff symbol from §**??** with* $a_{\mathrm{int}} \equiv 1$ *on a neighborhood of* $\operatorname{supp}(\varphi \circ q_0)$*. Then, for* $h \to 0$,

$$\operatorname{Tr}\Big[\mathrm{Op}_h^{\mathrm{W}}(a_{\mathrm{int}})\, \varphi(W_h)\, \mathrm{Op}_h^{\mathrm{W}}(a_{\mathrm{int}})\Big] = (2\pi h)^{-1} \iint_{T^*Y} \varphi(q_0)\, dt\, d\tau + \frac{1}{2\pi} \iint_{T^*Y} q_1\, \varphi'(q_0)\, dt\, d\tau + \mathcal{O}(h). \tag{8.9}$$

Proof. Insert (8.3) and the interior parametrix

$$(W_h - z)^{-1} = \mathrm{Op}_h^{\mathrm{W}}(b_0(z) + h b_1(z)) + \mathcal{O}_{\mathfrak{S}_1}(h^2 |\Im z|^{-M}) \tag{8.10}$$

valid microlocally on $\operatorname{supp} a_{\mathrm{int}}$ (for some fixed M), where

$$b_0(z) = (q_0 - z)^{-1}, \qquad b_1(z) = q_1\, (q_0 - z)^{-2}. \tag{8.11}$$

The absence of an h-term coming from the Weyl product is due to the identity $\{q_0, (q_0 - z)^{-1}\} = 0$.

Taking traces, we use the exact diagonal formula for Weyl kernels: for $a \in S^{-2}$ properly supported,

$$\operatorname{Tr}\, \mathrm{Op}_h^{\mathrm{W}}(a) = (2\pi h)^{-1} \iint a(t,\tau)\, dt\, d\tau.$$

Applying this with $a = a_{\mathrm{int}}^2 b_j(z)$ and using $a_{\mathrm{int}} \equiv 1$ on $\operatorname{supp}(\varphi \circ q_0)$ yields

$$\operatorname{Tr}\Big[\mathrm{Op}_h^{\mathrm{W}}(a_{\mathrm{int}}) (W_h - z)^{-1} \mathrm{Op}_h^{\mathrm{W}}(a_{\mathrm{int}})\Big] = (2\pi h)^{-1} \iint \big(b_0(z) + h b_1(z)\big)\, dt\, d\tau + \mathcal{O}(h\, |\Im z|^{-M}).$$

Finally integrate against $-\pi^{-1}\partial_{\bar z}\widetilde{\varphi}_J(z)$. Using the scalar identities

$$-\frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar z}\widetilde{\varphi}_J(z)\,(x-z)^{-1}\,dL(z)=\varphi(x), \qquad -\frac{1}{\pi}\int_{\mathbb{C}} \partial_{\bar z}\widetilde{\varphi}_J(z)\,(x-z)^{-2}\,dL(z)=\varphi'(x),$$

and Lemma 8.4 with $N_0=2$ gives (8.9). □

8.4. **Glancing diagonal extraction: boundary scattering as a log-derivative.** The glancing contribution is computed in the Airy chart and reduces to the reflected-trace analysis of §**??**. Combining Proposition **??** and Proposition **??**, the boundary contribution to the trace can be written directly in the invariant log-derivative form

$$\mathcal{A}_\partial(\varphi) := \frac{1}{2\pi i}\int_{\mathbb{R}} \varphi(\Lambda)\,\partial_\Lambda \log \mathcal{R}_0(\Lambda)\,d\Lambda \equiv -\frac{1}{\pi}\int_{\mathbb{R}} S_0(\Lambda)\,\varphi'(\Lambda)\,d\Lambda, \tag{8.12}$$

where $\mathcal{R}_0(\Lambda)=e^{2iS_0(\Lambda)}$ is the boundary reflection coefficient. The equality of the two expressions is integration by parts, and the log-derivative form is the one that matches the CCM functional.

8.5. **Density-level form and dispersion.** Define the bulk pushforward density ρ_{bulk} as the pushforward of Liouville measure by q_0:

$$\int_{\mathbb{R}} \varphi(\Lambda)\,\rho_{\mathrm{bulk}}(\Lambda)\,d\Lambda := \iint_{T^*Y} \varphi(q_0(t,\tau))\,dt\,d\tau. \tag{8.13}$$

Similarly define the boundary density distribution

$$\rho_\partial(\Lambda) := \frac{1}{2\pi i}\,\partial_\Lambda \log \mathcal{R}_0(\Lambda) = \frac{1}{\pi}\,\partial_\Lambda S_0(\Lambda), \tag{8.14}$$

so that the boundary pairing is equivalently

$$\langle \rho_\partial, \varphi\rangle := \int_{\mathbb{R}} \rho_\partial(\Lambda)\,\varphi(\Lambda)\,d\Lambda = -\frac{1}{\pi}\int_{\mathbb{R}} S_0(\Lambda)\,\varphi'(\Lambda)\,d\Lambda. \tag{8.15}$$

Corollary 8.6 (Target density-level decomposition)**.** *In the sense of tempered distributions on $\mathbb{R}$ (i.e. paired against $\varphi\in\mathcal{S}(\mathbb{R})$), the semiclassical spectral density ρ_h of W_h admits the expansion*

$$\rho_h(\Lambda) = (2\pi h)^{-1}\rho_{\mathrm{bulk}}(\Lambda) + \rho_\partial(\Lambda) + \rho_{\mathrm{sub}}(\Lambda) + \mathcal{O}(h), \tag{8.16}$$

where ρ_{sub} is the bulk subprincipal correction characterized by

$$\int_{\mathbb{R}} \varphi(\Lambda)\,\rho_{\mathrm{sub}}(\Lambda)\,d\Lambda := \frac{1}{2\pi}\iint_{T^*Y} q_1(t,\tau)\,\varphi'(q_0(t,\tau))\,dt\,d\tau. \tag{8.17}$$

The remainder $\mathcal{O}(h)$ means that there is $C<\infty$ and $J\in\mathbb{N}$ (independent of h) such that

$$\left|\langle \rho_h - (2\pi h)^{-1}\rho_{\mathrm{bulk}} - \rho_\partial - \rho_{\mathrm{sub}}, \varphi\rangle\right| \le C\,h\sum_{k=0}^{J}\|\varphi^{(k)}\|_{L^\infty(\mathbb{R})} \qquad (0<h\ll 1). \tag{8.18}$$

8.6. **Dispersion and Jacobian.** Define the bulk counting function

$$N_0(\Lambda) := \frac{1}{2\pi}\,\mathrm{vol}\big\{(t,\tau)\in T^*Y:\ q_0(t,\tau)\le\Lambda\big\} = \frac{1}{2\pi}\int_{-\infty}^{\Lambda} \rho_{\mathrm{bulk}}(\Lambda')\,d\Lambda'. \tag{8.19}$$

Since q_0 is proper with a unique nondegenerate minimum (quadratic confinement), N_0 is finite for each Λ and strictly increasing for $\Lambda>\Lambda_*$ (the minimum value of q_0). In particular the inverse function exists on (Λ_*,∞):

$$\Lambda(\nu) := N_0^{-1}(\nu), \qquad \nu > N_0(\Lambda_*). \tag{8.20}$$

Differentiating the identity $N_0(\Lambda(\nu)) \equiv \nu$ yields the Jacobian formula

$$\Lambda'(\nu) = \big(N_0'(\Lambda(\nu))\big)^{-1} = \frac{2\pi}{\rho_{\rm bulk}(\Lambda(\nu))}. \tag{8.21}$$

8.7. **Connector: boundary term as the CCM pairing.** The CCM Archimedean functional is naturally written in terms of the de Branges structure function E of the canonical system. Write $E(\nu) = |E(\nu)|e^{i\phi_E(\nu)}$ for $\nu \in \mathbb{R}$, and $E^\sharp(\nu) = \overline{E(\overline{\nu})}$.

Lemma 8.7 (Log-derivative connector and sign)**.** *For real ν one has*

$$\frac{1}{2\pi i}\,\partial_\nu \log\Big(\frac{E^\sharp(\nu)}{E(\nu)}\Big) = -\frac{1}{\pi}\,\partial_\nu\phi_E(\nu). \tag{8.22}$$

Proof. Since $E^\sharp/E = \overline{E}/E = e^{-2i\phi_E}$, one has $\log(E^\sharp/E) = -2i\phi_E$ (choosing a continuous branch of ϕ_E), and (8.22) follows by differentiation. □

Assuming the boundary pushforward phase identification of Proposition **??** (which relates $\mathcal{R}_0(\Lambda(\nu))$ to $E^\sharp(\nu)/E(\nu)$), (8.12) and (8.22) identify the boundary trace contribution with the CCM pairing

$$\mathcal{A}_\partial(\varphi) = \frac{1}{2\pi i}\int_{\mathbb{R}} F(\nu)\,\partial_\nu \log\Big(\frac{E^\sharp(\nu)}{E(\nu)}\Big)\,d\nu, \qquad F(\nu) := \varphi(\Lambda(\nu)). \tag{8.23}$$

8.8. **Archimedean phase identification in CCM normalization.** We now pin down the *derivative-level* phase matching which turns the invariant log–derivative boundary term (8.23) into the classical Γ'/Γ contribution of Weil's explicit formula.
Riemann–Siegel phase and the Archimedean de Branges function. Define the Riemann–Siegel phase function

$$\theta(\nu) := \Im \log\Gamma\Big(\frac{1}{4} + \frac{i\nu}{2}\Big) - \frac{\nu}{2}\log\pi, \qquad \nu \in \mathbb{R}, \tag{8.24}$$

where $\log\Gamma$ is the principal branch on $\{\Re w > 0\}$, so that $\theta(0) = 0$. In the CCM normalization, the Archimedean de Branges structure function is the entire function

$$E_{\rm Arch}(\nu) := \pi^{\frac{1}{4}+\frac{i\nu}{2}}\Big/\Gamma\Big(\frac{1}{4} + \frac{i\nu}{2}\Big), \qquad \nu \in \mathbb{C}. \tag{8.25}$$

Since $1/\Gamma$ is entire and its zeros are the nonpositive integers, the zeros of $E_{\rm Arch}$ lie at $\nu = -2i(n + \frac{1}{4})$ $(n \in \mathbb{N}_0)$, hence in the *lower* half-plane; in particular $E_{\rm Arch}(\nu) \neq 0$ for $\nu \in \mathbb{R}$. Writing $E_{\rm Arch}(\nu) = |E_{\rm Arch}(\nu)|e^{i\phi_{\rm Arch}(\nu)}$ on $\mathbb{R}$ with $\phi_{\rm Arch}(0) = 0$, we have the *exact* identity

$$\phi_{\rm Arch}(\nu) = \Im\log E_{\rm Arch}(\nu) = \frac{\nu}{2}\log\pi - \Im\log\Gamma\Big(\frac{1}{4} + \frac{i\nu}{2}\Big) = -\theta(\nu). \tag{8.26}$$

Lemma 8.8 (Archimedean log–derivative equals digamma)**.** *Let $\psi := \Gamma'/\Gamma$ be the digamma function. For $\nu \in \mathbb{R}$ one has*

$$\frac{1}{2\pi i}\,\partial_\nu \log\Big(\frac{E^\sharp_{\rm Arch}(\nu)}{E_{\rm Arch}(\nu)}\Big) = \frac{1}{2\pi}\Big(\Re\psi\Big(\frac{1}{4} + \frac{i\nu}{2}\Big) - \log\pi\Big) = \frac{1}{\pi}\,\theta'(\nu). \tag{8.27}$$

Proof. For $\nu \in \mathbb{R}$, $E^\sharp_{\rm Arch}(\nu) = \overline{E_{\rm Arch}(\nu)}$ and thus

$$\frac{E^\sharp_{\rm Arch}(\nu)}{E_{\rm Arch}(\nu)} = \frac{\pi^{\frac{1}{4}-\frac{i\nu}{2}}}{\Gamma(\frac{1}{4}-\frac{i\nu}{2})}\Big/\frac{\pi^{\frac{1}{4}+\frac{i\nu}{2}}}{\Gamma(\frac{1}{4}+\frac{i\nu}{2})} = \pi^{-i\nu}\,\frac{\Gamma(\frac{1}{4}+\frac{i\nu}{2})}{\Gamma(\frac{1}{4}-\frac{i\nu}{2})}.$$

Differentiating the logarithm gives

$$\partial_\nu \log\Big(\frac{E^\sharp_{\mathrm{Arch}}}{E_{\mathrm{Arch}}}\Big) = -i\log\pi + \frac{i}{2}\psi\Big(\frac{1}{4}+\frac{i\nu}{2}\Big) + \frac{i}{2}\psi\Big(\frac{1}{4}-\frac{i\nu}{2}\Big),$$

and dividing by $2\pi i$ yields the first equality in (8.27), since $\psi(\overline{w}) = \overline{\psi(w)}$ implies $\frac{1}{2}\big(\psi(w)+\psi(\overline{w})\big) = \Re\psi(w)$ for $w = \frac{1}{4}+\frac{i\nu}{2}$. Finally, (8.26) implies $\phi'_{\mathrm{Arch}}(\nu) = -\theta'(\nu)$, and inserting this into (8.22) (with $E = E_{\mathrm{Arch}}$) yields the last identity in (8.27). □

Corollary 8.9 (Boundary term equals the Γ'/Γ functional after dispersion)**.** *Assume the boundary pushforward phase identification of Proposition* **??** *and work in the CCM normalization $E = E_{\mathrm{Arch}}$ of* (8.25)*. Then the boundary functional in the two-chart trace theorem is*

$$\mathcal{A}_\partial(\varphi) = \frac{1}{2\pi}\int_{\mathbb{R}} F(\nu)\Big(\Re\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{i\nu}{2}\Big) - \log\pi\Big)\,d\nu, \qquad F(\nu) := \varphi(\Lambda(\nu)). \tag{8.28}$$

Equivalently, $\mathcal{A}_\partial$ is the invariant log–derivative scattering term

$$\mathcal{A}_\partial(\varphi) = \frac{1}{2\pi i}\int_{\mathbb{R}} F(\nu)\,\partial_\nu \log\Big(\frac{E^\sharp_{\mathrm{Arch}}(\nu)}{E_{\mathrm{Arch}}(\nu)}\Big)\,d\nu.$$

Proof. Insert (8.27) into (8.23). □

Remark 8.10 (Upgrade to constant difference)**.** By Proposition **??**, the principal boundary reflection coefficient satisfies $\mathcal{R}_0(\Lambda(\nu)) = E^\sharp(\nu)/E(\nu)$, hence $\mathcal{R}_0(\Lambda(\nu)) = \exp\big(2iS_0(\Lambda(\nu))\big)$ with $S_0(\Lambda(\nu)) = -\phi_E(\nu) \pmod{\pi}$. In CCM normalization $E = E_{\mathrm{Arch}}$, (8.26) gives

$$S_0(\Lambda(\nu)) = \theta(\nu) \pmod{\pi}, \qquad \partial_\nu S_0(\Lambda(\nu)) = \theta'(\nu).$$

Since both sides are continuous and $\theta(0) = 0$, fixing the normalization $S_0(\Lambda_*) = 0$ upgrades the congruence to an *exact* identity $S_0(\Lambda(\nu)) \equiv \theta(\nu)$ on the whole dispersion range.

9. The arithmetic deformation conjecture and evidence

The results above isolate the genuinely analytic (Archimedean) bottleneck and yield the uniform stability inputs needed for determinant control. What remains to prove the Riemann Hypothesis in the CCM framework is to identify the unique limit of the normalized determinants with Ξ. We state a clean conjecture in the "Prime Hamiltonian + resolvent" language.

Conjecture 9.1 (Uniform prime-Hamiltonian convergence)**.** Let T_N be the CCvS truncation matrix (equivalently, the finite-dimensional operator $D^{(\lambda,N)}_{\log}$ in a canonical basis) and let $H^{(N)}_{p^m}(z)$ be the corresponding prime Hamiltonians defined via resolvent matrix elements. For each fixed z with $\Re z = \frac{1}{2}$ and for each fixed vertical strip $|\Re z - 1/2| \le \sigma_0$, the series

$$\sum_p \sum_{m\ge 1} (\log p)\, p^{-m/2}\, H^{(N)}_{p^m}(z)$$

converges absolutely and uniformly in N, and its limit equals $\Xi'(z)/\Xi(z)$.

Remark 9.2 (Evidence-building)**.** Two complementary directions for evidence and partial results are suggested by recent work. First, the arithmetic spin-glass/prime-graph expansion framework develops unconditional variance bounds and a Guerra interpolation mechanism that, in principle, can control multi-prime interactions [5]. Second, the Gaussian approximation/Block Berry–Esseen approach pinpoints the need for global Dirichlet polynomial representations (Conjecture 7.1 therein) as a prerequisite for integration-by-parts identities in the arithmetic field [6]. These tools are natural candidates for attacking Conjecture 9.1 in model problems and for building numerical/structural evidence.

Appendix A. Rank–2 IIKS integrable kernel structure of CCvS truncations

We record the elementary, but structurally important, observation that the CCvS truncation matrices are rank–2 discrete IIKS integrable kernels.

Proposition A.1 (Divided differences as a rank–2 IIKS kernel)**.** *Let $I = \{-N, \dots, N\} \subset \mathbb{Z}$ and let $b : I \to \mathbb{R}$ be any function. Define a matrix $K = (K_{ij})_{i,j \in I}$ by*

$$K_{ij} := \begin{cases} \dfrac{b_i - b_j}{i - j}, & i \neq j, \\ 0, & i = j. \end{cases}$$

Then K is a rank–2 IIKS integrable kernel:

$$K_{ij} = \frac{f(i)^\top g(j)}{i - j}, \qquad f(i) = \begin{pmatrix} 1 \\ b_i \end{pmatrix}, \quad g(j) = \begin{pmatrix} -b_j \\ 1 \end{pmatrix},$$

and $f(i)^\top g(i) = 0$ for all $i \in I$.

Proof. For $i \neq j$,

$$f(i)^\top g(j) = (1, b_i) \cdot (-b_j, 1) = -b_j + b_i = b_i - b_j,$$

hence $K_{ij} = (b_i - b_j)/(i - j)$. For $i = j$ one has $f(i)^\top g(i) = 0$. □

Remark A.2. In CCM the truncation matrices for the Weil quadratic form have precisely the divided-difference structure, e.g. [1, Prop. 4.3] and the more general divided-difference framework of [2]. This IIKS form is the correct "integrable operator" lens for the arithmetic deformation layer; it is not a Toeplitz/Fisher–Hartwig problem.

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