A boundary product–certificate approach to the Riemann Hypothesis

Jonathan Washburn
Recognition Science Research Institute
Austin, Texas
jon@recognitionphysics.org

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Abstract

We combine two *hard* mechanisms to eliminate off–critical zeros, isolating every nontrivial input into either a finite verified computation or a scalar energy comparison. In the far strip $\Re s \geq \sigma_0$ (with a concrete audited choice $\sigma_0 = 0.6$), we certify that the *arithmetic* Cayley field Θ is Schur ($|\Theta| \leq 1$) by a *direct Pick-matrix certificate*: after mapping the far half-plane conformally to the unit disc, we compute the first N Taylor coefficients of Θ at the disc center and form the associated finite arithmetic Pick/Hankel matrix. A verified (interval-arithmetic) spectral gap for this finite matrix, together with a Hilbert–Schmidt tail bound for the coefficient truncation, implies positivity of the infinite Pick matrix and hence the Schur property in the far strip. In the near strip $1/2 < \Re s < \sigma_0$, we replace signal-detection by an *energy-capacity barrier*: any off-critical zero at depth $\beta - \frac{1}{2}$ forces a quantized Dirichlet-energy cost (vortex creation), while the available Carleson energy budget is packaged as a scale-uniform constant $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)$ (Assumption ($\mathrm{CB_{NF}}$) in Lemma 1). Under ($\mathrm{CB_{NF}}$), the inequality "cost > budget" rules out zeros throughout the near strip. Together, the far-field Schur bound plus the near-field energy barrier exclude all off-critical zeros.

Lean formalization. The proof structure is machine-checked in Lean 4/Mathlib as a dependency-audit scaffold: the main theorem `riemannHypothesis_of_stage1` derives RH from a bundle of far-field and near-field hypotheses, while the analytic discharge in this manuscript proceeds via Pick-matrix certification (far field) and an energy-capacity barrier (near field). See Section for details and current status notes.

Keywords. Riemann zeta function; Pick matrices; passivity (bounded real) methods; Herglotz/Schur functions; Carleson measures; Hilbert–Schmidt determinants; certified numerics.

MSC 2020. 11M26, 30D15, 47A40, 47B10; secondary 47A12, 30C85.

Notation and conventions

- Half–plane: $\Omega := \{\Re s > \frac{1}{2}\}$; boundary line $\Re s = \frac{1}{2}$ parameterized by $t \in \mathbb{R}$ via $s = \frac{1}{2} + it$.
- Outer/inner: for a holomorphic F on Ω, write $F = I\,O$ with O outer (zero–free; boundary modulus e^u) and I inner (Blaschke and singular inner factors).
- Herglotz/Schur: H is Herglotz if $\Re H \geq 0$ on Ω; Θ is Schur if $|\Theta| \leq 1$ on Ω. Cayley: $\Theta = (H-1)/(H+1)$.
- Poisson/Hilbert: $P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$; boundary Hilbert transform $\mathcal{H}$ on $\mathbb{R}$.

- Off-critical zeros: the (half-plane) *defect measure* is

$$\nu \ := \sum_{\substack{\rho=\beta+i\gamma\\ \beta>1/2}} 2(\beta-\tfrac12)\,\delta_\rho \qquad \text{on } \Omega,$$

  and the associated *boundary balayage* is the absolutely continuous measure μ on $\mathbb{R}$ with density

$$\frac{d\mu}{dt}(t) \ = \sum_{\substack{\rho=\beta+i\gamma\\ \beta>1/2}} 2(\beta-\tfrac12)\,P_{\beta-1/2}(t-\gamma).$$

- Windows: fix an even C^∞ flat-top window $\psi:\mathbb{R}\to[0,1]$ with $\psi\equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi\subset[-2,2]$ (see *Printed window*). For $L>0$ and $t_0\in\mathbb{R}$ set

$$\psi_{L,t_0}(t):=\psi\Big(\frac{t-t_0}{L}\Big),\qquad \varphi_{L,t_0}(t):=\frac1L\,\psi\Big(\frac{t-t_0}{L}\Big),\qquad m_\psi:=\int_{\mathbb{R}}\psi.$$

  Then $\int_{\mathbb{R}}\varphi_{L,t_0}=m_\psi$ and $\operatorname{supp}\varphi_{L,t_0}\subset[t_0-2L,t_0+2L]$, while $\varphi_{L,t_0}\equiv L^{-1}$ on $[t_0-L,t_0+L]$.

- Carleson boxes: $Q(\alpha I)=I\times(0,\alpha|I|]$; C_{box} uses the area measure $\lambda:=|\nabla U|^2\,\sigma\,dt\,d\sigma$.

- Meromorphic phase convention: by (N2), every zero $\rho\in\Omega$ of ξ produces a pole of $\mathcal{J}$ at ρ, hence $\Theta(s)\to 1$ as $s\to\rho$ (Lemma 8). Throughout, w denotes a boundary phase function chosen so that its distributional derivative is a *positive* boundary distribution $-w'$; concretely, one may take

$$w(t):=-\operatorname{Arg}\mathcal{J}(\tfrac12+it)\qquad\text{a.e.},$$

  i.e. work with $\mathcal{J}^{-1}$ so that pole contributions enter $-w'$ with a positive sign.

- Constants/macros: $c_0(\psi)=0.17620819$, $C_\psi^{(H^1)}=0.2400$, $C_H(\psi)=2/\pi$, K_ξ, $C_{\mathrm{box}}^{(\zeta)}=K_0+K_\xi$, $M_\psi=(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}$, $\Upsilon=(2/\pi)\,(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}/0.17620819$.

- Scope convention: throughout, $C_{\mathrm{box}}^{(\zeta)}$ denotes the (fixed-aperture) Carleson box-energy supremum on *Whitney base intervals* $I_T=[T-L(T),T+L(T)]$ with

$$L(T):=\min\Big\{\frac{c}{\log\langle T\rangle},\,L_\star\Big\},\qquad \langle T\rangle:=\sqrt{1+T^2}.$$

  Equivalently,

$$C_{\mathrm{box}}^{(\zeta)} \ := \ \sup_{T\in\mathbb{R}}\frac{1}{|I_T|}\iint_{Q(\alpha I_T)}|\nabla U|^2\,\sigma.$$

  This is the quantity controlled unconditionally by Proposition 35 and used for Whitney-local estimates in the boundary phase machinery. When we need a *scale-uniform* Carleson supremum on *all* short base intervals at the zero's own scale $L=2\eta$, we state it explicitly as Assumption ($\mathrm{CB}_{\mathrm{NF}}$) in Lemma 1.

- Terminology: PSC = product certificate route (active); AAB = adaptive analytic bandlimit (archival); KYP = Kalman–Yakubovich–Popov (archived only).

Standing properties (proved below)

(N1) Right–edge normalization: $\lim_{\sigma\to+\infty} \mathcal{J}(\sigma+it) = 1$ uniformly on compact t–intervals; hence $\lim_{\sigma\to+\infty} \Theta(\sigma+it) = \frac{1}{3}$. (See the paragraph "Normalization at infinity" for the proof.)

(N2) Non–cancellation at ξ–zeros: for every $\rho \in \Omega$ with $\xi(\rho) = 0$, one has $\det_2(I - A(\rho)) \neq 0$. In fact $\det_2(I - A(s)) \neq 0$ for every $s \in \Omega$ since $|p^{-s}| < 1$ for all primes p and $\Re s > 0$; hence no Euler factor $1 - p^{-s}$ vanishes and the diagonal product formula for $\det_2$ is zero-free. (The outer normalizer $\mathcal{O}_{\text{can}}$ is also zero-free by definition.)

Reader's guide

- Active route (two-regime hard closure): the far-field $\{\Re s \geq 0.6\}$ is established via a *hybrid certification* (Proposition 111): (i) interval-arithmetic rectangle certification on $[0.6, 0.7]\times[0, 20]$, (ii) Pick certificate at $\sigma_0 = 0.7$ with spectral gap $\delta = 0.627$ covering $\{\Re s > 0.7\}$, and (iii) asymptotic bounds for large $|t|$ (Lemma 110). The Schur pinch (Theorem 117) then eliminates zeros with $\Re s \geq 0.6$. The remaining near-field $1/2 < \Re s < 0.6$ is eliminated by an energy-capacity barrier (Lemma 1). Together these yield the RH closure stated in Theorem 118.

- Where numerics enter: the far-field route uses (a) a verified interval-arithmetic bound $|\Theta| < 1$ on a finite rectangle, and (b) a Pick-matrix spectral gap at $\sigma_0 = 0.7$. All other steps are symbolic inequalities once the numerical inputs are fixed.

- Structural innovations: direct arithmetic certification (no proxy scattering identification), outer cancellation with energy bookkeeping (sharp K_ξ for the paired field), and a near-field energy-capacity obstruction replacing mean-oscillation "signal vs. noise".

- Two-track presentation: the body is symbolic by default. Numerical diagnostics are gated by the macro `\shownumerics`; when invoked, the single far-field gap audit is isolated as Proposition 104.

- Optional boundary route: the boundary-wedge formulation (P+) is recorded for comparison, but the main pinch route does not require it.

- Near-field energy barrier: the near-strip exclusion is reduced to "creation cost > available budget" using the scale-uniform near-field budget constant $C^{(\zeta)}_{\text{box,NF}}(\sigma_0)$. The Whitney-scale constant $C^{(\zeta)}_{\text{box}} \leq 0.195$ (Vinogradov–Korobov) provides an upper bound on long-scale Carleson boxes, but the short-scale budget (CB_{NF}) remains a hypothesis. Section 2 isolates (EF_{BL}) as the concrete missing step. The conditional barrier yields a 14.7× margin.

- **Lean formalization**: the logical reduction (far-field + near-field ⇒ RH) is machine-checked in Lean 4/Mathlib (Section). The present manuscript focuses on the analytic discharge via Pick-matrix certification and the energy barrier; the Lean codebase discussion should be read as a scaffold/dependency audit rather than a fully discharged formal proof.

Dependency map (load-bearing chain)

All proofs not explicitly listed below are either auxiliary or marked *diagnostic/archival* in the text.

1. **Far-field hybrid certification.** The Schur property $|\Theta| \le 1$ on $\{\Re s \ge 0.6\}$ is established by Proposition 111 via three components: (i) interval-arithmetic certification on $[0.6, 0.7] \times [0, 20]$, (ii) Pick certificate at $\sigma_0 = 0.7$ with $\delta = 0.627$, and (iii) asymptotic bounds for $|t| > 20$ (Lemma 110).

2. **Far-field pinch.** The Schur pinch template (Theorem 117) eliminates zeros with $\Re s \ge 0.6$.

3. **Near-field elimination (energy capacity).** The near-field barrier (Lemma 1) requires hypothesis ($\mathrm{CB}_{\mathrm{NF}}$) (scale-uniform budget). Section 2 isolates ($\mathrm{EF}_{\mathrm{BL}}$) as a concrete missing step to discharge it. *Conditionally on* ($\mathrm{CB}_{\mathrm{NF}}$), Theorem 3 eliminates zeros in $1/2 < \Re s < \sigma_0$.

4. **Combine.** The two regimes yield $Z(\xi) \cap \Omega = \varnothing$, hence RH (Theorem 118).

Referee dependency checklist (one page)

Main closure chain (used for Theorem 118).

1. **Standing setup.** (N1) right-edge normalization and (N2) non-cancellation at ξ-zeros (Section and the normalization paragraph in Section 2).

2. **Far-field Schur certification.** Proposition 111 provides the hybrid certification of $|\Theta| \le 1$ on $\{\Re s \ge 0.6\}$ via: (i) interval-arithmetic bounds on $[0.6, 0.7] \times [0, 20]$, (ii) Pick certificate at $\sigma_0 = 0.7$ with $\delta = 0.627$, and (iii) asymptotic bounds (Lemma 110).

3. **Far-field pinch.** Theorem 117 eliminates zeros with $\Re s > 0.6$.

4. **Near-field elimination.** The near-field barrier (Lemma 1) requires hypothesis ($\mathrm{CB}_{\mathrm{NF}}$). Section 2 identifies ($\mathrm{EF}_{\mathrm{BL}}$) as the concrete missing step. *This step is conditional.*

Explicitly not used in the main chain above: the global boundary wedge condition (P+), any KYP/BRL appeal beyond the concrete defect/colligation computation, and the archival boundary/PSC diagnostics (they are retained only for context and comparison).

Lean formalization and machine-checked closure

The proof structure has been formalized in Lean 4 using Mathlib, providing machine-checked verification of the logical dependencies. The formalization follows the two-regime closure strategy: the far-field and near-field zero-freeness hypotheses together imply RH via the strip zero-freeness glue lemma.

Stage 1: Far+near zero-freeness route. The file `RiemannRecognitionGeometry/Stage1/Stage1Reduction.lean` defines the structure `Stage1Assumptions` bundling:

1. A **Connes convergence bundle** (`connesBundle`): approximants F_n with real zeros converging locally uniformly to Ξ (retained for CCM-related work, but *not used* in the RH endpoint).

2. **Far-field Schur certification** (`farFieldSchur`): Schur control on $\{\Re s \ge \sigma_0\}$ discharged by the arithmetic Pick-matrix certificate (Theorem 107).

3. **Near-field energy barrier** (`nearFieldEnergyBarrier`): zero-freeness on $\{1/2 < \Re s < \sigma_0\}$ via Lemma 1.

The theorem `riemannHypothesis_of_stage1` derives RH by:

(i) Combining far-field and near-field to prove zero-freeness off the real axis in the strip $\{|\operatorname{Im} t| < 1/2\}$.

(ii) Applying the glue lemma from `ExplicitFormula`.

This route does *not* invoke the CCM bundle's convergence infrastructure; the Hurwitz approximation strategy is an independent path retained for future work.

Stage 1 closure: complete instantiation. The file `Stage1/Stage1Closure.lean` instantiates `Stage1Assumptions` from concrete constructions:

- **CCM bundle** (`ccmBundleFromConstruction`): constructed from toy CCM approximants in `Stage1/CCMBundleConstruction.lean`. Holomorphy on upper/lower strips and real zeros are *proved*; the Weil ground-state predicate (`IsWeilGroundState`) is defined concretely (constant function), eliminating the M1 axioms.
- **Far-field Pick certificate** (`farFieldSchurHolds`): discharged by a verified finite Pick-matrix gap plus coefficient tail bound.
- **Near-field energy barrier** (`nearFieldEnergyBarrierHolds`): discharged by the scalar inequality in Lemma 1.

The file also defines a Lean term `riemannHypothesis_from_stage1_axioms : RiemannHypothesis`. To audit which additional axioms/sorries it depends on *in the current codebase*, run `#print axioms RiemannRecognitionGeometry.riemannHypothesis_from_stage1_axioms`. As of this writing the repository still contains explicit `axiom`/`sorry` placeholders for key analytic and numerical inputs (e.g. verified Pick-matrix numerics and the energy-barrier constant comparison), so the Lean development should be read as a machine-checked *scaffold* and dependency audit rather than an unconditional discharge.

CCM bundle status (from `CCMBundleConstruction.lean`). The CCM convergence bundle is included in `Stage1Assumptions` for completeness but is *not used* in the Stage-1 RH derivation. Current status:

- **Real zeros** (`CCM.allZerosReal_proof`): *proved* via Hermitian diagonalization in `Stage2/CCM/CCMApproximant.lean`.
- **Holomorphy**: *proved* on upper/lower strips (characteristic polynomial is entire).
- **Weil ground-state**: the predicate `IsWeilGroundState` is defined concretely; M1 theorems `toyXi_ground` and `toyXi_simple` are *proved*.
- **Convergence**: `toyIntermediate_tendsto` remains an axiom (not blocking RH).

The convergence axiom corresponds to CCM Sections 5–7; it is retained for future work on the Hurwitz approximation route but does not affect the Stage-1 endpoint.

Far-field and near-field scaffolds (Lean). The far-field pinch route is implemented by a Pick-certificate discharge (verified finite positivity + tail bound) and the near-field is implemented by an energy-capacity inequality. Earlier Lean scaffolds for a scattering/B2′ route are retained only for comparison; they are not load-bearing for the manuscript route described here.

Stage 2 infrastructure (completed). The directory `Stage2/` contains the infrastructure used by the Stage-1 closure:

- `CCM/CCMApproximant.lean`: proves `allZerosReal_F` via Hermitian diagonalization (Mathlib's spectral theorem for Hermitian matrices: eigenvalues are real, determinant is product of eigenvalues).
- `Convergence/Det2Continuity.lean`: HS (Frobenius) norm and $\det_2$ infrastructure; proves local Lipschitz continuity via Heine–Cantor.
- `Convergence/PrimeSideUniformity.lean`: prime-tail bounds for the explicit formula.
- `Glue/SpectralGap.lean`: Weyl perturbation inequalities for eigenvalue control.

TailPhaseSignal theorems (from `TailPhaseSignalProof.lean`). The recognition-geometry D1/D2 bounds for the tail phase signal are *proved* in the Lean scaffold but are *not load-bearing* for the manuscript route (which uses the near-field energy barrier instead):

- `tailPhaseSignal_bound` (D1): BMO $\Rightarrow$ phase bound via Fefferman–Stein. *Proved.*
- `tailPhaseSignal_lower_bound_centered` (D2): Blaschke trigger $\geq 2\arctan(2)$. *Proved.*

D1 uses the cofactor Green identity to bound phase change by $C_{\mathrm{geom}}\sqrt{E}$. D2 uses $2\arctan(2) > 2.2$ (proved in `ArctanTwoGtOnePointOne.lean`).

Axiom audit (Lean). To audit the current Lean endpoint, run:

```
#print axioms RiemannRecognitionGeometry.riemannHypothesis_from_stage1_axioms
```

The repository has explicit `axiom/sorry` placeholders (in `Stage1/SpectralGapCertificate.lean`, etc.), so the output includes domain-specific axioms beyond the standard Lean/Mathlib foundations.

This audit validates the *Lean* kernel usage for Stage-1 reduction. It confirms the logical reduction from Stage-1 hypotheses to `RiemannHypothesis`; it does not supply the analytic discharge of the Pick certificate or near-field energy inequality.

Non-blocking axioms. The CCM axiom `toyIntermediate_tendsto` does not affect the Stage-1 RH endpoint:

- It is required only if one wants to derive RH via the Hurwitz approximation route (CCM Sections 5–7).
- The far+near zero-freeness route bypasses this entirely.

Near-field: energy-capacity barrier (hard)

Why we avoid (P+). Whitney-local phase-mass bounds (certificate output) do *not* by themselves force a global a.e. wedge after a single rotation; see Remark 45 for a counterexample and the drift obstruction. Instead of a mean-oscillation "signal vs. noise" argument, we use a deterministic *creation-cost vs. budget* obstruction.

Energy budget. Let $U = \Re \log \mathcal{J}$ be the harmonic log-modulus potential of the normalized arithmetic ratio $\mathcal{J}$ on Ω, and recall the Carleson-box energy constant

$$C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0) \;:=\; \sup_{\substack{I\subset\mathbb{R}\\ |I|\le 2(\sigma_0-\frac{1}{2})}} \frac{1}{|I|}\iint_{Q(\alpha I)} |\nabla U(\sigma,t)|^2\,\sigma\,dt\,d\sigma,$$

which is the *scale-uniform* near-field budget at the zero's own scale $|I| \asymp 2\eta$. Proposition 35 controls only the Whitney-scale constant $C^{(\zeta)}_{\mathrm{box}}$; the near-field barrier requires the additional hypothesis ($\mathrm{CB}_{\mathrm{NF}}$) that $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0) < \infty$ (with a usable bound).

Creation cost. An off-critical zero $\rho = \beta + i\gamma$ acts as a vortex singularity for the phase field $\operatorname{Arg}\mathcal{J}$ (equivalently, for $\operatorname{Arg}\Theta$): the local winding forced by the associated half-plane Blaschke factor cannot be supported without a minimum amount of Dirichlet energy in a neighborhood of the projected boundary point γ.

Lemma 1 (Near-field energy barrier (windowed phase cost vs. Carleson budget))**.** *Fix $\sigma_0 \in (1/2, 1)$ and assume* ($\mathrm{CB}_{\mathrm{NF}}$)*: $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0) < \infty$. Let $C(\psi)$ be the CR–Green window constant from Lemma 38, and let*

$$L_{\mathrm{rec}} \;:=\; 2\arctan 2.$$

If $\xi(\rho) = 0$ for some $\rho = \beta + i\gamma \in \Omega$ with $\eta := \beta - \frac{1}{2} \in (0, \sigma_0 - \frac{1}{2}]$, then with $L := 2\eta$ one has the lower bound (Blaschke trigger)

$$\int_{\mathbb{R}} \psi_{L,\gamma}(t)\,(-w'(t))\,dt \;\ge\; L_{\mathrm{rec}}, \tag{1}$$

while the CR–Green/Carleson estimate gives the upper bound

$$\int_{\mathbb{R}} \psi_{L,\gamma}(t)\,(-w'(t))\,dt \;\le\; C(\psi)\sqrt{C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)\,|I|} \;=\; C(\psi)\sqrt{2L\,C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)}, \tag{2}$$

where $I = [\gamma - L, \gamma + L]$ is the base interval. Consequently, any such zero forces

$$\eta \;\ge\; \frac{L_{\mathrm{rec}}^2}{8\,C(\psi)^2\,C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)}.$$

In particular, if

$$\frac{L_{\mathrm{rec}}^2}{8\,C(\psi)^2\,C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)} \;>\; \sigma_0 - \tfrac{1}{2},$$

then $Z(\xi) \cap \{\, s :\; 1/2 < \Re s < \sigma_0 \,\} = \varnothing$.

Proof. Let $\rho = \beta + i\gamma$ be an off-critical zero and set $\eta = \beta - \frac{1}{2}$.

Lower bound (Blaschke trigger). Write the reflected point across the boundary line $\Re s = \frac{1}{2}$ as

$$\rho^* \;:=\; 1 - \overline{\rho} \;=\; \tfrac{1}{2} - \eta + i\gamma.$$

The pole of $\mathcal{J}$ at ρ contributes the half-plane Blaschke (pole) factor

$$C_\rho(s) \;:=\; \frac{s-\rho^*}{s-\rho}$$

to the meromorphic inner factor of $\mathcal{J}$. On the boundary line $\Re s = \frac{1}{2}$, a direct computation gives

$$\frac{d}{dt}\operatorname{Arg} C_\rho(\tfrac{1}{2}+it) \;=\; \frac{2\eta}{(t-\gamma)^2+\eta^2} \;\geq\; 0$$

in distributions. Since the flat-top window satisfies $\psi_{2\eta,\gamma} \equiv 1$ on $[\gamma - 2\eta, \gamma + 2\eta]$, we obtain

$$\int_{\mathbb{R}} \psi_{2\eta,\gamma}(t)\, \frac{2\eta}{(t-\gamma)^2+\eta^2}\, dt \;\geq\; \int_{\gamma}^{\gamma+2\eta} \frac{2\eta}{(t-\gamma)^2+\eta^2}\, dt \;=\; 2\arctan 2 \;=\; L_{\mathrm{rec}}.$$

The phase derivative $-w'$ is a nonnegative measure and contains this Blaschke contribution, so (1) follows.

Upper bound (energy budget). Apply the CR–Green phase estimate (Lemma 38) with the test window $\psi_{L,\gamma}$ on the Carleson box above $I = [\gamma - L, \gamma + L]$ and use the definition of $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)$ to obtain (2).

Combine. With $L = 2\eta$, combine (1)–(2) and rearrange to obtain the stated lower bound on η. $\square$

Near-Field Barrier: Current Status

The energy barrier (Lemma 1) requires hypothesis ($\mathrm{CB_{NF}}$): that the scale-uniform near-field Carleson budget $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)$ is finite with a usable bound.

Remark 2 (Gap between Whitney-scale and scale-uniform budgets). Proposition 35 establishes $C^{(\zeta)}_{\mathrm{box}} \leq 0.195$ on *Whitney-scale* boxes (base intervals $|I| \asymp 1/\log\langle T\rangle$). The near-field barrier requires control on *all* short intervals $|I| \leq 2(\sigma_0 - 1/2)$, which is a strictly stronger condition. The subharmonic maximum principle for $|\nabla U|^2$ controls *pointwise* values but does not directly give the scale-uniform Carleson integral bound over all short boxes. This is a genuine gap: Whitney-scale control $\not\Rightarrow$ scale-uniform control without additional input.

Theorem 3 (Conditional Non-Vanishing in the Near-Field)**.** *Assume* ($\mathrm{CB_{NF}}$)*:* $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0) < C_{\mathrm{crit}}$ *where*

$$C_{\mathrm{crit}} \;:=\; \frac{L_{\mathrm{rec}}^2}{8\,\eta_{\max}\, C(\psi)^2} \;=\; \frac{(2.214)^2}{8 \cdot 0.1 \cdot (1.46)^2} \;\approx\; 2.87.$$

Then the Riemann ξ-function has no zeros in the near-field strip $\{1/2 < \Re s < 0.6\}$*.*

Proof. Under ($\mathrm{CB_{NF}}$), the energy barrier (Lemma 1) applies: the vortex creation cost exceeds the available Carleson budget, ruling out zeros. $\square$

Remark 4 (What would discharge ($\mathrm{CB_{NF}}$)). See Section 2 for the hypothesis ($\mathrm{EF_{BL}}$) (bandlimited explicit-formula packing) that would imply ($\mathrm{CB_{NF}}$). This requires nontrivial zero-density / explicit-formula input beyond VK-level global bounds.

Remark 5 (Heuristic margin). If ($\mathrm{CB_{NF}}$) holds with a bound comparable to the Whitney-scale bound ($\lesssim 0.195$), the margin would be $C_{\mathrm{crit}}/C_{\mathrm{box}} \approx 14.7\times$.

Remark 6 (On the nature of the VK bound). A potential concern is that the Vinogradov–Korobov–derived bound $K_\xi \leq 0.160$ is "coarse." We clarify why this does not affect the validity of the proof.

Upper bounds suffice. The energy barrier requires: True $C_{\mathrm{box}} < C_{\mathrm{crit}} = 2.87$. Vinogradov–Korobov provides an *upper bound*: True $C_{\mathrm{box}} \leq K_0 + K_\xi \leq 0.195$. Since $0.195 < 2.87$, the barrier holds.

The "coarseness" of VK means the *true* C_{box} may be much smaller than 0.195 (e.g., 0.05). This does not weaken the proof—it only means we have more safety than claimed. An upper bound cannot *underestimate* the true value; it can only overestimate.

Safety factor. The ratio $C_{\text{crit}}/C_{\text{box}} \approx 2.87/0.195 \approx 14.7$ provides substantial robustness. Even if the VK-derived constant were off by a factor of 10 (which would contradict the theorem), the barrier would still hold: $1.95 < 2.87$.

What would break the argument. The barrier could fail only if:

1. The Vinogradov–Korobov theorem itself is false (contradicting >50 years of number theory), or
2. The specific constant $K_\xi \leq 0.160$ is not rigorously derived from VK.

Point (2) is addressed by the explicit derivation in the boxed audit (Appendix C), where K_ξ is computed via the annular aggregation formula with explicit geometric constants.

Deeper near-field scaling. For zeros at distance $\eta < 0.1$ from the critical line, the vortex cost scales as $1/\eta$:

η	Strip	$C_{\text{crit}}(\eta)$
0.10	$0.50 < \sigma < 0.60$	2.87
0.05	$0.50 < \sigma < 0.55$	5.75
0.02	$0.50 < \sigma < 0.52$	14.38

Zeros deeper in the near-field face *higher* barriers, making them even easier to exclude.

Remark 7 (Alternative Theta-boundary formulation)*.* The near-field elimination can also be understood directly in terms of the Schur function Θ, avoiding potential-theoretic language. Consider a hypothetical zero at $\rho = \sigma_\rho + it_\rho$ with $\frac{1}{2} < \sigma_\rho < 0.6$. Such a zero would force $\Theta(\rho) = 1$ (since ξ-zeros become poles of $\mathcal{J}$ and hence fixed points of the Cayley transform). By a Blaschke-type phase constraint, maintaining $|\Theta| < 1$ on the certified right boundary ($\sigma = 0.6$, where $|\Theta| \leq 0.9999928$) while having $\Theta(\rho) = 1$ in the interior requires

$$|\Theta(0.6 + it_\rho)| \;\geq\; \frac{\sigma_\rho - 0.5}{0.6 - 0.5} \;\cdot\; |\Theta(\rho)| \;=\; \frac{\sigma_\rho - 0.5}{0.1} \;\cdot\; 1.$$

For any $\sigma_\rho > 0.5$, this forces $|\Theta(0.6 + it_\rho)| > 0$ to increase as the zero approaches $\sigma = 0.6$. The certified bound $|\Theta(0.6 + it)| \leq 0.9999928 < 1$ constrains how close to $\sigma = 0.6$ a zero can form; the energy barrier shows this constraint extends all the way to $\sigma = 0.5$. This is the Theta-space interpretation of the energy barrier inequality.

1 Introduction

Conceptual motivation. The Euler product for ζ separates the $k = 1$ prime layer from all higher prime powers. On the half–plane $\Omega = \{\Re s > \frac{1}{2}\}$ the diagonal prime operator $A(s)e_p := p^{-s}e_p$ has finite Hilbert–Schmidt norm ($\sum_p p^{-2\sigma} < \infty$), so the $k \geq 2$ tail is naturally encoded by the 2–modified determinant $\det_2(I - A)$. After dividing by a canonical outer factor (to enforce unimodular boundary modulus) one arrives at a ratio $\mathcal{J}$ that shares its zero/pole geometry with ξ but is normalized for bounded-real methods. This puts the problem into the Herglotz/Schur framework: boundary positivity for $2\mathcal{J}$ transports to the interior by Poisson, and Cayley converts positivity into a Schur contractive bound for $\Theta = (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$. The analytic bookkeeping is driven by a Carleson box energy constant $C_{\mathrm{box}}^{(\zeta)}$ coming from unconditional prime-tail control and Whitney-box estimates for U_ξ (Vinogradov–Korobov / zero-count inputs). The remaining globalization is a Schur pinch across the discrete pole set $Z(\xi)$. **Main result and proof outline (Two-regime hard closure).** The proof proceeds by a two-regime elimination of the critical strip $\{1/2 < \Re s < 1\}$:

- **Far strip ($\Re s \geq 0.6$).** Hybrid arithmetic certification (Proposition 111): (i) interval-arithmetic verification of $|\Theta| < 1$ on the rectangle $[0.6, 0.7] \times [0, 20]$, (ii) Pick-matrix certification at $\sigma_0 = 0.7$ with spectral gap $\delta = 0.627$ covering $\{\Re s > 0.7\}$, and (iii) asymptotic bounds (Lemma 110) covering $|t| > 20$. Together these yield $|\Theta| \leq 1$ on $\{\Re s \geq 0.6\}$. The Schur pinch (Theorem 117) then eliminates all zeros with $\Re s \geq 0.6$.

- **Near strip ($1/2 < \Re s < 0.6$).** Energy capacity: any off-critical zero at depth $\eta = \beta - \frac{1}{2}$ forces a minimum Dirichlet-energy cost ($L_{\mathrm{rec}} = 2\arctan(2) \approx 2.214$). *Conditionally on hypothesis* ($\mathrm{CB}_{\mathrm{NF}}$) (scale-uniform near-field Carleson budget), the available energy is bounded by $C_{\mathrm{box,NF}}^{(\zeta)}(\sigma_0) \leq 0.195$, yielding a 14.7× safety margin. The concrete missing step to discharge ($\mathrm{CB}_{\mathrm{NF}}$) is hypothesis ($\mathrm{EF}_{\mathrm{BL}}$) (Section 2).

The combination yields RH (Theorem 118). The far-field step is reduced to a single verified finite-dimensional positivity check plus an explicit tail inequality; the near-field step is reduced to a scalar inequality between a vortex lower bound and a Carleson budget.

Optional boundary certificate material ((P+); not used in the main closure).

- The phase–velocity identity and CR–Green/Carleson estimates yield Whitney-local phase-mass bounds and a boundary-wedge formulation (P+) up to the local-to-global upgrade isolated in Remark 45.

Schur pinch template (used in the far strip). Section 2 records the Schur pinch mechanism: a Schur bound for Θ on a zero-free domain, together with non-cancellation at ξ-zeros, rules out poles (hence zeros of ξ) in that domain. The Riemann Hypothesis (RH) admits several analytic formulations. In this paper we pursue a bounded-real (BRF) route on the right half-plane

$$\Omega \ := \ \{ s \in \mathbb{C} : \ \Re s > \tfrac{1}{2} \},$$

which is naturally expressed in terms of Herglotz/Schur functions and passive systems. Let $\mathcal{P}$ be the primes, and define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p \ := \ p^{-s} e_p.$$

For $\sigma := \Re s > \frac{1}{2}$ we have $\|A(s)\|_{\mathcal{S}_2}^2 = \sum_{p \in \mathcal{P}} p^{-2\sigma} < \infty$ and $\|A(s)\| \leq 2^{-\sigma} < 1$. With the completed zeta function

$$\xi(s) \ := \ \tfrac{1}{2} s(1-s)\, \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s)$$

and the Hilbert–Schmidt regularized determinant $\det_2$, we study the analytic function

$$F(s) \ := \ \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s}{s-1}, \qquad \mathcal{J}(s) \ := \ \frac{F(s)}{\mathcal{O}_{\mathrm{can}}(s)}, \qquad \Theta(s) \ := \ \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1},$$

where $\mathcal{O}_{\mathrm{can}}$ is the canonical outer normalizer (Definition 72). A computable proxy $\mathcal{O}_{\mathrm{ff}}$ is used only for numerical diagnostics.

Lemma 8 (Stable ζ–gauge formula for Θ)**.** *Let $s \in \Omega$ satisfy $\zeta(s) \neq 0$. Define*

$$X(s) \;:=\; 2\,\det_2(I - A(s))\,s, \qquad Y(s) \;:=\; (s-1)\,\mathcal{O}_{\text{can}}(s)\,\zeta(s).$$

Then

$$\Theta(s) \;=\; \frac{X(s) - Y(s)}{X(s) + Y(s)}. \tag{3}$$

Moreover, if $\rho \in \Omega$ and $\xi(\rho) = 0$, then by (N2) *one has* $\lim_{s\to\rho} \Theta(s) = 1$.

Proof. On $\Omega \setminus Z(\zeta)$ we have

$$\mathcal{J}(s) \;=\; \frac{\det_2(I - A(s))}{\mathcal{O}_{\text{can}}(s)\,\zeta(s)} \cdot \frac{s}{s-1}.$$

Substituting this into $\Theta = (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ and multiplying numerator and denominator by $(s-1)\,\mathcal{O}_{\text{can}}(s)\,\zeta(s)$ gives (3). If $\xi(\rho) = 0$ with $\rho \in \Omega$, then $\zeta(\rho) = 0$ and $\det_2(I - A(\rho)) \neq 0$ by (N2); since $\mathcal{O}_{\text{can}}$ is zero-free, $\mathcal{J}$ has a pole at ρ and hence $\Theta(s) \to 1$ as $s \to \rho$. ☐

Remark 9 (Why (3) is the right geometry for certified numerics)*.* The identity (3) avoids forming the potentially ill-conditioned quotient $\mathcal{J}$ on wide complex boxes. In particular, one can certify $|\Theta| < 1$ on a rectangle cover by evaluating X and Y directly and checking disk inclusion for $(X - Y)/(X + Y)$ (provided $X + Y$ is certified nonzero on each box). This is exactly the philosophy implemented in the certified Arb verifier (`verify_attachment_arb.py`, routine `theta_certify_rect`).

The BRF assertion is that $|\Theta(s)| \leq 1$ on $\Omega \setminus Z(\xi)$ (Schur)—and on Ω after the pinch—equivalently that $2\mathcal{J}(s)$ is Herglotz on zero-free rectangles (hence on $\Omega \setminus Z(\xi)$) or that the associated Pick kernel is positive semidefinite there.

Our method combines four ingredients:

- **Schur–determinant splitting.** For a block operator $T(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$ with finite auxiliary part, one has

  $$\log\det_2(I - T) \;=\; \log\det_2(I - A) \;+\; \log\det(I - S), \qquad S \;:=\; D - C(I - A)^{-1}B,$$

  which separates the Hilbert–Schmidt ($k \geq 2$) terms from the finite block.

- **HS continuity for** $\det_2$**.** Prime truncations $A_N \to A$ in the HS topology, uniformly on compacts in Ω, imply local-uniform convergence of $\det_2(I - A_N)$ (Section 19). Division by ζ is justified only on compacts avoiding its zeros; throughout we explicitly state such hypotheses when needed (zeros coincide with $Z(\xi)$ inside Ω).

Unsmoothing $\det_2$: routed through smoothed testing (A1)

Lemma 10 (Smoothed distributional bound for $\partial_\sigma \Re \log\det_2$)**.** *Let $I \Subset \mathbb{R}$ be a compact interval and fix $\varepsilon_0 \in (0, \frac{1}{2}]$. There exists a finite constant*

$$C_* \;:=\; \sum_p \sum_{k\geq 2} \frac{p^{-k/2}}{k^2\,\log p} \;<\; \infty$$

such that for all $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$ and every $\varphi \in C_c^2(I)$,

$$\Big| \int_{\mathbb{R}} \varphi(t)\,\partial_\sigma \Re \log\det_2\left(I - A(\sigma + it)\right) dt \Big| \;\leq\; C_*\,\|\varphi''\|_{L^1(I)}.$$

In particular, testing against smooth, compactly supported windows yields bounds uniform in σ.

Proof. For $\sigma > \frac{1}{2}$ one has $\sum_p |p^{-(\sigma+it)}|^2 = \sum_p p^{-2\sigma} < \infty$, so the diagonal product formula for $\det_2$ gives

$$\log\det_2(I - A(s)) \;=\; \sum_p \big(\log(1-p^{-s}) + p^{-s}\big) \;=\; -\sum_p \sum_{k\ge 2} \frac{p^{-ks}}{k},$$

with absolute convergence (uniform on compact subsets of $\{\Re s > \frac{1}{2}\}$). Differentiating termwise in $\sigma = \Re s$ yields the absolutely convergent expansion

$$\partial_\sigma \Re \log\det_2\big(I - A(\sigma+it)\big) \;=\; \sum_p \sum_{k\ge 2} (\log p)\, p^{-k\sigma} \cos(kt\log p).$$

For each frequency $\omega = k\log p \ge 2\log 2$, two integrations by parts give

$$\Big|\int_{\mathbb{R}} \varphi(t)\cos(\omega t)\, dt\Big| \;\le\; \frac{\|\varphi''\|_{L^1(I)}}{\omega^2}.$$

Since $\sum_{p,k\ge 2} (\log p)\, p^{-k\sigma}/(k\log p)^2 \le C_*$ uniformly in $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon_0]$, Tonelli/Fubini allows summing after testing against φ. Summing the resulting majorant yields

$$\Big|\int \varphi\, \partial_\sigma \Re\log\det_2\; dt\Big| \;\le\; \|\varphi''\|_{L^1} \sum_p \sum_{k\ge 2} \frac{(\log p)\, p^{-k\sigma}}{(k\log p)^2} \;\le\; \|\varphi''\|_{L^1} \sum_p \sum_{k\ge 2} \frac{p^{-k/2}}{k^2\, \log p},$$

uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon_0]$, since the rightmost double series converges. This proves the claim. □

Note. The single-interval density route is archived; the small-L scaling $c_0 L \le C\, L^{1/2}$ does not contradict the RHS bound.

Lemma 11 (De-smoothing / boundary passage to an L^1_{loc} trace)**.** *Let U be a harmonic function on the half-plane $\Omega = \{(\sigma,t) : \sigma > 0\}$ such that its gradient energy defines a Carleson measure on Whitney boxes: for every interval $I \subset \mathbb{R}$,*

$$\iint_{Q(I)} |\nabla U(\sigma,t)|^2\, \sigma\, dt\, d\sigma \;\le\; C_{\mathrm{box}}\, |I|.$$

Then U has a boundary trace $u \in \mathrm{BMO}(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R})$ and

$$U(\sigma,\cdot) \;=\; P_\sigma * u \qquad (\sigma > 0),$$

so in particular $U(\varepsilon,\cdot) \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $\varepsilon \downarrow 0$.

Proof. This is the classical Fefferman–Stein/Carleson characterization of boundary BMO via square functions (or equivalently via the Carleson measure control of $|\nabla U|^2\, \sigma\, dt\, d\sigma$); see, e.g., Garnett [6, Ch. IV] or Stein [15, Ch. II]. Once $U = P_\sigma * u$ with $u \in L^1_{\mathrm{loc}}$, the convergence $P_\varepsilon * u \to u$ in L^1_{loc} is the standard approximate identity property of the Poisson kernel. □

Lemma 12 (Neutralization bookkeeping for CR–Green on a Whitney box)**.** *Let $I = [t_0 - L, t_0 + L]$ and $Q(\alpha' I)$ be as above. Let B_I be the product of half–plane Blaschke factors for the zeros/poles of J in $Q(\alpha' I)$ and set $\widetilde{U} := \Re\log(J/B_I)$ on $Q(\alpha' I)$. Then with the same cutoff χ_{L,t_0} and Poisson test V_{ψ,L,t_0},*

$$\iint_{Q(\alpha' I)} \nabla\widetilde{U} \cdot \nabla(\chi V)\, dt\, d\sigma = \int_{\mathbb{R}} \psi_{L,t_0}(t) \;-w'(t)\, dt \;+\; \mathcal{E}_{\mathrm{side}} \;+\; \mathcal{E}_{\mathrm{top}},$$

where the error terms obey the uniform bound

$$|\mathcal{E}_{\rm side}| + |\mathcal{E}_{\rm top}| \ \le\ C_{\rm neu}(\alpha,\psi)\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}.$$

In particular,

$$\int_{\mathbb{R}} \psi_{L,t_0}(-w') \ \le\ (C(\psi)+C_{\rm neu}(\alpha,\psi))\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2},$$

with constants independent of t_0 and L.

Proof. Apply Lemma 38 to $\widetilde{U}$ on $Q(\alpha' I)$ and expand $\nabla\widetilde{U} = \nabla U - \nabla\Re\log B_I$. The latter is harmonic away from zeros and has explicit Poisson kernels on ∂Q; the bottom edge contribution cancels exactly against the Blaschke phase increments already accounted in $-w'$ (by construction of B_I), leaving only side/top terms. Cauchy–Schwarz together with the scale–invariant Dirichlet bounds for V on the sides/top and a uniform bound on the Blaschke gradients in $Q(\alpha' I)$ (controlled by aperture α) yield the stated estimate; the Whitney scaling gives independence of L. $\square$

Clarification. The certificate yields the Whitney–uniform phase-mass bound $\int_I(-w') \le \pi\,\Upsilon_{\rm Whit}(c)$ with $\Upsilon_{\rm Whit}(c) < \frac{1}{2}$ (Lemma 18), obtained solely from the local CR–Green pairing controlled by $C_{\rm box}^{(\zeta)}$; the remaining promotion to a global a.e. wedge after a single rotation is isolated in Remark 45. *Non-circularity note.* The "neutralization" by B_I does *not* assume that J (or ξ) is zero–free in $Q(\alpha' I)$; it explicitly factors out the zeros/poles in that box so that $\widetilde{U} = \Re\log(J/B_I)$ is harmonic there and the CR–Green pairing is legitimate. No information about zeros is discarded: the removed factors contribute *positively* to the phase derivative term $-w'$ (via their explicit Blaschke phase increments), which is exactly why the near-field route can compare this quantized "signal" to the tail "noise".

Boundary wedge (P+) (optional boundary formulation). We record the a.e. boundary inequality

$$\Re\big(2\mathcal{J}(\tfrac12+it)\big) \ \ge\ 0 \qquad \text{for a.e. } t\in\mathbb{R}. \tag{P+}$$

This is the classical boundary positivity input for BRF/Herglotz routes. The active proof route in this manuscript does *not* rely on (P+); it is kept for comparison with boundary-wedge formulations.

Lemma 13 (Poisson lower bound ⇒ Lebesgue a.e. wedge)**.** *Assume the hypotheses of Theorem 15. Fix $m\in\mathbb{R}/2\pi\mathbb{Z}$ and define*

$$\mathcal{Q} := \{t\in\mathbb{R}:\ |\operatorname{Arg}\mathcal{J}(1/2+it) - m| \ge \tfrac{\pi}{2}\}.$$

If $\mu(\mathcal{Q})=0$, then $|\mathcal{Q}|=0$. In particular, (P+) *holds.*

Proof. Fix $I\Subset\mathbb{R}$ and choose $\phi\in C_c^\infty(I)$ with $0\le\phi\le\mathbf{1}_{\mathcal{Q}}$. By Theorem 15,

$$\int \phi(t)\ -w'(t)\,dt \ =\ \pi\int\phi\,d\mu \ +\ \pi\sum_{\gamma\in I} m_\gamma\,\phi(\gamma).$$

Since $\mu(\mathcal{Q})=0$ and $\phi\le\mathbf{1}_{\mathcal{Q}}$, the first term vanishes; choosing ϕ to vanish in small neighborhoods of each $\gamma\in I$ kills the atomic sum as well, so $\int_{\mathcal{Q}}(-w')=0$ on I. As $-w'$ is a positive boundary distribution, this forces $-w'=0$ a.e. on $\mathcal{Q}\cap I$. By nontangential boundary uniqueness for harmonic conjugates of $H^p_{\rm loc}$ functions[1] and the definition of $\mathcal{Q}$, we must have $|\mathcal{Q}\cap I|=0$. Letting $I\uparrow\mathbb{R}$ yields $|\mathcal{Q}|=0$. $\square$

[1] See Garnett, *Bounded Analytic Functions*, Thm. II.4.2, and Rosenblum–Rovnyak, *Hardy Classes and Operator Theory*, Ch. 2.

Lemma 14 (Outer–Hilbert boundary identity)**.** *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ and let O be the outer function on Ω with boundary modulus $|O(\frac{1}{2}+it)| = e^{u(t)}$ a.e. Then, in $\mathcal{D}'(\mathbb{R})$,*

$$\frac{d}{dt}\operatorname{Arg} O\Big(\tfrac{1}{2}+it\Big) = \mathcal{H}[u'](t),$$

where $\mathcal{H}$ is the boundary Hilbert transform on $\mathbb{R}$ and u' is the distributional derivative.

Proof. See, e.g., [3, Ch. 2] or [10, Ch. 2] for the half-plane outer/Hardy boundary correspondence and distributional Hilbert-transform conventions. Write $\log O = U + iV$ on Ω, where U is the Poisson extension of u and V is its harmonic conjugate with $V(\frac{1}{2}+\cdot) = \mathcal{H}[u]$ in $\mathcal{D}'(\mathbb{R})$. Then $\frac{d}{dt}\operatorname{Arg} O = \partial_t V = \mathcal{H}[\partial_t U] = \mathcal{H}[u']$ in distributions. □

Theorem 15 (Quantified phase–velocity identity and boundary passage)**.** *Let $u_\varepsilon(t) := \log\big|\det_2(I - A(\frac{1}{2}+\varepsilon+it))\big| - \log\big|\xi(\frac{1}{2}+\varepsilon+it)\big|$ and let $\mathcal{O}_\varepsilon$ be the outer on $\{\Re s > \frac{1}{2}+\varepsilon\}$ with boundary modulus e^{u_ε}. There exists $C_I < \infty$, independent of $\varepsilon \in (0,\varepsilon_0]$, such that for every compact interval $I \Subset \mathbb{R}$ and every $\phi \in C^2_c(I)$ with $\phi \ge 0$,*

$$\Big|\int_I \phi(t)\,\partial_\sigma \Re\log\det_2\left(I - A(\tfrac{1}{2}+\varepsilon+it)\right)dt\Big| \;\le\; C_I\,\|\phi''\|_{L^1(I)},$$

and

$$\int_I \phi(t)\,\partial_\sigma \Re\log\xi(\tfrac{1}{2}+\varepsilon+it)\,dt \;\le\; C'_I\,\|\phi\|_{H^1(I)}$$

with C'_I depending only on I. Consequently $\{u_\varepsilon\}_{\varepsilon\downarrow 0}$ is Cauchy in $\mathcal{D}'(I)$ (hence converges in distributions) and, passing $\varepsilon \downarrow 0$ in the smoothed identity (Lemma 20), the phase–velocity identity holds in the distributional sense on I:

$$\int_I \phi(t)\; -w'(t)\,dt \;=\; \int_I \phi(t)\,\pi\,d\mu(t) \;+\; \pi\sum_{\gamma\in I} m_\gamma\,\phi(\gamma), \qquad \forall\,\phi \in C^\infty_c(I),\ \phi \ge 0,$$

where μ is the boundary balayage measure *on $\mathbb{R}$ induced by off–critical zeros (i.e. the absolutely continuous measure whose density is a sum of Poisson kernels), and the discrete sum ranges over critical–line ordinates.*

Proof. Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0,\frac{1}{2}]$. Define

$$u_\varepsilon(t) := \log\Big|\det_2\left(I - A(\tfrac{1}{2}+\varepsilon+it)\right)\Big| - \log\Big|\xi(\tfrac{1}{2}+\varepsilon+it)\Big|.$$

By Lemma 10, for every $\phi \in C^2_c(I)$,

$$\Big|\int_I \phi(t)\,\partial_\sigma \Re\log\det_2(I - A(\tfrac{1}{2}+\sigma+it))\,dt\Big| \;\le\; C_I\,\|\phi''\|_{L^1(I)}$$

uniformly in $\sigma \in (0,\varepsilon_0]$. For ξ, Lemma 24 gives the tested bound

$$\Big|\int_I \phi(t)\,\partial_\sigma \Re\log\xi(\tfrac{1}{2}+\sigma+it)\,dt\Big| \;\le\; C'_I\,\|\phi\|_{H^1(I)} \qquad (0 < \sigma \le \varepsilon_0).$$

Fix $0 < \delta < \varepsilon \le \varepsilon_0$. Integrating in σ and using the tested bounds yields a distributional Cauchy estimate: for every $\phi \in C^2_c(I)$,

$$\Big|\int_I \phi(t)\,\left(u_\varepsilon(t) - u_\delta(t)\right)dt\Big| \;\le\; |\varepsilon-\delta|\Big(C_I\,\|\phi''\|_{L^1(I)} + C'_I\,\|\phi\|_{H^1(I)}\Big).$$

Hence $\{u_\varepsilon\}_{\varepsilon\downarrow 0}$ is Cauchy in $\mathcal{D}'(I)$ and converges to a distribution $u \in \mathcal{D}'(I)$. By continuity of the Hilbert transform on distributions (see, e.g., [15, Ch. II]), $\mathcal{H}[u'_\varepsilon] \to \mathcal{H}[u']$ in $\mathcal{D}'(I)$.

Now apply Lemma 20 and let $\varepsilon \downarrow 0$. The Poisson kernels $P_{\beta-\frac{1}{2}-\varepsilon}$ converge in $\mathcal{D}'(\mathbb{R})$ to $P_{\beta-\frac{1}{2}}$, and boundary atoms from critical-line zeros appear in the limit through the argument principle on the boundary. Passing to the limit in (4) yields the stated distributional identity for $-w'$ on I. □

Lemma 16 (Balayage density and consequence for Q)**.** *If there exists at least one off–critical zero $\rho = \beta + i\gamma$ of ξ with $\beta > \frac{1}{2}$, then the boundary balayage measure μ from Theorem 15 has an a.e. density $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ of the form*

$$f(t) \;=\; \sum_{\substack{\rho=\beta+i\gamma\\ \beta>1/2}} c_\rho\, P_{\beta-1/2}(t-\gamma), \qquad P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2},$$

which is strictly positive a.e. on $\mathbb{R}$ whenever at least one off–critical zero exists. Consequently, for any measurable set $E \subset \mathbb{R}$, $\mu(E) = 0$ implies $|E| = 0$. In particular, $\mu(Q) = 0$ forces $|Q| = 0$, hence (P+)*.*

Proof. For each finite subset of zeros $\mathcal{Z} \subset \{\rho : \Re\rho > 1/2\}$ the partial density $f_{\mathcal{Z}}(t) := \sum_{\rho\in\mathcal{Z}} c_\rho\, P_{\beta-1/2}(t-\gamma)$ is continuous and strictly positive for all t because each Poisson kernel is strictly positive on $\mathbb{R}$. The phase–velocity formula and the Carleson energy finiteness imply the balayage of zeros on any Whitney box is finite, so the monotone limit of the partial sums converges in L^1_{loc} to an a.e. finite function $f \ge 0$. Since the pointwise limit of strictly positive functions is nonnegative and cannot vanish on a set of positive measure unless all coefficients vanish, we obtain $f > 0$ a.e. whenever at least one off–critical zero exists. Moreover, by positivity and monotone convergence, $\mu(E) = \int_E f\,dt = 0$ forces $f = 0$ a.e. on E, whence $|E| = 0$. □

Certificate ⇒ (P+): narrative. The outer, boundary phase–velocity identity shows that $\int \varphi_{L,t_0}(-w')$ is the mass picked up by φ_{L,t_0} against a positive measure supported on off–critical zeros (with atoms on the line if they occur). The left plateau inequality therefore lower-bounds it by $c_0(\psi)\,\nu(Q(I))$, where ν is the defect measure on Ω (see Notation and conventions) and $Q(I)$ is the Carleson box. The CR–Green pairing controls the same integral from above by box energy, and the Carleson bound is uniform on Whitney boxes. This yields a Whitney–uniform *local* phase-drop bound $\int_I(-w') \le \pi\,\Upsilon_{\mathrm{Whit}}(c)$ with $\Upsilon_{\mathrm{Whit}}(c) < \frac{1}{2}$ for suitably small c (Lemma 18). The remaining upgrade from Whitney-local control to a global a.e. boundary wedge (P+) after a single rotation is a separate local-to-global step; see Remark 45.

Lemma 17 (Quantitative wedge criterion)**.** *Let $w \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ be a boundary phase function. For a measurable interval $I \subset \mathbb{R}$, write*

$$\operatorname{osc}_I w \;:=\; \operatorname*{ess\,sup}_I w \;-\; \operatorname*{ess\,inf}_I w$$

for the essential oscillation (with respect to Lebesgue measure).

1. ***Local-to-global from a centered exhaustion.*** *If there is a $D \ge 0$ such that*

$$\operatorname{osc}_{[-N,N]} w \;\le\; D \qquad \text{for every } N \ge 1,$$

then there exists a constant $c \in \mathbb{R}$ such that $|w(t) - c| \le D$ for a.e. $t \in \mathbb{R}$.

2. ***Windowed phase-mass ⇒ oscillation on an interval.*** *Assume $-w'$ is a positive Radon measure on $\mathbb{R}$ (in the sense of distributions). If $I = [a, b]$ and $\psi \geq \mathbf{1}_I$ is a nonnegative test function, then*
$$\int_I (-w') \;\leq\; \int_{\mathbb{R}} \psi\,(-w'),$$
and the phase drop (hence essential oscillation) on I is bounded by the left-hand side. In particular, if for some $\Upsilon \geq 0$ one has $\int_{\mathbb{R}} \psi\,(-w') \leq \pi\,\Upsilon$, then $\operatorname{osc}_I w \leq \pi\,\Upsilon$.

Proof. (1) For $N \geq 1$ set $a_N := \operatorname{ess\,inf}_{[-N,N]} w$ and $b_N := \operatorname{ess\,sup}_{[-N,N]} w$. Then a_N is nonincreasing, b_N is nondecreasing, and $b_N - a_N \leq D$ by hypothesis. Let
$$a_\infty := \lim_{N\to\infty} a_N \in [-\infty, \infty), \qquad b_\infty := \lim_{N\to\infty} b_N \in (-\infty, \infty].$$
Then $b_\infty - a_\infty \leq D$ and for each N we have $a_\infty \leq a_N \leq w(t) \leq b_N \leq b_\infty$ for a.e. $t \in [-N, N]$, hence for a.e. $t \in \mathbb{R}$. Choosing $c := (a_\infty + b_\infty)/2$ gives $|w(t) - c| \leq D$ a.e.

(2) The first inequality is immediate from $\psi \geq \mathbf{1}_I$ and positivity of the measure $-w'$. Since $-w'$ is the (distributional) derivative of a locally BV representative of w, its mass on I bounds the phase drop across I, which in turn bounds the essential oscillation on I. (See, e.g., [1, Ch. 3] for BV representatives and the identification of distributional derivatives with measures.) □

Lemma 18 (Whitney–uniform wedge)**.** *Fix the Whitney schedule and clip by $L_\star$: set $L_\star := c/\log 2$ and henceforth*
$$L(T) \;:=\; \min\Big\{\frac{c}{\log\langle T\rangle},\; L_\star\Big\}.$$
Then for every Whitney interval $I = [t_0 - L, t_0 + L]$ (so $L \leq L_\star$), with the printed flat–top window $\psi_{L,t_0}(t) = \psi((t - t_0)/L)$ one has
$$\int_I (-w')\,dt \;\leq\; \int_{\mathbb{R}} \psi_{L,t_0}(t)\,(-w'(t))\,dt \;\leq\; C(\psi)\,\sqrt{C_{\rm box}^{(\zeta)}}\,L_\star^{1/2} \;:=\; \pi\,\Upsilon_{\rm Whit}(c),$$
where $C(\psi)$ is the CR–Green window constant and $\Upsilon_{\rm Whit}(c)$ depends only on c, ψ and the fixed aperture. Choosing $c > 0$ sufficiently small so that $\Upsilon_{\rm Whit}(c) < \frac{1}{2}$ yields the Whitney-local phase-drop bound $\int_I (-w') \leq \pi\,\Upsilon_{\rm Whit}(c)$ on every Whitney interval. Promoting this Whitney-local bound to a global *a.e. boundary wedge* (P+) *requires an additional local-to-global step; see Remark 45.*

Proof. Since $-w'$ is a positive boundary distribution and $\psi_{L,t_0} \geq \mathbf{1}_I$ (because $\psi \equiv 1$ on $[-1, 1]$), we have
$$\int_I (-w') \;\leq\; \int_{\mathbb{R}} \psi_{L,t_0}\,(-w').$$
By Lemma 38,
$$\int_{\mathbb{R}} \psi_{L,t_0}\,(-w') \;\leq\; C(\psi)\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}.$$
Using the box constant $C_{\rm box}^{(\zeta)} = \sup_I |I|^{-1} \iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma$ and $|I| = 2L \leq 2L_\star$, we obtain
$$\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2} \;\leq\; \sqrt{C_{\rm box}^{(\zeta)}\,|I|} \;\leq\; \sqrt{2}\,\sqrt{C_{\rm box}^{(\zeta)}}\,L_\star^{1/2},$$
and we absorb the harmless factor $\sqrt{2}$ into the definition of $\Upsilon_{\rm Whit}(c)$. □

Clarification. The certificate yields the Whitney–uniform phase-mass bound $\int_I (-w') \leq \pi\,\Upsilon_{\rm Whit}(c)$ with $\Upsilon_{\rm Whit}(c) < \frac{1}{2}$ (Lemma 18), obtained solely from the local CR–Green pairing controlled by $C_{\rm box}^{(\zeta)}$; the remaining promotion to a global a.e. wedge after a single rotation is isolated in Remark 45.

Window constant. Set once and for all the window constant

$$C(\psi) \;:=\; C_{\mathrm{rem}}(\alpha,\psi)\,\mathcal{A}(\psi),$$

where $\mathcal{A}(\psi)$ is the fixed Poisson energy of the window and $C_{\mathrm{rem}}(\alpha,\psi)$ is the side/top remainder factor from Corollary 46. Then $C(\psi)$ is independent of L and t_0 and will be used uniformly below.

Proposition 19 (HS→$\det_2$ continuity)**.** *Let A_N, A be analytic $\mathcal{S}_2$-valued maps on Ω with $A_N \to A$ in the Hilbert–Schmidt norm uniformly on compact subsets of Ω. Then $\det_2(I - A_N) \to \det_2(I - A)$ locally uniformly on Ω.*

Proof. Fix a compact $K \Subset \Omega$. By hypothesis, $\sup_{s\in K} \|A_N(s) - A(s)\|_{\mathcal{S}_2} \to 0$, and in particular $\sup_N \sup_{s\in K} \|A_N(s)\|_{\mathcal{S}_2} < \infty$. We use the standard definition of the 2–modified determinant on $\mathcal{S}_2$:

$$\det_2(I - T) \;:=\; \det\big((I - T)e^{T}\big),$$

where the Fredholm determinant on the right is defined for trace-class perturbations of the identity. Indeed, for $T \in \mathcal{S}_2$ one has

$$(I - T)e^{T} - I \;=\; -\sum_{n\geq 2} \frac{n-1}{n!}\, T^{n},$$

and the series converges absolutely in trace norm because T^n is trace class for $n \geq 2$ and $\|T^n\|_1 \leq \|T\|^{n-2}\|T^2\|_1 \leq \|T\|^{n}_{\mathcal{S}_2}$. In particular, on any $\mathcal{S}_2$-ball $\{\|T\|_{\mathcal{S}_2} \leq M\}$, the map

$$T \;\longmapsto\; (I - T)e^{T} - I$$

is Lipschitz from $\mathcal{S}_2$ to trace class: writing the series termwise and using $T^n - S^n = \sum_{k=0}^{n-1} T^k(T - S)S^{n-1-k}$ together with $\|XY\|_1 \leq \|X\|_2\|Y\|_2$ and $\|T\| \leq \|T\|_{\mathcal{S}_2}$ gives

$$\|(I - T)e^{T} - (I - S)e^{S}\|_1 \;\leq\; C(M)\,\|T - S\|_{\mathcal{S}_2}.$$

Since the Fredholm determinant on trace-class perturbations of the identity is defined by an absolutely convergent trace-norm series (hence is continuous in $\|\cdot\|_1$), it follows that $\det_2(I - T)$ is continuous (indeed locally Lipschitz) with respect to $\|\cdot\|_{\mathcal{S}_2}$. Thus

$$\sup_{s\in K} \Big|\det_2\left(I - A_N(s)\right) - \det_2\left(I - A(s)\right)\Big| \;\longrightarrow\; 0,$$

which is local-uniform convergence on K. Since K was arbitrary, the convergence is locally uniform on Ω. □

Lemma 20 (Smoothed phase–velocity calculus)**.** *Fix $\varepsilon \in (0, \frac{1}{2}]$ and set*

$$u_\varepsilon(t) := \log\Big|\det_2(I - A(\tfrac{1}{2} + \varepsilon + it))\Big| - \log\Big|\xi(\tfrac{1}{2} + \varepsilon + it)\Big|.$$

Let $\mathcal{O}_\varepsilon$ be the outer on $\{\Re s > \frac{1}{2} + \varepsilon\}$ with boundary modulus e^{u_ε} and write $F_\varepsilon := \det_2/\xi$ and $O_\varepsilon := \mathcal{O}_\varepsilon$. Then for every $\phi \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \phi(t)\Big(\Im\frac{\xi'}{\xi} - \Im\frac{\det_2{}'}{\det_2} + \mathcal{H}[u_\varepsilon']\Big)(\tfrac{1}{2} + \varepsilon + it)\,dt \;=\; \sum_{\substack{\rho = \beta + i\gamma \\ \Re\rho > \frac{1}{2} + \varepsilon}} c_\rho\,\big(P_{\beta - \frac{1}{2} - \varepsilon} * \phi\big)(\gamma) \tag{4}$$

where $P_a(x) = \frac{1}{\pi}\frac{a}{a^2 + x^2}$ is the Poisson kernel, and the coefficients $c_\rho \geq 0$ are the pole multiplicities of F_ε (equivalently the zero multiplicities of ξ) in the half-plane $\{\Re s > \frac{1}{2} + \varepsilon\}$. In particular, the right-hand side is nonnegative.

Proof. Factor $F_\varepsilon = I_\varepsilon\, O_\varepsilon$ with O_ε outer on $\{\Re s > \frac{1}{2}+\varepsilon\}$ and I_ε inner (product of half-plane Blaschke factors for poles/zeros of F_ε in the open half-plane). By Lemma 14, on the boundary line $\Re s = \frac{1}{2}+\varepsilon$ one has $\frac{d}{dt}\operatorname{Arg} O_\varepsilon = \mathcal{H}[u_\varepsilon']$ in $\mathcal{D}'(\mathbb{R})$. Each pole of F_ε at $\rho = \beta+i\gamma$ with $\beta > \frac{1}{2}$ contributes a half-plane Blaschke factor of the form $C_{\rho,\varepsilon}(s) = \big(s-\rho_\varepsilon^*\big)/(s-\rho)$ with $\rho_\varepsilon^* := 1+2\varepsilon-\overline{\rho}$ (reflection across $\Re s = \frac{1}{2}+\varepsilon$), whose boundary phase derivative is a nonnegative multiple of the Poisson kernel $P_{\beta-\frac{1}{2}-\varepsilon}(t-\gamma)$. Summing these contributions and writing $\frac{d}{dt}\operatorname{Arg} F_\varepsilon = \Im(F_\varepsilon'/F_\varepsilon) = \Im(\det_2{}'\,/\det_2) - \Im(\xi'/\xi)$ yields (4) after testing against ϕ. $\square$

2 Globalization across $Z(\xi)$ via a Schur–Herglotz pinch

This section records the Schur pinch *template*: given a domain $D \subset \Omega$ on which Θ is Schur on $D \setminus Z(\xi)$, together with non-cancellation (N2) and the right-edge normalization (N1), one rules out zeros of ξ in D. In the far-field route, we apply this with $D = \{\, \Re s > \sigma_0 \,\}$ once the Schur bound is obtained there (Corollary 109 via the arithmetic Pick certificate).

Globalization and pinch: narrative. In particular, once Corollary 109 provides Θ Schur on $D \setminus Z(\xi)$, any putative zero $\rho \in D$ forces $\Theta(\rho) = 1$ by removability, hence Θ is constant unimodular on $D \setminus Z(\xi)$ by the Maximum Modulus Principle; the normalization (N1) forces $\Theta(\sigma + it) \to \frac{1}{3}$ as $\sigma \to +\infty$, contradicting a unimodular constant. **Standing setup.** Let

$$\Omega := \{s \in \mathbb{C} :\ \Re s > \tfrac{1}{2}\}, \qquad \xi(s) = \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s).$$

Clarification. Although the factor $(s-1)$ vanishes at $s = 1$, the zeta factor has a simple pole there and the product $(s-1)\zeta(s) \to 1$. Hence ξ is entire and $\xi(1) = \frac{1}{2}\,\pi^{-1/2}\Gamma(1/2)\cdot 1 = \frac{1}{2} \neq 0$. Define

$$F(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s}{s-1}, \qquad \mathcal{J}(s) := \frac{F(s)}{\mathcal{O}_{\mathrm{can}}(s)}, \qquad G(s) := 2\,\mathcal{J}(s), \qquad \Theta(s) := \frac{G(s)-1}{G(s)+1}.$$

Here $\mathcal{O}_{\mathrm{can}}$ is the canonical outer normalizer (Definition 72); it is holomorphic and zero–free on Ω, and $\det_2(I - A)$ is holomorphic and zero–free on Ω. We record the two normalization properties actually used below:

(N1) (*Right–edge normalization*) For each fixed t (indeed uniformly on compact t–intervals), $\lim\limits_{\sigma\to+\infty} \mathcal{J}(\sigma + it) = 1$; hence $\lim\limits_{\sigma\to+\infty} \Theta(\sigma + it) = \frac{1}{3}$.

(N2) (*Non–cancellation at ξ–zeros*) For every $\rho \in \Omega$ with $\xi(\rho) = 0$,

$$\det_2(I - A(\rho)) \neq 0.$$

Thus $\mathcal{J}$ has a pole at ρ of order $\operatorname{ord}_\rho(\xi)$ (since F has a pole there and $\mathcal{O}_{\mathrm{can}}$ is zero-free).

Schur bound on the far half-plane off $Z(\xi)$. By Corollary 109, the Cayley transform is Schur on $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$:

$$|\Theta(s)| \ \le\ 1 \qquad (s \in \{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)). \tag{Schur}$$

Local pinch at a putative off–critical zero. *We use (N2) for non–cancellation at ξ–zeros and (N1) for the right–edge limit* $\Theta \to \frac{1}{3}$. Fix $\rho \in \Omega$ with $\Re\rho > \sigma_0$ and $\xi(\rho) = 0$. By (N2) the function $\mathcal{J}$ has a pole at ρ (equivalently $G = 2\mathcal{J}$ has a pole), hence

$$\Theta(s) = \frac{G(s)-1}{G(s)+1} \ \longrightarrow\ 1 \qquad (s \to \rho).$$

By (Schur), Θ is bounded by 1 on $(\Omega \setminus Z(\xi))$, so the singularity of Θ at ρ is removable (Riemann's theorem), and the holomorphic extension satisfies

$$\Theta(\rho) = 1.$$

Because Θ is holomorphic on the connected domain $\{ \Re s > \sigma_0 \} \setminus (Z(\xi) \setminus \{\rho\})$ and $|\Theta| \le 1$ there, the Maximum Modulus Principle forces Θ to be a *constant unimodular* function on that domain (it attains its supremum 1 at an interior point). By analyticity, the same constant extends throughout $\{ \Re s > \sigma_0 \} \setminus Z(\xi)$.

Lemma 21 (Connectedness and isolation). *Since $Z(\xi) \cap \Omega$ is a discrete subset (zeros are isolated), one can choose a disc $D \subset \{ \Re s > \sigma_0 \}$ centered at ρ containing no other zeros. Moreover, $\{ \Re s > \sigma_0 \} \setminus Z(\xi)$ is (path-)connected. Hence in the argument above, $\{ \Re s > \sigma_0 \} \setminus (Z(\xi) \setminus \{\rho\})$ is connected and the Maximum Modulus Principle applies on this domain.*

Proof. Since ξ is holomorphic and not identically zero on Ω, its zeros are isolated; thus $Z(\xi) \cap \Omega$ is discrete and we may choose a disc $D \subset \{ \Re s > \sigma_0 \}$ around ρ containing no other zeros. For connectedness: given $z_0, z_1 \in \{ \Re s > \sigma_0 \} \setminus Z(\xi)$, join them by a polygonal path in $\{ \Re s > \sigma_0 \}$. A compact polygonal path meets only finitely many points of the discrete set $Z(\xi) \cap \Omega$, so we can locally perturb the path in small discs around those points to avoid them. This produces a path in $\{ \Re s > \sigma_0 \} \setminus Z(\xi)$, hence $\{ \Re s > \sigma_0 \} \setminus Z(\xi)$ is path-connected. The same argument applies to $\{ \Re s > \sigma_0 \} \setminus (Z(\xi) \setminus \{\rho\})$. ☐

Contradiction with right–edge normalization. By (N1), $\Theta(\sigma + it) \to \frac{1}{3}$ as $\sigma \to +\infty$ (uniformly for t in compact intervals). A constant unimodular function cannot have such a limit. Contradiction. **Conclusion of the pinch.** No $\rho \in \Omega$ with $\Re\rho > \sigma_0$ and $\xi(\rho) = 0$ can exist. **Connective summary (secondary BRF/pinch route).** This section records the Schur pinch argument: the Schur bound on $\{ \Re s > \sigma_0 \} \setminus Z(\xi)$ comes from the arithmetic Pick-matrix certification (Theorem 107 and Corollary 109), and the pinch uses only (N1)–(N2). A boundary-wedge route via (P+) is optional and recorded elsewhere for comparison, but is not required for the pinch. **Normalization at infinity (used in (N1)).** We record explicit bounds ensuring $\Theta(\sigma + it) \to \frac{1}{3}$ uniformly for t in compact t-intervals as $\sigma \to +\infty$.

- Zeta limit: For $\sigma \ge 2$ and all $t \in \mathbb{R}$, $|\zeta(\sigma + it) - 1| \le 2^{1-\sigma}$, hence $|\zeta(\sigma + it)| \to 1$ uniformly for t in compact intervals as $\sigma \to +\infty$. Also $(\sigma + it - 1)/(\sigma + it) \to 1$ uniformly on compact t-intervals.

- Det_2 limit: For $\sigma \ge 1$, $\|A(\sigma + it)\| \le 2^{-\sigma} \le \frac{1}{2}$. By the product representation in Lemma 26 and since $\sum_p p^{-2\sigma} \to 0$ as $\sigma \to \infty$, one has $|\det_2(I - A(\sigma + it)) - 1| \le C \sum_p p^{-2\sigma} \to 0$ (uniformly for t in compact intervals).

- Canonical outer normalizer: $\mathcal{O}_{\mathrm{can}}$ is an outer function on Ω with boundary modulus $|\mathcal{O}_{\mathrm{can}}(\frac{1}{2} + it)| = |F(\frac{1}{2} + it)|$ a.e. (Definition 72), uniquely determined up to a unimodular constant. We fix that constant so that $\mathcal{O}_{\mathrm{can}}(\sigma + it) \to 1$ as $\sigma \to +\infty$ (uniformly for t in compact intervals), which is the standard right-edge normalization for outers on Ω.

Combining, $F(\sigma + it) \to 1$ and $\mathcal{O}_{\mathrm{can}}(\sigma + it) \to 1$ uniformly for t in compact intervals, hence $\mathcal{J}(\sigma + it) = F/\mathcal{O}_{\mathrm{can}} \to 1$ and therefore $\Theta(\sigma + it) = (2\mathcal{J} - 1)/(2\mathcal{J} + 1) \to \frac{1}{3}$.

Lemma 22 (Carleson box energy: stable sum bound)**.** *For harmonic potentials U_1, U_2 on Ω, one has*

$$\sqrt{C_{\mathrm{box}}(U_1+U_2)} \;\le\; \sqrt{C_{\mathrm{box}}(U_1)} \;+\; \sqrt{C_{\mathrm{box}}(U_2)}.$$

Proof. Write $\mu_j := |\nabla U_j|^2\,\sigma\,dt\,d\sigma$ and $\mu_{12} := |\nabla(U_1{+}U_2)|^2\,\sigma\,dt\,d\sigma$. For any Carleson box B, by Cauchy–Schwarz,

$$\int_B |\nabla(U_1+U_2)|^2\,\sigma\,dt\,d\sigma \;\le\; \Big(\sqrt{\int_B |\nabla U_1|^2\,\sigma} \;+\; \sqrt{\int_B |\nabla U_2|^2\,\sigma}\Big)^2.$$

Taking supremum over Carleson boxes B and dividing by $|I_B|$ yields the claimed inequality. $\square$

Corollary 23 (Local Carleson energy for U_ξ on a fixed interval)**.** *For each compact interval $I \Subset \mathbb{R}$ there exists a finite constant $C_{\xi,I} < \infty$ such that*

$$\iint_{Q(I)} |\nabla U_\xi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \;\le\; C_{\xi,I}\,|I|.$$

In particular, on Whitney intervals $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ one may take $C_{\xi,I} = C_\xi$ from Lemma 34.

Proof. (*Sketch.*) Fix $I \Subset \mathbb{R}$. Cover I by finitely many Whitney intervals $I_j = [T_j - L(T_j), T_j + L(T_j)]$ with bounded overlap (since I is compact and $L(\cdot)$ is bounded below on I), so that $Q(I) \subset \bigcup_j Q(\alpha I_j)$. Apply Lemma 34 on each $Q(\alpha I_j)$ and sum; the overlap and the finiteness of the cover yield the stated bound with a constant depending on I (through the finite cover) and on the fixed aperture. $\square$

Lemma 24 (L^1-tested control for $\partial_\sigma \Re \log \xi$)**.** *For each compact $I \Subset \mathbb{R}$ there exists $C_I' < \infty$ such that for all $0 < \sigma \le \varepsilon_0$ and all $\phi \in C_c^2(I)$,*

$$\Big|\int_I \phi(t)\,\partial_\sigma \Re \log \xi(\tfrac{1}{2}+\sigma+it)\,dt\Big| \;\le\; C_I'\,\|\phi\|_{H^1(I)}.$$

Proof of Lemma 24. Let $I \Subset \mathbb{R}$ and $\phi \in C_c^2(I)$. Let V be the Poisson extension of ϕ on a fixed dilation $Q(\alpha I)$. Green's identity together with Cauchy–Riemann for $U_\xi = \Re \log \xi$ gives

$$\int_I \phi(t)\,\partial_\sigma \Re \log \xi(\tfrac{1}{2}+\sigma+it)\,dt \;=\; \iint_{Q(\alpha I)} \nabla U_\xi \cdot \nabla V\,dt\,d\sigma.$$

This is exactly the standard Carleson embedding / H^1–BMO pairing estimate for Poisson extensions (see Garnett [6, Thm. VI.1.1] or Stein [23, Ch. IV]): if $\lambda := |\nabla U_\xi|^2\,\sigma\,dt\,d\sigma$ is Carleson on boxes above I, then

$$\Big|\iint_{Q(\alpha I)} \nabla U_\xi \cdot \nabla V\,dt\,d\sigma\Big| \;\lesssim_{I,\alpha}\; \|\phi\|_{H^1(I)}.$$

Using the local Carleson bound from Corollary 23 gives the asserted constant $C_I' < \infty$ depending only on I (and the fixed aperture). $\square$

Corollary 25 (Conservative closure inequalities)**.** *Let K_0 be the arithmetic tail box-energy constant (Lemma 32) and let K_ξ be the neutralized ξ box-energy constant (Lemma 34). Define*

$$C_{\mathrm{box}}^{(\zeta)} \;:=\; K_0 + K_\xi.$$

Then one has the conservative subadditivity bound

$$\sqrt{C_{\mathrm{box}}^{(\zeta)}} \;\le\; \sqrt{K_0} + \sqrt{K_\xi}.$$

Moreover, for the printed window ψ one has the structural mean-oscillation bound

$$M_\psi \;\le\; \frac{4}{\pi}\, C_\psi^{(H^1)} \sqrt{C_{\mathrm{box}}^{(\zeta)}}.$$

Proof. The inequality $\sqrt{C_{\mathrm{box}}^{(\zeta)}} \le \sqrt{K_0} + \sqrt{K_\xi}$ is Lemma 22 applied to the decomposition of the paired potential into the arithmetic tail and the neutralized ξ-part (cf. Lemma 41). The bound on M_ψ follows from the H^1–BMO/Carleson embedding estimate (Lemma 54) together with the embedding normalization $C_{\mathrm{CE}}(\alpha) = 1$ (Lemma 120). □

Proof of (N2) (non–cancellation at ξ–zeros). For $s = \sigma + it$ with $\sigma > \frac{1}{2}$, define the diagonal operator $A(s)e_p = p^{-s}e_p$ on $\ell^2(\mathbb{P})$. Then $\|A(s)\| = 2^{-\sigma} < 1$ and $\|A(s)\|_{\mathrm{HS}}^2 = \sum_p p^{-2\sigma} < \infty$, so $A(s)$ is Hilbert–Schmidt. The 2–modified determinant for diagonal $A(s)$ is

$$\det{}_2\big(I - A(s)\big) \;=\; \prod_{p\in\mathbb{P}} (1 - p^{-s})\, e^{p^{-s}},$$

which converges absolutely and is nonzero because each factor is nonzero. Moreover, $I - A(s)$ is invertible with $\|(I - A(s))^{-1}\| \le (1 - 2^{-\sigma})^{-1}$ since $|1 - p^{-s}| \ge 1 - 2^{-\sigma} > 0$. Finally, the outer normalizer has the form $\mathcal{O}(s) = \exp H(s)$ with H analytic on Ω, hence $\mathcal{O}$ is zero–free on Ω. Thus if $\rho \in \Omega$ with $\xi(\rho) = 0$, then $\det_2(I - A(\rho)) \ne 0$ and $\mathcal{O}(\rho) \ne 0$, i.e. no cancellation can occur at ρ. Local-uniform analyticity on Ω follows from HS$\to$ $\det_2$ continuity (Proposition 19). which converges absolutely and is nonzero because each factor is nonzero. Moreover, $I - A(s)$ is invertible with $\|(I - A(s))^{-1}\| \le (1 - 2^{-\sigma})^{-1}$ since $|1 - p^{-s}| \ge 1 - 2^{-\sigma} > 0$. Finally, the canonical outer normalizer $\mathcal{O}_{\mathrm{can}}$ is an outer function on Ω (Definition 72), hence is zero–free on Ω. Thus if $\rho \in \Omega$ with $\xi(\rho) = 0$, then $\det_2(I - A(\rho)) \ne 0$ and $\mathcal{O}_{\mathrm{can}}(\rho) \ne 0$, i.e. no cancellation can occur at ρ. Local-uniform analyticity on Ω follows from HS$\to$ $\det_2$ continuity (Proposition 19).

Lemma 26 (Diagonal HS determinant is analytic and nonzero)**.** *For $s = \sigma + it$ with $\sigma > \frac{1}{2}$, the diagonal operator $A(s)e_p = p^{-s}e_p$ satisfies*

$$\sup_p |p^{-s}| = 2^{-\sigma} < 1, \qquad \sum_p |p^{-s}|^2 = \sum_p p^{-2\sigma} < \infty.$$

Hence $A(s) \in \mathrm{HS}$, $I - A(s)$ is invertible, and

$$\det{}_2\left(I - A(s)\right) = \prod_p (1 - p^{-s})\, e^{p^{-s}}$$

is analytic and nonzero on $\{\Re s > \frac{1}{2}\}$.

Proof. Immediate from the displayed bounds; invertibility follows since $|1 - p^{-s}| \ge 1 - 2^{-\sigma} > 0$, and the product defining $\det_2$ converges absolutely with nonzero factors. □

Normalization and finite port (eliminating C_P and C_Γ). We record the implementation details that ensure the product certificate contains no prime budget and no Archimedean term.

Lemma 27 (ζ–normalized outer and compensator)**.** *Define the outer $\mathcal{O}_\zeta$ on Ω with boundary modulus $\big|\det_2(I-A)/\zeta\big|$ and set*

$$J_\zeta(s) \ := \ \frac{\det_2(I-A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)} \cdot B(s), \qquad B(s) := \frac{s}{s-1}.$$

On $\Re s = \frac{1}{2}$ one has $|B| = 1$. The phase–velocity identity of Theorem 15 holds for J_ζ with the same Poisson/zero right-hand side. In particular, no separate Archimedean term enters the inequality used by the certificate.

Proof. Set $X := \xi$ and $Z := \zeta$, and let G denote the archimedean factor linking them,

$$X(s) \ = \ \tfrac{1}{2}s(1-s)\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,Z(s) \ =: \ G(s)\,Z(s).$$

Define $\mathcal{O}_X$ (resp. $\mathcal{O}_Z$) to be the outer on Ω with boundary modulus $\big|\det_2(I-A)/X\big|$ (resp. $\big|\det_2(I-A)/Z\big|$). Then, by construction,

$$\Big|\frac{\det_2(I-A)}{\mathcal{O}_X\,X}\Big| \equiv 1 \equiv \Big|\frac{\det_2(I-A)}{\mathcal{O}_Z\,Z}\Big| \quad \text{a.e. on } \{\Re s = \tfrac{1}{2}\}.$$

Consequently the phase–velocity identity (Theorem 15) applies to either unimodular ratio. Writing

$$\log\frac{\det_2(I-A)}{\mathcal{O}_X\,X} \ = \ \log\frac{\det_2(I-A)}{\mathcal{O}_Z\,Z} \ - \ \log\frac{\mathcal{O}_X}{\mathcal{O}_Z} \ - \ \log G,$$

and differentiating in σ on the boundary, the two outer terms contribute zero to the boundary phase derivative (by unimodularity and the outer/Poisson representation). The remaining difference is $-\partial_\sigma \Im \log G$.

On $\Re s = \frac{1}{2}$ we have $|O_X/O_Z| = |Z/X| = |1/G|$, hence (a.e.) $\Re \log(O_X/O_Z) = -\Re \log G$. Since both $\log(O_X/O_Z)$ and $\log G$ are analytic on Ω, Cauchy–Riemann gives on the boundary line (in $\mathcal{D}'(\mathbb{R})$)

$$\partial_\sigma \Im \log\Big(\frac{O_X}{O_Z}\Big) \ = \ -\partial_t \Re \log\Big(\frac{O_X}{O_Z}\Big) \ = \ -\partial_t(-\Re \log G) \ = \ -\partial_\sigma \Im \log G.$$

Compensating the simple zero at $s = 1$ of $\det_2(I-A)/\zeta$ by the half–plane compensator

$$B(s) \ = \ \frac{s}{s-1} \qquad (|B| \equiv 1 \text{ on } \Re s = \tfrac{1}{2})$$

accounts for the inner contribution at $s = 1$. Therefore, on the boundary,

$$\partial_\sigma \Im \log\Big(\frac{\det_2(I-A)}{\mathcal{O}_Z\,Z} \cdot B\Big) \ = \ \partial_\sigma \Im \log \frac{\det_2(I-A)}{\mathcal{O}_X\,X},$$

and the quantitative phase–velocity identity holds in the same form for $J_\zeta = (\det_2/(\mathcal{O}_\zeta\,\zeta))\,B$ as for $\mathcal{J} = \det_2/(\mathcal{O}\,\xi)$. In particular, no Archimedean term enters the certificate. ☐

Corollary 28 (No C_P/C_Γ in the certificate)**.** *With J_ζ and $\widehat{J}$ as above, the active CR–Green route uses $c_0(\psi)$ and the CR–Green constant $C(\psi)$ together with the box–energy constant $C_{\rm box}^{(\zeta)}$. In particular, $C_P = 0$ and $C_\Gamma = 0$ on the RHS; $C_H(\psi)$ and M_ψ are retained only as auxiliary/readability bounds.*

Proof. By construction of the ζ–normalized gauge and the compensator B (Lemma 27), the Archimedean factor contributes no boundary phase term and the simple pole/zero bookkeeping at $s=1$ is absorbed into B with $|B|=1$ on $\Re s=\frac{1}{2}$. Thus the product certificate has no C_Γ term and no separate prime-budget term C_P on the right-hand side; the remaining inputs are $c_0(\psi)$, the CR–Green constant $C(\psi)$, and the box-energy constant $C_{\mathrm{box}}^{(\zeta)}$. □

Active route. Throughout we use the ζ-normalized boundary gauge with the Blaschke compensator; the product certificate uses $c_0(\psi)$ and the CR–Green constant $C(\psi)$ together with $C_{\mathrm{box}}^{(\zeta)}$ (no C_P, no C_Γ). These inputs yield Whitney-local smallness $\Upsilon_{\mathrm{Whit}}(c)<\frac{1}{2}$ (Lemma 18); the remaining promotion to a global a.e. boundary wedge (P+) after a single rotation is isolated in Remark 45.

Lemma 29 (Derivative envelope for the printed window)**.** *Let ψ be the even C^∞ flat–top window from the "Printed window" paragraph (equal to* 1 *on* $[-1,1]$*, supported in* $[-2,2]$*, with monotone ramps on* $[-2,-1]$ *and* $[1,2]$*), and* $\varphi_L(t):=L^{-1}\psi((t-T)/L)$*. Then, for every* $L>0$*,*

$$\|(\mathcal{H}[\varphi_L])'\|_{L^\infty(\mathbb{R})}\ \le\ \frac{C_H(\psi)}{L}\qquad \textit{with}\qquad C_H(\psi)\ \le\ \frac{2}{\pi}\ <\ 0.65.$$

Proof. Step 1 (Scaling). By the standard scale/translation identity (recorded in the manuscript),

$$\mathcal{H}[\varphi_L](t)=H_\psi\Big(\frac{t-T}{L}\Big),\qquad H_\psi(x):=\frac{1}{\pi}\,\mathrm{p.v.}\!\int_{\mathbb{R}}\frac{\psi(y)}{x-y}\,dy,$$

we get

$$(\mathcal{H}[\varphi_L])'(t)=\frac{1}{L}\,H_\psi'\Big(\frac{t-T}{L}\Big)\quad\Longrightarrow\quad \|(\mathcal{H}[\varphi_L])'\|_\infty=\frac{1}{L}\,\|H_\psi'\|_\infty.$$

Thus it suffices to bound $\|H_\psi'\|_\infty$.

Step 2 (Structure and signs). Since $\psi'\equiv 0$ on $(-1,1)$ and the ramps are monotone,

$$\psi'(y)\ge 0 \text{ on } [-2,-1],\qquad \psi'(y)\le 0 \text{ on } [1,2],\qquad \int_{-2}^{-1}\psi'(y)\,dy=1=-\int_1^2\psi'(y)\,dy.$$

In distributions, $(H_\psi)'=\mathcal{H}[\psi']$, so for every $x\in\mathbb{R}$

$$H_\psi'(x)=\frac{1}{\pi}\,\mathrm{p.v.}\!\int_{-2}^{-1}\frac{\psi'(y)}{x-y}\,dy\ +\ \frac{1}{\pi}\,\mathrm{p.v.}\!\int_1^2\frac{\psi'(y)}{x-y}\,dy.$$

Step 3 (Worst case occurs between the ramps). Fix $x\in(-1,1)$. On $y\in[-2,-1]$ the kernel $y\mapsto 1/(x-y)$ is positive and strictly increasing; on $y\in[1,2]$ the kernel is negative and strictly decreasing. Since the ramp densities are monotone and have unit mass in absolute value, the rearrangement/endpoint principle (maximize a monotone–kernel integral by concentrating mass at an endpoint) gives the pointwise bound

$$\Big|\mathrm{p.v.}\!\int_{-2}^{-1}\frac{\psi'(y)}{x-y}\,dy\Big|\le\frac{1}{1+x},\qquad \Big|\mathrm{p.v.}\!\int_1^2\frac{\psi'(y)}{x-y}\,dy\Big|\le\frac{1}{1-x}.$$

Therefore, for every $x\in(-1,1)$,

$$|H_\psi'(x)|\ \le\ \frac{1}{\pi}\Big(\frac{1}{1+x}+\frac{1}{1-x}\Big)\ \le\ \frac{2}{\pi}\,\frac{1}{1-x^2}\ \le\ \frac{2}{\pi},$$

with the maximum at $x=0$. *Step 4 (Outside the plateau).* For $x\notin[-1,1]$ the two ramp contributions have opposite signs but larger denominators, hence smaller magnitude. More precisely, for $x > 1$, the left–ramp integral is a principal value on $[-2, -1]$ against a C^∞ density that vanishes at the endpoints; the standard C^1–vanishing at $y = -2, -1$ eliminates the endpoint singularity and keeps the PV finite and strictly smaller than its in–plateau counterpart (a short integration–by–parts argument on the left interval makes this explicit). By evenness, the same holds for $x < -1$. Consequently,

$$\sup_{x\in\mathbb{R}} |H'_\psi(x)| = \sup_{x\in(-1,1)} |H'_\psi(x)| \;\le\; \frac{2}{\pi}.$$

Putting Steps 1–4 together,

$$\big\|(\mathcal{H}[\varphi_L])'\big\|_\infty \;=\; \frac{1}{L}\,\|H'_\psi\|_\infty \;\le\; \frac{1}{L}\cdot\frac{2}{\pi}.$$

Hence we can take $C_H(\psi) \le 2/\pi < 0.65$. □

Corollary 30 (Boundary-uniform smoothed control)**.** *Let $I \Subset \mathbb{R}$, $\varepsilon_0 \in (0, \frac{1}{2}]$, and $\varphi \in C_c^2(I)$. Then, uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon_0]$,*

$$\Big|\int_{\mathbb{R}} \varphi(t)\,\partial_\sigma \Re \log \det{}_2\left(I - A(\sigma + it)\right) dt\Big| \;\le\; C_*\,\|\varphi''\|_{L^1(I)}.$$

In particular, the bound remains valid in the boundary limit $\sigma \downarrow \frac{1}{2}$ in the sense of distributions.

Proof. This is exactly the tested bound from Lemma 10 (uniform in $\sigma \in (0, \varepsilon_0]$ after the shift $\sigma \mapsto \frac{1}{2}+\sigma$). Since the right-hand side is uniform in σ, the family of distributions $\sigma \mapsto \partial_\sigma \Re \log \det_2(I - A(\frac{1}{2} + \sigma + it))$ is bounded in $\mathcal{D}'(I)$ and the estimate persists in the boundary limit $\sigma \downarrow \frac{1}{2}$ when tested against φ. □

Smoothed Cauchy and outer limit (A2)

Proposition 31 (Outer normalization: existence, boundary a.e. modulus, and limit)**.** *There exist outer functions $\mathcal{O}_\varepsilon$ on $\{\Re s > \frac{1}{2} + \varepsilon\}$ with a.e. boundary modulus*

$$\left|\mathcal{O}_\varepsilon(\tfrac{1}{2} + \varepsilon + it)\right| \;=\; \exp\left(u_\varepsilon(t)\right),$$

and $\mathcal{O}_\varepsilon \to \mathcal{O}$ locally uniformly on Ω as $\varepsilon \downarrow 0$, where $\mathcal{O}$ has boundary modulus $\exp u(t)$*. (Standard Poisson–outer representation; see, e.g., [3, Ch. 2] and [10, Ch. 2].) Consequently the outer-normalized ratio $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ has a.e. boundary values on $\Re s = \frac{1}{2}$ with $|\mathcal{J}(\frac{1}{2} + it)| = 1$.*

Proof. Existence of each outer $\mathcal{O}_\varepsilon$ with the stated boundary modulus is standard. The Carleson-energy control for the relevant harmonic log-modulus on Whitney boxes implies the existence of a boundary trace $u \in \mathrm{BMO}(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R})$ and convergence $u_\varepsilon \to u$ in L^1_{loc} (Lemma 11). The Poisson/outer representation then gives local-uniform convergence $\mathcal{O}_\varepsilon \to \mathcal{O}$ on Ω and the unimodularity $|\mathcal{J}(\frac{1}{2} + it)| = 1$ a.e. □

Carleson energy and boundary BMO (unconditional)

We record a direct Carleson–energy route to boundary BMO for the limit $u(t) = \lim_{\varepsilon\downarrow 0} u_\varepsilon(t)$.

Lemma 32 (Arithmetic Carleson energy)**.** *Let*

$$U_{\det_2}(\sigma,t) \;:=\; \sum_p \sum_{k\geq 2} \frac{(\log p)\,p^{-k/2}}{k\log p}\, e^{-k\log p\,\sigma}\, \cos\big(k\log p\, t\big), \qquad \sigma>0.$$

Then for every interval $I\subset\mathbb{R}$ with Carleson box $Q(I):=I\times(0,|I|]$

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2\,\sigma\,dt\,d\sigma \;\leq\; \frac{|I|}{4}\sum_p\sum_{k\geq 2}\frac{p^{-k}}{k^2} \;=:\; K_0\,|I|, \qquad K_0 := \frac{1}{4}\sum_p\sum_{k\geq 2}\frac{p^{-k}}{k^2} < \infty.$$

Proof. For a single mode $b\,e^{-\omega\sigma}\cos(\omega t)$ one has $|\nabla|^2 = b^2\omega^2 e^{-2\omega\sigma}$, hence

$$\int_0^{|I|}\int_I |\nabla|^2\,\sigma\,dt\,d\sigma \;\leq\; |I|\cdot\sup_{\omega>0}\int_0^{|I|}\sigma\,\omega^2 e^{-2\omega\sigma}d\sigma\cdot b^2 \;\leq\; \tfrac{1}{4}\,|I|\,b^2.$$

With $b=(\log p)\,p^{-k/2}/(k\log p)$ and $\omega = k\log p$, summing over (p,k) gives the claim and the finiteness of K_0. ☐

Whitney scale and short–interval zeros. Throughout we use the Whitney schedule clipped at $L_\star$:

$$L \;=\; L(T) \;:=\; \frac{c}{\log\langle T\rangle} \;\leq\; \frac{1}{\log\langle T\rangle}, \qquad \langle T\rangle := \sqrt{1+T^2},$$

for a fixed absolute $c\in(0,1]$; all boxes are $Q(\alpha I)$ with a uniform $\alpha\in[1,2]$. We work on Whitney boxes $Q(I)$ with

$$L = L(T) := \min\Big\{\frac{c}{\log\langle T\rangle},\; L_\star\Big\}, \qquad \langle T\rangle := \sqrt{1+T^2}, \quad c>0 \text{ fixed}.$$

There exist absolute $A_0, A_1>0$ such that for $T\geq 2$ and $0<H\leq 1$,

$$N(T;H) \;:=\; \#\{\rho=\beta+i\gamma:\; \gamma\in[T,T+H]\} \;\leq\; A_0 \;+\; A_1\,H\log\langle T\rangle.$$

Lemma 33 (Annular Poisson–balayage L^2 bound)**.** *Let $I=[T-L,T+L]$, $Q_\alpha(I)=I\times(0,\alpha L]$, and fix $k\geq 1$. For $\mathcal{A}_k := \{\rho=\beta+i\gamma:\; 2^kL<|T-\gamma|\leq 2^{k+1}L\}$ set*

$$V_k(\sigma,t) := \sum_{\rho\in\mathcal{A}_k}\frac{\sigma}{(t-\gamma)^2+\sigma^2}.$$

Then

$$\iint_{Q_\alpha(I)} V_k(\sigma,t)^2\,\sigma\,dt\,d\sigma \;\ll_\alpha\; |I|\,4^{-k}\,\nu_k,$$

where $\nu_k := \#\mathcal{A}_k$, and the implicit constant depends only on α.

Proof. Write $K_\sigma(x) := \sigma/(x^2+\sigma^2)$ and $V_k = \sum_{\rho\in\mathcal{A}_k}K_\sigma(\cdot-\gamma)$. For any finite index set $\mathcal{J}$,

$$V_k^2 \;\leq\; \sum_{j\in\mathcal{J}} K_\sigma(\cdot-\gamma_j)^2 \;+\; 2\sum_{i<j}K_\sigma(\cdot-\gamma_i)K_\sigma(\cdot-\gamma_j).$$

Integrate over $t\in I$ first. For the diagonal terms, using $|t-\gamma|\geq 2^kL - L \geq 2^{k-1}L$ for $t\in I$ and $k\geq 1$,

$$\int_I K_\sigma(t-\gamma)^2\,dt \;=\; \int_I \frac{\sigma^2}{\big((t-\gamma)^2+\sigma^2\big)^2}\,dt \;\leq\; \frac{\sigma}{(2^{k-1}L)^2}\int_I\frac{\sigma}{(t-\gamma)^2+\sigma^2}\,dt \;\leq\; \frac{\pi\,\sigma}{(2^{k-1}L)^2}.$$

Multiplying by the area weight σ and integrating $\sigma \in (0, \alpha L]$ gives

$$\int_0^{\alpha L}\Big(\int_I K_\sigma(t-\gamma)^2\,dt\Big)\,\sigma\,d\sigma \;\le\; \frac{\pi}{(2^{k-1}L)^2}\int_0^{\alpha L}\sigma^2 d\sigma \;=\; \frac{\pi\,\alpha^3}{3}\,\frac{L}{4^{k-1}} \;\le\; \frac{C_{\mathrm{diag}}(\alpha)}{4^k}\,|I|,$$

with $C_{\mathrm{diag}}(\alpha) := \frac{8\pi\alpha^3}{3}$ (using $|I| = 2L$). Summing over ν_k choices of γ contributes a factor ν_k.

For the off-diagonal terms, for $i \ne j$ one has on I that $K_\sigma(t-\gamma_j) \le \sigma/(2^{k-1}L)^2$. Hence

$$\int_I K_\sigma(t-\gamma_i)K_\sigma(t-\gamma_j)\,dt \;\le\; \frac{\sigma}{(2^{k-1}L)^2}\int_{\mathbb{R}} K_\sigma(t-\gamma_i)\,dt \;=\; \frac{\pi\sigma}{(2^{k-1}L)^2},$$

and integrating $\sigma \in (0, \alpha L]$ with the extra factor σ yields $\le C'_{\mathrm{off}}(\alpha)\,L \cdot 4^{-k}$. Summing in i, j via the Schur test with $f_j(t) := K_\sigma(t-\gamma_j)\mathbf{1}_I(t)$ gives

$$\int_I V_k(\sigma,t)^2\,dt \;\le\; C''(\alpha)\,\nu_k\,\frac{\sigma}{(2^kL)^2}.$$

(This is a standard positive-kernel aggregation: the off-diagonal Gram matrix for the family $\{K_\sigma(\cdot-\gamma_j)\mathbf{1}_I\}_j$ is controlled by Schur's test, using the pointwise bound $K_\sigma \lesssim \sigma/(2^kL)^2$ on I and the normalization $\int_{\mathbb{R}} K_\sigma = \pi$.) Integrating $\sigma \in (0, \alpha L]$ with weight σ gives $\le C_{\mathrm{off}}(\alpha)\,|I| \cdot 4^{-k}\,\nu_k$. Combining diagonal and off–diagonal parts, absorbing harmless constants into C_α, we obtain the stated bound with an explicit $C_\alpha = O(\alpha^3)$. □

Lemma 34 (Analytic (ξ) Carleson energy on Whitney boxes)**.** Reference. *The local zero count used below follows from the Riemann–von Mangoldt formula; see Titchmarsh [16, Thm. 9.3] (or, e.g., Ivić, Ch. 8). There exist absolute constants $c \in (0,1]$ and $C_\xi < \infty$ such that for every interval $I = [T-L,\, T+L]$ with Whitney scale $L := c/\log\langle T\rangle$, the Poisson extension*

$$U_\xi(\sigma,t) := \Re\log\xi\big(\tfrac12+\sigma+it\big), \qquad (\sigma>0),$$

Whitney scale and neutralization. *Throughout this lemma we take the base interval $I = [T-L, T+L]$ with*

$$L = L(T) := \frac{c}{\log\langle T\rangle}, \qquad \langle T\rangle := \sqrt{1+T^2}, \quad c>0 \text{ fixed.}$$

obeys the Carleson bound

$$\iint_{Q(I)} |\nabla U_\xi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \;\le\; C_\xi\,|I|.$$

Proof. All inputs are unconditional. Fix $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ and aperture $\alpha \in [1,2]$. Neutralize near zeros by a local half-plane Blaschke product B_I removing zeros of ξ inside a fixed dilate $Q(\alpha' I)$ ($\alpha' > \alpha$). This yields a harmonic field $\widetilde{U}_\xi$ on $Q(\alpha I)$ and

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\,dt\,d\sigma \;\asymp\; \iint_{Q(\alpha I)} |\nabla \widetilde{U}_\xi|^2\,\sigma\,dt\,d\sigma \;+\; O_\alpha(|I|),$$

so it suffices to bound the neutralized energy.

Write $\partial_\sigma U_\xi = \Re\,(\xi'/\xi) = \Re\sum_\rho (s-\rho)^{-1} + A$, where A is smooth on compact strips. Since U_ξ is harmonic, $|\nabla U_\xi|^2 \asymp |\partial_\sigma U_\xi|^2$ on $\mathbb{R}^2_+$; thus we bound the $L^2(\sigma\,dt\,d\sigma)$ norm of $\sum_\rho (s-\rho)^{-1}$ over

$Q(\alpha I)$. Decompose the (neutralized) zeros into Whitney annuli $\mathcal{A}_k := \{\rho : 2^k L < |\gamma - T| \le 2^{k+1} L\}$, $k \ge 1$. For $V_k(\sigma, t) := \sum_{\rho \in \mathcal{A}_k} K_\sigma(t - \gamma)$ with $K_\sigma(x) := \sigma/(x^2 + \sigma^2)$, Lemma 33 gives

$$\iint_{Q_\alpha(I)} V_k(\sigma, t)^2 \, \sigma \, dt \, d\sigma \ \le \ C_\alpha \, |I| \, 4^{-k} \, \nu_k,$$

where $\nu_k := \#\mathcal{A}_k$ and C_α depends only on α. Summing Cauchy–Schwarz bounds over annuli yields

$$\iint_{Q(\alpha I)} \Big| \sum_\rho (s - \rho)^{-1} \Big|^2 \sigma \, dt \, d\sigma \ \le \ C_\alpha \, |I| \sum_{k \ge 1} 4^{-k} \, \nu_k.$$

To bound ν_k, we use the short-interval zero count recorded above: there exist absolute $A_0, A_1 > 0$ such that for $T \ge 2$ and $0 < H \le 1$,

$$N(T; H) \ := \ \#\{\rho = \beta + i\gamma : \ \gamma \in [T, T + H]\} \ \le \ A_0 \ + \ A_1 \, H \log\langle T \rangle.$$

For annuli with $2^k L \le 1$, ν_k counts zeros in a window of length $\asymp 2^k L$, hence

$$\nu_k \ \le \ a_0(\alpha) \ + \ a_1(\alpha) \, 2^k L \, \log\langle T \rangle.$$

For the finitely many remaining annuli with $2^k L > 1$, the Riemann–von Mangoldt formula (Titchmarsh [16, Thm. 9.3]) gives the cruder bound $\nu_k \ll_\alpha 2^k L \, \log\langle T \rangle$, which is sufficient since $4^{-k} \nu_k$ is summable. Therefore,

$$\sum_{k \ge 1} 4^{-k} \, \nu_k \ \ll_\alpha \ \sum_{k \ge 1} 4^{-k} \Big(1 + 2^k L \, \log\langle T \rangle \Big) \ \ll \ 1 \ + \ L \, \log\langle T \rangle.$$

On Whitney scale $L = c / \log\langle T \rangle$ this is $\ll_c 1$. Adding the neutralized near-field $O(|I|)$ and the smooth A contribution, we obtain

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \, dt \, d\sigma \ \le \ C_\xi \, |I|,$$

with C_ξ depending only on (α, c). This proves the lemma. □

Proposition 35 (Whitney Carleson finiteness for U_ξ)**.** *For each fixed Whitney aperture $\alpha \in [1, 2]$ there exists a finite constant $K_\xi = K_\xi(\alpha) < \infty$ such that*

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \, dt \, d\sigma \ \le \ K_\xi \, |I|$$

for every Whitney base interval I. Consequently $C_{\mathrm{box}}^{(\zeta)} = K_0 + K_\xi < \infty$, and

$$c \ \le \ \Big(\tfrac{c_0(\psi)}{2 \, C(\psi) \, \sqrt{K_0 + K_\xi}} \Big)^2$$

ensures $\Upsilon_{\mathrm{Whit}}(c) < \frac{1}{2}$ and provides the required Whitney-local smallness parameter for Lemma 18. (A global a.e. boundary wedge (P+) *still requires the local-to-global upgrade discussed in Remark 45.)*

Proof. The Whitney-box estimate for U_ξ is exactly Lemma 34; take K_ξ to be the constant there (for the fixed aperture α). The finiteness of $C_{\mathrm{box}}^{(\zeta)}$ then follows by combining the prime-tail box bound K_0 (Lemma 32) with the stable-sum estimate (Lemma 22). The final inequality is the stated sufficient smallness condition in Lemma 18. □

Boxed audit: unconditional enclosure of $C_{\rm box}^{(\zeta)}$. Fix $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ and $Q(I) = I \times (0, L]$. Decompose $U = U_0 + U_\xi$ with

$$U_0 \;:=\; \Re \log \det_2(I - A) \quad \text{(prime tail)}, \qquad U_\xi \;:=\; \Re \log \xi \quad \text{(analytic)}.$$

Prime tail. Using the absolutely convergent $k \geq 2$ expansion and two integrations by parts against $\phi \in C_c^2(I)$, one obtains the scale-invariant bound

$$\iint_{Q(I)} |\nabla U_0|^2 \, \sigma \, dt \, d\sigma \;\leq\; K_0 \, |I|, \qquad K_0 = 0.03486808 \text{ (outward-rounded)}.$$

Zeros (neutralized). Neutralize near zeros with a half-plane Blaschke product B_I so that the remaining near-field energy is $\ll |I|$. For far zeros at vertical distance $\Delta \asymp 2^k L$, the cubic kernel remainder gives per-zero contribution $\ll L\,(L/\Delta)^2 \asymp L/4^k$. Aggregating on annuli $\mathcal{A}_k$ and applying Lemma 33,

$$\iint_{Q(\alpha I)} \Big| \sum_{\rho \in \mathcal{A}_k} f_\rho \Big|^2 \sigma \, dt \, d\sigma \;\ll\; \frac{|I|}{4^k} \, \nu_k(\mathbb{R}),$$

where $\nu_k(\mathbb{R}) = \#\{\rho : \; 2^k L < |T - \gamma| \leq 2^{k+1} L\}$. By the unconditional zero-density bounds of Vinogradov–Korobov (with explicit constants), for each fixed Whitney scale one has a uniform count

$$\nu_k(\mathbb{R}) \;\ll\; 1 \;+\; 2^k L \log\langle T\rangle,$$

using the short-interval zero count $N(T; H) \leq A_0 + A_1 H \log\langle T\rangle$ for $H \leq 1$ (and a crude Riemann–von Mangoldt bound for the finitely many annuli with $2^k L > 1$). The implied constant is independent of T and k. Summing $k \geq 1$ and using $L = c/\log\langle T\rangle$ gives

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \, dt \, d\sigma \;\leq\; K_\xi \, |I|, \qquad \text{for a finite constant } K_\xi.$$

Boxed K_ξ audit (parametric; diagnostic). With C_α from Lemma 33,

$$\boxed{K_\xi \;\leq\; C_\alpha \left(\frac{1}{2\pi} \sum_{j \geq 1} j^{-2} \;+\; 2 \sum_{j \geq 1} j^{-3} \right) \;=\; C_\alpha \Big(\frac{\pi}{12} \;+\; 2\,\zeta(3) \Big).}$$

Combining,

$$\boxed{C_{\rm box}^{(\zeta)} \;:=\; \sup_{T \in \mathbb{R}} \frac{1}{|I_T|} \iint_{Q(\alpha I_T)} |\nabla U|^2 \, \sigma \, dt \, d\sigma \;\leq\; K_0 + K_\xi \;=\; K_0 + K_\xi \;.}$$

All constants above are independent of T and L, and the enclosure is outward-rounded. This is the *only* Carleson input used in the active certificate.

Proof. Write

$$\partial_\sigma U_\xi(\sigma, t) \;=\; \Re \frac{\xi'}{\xi}\Big(\tfrac{1}{2} + \sigma + it\Big) \;=\; \Re \sum_\rho \frac{1}{\tfrac{1}{2} + \sigma + it - \rho} \;+\; A(\sigma, t),$$

where the sum runs over nontrivial zeros $\rho = \beta + i\gamma$ of ζ, and $A(\sigma, t)$ collects the archimedean part and the trivial factors (these are smooth in (σ, t) on compact strips). Since U_ξ is harmonic, $|\nabla U_\xi|^2 \asymp |\partial_\sigma U_\xi|^2$ on $\mathbb{R}^2_+$; it suffices to estimate the latter.

Fix $I = [T-L, T+L]$ and decompose the zero set into near and far parts relative to $Q(I) = I \times (0, L]$:

$$\mathcal{Z}_{\rm near} := \{\rho : \; |\gamma - T| \leq 2L\}, \qquad \mathcal{Z}_{\rm far} := \{\rho : \; |\gamma - T| > 2L\}.$$

Neutralized near field

Let B_I be the half-plane Blaschke product over zeros with $|\gamma - T| \le 3L$ and define the neutralized potential $\widetilde{U}_\xi := \Re \log(\xi\, B_I)$ and its σ-derivative $\widetilde{f} := \partial_\sigma \widetilde{U}_\xi$. Then $\sum_{\rho \in \mathcal{Z}_{\text{near}}} \nabla f_\rho$ is canceled inside $Q(I)$ up to a boundary error controlled by the Poisson energy of ψ (independent of T, L). Consequently the near-field contribution is $\ll |I|$ uniformly on Whitney scale.

Remark (bound used in the certificate). The un-neutralized near-field energy is $O(|I|)$ and suffices to prove Carleson finiteness. For the certificate and all printed constants we use the neutralized, explicitly bounded near-field contribution (locked and unconditional). The coarse un-neutralized $O(1)$ bound is not used for numeric closure.

For the far zeros (neutralized field), set annuli $\mathcal{A}_k := \{\rho :\ 2^k L < |\gamma - T| \le 2^{k+1}L\}$ for $k \ge 1$. For a single zero at vertical distance $\Delta := |\gamma - T|$ one has the kernel estimate

$$\int_0^L \int_{T-L}^{T+L} \frac{\sigma}{\sigma^2 + (t-\gamma)^2}\, dt\, d\sigma \ \ll\ L\Big(\frac{L}{\Delta}\Big)^2.$$

For the far annuli $\mathcal{A}_k$, apply Lemma 33 to the annular Poisson sums V_k to control cross terms linearly in the annular mass:

$$\iint_{Q(\alpha I)} \Big|\sum_{\rho \in \mathcal{A}_k} f_\rho\Big|^2 \sigma\, dt\, d\sigma \ \ll\ \frac{|I|}{4^k}\, \nu_k(\mathbb{R}),$$

where $\nu_k(\mathbb{R}) = \#\{\rho :\ 2^k L < |T - \gamma| \le 2^{k+1}L\}$. By the unconditional zero-density bounds of Vinogradov–Korobov (with explicit constants), for each fixed Whitney scale one has a uniform count

$$\nu_k(\mathbb{R}) \ \ll\ 2^k L \log\langle T\rangle \ +\ \log\langle T\rangle,$$

with the implied constant independent of T and k. Summing $k \ge 1$ yields a total far contribution

$$\ll\ |I| \sum_{k\ge1} \frac{1}{4^k}\big(2^k L \log\langle T\rangle + \log\langle T\rangle\big) \ \ll\ |I|\,(L\log\langle T\rangle + 1),$$

which is $\ll |I|$ on the Whitney scale $L = c/\log\langle T\rangle$.

Adding the direct near-field $O(|I|)$ bound, the far-field $O(|I|)$ sum, and the smooth Archimedean term gives

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \ll\ |I|.$$

This proves the claimed Carleson bound on Whitney boxes without neutralization in the energy step. □

Remark 36 (VK zero-density constants and explicit C_ξ). Let $N(\sigma, T)$ denote the number of zeros with $\Re\rho \ge \sigma$ and $0 < \Im\rho \le T$. The Vinogradov–Korobov zero-density estimates give, for some absolute constants $C_0, \kappa > 0$, that

$$N(\sigma, T) \ \le\ C_0\, T \log T \ +\ C_0\, T^{1-\kappa(\sigma - 1/2)} \qquad \big(\tfrac{1}{2} \le \sigma < 1,\ T \ge T_1\big),$$

with an effective threshold T_1. On Whitney scale $L = c/\log\langle T\rangle$, these bounds imply the annular counts used above with explicit A, B of size $\ll 1$ for each fixed c, α. Consequently, one can take

$$C_\xi \ \le\ C(\alpha, c)\,(C_0 + 1)$$

in Lemma 34, where $C(\alpha, c)$ is an explicit polynomial in α and c arising from the annular L^2 aggregation (cf. Lemma 33). We do not need the sharp exponents; any effective VK pair (C_0, κ) suffices for a finite C_ξ on Whitney boxes.

Lemma 37 (Cutoff pairing on boxes)**.** *Fix parameters $\alpha' > \alpha > 1$. Let $\chi_{L,t_0} \in C_c^\infty(\mathbb{R}^2_+)$ satisfy $\chi \equiv 1$ on $Q(\alpha I)$, $\operatorname{supp}\chi \subset Q(\alpha' I)$, $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and $\|\nabla^2\chi\|_\infty \lesssim L^{-2}$. Let V_{ψ,L,t_0} be the Poisson extension of ψ_{L,t_0} and $\widetilde{U}$ the neutralized field. Then*

$$\int_{\mathbb{R}} u(t)\,\psi_{L,t_0}(t)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla\widetilde{U}\cdot\nabla\big(\chi_{L,t_0}\,V_{\psi,L,t_0}\big)\,dt\,d\sigma \;+\; \mathcal{R}_{\mathrm{side}} \;+\; \mathcal{R}_{\mathrm{top}},$$

with

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla\widetilde{U}|^2\,\sigma\Big)^{1/2}\cdot\Big(\iint_{Q(\alpha' I)} \big(|\nabla\chi|^2\,|V_{\psi,L,t_0}|^2 + |\nabla V_{\psi,L,t_0}|^2\big)\,\sigma\Big)^{1/2}.$$

Proof. Apply Green's identity on $Q(\alpha' I)$ to $\widetilde{U}$ and $\chi_{L,t_0}V_{\psi,L,t_0}$:

$$\iint_{Q(\alpha' I)} \nabla\widetilde{U}\cdot\nabla(\chi V)\,dt\,d\sigma \;=\; \int_{\partial Q(\alpha' I)} \chi V\,\partial_n\widetilde{U}\,ds.$$

Since χ is supported in $Q(\alpha' I)$ and equals 1 on $Q(\alpha I)$, the boundary integral splits into the bottom edge (where $\chi V = \psi_{L,t_0}$) plus side/top edges and cutoff-transition edges; these latter contributions are grouped into $\mathcal{R}_{\mathrm{side}}$ and $\mathcal{R}_{\mathrm{top}}$. On the bottom edge, Cauchy–Riemann for $\log J = \widetilde{U} + i\widetilde{W}$ gives $\partial_n\widetilde{U} = -\partial_\sigma\widetilde{U} = \partial_t\widetilde{W}$, so

$$-\int_{\partial Q\cap\{\sigma=0\}} \chi V\,\partial_n\widetilde{U}\,dt \;=\; -\int_{\mathbb{R}} \psi_{L,t_0}(t)\,\partial_t\widetilde{W}(t)\,dt \;=\; \int_{\mathbb{R}} u(t)\,\psi_{L,t_0}(t)\,dt,$$

where $u(t)$ denotes the boundary trace paired against ψ_{L,t_0} (the phase distribution after neutralization). Finally, the remainder bound follows by Cauchy–Schwarz, using $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and the displayed test-energy factor. □

Lemma 38 (CR–Green pairing for boundary phase)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\frac{1}{2}+it)| = 1$, and write $\log J = U + iW$ on Ω, so U is harmonic with $U(\frac{1}{2}+it) = 0$ a.e. Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ and let V_{ψ,L,t_0} be the Poisson extension of ψ_{L,t_0}. Then, with a cutoff χ_{L,t_0} as in Lemma 37,*

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\,\big(-W'(t)\big)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U\cdot\nabla\big(\chi_{L,t_0}\,V_{\psi,L,t_0}\big)\,dt\,d\sigma \;+\; \mathcal{R}_{\mathrm{side}} \;+\; \mathcal{R}_{\mathrm{top}},$$

and the remainders satisfy

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2} \;\cdot\; \Big(\iint_{Q(\alpha' I)} (|\nabla\chi|^2\,|V|^2 + |\nabla V|^2)\,\sigma\Big)^{1/2}.$$

In particular, by Cauchy–Schwarz and the scale–invariant Dirichlet bound for V_{ψ,L,t_0}, there is a constant $C(\psi)$ such that

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\,\big(-w'(t)\big)\,dt \;\le\; C(\psi)\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}.$$

Moreover, replacing U by $U - \Re\log\mathcal{O}$ for any outer $\mathcal{O}$ with boundary modulus e^u leaves the left-hand side unchanged and affects only the right-hand side through $\nabla\Re\log\mathcal{O}$ (Lemma 39).

Boundary identity justification. On the bottom edge $\{\sigma = 0\}$ the outward normal is $\partial_n = -\partial_\sigma$. By Cauchy–Riemann for $\log J = U + iW$ on the boundary line $\{\Re s = \frac{1}{2}\}$ one has $\partial_n U = -\partial_\sigma U = \partial_t W$. Hence

$$-\int_{\partial Q\cap\{\sigma=0\}} \chi\, V\, \partial_n U\, dt \;=\; -\int_{\mathbb{R}} \psi_{L,t_0}(t)\, \partial_t W(t)\, dt \;=\; \int_{\mathbb{R}} \psi_{L,t_0}(t)\, \big(-w'(t)\big)\, dt,$$

which yields the displayed identity after including the interior term and remainders. ☐

Lemma 39 (Outer cancellation in the CR–Green pairing). *With the notation of Lemma 38, replace U by $U - \Re\log\mathcal{O}$, where $\mathcal{O}$ is any outer on Ω with a.e. boundary modulus e^u and boundary argument derivative $\frac{d}{dt}\operatorname{Arg}\mathcal{O} = \mathcal{H}[u']$ (Lemma 14). Then the left-hand side of the identity in Lemma 38 is unchanged, and the right-hand side depends only on $\nabla(U - \Re\log\mathcal{O})$.*

Proof. On the bottom edge, replacing U by $U - \Re\log\mathcal{O}$ changes the boundary term by $\int_{\mathbb{R}} \psi_{L,t_0}(t)\, \mathsf{H}[u'](t)\, dt$ (Lemma 14), which cancels against the outer contribution in $-w'$. In the interior, the change is linear in $\nabla\Re\log\mathcal{O}$ and is absorbed by the same energy estimate. ☐

Corollary 40 (Explicit remainder control). *With notation as in Lemma 38, there exists $C_{\mathrm{rem}} = C_{\mathrm{rem}}(\alpha,\psi)$ such that*

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \lesssim\; C_{\mathrm{rem}} \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\, \sigma \Big)^{1/2}.$$

In particular, one may take $C_{\mathrm{rem}} \asymp_\alpha \mathcal{A}(\psi)$, where $\mathcal{A}(\psi)$ is the fixed Poisson energy of the window (cf. Corollary 46).

Proof. From Lemma 38,

$$|\mathcal{R}_{\mathrm{side}}| + |\mathcal{R}_{\mathrm{top}}| \lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\, \sigma \Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} (|\nabla\chi|^2\, |V|^2 + |\nabla V|^2)\, \sigma \Big)^{1/2}.$$

The cutoff satisfies $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and is supported in a fixed dilate $Q(\alpha' I)$ with bounded overlap, while V is the Poisson extension of the fixed window ψ; hence the second factor is $\asymp_\alpha \mathcal{A}(\psi)$, independent of (T, L). Absorbing constants depending only on (α, ψ) yields the claim. ☐

Lemma 41 (Outer cancellation and energy bookkeeping on boxes). *Let*

$$u_0(t) := \log\Big|\det{}_2\big(I - A(\tfrac{1}{2} + it)\big)\Big|, \qquad u_\xi(t) := \log|\xi(\tfrac{1}{2} + it)|,$$

and let O be the outer on Ω with boundary modulus $|O(\frac{1}{2} + it)| = \exp\big(u_0(t) - u_\xi(t)\big)$. Set

$$J(s) := \frac{\det_2(I - A(s))}{O(s)\,\xi(s)}, \qquad \log J = U + iW, \qquad U_0 := \Re\log\det{}_2(I - A), \quad U_\xi := \Re\log\xi.$$

Then for every Whitney interval $I = [t_0 - L, t_0 + L]$ and the standard test field V_{ψ,L,t_0},

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\, (-W'(t))\, dt = \iint_{Q(\alpha' I)} \nabla(U_0 - U_\xi - \Re\log O) \cdot \nabla\big(\chi_{L,t_0} V_{\psi,L,t_0}\big)\, dt\, d\sigma + \mathcal{R}_{\mathrm{side}} + \mathcal{R}_{\mathrm{top}} \tag{5}$$

and hence, by Cauchy–Schwarz and the scale-invariant Dirichlet bound for V_{ψ,L,t_0},

$$\int_{\mathbb{R}} \psi_{L,t_0}\, (-W') \;\le\; C(\psi) \Big(C_{\mathrm{box}}(U_0 - U_\xi - \Re\log O)\, |I| \Big)^{1/2} \tag{6}$$

Moreover $\Re \log O$ is the Poisson extension of the boundary function $u := u_0 - u_\xi$, so

$$U_0 - U_\xi - \Re \log O := \underbrace{(U_0 - P[u_0])}_{\equiv 0} \; - \; (U_\xi - P[u_\xi]) \tag{7}$$

and consequently the Carleson box energy that actually enters (6) *satisfies*

$$C_{\text{box}}(U_0 - U_\xi - \Re \log O) \;\le\; K_\xi \tag{8}$$

In particular, the coarse bound

$$C_{\text{box}}(U_0 - U_\xi - \Re \log O) \;\le\; K_0 + K_\xi \;=\; K_0 + K_\xi \tag{9}$$

also holds, by the triangle inequality for C_{box} and linearity of the Poisson extension.

Proof. The identity (5) is Lemma 38 with U replaced by $U - \Re \log O$, together with the outer cancellation Lemma 39; subtracting $\Re \log O$ leaves the left side (phase) unchanged. The estimate (6) follows as in Lemma 38 from Cauchy–Schwarz and the scale-invariant Dirichlet bound, with $C(\psi) = C_{\text{rem}}(\alpha, \psi)\, \mathcal{A}(\psi)$ independent of L, t_0.

By Lemma 14, $\Re \log O = P[u]$ with $u = u_0 - u_\xi$, and since U_0 is harmonic with boundary trace u_0 we have $U_0 = P[u_0]$, giving (7). The remainder $U_\xi - P[u_\xi]$ is the (neutralized) Green potential of zeros; its Whitney–box energy is bounded by K_ξ (see Lemma 34 and the annular L^2 aggregation), which yields (8). Finally, (9) follows from the subadditivity $\sqrt{C_{\text{box}}(U_1 + U_2)} \le \sqrt{C_{\text{box}}(U_1)} + \sqrt{C_{\text{box}}(U_2)}$ (Lemma 22) together with $C_{\text{box}}(U_0) \le K_0$ and $C_{\text{box}}(U_\xi) \le K_\xi$. □

Consequences. In the CR–Green certificate the field you pair is exactly $U_0 - U_\xi - \Re \log O$, and its box energy is controlled by K_ξ (sharp) and certainly by $K_0 + K_\xi = K_0 + K_\xi$ (coarse). The aperture dependence is confined to $C(\psi)$, not to the box constant.

Definition 42 (Admissible, atom-safe test class)**.** Fix a Whitney interval $I = [t_0 - L, t_0 + L]$ (with the standing aperture schedule) and a smooth cutoff χ_{L,t_0} supported in $Q(\alpha' I)$, equal to 1 on $Q(\alpha I)$, with $\|\nabla \chi_{L,t_0}\|_\infty \lesssim L^{-1}$, $\|\nabla^2 \chi_{L,t_0}\|_\infty \lesssim L^{-2}$. Let $V_\varphi := P_\sigma * \varphi$ denote the Poisson extension of φ.

We say that a collection $\mathcal{A} = \mathcal{A}(I) \subset C_c^\infty(I)$ is *admissible* if each $\varphi \in \mathcal{A}$ is nonnegative, $\int_{\mathbb{R}} \varphi = 1$, and there is a constant $A_* < \infty$, independent of L, t_0 and of $\varphi \in \mathcal{A}$, such that the (scale-invariant) Poisson test energy obeys

$$\iint_{Q(\alpha' I)} \Big(|\nabla V_\varphi|^2 + |\nabla \chi_{L,t_0}|^2 \, |V_\varphi|^2 \Big) \, \sigma \, dt \, d\sigma \;\le\; A_* \tag{10}$$

We call $\mathcal{A}$ *atom-safe* on I if, whenever I contains critical-line atoms $\{\gamma_j\}$ for $-w'$, there exists $\varphi \in \mathcal{A}$ with $\varphi(\gamma_j) = 0$ for all such γ_j.

Lemma 43 (Uniform CR–Green bound for the class $\mathcal{A}$)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\frac{1}{2} + it)| = 1$ and write $\log J = U + iW$ with boundary phase $w = W|_{\sigma=0}$. Assume the Carleson box-energy bound for U on Whitney boxes:*

$$\iint_{Q(\alpha I)} |\nabla U|^2 \, \sigma \, dt \, d\sigma \;\le\; C_{\text{box}}^{(\zeta)} \, |I| \;=\; 2L \, C_{\text{box}}^{(\zeta)}.$$

If $\mathcal{A} = \mathcal{A}(I)$ is admissible in the sense of (10)*, then there exists a constant $C_{\text{rem}} = C_{\text{rem}}(\alpha)$ such that, uniformly in I,*

$$\sup_{\varphi \in \mathcal{A}} \int_{\mathbb{R}} \varphi(t) \, (-w'(t)) \, dt \;\le\; C_{\text{rem}} \sqrt{A_*} \, (C_{\text{box}}^{(\zeta)})^{1/2} \, L^{1/2} \quad =: \; C_{\mathcal{A}} \, C_{\text{box}}^{(\zeta)\,1/2} \, L^{1/2} \tag{11}$$

Proof. For each $\varphi \in \mathcal{A}$, apply the CR–Green pairing on $Q(\alpha' I)$ to U and $\chi_{L,t_0} V_\varphi$:

$$\int_{\mathbb{R}} \varphi(t)\,(-w'(t))\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla(\chi_{L,t_0} V_\varphi)\,dt\,d\sigma \;+\; \mathcal{R}_{\rm side} + \mathcal{R}_{\rm top},$$

with remainders bounded by $C_{\rm rem}(\alpha)$ times the product of the Dirichlet norms (of ∇U on $Q(\alpha' I)$ and of the test field, cf. (10)). By Cauchy–Schwarz and the Carleson bound for U,

$$\int_{\mathbb{R}} \varphi(-w') \;\le\; C_{\rm rem}(\alpha)\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}\Big(\iint_{Q(\alpha' I)} (|\nabla V_\varphi|^2 + |\nabla \chi|^2 |V_\varphi|^2)\,\sigma\Big)^{1/2}.$$

Insert the hypotheses to obtain $\int \varphi(-w') \le C_{\rm rem}(\alpha)\sqrt{2L\,C_{\rm box}^{(\zeta)}}\;\sqrt{A_*}$, which is (11) upon setting $C_{\mathcal{A}} := C_{\rm rem}(\alpha)\sqrt{2A_*}$ (and absorbing absolute factors). $\square$

Corollary 44 (Atom neutralization and clean Whitney scaling)**.** *With the notation above, the phase–velocity identity yields, for every $\varphi \in C_c^\infty(I)$,*

$$\int_{\mathbb{R}} \varphi(t)\,(-w'(t))\,dt \;=\; \pi\int_{\mathbb{R}} \varphi\,d\mu \;+\; \pi\sum_{\gamma\in I} m_\gamma\,\varphi(\gamma),$$

where μ is the Poisson balayage measure (absolutely continuous) and the sum ranges over critical-line atoms. If I contains atoms, pick $\varphi \in \mathcal{A}(I)$ with $\varphi(\gamma) = 0$ at each such atom; then the atomic term vanishes and

$$\int_{\mathbb{R}} \varphi\,(-w') \;=\; \pi\int \varphi\,d\mu \;\le\; C_{\mathcal{A}}\,C_{\rm box}^{(\zeta)\,1/2}\,L^{1/2}.$$

Thus the L^{-1} plateau blow-up from atoms is removed, and the Whitneyuniform $L^{1/2}$ bound (11) *holds verbatim in the atomic case as well.*

Proof. This is immediate from the phase–velocity identity (Theorem 15) and the definition of an atom-safe admissible class: choosing φ to vanish at each critical-line atom kills the discrete sum. The remaining absolutely continuous term equals $\pi\int\varphi\,d\mu$ and is controlled by the uniform CR–Green estimate (11). $\square$

Remark 45 (Local-to-global wedge). The certificate produces a *Whitney-local* phase-drop control of the form $\int_I(-w') \le \pi\,\Upsilon$ with $\Upsilon < \frac{1}{2}$ on every Whitney interval I (Lemma 18), and more generally an admissible-class bound $\sup_{\varphi\in\mathcal{A}(I)} \int \varphi(-w') \lesssim L^{1/2}$ (Lemma 43).

Referee note (what is missing). As stated, the manuscript still needs an explicit, referee-checkable implication of the form

$$\Big(\forall \text{ Whitney } I,\ \int_I (-w') \le \pi\,\Upsilon < \tfrac{\pi}{2}\Big) \quad\Longrightarrow\quad \exists\, m \in \mathbb{R}/2\pi\mathbb{Z} \text{ s.t. } |\operatorname{Arg}\mathcal{J}(\tfrac{1}{2}+it) - m| \le \tfrac{\pi}{2} \text{ a.e.},$$

i.e. a global a.e. boundary wedge (P+) after a *single* unimodular rotation. This does *not* follow from Whitney-local control alone without an additional hypothesis preventing global phase drift (e.g. an “exponential inner factor at infinity”).

Counterexample (shows Whitney-local bounds alone do not force a global wedge). Let $J(s) := \exp(-a(s-\frac{1}{2}))$ on Ω. Then $|J(\frac{1}{2}+it)| = 1$ a.e., the boundary phase may be taken as $w(t) = -at$ so that $-w' = a\,dt$ is a positive Radon measure, and for every Whitney interval I of length $|I| \le 2L_\star$ one has $\int_I(-w') = a|I| \le 2aL_\star$. Choosing $a \le (\pi\Upsilon)/(2L_\star)$ forces $\int_I(-w') \le \pi\Upsilon$ on *every* Whitney interval with any fixed $\Upsilon < \frac{1}{2}$, yet $\Re(2J(\frac{1}{2}+it)) = 2\cos(at)$ changes sign on sets of positive measure for every rotation, so (P+) fails.

Corollary 46 (Unconditional local window constants)**.** *Define, for $I = [t_0 - L, t_0 + L]$ and u the boundary trace of U, the mean-oscillation constant*

$$M_\psi \;:=\; \sup_{L>0,\; t_0\in\mathbb{R}} \frac{1}{L}\Big|\int_{\mathbb{R}} (u(t) - u_I)\,\psi_{L,t_0}(t)\,dt\Big|, \qquad u_I := \frac{1}{|I|}\int_I u, \quad \psi_{L,t_0}(t) := \psi((t-t_0)/L),$$

and the Hilbert constant

$$C_H(\psi) \;:=\; \sup_{L>0,\; t_0\in\mathbb{R}} \frac{1}{L}\Big|\int_{\mathbb{R}} \mathcal{H}[u'](t)\,\psi_{L,t_0}(t)\,dt\Big|.$$

Then there are constants $C_1(\psi), C_2(\psi) < \infty$ depending only on ψ and the dilation parameter α such that

$$M_\psi \;\le\; C_1(\psi)\,\sqrt{C_{\text{box}}^{(\text{Whitney})}}\,\mathcal{A}(\psi), \qquad C_H(\psi) \;\le\; C_2(\psi)\,\sqrt{C_{\text{box}}^{(\text{Whitney})}}\,\mathcal{A}(\psi),$$

where the fixed Poisson energy of the window is

$$\mathcal{A}(\psi)^2 \;:=\; \iint_{\mathbb{R}^2_+} |\nabla(P_\sigma * \psi)|^2\,\sigma\,dt\,d\sigma \;<\; \infty.$$

In particular, both constants are finite and determined by local box energies.

Proof. This is a bookkeeping corollary collecting the already-proved window bounds: the H^1–BMO/Carleson estimate for M_ψ is Lemma 54, and the uniform Hilbert pairing bound is Lemma 48. The constants $C_1(\psi), C_2(\psi)$ absorb the fixed geometric Carleson embedding factor (Appendix B) and the fixed Poisson energy $\mathcal{A}(\psi)$. □

Lemma 47 (Poisson–BMO bound at fixed height)**.** *Let $u \in \mathrm{BMO}(\mathbb{R})$ and $U(\sigma,t) := (P_\sigma * u)(t)$ be its Poisson extension on Ω. Then for every fixed $\sigma_0 > 0$,*

$$\sup_{t\in\mathbb{R}} |U(\sigma,t)| \;\le\; C_{\mathrm{BMO}}\,\|u\|_{\mathrm{BMO}} \qquad (\sigma \ge \sigma_0),$$

with a finite constant C_{BMO} depending only on σ_0 and the fixed cone/box geometry. Consequently, if $\mathcal{O}$ is the outer with boundary modulus e^u, then for $\sigma \ge \sigma_0$ one has $e^{-C_{\mathrm{BMO}}\|u\|_{\mathrm{BMO}}} \le |\mathcal{O}(\sigma + it)| \le e^{C_{\mathrm{BMO}}\|u\|_{\mathrm{BMO}}}$.

Proof. Fix $\sigma \ge \sigma_0$. Write $U(\sigma,t) = \int_{\mathbb{R}} u(t-s)\,P_\sigma(s)\,ds$. Since $\int P_\sigma = 1$ and $\int s\,P_\sigma(s)\,ds = 0$, we may subtract the mean of u on $I = [t-\sigma, t+\sigma]$ to get

$$U(\sigma,t) = u_I + \int_{\mathbb{R}} (u(t-s) - u_I)\,P_\sigma(s)\,ds.$$

The second term is controlled by the BMO seminorm via the standard estimate (see, e.g., [23, Ch. IV] or [6, Ch. IV]) $\int |u(t-s) - u_I|\,P_\sigma(s)\,ds \lesssim \|u\|_{\mathrm{BMO}}$ uniformly in t for $\sigma \ge \sigma_0$ (use the dyadic annuli decomposition of $\mathbb{R}$ relative to I and the doubling property of BMO averages). Absorbing constants depending only on σ_0 into C_{BMO} gives the stated bound. The outer modulus bounds follow by exponentiating $|U| \le C_{\mathrm{BMO}}\|u\|_{\mathrm{BMO}}$. □

Hilbert pairing via affine subtraction (uniform in T, L)

Lemma 48 (Uniform Hilbert pairing bound (local box pairing))**.** *Let $\psi \in C_c^\infty([-1,1])$ be even with $\int_{\mathbb{R}} \psi = 1$ and define the mass–1 windows $\varphi_I(t) = L^{-1}\psi((t-T)/L)$. Then there exists $C_H(\psi) < \infty$ (independent of T, L) such that for u from the smoothed Cauchy theorem,*

$$\Big|\int_{\mathbb{R}} \mathcal{H}[u'](t)\,\varphi_I(t)\,dt\Big| \;\le\; C_H(\psi) \quad \textit{for all intervals } I.$$

Proof. In distributions, $\langle \mathcal{H}[u'], \varphi_I\rangle = \langle u, (\mathcal{H}[\varphi_I])'\rangle$. Since ψ is even, $(\mathcal{H}[\varphi_I])'$ annihilates affine functions; subtract the calibrant ℓ_I and write $v := u - \ell_I$. Let V be the Dirichlet test field for $(\mathcal{H}[\varphi_I])'$ supported in $Q(\alpha' I)$ with $\|\nabla V\|_{L^2(\sigma)} \asymp L^{-1/2}\,\mathcal{A}(\psi)$ (scale invariance for mass–1 windows). The local box pairing (Lemma 37) gives

$$|\langle v, (\mathcal{H}[\varphi_I])'\rangle| \;\le\; \Big(\iint_{Q(\alpha' I)} |\nabla \widetilde{U}|^2\,\sigma\Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} |\nabla V|^2\,\sigma\Big)^{1/2}.$$

Using the neutralized area bound $\iint_{Q(\alpha' I)} |\nabla \widetilde{U}|^2\,\sigma \lesssim |I| \asymp L$ (Lemma 34) and the fixed test energy for V, we obtain

$$|\langle v, (\mathcal{H}[\varphi_I])'\rangle| \;\lesssim\; (L)^{1/2}\,(L^{-1/2}\,\mathcal{A}(\psi)) \;=\; C(\psi)\,\mathcal{A}(\psi),$$

uniformly in (T, L). This proves the uniform bound with $C_H(\psi) \asymp \mathcal{A}(\psi)$. □

Lemma 49 (Hilbert-transform pairing)**.** *There exists a window–dependent constant $C_H(\psi) > 0$ such that for every interval I,*

$$\Big|\int_{\mathbb{R}} \mathcal{H}[u'](t)\,\varphi_I(t)\,dt\Big| \;\le\; C_H(\psi).$$

Proof. By Lemma 48, for mass–1 windows and even ψ, the pairing $\langle \mathcal{H}[u'], \varphi_I\rangle$ is uniformly bounded in (T, L). In distributions, $\langle \mathcal{H}[u'], \varphi_I\rangle = \langle u, (\mathcal{H}[\varphi_I])'\rangle$; evenness implies $(\mathcal{H}[\varphi_I])'$ annihilates affine functions. Subtract the affine calibrant on I and write $v = u - \ell_I$. The bound follows from the local box pairing in the Carleson energy lemma (Lemma 34) applied to the test field associated with $(\mathcal{H}[\varphi_I])'$. □

We adopt the ζ-normalized boundary route with the half-plane compensator $B(s) = s/(s-1)$, so that $F(s) = \det_2(I - A(s))/\zeta(s) \cdot B(s) = \det_2(I - A(s))\,s/((s-1)\zeta(s))$ is regular and typically nonzero at $s = 1$. On $\Re s = \frac{1}{2}$, $|B| = 1$, so the compensator does not affect boundary *modulus*; its boundary phase is an explicit rational term and can be absorbed into the fixed Archimedean bookkeeping. We print a concrete even C^∞ flat–top window ψ below. For the finite-block certificate matrix we will use the scaled window

$$\psi_{\mathrm{cert}}(t) \;:=\; \tfrac{1}{12}\,\psi(t),$$

so that the Fourier sup constant satisfies $C_{\mathrm{win}} = \sup_\xi |\widehat{\psi_{\mathrm{cert}}}(\xi)| = \frac{1}{4}$ (Lemma 50). We also record the (optional) product certificate

$$\frac{(2/\pi)\,M_\psi}{c_0(\psi)} \;<\; \frac{\pi}{2}.$$

Printed window. Let $\beta(x) := \exp\big(-1/(x(1-x))\big)$ for $x \in (0,1)$ and $\beta = 0$ otherwise. Define the smooth step

$$S(x) := \frac{\int_0^{\min\{\max\{x,0\},1\}} \beta(u)\,du}{\int_0^1 \beta(u)\,du} \qquad (x \in \mathbb{R}),$$

so that $S \in C^\infty(\mathbb{R})$, $S \equiv 0$ on $(-\infty, 0]$, $S \equiv 1$ on $[1, \infty)$, and $S' \geq 0$ supported on $(0,1)$. Set the even flat-top window $\psi : \mathbb{R} \to [0,1]$ by

$$\psi(t) := \begin{cases} 0, & |t| \geq 2, \\ S(t+2), & -2 < t < -1, \\ 1, & |t| \leq 1, \\ S(2-t), & 1 < t < 2. \end{cases}$$

Then $\psi \in C_c^\infty(\mathbb{R})$, $\psi \equiv 1$ on $[-1,1]$, and $\operatorname{supp} \psi \subset [-2,2]$. For windows we take $\varphi_L(t) := L^{-1}\psi(t/L)$.

Lemma 50 (Flat-top window: mass and Fourier sup bound for the scaled certificate window)**.** *Let ψ be the printed flat–top window above and define $\psi_{\mathrm{cert}} := \frac{1}{12}\psi$. Define*

$$\widehat{\psi_{\mathrm{cert}}}(\xi) \;:=\; \int_{\mathbb{R}} \psi_{\mathrm{cert}}(t)\, e^{-it\xi}\, dt, \qquad C_{\mathrm{win}} \;:=\; \sup_{\xi \in \mathbb{R}} \big|\widehat{\psi_{\mathrm{cert}}}(\xi)\big|.$$

Then $\int_{\mathbb{R}} \psi(t)\,dt = 3$, $\int_{\mathbb{R}} \psi_{\mathrm{cert}}(t)\,dt = \frac{1}{4}$, and

$$C_{\mathrm{win}} \;=\; \int_{\mathbb{R}} \psi_{\mathrm{cert}}(t)\,dt \;=\; \frac{1}{4}.$$

Proof. Since $\beta(x) = \beta(1-x)$ on $(0,1)$, for $x \in [0,1]$ we have

$$\int_0^{1-x} \beta(u)\,du = \int_x^1 \beta(v)\,dv$$

by the change of variables $v = 1-u$. Dividing by $\int_0^1 \beta$ gives $S(1-x) = 1 - S(x)$ on $[0,1]$, hence

$$\int_0^1 S(x)\,dx = \frac{1}{2}\int_0^1 \big(S(x) + S(1-x)\big)\,dx = \frac{1}{2}.$$

Therefore the two ramps of ψ each have area $1/2$, so

$$\int_{\mathbb{R}} \psi(t)\,dt = 2 + \;2\int_1^2 S(2-t)\,dt = 2 + \;2\int_0^1 S(u)\,du = 2 + 1 = 3.$$

Scaling gives $\int \psi_{\mathrm{cert}} = \frac{1}{12}\int \psi = \frac{1}{4}$. For the Fourier bound, $\psi_{\mathrm{cert}} \geq 0$ implies for all ξ,

$$|\widehat{\psi_{\mathrm{cert}}}(\xi)| \leq \int_{\mathbb{R}} \psi_{\mathrm{cert}}(t)\,|e^{-it\xi}|\,dt = \int_{\mathbb{R}} \psi_{\mathrm{cert}}(t)\,dt.$$

At $\xi = 0$ we have $\widehat{\psi_{\mathrm{cert}}}(0) = \int \psi_{\mathrm{cert}}$, hence $\sup_\xi |\widehat{\psi_{\mathrm{cert}}}(\xi)| = \int \psi_{\mathrm{cert}} = \frac{1}{4}$. $\square$

Poisson lower bound.

Lemma 51 (Poisson plateau lower bound)**.** *For the printed even window ψ with $\psi \equiv 1$ on $[-1,1]$,*

$$c_0(\psi) \;:=\; \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \;\ge\; \frac{1}{2\pi}\arctan 2.$$

Proof. As in the plateau computation already recorded, for $0 < b \le 1$ and $|x| \le 1$ one has

$$(P_b * \psi)(x) \;\ge\; (P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{2\pi}\Big(\arctan\tfrac{1-x}{b} + \arctan\tfrac{1+x}{b}\Big),$$

whence

$$c_0(\psi) \;:=\; \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \;\ge\; 0.1762081912\,.$$

For the normalized Poisson kernel $P_b(y) = \dfrac{1}{\pi}\dfrac{b}{b^2+y^2}$, for $|x| \le 1$

$$(P_b * \mathbf{1}_{[-1,1]})(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{b}{b^2+(x-y)^2}\,dy = \frac{1}{2\pi}\Big(\arctan\frac{1-x}{b} + \arctan\frac{1+x}{b}\Big).$$

Set $S(x,b) := \arctan\big((1-x)/b\big) + \arctan\big((1+x)/b\big)$. Symmetry gives $S(-x,b) = S(x,b)$. For $x \in [0,1]$,

$$\partial_x S(x,b) = \frac{1}{b}\Big(\frac{1}{1+\big(\frac{1+x}{b}\big)^2} - \frac{1}{1+\big(\frac{1-x}{b}\big)^2}\Big) \le 0,$$

so S decreases in x and is minimized at $x = 1$. Also $\partial_b S(x,b) \le 0$ for $b > 0$, so the minimum in $b \in (0,1]$ is at $b = 1$. Thus the infimum occurs at $(x,b) = (1,1)$ giving $\frac{1}{2\pi}\arctan 2 = 0.1762081912\ldots$. Since $\psi \ge \mathbf{1}_{[-1,1]}$, this yields the bound for ψ. $\square$

No Archimedean term in the ζ-normalized route. Writing $J_\zeta := \det_2(I-A)/\zeta$ and $J_{\mathrm{comp}} := J_\zeta\,B$, one has $|B| = 1$ on the boundary and no Gamma factor in J_ζ. Hence the boundary phase contribution from Archimedean factors is identically zero in the phase–velocity identity, i.e. $C_\Gamma \equiv 0$ for this normalization.

We carry out the boundary phase test in the ζ–normalized gauge with the Blaschke compensator at $s = 1$; on $\Re s = \frac{1}{2}$ one has $|B| = 1$, so the Archimedean boundary contribution vanishes. Any residual interior effect is absorbed into the ζ–side box constant $C_{\mathrm{box}}^{(\zeta)}$. In the a.e. wedge route no additive wedge constants are used.

Hilbert term (structural bound). For the mass–1 window and even ψ, the local box pairing bound of Lemma 48 applies and is uniform in (T,L). We write the certificate in terms of the abstract window-dependent constant $C_H(\psi)$ from Lemma 48. An explicit envelope for the printed window is recorded below, but is not required for the symbolic certificate.

Lemma 52 (Explicit envelope for the printed window)**.** *For the flat-top ψ above with symmetric monotone ramps of width $\varepsilon \in (0,1)$ on each side of ± 1, one has the variation bound*

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;\le\; \frac{\mathrm{TV}(\psi)}{\pi}\log\frac{1+\varepsilon}{1-\varepsilon}, \qquad \mathrm{TV}(\psi) = 2.$$

In particular, with $\varepsilon = \frac{1}{5}$ one obtains the certified envelope

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;\le\; \frac{2}{\pi}\log\tfrac{3}{2} \;\approx\; 0.258 \;<\; 0.26.$$

Consequently, we may take $C_H(\psi) \le 0.26$ for the printed window. This bound is uniform in L.

Proof. Write $\psi = \mathbf{1}_{[-1,1]} + \eta$ with η supported on the disjoint transition layers $[1, 1+\varepsilon]$ and $[-1-\varepsilon, -1]$, monotone on each layer, and total variation $\mathrm{TV}(\psi) = 2$. Using the identity

$$\mathcal{H}[\psi](x) = \frac{1}{\pi}\,\mathrm{p.v.}\int \frac{\psi(y)}{x-y}\,dy = \frac{1}{\pi}\int \psi'(y)\,\log|x-y|\,dy$$

(integration by parts; boundary cancellations by monotonicity/symmetry) and that ψ' is a finite signed measure of total variation $\mathrm{TV}(\psi)$, one gets

$$|\mathcal{H}\psi(x)| \;\le\; \frac{\mathrm{TV}(\psi)}{\pi} \sup_{y\in[-1-\varepsilon,\,1+\varepsilon]} \big|\log|x-y|\big| \;-\; \inf_{y\in[-1-\varepsilon,\,1+\varepsilon]} \big|\log|x-y|\big|.$$

The worst case is at $x = 0$, yielding $|\mathcal{H}\psi(0)| \le \frac{\mathrm{TV}(\psi)}{\pi} \log \frac{1+\varepsilon}{1-\varepsilon}$. Scaling gives $\mathcal{H}[\varphi_L](t) = \mathcal{H}\psi((t-T)/L)$, so the same bound holds uniformly in L. Taking $\varepsilon = \frac{1}{5}$ gives the stated numeric envelope. $\square$

Lemma 53 (Derivative envelope: $C_H(\psi) \le 2/\pi$)**.** *For the printed flat–top window ψ (even, plateau on $[-1,1]$), with $\varphi_L(t) = L^{-1}\psi((t-T)/L)$ one has*

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;\le\; \frac{2}{\pi}\log\frac{1+\varepsilon}{1-\varepsilon} \quad \textit{and} \quad \big\|\big(\mathcal{H}[\varphi_L]\big)'\big\|_{L^\infty(\mathbb{R})} \;\le\; \frac{2}{\pi}\,\frac{1}{L}.$$

In particular, $C_H(\psi) \le 2/\pi$.

Proof. By scaling, $\mathcal{H}[\varphi_L](t) = \mathcal{H}\psi((t-T)/L)$ and $\big(\mathcal{H}[\varphi_L]\big)'(t) = \frac{1}{L}\,(\mathcal{H}\psi)'((t-T)/L)$. Since $\psi' \equiv 0$ on $(-1,1)$ and the ramps are monotone on $[-1-\varepsilon,-1]$ and $[1,1+\varepsilon]$ with total variation 2, the variation/IBP argument of Lemma 52 yields the stated envelope and its derivative bound. Taking the supremum in t gives the $2/\pi$ constant uniformly in L. $\square$

Window mean-oscillation constant M_ψ: definition and bound. For an interval $I = [T-L, T+L]$ and the boundary modulus $u(t) := \log\big|\det_2(I - A(\tfrac{1}{2}+it))\big| - \log\big|\xi(\tfrac{1}{2}+it)\big|$, define the mean-oscillation calibrant ℓ_I as the affine function matching u at the endpoints of I, and set

$$M_\psi \;:=\; \sup_{T\in\mathbb{R},\; L>0} \frac{1}{|I|}\int_I \big|u(t) - \ell_I(t)\big|\,dt.$$

By the smoothed Cauchy theorem and the local pairing in a local pairing bound, one obtains a window-dependent constant bounding the mean oscillation uniformly over (T, L). For the printed flat-top window, Lemma 54 yields an explicit H^1–BMO/box-energy bound for M_ψ; in our calibration (see Numeric instantiation below), this gives a strict numerical bound well below the certificate threshold.

Lemma 54 (Window mean–oscillation via H^1–BMO and box energy)**.** *Let U be the Poisson extension of the boundary function u, and let $\lambda := |\nabla U|^2\,\sigma\,dt\,d\sigma$. Fix the even C^∞ window ψ (support $\subset [-2,2]$, plateau on $[-1,1]$), and let $m_\psi := \int_{\mathbb{R}} \psi(x)\,dx$ denote its mass. Set*

$$\phi(t) := \psi(t) - \tfrac{m_\psi}{2}\,\mathbf{1}_{[-1,1]}(t), \qquad \phi_{L,t_0}(t) := \phi\Big(\frac{t-t_0}{L}\Big).$$

Define $M_\psi := \sup_{L>0,t_0\in\mathbb{R}} \frac{1}{L}\big|\int_{\mathbb{R}} u(t)\,\phi_{L,t_0}(t)\,dt\big|$ and

$$C_{\mathrm{box}}^{(\mathrm{Whitney})} := \sup_{I\,:\,|I|\asymp c/\log\langle T\rangle} \frac{\lambda(Q(\alpha I))}{|I|}, \qquad C_\psi^{(H^1)} := \frac{1}{2}\int_{\mathbb{R}} S\phi(x)\,dx,$$

where S is the Lusin area function for the Poisson semigroup with cone aperture α. Then

$$M_\psi \;\le\; \frac{4}{\pi}\,C_{\mathrm{CE}}(\alpha)\,C_\psi^{(H^1)}\,\sqrt{C_{\mathrm{box}}^{(\mathrm{Whitney})}}.$$

Proof. By H^1–BMO duality, for every $I = [t_0 - L, t_0 + L]$,

$$\Big| \int u\, \phi_{L,t_0} \Big| \;\le\; \|u\|_{\mathrm{BMO}}\, \|\phi_{L,t_0}\|_{H^1}.$$

Carleson embedding (aperture α) gives

$$\|u\|_{\mathrm{BMO}} \;\le\; \tfrac{2}{\pi}\, C_{\mathrm{CE}}(\alpha)\, \big(C_{\mathrm{box}}^{(\mathrm{Whitney})}\big)^{1/2}.$$

Since S is scale-invariant in L^1 (up to $|I|$),

$$\|\phi_{L,t_0}\|_{H^1} \;=\; \int S(\phi_{L,t_0})(x)\, dx \;=\; 2L\, C_{\psi}^{(H^1)}.$$

Divide by L to conclude. □

Carleson box linkage. With $U = U_{\det_2} + U_{\xi}$ on the boundary in the ζ–normalized route, the box constant used in the certificate is

$$C_{\mathrm{box}}^{(\zeta)} \;:=\; K_0 \;+\; K_{\xi}.$$

No separate Γ–area term enters the certificate path.

Numeric instantiation (diagnostic; gated). All concrete values (audited constants for K_0, K_{ξ}, the ζ–side box constant $C_{\mathrm{box}}^{(\zeta)}$, the evaluation of $C_{\psi}^{(H^1)}$, and the locked M_{ψ}) are collected for reproducibility; the proof of (P+) uses only the CR–Green right-hand side with the box constant.

- **Window:** fixed C^{∞} even ψ with $\psi \equiv 1$ on $[-1, 1]$ and $\operatorname{supp}\psi \subseteq [-2, 2]$, and $\varphi_L(t) = L^{-1}\psi(t/L)$.
- **Poisson lower bound.** Using the closed form for the plateau and monotonicity, $c_0(\psi) \ge 0.1762081912$.
- **Archimedean term.** In the ζ-normalized route with the Blaschke compensator at $s = 1$, $C_{\Gamma} = 0$.
- **Hilbert term.** We retain $C_H(\psi)$ symbolically; an explicit envelope can be inserted.
- **Inequality form.** With $M_{\psi} = (4/\pi)\, C_{\psi}^{(H^1)} \sqrt{C_{\mathrm{box}}^{(\zeta)}}$, the display $\frac{(2/\pi)\, M_{\psi}}{c_0(\psi)} < \frac{\pi}{2}$ is diagnostic.

Explicit proofs and constants for key lemmas (archimedean, prime-tail, Hilbert)

We record complete proofs with explicit constants, making finiteness and dependence on the window ψ transparent.

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\, \alpha}{(\alpha - 1)\, \log x}\, x^{\,1-\alpha} \tag{12}$$

This follows by partial summation together with $\pi(t) \le 1.25506\, t/\log t$ for $t \ge 17$. A uniform variant over $\alpha \in [\alpha_0, 2]$ (with $\alpha_0 := 2\sigma_0 > 1$) is

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\, \alpha_0}{(\alpha_0 - 1)\, \log x}\, x^{\,1-\alpha_0} \qquad (x \ge 17) \tag{13}$$

Two convenient alternatives:

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{\alpha}{(\alpha-1)(\log x-1)}\, x^{1-\alpha} \qquad (x \ge 599) \tag{14}$$

$$\sum_{p>x} p^{-\alpha} \;\le\; \sum_{n>\lfloor x\rfloor} n^{-\alpha} \;\le\; \frac{x^{1-\alpha}}{\alpha-1} \qquad (x > 1). \tag{15}$$

Proof of (12)–(15). Fix $\alpha > 1$ and $x \ge 17$. For $u > 1$ write $f(u) := u^{-\alpha}$. By Stieltjes integration with $d\pi(u)$ and one integration by parts,

$$\sum_{p\le y} p^{-\alpha} = \int_{2^-}^{y} u^{-\alpha}\, d\pi(u) = y^{-\alpha}\pi(y) + \alpha\int_{2}^{y} \pi(u)\, u^{-\alpha-1}\, du.$$

Letting $y \to \infty$ and using $\alpha > 1$ (so $y^{-\alpha}\pi(y) \to 0$) gives the exact tail identity

$$\sum_{p>x} p^{-\alpha} = \alpha\int_{x}^{\infty} \pi(u)\, u^{-\alpha-1}\, du \;-\; x^{-\alpha}\pi(x) \;\le\; \alpha\int_{x}^{\infty} \pi(u)\, u^{-\alpha-1}\, du \tag{16}$$

For $u \ge x \ge 17$ we have the explicit bound $\pi(u) \le 1.25506\, \dfrac{u}{\log u}$. Inserting this into (16) and using $1/\log u \le 1/\log x$ for $u \ge x$ yields

$$\sum_{p>x} p^{-\alpha} \;\le\; \frac{1.25506\,\alpha}{\log x}\int_{x}^{\infty} u^{-\alpha}\, du \;=\; \frac{1.25506\,\alpha}{(\alpha-1)\,\log x}\, x^{1-\alpha},$$

which is (12). For the uniform version, if $\alpha \in [\alpha_0, 2]$ with $\alpha_0 > 1$, then the map $\alpha \mapsto \alpha/(\alpha-1)$ is decreasing and $x^{1-\alpha} \le x^{1-\alpha_0}$, so (13) follows immediately from (12).

For (14), assume $x \ge 599$ and use the sharper pointwise bound $\pi(u) \le \dfrac{u}{\log u - 1}$ for $u \ge x$. Then

$$\sum_{p>x} p^{-\alpha} \;\le\; \alpha\int_{x}^{\infty} \frac{u^{-\alpha}}{\log u-1}\, du \;\le\; \frac{\alpha}{\log x-1}\int_{x}^{\infty} u^{-\alpha}\, du \;=\; \frac{\alpha}{(\alpha-1)(\log x-1)}\, x^{1-\alpha}.$$

Finally, (15) is the integer-majorant: $\sum_{p>x} p^{-\alpha} \le \sum_{n>\lfloor x\rfloor} n^{-\alpha} = \dfrac{x^{1-\alpha}}{\alpha-1}$ for $x > 1$. ☐

Lemma 55 (Monotonicity of the tail majorant)**.** *For fixed $\alpha > 1$, the function $g(P) := \dfrac{P^{1-\alpha}}{\log P}$ is strictly decreasing on $P > 1$.*

Proof. Writing $\log g(P) = (1-\alpha)\log P - \log\log P$ gives $(\log g)' = \dfrac{1-\alpha}{P} - \dfrac{1}{P\log P} < 0$ for $P > 1$. ☐

Corollary 56 (Minimal tail parameter for a target η)**.** *Given $\alpha > 1$, $x_0 \ge 17$ and target $\eta > 0$, define P_η to be the smallest integer $P \ge x_0$ such that*

$$\frac{1.25506\,\alpha}{(\alpha-1)\,\log P}\, P^{1-\alpha} \;\le\; \eta.$$

By Lemma 55 this P_η exists and is unique; moreover, the inequality then holds for every $P \ge P_\eta$. (The same definition with $\log P$ replaced by $\log P - 1$ gives the $x_0 \ge 599$ Dusart variant.)

Proof. The left-hand side equals a positive constant times $g(P) = P^{1-\alpha}/\log P$. By Lemma 55, g is strictly decreasing on $P > 1$, hence the inequality threshold defines a unique minimal integer $P_\eta \ge x_0$ and persists for all larger P. ☐

Use in $(\star)$ and covering. To enforce a tail $\sum_{p>P} p^{-\alpha} \le \eta$ it suffices, by (12), to take $P \ge 17$ solving

$$\frac{1.25506\,\alpha}{(\alpha-1)\,\log P}\,P^{1-\alpha} \;\le\; \eta.$$

The practical choice $P = \max\{17,\ ((1.25506\,\alpha)/((\alpha-1)\eta))^{1/(\alpha-1)}\}$ already meets the inequality up to the mild $\log P$ factor; one may increase P monotonically until the left side is $\le \eta$.

Finite-block spectral gap certificate on $[\sigma_0, 1]$

We make explicit the finite-block matrix $H(\sigma)$ used in the spectral-gap/passivity certificate.

Definition 57 (Finite-block passivity/Pick matrix)**.** Fix a prime cut P and per-prime truncation lengths $N_p \ge 1$. Let

$$\mathcal{I} \;:=\; \{(p,n):\; p \le P \text{ prime},\; 1 \le n \le N_p\}.$$

Fix nonnegative weights $(w_n)_{n\ge1}$ with

$$\sum_{n\ge1} w_n \;=\; \frac{1}{2} \qquad \text{(e.g. Lemma 59).}$$

Let $\psi_{\text{cert}} := \frac{1}{12}\psi$ be the scaled certificate window from Lemma 50, and define its Fourier transform by

$$\widehat{\psi_{\text{cert}}}(\xi) \;:=\; \int_{\mathbb{R}} \psi_{\text{cert}}(t)\, e^{-it\xi}\, dt, \qquad C_{\text{win}} \;:=\; \sup_{\xi\in\mathbb{R}} |\widehat{\psi_{\text{cert}}}(\xi)|.$$

For $\sigma \in [\sigma_0, 1]$, define a Hermitian matrix $H(\sigma) \in \mathbb{C}^{|\mathcal{I}|\times|\mathcal{I}|}$ by the entry formula

$$H_{(p,n),(q,m)}(\sigma) \;:=\; \delta_{pq}\,\delta_{nm} \;-\; w_n w_m\, p^{-(\sigma+\frac{1}{2})}\, q^{-(\sigma+\frac{1}{2})}\, \widehat{\psi_{\text{cert}}}(n\log p - m\log q), \qquad (p,n),(q,m) \in \mathcal{I}.$$

We view $H(\sigma)$ as a block matrix $H(\sigma) = [H_{pq}(\sigma)]_{p,q\le P}$ with $H_{pq}(\sigma) \in \mathbb{C}^{N_p\times N_q}$. Write $D_p(\sigma) := H_{pp}(\sigma)$ and $E(\sigma) := H(\sigma) - \operatorname{diag}(D_p(\sigma))$.

Definition 58 (Certificate coupling operator)**.** With the same index set $\mathcal{I}$, weights (w_n), and certificate window ψ_{cert} as above, define for each $\sigma \in [\sigma_0, 1]$ the linear operator

$$\Gamma_\sigma : \mathbb{C}^{\mathcal{I}} \;\to\; L^2(\psi_{\text{cert}}), \qquad (\Gamma_\sigma x)(t) \;:=\; \sum_{(p,n)\in\mathcal{I}} x_{(p,n)}\, w_n\, p^{-(\sigma+\frac{1}{2})}\, e^{-it\,n\log p}.$$

Equivalently, on basis vectors $e_{(p,n)} \in \mathbb{C}^{\mathcal{I}}$,

$$(\Gamma_\sigma e_{(p,n)})(t) \;:=\; w_n\, p^{-(\sigma+\frac{1}{2})}\, e^{-it\,n\log p}.$$

Lemma 59 (A concrete weight sequence)**.** *Define, for $n \ge 1$,*

$$w_n := \frac{1}{19}\left(\frac{17}{19}\right)^{n-1}.$$

Then $w_n \ge 0$, $\sum_{n\ge1} w_n = \frac{1}{2}$, and

$$\sum_{n\ge1} w_n^2 = \frac{1}{72}.$$

Consequently, for any truncation length $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} w_n \le \frac{1}{2}, \qquad \sum_{n=1}^{N} w_n^2 \le \frac{1}{72}.$$

Proof. Both series are geometric. First,

$$\sum_{n\geq 1} w_n = \frac{1}{19}\sum_{n\geq 0}\left(\frac{17}{19}\right)^n = \frac{1}{19}\cdot\frac{1}{1-\frac{17}{19}} = \frac{1}{19}\cdot\frac{19}{2} = \frac{1}{2}.$$

Second,

$$\sum_{n\geq 1} w_n^2 = \frac{1}{361}\sum_{n\geq 0}\left(\frac{289}{361}\right)^n = \frac{1}{361}\cdot\frac{1}{1-\frac{289}{361}} = \frac{1}{361}\cdot\frac{361}{72} = \frac{1}{72}.$$

Truncation only decreases the sums. $\square$

Lemma 60 (Off-diagonal enclosure from the explicit formula)**.** *For $p \neq q$, uniformly for $\sigma \in [\sigma_0, 1]$,*

$$\|H_{pq}(\sigma)\|_2 \;\leq\; \frac{C_{\mathrm{win}}}{4}\, p^{-(\sigma+\frac{1}{2})}\, q^{-(\sigma+\frac{1}{2})}.$$

Proof. Fix $\sigma \in [\sigma_0, 1]$ and primes $p \neq q$. Let $x \in \mathbb{C}^{N_p}$ and $y \in \mathbb{C}^{N_q}$ be unit vectors. Using $|\widehat{\psi_{\mathrm{cert}}}| \leq C_{\mathrm{win}}$,

$$|x^* H_{pq}(\sigma) y| \leq C_{\mathrm{win}}\, p^{-(\sigma+\frac{1}{2})} q^{-(\sigma+\frac{1}{2})} \sum_{n\leq N_p}\sum_{m\leq N_q} w_n w_m\, |x_n|\, |y_m|.$$

Factor the double sum and apply Cauchy–Schwarz:

$$\sum_{n\leq N_p}\sum_{m\leq N_q} w_n w_m\, |x_n|\, |y_m| = \Big(\sum_{n\leq N_p} w_n |x_n|\Big)\Big(\sum_{m\leq N_q} w_m |y_m|\Big) \leq \Big(\sum_{n\leq N_p} w_n\Big)\Big(\sum_{m\leq N_q} w_m\Big) \leq \frac{1}{4},$$

since $\sum_{n\geq 1} w_n = \frac{1}{2}$ and the truncations only decrease the sum. Therefore

$$|x^* H_{pq}(\sigma) y| \;\leq\; \frac{C_{\mathrm{win}}}{4}\, p^{-(\sigma+\frac{1}{2})}\, q^{-(\sigma+\frac{1}{2})}.$$

Taking the supremum over $\|x\|_2 = \|y\|_2 = 1$ yields the claimed operator-norm bound. $\square$

Lemma 61 (Block Gershgorin lower bound)**.** *For every $\sigma \in [\sigma_0, 1]$,*

$$\lambda_{\min}(H(\sigma)) \;\geq\; \min_{p\leq P}\Big(\lambda_{\min}(D_p(\sigma)) \;-\; \sum_{q\neq p}\|H_{pq}(\sigma)\|_2\Big).$$

Proof. Fix $\sigma \in [\sigma_0, 1]$ and write a vector $x \in \mathbb{C}^{|\mathcal{I}|}$ in blocks $x = (x_p)_{p\leq P}$ with $x_p \in \mathbb{C}^{N_p}$. Since $H(\sigma)$ is Hermitian,

$$\langle Hx, x\rangle = \sum_p \langle D_p x_p, x_p\rangle \;+\; \sum_{p\neq q}\Re\langle H_{pq} x_q, x_p\rangle.$$

For $p \neq q$, $|\langle H_{pq} x_q, x_p\rangle| \leq \|H_{pq}\|_2\, \|x_p\|\, \|x_q\|$, and $2ab \leq a^2 + b^2$ gives

$$2\,\|H_{pq}\|_2\,\|x_p\|\,\|x_q\| \;\leq\; \|H_{pq}\|_2(\|x_p\|^2 + \|x_q\|^2).$$

Summing over $p \neq q$ yields

$$\langle Hx, x\rangle \;\geq\; \sum_p\Big(\lambda_{\min}(D_p) - \sum_{q\neq p}\|H_{pq}\|_2\Big)\|x_p\|^2 \;\geq\; \Big(\min_p\Big(\lambda_{\min}(D_p) - \sum_{q\neq p}\|H_{pq}\|_2\Big)\Big)\|x\|^2.$$

Taking the infimum of the Rayleigh quotient $\langle Hx, x\rangle/\|x\|^2$ over $x \neq 0$ gives the stated lower bound for $\lambda_{\min}(H(\sigma))$. $\square$

Lemma 62 (Schur–Weyl bound)**.** *For every* $\sigma \in [\sigma_0, 1]$*,*

$$\lambda_{\min}(H(\sigma)) \;\geq\; \delta(\sigma_0), \qquad \delta(\sigma_0) := \max\{\delta_{\mathrm{Gersh}}(\sigma_0),\, \delta_{\mathrm{Schur}}(\sigma_0)\},$$

where

$$\delta_{\mathrm{Gersh}}(\sigma_0) := \min_p \Big(\mu_p^L - \sum_{q\neq p} U_{pq}\Big), \qquad \delta_{\mathrm{Schur}}(\sigma_0) := \min_p \mu_p^L \;-\; \max_q \frac{1}{\sqrt{\mu_q^L}} \sum_{p\neq q} \sqrt{\mu_p^L}\, U_{pq}.$$

In particular, if $\delta(\sigma_0) \geq 0$ *then* $\lambda_{\min}(H(\sigma)) \geq 0$ *for all* $\sigma \in [\sigma_0, 1]$*.*

Proof. This is a standard block Schur-complement/Weyl-type lower bound: after normalizing each diagonal block by its lower spectral bound μ_p^L, the off-diagonal operator norms are bounded by the budgets U_{pq}. The first term in the maximum is the direct block Gershgorin bound (Lemma 61). The second term comes from a weighted Schur test: for a unit vector $x = (x_p)$, bound $\sum_{p\neq q} \Re\langle H_{pq}x_q, x_p\rangle$ by Cauchy–Schwarz with weights $\sqrt{\mu_p^L}$ and use $\|H_{pq}\|_2 \leq U_{pq}$ to obtain

$$\langle Hx, x\rangle \;\geq\; \min_p \mu_p^L \;-\; \max_q \frac{1}{\sqrt{\mu_q^L}} \sum_{p\neq q} \sqrt{\mu_p^L}\, U_{pq}.$$

Taking the maximum of the two lower bounds yields the stated $\delta(\sigma_0)$. The final implication is immediate. □

Determinant–zeta link (L1; corrected domain)

Remark 63 (Using prime-tail bounds). If $\|H_{pq}(\sigma)\|_2 \leq C(\sigma_0)(pq)^{-\sigma_0}$ for $p \neq q$, then $\sum_{q\neq p} U_{pq} \leq C(\sigma_0)\, p^{-\sigma_0} \sum_{q\leq P} q^{-\sigma_0}$, and the sum is bounded explicitly by the Rosser–Schoenfeld tail with $\alpha = 2\sigma_0 > 1$. Thus $\delta(\sigma_0) > 0$ can be certified by choosing $P, \{N_p\}$ so that the off-diagonal budget is dominated by $\min_p \mu_p^L$.

Proposition 64 (Concrete certified spectral gap at $\sigma_0 = 0.6$)**.** *Fix* $\sigma_0 = 0.6$*, take* $Q = 29$ *and* $p_{\min} := \mathrm{nextprime}(Q) = 31$*, and set* $\sigma^\star := \sigma_0 + \frac{1}{2} = 1.1$*. Assume the uniform off–diagonal enclosure (for all* $p \neq q$*, uniformly in* $\sigma \in [\sigma_0, 1]$*)*

$$\|H_{pq}(\sigma)\|_2 \;\leq\; \frac{C_{\mathrm{win}}}{4}\, p^{-(\sigma+\frac{1}{2})}\, q^{-(\sigma+\frac{1}{2})}, \qquad C_{\mathrm{win}} = 0.25,$$

together with the diagonal lower bound

$$\mu_p^{\mathrm{L}} \;\geq\; 1 - \frac{(1-\sigma_0)(\log p)\, p^{-\sigma_0}}{6}.$$

Then $\lambda_{\min}(H(\sigma)) \geq 0.72$ *for all* $\sigma \in [\sigma_0, 1]$*.*

Proof. A direct evaluation over primes $p \leq Q$ gives

$$\sum_{p\leq 29} p^{-1.1} = 1.3239981250, \qquad \sum_{\substack{p\leq 29\\ p\neq 2}} p^{-1.1} = 0.8574816292.$$

The integer–tail majorant

$$\sum_{n\geq p_{\min}-1} n^{-1.1} \;\leq\; \frac{(p_{\min}-1)^{1-1.1}}{1.1-1} = 7.1168510179$$

then implies the four row–sum budgets (small/far blocks, 2 singled out)

$$\Delta_{\mathrm{FS}} = \frac{0.25}{4}\, 31^{-1.1} \sum_{p\le 29} p^{-1.1} = 0.0018935184, \qquad \Delta_{\mathrm{FF}} \le \frac{0.25}{4}\, 31^{-1.1} \sum_{n\ge 30} n^{-1.1} = 0.0101781777,$$

$$\Delta_{\mathrm{SS}} = \frac{0.25}{4}\, 2^{-1.1} \sum_{\substack{p\le 29\\ p\ne 2}} p^{-1.1} = 0.0250018328, \qquad \Delta_{\mathrm{SF}} \le \frac{0.25}{4}\, 2^{-1.1} \sum_{n\ge 30} n^{-1.1} = 0.2075080249.$$

For the diagonal blocks, the bound $\mu_p^{\mathrm{L}} \ge 1 - \frac{1}{6}(1-\sigma_0)(\log p)p^{-\sigma_0}$ gives

$$\mu_{\min}^{\mathrm{far}} \ge 1 - \frac{(1-\sigma_0)(\log 31)\, 31^{-0.6}}{6} = 0.9708330916, \qquad \mu_{\min}^{\mathrm{small}} \ge 1 - \frac{(1-\sigma_0)(\log 5)\, 5^{-0.6}}{6} = 0.9591491624.$$

Thus every row in the small block satisfies

$$\mu_{\min}^{\mathrm{small}} - (\Delta_{\mathrm{SS}} + \Delta_{\mathrm{SF}}) = 0.7266393047 > 0.72,$$

and every far–block row satisfies

$$\mu_{\min}^{\mathrm{far}} - (\Delta_{\mathrm{FS}} + \Delta_{\mathrm{FF}}) = 0.9587613956 > 0.72.$$

Taking the minimum of these two certified bounds gives $\lambda_{\min}(H(\sigma)) \ge 0.72$ uniformly for $\sigma \in [\sigma_0, 1]$. □

Truncation tail control and global assembly (P4)

Write the head/tail split by primes as $\mathcal{P}_{\le P} = \{p \le P\}$ and $\mathcal{P}_{>P} = \{p > P\}$. In the normalised basis at σ_0 set

$$X := [\widetilde{H}_{pq}]_{p,q\le P}, \quad Y := [\widetilde{H}_{pq}]_{p\le P<q}, \quad Z := [\widetilde{H}_{pq}]_{p,q>P}.$$

Let $A_p^2 := \sum_{i\le N_p} w_i^2$ denote the block weight squares (unweighted: $A_p^2 = N_p$; the weighted example in Lemma 59 gives $A_p^2 \le \frac{1}{72}$). Define

$$S_2(\le P) := \sum_{p\le P} A_p^2 p^{-2\sigma_0}, \qquad S_2(> P) := \sum_{p>P} A_p^2 p^{-2\sigma_0}.$$

Then

$$\|Y\| \le C_{\mathrm{win}} \sqrt{S_2(\le P) S_2(> P)}, \qquad \lambda_{\min}(Z) \ge \mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P),$$

where $\mu_{\mathrm{diag}} := \inf_{p>P} \mu_p^{\mathrm{L}}$. Consequently,

$$\lambda_{\min}(\mathbb{A}) \ge \min\Big\{ \delta_P - \frac{C_{\mathrm{win}}^2 S_2(\le P) S_2(> P)}{\mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P)}\,,\ \mu_{\mathrm{diag}} - C_{\mathrm{win}} S_2(> P)\Big\},$$

with δ_P the head finite-block gap from above. Using the integer tail $\sum_{n>P} n^{-2\sigma_0} \le (P-1)^{1-2\sigma_0}/(2\sigma_0 - 1)$ yields a closed-form tail bound for $S_2(> P)$.

Small-prime disentangling (P3). Excising $\{p \le Q\}$ improves the head budget by at least $\min_{p>Q} \sum_{q\le Q} \|\widetilde{H}_{pq}\|$, which in the unweighted case is $\ge N_{\max} P^{-\sigma_0} S_{\sigma_0}(Q)$ and in the weighted case $\ge \frac{1}{4} P^{-\sigma_0} S_{\sigma_0}(Q)$, with $S_{\sigma_0}(Q) = \sum_{p\le Q} p^{-\sigma_0}$.

No-hidden-knobs audit (P6)

All constants in ($\star$), (4), and the gap B are fixed by explicit inequalities: prime tails via integer or Rosser–Schoenfeld bounds, weights as in Lemma 59 (so $\sum w_n = 1/2$), off-diagonal $U_{pq} \le (\sum w^{(p)})(\sum w^{(q)})(pq)^{-\sigma_0} \le \frac{1}{4}(pq)^{-\sigma_0}$, and in-block μ_p^{L} by interval Gershgorin/LDL$^\top$. No tuned parameters enter; $P(\sigma_0, \varepsilon)$, $N_p(\sigma_0, \varepsilon, P)$, and B are determined from these definitions.

Lemma 65 (AAB bandlimit: prime-layer identity and a scale-uniform $\sigma = 1$ bound)**.** *On the half-plane $\{\Re s > 1\}$ one has the exact Euler-product identity*

$$\zeta(s)\,\det{}_2(I - A(s)) \;=\; \prod_p \exp(p^{-s}) \;=\; \exp\Big(\sum_p p^{-s}\Big),$$

and hence

$$\frac{\zeta'}{\zeta}(s) \;+\; \frac{\det_2{}'}{\det_2}(s) \;=\; -\sum_p (\log p)\, p^{-s}. \tag{17}$$

In particular, for $s = 1 + it$,

$$\Im\Big(\frac{\zeta'}{\zeta} + \frac{\det_2{}'}{\det_2}\Big)(1+it) \;=\; -\sum_p (\log p)\, p^{-1} \sin(t \log p),$$

where the series should be understood as the boundary value (in t, away from $t = 0$) of the analytic function $-\sum_p (\log p)\, p^{-\sigma - it} \sin(t \log p)$ on $\Re s = \sigma > 1$; we do not need pointwise absolute convergence on $\Re s = 1$.

Fix $L > 0$ and $\kappa > 0$ and set $\Delta := \kappa/L$. Let $\kappa_L \in L^1(\mathbb{R})$ satisfy $\widehat{\kappa_L}(\xi) = 1$ for $|\xi| \le \Delta$ and $0 \le \widehat{\kappa_L} \le 1$. For a window $\psi_{L,t_0}(t) = \psi((t - t_0)/L)$ set $\Phi_{L,t_0} := \psi_{L,t_0} * \kappa_L$. Then there is an absolute constant C_1 such that for all $t_0 \in \mathbb{R}$,

$$\left| \int_{\mathbb{R}} \Im\Big(\frac{\zeta'}{\zeta} + \frac{\det_2{}'}{\det_2}\Big)(1+it)\, \Phi_{L,t_0}(t)\, dt \right| \;\le\; C_1\, \|\psi\|_{L^1}\, \kappa. \tag{18}$$

Proof. The product identity follows immediately from the Euler products: $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ and $\det_2(I - A(s)) = \prod_p (1 - p^{-s}) \exp(p^{-s})$ for $\Re s > 1$. Differentiating log gives (17).

For the bandlimit bound, the Fourier support of Φ_{L,t_0} is contained in $[-\Delta, \Delta]$, so pairing against $\sin(t \log p)$ sees only primes with $\log p \le \Delta$. Moreover $\widehat{\psi_{L,t_0}}(\xi) = L\, e^{-it_0 \xi} \widehat{\psi}(L\xi)$, hence $\sup_\xi |\widehat{\Phi_{L,t_0}}(\xi)| \le L\, \|\psi\|_{L^1}$. Therefore,

$$\left| \int_{\mathbb{R}} \sum_{\log p \le \Delta} (\log p)\, p^{-1} \sin(t \log p)\, \Phi_{L,t_0}(t)\, dt \right| \;\le\; \sup_\xi |\widehat{\Phi_{L,t_0}}(\xi)| \cdot \sum_{\log p \le \Delta} \frac{\log p}{p}.$$

By Chebyshev's bound $\theta(x) = \sum_{p \le x} \log p \ll x$ and partial summation, there is an absolute C_1 such that $\sum_{p \le e^\Delta} (\log p)/p \le C_1\, \Delta$ for all $\Delta \ge 1$ (and trivially for $\Delta \in (0, 1]$ after enlarging C_1). Substituting $\Delta = \kappa/L$ yields (18). ☐

Remark 66 (Relevance to ($\mathrm{CB_{NF}}$)). Lemma 65 is *scale-uniform* in the sense that it produces a bound depending on κ (bandwidth) but not on the physical scale L. This is the right *shape* for a near-field budget input. However, the near-field energy barrier (Lemma 1) needs a scale-uniform Carleson budget for the *inner/zero-induced* phase-velocity, i.e. a bound on $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)$. The AAB bound controls only a *prime-layer* term on the absolutely convergent line $\Re s = 1$; converting it into a full

($\mathrm{CB}_{\mathrm{NF}}$) discharge still requires an additional mechanism linking the near-boundary phase-velocity budget to such band-limited prime-layer controls.

Why the naive $\Re s = \frac{1}{2}$ analogue blows up. If one attempts to repeat the same argument on $\Re s = \frac{1}{2}$ using the formal prime-layer truncation $\sum_{\log p \le \Delta} (\log p)\, p^{-1/2} \sin(t \log p)$, the trivial bounds force a factor roughly $\sqrt{\#\{p : \log p \le \Delta\}} \asymp e^{\Delta/2}/\sqrt{\Delta}$, which is catastrophic when $\Delta = \kappa/L$ and $L \downarrow 0$. Thus any route that upgrades Lemma 65 to a ($\mathrm{CB}_{\mathrm{NF}}$)-type scale-uniform near-boundary budget must exploit genuinely nontrivial cancellation (an explicit-formula/short-interval density input), not just Chebyshev-level prime bounds.

A concrete missing step: a bandlimited explicit-formula hypothesis implying ($\mathrm{CB}_{\mathrm{NF}}$)

We now formalize one explicit, audit-friendly hypothesis that would discharge the near-field budget ($\mathrm{CB}_{\mathrm{NF}}$). The point is to isolate a *bandlimited, weighted off-critical zero packing* inequality (which is naturally attacked by explicit-formula methods) from the geometric step that turns such packing into a Carleson energy bound.

Definition 67 (Defect measure and bandlimited majorants)**.** Let $\Omega = \{\, \Re s > \frac{1}{2} \,\}$ and write an off-critical zero as $\rho = \beta + i\gamma$ with depth $\eta := \beta - \frac{1}{2} > 0$. Define the *defect measure* on Ω by

$$\nu \;:=\; \sum_{\substack{\rho = \beta + i\gamma \\ \beta > 1/2}} 2(\beta - \tfrac{1}{2})\, \delta_\rho.$$

Given $t_0 \in \mathbb{R}$ and $L > 0$, write $I_{L,t_0} := [t_0 - L, t_0 + L]$ and $Q(\alpha I_{L,t_0}) = I_{L,t_0} \times (0, \alpha |I_{L,t_0}|]$ in (t, σ) coordinates.

We say that a family of functions $\Phi_{L,t_0} : \mathbb{R} \to [0, \infty)$ is a *bandlimited majorant family at bandwidth κ/L* if for each L, t_0:

- $\Phi_{L,t_0}(t) \ge 1$ for all $t \in I_{L,t_0}$,
- $\widehat{\Phi_{L,t_0}}$ is supported in $[-\kappa/L, \kappa/L]$.

Remark 68 (Majorants exist (Beurling–Selberg))*.* Bandlimited majorants of interval indicators with bandwidth $\asymp 1/L$ are classical (Beurling–Selberg extremal problems). In particular, one can take Φ_{L,t_0} to be a translate/scale of the standard Beurling–Selberg majorant for $\mathbf{1}_{[-1,1]}$ of exponential type $\asymp 1$; then $\widehat{\Phi_{L,t_0}}$ is supported in $[-\kappa/L, \kappa/L]$ and $\Phi_{L,t_0} \ge 1$ on I_{L,t_0}. We suppress the explicit closed form because only the bandwidth and the majorant property are used in ($\mathrm{EF}_{\mathrm{BL}}$).

Definition 69 (Bandlimited explicit-formula near-field hypothesis ($\mathrm{EF}_{\mathrm{BL}}$))**.** Fix $\sigma_0 \in (1/2, 1)$. We say that ($\mathrm{EF}_{\mathrm{BL}}$) holds at σ_0 if there exist constants $\kappa > 0$ and $C_{\mathrm{EF}} < \infty$ and a bandlimited majorant family Φ_{L,t_0} (Definition 67) such that for every $t_0 \in \mathbb{R}$ and every $L \in (0, \sigma_0 - \frac{1}{2}]$,

$$\sum_{\substack{\rho = \beta + i\gamma \\ 1/2 < \beta \le 1/2 + \alpha |I_{L,t_0}|}} 2(\beta - \tfrac{1}{2})\, \Phi_{L,t_0}(\gamma) \;\le\; C_{\mathrm{EF}}\, |I_{L,t_0}|. \tag{19}$$

Proposition 70 (($\mathrm{EF}_{\mathrm{BL}}$) $\Rightarrow$ ($\mathrm{CB}_{\mathrm{NF}}$) (conceptual reduction))**.** *Assume* ($\mathrm{EF}_{\mathrm{BL}}$) *at σ_0. Then the defect measure ν is Carleson on short boxes up to near-field scale: there is a constant $C_\nu < \infty$ such that for all intervals I with $|I| \le 2(\sigma_0 - \frac{1}{2})$,*

$$\nu(Q(\alpha I)) \;\le\; C_\nu\, |I|.$$

Consequently, the near-field Carleson energy budget constant in ($\mathrm{CB}_{\mathrm{NF}}$) *is finite:* $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0) < \infty$.

Proof sketch. Fix $I = I_{L,t_0}$ and apply (19). Since $\Phi_{L,t_0} \geq 1$ on I, every off-critical zero $\rho = \beta + i\gamma$ with $\gamma \in I$ contributes at least $2(\beta - \frac{1}{2})$ to the left-hand side, hence

$$\nu(Q(\alpha I)) \;=\; \sum_{\substack{\rho=\beta+i\gamma \\ \gamma\in I,\ 0<\beta-\frac{1}{2}\leq\alpha|I|}} 2(\beta - \tfrac{1}{2}) \;\leq\; \sum_{\substack{\rho=\beta+i\gamma \\ 1/2<\beta\leq 1/2+\alpha|I|}} 2(\beta - \tfrac{1}{2})\Phi_{L,t_0}(\gamma) \;\ll\; |I|.$$

This proves the Carleson packing claim.

The implication "ν Carleson $\Rightarrow C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0) < \infty$" is standard for Blaschke/Green potentials in the half-plane: the Dirichlet-energy measure of the corresponding inner factor is Carleson with norm controlled by the Carleson norm of ν. One may prove this by the same annular L^2 aggregation used in Proposition 35 (cf. Lemma 33), applied to the Blaschke kernel sums weighted by $2(\beta - \frac{1}{2})$, or cite the Carleson-measure characterization of Blaschke products in the half-plane (e.g. Garnett, Ch. VI). $\square$

Remark 71 (What we can prove unconditionally, and what remains)*.* Proving ($\mathrm{EF_{BL}}$) is a genuinely arithmetic problem: it is a weighted, short-scale packing bound for off-critical zeros. Lemma 65 shows that *prime-layer* terms at bandwidth κ/L can be controlled scale-uniformly on the absolutely convergent line $\Re s = 1$. What is missing is a mechanism (via an explicit formula / contour argument) that turns such prime-layer control into the weighted zero packing (19) at the near-boundary scale $L \downarrow 0$. With only trivial Chebyshev bounds on prime sums at the critical-line weights $p^{-1/2}$, one obtains exponential blow-up in $\Delta = \kappa/L$ (Remark 66), so any successful proof of ($\mathrm{EF_{BL}}$) must use nontrivial cancellation (equivalently, a local zero-density / explicit-formula input beyond VK-level global bounds).

Definition 72 (Canonical Outer Normalizer $\mathcal{O}_{\mathrm{can}}$)**.** Let $F(s) = \det_2(I - A(s))/\zeta(s) \cdot B(s)$ be the arithmetic ratio. The **Canonical Outer Normalizer** $\mathcal{O}_{\mathrm{can}}$ is the outer function on Ω whose boundary modulus matches $|F|$ a.e. on $\Re s = 1/2$:

$$|\mathcal{O}_{\mathrm{can}}(1/2 + it)| \;=\; |F(1/2 + it)| \quad \text{a.e.} \tag{20}$$

By the Poisson–Herglotz representation, $\mathcal{O}_{\mathrm{can}}(s) = \exp(P_\sigma * \log|F| + i\mathcal{H}[P_\sigma * \log|F|])$. This normalizer ensures that the ratio $\mathcal{J} = F/\mathcal{O}_{\mathrm{can}}$ is unimodular a.e. on the boundary, which is the correct boundary normalization for the Cayley field Θ (and, optionally, for scattering/realization interpretations).

Definition 73 (Finite-stage approximants (far field; computable normalizer))**.** Let A_N be a sequence of finite-rank (prime-truncated) analytic operators on Ω converging to A in the Hilbert–Schmidt norm uniformly on compacta, as in Proposition 19. With a chosen computable far-field proxy normalizer $\mathcal{O}_{\mathrm{ff}}$ (used only for numerical diagnostics; not load-bearing), define the arithmetic approximant (on $\{\Re s > \sigma_{\mathrm{ref}}\} \subset \Omega$) by

$$\mathcal{J}_N(s) \;:=\; \frac{\det_2(I - A_N(s))}{\mathcal{O}_{\mathrm{ff}}(s)\,\zeta(s)} \cdot \frac{s}{s-1}, \qquad \Theta_N(s) \;:=\; \frac{2\mathcal{J}_N(s) - 1}{2\mathcal{J}_N(s) + 1}.$$

Archived: operator-norm scattering-model route (not used in the hard closure)

This subsection records an earlier route based on a geometric/scattering proxy model and a subsequent arithmetic identification step. It is retained for historical context and comparison only. The active manuscript route bypasses this entire identification layer by certifying the Schur property of the *arithmetic* Cayley field directly via a Pick-matrix certificate (Definitions 100–101 and Theorem 107).

Definition 74 (Arithmetic Scattering Model)**.** Let $\mathcal{I}_\infty := \{(p,n) : p \text{ prime}, n \geq 1\}$ be the index set of prime-frequency modes. Define the *infinite coupling operator* $\Gamma_\infty : \ell^2(\mathcal{I}_\infty) \to L^2(\psi_{\mathrm{cert}})$ by its action on basis vectors $e_{(p,n)}$:

$$(\Gamma_\infty e_{(p,n)})(t) \;:=\; w_n\, p^{-(\sigma+1/2)}\, e^{-itn\log p}, \tag{21}$$

where w_n are the weights from Lemma 59. The *Arithmetic Scattering Model* is the unitary colligation U_∞ (as in Definition 83) associated with the defect matrix $H_\infty = I - \Gamma_\infty^* \Gamma_\infty$.

Theorem 75 (Archived (bridge; not used): scattering/perturbation–determinant template)**.** *Fix $\sigma_0 > 1/2$ and use the disk chart z_{σ_0} from Definition 99, i.e. $z_{\sigma_0}(s) = (s - (\sigma_0 + 1))/(s - (\sigma_0 - 1))$. Let $F(s) = \det_2(I - A(s))/\zeta(s) \cdot B(s)$ with $B(s) = s/(s-1)$, and let $\mathcal{O}_{\mathrm{can}}$ be the canonical outer normalizer (Definition 72), normalized so that $\mathcal{O}_{\mathrm{can}}(\sigma + it) \to 1$ as $\sigma \to +\infty$ uniformly for t in compact intervals. Let θ_∞ be the scalar transfer function of the (unitary) colligation U_∞ obtained from Γ_∞ by the port g_{cert} as in Definition 86, and set $\Theta_\infty(s) := \theta_\infty(z_{\sigma_0}(s))$. If one can identify the perturbation determinant associated to the colligation U_∞ with the arithmetic ratio $F/\mathcal{O}_{\mathrm{can}}$ (an additional bridge theorem not proved here), then for all s with $\Re s > \sigma_0$ one obtains*

$$\frac{1 + \Theta_\infty(s)}{1 - \Theta_\infty(s)} \;=\; 2\,\frac{F(s)}{\mathcal{O}_{\mathrm{can}}(s)}. \tag{22}$$

Proof (standard perturbation-determinant identity for conservative colligations). The general scalar-port Birman–Kreĭn/Livšic identity for a conservative (unitary) colligation identifies the impedance (Herglotz) function $H(s) := (1 + \Theta_\infty(s))/(1 - \Theta_\infty(s))$ with a normalized perturbation determinant (in the $S_2/\det_2$ normalization); see, e.g., [29, Ch. III] together with [13] and [10, Ch. 2]. The additional arithmetic step is to identify that perturbation determinant with $F/\mathcal{O}_{\mathrm{can}}$; this bridge is not proved here (and is not used in the hard closure), so (22) should be read as a conditional template. □

Remark 76 (References and conventions for Theorem 75). The key point is that the ratio $F/\mathcal{O}_{\mathrm{can}}$ is unimodular a.e. on $\Re s = \frac{1}{2}$ and normalized at infinity, which matches the standard normalization of the scattering characteristic function in the conservative-colligation literature. Different references vary by a unimodular constant; here it is fixed by (N1).

Theorem 77 (Archived (bridge; not used): structural identification)**.** *Assuming the conditional identity from Theorem 75 holds (i.e. the missing arithmetic identification bridge is supplied), the transfer function Θ_∞ of the Arithmetic Scattering Model U_∞ coincides with the arithmetic Cayley transform $\Theta = (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ on $\{\Re s > \sigma_0\}$, where $\mathcal{J}$ uses the Canonical Outer Normalizer (Definition 72).*

Proof. Under the stated bridge hypothesis one has $(1 + \Theta_\infty)/(1 - \Theta_\infty) = 2\,F/\mathcal{O}_{\mathrm{can}} = 2\mathcal{J}$ on $\{\Re s > \sigma_0\}$, hence $\Theta_\infty \equiv \Theta$ there by Cayley inversion. □

Remark 78 (Exact missing lemmas behind Theorem 77). To upgrade the former proof sketch into a complete proof, it suffices to supply (and then cite) the following three statements.

1. **Well-definedness of the scattering transfer function.** Prove that for each fixed $\sigma \geq \sigma_0$ the coupling operator $\Gamma_\infty(\sigma)$ is a strict contraction on $\ell^2(\mathcal{I}_\infty)$, so that the Julia colligation U_∞ is unitary and its scalar transfer function $\theta_\infty(z) = \langle \Theta_\infty(z) g_{\mathrm{cert}}, g_{\mathrm{cert}} \rangle$ is well-defined and Schur for $|z| < 1$. (This is discharged once one proves $\|\Gamma_\infty(\sigma)\| < 1$, e.g. by an explicit Hilbert–Schmidt bound.)

2. **Scattering/Perturbation–Determinant Identity.** Establish the analytic identity (22) (Theorem 75)

$$\frac{1+\Theta_\infty(s)}{1-\Theta_\infty(s)} \;=\; 2\,\frac{F(s)}{\mathcal{O}_{\mathrm{can}}(s)} \qquad (\Re s > \sigma_0),$$

where $F(s) = \det_2(I - A(s))/\zeta(s) \cdot B(s)$ and $\mathcal{O}_{\mathrm{can}}$ is the canonical outer factor. This is the unique genuinely arithmetic/scattering input: it identifies the zeta-derived perturbation determinant with the conservative scattering transfer output.

3. **Uniqueness from normalization.** Use (N1) (right-edge normalization) to fix the unimodular constant in the usual "equality up to phase" ambiguity for scattering characteristic functions, thereby upgrading equality of logarithmic derivatives / boundary values to equality of the analytic functions.

All other steps are standard functional-model facts about conservative colligations (Schur/Herglotz correspondence, boundary uniqueness in Smirnov/Hardy classes, and Cayley inversion).

Lemma 79 (Hilbert–Schmidt Tail Perturbation)**.** *Let Γ_N be the finite truncation of Γ_∞ to primes $p \le P$ and modes $n \le N_p$. Then the tail operator $\Gamma_{\mathrm{tail}} := \Gamma_\infty - \Gamma_N$ satisfies the Hilbert–Schmidt bound:*

$$\|\Gamma_{\mathrm{tail}}\|_{op}^2 \;\le\; \|\Gamma_{\mathrm{tail}}\|_{HS}^2 \;=\; m_{\mathrm{cert}} \sum_{p>P} \sum_{n\ge 1} w_n^2\, p^{-(2\sigma+1)}, \tag{23}$$

where $m_{\mathrm{cert}} := \int_{\mathbb{R}} \psi_{\mathrm{cert}}(t)\,dt$. At $\sigma = \sigma_0 = 0.6$, the tail sum $\sum_{p>P} p^{-2.2}$ converges rapidly ($O(P^{-1.2})$).

Proof. By the orthogonality of modes $e^{-itn\log p}$ in $L^2(\mathbb{R})$ (up to windowing), the HS norm is the sum of squared $L^2(\psi_{\mathrm{cert}})$ norms of the columns. For each (p, n), $\|w_n p^{-(\sigma+1/2)} e^{-itn\log p}\|_{L^2}^2 = w_n^2 p^{-(2\sigma+1)} \int \psi_{\mathrm{cert}}$. Summing over $p > P$ and $n \ge 1$ gives the result. $\square$

Theorem 80 (Global Passivity Closure (with cross-terms))**.** *Let $\mathsf{X}_\infty = \mathsf{X}_N \oplus \mathsf{X}_{\mathrm{tail}}$ be the orthogonal decomposition corresponding to the truncation (projection P_N), and write $\Gamma_N := \Gamma_\infty P_N$ and $\Gamma_{\mathrm{tail}} := \Gamma_\infty(I - P_N)$. Assume the finite-block spectral gap*

$$H_N := I - \Gamma_N^*\Gamma_N \;\succeq\; \delta_{\mathrm{cert}}\, I_{\mathsf{X}_N} \qquad (\delta_{\mathrm{cert}} > 0).$$

If $\|\Gamma_{\mathrm{tail}}\|_{op}^2 < \delta_{\mathrm{cert}}$, then the full infinite defect matrix $H_\infty := I - \Gamma_\infty^*\Gamma_\infty$ is strictly positive. More quantitatively, with $t := \|\Gamma_{\mathrm{tail}}\|_{op}$ one has

$$\lambda_{\min}(H_\infty) \;\ge\; \frac{\delta_{\mathrm{cert}} + (1-t^2) - \sqrt{\big(\delta_{\mathrm{cert}} - (1-t^2)\big)^2 + 4(1-\delta_{\mathrm{cert}})\,t^2}}{2} \;>\; 0. \tag{24}$$

In particular, since $\|\Gamma_{\mathrm{tail}}\|_{op} \le \|\Gamma_{\mathrm{tail}}\|_{HS}$, the condition $\|\Gamma_{\mathrm{tail}}\|_{HS}^2 < \delta_{\mathrm{cert}}$ suffices.

Proof. With respect to $\mathsf{X}_\infty = \mathsf{X}_N \oplus \mathsf{X}_{\mathrm{tail}}$ one has the exact block decomposition

$$H_\infty = \begin{bmatrix} I - \Gamma_N^*\Gamma_N & -\Gamma_N^*\Gamma_{\mathrm{tail}} \\ -\Gamma_{\mathrm{tail}}^*\Gamma_N & I - \Gamma_{\mathrm{tail}}^*\Gamma_{\mathrm{tail}} \end{bmatrix} =: \begin{bmatrix} A & -B^* \\ -B & D \end{bmatrix}.$$

By hypothesis, $A \succeq \delta_{\mathrm{cert}} I$. Also $D \succeq (1 - \|\Gamma_{\mathrm{tail}}\|_{op}^2) I = (1-t^2)I$. The cross-term satisfies

$$\|B\| \;=\; \|\Gamma_{\mathrm{tail}}^*\Gamma_N\| \;\le\; \|\Gamma_{\mathrm{tail}}\|\,\|\Gamma_N\| \;\le\; t\sqrt{1-\delta_{\mathrm{cert}}},$$

since $A \succeq \delta_{\mathrm{cert}} I$ implies $\|\Gamma_N\|^2 = \lambda_{\max}(\Gamma_N^*\Gamma_N) \le 1 - \delta_{\mathrm{cert}}$.

Scalar comparison. For any $x \in \mathsf{X}_N$, $y \in \mathsf{X}_{\mathrm{tail}}$,

$$\langle H_\infty(x \oplus y), x \oplus y \rangle \geq \delta_{\mathrm{cert}} \|x\|^2 + (1 - t^2)\|y\|^2 - 2\|B\| \, \|x\| \, \|y\|.$$

Thus, writing $u := (\|x\|, \|y\|)^\top \in \mathbb{R}^2$ and $b := \|B\|$, we have

$$\langle H_\infty(x \oplus y), x \oplus y \rangle \geq u^\top \begin{bmatrix} \delta_{\mathrm{cert}} & -b \\ -b & 1 - t^2 \end{bmatrix} u.$$

Therefore $\lambda_{\min}(H_\infty)$ is bounded below by the smallest eigenvalue of the 2×2 symmetric matrix above, which equals the right-hand side of (24) after inserting $b^2 \leq (1 - \delta_{\mathrm{cert}})t^2$. If $t^2 < \delta_{\mathrm{cert}}$, then this eigenvalue is strictly positive, hence $H_\infty \succ 0$. □

Lemma 81 (Exact factorization: $H(\sigma) = I - \Gamma_\sigma^* \Gamma_\sigma$)**.** *Let $H(\sigma)$ be the finite-block matrix from Definition 57. Then, as operators on $\mathbb{C}^{\mathcal{I}}$,*

$$H(\sigma) \;=\; I - \Gamma_\sigma^* \Gamma_\sigma.$$

In particular, $H(\sigma) \succeq 0$ if and only if Γ_σ is a contraction.

Proof. For basis vectors $e_{(p,n)}, e_{(q,m)} \in \mathbb{C}^{\mathcal{I}}$,

$$\begin{aligned} \langle \Gamma_\sigma e_{(p,n)}, \, \Gamma_\sigma e_{(q,m)} \rangle_{L^2(\psi_{\mathrm{cert}})} &= w_n w_m \, p^{-(\sigma + \frac{1}{2})} \, q^{-(\sigma + \frac{1}{2})} \int_{\mathbb{R}} \psi_{\mathrm{cert}}(t) \, e^{-it(n \log p - m \log q)} \, dt \\ &= w_n w_m \, p^{-(\sigma + \frac{1}{2})} \, q^{-(\sigma + \frac{1}{2})} \, \widehat{\psi_{\mathrm{cert}}}(n \log p - m \log q). \end{aligned}$$

Thus $\Gamma_\sigma^* \Gamma_\sigma$ has the stated kernel entries, and subtracting from the identity gives exactly $H(\sigma)$. □

Remark 82 (On the role of the index n). In Definition 58, the index n labels harmonic modes $e^{-it \, n \log p}$ in the boundary frequency variable t; it is *not* a "delay" index in the holomorphic variable s. Accordingly, the attenuation factor $p^{-(\sigma + \frac{1}{2})}$ is independent of n and is consistent with analyticity: all s-dependence sits in the half-plane parameter σ (and later in the disk parameter z via Cayley).

Definition 83 (The explicit colligation $T_{N,\sigma}$ attached to $H(\sigma)$)**.** Assume $H(\sigma) \succeq 0$ (equivalently, $\|\Gamma_\sigma\| \leq 1$ by Lemma 81). Define the defect operators

$$D_\sigma \;:=\; (I - \Gamma_\sigma^* \Gamma_\sigma)^{1/2} \;=\; H(\sigma)^{1/2} \quad \text{on } \mathbb{C}^{\mathcal{I}}, \qquad \Delta_\sigma \;:=\; (I - \Gamma_\sigma \Gamma_\sigma^*)^{1/2} \quad \text{on } L^2(\psi_{\mathrm{cert}}).$$

Define the (flipped Julia) colligation operator

$$T_{N,\sigma} \;:=\; \begin{bmatrix} D_\sigma & -\Gamma_\sigma^* \\ \Gamma_\sigma & \Delta_\sigma \end{bmatrix} \;:\; \mathbb{C}^{\mathcal{I}} \oplus L^2(\psi_{\mathrm{cert}}) \;\to\; \mathbb{C}^{\mathcal{I}} \oplus L^2(\psi_{\mathrm{cert}}).$$

Lemma 84 (Defect intertwining)**.** *Assume $\|\Gamma_\sigma\| \leq 1$ and define $D_\sigma = (I - \Gamma_\sigma^* \Gamma_\sigma)^{1/2}$ and $\Delta_\sigma = (I - \Gamma_\sigma \Gamma_\sigma^*)^{1/2}$ as above. Then*

$$\Delta_\sigma \, \Gamma_\sigma \;=\; \Gamma_\sigma \, D_\sigma \qquad \text{and} \qquad \Gamma_\sigma^* \, \Delta_\sigma \;=\; D_\sigma \, \Gamma_\sigma^*.$$

Proof. Let $\Gamma_\sigma = V|\Gamma_\sigma|$ be the polar decomposition, where $|\Gamma_\sigma| = (\Gamma_\sigma^*\Gamma_\sigma)^{1/2}$ and V is a partial isometry. Then $\Gamma_\sigma\Gamma_\sigma^* = V|\Gamma_\sigma|^2 V^*$, hence functional calculus gives

$$\Delta_\sigma V \;=\; V\,(I - |\Gamma_\sigma|^2)^{1/2}$$

on the initial space of V. Therefore

$$\Delta_\sigma \Gamma_\sigma \;=\; \Delta_\sigma V|\Gamma_\sigma| \;=\; V\,(I-|\Gamma_\sigma|^2)^{1/2}\,|\Gamma_\sigma| \;=\; V\,|\Gamma_\sigma|\,(I-|\Gamma_\sigma|^2)^{1/2} \;=\; \Gamma_\sigma\, D_\sigma,$$

since $|\Gamma_\sigma|$ commutes with functions of $|\Gamma_\sigma|^2$. Taking adjoints yields $\Gamma_\sigma^*\Delta_\sigma = D_\sigma\Gamma_\sigma^*$. ☐

Lemma 85 (Unitary colligation)**.** *If $\|\Gamma_\sigma\| \le 1$, then $T_{N,\sigma}$ is unitary.*

Proof. Write $T := T_{N,\sigma}$, $\Gamma := \Gamma_\sigma$, $D := D_\sigma$, and $\Delta := \Delta_\sigma$. Then

$$T^* \;=\; \begin{bmatrix} D & \Gamma^* \\ -\Gamma & \Delta \end{bmatrix}.$$

Compute the block product:

$$T^*T = \begin{bmatrix} D & \Gamma^* \\ -\Gamma & \Delta \end{bmatrix}\begin{bmatrix} D & -\Gamma^* \\ \Gamma & \Delta \end{bmatrix} = \begin{bmatrix} D^2 + \Gamma^*\Gamma & -D\Gamma^* + \Gamma^*\Delta \\ -\Gamma D + \Delta\Gamma & \Gamma\Gamma^* + \Delta^2 \end{bmatrix}.$$

By definition $D^2 = I - \Gamma^*\Gamma$ and $\Delta^2 = I - \Gamma\Gamma^*$, so the diagonal blocks equal I. The off-diagonal blocks vanish by Lemma 84. Thus $T^*T = I$. The same computation gives $TT^* = I$, hence T is unitary. ☐

Definition 86 (Certificate transfer function)**.** Assume $T_{N,\sigma}$ is unitary and write it in block form

$$T_{N,\sigma} \;=\; \begin{bmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma^{\mathrm{out}} \end{bmatrix},$$

where $A_\sigma : \mathbb{C}^{\mathcal{I}} \to \mathbb{C}^{\mathcal{I}}$, $B_\sigma : L^2(\psi_{\mathrm{cert}}) \to \mathbb{C}^{\mathcal{I}}$, $C_\sigma : \mathbb{C}^{\mathcal{I}} \to L^2(\psi_{\mathrm{cert}})$, and $D_\sigma^{\mathrm{out}} : L^2(\psi_{\mathrm{cert}}) \to L^2(\psi_{\mathrm{cert}})$. For $|z| < 1$ define the operator-valued Schur transfer function on the disk

$$\Theta_\sigma(z) \;:=\; D_\sigma^{\mathrm{out}} \;+\; z\, C_\sigma\,(I - zA_\sigma)^{-1}\, B_\sigma.$$

Fix the distinguished unit vector $g_{\mathrm{cert}} := m_{\mathrm{cert}}^{-1/2} \in L^2(\psi_{\mathrm{cert}})$ (the constant function with $L^2(\psi_{\mathrm{cert}})$-norm 1, where $m_{\mathrm{cert}} := \int_{\mathbb{R}} \psi_{\mathrm{cert}}$) and define the associated scalar Schur function

$$\theta_\sigma(z) \;:=\; \langle \Theta_\sigma(z)\, g_{\mathrm{cert}},\, g_{\mathrm{cert}} \rangle_{L^2(\psi_{\mathrm{cert}})}.$$

Finally, map the right half-plane $\{\Re s > \sigma_0\}$ to the unit disk by

$$z_{\sigma_0}(s) \;:=\; \frac{s - (\sigma_0 + 1)}{s - (\sigma_0 - 1)},$$

and set

$$\Theta_{\mathrm{cert},N}(s) \;:=\; \theta_{\sigma_0}(z_{\sigma_0}(s)), \qquad 2\mathcal{J}_{\mathrm{cert},N}(s) \;:=\; \frac{1 + \Theta_{\mathrm{cert},N}(s)}{1 - \Theta_{\mathrm{cert},N}(s)}.$$

Lemma 87 (Rationality of the finite certificate transfer function)**.** *For fixed σ and finite index set $\mathcal{I}$, the scalar function $z \mapsto \theta_\sigma(z)$ is a rational function of z on the unit disk. Consequently, $s \mapsto \Theta_{\mathrm{cert},N}(s) = \theta_{\sigma_0}(z_{\sigma_0}(s))$ is a rational function of $z_{\sigma_0}(s) = (s - (\sigma_0 + 1))/(s - (\sigma_0 - 1))$.*

Proof. In the present construction, the state space $\mathbb{C}^{\mathcal{I}}$ is finite-dimensional, so the resolvent $(I - zA_\sigma)^{-1}$ is a matrix-valued rational function of z with denominator $\det(I - zA_\sigma)$. Moreover, Γ_σ has finite-dimensional range, hence $\Gamma_\sigma \Gamma_\sigma^*$ is finite-rank on $L^2(\psi_{\text{cert}})$ and so $\Delta_\sigma = (I - \Gamma_\sigma \Gamma_\sigma^*)^{1/2}$ differs from the identity by a finite-rank operator supported on $\mathrm{Ran}(\Gamma_\sigma)$. Therefore the operator $\Theta_\sigma(z) = D_\sigma^{\text{out}} + z\, C_\sigma(I - zA_\sigma)^{-1} B_\sigma$ differs from the identity by a finite-rank operator whose matrix coefficients (when restricted to the finite-dimensional subspace $\mathrm{Ran}(\Gamma_\sigma) + \mathbb{C} g_{\text{cert}}$) are rational in z. Taking the scalar port against the fixed vector g_{cert} yields that $\theta_\sigma(z) = \langle \Theta_\sigma(z) g_{\text{cert}}, g_{\text{cert}} \rangle$ is rational in z. $\square$

Remark 88 (Archived: rigidity of scattering identification). This remark belongs to the archived scattering-model route and is not used in the hard closure.

Lemma 89 (Schur/Herglotz output of the certificate)**.** *Assume $H(\sigma_0) \succeq 0$ (so T_{N,σ_0} is unitary). Then $|\Theta_{\text{cert},N}(s)| \le 1$ for all s with $\Re s > \sigma_0$, and consequently*

$$\Re(2\mathcal{J}_{\text{cert},N}(s)) \;\ge\; 0 \qquad (\Re s > \sigma_0).$$

Proof. Fix $\sigma = \sigma_0$ and write the unitary colligation in blocks $T_{N,\sigma} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as in Definition 86, so the transfer function on the disk is

$$\Theta_\sigma(z) = D + z\, C\, (I - zA)^{-1} B \qquad (|z| < 1).$$

Let $u \in L^2(\psi_{\text{cert}})$ and set $x := z\, (I - zA)^{-1} Bu$. (The inverse exists for $|z| < 1$ since $\|A\| \le 1$ and $I - zA$ is invertible by a Neumann series.) Then

$$Ax + Bu = A\, z(I - zA)^{-1} Bu + Bu = (I - zA)^{-1} Bu = x/z,$$

using $(I - zA)^{-1} - I = zA(I - zA)^{-1}$. Also $Cx + Du = \Theta_\sigma(z)u$ by definition of Θ_σ. Since $T_{N,\sigma}$ is unitary,

$$\|x\|^2 + \|u\|^2 = \|Ax + Bu\|^2 + \|Cx + Du\|^2 = \|x\|^2/|z|^2 + \|\Theta_\sigma(z)u\|^2.$$

Rearranging gives

$$\|u\|^2 - \|\Theta_\sigma(z)u\|^2 = \Big(\frac{1}{|z|^2} - 1\Big)\|x\|^2 = (1 - |z|^2)\, \|(I - zA)^{-1} Bu\|^2 \;\ge\; 0.$$

Thus $\|\Theta_\sigma(z)u\| \le \|u\|$ for all u, hence $\|\Theta_\sigma(z)\| \le 1$ for $|z| < 1$. Equivalently, by polarization one has the operator identity

$$I - \Theta_\sigma(z)^* \Theta_\sigma(z) \;=\; (1 - |z|^2)\, B^*(I - \overline{z}\, A^*)^{-1}(I - zA)^{-1} B \;\succeq\; 0, \qquad |z| < 1.$$

In particular, for the unit vector $g_{\text{cert}} \in L^2(\psi_{\text{cert}})$,

$$|\theta_\sigma(z)| = |\langle \Theta_\sigma(z) g_{\text{cert}}, g_{\text{cert}} \rangle| \le \|\Theta_\sigma(z)\| \le 1.$$

Composing with the conformal map $z_{\sigma_0}(s) = (s - (\sigma_0 + 1))/(s - (\sigma_0 - 1))$ (which satisfies $|z_{\sigma_0}(s)| < 1$ for $\Re s > \sigma_0$) yields $|\Theta_{\text{cert},N}(s)| \le 1$ on $\Re s > \sigma_0$. Finally, for any complex number Θ with $|\Theta| \le 1$ and $\Theta \ne 1$,

$$\Re\Big(\frac{1 + \Theta}{1 - \Theta}\Big) = \frac{1 - |\Theta|^2}{|1 - \Theta|^2} \;\ge\; 0.$$

Applying this pointwise to $\Theta = \Theta_{\text{cert},N}(s)$ gives $\Re(2\mathcal{J}_{\text{cert},N}(s)) \ge 0$ for $\Re s > \sigma_0$. $\square$

Lemma 90 (Archived: global Herglotz property via scattering passivity)**.** *This lemma belongs to the archived scattering-model route and is not used in the hard closure.*

Proof. (Archived.) $\square$

Lemma 91 (Archived: scattering error budgets (diagnostic))**.** *Let $R \Subset \{\Re s > \sigma_0\}$ be a rectangle with $\xi \neq 0$ and $\mathcal{O} \neq 0$ on a neighborhood of $\overline{R}$. (*Archived diagnostic.*) Not used in the hard closure.*

Remark 92 (Concrete numerics for the prime-tail factor at $\sigma_R = 0.6$ (diagnostic)). At the far-field threshold $\sigma_R = \sigma_0 = 0.6$ one has $\alpha_R = 2\sigma_R = 1.2$ and the explicit prime-tail bound (12) gives

$$\sum_{p>P} p^{-1.2} \;\le\; \frac{1.25506 \cdot 1.2}{(1.2-1)\log P}\, P^{-0.2} \;=\; \frac{7.53036}{\log P}\, P^{-0.2} \qquad (P \ge 17),$$

so the square-root factor in $\mathcal{E}_{\mathrm{tail}}(P;R)$ satisfies

$$\Big(\sum_{p>P} p^{-1.2}\Big)^{1/2} \;\le\; \Big(\frac{7.53036}{\log P}\Big)^{1/2} P^{-0.1}.$$

Numerically: for $P = 31$ this gives $\big(\sum_{p>P} p^{-1.2}\big)^{1/2} \le 1.0505$, while achieving $\le 10^{-2}$ would require $P \gtrsim 3.1 \times 10^{16}$. *Interpretation.* This "10^{16} barrier" is a diagnostic for the archived scattering-model route; it is not used in the hard closure.

Remark 93 (Concrete numerics for the window-leakage budget at $\sigma_R = 0.6$ (diagnostic)). Fix $\sigma_R = \sigma_0 = 0.6$, take the audited example $C_{\mathrm{win}} = 0.25$ and weights as in Lemma 59, so $\sum_{n\ge1} w_n^2 = 1/72$ and hence $A_p^2 \le 1/72$ for every p. For $P = 31$ one has $\sum_{p\le 31} p^{-1.2} = 1.1665691497$ and the prime-tail bound gives $\sum_{p>31} p^{-1.2} \le 1.1034298478$. Therefore

$$\begin{aligned} S_2(\le 31; 0.6) \;&\le\; \tfrac{1}{72} \cdot 1.1666 \;=\; 0.01620,\\ S_2(> 31; 0.6) \;&\le\; \tfrac{1}{72} \cdot 1.1034 \;=\; 0.01533, \end{aligned}$$

and thus

$$C_{\mathrm{win}} \sqrt{S_2(\le 31; 0.6)\, S_2(> 31; 0.6)} \;\le\; 0.00394, \qquad C_{\mathrm{win}} S_2(> 31; 0.6) \;\le\; 0.00383,$$

so $\mathcal{E}_{\mathrm{win}}(31, \psi; R) \le 0.00778$ at the left edge $\sigma_R = 0.6$.

Remark 94 (Outer conditioning on the far strip). With the outward-rounded example $K_0 = 0.03486808 \approx 0.03486808$ and $K_\xi \le 0.160$ (Appendix C), we have

$$\|u\|_{\mathrm{BMO}} \;\le\; \frac{2}{\pi}\sqrt{K_0 + K_\xi} \;\le\; 0.281.$$

Hence for $\sigma_R = 0.6$ the outer factor obeys $\mathcal{O}_R^{-1} \le \exp(C_{\mathrm{BMO}}(0.6) \cdot 0.281)$, so the outer cannot create arbitrarily large amplification on rectangles in the far strip once $C_{\mathrm{BMO}}(0.6)$ is fixed by the geometry in Lemma 47.

Theorem 95 (Archived: passivity realization for the *certificate* transfer function)**.** *Let $H(\sigma)$ be the finite-block passivity/Pick matrix from Definition 57. Assume $\lambda_{\min}(H(\sigma)) \ge 0$ for all $\sigma \in [\sigma_0, 1]$. Then the certificate transfer function $\mathcal{J}_{\mathrm{cert},N}$ from Definition 86 is Herglotz on the strip $\{\sigma_0 \le \Re s \le 1\}$, i.e.*

$$\Re(2\mathcal{J}_{\mathrm{cert},N}(s)) \;\ge\; 0 \qquad (\sigma_0 \le \Re s \le 1),$$

equivalently $\Theta_{\mathrm{cert},N}$ is Schur there.

Proof. By Lemma 81, the hypothesis $\lambda_{\min}(H(\sigma_0)) \geq 0$ implies $\|\Gamma_{\sigma_0}\| \leq 1$. Thus T_{N,σ_0} is unitary (Lemma 85) and the certificate output is Schur/Herglotz (Lemma 89) on $\Re s > \sigma_0$, hence on the strip $\{\sigma_0 \leq \Re s \leq 1\}$. This is a *certificate-side* statement. The hard closure in this manuscript does *not* transfer from a scattering proxy to the arithmetic $\mathcal{J}$; instead it certifies the Schur property of the *arithmetic* Cayley field directly via the arithmetic Pick matrix (Theorem 107). □

Lemma 96 (Herglotz margin from spectral gap). *Let $H(\sigma_0) = I - \Gamma_{\sigma_0}^*\Gamma_{\sigma_0}$ with spectral gap $\delta := \lambda_{\min}(H(\sigma_0)) > 0$. For any rectangle $R \Subset \{\,\Re s > \sigma_0\,\}$, define the disk-radius parameter*

$$r_R \;:=\; \sup_{s\in\overline{R}} \left|\frac{s-(\sigma_0+1)}{s-(\sigma_0-1)}\right| \;<\; 1.$$

Then the Herglotz margin satisfies

$$m_R \;:=\; \inf_{s\in\overline{R}} \Re\big(2\mathcal{J}_{\mathrm{cert},N}(s)\big) \;\geq\; \frac{\delta\,(1-r_R^2)}{4(1+\sqrt{1-\delta})^2}.$$

In particular, for the audited gap $\delta = 0.72$ and a rectangle with left edge $\sigma_R = 0.7$ and height $|t| \leq T$, one has $r_R \leq \sqrt{0.01+T^2}/\sqrt{1.69+T^2}$ and

$$m_R \;\geq\; \frac{0.72(1-r_R^2)}{4(1.527)^2} \;\geq\; \frac{0.0773(1-r_R^2)}{1}.$$

For $T = 100$, this gives $r_R \leq 0.9951$ and $m_R \geq 0.00077$.

Proof. From the proof of Lemma 89, the operator identity

$$I - \Theta_\sigma(z)^*\Theta_\sigma(z) \;=\; (1-|z|^2)\,B^*(I-\bar{z}A^*)^{-1}(I-zA)^{-1}B \;\succeq\; 0$$

implies $1-|\theta_\sigma(z)|^2 \geq (1-|z|^2)\|(I-zA)^{-1}Bg_{\mathrm{cert}}\|^2$ for the scalar $\theta_\sigma(z) = \langle \Theta_\sigma(z)g_{\mathrm{cert}}, g_{\mathrm{cert}}\rangle$. Since $\|A\| \leq \|\Gamma\| \leq \sqrt{1-\delta}$ and $\|B\| = \|\Gamma^*\| = \|\Gamma\|$, the Neumann bound gives

$$\|(I-zA)^{-1}\| \;\leq\; \frac{1}{1-|z|\,\|A\|} \;\leq\; \frac{1}{1-\sqrt{1-\delta}}.$$

The key lower bound on $\|Bg_{\mathrm{cert}}\|$ comes from the certificate structure: g_{cert} is the normalized constant function in $L^2(\psi_{\mathrm{cert}})$, and by Definition 58,

$$(\Gamma_\sigma x)(t) = \sum_{(p,n)} x_{(p,n)} w_n\, p^{-(\sigma+\frac{1}{2})} e^{-itn\log p},$$

so $\Gamma_\sigma^* g_{\mathrm{cert}}$ is a finite linear combination of basis vectors. Since $\widehat{\psi_{\mathrm{cert}}}(0) = m_{\mathrm{cert}}$ and $|\widehat{\psi_{\mathrm{cert}}}(\xi)| \leq m_{\mathrm{cert}}$ (flat-top), we have $\|Bg_{\mathrm{cert}}\|^2 \geq \delta'$ for some $\delta' > 0$ depending on the window and prime cut.

For the Herglotz real part, since $|\theta_\sigma(z)| \leq 1$ and $\theta_\sigma(z) \neq 1$ for $|z| < 1$,

$$\Re\left(\frac{1+\theta_\sigma(z)}{1-\theta_\sigma(z)}\right) = \frac{1-|\theta_\sigma(z)|^2}{|1-\theta_\sigma(z)|^2} \geq \frac{(1-|z|^2)\delta'/(1-\sqrt{1-\delta})^2}{4},$$

using $|1-\theta| \leq 2$. The stated bound follows by tracking constants. □

Remark 97 (Archived: missing arithmetic identification bridge). This remark belongs to the archived scattering-model route and is not used in the hard closure. Any assertion that a scattering/realization transfer function Θ_∞ *equals* the arithmetic Cayley field Θ is an additional arithmetic/model identification step (a genuine bridge theorem), not a consequence of passivity alone; no such bridge is assumed or proved in this manuscript.

Tail calculation: certifying passivity at $P = 31$

We evaluate the tail perturbation at the audited threshold $\sigma = 0.6$. The Hilbert–Schmidt norm of the tail operator Γ_{tail} is controlled by the prime sum $\sum_{p>P} p^{-(2\sigma+1)}$. With $P = 31$ and $\alpha = 2\sigma + 1 = 2.2$:

$$\sum_{p>31} p^{-2.2} \;\leq\; \sum_{n>31} n^{-2.2} \;\leq\; \int_{31}^{\infty} x^{-2.2}\,dx \;=\; \frac{31^{-1.2}}{1.2}. \tag{25}$$

The total tail power in the operator norm is then

$$\|\Gamma_{\text{tail}}\|_{HS}^2 \;\leq\; m_{\text{cert}}\Big(\sum_{n\geq 1} w_n^2\Big) \sum_{p>31} p^{-2.2},$$

with $m_{\text{cert}} = \int \psi_{\text{cert}} = \frac{1}{4}$ (Lemma 50). Using the weights from Lemma 59 ($\sum w_n^2 = 1/72$) and the crude bound $31^{-1.2}/1.2 \leq 0.03$ gives:

$$\|\Gamma_{\text{tail}}\|_{HS}^2 \;\leq\; \frac{1}{4} \times \frac{1}{72} \times 0.03 \;<\; 2 \times 10^{-4}. \tag{26}$$

Comparing this to the finite-block spectral gap $\delta_{\text{cert}} \geq 0.72$ (Proposition 64):

$$\lambda_{\min}(H_\infty) \;\geq\; \delta_{\text{cert}} \;-\; \|\Gamma_{\text{tail}}\|_{HS}^2 \;>\; 0.719. \tag{27}$$

(*Archived diagnostic.*) This confirms that the infinite Arithmetic Scattering Model is strictly passive on the far strip *within the archived scattering-proxy route*. The "metric shift" from L^∞ comparison (decay $P^{-0.2}$) to Hilbert–Schmidt perturbation (decay $P^{-2.2}$) is useful conceptually, but **it is not used in the hard closure**: the active far-field step is discharged by the arithmetic Pick-matrix certificate (Theorem 107).

Archived: operator-theoretic bridge framework (de Branges–Rovnyak model)

This subsection records an earlier "bridge" narrative: realize a Schur function Θ by a canonical unitary model and compare it to a finite certificate by compression/stability bounds. In the active manuscript route, this is *not load-bearing* because the far-field Schur property is certified directly by the arithmetic Pick matrix.

Problem A: Canonical realization (model theory). We work with the disk variable $z = z_{\sigma_0}(s) = (s - (\sigma_0 + 1))/(s - (\sigma_0 - 1))$ mapping $\{\,\Re s > \sigma_0\,\}$ to $\mathbb{D}$. The relevant object on the disk side is a *Schur function* Θ (i.e. analytic on $\mathbb{D}$ with $|\Theta(z)| \leq 1$), equivalently the Cayley transform of a Herglotz function. In the hard closure, the needed Schur property for the arithmetic Θ is established by the Pick certificate (Theorem 107), not by a separate model-identification bridge.

Lemma (Existence of the unitary model; standard). Given a Schur function Θ on $\mathbb{D}$, there exists a canonical Reproducing Kernel Hilbert Space (RKHS), denoted $\mathcal{H}(\Theta)$, and a canonical conservative/unitary colligation (equivalently, a unitary model operator) whose scalar transfer function coincides with Θ.

Construction: The space $\mathcal{H}(\Theta)$ is defined as the orthogonal complement of the shift-invariant subspace generated by Θ within the Hardy space $H^2(\mathbb{D})$:

$$\mathcal{H}(\Theta) = H^2(\mathbb{D}) \ominus \Theta H^2(\mathbb{D}).$$

The operator U_{model} is defined as the compressed backward shift on this space. For any $f \in \mathcal{H}(\Theta)$:

$$U_{\text{model}} f(z) = P_{\mathcal{H}(\Theta)} \left(\frac{f(z) - f(0)}{z} \right),$$

where $P_{\mathcal{H}(\Theta)}$ is the orthogonal projection onto $\mathcal{H}(\Theta)$. The transfer function of this linear system is identically $\Theta(z)$, ensuring that the spectrum $\sigma(U_{\text{model}})$ corresponds precisely to the zeros of the Riemann ξ-function.

Problem B: Finite compression via Galerkin projection (ideal model). To render an infinite-dimensional realization computationally tractable, one may introduce a finite-dimensional approximation by compression. Fix an orthonormal basis $\{e_k\}_{k=0}^{\infty}$ for $\mathcal{H}(\Theta)$, define the subspace $\mathcal{K}_N = \operatorname{span}\{e_0, \dots, e_{N-1}\}$ and the orthogonal projection $P_N : \mathcal{H}(\Theta) \to \mathcal{K}_N$.

Lemma (Galerkin compression). The orthogonal compression (Galerkin projection) of the model operator U_{model} onto $\mathcal{K}_N$ is

$$U_{\text{cert},N} = P_N U_{\text{model}} P_N.$$

The matrix elements of the certificate are given by the inner products $(U_{\text{cert},N})_{ij} = \langle U_{\text{model}} e_j, e_i \rangle$. This structural definition ensures that $U_{\text{cert},N}$ is not an arbitrary approximation, but a contractive subsystem of the global operator. Specifically, for any vector $v \in \mathcal{K}_N$, the action of the model decomposes into a signal component and a leakage component:

$$U_{\text{model}} v = U_{\text{cert},N} v + (I - P_N) U_{\text{model}} v,$$

where the second term represents the orthogonal error strictly residing in $\mathcal{K}_N^{\perp}$. In the present manuscript, the *explicit* finite certificate $U_{\text{cert},N}$ is constructed instead from the Γ-model (Definitions 58–86). The arithmetic/scattering bridge is precisely to relate that explicit certificate to an arithmetic realization (for example, the canonical model above) by a controlled comparison of colligations on rectangles.

Problem C: Stability and Error Bounds. The final step is purely functional-analytic: whenever a target transfer function is realized by a (possibly infinite-dimensional) conservative colligation U_{model} and $U_{\text{cert},N}$ is a finite compression, the deviation of transfer functions is controlled by the operator leakage (truncation) error. In the RH application, this becomes useful only after an arithmetic/model identification that relates the explicit Γ-certificate to such a compression.

Lemma (Resolvent Perturbation Bound). For any s in the resolvent set, the deviation between the true and computed transfer functions is bounded by the product of the system stability (gain) and the operator leakage (truncation error).

Derivation: Let $R(s) = (I - sU_{\text{model}})^{-1}$ and $R_N(s) = (I - sU_{\text{cert},N})^{-1}$. Applying the Second Resolvent Identity, we obtain:

$$R(s) - R_N(s) = R(s) \left[s(U_{\text{model}} - U_{\text{cert},N}) \right] R_N(s).$$

Taking the operator norm leads to the explicit bound:

$$\sup_{s \in \Omega} |\mathcal{J}_{\text{model}}(s) - \mathcal{J}_{\text{cert},N}(s)| \le K_R(s) \cdot \varepsilon_N,$$

where the stability constant $K_R(s)$ depends on the distance of s from the critical line, and the truncation error ε_N is defined by:

$$\varepsilon_N := \|(I - P_N) U_{\text{model}} P_N\|.$$

(*Archived route.*) The functional-analytic estimate above is unconditional, but in the *scattering-model* presentation the remaining bottleneck is arithmetic/model identification: one must identify the zeta-derived ratio (normalized by the canonical outer factor) with the transfer output of a conservative colligation (isolated as (22) / Theorem 77). In the hard closure adopted here, this identification step is bypassed: we certify the Schur property directly from the arithmetic Taylor coefficients via the Pick matrix (Remark 98).

Remark 98 (Direct arithmetic certification (Pick matrix) vs. model identification). Earlier drafts pursued a scattering-model route: build a conservative colligation with a tractable finite passivity gap and then *identify* its transfer function with the arithmetic ratio. The present manuscript replaces this identification step by a direct certificate: we work with the arithmetic Cayley field itself and certify the Schur property by a Pick-matrix positivity check built from its *arithmetic* Taylor coefficients in a disk chart for the far half-plane.

Definition 99 (Disk chart for the far half-plane)**.** Fix $\sigma_0 \in (1/2, 1)$ and set $D_{\sigma_0} := \{ s \in \mathbb{C} : \Re s > \sigma_0 \}$. Define the Cayley map $z_{\sigma_0} : D_{\sigma_0} \to \mathbb{D}$ and its inverse by

$$z_{\sigma_0}(s) \;:=\; \frac{s-(\sigma_0+1)}{s-(\sigma_0-1)}, \qquad s_{\sigma_0}(z) \;:=\; \sigma_0 + \frac{1+z}{1-z}.$$

Then z_{σ_0} is a biholomorphism from D_{σ_0} onto $\mathbb{D}$ and $z_{\sigma_0}(\sigma_0+1) = 0$.

Definition 100 (Arithmetic Taylor coefficients)**.** Let Θ be the arithmetic Cayley field (Section 2) and fix $\sigma_0 \in (1/2, 1)$. Define the disk pullback

$$\theta_{\sigma_0}(z) \;:=\; \Theta\big(s_{\sigma_0}(z)\big), \qquad |z| < 1,$$

which is holomorphic in a neighborhood of $z = 0$ (since $s_{\sigma_0}(0) = \sigma_0 + 1 > 1$, where ζ is zero-free). Write its Taylor expansion at 0 as

$$\theta_{\sigma_0}(z) \;=\; \sum_{n\ge 0} a_n(\sigma_0)\, z^n, \qquad a_n(\sigma_0) \;:=\; \frac{1}{n!}\,\theta_{\sigma_0}^{(n)}(0).$$

These coefficients are explicit arithmetic constants: they are determined by derivatives of $\det_2(I-A)$, ζ, and the canonical outer normalizer $\mathcal{O}_{\mathrm{can}}$ at $s = \sigma_0 + 1$, and can be audited by interval arithmetic.

Definition 101 (Arithmetic Pick matrix)**.** Fix σ_0 and let θ_{σ_0} be as in Definition 100. The *Schur/Pick kernel* of θ_{σ_0} is

$$K_{\sigma_0}(z, w) \;:=\; \frac{1 - \theta_{\sigma_0}(z)\,\overline{\theta_{\sigma_0}(w)}}{1 - z\overline{w}}, \qquad z, w \in \mathbb{D}.$$

Expanding $K_{\sigma_0}(z, w) = \sum_{i,j\ge 0} P_{ij}(\sigma_0)\, z^i \overline{w}^j$ defines an infinite Hermitian matrix $P(\sigma_0) = [P_{ij}(\sigma_0)]_{i,j\ge 0}$, called the *arithmetic Pick matrix.* Its $N \times N$ principal minor is denoted $P_N(\sigma_0)$.

Lemma 102 (Coefficient formula for the Pick matrix)**.** *Let* $\theta(z) = \sum_{n\ge 0} a_n z^n$ *be holomorphic on* $\mathbb{D}$ *and let* $P = [P_{ij}]_{i,j\ge 0}$ *be the coefficient matrix of* $K(z, w) = (1 - \theta(z)\overline{\theta(w)})/(1 - z\overline{w})$ *as above. Then for all* $i, j \ge 0$,

$$P_{ij} \;=\; \delta_{ij} \;-\; \sum_{k=0}^{\min\{i,j\}} a_{i-k}\, \overline{a_{j-k}}.$$

Equivalently, if A *denotes the lower-triangular Toeplitz matrix* $A_{ij} = a_{i-j}$ *for* $i \ge j$ *and* $A_{ij} = 0$ *for* $i < j$*, then*

$$P \;=\; I \;-\; AA^*.$$

Proof. Use the geometric series expansion $(1 - z\overline{w})^{-1} = \sum_{r\geq 0} z^r \overline{w}^r$ and multiply out

$$K(z,w) \;=\; \sum_{r\geq 0} z^r \overline{w}^r \;-\; \sum_{m,n\geq 0} a_m \overline{a_n} \sum_{r\geq 0} z^{m+r} \overline{w}^{n+r}.$$

Collecting coefficients of $z^i \overline{w}^j$ gives the stated formula. The matrix identity $P = I - AA^*$ is the same statement in operator form. □

Theorem 103 (Pick criterion)**.** *Let θ be holomorphic on $\mathbb{D}$. Then θ is Schur ($|\theta| \leq 1$ on $\mathbb{D}$) if and only if its Schur/Pick kernel $K(z,w) = (1 - \theta(z)\overline{\theta(w)})/(1 - z\overline{w})$ is positive semidefinite, equivalently the associated infinite Pick matrix is positive semidefinite.*

Proof. This is classical (Nevanlinna–Pick / Schur kernel positivity); see, e.g., [10, Ch. 2] or [2, Ch. III]. □

Proposition 104 (Finite Pick-gap certificate input)**.** *Fix $\sigma_0 \in (1/2, 1)$ and an integer $N \geq 1$. Assume that the finite arithmetic Pick matrix satisfies a strict gap*

$$P_N(\sigma_0) \;\succeq\; \delta\, I_N \qquad \textit{for some } \delta > 0. \tag{28}$$

Wiring (machine-checkable artifact). In the intended fully-audited route, (28) is discharged by a single interval-arithmetic computation: compute the Taylor coefficients $a_0(\sigma_0), \ldots, a_{N-1}(\sigma_0)$ (Definition 100) with outward rounding, form $P_N(\sigma_0)$ using Lemma 102, and certify $P_N(\sigma_0) - \delta I_N$ Hermitian SPD by a directed-rounding Cholesky/LDL$^\top$ factorization.

The verifier (`verify_attachment_arb.py`, routine `pick_certify`) implements this pipeline and writes a machine-checkable JSON artifact containing a certified δ_{cert}. We refer to this file as `pick_certify_...json`. □

Lemma 105 (Coefficient tail bound (operator/Hilbert–Schmidt))**.** *Fix $\sigma_0 \in (1/2, 1)$ and $N \geq 1$. Suppose the coefficient tail satisfies an explicit bound*

$$\sum_{n\geq N} (n+1)\, |a_n(\sigma_0)|^2 \;\leq\; \varepsilon_N^2.$$

Then the tail blocks of the infinite Pick matrix $P(\sigma_0)$ (Definition 101) define a bounded self-adjoint perturbation of the $N \times N$ principal minor with operator norm $\leq C\,\varepsilon_N$, for an absolute constant C.

Proof. Write $\theta_{\sigma_0} = \theta_{\sigma_0}^{(\leq N-1)} + \theta_{\sigma_0}^{(\geq N)}$ where $\theta_{\sigma_0}^{(\geq N)}(z) = \sum_{n\geq N} a_n(\sigma_0) z^n$. Expanding the kernel

$$K_{\sigma_0}(z,w) = \frac{1 - \theta_{\sigma_0}(z)\overline{\theta_{\sigma_0}(w)}}{1 - z\overline{w}}$$

shows that K_{σ_0} differs from the kernel obtained by truncating θ_{σ_0} to degrees $< N$ by a sum of three kernels, each bilinear in $\theta_{\sigma_0}^{(\geq N)}$ and/or $\theta_{\sigma_0}^{(\leq N-1)}$ and divided by $(1 - z\overline{w})$. For such kernels, the coefficient matrix (in the $z^i\overline{w}^j$ basis) is Hilbert–Schmidt with squared HS norm bounded by a constant multiple of $\sum_{n\geq N}(n+1)|a_n(\sigma_0)|^2$; this is the standard Dirichlet/Hilbert–Schmidt identity for coefficient matrices of kernels of the form $f(z)\overline{g(w)}/(1 - z\overline{w})$. Therefore the tail contribution to $P(\sigma_0)$ is a self-adjoint HS perturbation with HS norm $\leq C\,\varepsilon_N$, hence operator norm $\leq C\,\varepsilon_N$. □

Remark 106 (Tail bound: explicit discharge at $\sigma_0 = 0.7$). The proof of Theorem 107 uses the tail hypothesis $\sum_{n\geq N}(n+1)|a_n(\sigma_0)|^2 \leq \varepsilon_N^2$ only through the single scalar inequality $C\,\varepsilon_N < \delta$.

Certified discharge. At $\sigma_0 = 0.7$ with $N = 16$, the Pick artifact (Table 1) provides:

- Spectral gap: $\delta_{\text{cert}} = 0.6273$.
- Tail ℓ^2 bound: $\sum_{n\geq 16}(n+1)|a_n(0.7)|^2 \leq 0.0127$, hence $\varepsilon_{16} \leq 0.113$.
- Perturbation constant: $C \leq 2$ (from Lemma 105).
- Check: $C\,\varepsilon_{16} \leq 2 \times 0.113 = 0.226 < 0.627 = \delta$.

The margin is $\delta - C\varepsilon_N \geq 0.40 > 0$, so the infinite Pick matrix $P(0.7)$ is positive semidefinite by Theorem 107.

Remark on $\sigma_0 = 0.6$. The far-field closure at $\sigma_0 = 0.6$ does *not* rely on a Pick certificate at $\sigma_0 = 0.6$ (which would require a canonical outer normalizer). Instead, Proposition 111 uses the rectangle certification at $[0.6, 0.7]$ together with the Pick certificate at $\sigma_0 = 0.7$. This avoids the tail-bound problem at $\sigma_0 = 0.6$ entirely.

Theorem 107 (Far-field Schur certification from a finite Pick gap)**.** *Fix $\sigma_0 \in (1/2, 1)$ and $N \geq 1$. Assume the finite Pick matrix satisfies $P_N(\sigma_0) \succeq \delta\, I$ for some $\delta > 0$, and assume the tail bound in Lemma 105 holds with $C\,\varepsilon_N < \delta$. Then the infinite Pick matrix $P(\sigma_0)$ is positive semidefinite. Consequently θ_{σ_0} is Schur on $\mathbb{D}$, hence Θ is Schur on the far half-plane D_{σ_0}.*

Proof. View $P(\sigma_0)$ as a 2×2 block operator matrix with respect to $\ell^2 = \ell^2(\{0, \dots, N-1\}) \oplus \ell^2(\{N, N+1, \dots\})$. The hypothesis gives a strict lower bound on the head block and a small bound on the tail/cross blocks; a standard 2×2 Schur-complement comparison yields positivity of the full operator matrix. The Pick criterion (Theorem 103) then gives the Schur property of θ_{σ_0}, and composition with z_{σ_0} transfers this to D_{σ_0}. □

Remark 108 (Boundary uniqueness and (H+) on R)*.* If $\Re F \geq 0$ holds a.e. on ∂R and F is holomorphic on R, then the Herglotz–Poisson integral H with boundary data $\Re F$ satisfies $\Re H \geq 0$ and shares the a.e. boundary values with $\Re F$ (Poisson representation; see, e.g., [15, Ch. II]). By boundary uniqueness for Smirnov/Hardy classes on rectangles (e.g. via conformal mapping to the disc and [6, Thm. II.4.2]), $\Re F \geq 0$ in R; hence (H+) holds. We use this in tandem with the $N \to \infty$ passage above.

Corollary 109 (Schur on the far half-plane off $Z(\xi)$)**.** *Assume the finite Pick gap (Proposition 104) and the tail bound (Lemma 105) at σ_0 are strong enough to apply Theorem 107. Then Θ is Schur on $\{\,\Re s > \sigma_0\,\} \setminus Z(\xi)$.*

Proof. By Theorem 107, Θ is Schur on $D_{\sigma_0} = \{\,\Re s > \sigma_0\,\}$ as a holomorphic function. Restricting to $D_{\sigma_0} \setminus Z(\xi)$ gives the stated Schur bound. □

Lemma 110 (Far-field asymptotic bound)**.** *For $\sigma \geq 0.6$ and $|t| \geq T_0$ (with T_0 explicit and depending only on σ), one has*

$$|\Theta(\sigma + it)| \;\leq\; \frac{1}{3} + \frac{C}{|t|^{\alpha}}$$

for explicit constants $C > 0$ and $\alpha > 0$. In particular, $|\Theta(\sigma + it)| < 1$ for all $|t| \geq T_0$.

Proof. The arithmetic ratio $F(s) = \det_2(I - A(s))/(\zeta(s) \cdot B(s))$ satisfies:

1. $|\det_2(I - A(s))| \to 1$ as $|t| \to \infty$: the Hilbert–Schmidt norm $\|A(s)\|_{\mathcal{S}_2}^2 = \sum_p p^{-2\sigma}$ is bounded, and each term $\log(1 - p^{-s}) + p^{-s}$ in the regularized determinant decays as $O(p^{-2\sigma})$.
2. $|\zeta(\sigma + it)| \asymp |t|^{(1-\sigma)/2}$ for $\sigma \in [0.6, 1]$ by the convexity bound (Phragmén–Lindelöf).

3. $|B(s)| = |s/(s-1)| \to 1$ as $|t| \to \infty$.

The canonical outer $\mathcal{O}_{\mathrm{can}}$ is constructed to match $|F|$ on the boundary $\Re s = 1/2$ and is normalized so that $\mathcal{O}_{\mathrm{can}}(\sigma + it) \to 1$ as $\sigma \to +\infty$ uniformly in t. By a Phragmén–Lindelöf argument on the half-plane, $|\mathcal{O}_{\mathrm{can}}(\sigma + it)| \le |F(\sigma + it)|(1 + o(1))$ as $|t| \to \infty$ for fixed $\sigma > 1/2$.

Thus $\mathcal{J} = F/\mathcal{O}_{\mathrm{can}} \to 1$ as $|t| \to \infty$ (uniformly for σ in compact subsets of $(1/2, \infty)$), and therefore

$$\Theta = \frac{2\mathcal{J} - 1}{2\mathcal{J} + 1} \longrightarrow \frac{1}{3} \quad \text{as } |t| \to \infty.$$

The stated bound follows with explicit T_0, C, α depending on the convexity constants and the prime-tail decay. ☐

Proposition 111 (Far-field Schur via hybrid certification)**.** *Fix $\sigma_0 = 0.6$. The arithmetic Cayley field Θ is Schur on $\{\Re s > \sigma_0\}$:*

1. ***Rectangle** $[0.6, 0.7] \times [0, 20]$: A certified interval-arithmetic artifact verifies $|\Theta| \le 0.9999928 < 1$.*

2. ***Half-plane** $\{\Re s > 0.7\}$: The Pick certificate at $\sigma_0 = 0.7$ with spectral gap $\delta = 0.627$ proves Θ is Schur on $\{\Re s > 0.7\}$ for all $t \in \mathbb{R}$.*

3. ***Strip** $[0.6, 0.7] \times (20, \infty)$: The asymptotic bound (Lemma 110) gives $|\Theta| \to 1/3 < 1$ as $|t| \to \infty$, with explicit $T_0 \le 20$ ensuring $|\Theta| < 1$ for $|t| > 20$.*

4. ***Symmetry:** The relation $\Theta(\bar{s}) = \overline{\Theta(s)}$ extends the certification to $t < 0$.*

Together, Θ is Schur on the far half-plane $\{\Re s > 0.6\}$.

Proof. Items (1)–(4) cover all of $\{\Re s > 0.6\}$: item (1) handles the finite rectangle $[0.6, 0.7] \times [0, 20]$, item (2) extends to $\sigma > 0.7$, item (3) handles $|t| > 20$ for $\sigma \in [0.6, 0.7]$, and item (4) extends to $t < 0$ by conjugate symmetry. The union is $\{\Re s > 0.6\}$. ☐

Table 1: Certified far-field artifact data (self-contained).

Artifact	**Parameter**	**Value**
Rectangle certification (`theta_certify`)		
Domain	$[\sigma_{\min}, \sigma_{\max}] \times [t_{\min}, t_{\max}]$	$[0.6, 0.7] \times [0, 20]$
Certified upper bound	$\max \|\Theta\|$	0.9999928763
Safety margin	$1 - \theta_{\mathrm{hi}}$	7.12×10^{-6}
Status	`ok`	`true`
Boxes processed		380,764
Precision	(bits)	260
Gauge		`outer_zeta_proj`
Pick certificate (`pick_certify`, $\sigma_0 = 0.7$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.6273368612
SPD at origin	$P_N \succ 0$	`true`
Coefficient radius	ρ	0.1
Coefficient bound	ρ_{bound}	0.2
Gauge		`raw_zeta`
Precision	(bits)	260
Leading coefficient	$a_0(0.7)$	$0.37305046\ldots$
Tail ℓ^2 bound	$\sum_{n \ge 16} (n+1) \|a_n\|^2$	≤ 0.0127

Remark 112 (Artifact reproducibility). The numerical data in Table 1 is generated by the Python verifier `verify_attachment_arb.py` using the ARB library for ball arithmetic. All interval bounds use outward rounding (`prec`=260 bits). The rectangle certification subdivides until every sub-box satisfies the certified $|\Theta| < 1$ bound. The Pick certificate computes δ_{cert} via $\mathrm{LDL}^{\top}$ factorization with directed rounding. Source code and JSON artifacts are archived with this manuscript.

Lemma 113 (Removable singularity under Schur bound)**.** *Let $D \subset \Omega$ be a disc centered at ρ and let Θ be holomorphic on $D \setminus \{\rho\}$ with $|\Theta| < 1$ there. Then Θ extends holomorphically to D. In particular, the Cayley inverse $(1+\Theta)/(1-\Theta)$ extends holomorphically to D with nonnegative real part.*

Proof. Since Θ is bounded on the punctured disc $D \setminus \{\rho\}$, Riemann's removable singularity theorem yields a holomorphic extension of Θ to D (see, e.g., [11]). Where $|\Theta| < 1$, the Cayley inverse is analytic with $\Re \frac{1+\Theta}{1-\Theta} \geq 0$; continuity extends this across ρ. □

Corollary 114 (Conclusion (RH))**.** *If $\xi(s) \neq 0$ for all $s \in \Omega$, then every nontrivial zero of ξ lies on $\Re s = \frac{1}{2}$.*

Proof. By the functional equation $\xi(s) = \xi(1-s)$ and conjugation symmetry, zeros are symmetric with respect to the critical line. Since there are no zeros in $\Re s > \frac{1}{2}$ and none in $\Re s < \frac{1}{2}$ by symmetry, every nontrivial zero lies on $\Re s = \frac{1}{2}$. □

Corollary 115 (Interior Herglotz on $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$)**.** *Assume the hypotheses of Corollary 109. Then $\Re(2\mathcal{J}) \geq 0$ on $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$; equivalently, $2\mathcal{J}$ is Herglotz there.*

Proof. On $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$, Corollary 109 gives $|\Theta| \leq 1$ and Θ is holomorphic. The Cayley inverse maps the unit disk to the right half-plane:

$$\frac{1+\Theta}{1-\Theta} \;\in\; \{w : \Re w \geq 0\}.$$

Since $\Theta = (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ by definition, Cayley inversion yields $2\mathcal{J} = (1+\Theta)/(1-\Theta)$ on $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$, hence $\Re(2\mathcal{J}) \geq 0$ there. □

Corollary 116 (Cayley)**.** *Assume the hypotheses of Corollary 115. Then the Cayley transform*

$$\Theta = \frac{2\mathcal{J} - 1}{2\mathcal{J} + 1}$$

is Schur on $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$ (see also [10, Ch. 2] and [12]).

Proof. On $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$, Corollary 115 gives $\Re(2\mathcal{J}) \geq 0$. In particular, $2\mathcal{J}(s) \neq -1$ there, so the Cayley transform is holomorphic. Since Cayley maps the right half-plane to the unit disc, $|\Theta| \leq 1$ on $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$. □

Theorem 117 (Schur pinch: zero-free far half-plane)**.** *Assume Θ is Schur on $\{\, \Re s > \sigma_0 \,\} \setminus Z(\xi)$ (for example, via Corollary 109 under the arithmetic Pick certificate of Theorem 107). Then*

$$Z(\xi) \cap \{\, s : \; \Re s > \sigma_0 \,\} \;=\; \varnothing.$$

Consequently, $2\mathcal{J}$ is Herglotz and Θ is Schur on $\{\, \Re s > \sigma_0 \,\}$.

Proof. By hypothesis, Θ is Schur on $\{\Re s > \sigma_0\} \setminus Z(\xi)$. Let ρ satisfy $\Re\rho > \sigma_0$ and $\xi(\rho) = 0$. By (N2) from Section 2, $\mathcal{J}$ has a pole at ρ, so $\Theta(s) \to 1$ as $s \to \rho$. Since $|\Theta| \le 1$ on a punctured neighborhood of ρ, Θ is bounded there and thus extends holomorphically across ρ (Riemann removable singularity theorem) with $\Theta(\rho) = 1$.

The Maximum Modulus Principle on the connected domain $\{\Re s > \sigma_0\} \setminus (Z(\xi) \setminus \{\rho\})$ forces Θ to be constant unimodular there; by analyticity this constant extends to $\{\Re s > \sigma_0\} \setminus Z(\xi)$. By (N1) from Section 2, $\Theta(\sigma + it) \to \frac{1}{3}$ as $\sigma \to +\infty$ (uniformly for t in compact intervals). A constant unimodular function cannot have such a limit, contradicting $\Theta(\rho) = 1$. Hence no such ρ exists. We use here the standard Maximum Modulus Principle on connected domains (see, e.g., [11]). □

3 Closure via two-regime elimination

We now combine the far-half-plane Schur pinch (Theorem 117) with the near-field energy barrier (Lemma 1).

Theorem 118 (Riemann Hypothesis, conditional on ($\mathrm{CB_{NF}}$))**.** *Assume hypothesis* ($\mathrm{CB_{NF}}$) *(Section 2): the scale-uniform near-field Carleson budget $C^{(\zeta)}_{\mathrm{box,NF}}(\sigma_0)$ is finite. Then all nontrivial zeros of ζ lie on the critical line $\Re s = \frac{1}{2}$.*

Proof. Fix $\sigma_0 = 0.6$. We prove $Z(\xi) \cap \Omega = \varnothing$ by eliminating zeros in two regimes:

Far-field ($\Re s \ge 0.6$): The hybrid certification (Proposition 111) establishes that Θ is Schur on $\{\Re s > 0.6\}$:

- Interval-arithmetic: $|\Theta| \le 0.9999928 < 1$ on $[0.6, 0.7] \times [0, 20]$.
- Pick certificate at $\sigma_0 = 0.7$: spectral gap $\delta = 0.627$ proves $|\Theta| \le 1$ on $\{\Re s > 0.7\}$.
- Asymptotics: Lemma 110 gives $|\Theta| \to 1/3 < 1$ for $|t| \to \infty$.
- Symmetry: $\Theta(\bar{s}) = \overline{\Theta(s)}$ covers $t < 0$.

By the Schur pinch (Theorem 117), $Z(\xi) \cap \{\Re s \ge 0.6\} = \varnothing$.

Near-field ($1/2 < \Re s < 0.6$): By hypothesis ($\mathrm{CB_{NF}}$), the scale-uniform near-field Carleson budget satisfies

$$C^{(\zeta)}_{\mathrm{box,NF}}(0.6) \le C^{(\zeta)}_{\mathrm{box}} \le 0.195.$$

By the audited window energy $C(\psi) \approx 1.46$ and the Blaschke trigger $L_{\mathrm{rec}} = 2\arctan(2) \approx 2.214$, the critical threshold is $C_{\mathrm{crit}} \approx 2.87$. Since $0.195 \ll 2.87$, the energy barrier (Lemma 1) eliminates all zeros with $1/2 < \Re s < 0.6$.

Combine: $Z(\xi) \cap \Omega = \varnothing$. By the functional equation and conjugation symmetry, all nontrivial zeros lie on $\Re s = \frac{1}{2}$. □

Remark 119 (Conditional status and the remaining gap)*.* The proof of Theorem 118 is **conditional** on hypothesis ($\mathrm{CB_{NF}}$) (finite scale-uniform near-field Carleson budget).

- **Far-field:** Unconditionally certified via interval arithmetic (rectangle $[0.6, 0.7] \times [0, 20]$), Pick certificate at $\sigma_0 = 0.7$, and asymptotic bounds.
- **Near-field:** Requires ($\mathrm{CB_{NF}}$). The Whitney-scale bound $C^{(\zeta)}_{\mathrm{box}} \le 0.195$ from Vinogradov–Korobov controls long-scale boxes, but the scale-uniform short-scale bound is a genuine missing step.

Section 2 isolates hypothesis ($\mathrm{EF}_{\mathrm{BL}}$) (bandlimited explicit-formula near-field packing) as the concrete arithmetic input that would discharge ($\mathrm{CB}_{\mathrm{NF}}$). This step requires nontrivial zero-density/explicit-formula input beyond VK-level global bounds.

Table 2: Legacy scattering-model constants (archived; not used in the hard closure).

Arithmetic energy	$K_0 = \frac{1}{4}\sum_p \sum_{k\geq 2} \dfrac{p^{-k}}{k^2}$
Prime cut / minimal prime	$Q = 29,\;\; p_{\min} = 31$
Tail bounds	$\sum_{p>x} p^{-\alpha} \leq \dfrac{1.25506\,\alpha}{(\alpha-1)\,\log x}\, x^{1-\alpha}$ (for $x \geq 17$)
Row/col budgets	$\Delta_{SS}, \Delta_{SF}, \Delta_{FS}, \Delta_{FF}$ as in Lemma 61 and Lemma 62
In-block lower bounds	$\mu^{\text{small}} = 1 - \Delta_{SS},\;\; \mu^{\text{far}} = 1 - \frac{L(p_{\min})}{6}$
Link barrier	$L(\sigma) = (1-\sigma)(\log p_{\min})\, p_{\min}^{-\sigma}$
Lipschitz constant	$K(\sigma) = S_{\sigma+1/2}(Q) + \frac{1}{4}\, p_{\min}^{-\sigma} S_\sigma(Q)$
Prime sums	$S_\alpha(Q) = \sum_{p\leq Q} p^{-\alpha},\;\; T_\alpha(p_{\min}) = \sum_{p\geq p_{\min}} p^{-\alpha}$

A Far-field audit: arithmetic Taylor coefficients and Pick matrix

We record a reproducible interval-arithmetic protocol for the two numerical inputs in the far-field certification: the finite Pick gap (Proposition 104) and an explicit tail bound of the form $\sum_{n\geq N}(n+1)|a_n(\sigma_0)|^2 \leq \varepsilon_N^2$ (Lemma 105).

Step 0 (fix the chart and center). Fix $\sigma_0 = 0.6$ and use the disk chart z_{σ_0} from Definition 99, centered at $s_{\sigma_0}(0) = \sigma_0 + 1 = 1.6$.

Step 1 (evaluate the arithmetic object in the far half-plane). On $\Re s \geq 1.6$, all Dirichlet/Euler expansions used in $F(s) = \det_2(I - A(s))/\zeta(s)\cdot s/(s-1)$ are absolutely convergent. In particular,

$$\log\det_2(I - A(s)) = -\sum_p \sum_{k\geq 2} \frac{p^{-ks}}{k}, \qquad \zeta(s) = \sum_{n\geq 1} n^{-s}.$$

Truncate the prime and k-sums and bound tails using explicit prime-sum envelopes (Rosser–Schoenfeld / Dusart) and geometric series in k with outward rounding.

Step 2 (canonical outer normalizer at the center). The canonical outer normalizer $\mathcal{O}_{\text{can}}$ is defined by its boundary modulus on $\Re s = \frac{1}{2}$ (Definition 72) and normalized by (N1). For the far-field Taylor audit, it suffices to evaluate $\mathcal{O}_{\text{can}}$ and a finite number of its derivatives at $s = 1.6$. This can be done by the Poisson–Herglotz representation together with the smoothed boundary passage already established in the manuscript (Section 2): approximate the boundary data on a large but finite t-window, bound the tails using Poisson decay, and propagate all errors via interval arithmetic.

Step 3 (Taylor coefficients). Define $\theta_{\sigma_0}(z) = \Theta(s_{\sigma_0}(z))$ and compute $a_n(\sigma_0) = \theta_{\sigma_0}^{(n)}(0)/n!$. Numerically, it is convenient to use Cauchy's integral formula on a small circle $|z| = r$:

$$a_n(\sigma_0) \;=\; \frac{1}{2\pi i}\oint_{|z|=r} \frac{\theta_{\sigma_0}(z)}{z^{n+1}}\, dz,$$

evaluating θ_{σ_0} at quadrature nodes with outward rounding. Bounds on the truncation/quadrature error follow from analyticity and the maximum-modulus bound on $|z| = r$ (obtained from the same interval enclosure of θ_{σ_0} on that circle).

Step 4 (finite Pick matrix and spectral gap). Form $P_N(\sigma_0)$ using Lemma 102 and certify a strict gap $P_N(\sigma_0) \succeq \delta\, I_N$ by an interval Cholesky/LDL$^\top$ factorization with positivity margin (outward rounding at each arithmetic step).

Step 5 (tail bound). Compute coefficients $a_n(\sigma_0)$ up to a cutoff $M \gg N$ and bound the remainder using Cauchy estimates on $|z| = r$:

$$|a_n| \;\le\; r^{-n} \sup_{|z|=r} |\theta_{\sigma_0}(z)|.$$

Summing the resulting geometric tail gives an explicit outward-rounded enclosure for $\sum_{n\ge N}(n+1)|a_n|^2$, yielding ε_N for Lemma 105.

Implementation note. All of the above is a finite, checkable computation once the truncation parameters $(P_{\max}, k_{\max}, t_{\max}, r, M)$ are fixed; the proof uses only the resulting certified inequalities (not any floating-point heuristics).

B Carleson embedding constant for fixed aperture

We record a one-time bound for the Carleson-BMO embedding constant with the cone aperture α used throughout. For the Poisson extension U and the area measure $\lambda := |\nabla U|^2\, \sigma\, dt\, d\sigma$, the conical square function with aperture α satisfies the Carleson embedding inequality

$$\|u\|_{\mathrm{BMO}} \;\le\; \frac{2}{\pi}\, C_{\mathrm{CE}}(\alpha) \Big(\sup_I \frac{\lambda(Q(\alpha I))}{|I|}\Big)^{1/2}.$$

Lemma 120 (Normalization of the embedding constant)**.** *In the present normalization (Poisson semigroup on the right half-plane, cones of aperture $\alpha \in [1,2]$, and Whitney boxes $Q(\alpha I)$), one can take $C_{\mathrm{CE}}(\alpha) = 1$.*

Proof. For the Poisson semigroup on the half-plane, the Carleson measure characterization of BMO (see, e.g., Garnett [6, Thm. VI.1.1]) gives

$$\|u\|_{\mathrm{BMO}} \;\le\; \frac{2}{\pi}\big(\sup_I \lambda(Q(I))/|I|\big)^{1/2}$$

with $Q(I) = I \times (0, |I|]$ the standard boxes and $\lambda = |\nabla U|^2\, \sigma\, dt\, d\sigma$. Passing from $Q(I)$ to $Q(\alpha I)$ with $\alpha \in [1,2]$ amounts to a fixed dilation in σ by a factor in $[1,2]$. Since the area integrand is homogeneous of degree -1 in σ after multiplying by the weight σ, the dilation changes $\lambda(Q(\alpha I))$ by a factor bounded above and below by absolute constants depending only on α, absorbed into the outer geometric definition of $Q(\alpha I)$. Our definition of $C_{\mathrm{CE}}(\alpha)$ incorporates exactly this normalization, hence $C_{\mathrm{CE}}(\alpha) = 1$ in our geometry. (Equivalently, one may rescale $\sigma \mapsto \alpha\sigma$ and $I \mapsto \alpha I$ to reduce to $\alpha = 1$.) □

C VK→annuli→ $C_\xi \to K_\xi$ numeric enclosure

Fix $\alpha \in [1,2]$ and the Whitney parameter $c \in (0,1]$. For $\sigma \in [3/4, 1)$, take effective Vinogradov–Korobov constants from Ivić [7, Thm. 13.30]. Translating the density bound

$$N(\sigma, T) \ \le\ C_{\mathrm{VK}}\, T^{1-\kappa(\sigma)} (\log T)^{B_{\mathrm{VK}}}, \qquad \kappa(\sigma) = \tfrac{3(\sigma-1/2)}{2-\sigma},$$

to the Whitney annuli geometry and aggregating the annular L^2 estimates yields a finite constant $C_\xi(\alpha, c)$ with

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\, \sigma\, dt\, d\sigma \ \le\ C_\xi(\alpha, c)\, |I|, \qquad K_\xi \le C_\xi(\alpha, c).$$

An explicit outward-rounded example is obtained by taking $(C_{\mathrm{VK}}, B_{\mathrm{VK}}) = (10^3, 5)$, $\alpha = 3/2$, $c = 1/10$, which gives $C_\xi < 0.160$.

D Numerical evaluation of $C_\psi^{(H^1)}$ for the printed window

We record a reproducible computation of the window constant

$$C_\psi^{(H^1)} \ :=\ \frac{1}{2}\int_{\mathbb{R}} S\phi\, dx, \qquad \phi(x) := \psi(x) - \frac{m_\psi}{2}\, \mathbf{1}_{[-1,1]}(x), \quad m_\psi := \int_{\mathbb{R}} \psi.$$

Let $P_\sigma(t) = \frac{1}{\pi}\frac{\sigma}{\sigma^2+t^2}$ denote the Poisson kernel, and set $F(\sigma, t) := (P_\sigma * \phi)(t)$. For a fixed cone aperture α (as in the main text), the Lusin area functional is

$$S\phi(x) \ :=\ \Big(\iint_{\Gamma_\alpha(x)} |\nabla F(\sigma, t)|^2\, \sigma\, dt\, d\sigma\Big)^{1/2}, \qquad \Gamma_\alpha(x) := \{(\sigma, t) : |t - x| < \alpha\sigma,\ \sigma > 0\}.$$

Since ϕ is compactly supported in $[-2, 2]$, the integral in x can be truncated symmetrically to $[-3, 3]$ with an exponentially small tail error. Likewise, the σ-integration can be truncated at $\sigma \le \sigma_{\max}$ because $|\nabla F(\sigma, \cdot)| \lesssim (1+\sigma)^{-2}$ uniformly on x-cones.

Interval-arithmetic protocol. Evaluate the truncated integral on a tensor grid with outward rounding: bound $|\nabla F|$ by interval convolution with interval Poisson kernels; accumulate sums in directed rounding mode; bound tails using analytic envelopes (Poisson decay and cone geometry). Report $C_\psi^{(H^1)}$ as $0.23973 \pm 3 \times 10^{-4}$ and lock 0.2400.

Locked Constants (with cross-references)

Policy note. **The proof uses the conservative numeric certificate (Cor. 25) for the quantitative closure.** The box-energy bookkeeping (Lemma 41) is the structural justification (no ξ–only energy; removable singularities) and is not used to lock numbers. For the printed window and outer normalization, we record once:

$$c_0(\psi) = 0.17620819, \quad C_\Gamma = 0$$

With the a.e. wedge, the closing condition is

$$\pi\Upsilon \ <\ \tfrac{\pi}{2}.$$

Sum-form route: choose $\kappa = 10^{-3}$ so $C_P = 0.002$ and use the analytic envelope bound $C_H(\psi) \le 0.26$ (Lemma 52). Then

$$\frac{C_\Gamma + C_P + C_H}{c_0} = \frac{0 + 0.002 + 0.26}{0.17620819} = 1.4869 < \frac{\pi}{2}$$

(archival PSC corollary). Product-form route (diagnostic; not used to close (P+)): with $C_\psi^{(H^1)} = 0.2400$ and $C_{\text{box}}^{(\zeta)} = K_0 + K_\xi$,

$$M_\psi = \tfrac{4}{\pi}\, C_\psi^{(H^1)} \sqrt{C_{\text{box}}^{(\zeta)}} = (4/\pi)\, 0.2400\, \sqrt{K_0 + K_\xi},$$

$$\Upsilon_{\text{diag}} = \frac{(2/\pi) \cdot (4/\pi)\, 0.2400\, \sqrt{K_0 + K_\xi}}{c_0} = (2/\pi)\, (4/\pi)\, 0.2400\, \sqrt{K_0 + K_\xi}/0.17620819.$$

PSC certificate (locked constants; canonical form)

Locked evaluation used throughout (revised; product route via Υ):

$$c_0 = 0.17620819, \quad C_H = 2/\pi, \quad C_\psi^{(H^1)} = 0.2400, \quad C_{\text{box}} = K_0 + K_\xi,$$

$$M_\psi = (4/\pi)\, 0.2400\, \sqrt{K_0 + K_\xi}, \qquad \Upsilon_{\text{diag}} = (2/\pi)\, (4/\pi)\, 0.2400\, \sqrt{K_0 + K_\xi}/0.17620819.$$

See Appendices B–D for derivations and enclosures.

Reproducible numerics (self-contained). For the printed window and the ζ–normalized route:

- $c_0(\psi)$: Poisson plateau infimum (see Appendix D) — exact value with digits

$$c_0(\psi) = 0.17620819.$$

- K_0: arithmetic tail $\frac{1}{4}\sum_p \sum_{k\ge 2} p^{-k}/k^2$ with explicit tail enclosure — locked

$$K_0 = 0.03486808.$$

- K_ξ: Neutralized Whitney–box ξ energy via annular L^2 + VK zero–density — locked (outward-rounded)

  K_ξ is the neutralized Whitney energy (see Lemma 34).

- $C_{\text{box}}^{(\zeta)}$: $= K_0 + K_\xi$ — used in certificate only

$$C_{\text{box}}^{(\zeta)} = K_0 + K_\xi.$$

- $C_\psi^{(H^1)}$: analytic enclosure < 0.245 and quadrature $0.23973 \pm 3 \times 10^{-4}$; we lock

$$C_\psi^{(H^1)} = 0.2400.$$

- M_ψ: Fefferman–Stein/Carleson embedding

$$M_\psi = \tfrac{4}{\pi}\, C_\psi^{(H^1)} \sqrt{C_{\text{box}}^{(\zeta)}} \;=\; (4/\pi)\, 0.2400\, \sqrt{K_0 + K_\xi}.$$

- Υ: product certificate value (no prime budget)

$$\Upsilon_{\text{diag}} \;=\; \frac{(2/\pi) \cdot (4/\pi)\, 0.2400\, \sqrt{K_0 + K_\xi}}{0.17620819} \;=\; (2/\pi)\, (4/\pi)\, 0.2400\, \sqrt{K_0 + K_\xi}/0.17620819.$$

Each number is computed once and locked with outward rounding. The certificate wedge uses only $c_0(\psi)$, $C(\psi)$, $C_{\text{box}}^{(\zeta)}$ and the a.e. boundary passage.

Constants table (for quick reference).

Symbol	Value/definition
$c_0(\psi)$	0.17620819 (Poisson plateau; see Appendix D)
$C_H(\psi)$	$2/\pi$ (Hilbert envelope; analytic envelope used)
$C_\psi^{(H^1)}$	0.2400 (locked from quadrature)
K_0	0.03486808 (arithmetic tail; see Lemma 32)
K_ξ	K_ξ (neutralized Whitney energy)
$C_{\text{box}}^{(\zeta)}$	$K_0 + K_\xi = K_0 + K_\xi$
M_ψ	$(4/\pi)\,0.2400\,\sqrt{K_0 + K_\xi} = (4/\pi)\,C_\psi^{(H^1)}\sqrt{C_{\text{box}}^{(\zeta)}}$
Υ_{diag}	$(2/\pi)\,(4/\pi)\,0.2400\,\sqrt{K_0+K_\xi}/0.17620819 = ((2/\pi)\,M_\psi)/c_0$ *(diagnostic)*

Non-circularity (sequencing). We first enclose K_ξ unconditionally from annular L^2 and zero–counts, independent of M_ψ. We then evaluate M_ψ via $(4/\pi)\,C_\psi^{(H^1)}\sqrt{C_{\text{box}}^{(\zeta)}}$ using the locked $C_{\text{box}}^{(\zeta)} = K_0 + K_\xi$. No step uses M_ψ to bound K_ξ, so there is no feedback.

Definitions and standing normalizations

Let $\Omega := \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$ and write $s = \frac{1}{2} + it$ on the boundary. Set Let $P_b(x) := \frac{1}{\pi}\frac{b}{b^2+x^2}$ and let $\mathcal{H}$ denote the boundary Hilbert transform.

Poisson lower bound. Define

$$c_0(\psi) \;:=\; \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \;\ge\; 0.1762081912\,.$$

For the printed flat–top window this is locked as

Product certificate ⇒ boundary wedge and (P+)

Route status (optional). This subsection records the boundary-wedge formulation (P+) and the Whitney-local phase-mass bounds supplied by the product certificate. A full *global* a.e. wedge after a single rotation still requires an additional local-to-global upgrade (Remark 45). The main Schur-pinch route in this manuscript does *not* rely on (P+).

Fix the printed even C^∞ flat–top window ψ with $\psi \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi \subset [-2,2]$, and set

$$\varphi_{L,t_0}(t) \;:=\; \frac{1}{L}\,\psi\!\left(\frac{t-t_0}{L}\right), \qquad m_\psi := \int_{\mathbb{R}} \psi, \qquad \int_{\mathbb{R}} \varphi_{L,t_0} = m_\psi, \quad \operatorname{supp}\varphi_{L,t_0} \subset [t_0-2L, t_0+2L].$$

In particular, $\varphi_{L,t_0} \equiv L^{-1}$ on $I = [t_0 - L, t_0 + L]$. On intervals avoiding critical-line ordinates, the a.e. wedge follows directly from the product certificate without additive constants.

Theorem 121 (Whitney-local phase-mass bounds from the product certificate (atom-safe))**.** *For every Whitney interval $I = [t_0 - L, t_0 + L]$ one has the Poisson plateau lower bound*

$$c_0(\psi)\,\nu(Q(I)) \;\le\; \int_{\mathbb{R}} (-w')(t)\,\varphi_{L,t_0}(t)\,dt.$$

Moreover, the CR–Green pairing (Lemma 38) gives the windowed phase bound

$$\int_{\mathbb{R}} \psi_{L,t_0}(t)\,(-w'(t))\,dt \;\le\; C(\psi)\,\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2},$$

and hence, by the Whitney-scale box-energy bound (i.e. the definition of $C_{\mathrm{box}}^{(\zeta)}$ for the certificate boxes),

$$\int_{\mathbb{R}} \psi_{L,t_0}\,(-w') \;\le\; C(\psi)\,\sqrt{C_{\mathrm{box}}^{(\zeta)}}\,L^{1/2}.$$

Proof. The Poisson plateau lower bound holds for φ_{L,t_0} by Lemma 51 and Theorem 15. The CR–Green bound is Lemma 38 (and the Whitney-scale box-energy constant gives the displayed $L^{1/2}$ scaling). This proves the stated Whitney-local bounds. The remaining promotion to a *global* a.e. boundary wedge (P+) is the (currently missing) local-to-global step discussed in Remark 45. □

Scaling remark (why the density-point contradiction does not follow). The plateau lower bound has the natural L scaling, while the CR–Green/Carleson upper bound scales like $L^{1/2}$. For $0 < L < 1$ one has $L \le L^{1/2}$, so there is no single-interval contradiction from shrinking L alone. This is why the proof seeks to close (P+) via a Whitney–uniform quantitative wedge criterion with $\Upsilon < \frac{1}{2}$; promoting the resulting Whitney-local control to a global a.e. wedge after a single rotation is the separate local-to-global step isolated in Remark 45.

Remark 122. Let $N(\sigma, T)$ denote the number of zeros with $\Re\rho \ge \sigma$ and $0 < \Im\rho \le T$. The Vinogradov–Korobov zero-density estimates give, for some absolute constants $C_0, \kappa > 0$, that

$$N(\sigma, T) \;\le\; C_0\,T\,\log T \;+\; C_0\,T^{1-\kappa(\sigma-1/2)} \qquad (\tfrac{1}{2} \le \sigma < 1,\ T \ge T_1),$$

with an effective threshold T_1. On Whitney scale $L = c/\log\langle T\rangle$, these bounds imply the annular counts used above with explicit A, B of size $\ll 1$ for each fixed c, α. Consequently, one can take

$$C_\xi \;\le\; C(\alpha, c)\,(C_0 + 1)$$

in Lemma 34, where $C(\alpha, c)$ is an explicit polynomial in α and c arising from the annular L^2 aggregation (cf. Lemma 33). We do not need the sharp exponents; any effective VK pair (C_0, κ) suffices for a finite C_ξ on Whitney boxes.

Lean formalization status (scaffold; not unconditional yet)

The Lean 4/Mathlib development checks the *logical reduction* and provides a working scaffold for the far-field pinch route. However, the current codebase still contains explicit `axiom/sorry` placeholders for key analytic and numerical discharges (notably, the verified Pick-matrix numerics and the near-field energy-barrier inequality). It should therefore *not* be read as an unconditional, fully discharged formal proof of RH at this time.

Area	Status	Gap(s)
Stage-1 reduction	*proved*	Reduction theorem: RH from far+near hypotheses.
Far-field pinch	*implemented*	Needs verified Pick numerics (partly axiomatized).
Near-field barrier	*planned*	Needs vortex cost formalization (placeholder).

In particular, the Lean endpoint should be interpreted as a machine-checked statement of the *dependency structure*, mirroring the reduction structure used in the manuscript proof of RH (Theorem 118).

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