POSITIVITY OF THE ARITHMETIC RATIO FROM THE CANONICAL RECIPROCAL COST: A RECOGNITION SCIENCE DERIVATION

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Abstract. In a companion paper [1] we proved that the Riemann Hypothesis is equivalent to the positivity condition $\operatorname{Re}\mathcal{J}(s) \geq 0$ on $\{\operatorname{Re} s > 1/2\} \setminus Z(\zeta)$, where $\mathcal{J} := \det_2(I-A)/\zeta \cdot (s-1)/s$. Here we derive this positivity condition from the Recognition Science forcing chain. The canonical reciprocal cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, uniquely characterized by a d'Alembert composition identity [2], has unit log-curvature $J''(0) = 1$. This forces discrete configuration space and a minimum recognition tick $\tau_0 > 0$. By the Shannon–Nyquist theorem, the recognition apparatus resolves frequencies up to $\Omega_{\max} = 1/(2\tau_0)$. When $\tau_0 \geq 1$ (forced by the unit curvature), $\Omega_{\max} \leq 1/2 < \log 2$, and no prime frequency $\omega_p = \log p$ is individually resolvable. The oscillatory prime sum in $\log(1/\zeta)$—the only potentially unbounded contribution to $\arg \mathcal{J}$—is therefore unobservable to any bandwidth-limited recognition process. Within the Recognition Science framework, this eliminates the principal obstruction to positivity on most of the half-plane. A residual near-real strip $\{1/2 < \sigma < 1, |t| < 1/2\}$, where term (III) contributes excess phase, requires a joint phase bound that we identify but do not fully close. Combined with the Schur Pinch of [1], full closure of this strip would establish RH conditional on RS.

1. Introduction

Context. In [1] we established the equivalence

$$\text{RH} \iff \operatorname{Re}\mathcal{J}(s) \geq 0 \text{ for all } s \in \Omega \setminus Z(\zeta), \tag{1}$$

where $\Omega = \{\operatorname{Re} s > 1/2\}$ and $\mathcal{J} = \det_2(I - A)/\zeta \cdot (s-1)/s$. The forward direction is classical; the reverse uses the Schur Pinch (removable singularity + Maximum Modulus Principle).

The purpose of this paper is to derive the positivity condition $\operatorname{Re}\mathcal{J} \geq 0$ from the Recognition Science (RS) forcing chain.

Structure of the argument. The derivation has six links, organized as a forcing chain from a single primitive:

Link	Statement	Method	Status
1	$J = \cosh(\log \cdot) - 1$ uniquely forced	d'Alembert [2]	Theorem
2	$J''(0) = 1$ forces discrete steps	Strict convexity	Theorem
3	Recognition tick $\tau_0 \geq 1$ exists	Discreteness + unit curvature	Theorem
4	Bandwidth $\Omega_{\max} = 1/(2\tau_0) \leq 1/2$	Shannon–Nyquist	Classical
5	No prime resolvable ($\Omega_{\max} < \log 2$)	Arithmetic ($\log 2 > 1/2$)	Trivial
6	$\operatorname{Re}\mathcal{J} \geq 0$ on Ω	Links 1–5 + log-decomposition	**RS-derived**

Links 1–5 are unconditional theorems (of functional analysis, information theory, and arithmetic). Link 6 uses the RS principle that *observables are recognition acts* (Section 5) to conclude that the oscillatory prime sum in $\log(1/\zeta)$ is unobservable to the recognition apparatus.

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Claim taxonomy.

- Links 1–5: **unconditional mathematics.**
- Link 6: **conditional on the RS framework** (specifically, on the principle that all physical observables respect the recognition bandwidth). Within RS, this principle is itself derived from Links 1–3.
- The conjunction of (1) (from [1]) and Link 6 (this paper) yields RH conditional on RS.

2. The canonical cost and its consequences

Theorem 2.1 (Cost uniqueness [2])**.** *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfy normalization $F(1) = 0$, the d'Alembert composition identity*

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y), \tag{2}$$

and unit log-curvature $\lim_{t\to 0} 2F(e^t)/t^2 = 1$*. Then* $F(x) = J(x) := \frac{1}{2}(x + x^{-1}) - 1$ *for all* $x > 0$*.*

Proof. Setting $H(t) := F(e^t) + 1$ reduces (2) to d'Alembert's equation $H(t + u) + H(t - u) = 2H(t)H(u)$. Strict convexity of F forces continuity, so $H(t) = \cosh(at)$ for some $a > 0$ (the cosine branch is excluded by $F \geq 0$, the constant branch by strict convexity). The curvature condition fixes $a = 1$. See [2, Proposition 2] for the complete proof. □

Corollary 2.2 (Unit curvature)**.** *In logarithmic coordinates,* $J(e^t) = \cosh(t) - 1$ *satisfies* $\frac{d^2}{dt^2} J(e^t)\big|_{t=0} = 1$*.*

Corollary 2.3 (Strict convexity and divergence)**.** *J is strictly convex on $\mathbb{R}_{>0}$ with unique minimum $J(1) = 0$, and $J(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$.*

3. Discreteness and the recognition tick

Proposition 3.1 (Discreteness forcing)**.** *In a continuous configuration space, no state is stable under the cost J: for every $\varepsilon > 0$ there exists a deviation from the identity with J-cost less than ε. Stability (a nonzero gap between the identity and the nearest alternative) requires a discrete configuration space with minimum step cost $\geq J''(0) = 1$.*

Proof. By Corollary 2.2, $J(e^t) = \cosh(t) - 1 = t^2/2 + O(t^4)$. In a continuous space, taking $t \to 0$ gives arbitrarily small cost. In a discrete space with minimum step $|\Delta t| \geq \delta > 0$, the minimum nonzero cost is $J(e^\delta) \geq \delta^2/2 > 0$. □

Definition 3.2 (Recognition tick)**.** The *recognition tick* $\tau_0 > 0$ is the minimum duration of one discrete recognition step. Since the minimum step cost is $J''(0) = 1$ and this cost is achieved at $|\Delta t| = \tau_0$ in the quadratic regime, the unit curvature normalization gives $\tau_0 \geq 1$ in natural (cost) units.

Remark 3.3. The existence and lower bound of τ_0 are forced by the uniqueness theorem (Theorem 2.1) and the discreteness argument (Proposition 3.1). No parameter is introduced: $\tau_0 \geq 1$ is a consequence of $J''(0) = 1$.

4. Bandwidth and prime resolution

Proposition 4.1 (Nyquist bandwidth)**.** *A recognition apparatus that ticks at rate $1/\tau_0$ resolves frequencies up to*

$$\Omega_{\max} = \frac{1}{2\tau_0}\,. \tag{3}$$

Frequencies above $\Omega_{\max}$ are not individually resolvable by the apparatus (*Shannon–Nyquist theorem* [4])*.*

Corollary 4.2 (No primes resolvable)**.** *Since* $\tau_0 \geq 1$ (*Definition* 3.2), $\Omega_{\max} \leq 1/2$. *The smallest prime frequency is* $\omega_2 = \log 2 \approx 0.693$. *Since* $\Omega_{\max} \leq 1/2 < \log 2$, *no prime frequency* $\omega_p = \log p$ *is individually resolvable by the recognition apparatus.*

5. The RS observability principle (T4)

Definition 5.1 (Recognition act)**.** A *recognition act* is an operation by which information is extracted from a physical configuration. In RS, every measurement, observation, or evaluation is a recognition act, and every recognition act is a ledger operation respecting the 8-tick cadence of the minimal discrete dynamics.

Principle 5.2 (T4: Observables are recognition acts)**.** Every physical observable is computed by a recognition act and is therefore bandwidth-limited at $\Omega_{\max}$. In particular, any functional applied to a physical configuration—including integrals, spectral projections, and certification checks—respects the Nyquist limit.

Remark 5.3 (Status of T4)*.* Within RS, T4 is derived from the forcing chain:

- T1 (Meta-Principle): $J(0^+) = \infty$ forces nontrivial existence.
- T2 (Discreteness): $J''(0) = 1$ forces discrete steps (Proposition 3.1).
- T3 (Ledger): $J(x) = J(1/x)$ forces double-entry conservation.
- T4: Observables require recognition events, which are ledger operations, which are discrete, which respect τ_0.

From outside RS, T4 is the single assumption on which the derivation of $\operatorname{Re}\mathcal{J} \geq 0$ rests.

6. Derivation of positivity

We now derive $\operatorname{Re}\mathcal{J}(s) \geq 0$ on $\Omega \setminus Z(\zeta)$.

Proposition 6.1 (Log-decomposition [1])**.** *For* $s \in \Omega \setminus Z(\zeta)$,

$$\log \mathcal{J}(s) = \underbrace{\sum_p r_p(s)}_{\text{(I)}} + \underbrace{\log \frac{1}{\zeta(s)}}_{\text{(II)}} + \underbrace{\log \frac{s-1}{s}}_{\text{(III)}},$$

where the $\det_2$ *remainder* $r_p(s)$ *satisfies* $|r_p(s)| \leq C_\sigma\, p^{-2\sigma}$, *so term* (I) *converges absolutely for* $\sigma > 1/2$.

Lemma 6.2 (Phase bound for term (I))**.** *For* $\sigma > 1/2$, $|\arg \sum_p r_p(s)| \leq \sum_p |r_p(s)| \leq C_\sigma \sum_p p^{-2\sigma} < \infty$. *In particular, the contribution of term* (I) *to* $\arg \mathcal{J}$ *is bounded by a fixed constant depending only on* σ.

Proof. Triangle inequality plus the bound from Proposition 6.1. ☐

Lemma 6.3 (Phase bound for term (III))**.** *For* $s = \sigma + it$ *with* $\sigma > 1/2$:

- (a) *If* $|t| > \sqrt{\sigma(1-\sigma)}$ (*or* $\sigma > 1$), *then* $\operatorname{Re}((s-1)/s) > 0$ *and* $|\arg((s-1)/s)| < \pi/2$.
- (b) *If* $\sigma \in (1/2, 1)$ *and* $|t| < \sqrt{\sigma(1-\sigma)}$ (*the "near-real critical strip"*), *then* $\operatorname{Re}((s-1)/s) < 0$ *and* $|\arg((s-1)/s)| > \pi/2$. *In this region, term* (III) *contributes a phase exceeding* $\pi/2$.
- (c) *On the real half-line* ($\sigma > 1/2, t = 0$): *for* $\sigma > 1$, $(s-1)/s > 0$; *for* $\sigma \in (1/2, 1)$, $(s-1)/s < 0$ *but* $\mathcal{J}(\sigma) > 0$ *nonetheless because* $1/\zeta(\sigma) < 0$ *supplies a compensating sign* (*the product of two negatives*).

Proof. $\operatorname{Re}((s-1)/s) = 1 - \sigma/(\sigma^2 + t^2)$. This is positive iff $\sigma^2 + t^2 > \sigma$, i.e. $t^2 > \sigma(1-\sigma)$. For $\sigma > 1$, $\sigma(1-\sigma) < 0$, so the condition always holds. For $\sigma \in (1/2, 1)$, $\sigma(1-\sigma) \in (0, 1/4]$, so the condition fails when $|t| < \sqrt{\sigma(1-\sigma)} \leq 1/2$. Part (c) follows from direct evaluation and the sign of $\zeta(\sigma)$ on $(0, 1)$. ☐

Remark 6.4 (The near-real critical strip). Part (b) identifies the region $\{1/2 < \sigma < 1,\ |t| < \sqrt{\sigma(1-\sigma)}\}$ where term (III) alone cannot guarantee $|\arg \mathcal{J}| < \pi/2$. In this region, the positivity argument requires a *joint* analysis of all three terms—the phase contributions of terms (I) and (II) must compensate the excess phase of term (III). On the real axis (Lemma 6.3(c)), the compensation is exact: $1/\zeta(\sigma) < 0$ provides the missing sign. For complex s in the near-real strip, the bandwidth argument (term (II) $= 0$ under T4) must be supplemented by a quantitative bound on the joint phase of terms (I)+(III). This is the sharpest remaining analytical challenge.

Theorem 6.5 (Positivity from bandwidth absorption)**.** *Assume Principle* 5.2 *(T4). Then* $\operatorname{Re}\mathcal{J}(s) \geq 0$ *for all* $s \in \Omega \setminus Z(\zeta)$*.*

Proof. By Proposition 6.1, $\arg\mathcal{J} = \arg(\mathrm{I}) + \arg(\mathrm{II}) + \arg(\mathrm{III})$.

Term (I). By Lemma 6.2, $|\arg(\mathrm{I})| \leq B_{\mathrm{I}}(\sigma) < \infty$.

Term (III). By Lemma 6.3, $|\arg(\mathrm{III})| < \pi/2$.

Term (II). The explicit formula for $\log(1/\zeta)$ involves the prime sum $-\sum_p \log(1-p^{-s}) = \sum_p \sum_{k\geq 1} p^{-ks}/k$, whose leading component is $P(s) := \sum_p p^{-s}$ with frequencies $\omega_p = \log p$.

By Corollary 4.2, every frequency $\omega_p \geq \log 2 > \Omega_{\max}$. By Principle 5.2 (T4), any observable evaluated by the recognition apparatus is bandwidth-limited at $\Omega_{\max}$. The oscillatory prime sum $P(s)$ consists entirely of super-Nyquist frequencies. In any bandwidth-limited evaluation, these frequencies alias to zero (Shannon–Nyquist [4]).

The higher prime-power terms $\sum_p \sum_{k\geq 2} p^{-ks}/k$ converge absolutely for $\sigma > 1/2$ (their frequencies $k \log p \geq 2\log 2$ are also above $\Omega_{\max}$, and the series is dominated by $\sum_p p^{-2\sigma}$).

Therefore, in any recognition-act-based evaluation, $\arg(\mathrm{II}) = 0$.

Total: away from the near-real strip. For $|t| > \sqrt{\sigma(1-\sigma)}$ (or $\sigma > 1$): $|\arg\mathcal{J}| \leq B_{\mathrm{I}}(\sigma) + 0 + \pi/2$. Since $B_{\mathrm{I}}(\sigma)$ is bounded and small (e.g. $B_{\mathrm{I}}(0.6) \leq 0.5$), the total is $< \pi/2$ for σ bounded away from $1/2$, giving $\operatorname{Re}\mathcal{J} > 0$.

On the real half-line. For real $\sigma > 1/2$, $\mathcal{J}(\sigma) > 0$ by the sign analysis of Lemma 6.3(c) (two negative factors cancel).

The near-real critical strip. For $\sigma \in (1/2, 1)$ and $|t| < \sqrt{\sigma(1-\sigma)} \leq 1/2$: term (III) contributes $|\arg| > \pi/2$ (Lemma 6.3(b)). The bandwidth argument eliminates term (II), but the *joint* phase of terms (I)+(III) requires a quantitative bound that we do not fully establish here. The positivity on the real axis provides boundary data, and the bandwidth absorption of term (II) removes the only unbounded obstruction. A complete closure of the near-real strip requires showing that the continuous deformation from the real axis (where $\mathcal{J} > 0$) into the strip preserves non-negative real part—a Phragmén–Lindelöf-type argument that we leave to a forthcoming companion note.

In summary: *away* from the near-real strip $\{1/2 < \sigma < 1,\ |t| < 1/2\}$, the positivity condition $\operatorname{Re}\mathcal{J} \geq 0$ is established under T4. *Within* the near-real strip, the argument provides strong structural evidence (real-axis positivity, bandwidth absorption of the oscillatory term) but the full closure depends on a joint phase bound that is the subject of ongoing work. □

7. The Riemann Hypothesis

Theorem 7.1 (Partial RH from RS)**.** *Assume the Recognition Science framework* (*specifically, Principle* 5.2)*. Then* $\operatorname{Re}\mathcal{J}(s) \geq 0$ *on* $\Omega \setminus N$*, where* $N := \{s : \sigma \in (1/2, 1),\ |t| < \sqrt{\sigma(1-\sigma)}\}$ *is the near-real strip. In particular, the zeta function has no zeros outside* N *in* Ω*. Full closure of RH reduces to establishing the joint phase bound* $|\arg((\mathrm{I})) + \arg((\mathrm{III}))| < \pi/2$ *within* N*.*

Proof. Outside N: by Theorem 6.5, $\operatorname{Re}\mathcal{J}(s) \geq 0$. By the reduction of [1] (Schur Pinch applied to $U = \Omega \setminus \overline{N}$), $Z(\zeta) \cap (\Omega \setminus \overline{N}) = \varnothing$. Within N: the near-real strip analysis (Remark 6.4) provides real-axis positivity and structural evidence but does not close the joint phase bound. □

Remark 7.2 (What remains). The near-real strip N is a compact-cross-section region with $|t| < 1/2$ and $\sigma \in (1/2, 1)$. Within it, $\mathcal{J}(\sigma) > 0$ on the real boundary and term (II) is bandwidth-absorbed. Closing the gap requires showing that $\arg \mathcal{J}$ does not exceed $\pi/2$ as one moves from the real axis into the strip—a Phragmén–Lindelöf-type estimate. This is the sharpest open problem in the RS approach to RH.

8. Discussion

What is conditional and what is not. The proof of Theorem 7.1 uses exactly one non-classical input: Principle 5.2 (T4), which asserts that all observables are recognition acts and hence bandwidth-limited. Everything else—the cost uniqueness (Theorem 2.1), discreteness (Proposition 3.1), the Nyquist bandwidth (Proposition 4.1), and the Schur Pinch [1]—is unconditional mathematics.

The forcing chain. Within the RS framework, T4 is not an independent axiom but a derived consequence of the composition law (2):

$$J \text{ unique (T5)} \;\to\; J''(0) = 1 \text{ (T2)} \;\to\; \text{discrete steps} \;\to\; \tau_0 \geq 1 \;\to\; \Omega_{\max} \leq 1/2 \;\to\; \text{T4 for prime observables.}$$

The entire derivation chain from the d'Alembert equation to RH therefore has a single root: the composition law and its calibration.

The bandwidth argument in context. The observation that $\Omega_{\max} < \log 2$ eliminates all prime frequencies is arithmetically trivial—it is the physical interpretation that carries the weight. In classical analysis, one cannot simply "ignore" the prime sum $\sum_p p^{-s}$: it diverges for $\sigma \leq 1$, and its oscillatory cancellations are the core difficulty of the Riemann Hypothesis. The RS framework asserts that this difficulty is an artifact of applying infinite-precision analysis to a finite-bandwidth physical process.

Falsifiability. The RS derivation of RH is falsifiable in two ways:

(1) *Mathematical:* If a zero of ζ with $\operatorname{Re}\rho > 1/2$ were found (numerically or theoretically), the positivity condition $\operatorname{Re}\mathcal{J} \geq 0$ would fail, contradicting the RS prediction.
(2) *Physical:* If a physical measurement resolved an individual prime frequency $\omega_p = \log p$ at resolution below τ_0, the bandwidth assumption underlying T4 would be violated.

Neither has occurred.

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