THE RIEMANN HYPOTHESIS VIA SCHUR CERTIFICATION OF THE ARITHMETIC CAYLEY FIELD

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ABSTRACT. In a companion paper [1] we proved that the nontrivial zeros of the Riemann zeta function in the half-plane $\{\Re s > \frac{1}{2}\}$ are encoded as a pure Blaschke product $\mathcal{I}$ (the singular inner factor is trivial), and that the Riemann Hypothesis is equivalent to the statement that this Blaschke product has no zeros. In this paper we prove the Riemann Hypothesis by showing the Blaschke product is empty. The argument uses the Cayley transform $\Xi := (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ of the arithmetic ratio $\mathcal{J} := \det_2(I - A)/\zeta \cdot (s-1)/s$, together with the Nevanlinna–Pick criterion for Schur functions. The sole non-classical input is the Nyquist bandwidth hypothesis (T7-Hyp), which asserts that prime-frequency observables in the Guinand–Weil explicit formula are bandlimited by a fixed cutoff $\Omega_{\max}$. Under this hypothesis the windowed prime sum becomes a finite Dirichlet polynomial, the Carleson energy of $\log|\mathcal{J}|$ is uniformly bounded, the Pick spectral gap persists as $\sigma_0 \to (\frac{1}{2})^+$, and the Schur bound $|\Xi| \le 1$ closes on all of $\{\Re s > \frac{1}{2}\}$. No Cauchy–Schwarz inequality or energy pairing is used.

1. INTRODUCTION

Let $\Omega := \{\, s \in \mathbb{C} : \Re s > \frac{1}{2} \,\}$. In the companion paper [1] we established:

Theorem 1 ([1, Theorem 1]). *There exists an inner function $\mathcal{I}$ on Ω, constructed explicitly from ζ, the regularized determinant $\det_2(I - A(s))$, and an outer normalizer $\mathcal{O}_\zeta$, such that:*

(a) *$\mathcal{I}$ is holomorphic on Ω with $|\mathcal{I}| \le 1$;*
(b) *$|\mathcal{I}(\frac{1}{2} + it)| = 1$ for a.e. t;*
(c) *the zeros of $\mathcal{I}$ in Ω are exactly the nontrivial zeros of ζ in Ω;*
(d) *$\mathcal{I}$ is a pure Blaschke product ($S \equiv 1$).*

In particular, the Riemann Hypothesis is equivalent to $\mathcal{I} \equiv e^{i\theta}$.

The present paper proves:

Theorem 2 (Riemann Hypothesis). *Assume Hypothesis 3 below. Then $\zeta(s) \ne 0$ for all $s \in \Omega$.*

Hypothesis 3 (Nyquist bandwidth cutoff (T7-Hyp)). There exists $\Omega_{\max} < \infty$ such that for every admissible test function Φ in the Guinand–Weil explicit formula, the Fourier transform satisfies $\widehat{\Phi}(\xi) = 0$ for $|\xi| > \Omega_{\max}$.

Remark 4 (Status of T7-Hyp). T7-Hyp is a prediction of Recognition Science (derived from the eight-tick ledger structure; see [2]). It is not a theorem of classical analysis. Under T7-Hyp, the windowed prime sum in the explicit formula has at most $\pi(e^{\Omega_{\max}})$ terms—a fixed finite number. Every other ingredient of the proof is classical.

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Strategy. The proof avoids all Cauchy–Schwarz/energy pairings (which encounter a scaling obstruction; see [1, Remark A.15]). Instead, we use the *Schur/Nevanlinna–Pick pathway*:

(1) Form the *Cayley field* $\Xi := (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$ from the arithmetic ratio $\mathcal{J}$ of [1]. A pole of $\mathcal{J}$ (= zero of ζ) forces $\Xi \to 1$.

(2) A global *Schur bound* $|\Xi| \le 1$ makes every such singularity removable (Riemann's theorem), hence $\mathcal{J}$ has no poles and ζ has no zeros.

(3) The Schur bound is certified via the *Nevanlinna–Pick criterion*: a finite Pick matrix with positive spectral gap, plus a geometric Taylor tail bound.

(4) Under T7-Hyp, the windowed prime sum is a finite Dirichlet polynomial. By Montgomery–Vaughan, this gives a uniform Carleson energy bound for $\log|\mathcal{J}|$. The uniform bound prevents the Pick gap from degrading as $\sigma_0 \to (\frac{1}{2})^+$, closing the Schur certificate on all of Ω.

2. The Cayley field and the Schur pinch

We recall from [1] the arithmetic ratio

$$\mathcal{J}(s) \;:=\; \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}\,, \qquad s \in \Omega, \tag{2.1}$$

which is meromorphic on Ω with poles exactly at the nontrivial zeros of ζ, and satisfies $\mathcal{J}(s) \to 1$ as $\Re s \to +\infty$.

Definition 5 (Cayley field)**.** Define

$$\Xi(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}\,. \tag{2.2}$$

Lemma 6 (Pole-to-boundary behavior)**.** *If $\mathcal{J}$ has a pole at ρ, then $\Xi(\rho) \to 1$. If $\Re\mathcal{J}(s) > 0$ at a point, then $|\Xi(s)| < 1$.*

Proof. Write $\Xi - 1 = -2/(2\mathcal{J} + 1)$; a pole of $\mathcal{J}$ sends the denominator to ∞, so $\Xi \to 1$. For the second claim: $|\Xi| < 1 \iff |2\mathcal{J} - 1| < |2\mathcal{J} + 1| \iff \Re\mathcal{J} > 0$. □

Lemma 7 (Schur pinch)**.** *Let $U \subset \Omega$ be a domain. If $|\Xi(s)| \le 1$ on U (away from poles) and $\Xi \not\equiv 1$ on U, then Ξ extends holomorphically to U and $\mathcal{J}$ has no poles in U. In particular, ζ has no zeros in U.*

Proof. On a punctured disc around any pole of Ξ, the bound $|\Xi| \le 1$ makes Ξ bounded, hence the singularity is removable by Riemann's theorem. Thus Ξ extends holomorphically. Since $\Xi \not\equiv 1$, the Maximum Modulus Principle gives $|\Xi| < 1$ in the interior, so $1 - \Xi \neq 0$ and $\mathcal{J} = (1 + \Xi)/(2(1 - \Xi))$ is holomorphic on U. □

Remark 8 (Why this avoids the Cauchy–Schwarz obstruction). The CR–Green pathway of [1] pairs the field energy against a test-function energy via Cauchy–Schwarz, and the two scale differently in L (see [1, Remark A.15]). The Schur/Pick pathway never forms such a pairing. The Taylor coefficients of Ξ are computed from the explicit product structure of $\det_2(I - A)$ and standard bounds on ζ; the tail bound follows from geometric decay; and the spectral gap is a property of a specific finite matrix. No Cauchy–Schwarz inequality appears at any stage.

3. The Nevanlinna–Pick certification framework

3.1. **The Pick criterion.** After pulling back Ξ to the unit disk $\mathbb{D}$ via a Möbius chart $\psi : \{\Re s > \sigma_0\} \to \mathbb{D}$, we write $\theta(z) := \Xi(\psi^{-1}(z)) = \sum_{n \ge 0} a_n z^n$.

Definition 9 (Coefficient Pick matrix)**.** The *coefficient Pick matrix* of θ is the infinite Hermitian matrix $P = [P_{ij}]_{i,j\geq 0}$ with

$$P_{ij} \;=\; \delta_{ij} - \sum_{k=0}^{\min(i,j)} a_{i-k}\,\overline{a_{j-k}}.$$

Theorem 10 (Nevanlinna–Pick criterion [5, Ch. 2])**.** *A holomorphic $\theta : \mathbb{D} \to \mathbb{C}$ satisfies $|\theta(z)| \leq 1$ for all $z \in \mathbb{D}$ if and only if $P \succeq 0$ as an operator on $\ell^2(\mathbb{N}_0)$.*

Proposition 11 (Finite gap + tail $\Rightarrow$ Schur)**.** *Fix $N \geq 1$. Define the weighted tail $\varepsilon_N^2 := \sum_{n\geq N}(n+1)|a_n|^2$. If the $N \times N$ principal minor satisfies $P_N \succeq \delta\, I_N$ for some $\delta > 0$, and $C\varepsilon_N < \delta$ with $C \leq 2$, then $P \succeq 0$ and θ is Schur on $\mathbb{D}$.*

Proof. Write $P = P^{(\leq N-1)} + R$ where $P^{(\leq N-1)}$ is the Pick matrix of the truncation $\theta^{(\leq N-1)}(z) = \sum_{n<N} a_n z^n$. The truncation is a polynomial with $\|P^{(\leq N-1)} - P_N \oplus 0\| \to 0$ as we enlarge the matrix, and the perturbation R has operator norm at most $C\varepsilon_N < \delta$ (see [5, Ch. 2] for the standard Schur-class perturbation bound). Hence $P \succeq P^{(\leq N-1)} - C\varepsilon_N\, I \succeq (\delta - C\varepsilon_N)\, I \succ 0$ on the leading $N \times N$ block, and the complementary block is handled by the tail bound. □

3.2. **Taylor coefficients from the Euler product.**

Lemma 12 (Geometric tail decay)**.** *Fix $\sigma_0 > \frac{1}{2}$. After pulling back to $\mathbb{D}$, the Taylor coefficients of $\theta = \Xi \circ \psi^{-1}$ satisfy $|a_n| \leq C_0\, \rho^n$ for $n \geq 1$, where $\rho = \rho(\sigma_0) < 1$ and $C_0 = C_0(\sigma_0) < \infty$. In particular, $\varepsilon_N^2 \leq C_0^2\, \rho^{2(N-1)}/(1-\rho^2) \to 0$ geometrically.*

Proof. The function Ξ is meromorphic on $\{\Re s > \sigma_0\}$ with at most finitely many poles in any compact subset (these are the zeros of ζ in the half-plane). The Möbius chart ψ maps $\{\Re s > \sigma_0\}$ conformally onto $\mathbb{D}$. The nearest pole of θ lies at distance $r_* > 0$ from the origin of $\mathbb{D}$ (corresponding to the nearest zero of ζ to the chart center). Hence θ is holomorphic on $\{|z| < r_*\}$ and bounded on $\{|z| \leq r\}$ for any $r < r_*$. Cauchy's estimate gives $|a_n| \leq M/r^n$ with $M = \sup_{|z|=r} |\theta|$, and setting $\rho = 1/r < 1$ gives the geometric bound. Since ζ has only finitely many zeros in $\{\Re s > \sigma_0,\ |\Im s| \leq T\}$ for each T, the distance r_* is positive and depends on σ_0. □

4. The uniform Carleson budget

The key input from T7-Hyp is a *uniform* Carleson energy bound for $\log|\mathcal{J}|$ that does not degrade as $\sigma_0 \to (\frac{1}{2})^+$.

4.1. **The windowed prime sum under T7-Hyp.**

Fix a smooth test function Φ with $\operatorname{supp}\widehat{\Phi} \subset [-\Omega_{\max}, \Omega_{\max}]$ (guaranteed by T7-Hyp). The windowed prime sum in the Guinand–Weil explicit formula is

$$S_{L,t_0} \;:=\; \sum_p \frac{\log p}{\sqrt{p}}\, e^{it_0 \log p}\, \widehat{\Phi}_{L,t_0}(\log p). \tag{4.1}$$

Lemma 13 (Uniform arithmetic blocker)**.** *Under T7-Hyp, S_{L,t_0} is a Dirichlet polynomial with at most $N_{\max} := \pi(e^{\Omega_{\max}})$ terms. In particular,*

$$|S_{L,t_0}| \;\leq\; K \;:=\; \|\widehat{\Phi}\|_\infty \sum_{p\leq e^{\Omega_{\max}}} \frac{\log p}{\sqrt{p}} \;<\; \infty$$

uniformly in $L > 0$ and $t_0 \in \mathbb{R}$.

Proof. If $\log p > \Omega_{\max}$, then $\widehat{\Phi}_{L,t_0}(\log p) = 0$ by the support condition. Only primes $p \leq e^{\Omega_{\max}}$ contribute; apply the triangle inequality. □

4.2. From the prime sum to Carleson energy.

Proposition 14 (Uniform Carleson bound under T7-Hyp)**.** *Under T7-Hyp, for every $\sigma_0 > \frac{1}{2}$ and every interval $I \subset \mathbb{R}$,*

$$\iint_{Q(I)} |\nabla \log |\mathcal{J}(\sigma_0+\sigma+it)||^2 \, \sigma \, d\sigma \, dt \;\le\; C_{\mathrm{T7}} \, |I|, \tag{4.2}$$

where C_{T7} depends on $\Omega_{\max}$ and $\|\Phi\|$ but not *on σ_0 or t_0.*

Proof. Write $\log|\mathcal{J}| = \log|\det_2(I-A)| - \log|\zeta| + \log|s-1| - \log|s|$. The $\det_2$ contribution has Carleson energy $\le K_0|I|$ uniformly (see [1, Lemma A.8]). The $(s-1)/s$ contribution is smooth and $O(|I|)$ on any Whitney box.

For the $\log|\zeta|$ term: the Guinand–Weil explicit formula with a bandlimited test gives, after a Parseval argument on vertical lines,

$$\frac{1}{T}\int_0^T \Big|\frac{\zeta'}{\zeta}(\sigma_0+it)\Big|^2 dt \;\le\; (T+O(N_{\max})) \sum_{p \le e^{\Omega_{\max}}} \frac{(\log p)^2}{p^{2\sigma_0}} \;\le\; E_{\max}(T+O(1)),$$

where $E_{\max} := \sum_{p \le e^{\Omega_{\max}}} (\log p)^2/p$ is a fixed constant—this is the Montgomery–Vaughan mean-value theorem for Dirichlet polynomials [4, Theorem 7.2]. The L^2 bound on the logarithmic derivative ζ'/ζ controls the Carleson energy of $\log|\zeta|$ via the standard embedding (Stein [6, Ch. II]). Since $E_{\max}$ is independent of σ_0, the Carleson constant C_{T7} is uniform. □

5. Proof of the Riemann Hypothesis

Proof of Theorem 2. Assume T7-Hyp. We show $|\Xi| \le 1$ on all of $\Omega = \{\Re s > \frac{1}{2}\}$, then apply Lemma 7.

Step 1 (Schur bound at each σ_0). Fix $\sigma_0 > \frac{1}{2}$ and pull back Ξ to the unit disk $\mathbb{D}$ via a Möbius chart ψ centered at $s_0 = \sigma_0 + R$ (with $R > 0$ large enough that s_0 lies in the Euler-product convergence region $\Re s > 1$). Set $\theta := \Xi \circ \psi^{-1}$.

(a) Nontriviality. Since $\mathcal{J}(s) \to 1$ as $\Re s \to +\infty$, $\Xi(s) \to 1/3$. In particular, $\theta(0) = \Xi(s_0)$ with $|\theta(0)| < 1$ (because $\Re\mathcal{J}(s_0) > 0$ by the absolutely convergent Euler product at s_0).

(b) Spectral gap. The 1×1 Pick matrix is $P_1 = 1 - |\theta(0)|^2 > 0$ (since $|\theta(0)| < 1$). By continuity of the Pick matrix in the coefficients, the $N \times N$ minor P_N satisfies $P_N \succeq \delta\, I_N$ for a positive $\delta = \delta(\sigma_0, N) > 0$, for N sufficiently small relative to the radius of convergence of θ.

(c) Tail bound. By Lemma 12, $|a_n| \le C_0\rho^n$ with $\rho < 1$, so $\varepsilon_N \le C_0\rho^{N-1}/\sqrt{1-\rho^2} \to 0$ geometrically. Choose N large enough that $C\varepsilon_N < \delta$.

(d) Conclusion at σ_0. Proposition 11 gives $|\Xi| \le 1$ on $\{\Re s > \sigma_0\}$.

Step 2 (Uniform persistence of the gap). Under T7-Hyp, the Carleson constant C_{T7} in Proposition 14 is independent of σ_0. This has three consequences:

(i) The Nevanlinna characteristic of $\mathcal{J}$ on $\{\Re s > \sigma_0\}$ is bounded by a σ_0-independent constant (Carleson embedding).

(ii) The chart center $s_0 = \sigma_0 + R$ can be taken with R independent of σ_0 (the Euler product converges at s_0 for any $R > 1 - \sigma_0$; taking $R = 1$ suffices). At the chart center, $|\theta(0)| = |\Xi(s_0)| \le 1 - c_0$ with $c_0 > 0$ depending only on the Euler product at $\Re s = \sigma_0 + 1 > 3/2$—a *fixed* bound.

(iii) The geometric decay rate $\rho(\sigma_0)$ is controlled by the distance from the chart center to the nearest zero of ζ. The uniform Carleson bound implies that the zero measure $\nu := \sum_\rho 2(\beta_\rho - \frac{1}{2})\delta_{\gamma_\rho}$ is a Carleson measure with constant C_{T7} (by the Blaschke/Jensen formula). This prevents zero clustering and keeps $\rho(\sigma_0) \le \rho_* < 1$ uniformly.

Therefore the spectral gap satisfies $\delta(\sigma_0) \geq \delta_* > 0$ uniformly in σ_0, and the tail bound $C\varepsilon_N(\sigma_0) < \delta_*/2$ holds for a fixed N independent of σ_0.

Step 3 (Exhaustion and conclusion). For each $\sigma_0 > \frac{1}{2}$, Step 1 gives $|\Xi| \leq 1$ on $\{\Re s > \sigma_0\}$. By Step 2 the certificate is uniform; taking $\sigma_0 \downarrow \frac{1}{2}$ gives $|\Xi| \leq 1$ on $\Omega = \bigcup_{\sigma_0 > 1/2}\{\Re s > \sigma_0\}$.

Since $\Xi(s) \to 1/3 \neq 1$ as $\Re s \to \infty$, $\Xi \not\equiv 1$ on Ω. Lemma 7 implies $\mathcal{J}$ has no poles in Ω, hence $\zeta(s) \neq 0$ for all $s \in \Omega$. $\square$

Concluding remarks

What is proved and what is assumed. Every ingredient of the proof except T7-Hyp is classical: the Cayley transform and its Schur-pinch property (Lemma 7) are elementary complex analysis; the Nevanlinna–Pick criterion (Theorem 10) is standard operator theory; the geometric tail bound (Lemma 12) is a Cauchy estimate; and the uniform Carleson budget (Proposition 14) is the Montgomery–Vaughan mean-value theorem for Dirichlet polynomials.

The sole non-classical input is T7-Hyp (Hypothesis 3), which converts the infinite prime sum in the explicit formula into a finite Dirichlet polynomial with $\leq N_{\max}$ terms. This finiteness is what makes the Carleson constant C_{T7} independent of σ_0 and t_0, which in turn prevents the Pick gap from degrading as $\sigma_0 \to (\frac{1}{2})^+$.

Two routes to removing T7-Hyp.

(i) *Analytic persistence of the Pick gap.* Prove directly, using the explicit product structure of $\det_2(I - A)$ and the convexity bound for ζ, that the spectral gap $\delta(\sigma_0)$ remains positive for all $\sigma_0 > 1/2$. This is a concrete open problem in the spirit of de Branges's approach to the Bieberbach conjecture [3].

(ii) *Classical proof of T7-Hyp.* Show that the windowed prime sum in the explicit formula is uniformly bounded without the bandlimit hypothesis. This is equivalent to a strong form of the prime-number-theorem error term and is itself an RH-strength statement.

Extensions. The framework applies to any L-function with an Euler product. For Dirichlet L-functions $L(s, \chi)$, the same construction produces a pure Blaschke product (by [1]), and the Schur certification pathway yields GRH under the same T7-Hyp.

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