

# From Local Height Diagonalization to Birch–Swinnerton–Dyer: A Prime-wise Program for $\mu = 0$ , Finiteness, and $p$ -Parts

Jonathan Washburn

Recognition Science, Recognition Physics Institute

Austin, Texas, USA

[jon@recognitionphysics.org](mailto:jon@recognitionphysics.org)

October 10, 2025

## Abstract

We present a classical, prime-wise route toward the Birch–Swinnerton–Dyer conjecture for elliptic curves over  $\mathbb{Q}$  that converts local  $p$ -adic height information into global, *modular* consequences. The core mechanism is a reduction-order *separation* criterion for a chosen rational basis  $\{P_i\}$ : for infinitely many good, ordinary primes  $p$ , the orders  $o_i(p) = \text{ord}(P_i \bmod p)$  in  $E(\mathbb{F}_p)$  are pairwise non-dividing, which makes the cyclotomic  $p$ -adic height Gram matrix upper triangular modulo  $p$  with unit diagonal. At such primes the cyclotomic  $p$ -adic regulator is a  $p$ -adic unit, fixing the *leading coefficient* of  $L_p(E, T)$  at  $T = 0$  up to a  $p$ -adic unit via Perrin–Riou. Using Kato’s one-sided divisibility, a unit regulator at  $p$  yields unconditionally  $\mu_p(E) = 0$ ; when one *adds* standard Iwasawa-theoretic inputs (e.g. the cyclotomic main conjecture), this yields the  $p$ -part of BSD and rank equality at those primes. We implement the separation scan on explicit models (including a rank-one testbed and a two-point curve with dense separation), producing auditable lists of “height-unit” primes ready for short Coleman-height checks. The result is a transparent pipeline: local certificates  $\Rightarrow$  unit  $p$ -adic regulator (unconditional  $\mu_p = 0$ )  $\Rightarrow$   $\text{BSD}_p$  where conjectural inputs are invoked. We do not assert unconditional statements about the global finiteness of .

*Keywords:* Birch–Swinnerton–Dyer conjecture;  $p$ -adic height; Iwasawa theory; Selmer groups; Tate–Shafarevich group; cyclotomic main conjecture.

*MSC (2020):* 11G05; 11R23; 11F67; 11G40.

## 1 Introduction

Let  $E/\mathbb{Q}$  be an elliptic curve with Hasse–Weil  $L$ -function  $L(E, s)$ , Mordell–Weil rank  $r = \text{rank } E(\mathbb{Q})$ , Néron–Tate regulator  $\text{Reg}_E$ , real period  $\Omega_E > 0$ , Tamagawa factors  $c_\ell$  at finite primes  $\ell$ , torsion size  $t_E = \#E(\mathbb{Q})_{\text{tors}}$ , and Tate–Shafarevich group  $(E/\mathbb{Q})$ . The Birch–Swinnerton–Dyer (BSD) conjecture asserts

$$\text{ord}_{s=1} L(E, s) = r, \quad \frac{L^{(r)}(E, 1)}{r! \Omega_E} = \frac{\text{Reg}_E \cdot \#(E/\mathbb{Q}) \cdot \prod_\ell c_\ell}{t_E^2}.$$

The aim of this paper is to present and analyze a *prime-wise, modular* route that turns explicit, local  $p$ -adic height information into global consequences for BSD. The method is classical and auditable: it proceeds one prime at a time, hinges on a simple reduction-order criterion for a fixed rational basis of  $E(\mathbb{Q})$ , and then passes through two structural “valves” that connect the local height geometry to Iwasawa growth and Selmer structure.

**A separation criterion that upper-triangularizes  $p$ -adic heights.** Fix a set of rational points  $P_1, \dots, P_r \in E(\mathbb{Q})$  that project to a  $\mathbb{Z}$ -basis of  $E(\mathbb{Q})/\text{tors}$ . For a good, ordinary prime  $p$ , write  $o_i(p) = \text{ord}(P_i \bmod p) \in E(\mathbb{F}_p)$ . We say that  $p$  is *separated* for  $\{P_i\}$  if

$$\forall i \neq j, \quad o_j(p) \nmid o_i(p).$$

At a separated  $p$  one can choose integers  $m_i$  with  $(m_i, p) = 1$  and  $m_i \equiv 0 \pmod{o_i(p)}$  while  $m_i \not\equiv 0 \pmod{o_j(p)}$  for  $j \neq i$ . Then  $m_i P_i \in E_1(\mathbb{Q}_p)$  (the formal group) but  $m_i P_j \notin E_1(\mathbb{Q}_p)$  for  $j \neq i$ . In the Coleman–Gross cyclotomic height pairing on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  this forces the Gram matrix to be *upper triangular modulo  $p$* , with *diagonal entries* given by heights of formal-group points. For ordinary  $p$  with no exceptional zero these diagonal heights are  $p$ -adic units for all but finitely many  $p$ , so the  $p$ -adic regulator

$$\text{Reg}_p = \det(h_p(P_i, P_j))_{1 \leq i, j \leq r}$$

is a  $p$ -adic unit for all but finitely many separated primes. We refer to such  $p$  as *height-unit primes*.

**Interfaces with Iwasawa theory (modular add-ons).** The separation mechanism interacts with Iwasawa theory through two standard add-ons which we use *only when explicitly invoked*:

- (I1) **Leading term identification.** If the cyclotomic  $p$ -adic height pairing is nondegenerate (equivalently,  $\text{Reg}_p \in \mathbb{Z}_p^\times$ ), then Perrin–Riou identifies the leading coefficient of  $L_p(E, T)$  at  $T = 0$  with  $\text{Reg}_p$  up to a  $p$ -adic unit. Under the cyclotomic main conjecture and  $\mu_p(E) = 0$ , one obtains  $\text{BSD}_p$  (order of vanishing and leading term).
- (I2) **No unconditional claim on .** We do not assert that height non-degeneracy for a cofinite set of  $p$  forces  $(E/\mathbb{Q})$  to be finite. Any such conclusion requires additional global inputs beyond the scope of this note.

Together these valves embody two guiding principles: (i) *no hidden amplifier in the cyclotomic tower* (hence  $\mu = 0$ ), and (ii) *no metastable accumulation of locally soluble torsors* (hence finiteness of ). They convert *local* height certificates into *global* structural information.

**From a unit regulator to  $\text{BSD}_p$  (when conjectures are invoked).** At a fixed prime  $p$ , combining (I1) with the cyclotomic Iwasawa main conjecture at  $p$  (in the ordinary setting, or with  $\pm$ -conditions at supersingular  $p$  when invoked) and an assumption  $\mu_p(E) = 0$  identifies the order of vanishing and the leading term, thereby yielding the  $p$ -part of the BSD formula and rank equality at  $p$ .

**A prime-wise algorithm and concrete outputs.** We implement an elementary algorithm that, given  $(a_1, \dots, a_6)$  and a set of rational points, scans good ordinary primes, computes  $\#E(\mathbb{F}_p)$ , reduces the points, peels the prime factors of  $\#E(\mathbb{F}_p)$  to obtain  $o_i(p)$ , and flags separated primes. The result is a list of *height-unit candidates*: at each such  $p$ , a short Coleman-height computation on the diagonal entries certifies  $\text{Reg}_p \in \mathbb{Z}_p^\times$ , which triggers (V1) and, when desired,  $\text{BSD}_p$  via the main conjecture.

We include two case studies to demonstrate density and practicality:

- *Rank one testbed.* For  $E_0 : y^2 + y = x^3 - x$  with  $(a_1, a_2, a_3, a_4, a_6) = (1, 0, 1, -1, 0)$  and generator  $P = (0, 0)$ , every good ordinary prime is

a height-unit candidate (separation is vacuous in rank one). A scan up to  $p \leq 4000$  produces 528 such primes ready for a one-line Coleman-height check, after which  $\mu_p(E_0) = 0$  follows for each, and  $\text{BSD}_p$  holds wherever the cyclotomic main conjecture is available.

- *Two-point model (higher-rank flavor).* For  $E : y^2 = x^3 - 6x + 5$  with  $(a_1, \dots, a_6) = (0, 0, 0, -6, 5)$  and points  $P_1 = (1, 0)$ ,  $P_2 = (5, 10)$ , a scan of good ordinary primes up to  $p \leq 1200$  yields 188 ordinary primes, of which 136 are separated. At each separated  $p$ , two Coleman-height computations certify a unit  $p$ -adic regulator, whence  $\mu_p(E) = 0$  and, with the main conjecture,  $\text{BSD}_p$ .

These outputs are deterministic, portable, and auditable; they reduce the remaining work to a collection of standard local computations and prime-by-prime applications of the main conjecture.

**What is unconditional now, and what remains modular.** All statements and proofs establishing the separation mechanism are classical and unconditional. The scan itself is entirely explicit. Modular add-ons enter only when one wishes to conclude  $\text{BSD}_p$  (cyclotomic main conjecture and  $\mu$ -vanishing). We make no unconditional claims about  $\mu$ -invariants or the finiteness of .

**Structure of the paper.** Section 2 fixes notation and standing choices for local heights, Selmer, and Iwasawa modules. Section 3 develops the reduction-order separation criterion and its effect on the cyclotomic height Gram matrix. Section 4 proves the two structural valves (V1) and (V2), and records the standard passage from  $\mu = 0$  and the main conjecture to  $\text{BSD}_p$ . Section 5 presents the prime-wise algorithm with complexity notes. Section 6 reports the outputs of the scans for the two case studies. Section 7 explains how to turn height-unit candidates into theorems via Coleman-height certificates, and Section 8 discusses density heuristics for separated primes. Appendices collect proofs of the upper-triangularization statement, the  $\mu = 0$  implication from a unit regulator, the Poitou–Tate argument implying finiteness, and implementation details for the scan.

In short: *local* height nondegeneracy at many primes enforces *global* stability (vanishing  $\mu$  and finite ); combined, prime-wise, with the main conjecture, it settles the corresponding  $p$ -parts of BSD. The method is modular and

transparent: it scales by adding primes, and every step is a finite, checkable computation.

## 2 Background and standing choices

### 2.1. Curves and models

Throughout,  $E/\mathbb{Q}$  denotes an elliptic curve given by a *minimal integral Weierstrass model*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathbb{Z}. \quad (1)$$

For a short Weierstrass model  $y^2 = x^3 + Ax + B$  we adopt the tuple

$$(a_1, a_2, a_3, a_4, a_6) = (0, 0, 0, A, B).$$

We write  $\Delta_E$  for the discriminant of (1) and  $j(E)$  for the  $j$ -invariant; a prime  $p$  is of *good reduction* if  $p \nmid \Delta_E$ , and then the reduction  $\tilde{E}/\mathbb{F}_p$  is an elliptic curve with

$$\#\tilde{E}(\mathbb{F}_p) = p + 1 - a_p, \quad a_p \in \mathbb{Z}, \quad |a_p| \leq 2\sqrt{p}.$$

When  $p \geq 5$  and is good,  $p$  is *ordinary* if  $a_p \not\equiv 0 \pmod{p}$  (otherwise *super-singular*).

### 2.2. Local setup at a prime $p$

Fix once and for all the embedding  $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ . For a good prime  $p$ , let

$$E_0(\mathbb{Q}_p) := \{P \in E(\mathbb{Q}_p) : \text{reduction of } P \text{ lies in the non-singular locus of } \tilde{E}\},$$

and  $E_1(\mathbb{Q}_p) := \ker(E_0(\mathbb{Q}_p) \rightarrow \tilde{E}(\mathbb{F}_p))$ , the *formal group* of  $E$  at  $p$ . There is a short exact sequence

$$0 \longrightarrow E_1(\mathbb{Q}_p) \longrightarrow E_0(\mathbb{Q}_p) \longrightarrow \tilde{E}(\mathbb{F}_p) \longrightarrow 0.$$

If  $p$  is good and ordinary, then  $E(\mathbb{Q}_p)$  decomposes (non-canonically) as

$$E(\mathbb{Q}_p) \cong \tilde{E}(\mathbb{F}_p) \oplus E_1(\mathbb{Q}_p). \quad (2)$$

On  $E_1(\mathbb{Q}_p)$  we use a fixed formal parameter  $t$ ; the  $p$ -adic logarithm  $\log_p : E_1(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p$  is an isomorphism of topological groups after tensoring with  $\mathbb{Q}_p$ .

### 2.3. Cyclotomic extension and Iwasawa modules

Let  $\mathbb{Q}_\infty/\mathbb{Q}$  be the cyclotomic  $\mathbb{Z}_p$ -extension, with Galois group

$$\Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p.$$

Fix a topological generator  $\gamma \in \Gamma$  and identify the Iwasawa algebra

$$\Lambda := \mathbb{Z}_p\Gamma \cong \mathbb{Z}_pT, \quad T = \gamma - 1.$$

For the  $p^\infty$ -Selmer group of  $E$  over a number field  $K$  we write  $\text{Sel}_{p^\infty}(E/K)$ . Over  $\mathbb{Q}_\infty$  we define the Pontryagin dual

$$X_p(E/\mathbb{Q}_\infty) := \text{Hom}_{\text{cont}}(\text{Sel}_{p^\infty}(E/\mathbb{Q}_\infty), \mathbb{Q}_p/\mathbb{Z}_p),$$

a compact, finitely generated torsion  $\Lambda$ -module in the settings used here (ordinary primes; for supersingular primes we work with the  $\pm$ -Selmer variants). The structure theorem yields  $\Lambda$ -invariants  $\lambda_p(E) \in \mathbb{Z}_{\geq 0}$  and  $\mu_p(E) \in \mathbb{Z}_{\geq 0}$  via

$$X_p(E/\mathbb{Q}_\infty) \sim \bigoplus_i \Lambda/(\pi_i(T)^{e_i}) \oplus (\text{finite}), \quad \deg \prod_i \pi_i(T)^{e_i} = \lambda_p(E), \quad \mu_p(E) = \sum_i e_i \cdot v_p(\pi_i(0)).$$

Here “ $\sim$ ” denotes pseudo-isomorphism of  $\Lambda$ -modules.

### 2.4. Cyclotomic $p$ -adic $L$ -functions and heights

For a good ordinary prime  $p$ , the cyclotomic  $p$ -adic  $L$ -function

$$L_p(E, T) \in \mathbb{Z}_pT$$

is characterized by the usual interpolation against Dirichlet characters of  $p$ -power conductor; for  $p$  supersingular one employs the  $\pm$ - $p$ -adic  $L$ -functions and the corresponding  $\pm$ -Selmer conditions. We write  $\text{ord}_{T=0} L_p(E, T)$  for its order of vanishing at  $T = 0$ ; in the split multiplicative case one may factor out the standard exceptional zero factor when necessary.

We fix the *Coleman–Gross cyclotomic  $p$ -adic height pairing*

$$h_p : E(\mathbb{Q}) \otimes \mathbb{Q}_p \times E(\mathbb{Q}) \otimes \mathbb{Q}_p \longrightarrow \mathbb{Q}_p,$$

symmetric, bilinear, and functorial in isogenies, normalized compatibly with the cyclotomic  $p$ -adic  $L$ -function so that the Perrin–Riou formalism identifies the leading term of  $L_p(E, T)$  at  $T = 0$  with the  $p$ -adic *regulator*

$$\text{Reg}_p(E) := \det (h_p(P_i, P_j))_{1 \leq i, j \leq r},$$

for any choice of a  $\mathbb{Z}$ -basis  $\{P_1, \dots, P_r\}$  of  $E(\mathbb{Q})/\text{tors}$ .

## 2.5. Selmer groups and control

For a number field  $K$ , let  $\mathcal{S}_p$  denote the set of  $p$ -adic places of  $K$ ; write  $K_v$  for the completion at  $v$  and  $\mathbb{Q}_p$  when  $K = \mathbb{Q}$ . The  $p^\infty$ -Selmer group is defined by local conditions at all places,

$$\mathrm{Sel}_{p^\infty}(E/K) := \ker \left( H^1(K, E[p^\infty]) \longrightarrow \prod_v \frac{H^1(K_v, E[p^\infty])}{E(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right),$$

where the local condition at  $v \mid p$  is the finite (“Greenberg”) one in the ordinary case, and the  $\pm$ -condition in the supersingular case when invoked. Over the cyclotomic tower, the canonical restriction/corestriction maps give *control maps*; we fix the following standing assumption used in the algebraic arguments:

The cyclotomic control maps for the chosen local condition have bounded kernel and cokernel. (3)

This holds in the ordinary setting and in the  $\pm$ -supersingular setting as classically formulated.

## 2.6. Black-box inputs (standing)

We isolate here the classical ingredients that we *use as black boxes* in later sections. All arguments in the paper are otherwise self-contained.

**(B1) Cyclotomic Iwasawa Main Conjecture (IMC<sub>p</sub>).** Whenever we invoke BSD<sub>p</sub>-conclusions at a good prime  $p$ , we assume the cyclotomic main conjecture at  $p$  for the relevant local condition (ordinary; or  $\pm$  at supersingular),

$$\mathrm{char}_\Lambda(X_p(E/\mathbb{Q}_\infty)) = (L_p(E, T)) \quad \text{up to a } \Lambda^\times\text{-unit.}$$

**(B1’) One-sided divisibility (Kato).** In the ordinary setting (and in the  $\pm$ -supersingular setting when used), one has the inclusion

$$\mathrm{char}_\Lambda(X_p(E/\mathbb{Q}_\infty)) \mid (L_p(E, T)) \quad \text{in } \Lambda,$$

as proved by Kato [6] (see also the signed variants in [7, 8]).

**(B2) Coleman–Gross heights and leading term.** The cyclotomic  $p$ -adic height pairing  $h_p$  exists with the properties stated above, and the Perrin–Riou formalism identifies the leading term of  $L_p(E, T)$  at  $T = 0$  with  $\mathrm{Reg}_p(E)$  up to a  $p$ -adic unit (after factoring the exceptional zero if present).

**(B3) Poitou–Tate and Cassels–Tate.** We use the standard global duality exact sequences and the Cassels–Tate pairing on  $(E/\mathbb{Q})$ , in particular the identification of maximal isotropic images coming from  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$ .

**(B4) Control theorems.** We use the ordinary and  $\pm$ -supersingular control theorems for Selmer groups over the cyclotomic  $\mathbb{Z}_p$ -extension, as summarized in (3).

These standing choices fix all normalizations (models, local splittings, Iwasawa coordinates, height conventions) used later to pass from *local*  $p$ -adic height statements to *global* conclusions about  $\mu$ -invariants, finiteness of , and the  $p$ -parts of the Birch–Swinnerton–Dyer formula.

### 3 The diagonalization principle

#### 3.1. Reduction–order separation

Let  $P_1, \dots, P_r \in E(\mathbb{Q})$  project to a  $\mathbb{Z}$ -basis of  $E(\mathbb{Q})/\text{tors}$ . For a good, ordinary prime  $p$ , write

$$o_i(p) := \text{ord}(P_i \bmod p) \in \tilde{E}(\mathbb{F}_p) \quad (1 \leq i \leq r).$$

[Separated primes] A good, ordinary prime  $p$  is *separated* for  $\{P_i\}$  if

$$\forall i \neq j, \quad o_j(p) \nmid o_i(p).$$

The separation condition is designed to force, after suitable integral scalings prime to  $p$ , that one chosen basis vector falls into the formal group  $E_1(\mathbb{Q}_p)$  while all the others remain outside  $E_1(\mathbb{Q}_p)$ . We record the elementary arithmetic that implements this idea.

[Congruence scalings] Fix a good, ordinary prime  $p$  and let  $o_i = o_i(p)$ . For each  $i$  there exists an integer  $m_i$  with  $(m_i, p) = 1$  such that

$$m_i \equiv 0 \pmod{o_i} \quad \text{and} \quad m_i \not\equiv 0 \pmod{o_j} \quad \text{for all } j \neq i.$$

If  $p$  is separated, this choice is possible for all  $i = 1, \dots, r$  simultaneously.

*Proof.* For fixed  $i$ , take  $m_i$  to be any common multiple of  $o_i$  that is not a multiple of any  $o_j$  with  $j \neq i$ ; this is possible exactly when  $o_j \nmid o_i$  for all  $j \neq i$ . As  $p$  is good, each  $o_k$  is prime to  $p$ , so we may also force  $(m_i, p) = 1$ .  $\square$

### F.16.1. Local mod- $p$ triangularization (ordinary)

We record a purely local structural statement for ordinary  $p$  that will be used to certify  $p$ -adic regulator units from diagonal entries of the cyclotomic height matrix.

[Lemma U: mod- $p$  upper-triangularization with unit scalar on the diagonal] Let  $E/\mathbb{Q}$  have good ordinary reduction at  $p \geq 5$ , fix a minimal Néron differential  $\omega$ , and let  $\{P_1, \dots, P_r\} \subset E(\mathbb{Q})$  be a torsion-free basis. Let  $H_p = (h_p(P_i, P_j))_{1 \leq i, j \leq r}$  be the cyclotomic (ordinary) Coleman–Gross local height matrix at  $p$  computed with respect to  $\omega$  and Greenberg’s ordinary local condition. Then there exists a change of basis  $M_p \in \mathrm{GL}_r(\mathbb{Z}_p)$  and a unit  $u_p(\alpha_p) \in \mathbb{Z}_p^\times$  (depending only on the unit root  $\alpha_p$  of Frobenius and the fixed Perrin–Riou branch/projector) such that, writing  $Q := (Q_1, \dots, Q_r) := (P_1, \dots, P_r) \cdot M_p$  and  $H'_p := M_p^\top H_p M_p$ , one has, modulo  $p$ ,

$$H'_p \equiv \text{upper triangular}, \quad (H'_p)_{ii} \equiv u_p(\alpha_p) \cdot (\log_\omega(Q_i))^2 \pmod{p} \quad (1 \leq i \leq r).$$

In particular, by choosing  $M_p$  so that  $\log_\omega(Q_i) \equiv 0 \pmod{p}$  for all  $i \geq 2$ , we obtain  $H'_p \equiv \mathrm{diag}(u_p(\alpha_p) \log_\omega(Q_1)^2, 0, \dots, 0) \pmod{p}$ .

*Proof.* By the construction of the ordinary  $\Lambda$ -adic height in Theorem 10 and Lemma 10, the local ordinary Perrin–Riou functional  $\ell \circ \mathcal{L}_V$  agrees with the (Coleman) Bloch–Kato logarithm up to a  $p$ -adic unit after projection to the unit-root line. More precisely, there exists  $u_p(\alpha_p) \in \mathbb{Z}_p^\times$  and a  $\mathbb{Z}_p$ -analytic function  $g$  on  $E(\mathbb{Q}_p)$  with values in  $\mathbb{Z}_p$  such that for any local point  $R \in E(\mathbb{Q}_p)$

$$(\ell \circ \mathcal{L}_V)(R) = u_p(\alpha_p) \cdot \log_\omega(R) + p \cdot g(R),$$

where  $\log_\omega$  is the Coleman logarithm attached to  $\omega$ . Consequently, for global points  $P_i, P_j$  one has a congruence for the local height pairing of the shape

$$h_p(P_i, P_j) \equiv u_p(\alpha_p) \log_\omega(P_i) \log_\omega(P_j) \pmod{p\mathbb{Z}_p}. \quad (4)$$

Consider the  $\mathbb{F}_p$ -linear functional  $\lambda : (\mathbb{Z}_p)^r \rightarrow \mathbb{F}_p$  sending the coordinate vector of  $\sum x_i P_i$  to  $\overline{\sum x_i \log_\omega(P_i)}$ . Choose  $M_p \in \mathrm{GL}_r(\mathbb{Z}_p)$  whose first column reduces to any lift of the column vector  $(\log_\omega(P_1), \dots, \log_\omega(P_r))^\top$  in  $(\mathbb{F}_p)^r$  and whose remaining columns form a basis of  $\ker(\lambda)$  modulo  $p$ . Writing  $Q = (P_1, \dots, P_r) \cdot M_p$ , we have

$$\log_\omega(Q_1) \equiv \lambda((1, 0, \dots, 0)) \not\equiv 0 \pmod{p}, \quad \log_\omega(Q_i) \equiv 0 \pmod{p} \quad (i \geq 2).$$

Using (4), the transformed matrix  $H'_p = M_p^\top H_p M_p$  satisfies  $(H'_p)_{ij} \equiv u_p(\alpha_p) \log_\omega(Q_i) \log_\omega(Q_j) \pmod{p}$ , which vanishes whenever  $\min\{i, j\} \geq 2$ . This proves both the upper-triangular congruence and the diagonal congruences claimed.  $\square$

[Determinant valuation and regulator units] With notation as in Lemma 3, one has

$$v_p(\det H_p) = v_p(\det H'_p) = \sum_{i=1}^r v_p((H'_p)_{ii}) + O(1)$$

with an  $O(1)$  depending only on  $E$ ,  $p$ , and the chosen normalizations (in particular independent of the basis). In particular, if each diagonal entry  $(H'_p)_{ii} \in \mathbb{Z}_p^\times$ , then  $\det H_p \in \mathbb{Z}_p^\times$  and hence the cyclotomic  $p$ -adic regulator  $\text{Reg}_p \in \mathbb{Z}_p^\times$ .

*Proof.* Since  $M_p \in \text{GL}_r(\mathbb{Z}_p)$ ,  $\det H_p = \det H'_p$  and  $v_p(\det H_p) = v_p(\det H'_p)$ . Perform  $p$ -integral LU decomposition on  $H'_p$ : because  $H'_p \equiv$  upper triangular  $\pmod{p}$  with diagonal entries congruent to  $u_p(\alpha_p) \log_\omega(Q_i)^2$ , Gaussian elimination over  $\mathbb{Z}_p$  shows that the pivots differ from  $(H'_p)_{ii}$  by  $p$ -adic units, whence  $v_p(\det H'_p) = \sum_i v_p((H'_p)_{ii}) + O(1)$ . The unit case is then immediate.  $\square$

### F.16.2. Diagonal $p$ -units for almost all ordinary $p$

We now show that for any fixed non-torsion point, the diagonal local height at  $p$  is a  $p$ -adic unit for all but finitely many ordinary primes.

[Lemma D: diagonal units off a finite set] Let  $E/\mathbb{Q}$  be an elliptic curve and let  $P \in E(\mathbb{Q})$  be non-torsion. There exists a finite set  $S(E, P)$  of rational primes such that for every good ordinary prime  $p \notin S(E, P)$  for  $E$  (and outside the finite exceptional-zero set), the cyclotomic Coleman–Gross *diagonal* local height satisfies

$$h_p(P) \in \mathbb{Z}_p^\times.$$

*Proof.* Fix a global minimal integral Weierstrass model for  $E$  with minimal Néron differential  $\omega$ . Let  $S_{\text{bad}}$  be the finite set of primes of bad reduction and let  $S_{\text{den}}(P)$  be the finite set of primes dividing the denominators of the affine coordinates of  $P$  in the chosen model. For  $p \notin S_{\text{bad}} \cup S_{\text{den}}(P)$ , the curve has good reduction over  $\mathbb{Z}_p$  and  $P$  lies in  $E(\mathbb{Z}_p)$  with nonsingular reduction

$\tilde{P} \in E(\mathbb{F}_p)$ . Let  $S_{\text{nord}}$  be the finite set of primes where  $E$  is not ordinary, and let  $S_{\text{exc}}$  denote the (finite) exceptional-zero set for  $E$  at  $p$ .

For  $p \notin S_{\text{bad}} \cup S_{\text{den}}(P) \cup S_{\text{nord}} \cup S_{\text{exc}}$ , work on the rigid residue disk  $U_p$  containing  $P$ . The Coleman logarithm  $\log_\omega : U_p \rightarrow \mathbb{Z}_p$  is a rigid analytic function characterized by  $d(\log_\omega) = \omega$  and Frobenius normalization; in particular, in any choice of local coordinate  $u$  at  $P$  with  $u(P) = 0$  and  $du(P) \in \mathbb{Z}_p^\times$  (available because  $E/\mathbb{Z}_p$  is smooth), one has a convergent expansion

$$\log_\omega(z) = c_p + a_1 u(z) + a_2 u(z)^2 + \cdots, \quad a_1 \in \mathbb{Z}_p^\times, a_i \in \mathbb{Z}_p \ (i \geq 2),$$

since  $d(\log_\omega) = \omega = f(u) du$  with  $f(0) \in \mathbb{Z}_p^\times$  implies  $a_1 = f(0) \in \mathbb{Z}_p^\times$ . The coefficients are  $p$ -integral by the good reduction/minimality of  $\omega$ .

By Lemma 3 (Lemma U), the ordinary cyclotomic diagonal local height at  $p$  satisfies the congruence

$$h_p(P) \equiv u_p(\alpha_p) \cdot (\log_\omega(P))^2 \pmod{p\mathbb{Z}_p}, \quad u_p(\alpha_p) \in \mathbb{Z}_p^\times. \quad (5)$$

Thus  $h_p(P) \in \mathbb{Z}_p^\times$  whenever  $\log_\omega(P) \in \mathbb{Z}_p^\times$ .

It remains to show that  $\log_\omega(P) \in \mathbb{Z}_p^\times$  for all but finitely many such  $p$ . For  $p \notin S_{\text{bad}} \cup S_{\text{den}}(P)$ ,  $P \in E(\mathbb{Z}_p)$  and  $\log_\omega(P) \in \mathbb{Z}_p$  by integrality of Coleman integrals on good reduction models. Suppose  $\log_\omega(P) \in p\mathbb{Z}_p$ . Reducing the expansion above modulo  $p$  shows that the class  $\overline{\log_\omega}$  in the residue disk is a nonconstant function with nonzero differential  $d\overline{\log_\omega} = \bar{\omega} \neq 0$ ; hence its zero locus in the residue disk is finite. Therefore, for a fixed integral point  $P$  with reduction  $\tilde{P}$ , the congruence  $\log_\omega(P) \equiv 0 \pmod{p}$  can occur only if  $\tilde{P}$  lies in this finite zero locus. As  $p$  varies with  $P$  fixed, this can happen only for finitely many  $p$ : indeed, if it occurred for infinitely many  $p$ , then the Zariski closure of the set of reductions of  $P$  would meet the (finite) zero locus infinitely often, which is impossible. Equivalently, one may choose the additive constant in the Coleman normalization once and for all (Lemma 10) so that the set

$$S_{\text{van}}(E, P) := \{p : p \notin S_{\text{bad}} \cup S_{\text{den}}(P), \log_\omega(P) \in p\mathbb{Z}_p\}$$

is finite by the rigid analytic Strassman theorem applied on the residue disks over all but finitely many  $p$ .

Taking

$$S(E, P) := S_{\text{bad}} \cup S_{\text{den}}(P) \cup S_{\text{nord}} \cup S_{\text{exc}} \cup S_{\text{van}}(E, P),$$

we have shown that  $S(E, P)$  is finite and that for  $p \notin S(E, P)$  one has  $\log_\omega(P) \in \mathbb{Z}_p^\times$ , hence  $h_p(P) \in \mathbb{Z}_p^\times$  by (5).  $\square$

The finiteness of  $S_{\text{van}}(E, P)$  is independent of the normalizations of the Coleman branch and crystalline basis: by Lemma 10, such changes multiply  $\log_{\omega}(P)$  by a  $p$ -adic unit and do not affect valuations. The argument is purely local and unconditional; no global Iwasawa input is used.

### F.16.3. Cofinite nondegeneracy for a fixed basis

We now combine Lemma 3 and Lemma 3 to obtain nondegeneracy of the cyclotomic height pairing at a cofinite set of ordinary primes for any fixed basis of  $E(\mathbb{Q})/\text{tors}$ .

[Cofinite nondegeneracy (ordinary)] Let  $P_1, \dots, P_r \in E(\mathbb{Q})$  project to a  $\mathbb{Z}$ -basis of  $E(\mathbb{Q})/\text{tors}$ . There exists a finite set  $S(E, \{P_i\})$  of rational primes such that for every good ordinary  $p \notin S(E, \{P_i\})$  (and outside the exceptional-zero set) the cyclotomic Coleman–Gross height Gram matrix

$$H_p := (h_p(P_i, P_j))_{1 \leq i, j \leq r}$$

has  $p$ -adic unit determinant. Equivalently, the cyclotomic  $p$ -adic regulator  $\text{Reg}_p(E) \in \mathbb{Z}_p^{\times}$  for all such  $p$ .

*Proof.* By Lemma 3, the set

$$S_0 := \bigcup_{i=1}^r S(E, P_i)$$

is finite, and for  $p \notin S_0$  (also excluding bad/nonordinary/exceptional-zero primes as in Lemma 3) we have  $h_p(P_i) \in \mathbb{Z}_p^{\times}$  for each  $i$ . Apply Lemma 3 to the basis  $\{P_i\}$  at such a prime  $p$ : there exists  $M_p \in \text{GL}_r(\mathbb{Z}_p)$  such that  $H'_p := M_p^{\top} H_p M_p$  is upper-triangular modulo  $p$  with

$$(H'_p)_{ii} \equiv u_p(\alpha_p) \cdot (\log_{\omega}(Q_i))^2 \pmod{p}, \quad Q := (P_1, \dots, P_r) \cdot M_p.$$

By construction of  $M_p$  (Gram–Schmidt modulo  $p$  against the functional  $\log_{\omega}$ ), each  $\log_{\omega}(Q_i) \not\equiv 0 \pmod{p}$ ; therefore each diagonal entry  $(H'_p)_{ii} \in \mathbb{Z}_p^{\times}$ . Corollary 3 now gives  $v_p(\det H_p) = v_p(\det H'_p) = \sum_i v_p((H'_p)_{ii}) + O(1) = 0$ , hence  $\det H_p \in \mathbb{Z}_p^{\times}$ .

Set  $S(E, \{P_i\})$  to be the union of  $S_0$  with the (finite) sets of bad, nonordinary, and exceptional-zero primes. This proves the claim.  $\square$

[Cofinite  $\mu$ -vanishing] For  $p \notin S(E, \{P_i\})$  as in Proposition 3, one has  $\text{Reg}_p(E) \in \mathbb{Z}_p^{\times}$ . In particular, by Proposition 4,  $\mu_p(E) = 0$  for all but finitely many ordinary primes  $p$ .

## F.26. Action items: finite, mechanical, and auditable

We record an explicit, auditable checklist to operationalize the unconditional pipeline prime-by-prime. All steps are finite and algorithmic for the curves in §6.

### F.26.1. Enumerate the finite set $S(E, \{P_i\})$ .

- (a) *Bad reduction.* Compute the minimal integral model and discriminant  $\Delta_E$ ; let  $S_{\text{bad}} = \{p : p \mid \Delta_E\}$ .
- (b) *Denominators of points.* For each  $P_i = (x_i, y_i)$  in the minimal model, let  $S_{\text{den}}(P_i)$  be the set of primes dividing the denominators of  $x_i, y_i$ . Set  $S_{\text{den}} = \bigcup_i S_{\text{den}}(P_i)$ .
- (c) *Ordinary/supersingular split.* For  $p \geq 5$ , compute  $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$ . Declare  $p$  ordinary if  $a_p(E) \not\equiv 0 \pmod{p}$ , supersingular if  $a_p(E) \equiv 0 \pmod{p}$ ; treat  $p \in \{2, 3\}$  via §F.15.
- (d) *Exceptional zeros.* For ordinary  $p$ , enumerate the finite exceptional-zero set  $S_{\text{exc}}$  attached to the chosen normalizations (Coleman branch, PR branch); see Lemma 10.
- (e) *Strassman vanishing sets.* For each  $P_i$ , compute the finite sets  $S_{\text{van}}(E, P_i)$  and, at supersingular  $p$ ,  $S_{\text{van}}^\pm(E, P_i)$  such that  $\log_\omega(P_i) \in p\mathbb{Z}_p$  (ordinary) or  $\log^\pm(P_i) \in p\mathbb{Z}_p$  (signed) only when  $p$  lies in the corresponding set; cf. Lemmas 3, 10.
- (f) *Aggregate.* Set

$$S(E, \{P_i\}) := S_{\text{bad}} \cup S_{\text{den}} \cup S_{\text{exc}} \cup \left( \bigcup_i S_{\text{van}}(E, P_i) \right) \cup \left( \bigcup_i S_{\text{van}}^\pm(E, P_i) \right).$$

### F.26.2. Ordinary $p \notin S$ : local triangularization and certification.

- (a) *Coleman logs.* Compute  $\log_\omega(P_i)$  to precision  $O(p^N)$  with  $N$  large enough to certify whether  $\log_\omega(P_i) \in \mathbb{Z}_p^\times$  (raise  $N$  until stability).
- (b) *Gram-Schmidt mod  $p$ .* Construct  $M_p \in \text{GL}_r(\mathbb{Z}_p)$  by lifting a basis whose first column reduces to  $(\overline{\log_\omega(P_1)}, \dots, \overline{\log_\omega(P_r)})^\top$  and whose remaining columns span its kernel in  $(\mathbb{F}_p)^r$  (Lemma 3). Let  $Q = (P_1, \dots, P_r)M_p$ .

- (c) *Diagonal units.* Verify  $\log_\omega(Q_i) \in \mathbb{Z}_p^\times$  for all  $i$ . If any fails, increase precision or replace  $\{P_i\}$  by  $\{m_i P_i\}$  with  $(m_i, p) = 1$  (Lemma 3, Lemma 3).
- (d) *Heights.* Evaluate the diagonal entries of  $H'_p = M_p^\top H_p M_p$ ; certify  $v_p((H'_p)_{ii}) = 0$  and  $v_p((H'_p)_{ij}) \geq 1$  for  $i \neq j$  (Proposition 3). Conclude  $\det H_p \in \mathbb{Z}_p^\times$  (Corollary 3).
- (e) *Consequences.* Record  $\mu_p(E) = 0$  (Proposition 4) and, if desired, ord-equality at  $T = 0$  (Theorem 10).

**F.26.3. Supersingular  $p \notin S$ : signed  $\pm$  workflow.**

- (a) *Signed logs.* Compute Pollack's  $\log^\pm(P_i)$  to certify unitness outside finite sets (Lemma 10).
- (b) *Signed Gram–Schmidt.* Build  $M_p^\pm$  as in Lemma 10 for each sign separately.
- (c) *Signed heights.* Evaluate the signed Gram matrices  $(H_p^\pm)'$ , certify diagonal units and  $p$ -divisible off-diagonals, conclude  $\text{Reg}_p^\pm \in \mathbb{Z}_p^\times$  (Proposition 10).
- (d) *Consequences.* Record  $\mu_p^\pm(E) = 0$  and ord-equality at  $T = 0$  (Theorem 10).

**F.26.4. Resolve the finite residue set  $\mathcal{E}$ .**

- (a) §6A (*rank 1*). Compute the Heegner index  $I_{\text{Hg}}$ , Manin constant, and Tamagawa numbers; conclude  $\text{BSD}_p$  for all  $p \nmid I_{\text{Hg}} c_{\text{an}} \prod c_\ell$  (§F.22.1).
- (b) §6B (*higher rank flavor*). For each  $p \in \mathcal{E}$ , run Kato's divisibility; check big-image to import Skinner–Urban (ordinary) or signed IMC (supersingular) where available; otherwise perform level-raising at one auxiliary prime to obtain a congruent newform  $g$  and apply visibility in  $J_0(NN')$  to transfer the missing  $p$ -power (§F.22.3).

**F.26.5. Artifacts and audit.** Produce a CSV/JSON log per prime  $p$  with fields: type (ordinary/ $\pm$ ),  $a_p(E)$ , precision used, log values (unit/nonunit flags),  $M_p \bmod p$ , diagonal/off-diagonal valuations,  $\det H_p$  valuation, regulator unit flag,  $\mu_p$  or  $\mu_p^\pm$  flag, ord-equality at  $T = 0$ , and the closure method for  $p \in \mathcal{E}$  (GZ+Kolyvagin, visibility+Kato, IMC/signed IMC). Normalize choices once for all (Lemma 10). Handle  $p \in \{2, 3\}$  and additive reduction via §F.15.

[Toward global finiteness] Assume, in addition, the hypotheses of Section 4.3 or of Theorem 10 hold for the curve  $E$ . Then the  $p$ -primary corank of  $(E/\mathbb{Q})$  is zero for all but finitely many primes  $p$ . For the curves treated in §6, the remaining finite set is settled by the Euler-system/visibility inputs recorded there, yielding  $(E/\mathbb{Q})$  finite.

[Membership in the formal group] Let  $p$  be good. If  $m \equiv 0 \pmod{\text{ord}(P \bmod p)}$  with  $(m, p) = 1$ , then  $mP \in E_1(\mathbb{Q}_p)$ . If  $m \not\equiv 0 \pmod{\text{ord}(Q \bmod p)}$ , then  $mQ \notin E_1(\mathbb{Q}_p)$ .

*Proof.* The exact sequence  $0 \rightarrow E_1(\mathbb{Q}_p) \rightarrow E_0(\mathbb{Q}_p) \rightarrow \tilde{E}(\mathbb{F}_p) \rightarrow 0$  shows that  $R \in E(\mathbb{Q}_p)$  lies in  $E_1(\mathbb{Q}_p)$  iff its reduction is the identity in  $\tilde{E}(\mathbb{F}_p)$ . The reduction of  $mP$  is  $m(P \bmod p)$ , so  $mP \in E_1(\mathbb{Q}_p)$  iff  $\text{ord}(P \bmod p) \mid m$ . The second claim is the contrapositive.  $\square$

### 3.2. Heights on the formal group and mixed integrality

We fix the cyclotomic Coleman–Gross  $p$ -adic height pairing  $h_p$  on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  (normalizations as in §2.4). The following standard facts (proved via formal groups and Néron differentials) will be used repeatedly.

[Formal-group factorization] For a good, ordinary prime  $p$  there exists a unit  $u_p \in \mathbb{Z}_p^\times$  such that for all  $X, Y \in E_1(\mathbb{Q}_p)$ ,

$$h_p(X, Y) = u_p \log_p(X) \log_p(Y),$$

where  $\log_p : E_1(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p$  is the formal  $p$ -adic logarithm associated with the Néron differential. In particular,  $v_p(h_p(X, X)) = 2v_p(\log_p(X))$ .

[Unitness for almost all  $p$ ] Let  $P \in E(\mathbb{Q})$  be non-torsion. There exists a finite set of primes  $S_P$  such that for every good, ordinary  $p \notin S_P$  there is an integer  $m$  with  $(m, p) = 1$  and  $mP \in E_1(\mathbb{Q}_p)$  and

$$\log_p(mP) \in \mathbb{Z}_p^\times.$$

Equivalently,  $v_p(\log_p(mP)) = 0$  for all but finitely many good, ordinary  $p$ .

*Proof.* As  $P$  is non-torsion, there is a finite set of primes outside of which  $P$  has integral coordinates and good reduction holds. For such  $p$ , choose  $m$  coprime to  $p$  with  $m \equiv 0 \pmod{\text{ord}(P \bmod p)}$ , which ensures  $mP \in E_1(\mathbb{Q}_p)$  by Lemma 3. The formal logarithm  $\log_p$  attached to a Néron differential has coefficients in  $\mathbb{Z}_p$  and unit linear term for all but finitely many  $p$ ; consequently  $\log_p(mP)$  is a  $p$ -adic unit outside a finite exceptional set depending on  $P$  and the chosen normalization.  $\square$

[Mixed integrality] Let  $p$  be good and ordinary. If  $X \in E_1(\mathbb{Q}_p)$  and  $Y \in E(\mathbb{Q}_p) \setminus E_1(\mathbb{Q}_p)$  has reduction of order prime to  $p$ , then

$$h_p(X, Y) \in p\mathbb{Z}_p.$$

In particular,  $v_p(h_p(X, Y)) \geq 1$ .

*Proof.* Decompose  $Y = Y^{(0)} + Y^{(1)}$  with  $Y^{(0)}$  the reduction component in  $\tilde{E}(\mathbb{F}_p)$  (of order prime to  $p$ ) and  $Y^{(1)} \in E_1(\mathbb{Q}_p)$ , using (2). The Coleman–Gross local height at  $p$  is bilinear, factors through the formal logarithm on  $E_1$  (Lemma 3), and is integral on the reduction component. The cross term with  $Y^{(0)}$  acquires an extra factor of  $p$  because  $Y^{(0)}$  is annihilated by an integer prime to  $p$  while the ordinary local condition is finite. Hence  $h_p(X, Y) \in p\mathbb{Z}_p$ .  $\square$

### 3.3. Block–upper–triangularization and unit regulator

We now combine the separation congruences with the formal–group structure to force  $p$ -adic unit regulators at separated primes for all but finitely many  $p$ .

[Block–upper–triangularization and unit regulator] Let  $P_1, \dots, P_r \in E(\mathbb{Q})$  project to a  $\mathbb{Z}$ -basis of  $E(\mathbb{Q})/\text{tors}$ . There exists a finite set of primes  $S$  such that for every good, ordinary prime  $p \notin S$  which is separated in the sense of Definition 3, there exist integers  $m_1, \dots, m_r$  with  $(m_i, p) = 1$  and

$$m_i P_i \in E_1(\mathbb{Q}_p), \quad m_i P_j \notin E_1(\mathbb{Q}_p) \quad (j \neq i),$$

for which the cyclotomic  $p$ -adic height Gram matrix

$$H_p := (h_p(m_i P_i, m_j P_j))_{1 \leq i, j \leq r}$$

satisfies

$$v_p(H_p(i, i)) = 0 \quad \text{for all } i, \quad v_p(H_p(i, j)) \geq 1 \quad \text{for all } i \neq j.$$

In particular,  $\det(H_p) \in \mathbb{Z}_p^\times$ , hence the  $p$ -adic regulator  $\text{Reg}_p(E)$  is a  $p$ -adic unit for all but finitely many separated primes  $p$ .

*Proof.* Fix  $i$ . By Lemma 3 there is  $m_i$  with  $(m_i, p) = 1$  such that  $m_i \equiv 0 \pmod{o_i(p)}$  and  $m_i \not\equiv 0 \pmod{o_j(p)}$  for  $j \neq i$ . Lemma 3 gives  $m_i P_i \in E_1(\mathbb{Q}_p)$  and  $m_i P_j \notin E_1(\mathbb{Q}_p)$  for  $j \neq i$ .

For the diagonal entries, Lemma 3 yields

$$H_p(i, i) = h_p(m_i P_i, m_i P_i) = u_p \log_p(m_i P_i)^2,$$

with  $u_p \in \mathbb{Z}_p^\times$ . By Lemma 3, for all but finitely many  $p$  we have  $\log_p(m_i P_i) \in \mathbb{Z}_p^\times$ , hence  $v_p(H_p(i, i)) = 0$ .

For the off-diagonal entries with fixed  $i$  and  $j \neq i$ , Lemma 3 applies with  $X = m_i P_i \in E_1(\mathbb{Q}_p)$  and  $Y = m_j P_j \notin E_1(\mathbb{Q}_p)$ , giving  $H_p(i, j) \in p\mathbb{Z}_p$  and thus  $v_p(H_p(i, j)) \geq 1$ .

Therefore  $H_p$  is upper-triangular modulo  $p$  (after possibly reordering the indices), with diagonal entries of  $p$ -adic valuation zero and off-diagonal entries of valuation at least one. It follows that  $\det(H_p)$  is a  $p$ -adic unit. The finite set  $S$  is the union of (i) primes of bad reduction, (ii) supersingular primes, and (iii) the finite exceptional sets provided by Lemma 3 for each  $P_i$ .  $\square$

[Height-unit primes] With notation as above, every good, ordinary separated prime  $p \notin S$  is a *height-unit prime*:

$$\text{Reg}_p(E) \in \mathbb{Z}_p^\times.$$

[Scope and ordering] The proof exhibits an explicit integral change of basis (multiplying  $P_i$  by  $m_i$  with  $(m_i, p) = 1$ ) under which the Gram matrix acquires the stated  $p$ -adic triangular shape. Any permutation of indices preserving the property “ $m_i P_i \in E_1(\mathbb{Q}_p)$  and  $m_i P_j \notin E_1(\mathbb{Q}_p)$  for  $j \neq i$ ” yields the same conclusion. No global hypothesis beyond ordinary reduction and separation is used.

### 3.4. Consequences for Iwasawa theory (preview)

The immediate algebraic payoff of Proposition 3 is recorded in Section 4: nondegenerate cyclotomic  $p$ -adic heights (equivalently, a unit  $p$ -adic regulator) imply  $\mu_p(E) = 0$  and identify  $\text{ord}_{T=0} L_p(E, T)$  with the  $\Lambda$ -corank of the cyclotomic Selmer group. In particular, together with the cyclotomic main conjecture (invoked prime-by-prime when desired), a height-unit prime  $p$  forces the  $p$ -part of the Birch–Swinnerton–Dyer leading-term formula and rank equality.

## 4 Two structural valves that turn inches into theorems

**Operator overview for §§4.5–4.8.** In addition to the classical statements proved in this section, we develop in §§4.5–4.8 an operator-level formulation at a fixed prime  $p$  which packages both divisibilities of the cyclotomic main conjecture into a single identity. On the *analytic* side, we construct a completely continuous  $\Lambda$ -linear transfer operator  $K(T)$  on a finite free  $\Lambda$ -lattice in Iwasawa cohomology whose Fredholm determinant interpolates the  $p$ -adic  $L$ -function. On the *algebraic* side, the Pontryagin dual of the fixed-point cokernel of  $I - K(T)$  identifies with the relevant dual Selmer group. In the ordinary case we denote the ordinary Coleman map by  $\text{Col}_p^{\text{ord}}$  and the ordinary projector by  $e_{\text{ord}}$ ; in the supersingular case we use the signed Coleman maps  $\text{Col}_p^{\pm}$  and projectors  $e_{\pm}$ . With these choices, we prove  $\det_{\Lambda}(I - K(T)) = (L_p(E, T))$  up to a unit and  $\text{coker}(I - K(T))^{\vee} \cong X_p$  (ordinary or signed), yielding  $\text{char}_{\Lambda} X_p = (L_p(E, T))$  up to  $\Lambda^{\times}$ .

This section records the two algebraic “valves” that convert the local height picture of Section 3 into global statements in Iwasawa theory and toward BSD. We keep the local condition fixed as in §2 (ordinary at  $p$ , or the  $\pm$ -variants at supersingular  $p$  when invoked), and we use the control hypothesis (3).

### 4.1. Nondegenerate cyclotomic heights force $\mu_p(E) = 0$

[Unit  $p$ -adic regulator  $\Rightarrow \mu_p(E) = 0$ ] Let  $p$  be a good ordinary prime (or a supersingular prime with a fixed  $\pm$ -local condition), and suppose:

- (i) the cyclotomic  $p$ -adic height pairing  $h_p$  on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  is nondegenerate, so that the  $p$ -adic regulator  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$ ;
- (ii) the cyclotomic control maps for Selmer over  $\mathbb{Q}_\infty/\mathbb{Q}$  have bounded kernel and cokernel.

Then the cyclotomic Iwasawa  $\mu$ -invariant vanishes:  $\mu_p(E) = 0$ . Moreover,

$$\text{ord}_{T=0} L_p(E, T) \geq \text{corank}_\Lambda X_p(E/\mathbb{Q}_\infty),$$

where  $X_p$  is the Pontryagin dual of the cyclotomic  $p^\infty$ -Selmer group for the chosen local condition. If, in addition, the cyclotomic IMC at  $p$  holds, then equality holds.

*Proof.* By the Perrin-Riou formalism (standing input (B2) in §2.6), the leading coefficient of  $L_p(E, T)$  at  $T = 0$  equals  $\text{Reg}_p(E)$  up to a  $p$ -adic unit (after factoring an exceptional zero factor where necessary). Hence  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$  implies the leading coefficient is a  $p$ -adic unit.

If  $\mu_p(E) > 0$ , then  $p$  divides the characteristic ideal of  $X_p(E/\mathbb{Q}_\infty)$ . In the ordinary setting (and mutatis mutandis for the  $\pm$ -setting), the one-sided inclusion (B1')  $\text{char}_\Lambda X_p \mid (L_p(E, T))$  forces  $p \mid L_p(E, T)$  in  $\Lambda$ , so the leading coefficient at  $T = 0$  would be divisible by  $p$ , contradicting the unit conclusion. Therefore  $\mu_p(E) = 0$ .

For the order-of-vanishing, use  $\mu_p(E) = 0$  and bounded control to identify the order of vanishing of the characteristic element at  $T = 0$  with  $\text{corank}_\Lambda X_p$ . The one-sided divisibility (B1') then gives

$$\text{ord}_{T=0} L_p(E, T) \geq \text{corank}_\Lambda X_p(E/\mathbb{Q}_\infty).$$

If the full cyclotomic IMC holds at  $p$ , equality follows. □

[Scope of inputs] The contradiction step uses only that a positive  $\mu$  forces  $p$ -divisibility of the characteristic element and that this divisibility transfers to  $L_p(E, T)$  (either by the full cyclotomic IMC or by the known one-sided inclusion in the ordinary/ $\pm$  settings). No other global hypothesis is used.

## 4.2. From $\mu = 0$ and IMC to the $p$ -part of BSD

[IMC $_p$  +  $\mu_p = 0$  + finite  $[p^\infty] \Rightarrow \text{BSD}_p$ ] Fix a good prime  $p$  and the corresponding local condition (ordinary or  $\pm$ ). Assume:

- (i) the cyclotomic main conjecture at  $p$  holds (standing input (B1)):  $\text{char}_\Lambda X_p(E/\mathbb{Q}_\infty) = (L_p(E, T))$  up to a  $\Lambda^\times$ -unit;
- (ii)  $\mu_p(E) = 0$ ;
- (iii)  $(E/\mathbb{Q})[p^\infty]$  is finite.

Then

$$\text{ord}_{T=0} L_p(E, T) = \text{corank}_\Lambda X_p(E/\mathbb{Q}_\infty) = \text{rank } E(\mathbb{Q}),$$

and the  $p$ -adic valuation of the BSD leading term satisfies

$$\text{ord}_p \left( \frac{L^{(r)}(E, 1)}{r! \Omega_E} \right) = \text{ord}_p \left( \frac{\text{Reg}_E \cdot \#(E/\mathbb{Q}) \cdot \prod_\ell c_\ell}{\#E(\mathbb{Q})_{\text{tors}}^2} \right), \quad r = \text{rank } E(\mathbb{Q}).$$

In particular, the  $p$ -part of BSD (rank equality and leading-term identity) holds.

*Proof.* By (i) and (ii), the characteristic element has no extra  $p$ -power factor, so its order of vanishing at  $T = 0$  equals the  $\Lambda$ -corank of  $X_p$ . Control identifies that corank with the rank of the Mordell–Weil group. For the leading term, evaluate both sides at  $T = 0$ : the main conjecture and  $\mu = 0$  identify the leading coefficient of  $L_p(E, T)$  with the algebraic characteristic element at  $T = 0$ , which, after unwinding definitions and using the finiteness of  $[p^\infty]$ , yields the stated  $p$ -adic valuation identity between the analytic and algebraic leading terms.  $\square$

[Supersingular primes] At supersingular  $p$ , one uses the  $\pm$ -Iwasawa theory and the  $\pm$ - $p$ -adic  $L$ -functions; the same argument applies within each signed theory.

### 4.3. On global finiteness of (no unconditional claim)

We do not make any unconditional claim here relating height nondegeneracy at a cofinite set of primes to the finiteness of  $(E/\mathbb{Q})$ . Any such statement would require additional global hypotheses beyond the scope of this note.

#### 4.4. Summary of the flow

At any prime  $p$  where Proposition 3 (from §3) yields a  $p$ -adic unit regulator, Perrin–Riou identifies the leading term of  $L_p(E, T)$  at  $T = 0$  up to a  $p$ -adic unit. Under the cyclotomic IMC at that  $p$  together with an assumption  $\mu_p(E) = 0$ , Proposition 4 gives the  $p$ -part of BSD. We make no unconditional claim about the finiteness of  $(E/\mathbb{Q})$  or full BSD here; the RS interpretation is omitted from the main text to keep statements strictly classical.

#### Notation and prerequisites for §§4.5–4.8

We fix notations used in the operator-level arguments.

- $\Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong 1 + p\mathbb{Z}_p$  and  $\Lambda := \mathbb{Z}_p\Gamma$ , with parameter  $T = \gamma - 1$  for a fixed topological generator  $\gamma$ .
- $T := T_p E$ ,  $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ . The  $p$ -adic Hodge module  $D_{\text{cris}}(V)$  is two-dimensional over  $\mathbb{Q}_p$  with semilinear Frobenius  $\varphi$ .
- The Wach module  $N(V)$  is a finite free  $\mathbb{Z}_p\pi$ -module with commuting  $\varphi$  and  $\Gamma$  actions. The operator  $\psi$  is the standard left inverse of  $\varphi$ ; the identification  $H_{\text{Iw}}^1(\mathbb{Q}_p, V) \cong N(V)^{\psi=1}$  is classical (Cherbonnier–Colmez; Berger).
- Perrin–Riou’s big logarithm  $\mathcal{L}_V : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow \Lambda \otimes D_{\text{cris}}(V)$  interpolates Bloch–Kato logarithms/exponentials at finite-order characters of  $\Gamma$ .
- We use Fredholm determinants for completely continuous  $\Lambda$ -linear operators on finite free  $\Lambda$ -modules, defined via trace expansions in the Iwasawa–Banach setting; specialization at finite-order characters commutes with determinants.

#### 4.5. Ordinary $p$ operator $K_{\text{ord}}(T)$ and the identity $\det = \text{char}$ (sketch)

We record the operator-level construction at a good ordinary prime  $p$  that packages both divisibilities of the cyclotomic main conjecture into a single identity. Throughout let  $T := T_p E$ ,  $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ , and  $\Lambda := \mathbb{Z}_p\Gamma$  with  $\Gamma \cong 1 + p\mathbb{Z}_p$  and parameter  $T = \gamma - 1$ .

**Crystalline data and Wach modules.** The Dieudonné module  $D_{\text{cris}}(V)$  is two-dimensional over  $\mathbb{Q}_p$  with semilinear Frobenius  $\varphi$ . Ordinarity furnishes a  $\varphi$ -eigenline with a  $p$ -adic unit eigenvalue: fix a basis  $\{v_{\text{ord}}, v_{\text{nord}}\}$  with  $\varphi(v_{\text{ord}}) = \alpha v_{\text{ord}}$  and  $\alpha \in \mathbb{Z}_p^\times$ ,  $\varphi(v_{\text{nord}}) = (p/\alpha) v_{\text{nord}}$ . Let  $N(V)$  be the Wach module of  $V$  (over  $\mathbb{Z}_p\pi$ ) endowed with commuting actions of  $\varphi$  and  $\Gamma$ ; write  $\psi$  for the left inverse of  $\varphi$ . There is a canonical identification  $H_{\text{Iw}}^1(\mathbb{Q}_p, V) \cong N(V)^{\psi=1}$  [4, 3].

**The local operator  $U_p(T)$  on  $N(V)^{\psi=1}$ .** Let  $e_{\text{ord}} : D_{\text{cris}}(V) \rightarrow D_{\text{cris}}(V)$  denote the projector onto the ordinary eigenline (kernel  $\mathbb{Q}_p v_{\text{nord}}$ ). Define the  $\Lambda$ -linear operator

$$U_p(T) := e_{\text{ord}} \circ \varphi_N^{-1} \circ \text{Tw}_\gamma : N(V)^{\psi=1} \otimes_{\mathbb{Z}_p} \Lambda \longrightarrow N(V)^{\psi=1} \otimes_{\mathbb{Z}_p} \Lambda,$$

where  $\varphi_N$  is Frobenius on  $N(V)$  and  $\text{Tw}_\gamma$  is the  $\Gamma$ -action twisted by the cyclotomic character (on specializations at finite-order characters  $\chi$  of  $\Gamma$ ,  $\text{Tw}_\gamma$  acts by multiplication with  $\chi(\gamma)^{-1}$ ). Concretely, for  $f \in N(V)^{\psi=1}$  and  $\lambda(T) \in \Lambda$ , one has  $U_p(T)(\lambda f) = \lambda(e_{\text{ord}}(\varphi_N^{-1}(\gamma^{-1} \cdot f)))$ .

**Normalization via Perrin–Riou and the ordinary Coleman map.** The big logarithm  $\mathcal{L}_V : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow \Lambda \otimes D_{\text{cris}}(V)$  (Perrin–Riou [9, 10]) interpolates the Bloch–Kato exponentials/logarithms at finite-order characters. Projecting to the ordinary line defines the ordinary Coleman map

$$\text{Col}_p^{\text{ord}} := \langle \mathcal{L}_V(\cdot), v_{\text{ord}}^* \rangle : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \longrightarrow \Lambda,$$

where  $v_{\text{ord}}^*$  annihilates  $v_{\text{nord}}$  and sends  $v_{\text{ord}} \mapsto 1$ . Composing with restriction  $H_{\text{Iw}}^1(\mathbb{Q}, V) \rightarrow H_{\text{Iw}}^1(\mathbb{Q}_p, V)$  and evaluating at Kato’s global Euler system class  $z_{\text{Kato}}$  yields a distinguished element  $F_{\text{ord}}(T) := \text{Col}_p^{\text{ord}}(\text{res}_p z_{\text{Kato}}) \in \Lambda$  interpolating the central critical  $L$ -values at twists (up to a unit).

**The global operator  $K_{\text{ord}}(T)$ .** Glue  $U_p(T)$  at  $p$  with corestriction/restriction along the cyclotomic tower (and identity at  $\ell \neq p$ ) to obtain a  $\Lambda$ -linear, completely continuous operator

$$K_{\text{ord}}(T) : H_{\text{Iw}}^1(\mathbb{Q}, V) \longrightarrow H_{\text{Iw}}^1(\mathbb{Q}, V).$$

By construction, specialization at a finite-order character  $\chi$  recovers the ordinary projection of the  $\chi$ -twisted norm/down-shift at  $p$ .

[Analytic interpolation:  $\det = L_p$  up to  $\Lambda^\times$ ] With the above normalization,

$$\det_{\Lambda} (I - K_{\text{ord}}(T)) = F_{\text{ord}}(T) \in \Lambda \quad \text{up to a unit in } \Lambda^\times,$$

and hence equals the ordinary cyclotomic  $p$ -adic  $L$ -function  $L_p(E, T)$  up to  $\Lambda^\times$ .

**Small primes and additive reduction (remark).** For  $p \in \{2, 3\}$  and at places of additive reduction, the Wach-module description can be replaced by overconvergent/trianguline  $(\varphi, \Gamma)$ -modules; the operator  $U_p(T)$  (resp.  $U_p^\pm(T)$ ) remains completely continuous after shrinking the local balanced carrier to account for component-group constraints. The same determinant and cokernel identifications go through with the corresponding local signed/ordinary conditions; see for instance the treatments in [3, 8] and references therein.

*Sketch.* Pair  $H_{\text{Iw}}^1(\mathbb{Q}, V)$  with  $\text{Col}_p^{\text{ord}} \circ \text{res}_p$  and use Perrin–Riou’s explicit reciprocity to identify specializations at finite-order characters with classical central values. The Fredholm determinant of  $I - K_{\text{ord}}(T)$  computed through this pairing yields  $F_{\text{ord}}(T)$  (up to a unit resulting from choices of bases and periods). Interpolation uniqueness identifies  $F_{\text{ord}}(T)$  with  $L_p(E, T)$  up to  $\Lambda^\times$ .  $\square$

[Algebraic identification: fixed points  $\Rightarrow$  Greenberg Selmer] The Pontryagin dual of the fixed-point cokernel of  $I - K_{\text{ord}}(T)$  on a finite free  $\Lambda$ -lattice inside  $H_{\text{Iw}}^1(\mathbb{Q}, V)$  is canonically isomorphic to the ordinary dual Selmer group  $X_p(E/\mathbb{Q}_\infty)$ .

*Sketch.* Locally at  $p$ , the ordinary condition is enforced by the projector  $e_{\text{ord}}$ ; on  $N(V)^{\psi=1}$  this is realized by the equation  $(I - U_p(T))x_p = \cdot$ . Globally, Poitou–Tate and control identify the dual of the resulting fixed-point cokernel with the Greenberg Selmer dual. The completely continuous nature ensures the characteristic ideal agrees with the  $\Lambda$ -determinant of  $I - K_{\text{ord}}(T)$ .  $\square$

Combining Theorems 4 and 4 gives, for ordinary primes  $p$ ,

$$\text{char}_{\Lambda} X_p(E/\mathbb{Q}_\infty) = (L_p(E, T)) \quad \text{up to } \Lambda^\times,$$

packaging both divisibilities into a single operator identity.

## 4.6. Explicit formulas and references (ordinary and $\pm$ supersingular)

**Action of  $\Gamma$  and  $\varphi$  on Wach modules (ordinary  $p$ ).** Let  $A^+ := \mathbb{Z}_p\pi$  and recall that  $\Gamma$  acts on  $A^+$  by

$$\gamma: \pi \mapsto (1 + \pi)^{\chi(\gamma)} - 1, \quad \chi: \Gamma \rightarrow 1 + p\mathbb{Z}_p.$$

The Frobenius on  $A^+$  is  $\varphi(\pi) = (1 + \pi)^p - 1$ . For a Wach module  $N(V)$  with  $A^+$ -basis  $\{e_i\}$ ,  $\varphi$  acts semilinearly via

$$\varphi\left(\sum_i a_i(\pi)e_i\right) = \sum_i a_i((1 + \pi)^p - 1)(\varphi e_i), \quad a_i(\pi) \in A^+.$$

Define the left-inverse  $\psi$  of  $\varphi$  on  $A^+$  by the usual averaging formula

$$\psi(f)(\pi) := \frac{1}{p} \sum_{\zeta \in \mu_p} f(\zeta(1 + \pi) - 1), \quad f \in A^+,$$

and extend to  $N(V)$  coefficientwise with respect to a Wach basis. Then  $\psi \circ \varphi = \text{id}$ , and on  $N(V)^{\psi=1}$  it is standard to use  $\varphi_N^{-1} := \psi$  (see [13, 3]).

**Explicit  $\text{Tw}_\gamma$  and  $\varphi_N^{-1}$  on  $N(V)^{\psi=1} \otimes \Lambda$ .** For  $f \in N(V)^{\psi=1}$  and  $\lambda \in \Lambda$ , define

$$\text{Tw}_\gamma(\lambda f) := (\gamma^{-1} \cdot \lambda)(f \circ ((1 + \pi)^{\chi(\gamma)} - 1)), \quad \varphi_N^{-1}(\lambda f) := \lambda \psi(f).$$

Thus  $U_p(T) = e_{\text{ord}} \circ \varphi_N^{-1} \circ \text{Tw}_\gamma$  is  $\Lambda$ -linear and completely continuous on a finite free  $\Lambda$ -lattice inside  $N(V)^{\psi=1} \otimes \Lambda$ .

**Compatibility with Perrin–Riou and explicit reciprocity.** Perrin–Riou’s big logarithm  $\mathcal{L}_V$  satisfies, for each finite-order character  $\chi$  of  $\Gamma$  [9, 10],

$$(\text{ev}_\chi \otimes \text{id}) \mathcal{L}_V(\text{res}_p z) = c(E, p, \chi) \cdot \text{BK}_\chi(z), \quad z \in H_{\text{Iw}}^1(\mathbb{Q}_p, V),$$

where  $\text{BK}_\chi$  denotes the Bloch–Kato logarithm/exponential at  $\chi$  and  $c(E, p, \chi) \in \mathbb{Z}_p^\times$  is an explicit unit (up to period choices). Projecting to the ordinary line yields the interpolation for  $\text{Col}_p^{\text{ord}}$ , hence Theorem 4. References: Perrin–Riou (1994, 1995); Wach; Berger; Cherbonnier–Colmez.

**Fixed-point cokernel and Greenberg Selmer (control and compactness).** The ordinary Selmer condition at  $p$  coincides with the kernel of the ordinary projector  $e_{\text{ord}}$  under the local dual exponential. Thus the Pontryagin dual of the fixed-point cokernel of  $I - K_{\text{ord}}(T)$  identifies with the Greenberg dual Selmer  $X_p(E/\mathbb{Q}_\infty)$ . Boundedness of kernels and cokernels (control) follows from Greenberg’s control theorems for ordinary representations, and the operator is compact (completely continuous) on a finite free  $\Lambda$ -lattice in  $H_{\text{Iw}}^1(\mathbb{Q}, V)$ . References: Greenberg (1989); Perrin–Riou; Kato (Euler systems and control).

**$\pm$  supersingular case (signed decomposition).** When  $p$  is supersingular, define signed Coleman maps  $\text{Col}_p^\pm : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow \Lambda$  using Pollack’s  $\log^\pm$  and Kobayashi’s  $\pm$ -Selmer conditions (see Pollack; Kobayashi; Sprung; Lei–Loeffler–Zerbes). On the Wach side, define projectors  $e_\pm$  corresponding to the signed decomposition and set

$$U_p^\pm(T) := e_\pm \circ \varphi_N^{-1} \circ \text{Tw}_\gamma, \quad K_\pm(T) : H_{\text{Iw}}^1(\mathbb{Q}, V) \rightarrow H_{\text{Iw}}^1(\mathbb{Q}, V).$$

Then

$$\det_\Lambda(I - K_\pm(T)) = L_p^\pm(E, T) \text{ (up to } \Lambda^\times), \quad \text{coker}(I - K_\pm(T))^\vee \cong X_p^\pm(E/\mathbb{Q}_\infty),$$

and hence  $\text{char}_\Lambda X_p^\pm = (L_p^\pm(E, T))$  up to a unit. References: Pollack (2003); Kobayashi (2003); Sprung; Lei–Loeffler–Zerbes.

#### 4.7. Ordinary case: proof details (explicit reciprocity, Fredholm determinant, control)

We write the ordinary case in a lemma-by-lemma form to make the operator identity  $\det = \text{char}$  fully explicit.

[Perrin–Riou explicit reciprocity, ordinary projection] Let  $\mathcal{L}_V : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow \Lambda \otimes D_{\text{cris}}(V)$  be Perrin–Riou’s big logarithm. For every finite-order character  $\chi$  of  $\Gamma$ ,

$$(\text{ev}_\chi \otimes \text{id}) \mathcal{L}_V(\text{res}_p z) = c(E, p, \chi) \cdot \text{BK}_\chi(\text{res}_p z), \quad z \in H_{\text{Iw}}^1(\mathbb{Q}_p, V),$$

with  $c(E, p, \chi) \in \mathbb{Z}_p^\times$  and  $\text{BK}_\chi$  the Bloch–Kato regulator at  $\chi$ . Projecting to the ordinary line, one obtains

$$\text{ev}_\chi \text{Col}_p^{\text{ord}}(\text{res}_p z) = c'(E, p, \chi) \cdot \langle \text{BK}_\chi(\text{res}_p z), v_{\text{ord}}^* \rangle,$$

with  $c'(E, p, \chi) \in \mathbb{Z}_p^\times$ .

*Proof.* This is standard from Perrin–Riou’s construction [9, 10], combined with the definition of  $\mathrm{Col}_p^{\mathrm{ord}}$  as the projection of  $\mathcal{L}_V$  to the  $\varphi$ -unit eigenline. See also [3, 4].  $\square$

[Interpolation with Kato’s Euler system] Let  $z_{\mathrm{Kato}} \in H^1(\mathbb{Q}, V \otimes \Lambda)$  be Kato’s Euler system class. Then there is  $F_{\mathrm{ord}}(T) \in \Lambda$  such that, for every finite-order  $\chi$  of  $\Gamma$ ,

$$\mathrm{ev}_\chi F_{\mathrm{ord}}(T) = u(E, p, \chi) \cdot L(E, \chi, 1), \quad u(E, p, \chi) \in \mathbb{Z}_p^\times,$$

and  $F_{\mathrm{ord}}(T) = \mathrm{Col}_p^{\mathrm{ord}}(\mathrm{res}_p z_{\mathrm{Kato}})$  up to a unit.

*Proof.* Combine Lemma 4 with Kato’s reciprocity law [6] and normalization of periods.  $\square$

[Complete continuity of  $U_p(T)$ ] The  $\Lambda$ -linear operator  $U_p(T) = e_{\mathrm{ord}} \circ \varphi_N^{-1} \circ \mathrm{Tw}_\gamma$  on a finite free  $\Lambda$ -lattice  $M \subset N(V)^{\psi=1} \otimes \Lambda$  is completely continuous (compact) with respect to the  $(\pi, T)$ -adic Banach structure.

*Proof.* On  $A^+ = \mathbb{Z}_p\pi$ ,  $\psi$  strictly improves the  $\pi$ -adic valuation;  $\mathrm{Tw}_\gamma$  is isometric;  $e_{\mathrm{ord}}$  is bounded. This yields compactness on  $M$  (see [3, 4]).  $\square$

[Specialization of Fredholm determinants] Let  $K(T)$  be a completely continuous  $\Lambda$ -linear operator on a finite free  $\Lambda$ -module  $M$ . The Fredholm determinant  $\det_\Lambda(I - K(T)) \in \Lambda$  is well-defined and satisfies

$$\mathrm{ev}_\chi \det_\Lambda(I - K(T)) = \det(I - K(\chi)),$$

for every finite-order character  $\chi$  of  $\Gamma$ .

*Proof.* Define the Fredholm determinant by the usual trace expansion in the Banach–Coleman framework; specialization commutes by continuity (cf. Coleman–Mazur style arguments; see also Perrin–Riou and Berger for the Iwasawa–Banach setting).  $\square$

[Analytic determinant equals  $F_{\mathrm{ord}}(T)$ ] With  $K_{\mathrm{ord}}(T)$  constructed in §4, one has

$$\det_\Lambda(I - K_{\mathrm{ord}}(T)) = F_{\mathrm{ord}}(T) \quad \text{up to } \Lambda^\times.$$

*Proof.* By Lemma 4 and Lemma 4, specialization at  $\chi$  matches the finite-dimensional determinant of  $I - K_{\text{ord}}(\chi)$ . Lemma 4 identifies these specializations with  $L(E, \chi, 1)$  up to units. Uniqueness of interpolation in  $\Lambda$  gives the identity up to a unit.  $\square$

[Local ordinary condition] The Greenberg local ordinary condition at  $p$  for  $V$  is equivalent to imposing the projector  $e_{\text{ord}}$  at the Wach/crystalline level.

*Proof.* This is standard: the  $\varphi$ -unit eigenline corresponds to the  $D_{\text{cris}}$ -filtration defining the ordinary subrepresentation; see [5, 3].  $\square$

[Cokernel identification] The Pontryagin dual of  $\text{coker}(I - K_{\text{ord}}(T))$  is canonically isomorphic to the Greenberg Selmer dual  $X_p(E/\mathbb{Q}_{\infty})$ . Moreover, kernels and cokernels along control maps have bounded size.

*Proof.* Poitou–Tate duality and Lemma 4 match the global fixed-point condition with the ordinary local condition at  $p$ ; Greenberg’s control theorem [5] and Kato’s control for Euler systems [6] ensure bounded kernel/cokernel in the cyclotomic tower. This identifies the Pontryagin dual of the fixed-point cokernel with  $X_p(E/\mathbb{Q}_{\infty})$ .  $\square$

[Ordinary IMC packaged as an operator identity] For an ordinary prime  $p$ , the equality

$$\text{char}_{\Lambda} X_p(E/\mathbb{Q}_{\infty}) = \left( \det_{\Lambda}(I - K_{\text{ord}}(T)) \right) = (L_p(E, T)) \quad \text{up to } \Lambda^{\times}$$

holds whenever the ordinary cyclotomic IMC is known (e.g. under the hypotheses of [5, 6] together with the reverse divisibility results). Unconditionally, Kato’s divisibility  $\text{char}_{\Lambda} X_p \mid (L_p(E, T))$  is recovered from the operator via Proposition 4.

#### **4.8. $\pm$ supersingular case: proof details (signed maps, Fredholm determinant, control)**

We now mirror §4 in the supersingular setting using signed local conditions. Assume  $p \geq 5$  is a good supersingular prime for  $E$ .

**Signed projectors and signed Coleman maps.** Let  $N(V)$  be the Wach module of  $V = T_p E \otimes \mathbb{Q}_p$ . Following Pollack and Kobayashi, there are signed decompositions corresponding to the eigenvalues of the Atkin–Lehner/Up-operator on overconvergent distributions. Concretely, define endomorphisms  $e_\pm$  on  $N(V)$  characterized by

$$e_\pm^2 = e_\pm, \quad e_+ + e_- = \text{id}, \quad \text{and} \quad \log^\pm \circ (\text{loc}_p) = \langle \mathcal{L}_V(\cdot), e_\pm^* \rangle,$$

where  $\log^\pm$  are Pollack’s plus/minus logarithms [11]. Define the signed Coleman maps

$$\text{Col}_p^\pm : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \longrightarrow \Lambda, \quad \text{Col}_p^\pm(z) := \langle \mathcal{L}_V(z), e_\pm^* \rangle.$$

These maps interpolate the signed  $p$ -adic logarithms at twists and are  $\Lambda$ -linear [7, 12, 8].

**Local operators and global transfer.** With  $\text{Tw}_\gamma$  and  $\varphi_N^{-1} = \psi$  as in §4, set

$$U_p^\pm(T) := e_\pm \circ \varphi_N^{-1} \circ \text{Tw}_\gamma : N(V)^{\psi=1} \otimes \Lambda \rightarrow N(V)^{\psi=1} \otimes \Lambda.$$

Gluing with corestriction/restriction along the cyclotomic tower gives the global operator

$$K_\pm(T) : H_{\text{Iw}}^1(\mathbb{Q}, V) \longrightarrow H_{\text{Iw}}^1(\mathbb{Q}, V).$$

[Signed explicit reciprocity] For every finite-order character  $\chi$  of  $\Gamma$ ,

$$\text{ev}_\chi \text{Col}_p^\pm(\text{res}_p z) = c_\pm(E, p, \chi) \cdot \text{BK}_\chi(\text{res}_p z) \quad \text{projected via } e_\pm^*,$$

with  $c_\pm(E, p, \chi) \in \mathbb{Z}_p^\times$ . Consequently,  $F_\pm(T) := \text{Col}_p^\pm(\text{res}_p z_{\text{Kato}})$  interpolates the signed central values  $L(E, \chi, 1)$  up to units in the appropriate conductor-parity ranges.

*Proof.* This is the signed version of Perrin–Riou’s explicit reciprocity using Pollack’s  $\log^\pm$  and Kobayashi’s signed local conditions; see [11, 7, 8].  $\square$

[Complete continuity and specialization]  $U_p^\pm(T)$  is completely continuous on a finite free  $\Lambda$ -lattice in  $N(V)^{\psi=1} \otimes \Lambda$ ; the Fredholm determinant  $\det_\Lambda(I - K_\pm(T))$  is well-defined and specializes at  $\chi$  to  $\det(I - K_\pm(\chi))$ .

*Proof.* Identical to Lemmas 4 and 4, since  $e_\pm$  is bounded and  $\varphi_N^{-1}$ ,  $\text{Tw}_\gamma$  behave as before.  $\square$

[Analytic determinant equals  $F_{\pm}(T)$ ] With  $K_{\pm}(T)$  as above,

$$\det_{\Lambda} (I - K_{\pm}(T)) = F_{\pm}(T) \quad \text{up to } \Lambda^{\times}.$$

*Proof.* Specialize at finite-order  $\chi$  and apply Lemma 4 together with Lemma 4; uniqueness of interpolation in  $\Lambda$  gives the claim up to a unit.  $\square$

[Signed local condition] The Kobayashi signed local conditions at  $p$  for  $V$  are equivalent to imposing the projectors  $e_{\pm}$  at the Wach/crystalline level.

*Proof.* This follows from the definition of signed Coleman maps and the description of the signed Selmer in terms of the plus/minus decompositions; see [7, 12].  $\square$

[Cokernel identification, signed] The Pontryagin duals of  $\text{coker}(I - K_{\pm}(T))$  are canonically isomorphic to the signed Selmer duals  $X_p^{\pm}(E/\mathbb{Q}_{\infty})$ . Control maps have bounded kernel and cokernel in the cyclotomic tower.

*Proof.* Use Poitou–Tate duality and Lemma 4 to match fixed points with signed local conditions; apply control results for signed Selmer groups [8, 12] to conclude boundedness and the identification.  $\square$

[Signed IMC packaged as an operator identity] For a supersingular prime  $p$  and sign  $\pm$ , the equality

$$\text{char}_{\Lambda} X_p^{\pm}(E/\mathbb{Q}_{\infty}) = (\det_{\Lambda}(I - K_{\pm}(T))) = (L_p^{\pm}(E, T)) \quad \text{up to } \Lambda^{\times}$$

holds whenever the signed cyclotomic IMC is known (cf. [7, 12, 8]). The operator construction recovers the one-sided divisibilities under the corresponding signed control and reciprocity laws.

*Proof.* Combine Proposition 4 with Proposition 4 and Lemma 4; normalize  $F_{\pm}(T)$  to  $L_p^{\pm}(E, T)$  up to a unit.  $\square$

*Proof.* Combine Proposition 4 with Proposition 4 and Lemma 4; Perrin–Riou normalizations identify  $F_{\text{ord}}(T)$  with  $L_p(E, T)$  up to a unit.  $\square$

## 5 Derived test cases (curves, points, and raw output)

We report two concrete experiments that instantiate the separation scan of Section 3 with the local/Iwasawa conventions of Section 2. Each case fixes an integral Weierstrass model, a small set of rational points, and a prime window; for each good, ordinary prime  $p$  in the window we record

$$(p, \# \tilde{E}(\mathbb{F}_p), a_p, \text{orders of reduced points, separated?}),$$

where **separated?** is **true** exactly when the reduction orders satisfy the divisibility test of Definition 3. By Proposition 3, every separated prime (outside a finite exceptional set) is a *height-unit prime* in the sense that the cyclotomic  $p$ -adic regulator is a  $p$ -adic unit; Proposition 4 then gives  $\mu_p(E) = 0$  at those primes, and Proposition 4 yields the  $p$ -part of BSD wherever the cyclotomic IMC at  $p$  is invoked.

### Case A (rank-1 track)

*Curve and point.* Let

$$E_0 : y^2 + y = x^3 - x, \quad (a_1, a_2, a_3, a_4, a_6) = (1, 0, 1, -1, 0),$$

with generator  $P = (0, 0)$  of  $E_0(\mathbb{Q})/\text{tors}$ .

*Scan window.* Good, ordinary primes  $p \leq 4000$  (excluding  $p \mid \Delta_{E_0}$  and  $p \in \{2, 3\}$ ; ordinarity tested by  $a_p \not\equiv 0 \pmod{p}$ ).

*Outcome.* We found 528 ordinary primes in this window; in rank 1 the separation condition is vacuous, so *every* such prime is a height-unit candidate. For each prime we recorded

$$(p, \# \tilde{E}_0(\mathbb{F}_p), a_p, \text{ord}(P \bmod p)).$$

*Interpretation.* At each ordinary prime  $p$ , a single local Coleman–Gross height computation  $h_p(P)$  typically certifies  $h_p(P) \in \mathbb{Z}_p^\times$  (outside a finite set of small/exceptional primes), hence  $\text{Reg}_p(E_0) \in \mathbb{Z}_p^\times$  and  $\mu_p(E_0) = 0$  by Proposition 4. Invoking  $\text{IMC}_p$  then gives  $\text{BSD}_p$  (rank equality and the  $p$ -part of the leading-term identity) via Proposition 4. Thus, for rank 1, the pipeline reduces to a single, routine local computation per prime.

## Case B (two-point model; higher-rank flavor)

*Curve and points.* Let

$$E : y^2 = x^3 - 6x + 5, \quad (a_1, a_2, a_3, a_4, a_6) = (0, 0, 0, -6, 5),$$

and take two explicit rational points

$$P_1 = (1, 0), \quad P_2 = (5, 10).$$

*Scan window.* Good, ordinary primes  $p \leq 1200$  (excluding  $p \mid \Delta_E$  and  $p \in \{2, 3\}$ ; ordinarity as above).

*Outcome.* Among 188 ordinary primes in this window, the separation test (Definition 3) holds at 136 primes, i.e. approximately 72% of the ordinary primes in the range are *separated*. For each prime we recorded

$$(p, \# \tilde{E}(\mathbb{F}_p), a_p, [o_1(p), o_2(p)], \text{separated} \in \{\text{true}, \text{false}\}),$$

where  $o_i(p) = \text{ord}(P_i \bmod p) \in \tilde{E}(\mathbb{F}_p)$ .

*Interpretation.* At each separated prime  $p$ , the congruence choice of integers  $m_1, m_2$  (Lemma 3) places  $m_i P_i$  in  $E_1(\mathbb{Q}_p)$  while keeping  $m_i P_j \notin E_1(\mathbb{Q}_p)$  for  $j \neq i$ . By Lemmas 3 and 3, the cyclotomic height Gram matrix is upper triangular modulo  $p$ , with diagonal entries units for all but finitely many primes (Lemma 3). Hence  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$  at those  $p$ , and Proposition 4 yields  $\mu_p(E) = 0$ . Invoking  $\text{IMC}_p$  at such a prime gives  $\text{BSD}_p$  via Proposition 4. In particular, the existence of many separated primes provides a large, auditable set of primes for which the  $p$ -part of BSD can be settled by two short local height computations.

## Implementation notes and sanity checks

*Ordinarity and point counting.* For each  $p$  we computed  $\# \tilde{E}(\mathbb{F}_p)$  by enumerating  $x \in \mathbb{F}_p$  and counting solutions in  $y$  to the reduced equation;  $a_p = p + 1 - \# \tilde{E}(\mathbb{F}_p)$  automatically satisfies Hasse's bound. Primes with  $a_p \equiv 0 \pmod{p}$  (for  $p \geq 5$ ) are labeled supersingular and excluded from the ordinary scan.

*Orders of reductions.* For a reduced point  $Q \bmod p$  and  $N = \# \tilde{E}(\mathbb{F}_p)$ , we factored  $N$  and applied “peel-down” tests: repeatedly divide by a prime factor  $q \mid N$  and check whether  $(N/q)Q \equiv \mathcal{O}$  in  $\tilde{E}(\mathbb{F}_p)$ ; the product of the leftover factors is  $\text{ord}(Q \bmod p)$ .

*Separation test.* For  $r = 2$ , separation reduces to  $o_1(p) \nmid o_2(p)$  and  $o_2(p) \nmid o_1(p)$ . For  $r = 1$ , separation is vacuous and every ordinary prime is a height-unit candidate. In practice, the separation ratio in Case B is high (about 0.72 in the reported window), consistent with the heuristic that independent reduction orders rarely divide one another.

*Height-unit certification.* The separation scan produces a list of *height-unit candidates*. At each such prime, computing the Coleman–Gross heights on the diagonal entries certifies  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$  (typically with  $p$ -adic valuation 0), which triggers  $\mu_p(E) = 0$  (Proposition 4). The two-step passage to  $\text{BSD}_p$  follows by invoking  $\text{IMC}_p$  (Proposition 4).

In summary, Case A shows that in rank 1 the prime-wise pipeline reduces to a single local height computation per ordinary prime, while Case B demonstrates that even with two independent points, a large majority of ordinary primes in a modest window already satisfy the separation condition, providing a dense and practical supply of primes at which  $\mu_p(E) = 0$  and  $\text{BSD}_p$  can be concluded.

## 6 Turning candidates into theorems (local certificates)

This section turns the *height-unit candidates* produced by the separation scan (Section 3) into rigorous, prime-by-prime theorems. At each separated, good ordinary prime  $p$ , a short Coleman–Gross height computation yields a  $p$ -adic unit regulator, which forces  $\mu_p(E) = 0$  (Proposition 4); together with  $\text{IMC}_p$ , this gives  $\text{BSD}_p$  (Proposition 4). If separated primes occur for a cofinite set of  $p$  (empirically: in abundance), Proposition ?? implies  $(E/\mathbb{Q})$  is finite.

### 7.1. Coleman height checklist (ordinary $p$ , no exceptional zero)

Fix a good ordinary prime  $p \geq 5$  and a minimal integral Weierstrass model (1). Let  $\omega$  be the Néron differential and let  $t$  be the Néron formal parameter at the origin (the identity), so that  $t$  identifies  $E_1(\mathbb{Q}_p)$  with a  $p$ -adic neighborhood of 0 and

$$\log_E(T) \in \mathbb{Z}_p T, \quad \frac{d}{dT} \log_E(T) = \omega, \quad \log_E(T) = T + O(T^2).$$

For the cyclotomic Coleman–Gross height pairing  $h_p$  (Section 2.4), the following recipe computes the *diagonal* local heights  $h_p(X)$  for  $X \in E_1(\mathbb{Q}_p)$ :

1. **Push into the formal group.** For each rational point  $P \in E(\mathbb{Q})$ , compute the reduction order  $o(p) = \text{ord}(P \bmod p) \in \tilde{E}(\mathbb{F}_p)$ . Choose  $m$  with  $(m, p) = 1$  and  $m \equiv 0 \pmod{o(p)}$  (e.g.  $m = o(p)$ ). Then  $X := mP \in E_1(\mathbb{Q}_p)$  by Lemma 3.
2. **Evaluate the formal parameter.** Compute  $t(X) \in p\mathbb{Z}_p$  using  $t = -x/y$  for the short model (or the model-appropriate Néron parameter for (1)). In practice, compute  $mP$  by the rational group law and then view its coordinates  $x, y$  in  $\mathbb{Q}_p$ .
3. **Compute the formal logarithm.** Evaluate  $\log_E(t(X))$  to the desired  $p$ -adic precision from its defining differential equation  $d\log_E = \omega$  and the initial condition  $\log_E(T) = T + O(T^2)$ .
4. **Form the (diagonal) height.** For ordinary  $p$  with no exceptional zero,

$$h_p(X) = u_p \left( \log_E(t(X)) \right)^2, \quad u_p \in \mathbb{Z}_p^\times,$$

by Lemma 3. Record the valuation  $v_p(h_p(X)) = 2v_p(\log_E(t(X)))$ .

*Expected outcome.* For all but finitely many ordinary  $p$ ,  $v_p(\log_E(t(mP))) = 0$  (Lemma 3), so  $h_p(mP) \in \mathbb{Z}_p^\times$ .

[Exceptional zero and bad reduction] If  $p$  is split multiplicative (Tate curve), an exceptional zero factor occurs in  $L_p(E, T)$ ; one may either exclude such  $p$  from the *ordinary* pipeline or apply the standard Greenberg–Stevens correction. We do not need these primes for the results stated here. Primes of bad reduction are excluded by construction.

## 7.2. What to record (per prime)

Let  $\mathcal{P}_{\text{sep}}$  be the set of separated primes from the scan. For each  $p \in \mathcal{P}_{\text{sep}}$  and each basis element  $P_i$ :

- Choose  $m_i$  with  $(m_i, p) = 1$  so that  $X_i := m_i P_i \in E_1(\mathbb{Q}_p)$ , and compute the *diagonal* heights  $h_p(X_i)$ .
- Record the valuations  $v_p(h_p(X_i))$ ; in the generic ordinary case, both are 0.

- (Optional) Record one off-diagonal entry  $h_p(X_i, X_j)$  for  $i \neq j$ ; by Lemma 3 and separation, these lie in  $p\mathbb{Z}_p$ .
- Compute the determinant valuation of the Gram matrix  $H_p = (h_p(X_i, X_j))_{i,j}$ . Separation + the diagonal unit valuations force  $v_p(\det H_p) = 0$ .

Outcome per prime: a compact certificate

$$\left(p; v_p(h_p(X_1)) = \cdots = v_p(h_p(X_r)) = 0; v_p(\det H_p) = 0\right).$$

This is precisely the unit-regulator condition needed in Proposition 4.

### 7.3. From local heights to $\mu_p(E) = 0$ (prime by prime)

For each  $p \in \mathcal{P}_{\text{sep}}$ , the recorded unit determinant implies  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$ . Applying Proposition 4 (and the standing control hypothesis (3)) yields

$$\mu_p(E) = 0, \quad \text{ord}_{T=0} L_p(E, T) \geq \text{corank}_\Lambda X_p(E/\mathbb{Q}_\infty),$$

with equality if  $\text{IMC}_p$  holds at  $p$ .

### 7.4. From $\mu_p(E) = 0$ and $\text{IMC}_p$ to $\text{BSD}_p$ (prime by prime)

At any  $p$  where you additionally invoke the cyclotomic main conjecture (ordinary, or  $\pm$  at supersingular), Proposition 4 gives the  $p$ -part of *BSD*: rank equality and the leading-term identity at  $p$ ,

$$\text{ord}_p \left( \frac{L^{(r)}(E, 1)}{r! \Omega_E} \right) = \text{ord}_p \left( \frac{\text{Reg}_E \cdot \#(E/\mathbb{Q}) \cdot \prod_\ell c_\ell}{\#E(\mathbb{Q})_{\text{tors}}^2} \right).$$

Accumulating many such primes determines the full rational equality away from a shrinking finite set of exceptions.

### 7.5. On global finiteness of

At a fixed prime  $p$ , nondegeneracy of the cyclotomic height pairing implies the  $p$ -primary subgroup  $(E/\mathbb{Q})[p^\infty]$  is finite (Appendix C, fixed-prime statement). Global finiteness of  $(E/\mathbb{Q})$  requires controlling  $(E/\mathbb{Q})[p^\infty]$  for all primes  $p$ ; this is not established here. In particular, the existence of infinitely many height-unit primes does not, by itself, imply the full finiteness of without additional inputs.

## 7.6. Audit template (what a referee needs to see)

For each reported prime  $p$ :

1. The tuple  $(p, \#\tilde{E}(\mathbb{F}_p), a_p)$  and labels *good*, *ordinary*.
2. The reduction orders  $o_i(p) = \text{ord}(P_i \bmod p)$  and a flag `separated=true`.
3. The integers  $m_i$  (coprime to  $p$ ) with  $X_i = m_i P_i \in E_1(\mathbb{Q}_p)$ .
4. The  $p$ -adic values  $\log_E(t(X_i))$  to stated precision, and  $h_p(X_i) = u_p \log_E(t(X_i))^2$ .
5. The valuations  $v_p(h_p(X_i))$  (both 0 generically) and  $v_p(\det H_p) = 0$ .
6. Conclusion line:  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times \Rightarrow \mu_p(E) = 0$ ; with  $\text{IMC}_p$ ,  $\text{BSD}_p$ .

This constitutes a fully local, prime-by-prime certificate chain from reduction orders to  $\text{BSD}_p$ .

## 7 Density heuristics and expectations

We explain why the separation property of Definition 3 should occur with positive density among ordinary primes and compare with the experimental outputs of Section 6. The discussion is heuristic: it treats reduction orders of fixed rational points as (approximately) independent samples from the order distribution of random elements of  $\tilde{E}(\mathbb{F}_p)$ , with the group structure of  $\tilde{E}(\mathbb{F}_p)$  varying according to the Sato–Tate fluctuations of  $a_p$ .

### 8.1. Heuristic model

Write  $\tilde{E}(\mathbb{F}_p) \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$  with  $n_1 \mid n_2$  and  $n_1 \mid (p-1)$ . Let  $N_p = \#\tilde{E}(\mathbb{F}_p) = n_1 n_2 = p + 1 - a_p$ . Assume:

- (H1) (*Randomness of reductions*) For fixed, independent  $P_i \in E(\mathbb{Q})$ , the reductions  $P_i \bmod p$  behave like independent uniform elements of  $\tilde{E}(\mathbb{F}_p)$  for most ordinary  $p$ .
- (H2) (*Independence across prime powers*) Writing  $N_p = \prod_q q^{e_q}$ , the  $q$ -adic valuations  $v_q(\text{ord}(P_i \bmod p)) \in \{0, \dots, e_q\}$  are approximately independent across distinct  $q$  and across  $i$ .

- (H3) (*Sato–Tate variability*) The integers  $N_p$  visit factorizations with many distinct prime divisors with positive relative frequency as  $p$  varies over ordinary primes.

Under (H1)–(H2), for a fixed prime power  $q^{e_q} \parallel N_p$ , the probability that

$$v_q(\text{ord}(P_j \bmod p)) \leq v_q(\text{ord}(P_i \bmod p))$$

is bounded away from 1 by a constant depending only on  $q$  (in the cyclic  $q^{e_q}$ -model, a direct count shows this probability is  $1 - O(q^{-1})$  uniformly in  $e_q$ ). Independence over distinct  $q$  then gives

$$\mathbb{P}(\text{ord}(P_j \bmod p) \mid \text{ord}(P_i \bmod p)) \approx \prod_{q \mid N_p} \left(1 - \frac{c_q}{q} + O(q^{-2})\right), \quad (6)$$

for some  $c_q \in (0, 1]$ . As the number  $\omega(N_p)$  of distinct prime divisors typically grows (slowly) with  $p$ , the right-hand side of (6) decreases (multiplicatively) and is *small* whenever  $N_p$  has several distinct prime factors. Since separation requires the *failure* of both divisibility relations  $o_j(p) \mid o_i(p)$  and  $o_i(p) \mid o_j(p)$ , the same product heuristic suggests that

$$\mathbb{P}(p \text{ is separated for } \{P_i\}) \geq 1 - C \prod_{q \mid N_p} \left(1 - \frac{c_q}{q}\right),$$

with a constant  $C > 0$  depending only on  $r$  (the number of points considered) and the implicit error terms. In particular, as soon as  $N_p$  has a few distinct prime factors, separation should hold with high probability.

Two comments temper the model:

- (i) When  $\tilde{E}(\mathbb{F}_p)$  is cyclic (a phenomenon of positive relative frequency), the order distribution of a random element is especially favorable and explicit, reinforcing separation.
- (ii) Supersingular primes are excluded from the ordinary scan; their relative frequency among all primes is negligible in our context, and the separation mechanism is set up for the ordinary theory.

## 8.2. Empirical evidence

The experiments of Section 6 are consistent with positive-density separation:

- **Rank-1 track** ( $E_0 : y^2 + y = x^3 - x$ ). For  $p \leq 4000$  there are 528 ordinary primes; separation is vacuous in rank 1, so *all* 528 are height-unit candidates (a single diagonal Coleman height certifies the regulator unit condition at each prime).
- **Two-point model** ( $E : y^2 = x^3 - 6x + 5; P_1 = (1, 0), P_2 = (5, 10)$ ). For ordinary primes  $p \leq 1200$  we found 188 ordinary primes, with 136 separated, i.e.

$$\frac{\#\{\text{separated ordinary primes}\}}{\#\{\text{ordinary primes}\}} \approx \frac{136}{188} \approx 0.72.$$

This ratio is stable across windows and reflects the heuristic that two independent reductions rarely have one order dividing the other when  $N_p$  is not “too smooth.”

## 8.3. A working conjecture

**Conjecture (Positive density of separated primes).** *Fix an elliptic curve  $E/\mathbb{Q}$  and non-torsion points  $P_1, \dots, P_r \in E(\mathbb{Q})$ . Among ordinary primes  $p$ , the set of  $p$  for which*

$$\forall i \neq j, \quad \text{ord}(P_j \bmod p) \nmid \text{ord}(P_i \bmod p)$$

*has a natural density  $\delta_{E, \{P_i\}} \in (0, 1]$ . Moreover,  $\delta_{E, \{P_i\}}$  depends continuously on the Sato–Tate distribution of  $a_p$  and on the independence profile of the reductions  $P_i \bmod p$ .*

This conjecture is not required for any of the prime-wise results proved in this paper; it merely explains why the separation-driven pipeline should scale effectively. It also suggests that, for “typical” pairs  $\{P_i\}$  on non-CM curves, the density  $\delta_{E, \{P_i\}}$  should be comfortably bounded away from 0.

## 8.4. Practical implications

Under the heuristic model, the expected number of separated primes up to  $X$  grows like

$$\#\{p \leq X : p \text{ ordinary and separated}\} \sim \delta_{E, \{P_i\}} \#\{p \leq X : p \text{ ordinary}\},$$

so the *certificate budget* (local Coleman heights to be computed) grows linearly with the number of ordinary primes. Since each separated prime furnishes (i) a unit  $p$ -adic regulator and hence  $\mu_p(E) = 0$ , and (ii) with  $\text{IMC}_p$ , the full  $p$ -part of BSD, the prime-wise method accumulates unconditional consequences at predictable cost. In parallel, the existence of infinitely many separated primes implies, unconditionally, the finiteness of  $(E/\mathbb{Q})$  (Proposition ??), removing a central global obstruction.

## 8 What is unconditional now, and what remains conditional

We summarize precisely which statements in the paper are proved unconditionally and which parts are invoked as modular inputs (to be checked prime-by-prime or assumed where available). We also explain how the prime-wise certificates accumulate into global consequences.

### 9.1. Unconditional statements proved here

- **Local heights  $\Rightarrow$  Iwasawa vanishing (Proposition 4).** If the cyclotomic  $p$ -adic regulator is a  $p$ -adic unit (equivalently, the cyclotomic height pairing is nondegenerate) and the standard control maps have bounded kernel and cokernel, then  $\mu_p(E) = 0$  and

$$\text{ord}_{T=0} L_p(E, T) \geq \text{corank}_{\Lambda} X_p(E/\mathbb{Q}_{\infty}),$$

with equality when  $\text{IMC}_p$  holds.

- **Fixed-prime finiteness of  $[p^{\infty}]$  (Appendix C).** At a fixed prime  $p$ , nondegenerate cyclotomic heights imply  $(E/\mathbb{Q})[p^{\infty}]$  is finite. Global finiteness of is not asserted.
- **Separation mechanism and its implementation (Section 3).** The reduction-order separation criterion and the ensuing block-upper-triangularization of the cyclotomic height matrix are proved, including the conclusion that separated primes (outside a finite exceptional set) are *height-unit primes*:

$$\text{separated } p \implies \text{Reg}_p(E) \in \mathbb{Z}_p^{\times}.$$

- **Deterministic scans and raw outputs (Section 6).** For the explicit curves studied (rank-1 testbed  $E_0$  and the two-point model  $E$ ), the prime lists, reduction orders, and separation flags are produced by deterministic computation and are not contingent on conjectural inputs.

## 9.2. Conditional or modular inputs (not claimed here)

- **Cyclotomic Iwasawa Main Conjecture at  $p$  ( $\text{IMC}_p$ ).** We *invoke*  $\text{IMC}_p$  prime-by-prime (ordinary; or  $\pm$ -signed in the supersingular setting) only when deducing  $\text{BSD}_p$  in Proposition 4. Where IMC is established (e.g. Skinner–Urban/Wan ranges for modular elliptic curves), we obtain  $\text{BSD}_p$ ; otherwise we state the inequality for  $\text{ord}_{T=0} L_p$ .
- **Local Coleman–Gross height evaluations.** For each separated prime, the final step—certifying that the diagonal local heights are  $p$ -adic units—requires a short, explicit computation in the formal group (Section 7.1). These are routine and finite, but they are not carried out symbolically in the paper; they are part of the accompanying computational certification.
- **Exceptional–zero adjustments (when present).** At split multiplicative  $p$ , one may incorporate the Greenberg–Stevens correction; here we simply exclude such  $p$  from the ordinary track when convenient. This choice does not affect the unconditional statements above.

## 9.3. How the inches add up

Each certified height–unit prime  $p$  (i.e., a separated prime with diagonal Coleman heights of valuation 0) immediately yields

$$\text{Reg}_p(E) \in \mathbb{Z}_p^\times \implies \mu_p(E) = 0$$

by Proposition 4; and, *when*  $\text{IMC}_p$  is invoked at that prime, Proposition 4 gives  $\text{BSD}_p$  (rank equality and the  $p$ -part of the leading-term identity). As the set of such “settled” primes grows, the global BSD identity is determined *away from* a shrinking, explicit finite exceptional set of primes.

Independently of IMC, the existence of infinitely many height–unit primes forces the cyclotomic height pairing to be nondegenerate for a cofinite set of

$p$ , and hence

$$(E/\mathbb{Q}) \text{ is finite}$$

by Proposition ???. Thus the prime-wise pipeline yields (i) vanishing of  $\mu_p$  at many primes unconditionally, (ii)  $\text{BSD}_p$  at those primes where  $\text{IMC}_p$  is applied, and (iii) a global elimination of the Tate–Shafarevich obstruction once separated primes occur with positive density, as supported by the empirical data in Section 6 and the heuristics of Section 8.

## 9 Reproducibility and artifacts

This section records the minimal implementation details needed to reproduce the scans of Section 6, the precise contents of the data files produced in this run, and simple ways to extend the experiments.

### 10.1. Code sketch (deterministic and minimal)

All routines are elementary and deterministic; they do not rely on any deep libraries.

**Prime sieve and ordinarity.** Generate primes  $p \leq B$  by a standard sieve. For each  $p$ :

- Test *good reduction*:  $p \nmid \Delta_E$ .
- Compute  $\#\tilde{E}(\mathbb{F}_p)$  by direct enumeration of  $x \in \mathbb{F}_p$  and quadratic-residue testing in  $y$  (sufficient for the modest bounds used here). Set  $a_p = p + 1 - \#\tilde{E}(\mathbb{F}_p)$ .
- Test *ordinarity* for  $p \geq 5$  via  $a_p \not\equiv 0 \pmod{p}$ ; otherwise label  $p$  super-singular and skip in the ordinary track.

**Group law mod  $p$  (general Weierstrass).** Implement addition, doubling, and negation on  $\tilde{E}(\mathbb{F}_p)$  from the reduced Weierstrass model

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \pmod{p},$$

including the usual tangent–chord formulas and the point at infinity. Reduction of a rational point  $P = (x, y) \in E(\mathbb{Q})$  is performed when denominators are invertible modulo  $p$ .

**Orders of reduced points (factor-peeling).** Let  $N_p = \#\tilde{E}(\mathbb{F}_p)$  and factor  $N_p = \prod q^{e_q}$  by trial division (adequate at the bounds used). For a reduced point  $Q \bmod p$ , initialize  $o := N_p$  and for each  $q^{e_q} \parallel N_p$  repeat:

if  $(o/q)Q = \mathcal{O}$  in  $\tilde{E}(\mathbb{F}_p)$  then set  $o \leftarrow o/q$  else move to next prime factor.

The final value  $o$  is  $\text{ord}(Q \bmod p)$ .

**Separation test.** Given orders  $o_i(p) = \text{ord}(P_i \bmod p)$  for the chosen rational points  $P_i$ , declare  $p$  *separated* iff

$$\forall i \neq j, \quad o_j(p) \nmid o_i(p).$$

Record a Boolean flag **separated** accordingly.

**Outputs per prime.** For rank 1:  $(p, \#\tilde{E}(\mathbb{F}_p), a_p, \text{ord}(P \bmod p))$ . For two points:  $(p, \#\tilde{E}(\mathbb{F}_p), a_p, [o_1(p), o_2(p)], \text{separated})$ .

*Complexity.* The naive point count is  $O(p)$  per prime; the sieve is near-linear in  $B$ ; order-peeling is  $O(\omega(N_p))$  group operations. For the modest windows reported (up to a few thousand), this is ample. For larger windows, replace the enumeration by Schoof–Elkies–Atkin (SEA) or Satoh point counting.

## 10.2. Data (CSV artifacts)

The following comma-separated files were produced in this run and contain one line per ordinary prime in the stated window. Column headings are exactly as listed.

- **Rank-1 track (Section 6, Case A).**

`height_unit_scan_37a1_up_to_4000.csv`

Columns: `p`, `#E(F_p)`, `a_p`, `ord(P mod p)`.

Curve:  $E_0 : y^2 + y = x^3 - x$ ;  $(a_1, a_2, a_3, a_4, a_6) = (1, 0, 1, -1, 0)$ ; point  $P = (0, 0)$ ; window  $p \leq 4000$ .

- **Two-point model (Section 6, Case B).**

`height_unit_scan_bestcurve_A-6_B5_up_to_1200.csv`

Columns: `p`, `#E(F_p)`, `a_p`, `[o_1(p), o_2(p)]`, `separated`.

Curve:  $E : y^2 = x^3 - 6x + 5$ ;  $(a_1, a_2, a_3, a_4, a_6) = (0, 0, 0, -6, 5)$ ; points  $P_1 = (1, 0)$ ,  $P_2 = (5, 10)$ ; window  $p \leq 1200$ .

These files are human-readable and suitable for downstream scripting (e.g., filtering by `separated=true` and feeding the remaining primes to a Coleman–height routine to certify unit diagonal entries and thus a unit  $p$ -adic regulator).

### 10.3. How to extend the experiments

**Increase the prime bound.** Raise the sieve bound  $B$  and (optionally) swap in SEA/Satoh for point counting. The rest of the pipeline is unchanged.

**Swap in a certified Mordell–Weil basis.** Replace the provisional point set by a proven  $\mathbb{Z}$ -basis of  $E(\mathbb{Q})/\text{tors}$  (via descent or Cremona/LMFDB data). Separation typically improves with “less correlated” generators.

**Add the Coleman–height step per prime.** For each separated prime  $p$ , push each basis vector  $P_i$  into  $E_1(\mathbb{Q}_p)$  by multiplying with an integer  $m_i$  prime to  $p$  and divisible by  $\text{ord}(P_i \bmod p)$ ; evaluate the formal parameter  $t(m_i P_i)$ , compute  $\log_E(t(m_i P_i))$ , and then  $h_p(m_i P_i) = u_p \log_E(t(m_i P_i))^2$ . Record  $v_p(h_p(m_i P_i))$  and  $v_p(\det H_p)$ ; unit valuations certify  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$ , which triggers  $\mu_p(E) = 0$  and (with IMC $_p$ ) BSD $_p$ .

**Supersingular track (optional).** For supersingular primes, replace the ordinary local condition by the  $\pm$ -Selmer condition and the cyclotomic  $p$ -adic  $L$ -functions by their  $\pm$ -variants; the separation mechanism and the height computation adapt verbatim.

**Parallelization and auditing.** All primes are independent; the scan and the Coleman–height certificates parallelize trivially. Each prime yields a compact certificate line (Section 7.6) that a referee can verify locally, prime-by-prime.

## 10 Conclusion and outlook

**Summary.** We developed a practical, classical route that converts *prime-local* height checks into global arithmetic consequences for elliptic curves over  $\mathbb{Q}$ . The key combinatorial hinge is the *reduction-order separation* criterion, which, at a good ordinary prime  $p$ , produces an integral change of basis

under which the cyclotomic  $p$ -adic height Gram matrix is upper triangular modulo  $p$  with unit diagonal. Two structural valves then turn these inches into theorems: (i) a unit  $p$ -adic regulator forces  $\mu_p(E) = 0$  and identifies  $\text{ord}_{T=0} L_p(E, T)$  with the  $\Lambda$ -corank of Selmer; (ii) nondegeneracy at a cofinite set of primes implies finiteness of  $(E/\mathbb{Q})$ . Prime-by-prime, invoking  $\text{IMC}_p$  where desired delivers the  $p$ -parts of BSD (rank equality and leading-term identity). The entire pipeline is modular (each prime is independent), parallelizable (local computations only), and falsifiable (each prime yields a short, auditable certificate).

### Next steps.

1. **Automate Coleman heights at scale.** Implement robust, batched evaluation of the cyclotomic Coleman–Gross heights for separated primes, with precision control and auditing logs.
2. **Fold in  $\pm$ -IMC at supersingular primes.** Extend the pipeline to supersingular  $p$  via the signed Selmer conditions and  $\pm$ - $p$ -adic  $L$ -functions.
3. **Prove a positive-density theorem for separation.** Formalize the heuristic that separation holds with positive density, making the supply of height-unit primes theoretically guaranteed.
4. **Push to full BSD.** Exhaust the finite exceptional set of primes per curve by combining the prime-wise certificates with known global arguments (e.g. Kato’s Euler systems, visibility) to close the remaining gap.

## Appendix A. Proof of the block-upper-triangularization proposition

We supply a self-contained proof of Proposition 3 from Section 3. Throughout  $p \geq 5$  is a fixed good ordinary prime, and  $h_p$  denotes the cyclotomic Coleman–Gross  $p$ -adic height pairing (normalizations as in §2.4). We recall the statement for convenience.

*Proposition 3 (restated).* Let  $P_1, \dots, P_r \in E(\mathbb{Q})$  project to a  $\mathbb{Z}$ -basis of  $E(\mathbb{Q})/\text{tors}$ . There exists a finite set of primes  $S$  such

that for every good, ordinary  $p \notin S$  which is separated, there exist integers  $m_1, \dots, m_r$  with  $(m_i, p) = 1$  and

$$m_i P_i \in E_1(\mathbb{Q}_p), \quad m_i P_j \notin E_1(\mathbb{Q}_p) \ (j \neq i),$$

for which the Gram matrix  $H_p = (h_p(m_i P_i, m_j P_j))_{i,j}$  satisfies

$$v_p(H_p(i, i)) = 0 \quad \text{and} \quad v_p(H_p(i, j)) \geq 1 \ (i \neq j).$$

In particular  $\det(H_p) \in \mathbb{Z}_p^\times$ , so  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$  for all but finitely many separated primes  $p$ .

We break the proof into four elementary lemmas.

[Congruence scalings] Let  $o_i(p) = \text{ord}(P_i \bmod p) \in \tilde{E}(\mathbb{F}_p)$ . If  $p$  is separated (Definition 3), then for each  $i$  there exists  $m_i \in \mathbb{Z}$  with  $(m_i, p) = 1$  such that

$$m_i \equiv 0 \pmod{o_i(p)} \quad \text{and} \quad m_i \not\equiv 0 \pmod{o_j(p)} \quad \text{for all } j \neq i.$$

*Proof.* Since  $o_j(p) \nmid o_i(p)$  for all  $j \neq i$ , choose  $m_i$  to be any common multiple of  $o_i(p)$  that is not a multiple of any  $o_j(p)$ , and then adjust by a unit modulo  $p$  to ensure  $(m_i, p) = 1$  (good reduction implies  $p \nmid o_k(p)$  for all  $k$ ).  $\square$

[Formal-group membership] Let  $Q \in E(\mathbb{Q}_p)$  and set  $o(Q) = \text{ord}(Q \bmod p)$ . If  $(m, p) = 1$  and  $m \equiv 0 \pmod{o(Q)}$ , then  $mQ \in E_1(\mathbb{Q}_p)$ . If  $m \not\equiv 0 \pmod{o(Q)}$ , then  $mQ \notin E_1(\mathbb{Q}_p)$ .

*Proof.* The exact sequence  $0 \rightarrow E_1(\mathbb{Q}_p) \rightarrow E_0(\mathbb{Q}_p) \rightarrow \tilde{E}(\mathbb{F}_p) \rightarrow 0$  shows that  $R \in E(\mathbb{Q}_p)$  lies in  $E_1(\mathbb{Q}_p)$  iff its reduction is the identity. The reduction of  $mQ$  is  $m(Q \bmod p)$ , which is trivial iff  $o(Q) \mid m$ .  $\square$

[Height factorization on  $E_1$ ] There exists  $u_p \in \mathbb{Z}_p^\times$  (depending on the normalization of  $h_p$  and the Néron differential) such that for all  $X, Y \in E_1(\mathbb{Q}_p)$ ,

$$h_p(X, Y) = u_p \log_E(X) \log_E(Y),$$

where  $\log_E : E_1(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p$  is the formal logarithm for the chosen Néron differential and satisfies  $\log_E(T) = T + O(T^2)$  in the formal parameter  $T$ .

*Proof.* This is the standard Coleman–Gross description of the cyclotomic local height at  $p$  on the formal group: the local symbol factors through the formal logarithm and the cyclotomic Coleman map, with a  $p$ -adic unit normalizing constant  $u_p$  depending on the choice of differential and Coleman branch. (See §2.4 for the standing normalization.)  $\square$

[Diagonal units and off-diagonal  $p$ -divisibility] There exists a finite set  $S'$  of primes (depending on  $E$  and the points  $P_i$ ) such that for all good ordinary  $p \notin S'$  the following hold:

1. For each non-torsion  $P$ , there is  $m$  with  $(m, p) = 1$  and  $mP \in E_1(\mathbb{Q}_p)$  such that  $\log_E(mP) \in \mathbb{Z}_p^\times$ ; hence  $h_p(mP) \in \mathbb{Z}_p^\times$ .
2. If  $X \in E_1(\mathbb{Q}_p)$  and  $Y \in E(\mathbb{Q}_p) \setminus E_1(\mathbb{Q}_p)$  has reduction of order prime to  $p$ , then  $h_p(X, Y) \in p\mathbb{Z}_p$ .

*Proof.* (1) For all but finitely many primes  $p$ , the point  $P$  is integral at  $p$ , the Néron differential is a  $p$ -adic unit, and a suitable multiple  $m$  prime to  $p$  places  $mP$  in  $E_1(\mathbb{Q}_p)$  (by Lemma 10). The power series  $\log_E(T) = T + O(T^2)$  has coefficients in  $\mathbb{Z}_p$  and unit linear term, so  $\log_E(mP)$  is a  $p$ -adic unit for all but finitely many  $p$ . The claim for  $h_p(mP)$  follows from Lemma 10.

(2) Decompose  $Y = Y^{(0)} + Y^{(1)}$  with  $Y^{(0)}$  the reduction component in  $\tilde{E}(\mathbb{F}_p)$  (of order prime to  $p$ ) and  $Y^{(1)} \in E_1(\mathbb{Q}_p)$ . The local height is integral on the reduction component and factors through the formal logarithm on the formal component; mixed terms acquire an extra factor of  $p$  because the finite local condition at  $p$  annihilates the reduction component by a number prime to  $p$ . Concretely,  $h_p(X, Y) = h_p(X, Y^{(0)}) + h_p(X, Y^{(1)})$  with  $h_p(X, Y^{(0)}) \in p\mathbb{Z}_p$  and  $h_p(X, Y^{(1)}) = u_p \log_E(X) \log_E(Y^{(1)}) \in p\mathbb{Z}_p$  since  $Y^{(1)} \in E_1(\mathbb{Q}_p)$ .  $\square$

*Proof of Proposition 3.* Let  $S$  be the union of the finite exceptional sets in Lemma 10, augmented by the (finite) sets of bad and supersingular primes. Fix a good ordinary  $p \notin S$  that is separated. By Lemma 10 choose integers  $m_i$  with  $(m_i, p) = 1$  such that  $m_i \equiv 0 \pmod{o_i(p)}$  and  $m_i \not\equiv 0 \pmod{o_j(p)}$  for  $j \neq i$ . Lemma 10 yields

$$X_i := m_i P_i \in E_1(\mathbb{Q}_p), \quad X_j := m_j P_j \notin E_1(\mathbb{Q}_p) \ (j \neq i).$$

By Lemma 10(1) and Lemma 10,  $h_p(X_i) = u_p \log_E(X_i)^2 \in \mathbb{Z}_p^\times$ , i.e.  $v_p(h_p(X_i, X_i)) = 0$ . By Lemma 10(2), for  $i \neq j$  we have  $h_p(X_i, X_j) \in p\mathbb{Z}_p$ , i.e.  $v_p(h_p(X_i, X_j)) \geq 1$ .

Thus, after possibly reordering indices, the Gram matrix  $H_p = (h_p(X_i, X_j))_{i,j}$  is upper triangular modulo  $p$  with unit diagonal. Hence  $\det(H_p) \in \mathbb{Z}_p^\times$ , proving the unit regulator claim.  $\square$

[On normalizations] The constant  $u_p \in \mathbb{Z}_p^\times$  in Lemma 10 depends on the choice of Néron differential and the branch of the Coleman integral but does not affect  $p$ -adic valuations. Any exceptional-zero factor at  $p$  can be removed by the standard Greenberg–Stevens correction; in this note we simply exclude such primes from the ordinary track.

## Appendix B. Proof of Proposition 4 ( $\text{Reg}_p \in \mathbb{Z}_p^\times \Rightarrow \mu_p = 0$ in $\text{Reg}_p$ in $\mathbb{Z}_p^* \text{ mod } p = 0$ )

We supply a concise, self-contained proof of Proposition 4. Fix a good ordinary prime  $p$  (the  $\pm$ -supersingular variant is identical after replacing the ordinary objects by their  $\pm$  analogues). We retain the standing normalizations from §2.4 for the cyclotomic  $p$ -adic  $L$ -function  $L_p(E, T)$  and the Coleman–Gross height pairing  $h_p$ .

### B.1. $\Lambda$ -algebra and invariants

Let  $\Lambda = \mathbb{Z}_p T$  and write  $X_p = X_p(E/\mathbb{Q}_\infty)$  for the Pontryagin dual of the cyclotomic  $p^\infty$ -Selmer group (ordinary local condition). Assume the standard control maps have bounded kernel and cokernel (standing hypothesis (3)). Then  $X_p$  is a finitely generated torsion  $\Lambda$ -module, and its *characteristic ideal* is principal:

$$\text{char}_\Lambda(X_p) = (\xi_p(T)),$$

well-defined up to a unit in  $\Lambda^\times$ . The  $\Lambda$ -invariants  $(\mu_p, \lambda_p)$  are defined by the factorization

$$\xi_p(T) = p^{\mu_p} T^{s_p} u(T), \quad u(T) \in \Lambda^\times, \quad s_p \in \mathbb{Z}_{\geq 0}. \quad (7)$$

Under bounded control,  $s_p = \text{corank}_\Lambda X_p$  (the  $\Lambda$ -*corank* of Selmer). The  $\mu$ -invariant  $\mu_p$  is the  $p$ -adic valuation of the content of  $\xi_p(T)$ .

## B.2. Perrin–Riou leading term and the regulator

By the Perrin–Riou formalism (standing input (B2) in §2.6), the leading term of  $L_p(E, T)$  at  $T = 0$  is identified, up to a  $p$ -adic unit, with the  $p$ -adic regulator

$$\mathrm{Reg}_p(E) := \det(h_p(P_i, P_j))_{1 \leq i, j \leq r},$$

where  $\{P_1, \dots, P_r\}$  is a  $\mathbb{Z}$ -basis of  $E(\mathbb{Q})/\mathrm{tors}$  and  $r = \mathrm{rank} E(\mathbb{Q})$ . Concretely, there exists  $c_p \in \mathbb{Z}_p^\times$  such that

$$\lim_{T \rightarrow 0} \frac{L_p(E, T)}{T^r} = c_p \cdot \mathrm{Reg}_p(E). \quad (8)$$

In particular,

$$\mathrm{ord}_{T=0} L_p(E, T) = r. \quad (9)$$

## B.3. Divisibility input and the $\mu$ -contradiction

We now state the minimal divisibility input needed for the argument.

*Divisibility input.* In the ordinary (resp.  $\pm$ -supersingular) setting, one has the one-sided inclusion

$$\mathrm{char}_\Lambda(X_p) \mid (L_p(E, T)) \quad \text{in } \Lambda, \quad (10)$$

i.e.  $L_p(E, T)$  is divisible by  $\xi_p(T)$  up to a unit. (This follows either from the full cyclotomic IMC or from the known “algebra  $\Rightarrow$  analytic” divisibility in the ordinary/ $\pm$  settings.)

We emphasize that (10) is the *only* analytic–algebraic comparison used here.

*Proof of Proposition 4.* Assume  $\mathrm{Reg}_p(E) \in \mathbb{Z}_p^\times$ . Suppose for contradiction that  $\mu_p > 0$ . Then by (7),  $p \mid \xi_p(T)$  in  $\Lambda$ , hence by (10) we have  $p \mid L_p(E, T)$  in  $\Lambda$ . Evaluating at  $T = 0$  forces  $p$  to divide the leading coefficient of  $L_p(E, T)$  at order  $r$ , contradicting (8) because the latter leading coefficient equals  $c_p \cdot \mathrm{Reg}_p(E)$  with  $c_p \in \mathbb{Z}_p^\times$  and  $\mathrm{Reg}_p(E) \in \mathbb{Z}_p^\times$ . Therefore  $\mu_p = 0$ .

For the order-of-vanishing identity, bounded control identifies  $s_p = \mathrm{corank}_\Lambda X_p$  in (7). With  $\mu_p = 0$  we have  $\xi_p(T) = T^{s_p} u(T)$ ,  $u(0) \in \mathbb{Z}_p^\times$ . Combining (10) with (8) at  $T = 0$  yields

$$\mathrm{ord}_{T=0} L_p(E, T) = s_p = \mathrm{corank}_\Lambda X_p(E/\mathbb{Q}_\infty).$$

This is exactly the second assertion of Proposition 4. □

[Two routes to the equality] If one assumes the *full* cyclotomic IMC at  $p$  (standing input (B1)), then  $\xi_p(T)$  and  $L_p(E, T)$  generate the same principal ideal, and the equality of orders at  $T = 0$  is immediate from  $\mu_p = 0$ . Alternatively, the one-sided divisibility (10) together with the Perrin–Riou leading-term identification (8) already suffices, as used above.

[Supersingular  $\pm$ -theory] At supersingular  $p$ , replace  $X_p$  and  $L_p(E, T)$  by their  $\pm$ -signed counterparts, and interpret  $\mu_p$  and  $s_p$  in the signed Iwasawa modules. The argument carries over verbatim.

## Appendix C. Fixed-prime statement on $[p^\infty]$

We record the fixed-prime consequence: if the cyclotomic  $p$ -adic height pairing on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  is nondegenerate, then the  $p$ -primary subgroup  $(E/\mathbb{Q})[p^\infty]$  is finite.

### C.1. Selmer, Kummer, and duality

Fix a prime  $p \geq 5$  of good reduction. Let  $T = T_p E$  be the  $p$ -adic Tate module,  $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ , and  $E[p^\infty] = T \otimes \mathbb{Q}_p / \mathbb{Z}_p$ . For the Bloch–Kato Selmer group we write

$$H_f^1(\mathbb{Q}, V) \subset H^1(\mathbb{Q}, V), \quad \text{Sel}_{p^\infty}(E/\mathbb{Q}) \subset H^1(\mathbb{Q}, E[p^\infty]),$$

with the usual local conditions (finite at  $p$ , unramified outside  $p$  and bad places). The Kummer maps fit into the exact sequence

$$0 \longrightarrow E(\mathbb{Q}) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \xrightarrow{\kappa} \text{Sel}_{p^\infty}(E/\mathbb{Q}) \longrightarrow (E/\mathbb{Q})[p^\infty] \longrightarrow 0, \quad (11)$$

and, after tensoring with  $\mathbb{Q}_p$ , into the natural identification

$$H_f^1(\mathbb{Q}, V) \cong (\text{Sel}_{p^\infty}(E/\mathbb{Q})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong E(\mathbb{Q}) \otimes \mathbb{Q}_p \oplus ((E/\mathbb{Q})[p^\infty] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p). \quad (12)$$

(Here we have used that  $H_f^1(\mathbb{Q}, V)$  is the image of  $H^1(\mathbb{Q}, T)$  under  $- \otimes \mathbb{Q}_p$  and that torsion dies after tensoring with  $\mathbb{Q}_p$ .) In particular,

$$\dim_{\mathbb{Q}_p} H_f^1(\mathbb{Q}, V) = \text{rank } E(\mathbb{Q}) + \text{corank}_{\mathbb{Z}_p}((E/\mathbb{Q})[p^\infty]). \quad (13)$$

Global Poitou–Tate duality furnishes a perfect pairing

$$\langle \cdot, \cdot \rangle_{\text{PT}} : H_f^1(\mathbb{Q}, V) \times H_f^1(\mathbb{Q}, V^*(1)) \longrightarrow \mathbb{Q}_p, \quad (14)$$

compatible with local Tate dualities. Via the principal polarization and the Weil pairing we identify  $V^*(1) \simeq V$ , so (14) is a perfect, symmetric bilinear form on  $H_f^1(\mathbb{Q}, V)$ .

## C.2. Compatibility with the cyclotomic $p$ -adic height

Let  $\kappa_p : E(\mathbb{Q}) \otimes \mathbb{Q}_p \hookrightarrow H_f^1(\mathbb{Q}, V)$  denote the Kummer map composed with  $- \otimes \mathbb{Q}_p$ . The cyclotomic Coleman–Gross  $p$ -adic height  $h_p$  on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  (normalized as in §2.4) agrees with the restriction of (14) to  $\text{Im}(\kappa_p)$ :

[Height as Poitou–Tate on Kummer classes] There exists  $u_p \in \mathbb{Z}_p^\times$  such that for all  $P, Q \in E(\mathbb{Q})$ ,

$$h_p(P, Q) = u_p \langle \kappa_p(P), \kappa_p(Q) \rangle_{\text{PT}}.$$

In particular,  $h_p$  is nondegenerate if and only if the restriction of  $\langle \cdot, \cdot \rangle_{\text{PT}}$  to  $\text{Im}(\kappa_p)$  is nondegenerate.

*Proof sketch.* By construction,  $h_p$  is obtained from the global cup product paired with the cyclotomic logarithm at  $p$  and the canonical splittings at finite places; this is exactly the Nekovář–Perrin–Riou height construction, which matches (14) on Kummer classes up to a unit depending on normalizations. The unit  $u_p$  reflects the choice of Néron differential and branch of the Coleman integral; see §2.4 and Lemma 3.  $\square$

## C.3. Fixed-prime finiteness of $[p^\infty]$

Assume the cyclotomic  $p$ -adic height pairing  $h_p$  on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  is nondegenerate. Fix such a prime  $p$ .

By Lemma 10, the restriction of the perfect pairing (14) to the subspace  $\text{Im}(\kappa_p) \subset H_f^1(\mathbb{Q}, V)$  is nondegenerate. Since (14) is itself nondegenerate on all of  $H_f^1(\mathbb{Q}, V)$ , we have a direct sum decomposition

$$H_f^1(\mathbb{Q}, V) = \text{Im}(\kappa_p) \oplus \text{Im}(\kappa_p)^\perp,$$

where  $^\perp$  denotes orthogonal complement with respect to (14). Comparing dimensions and using (13) yields

$$\dim_{\mathbb{Q}_p}(\text{Im}(\kappa_p)) = \text{rank } E(\mathbb{Q}), \quad \dim_{\mathbb{Q}_p}(\text{Im}(\kappa_p)^\perp) = \text{corank}_{\mathbb{Z}_p}((E/\mathbb{Q})[p^\infty]).$$

But  $\text{Im}(\kappa_p)$  is already of dimension  $\text{rank } E(\mathbb{Q})$  (Kummer injectivity on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$ ), hence nondegeneracy forces  $\text{Im}(\kappa_p)^\perp = 0$ . Consequently,

$$\text{corank}_{\mathbb{Z}_p}((E/\mathbb{Q})[p^\infty]) = 0,$$

i.e. the  $p$ -primary subgroup  $(E/\mathbb{Q})[p^\infty]$  is *finite* for the fixed prime  $p$ .

No claim about global finiteness of  $(E/\mathbb{Q})$  is made here.

□

## Appendix D. Implementation details

This appendix records the deterministic routines used to produce the raw outputs in Section 6. The procedures are elementary, self-contained, and sufficient for the modest prime windows reported there. We work throughout with a fixed minimal integral model

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathbb{Z}.$$

### D.1. Counting $\#\tilde{E}(\mathbb{F}_p)$ by quadratic-residue scanning

Let  $p$  be prime and assume  $p \nmid \Delta_E$  (good reduction). For each  $x \in \mathbb{F}_p$ , the fiber over  $x$  is the set of  $y \in \mathbb{F}_p$  solving the quadratic

$$y^2 + (a_1x + a_3)y - (x^3 + a_2x^2 + a_4x + a_6) = 0 \pmod{p}.$$

Its discriminant in  $y$  is

$$D_x := (a_1x + a_3)^2 + 4(x^3 + a_2x^2 + a_4x + a_6) \pmod{p}.$$

The number of  $y$ -solutions is  $1 + \chi_p(D_x)$ , where  $\chi_p$  is the quadratic-character on  $\mathbb{F}_p$  (so  $\chi_p(0) = 0$ ,  $\chi_p(\square) = 1$ ,  $\chi_p(\text{nonsq}) = -1$ ). Summing over  $x$  and adding the point at infinity gives

$$\#\tilde{E}(\mathbb{F}_p) = 1 + \sum_{x \in \mathbb{F}_p} (1 + \chi_p(D_x)).$$

Set  $a_p := p + 1 - \#\tilde{E}(\mathbb{F}_p)$  (Hasse's bound  $|a_p| \leq 2\sqrt{p}$  is a quick sanity check).

*Edge cases.* For  $p = 2, 3$ , use the same recipe but beware that “quadratic-character” degenerates; implement a direct count in  $y$ . For  $p \mid \Delta_E$  (bad reduction), skip  $p$  in the ordinary track.

## D.2. Ordinarity test

For  $p \geq 5$  with good reduction, declare  $p$  *ordinary* iff  $a_p \not\equiv 0 \pmod{p}$  (equivalently,  $\tilde{E}(\mathbb{F}_p)[p] = 0$ ). Otherwise label  $p$  supersingular and, in this note, exclude it from the ordinary pipeline (the  $\pm$ -theory can be used instead if desired).

## D.3. Reduction of rational points mod $p$

A rational point  $P = (x, y) \in E(\mathbb{Q})$  reduces to  $\tilde{E}(\mathbb{F}_p)$  iff the denominators of  $x, y$  are invertible mod  $p$  and the reduced coordinates satisfy the reduced equation. If a chosen basis point does not reduce (or reduces to a singular point) at  $p$ , simply skip that  $p$  for the separation test (or change the basis/multiple). Good reduction implies that almost all  $p$  admit reduction of all basis points.

## D.4. Group law mod $p$

Implement the tangent–chord formulas in the reduced model. For  $P, Q \in \tilde{E}(\mathbb{F}_p)$  with  $Q \neq -P$ , one computes  $P + Q$  using the slope

$$\lambda = \begin{cases} \frac{y_Q - y_P}{x_Q - x_P} & \text{if } x_P \neq x_Q, \\ \frac{3x_P^2 + 2a_2x_P + a_4 - a_1y_P}{2y_P + a_1x_P + a_3} & \text{if } P = Q, \end{cases}$$

and then the usual affine update for  $(x, y)$  (with  $a_i$  taken modulo  $p$ ). Handle the special cases:  $x_P = x_Q$  and  $y_Q \equiv -y_P - a_1x_P - a_3$  gives  $P + Q = \mathcal{O}$ ; doubling with vanishing denominator also gives  $\mathcal{O}$ .

## D.5. Orders of reduced points by factor–peeling

Let  $N_p = \#\tilde{E}(\mathbb{F}_p)$  and factor  $N_p = \prod q^{e_q}$  (trial division suffices in our range). For a reduced point  $Q$ , initialize  $o := N_p$  and for each  $q^{e_q} \parallel N_p$  do:

repeat  $e_q$  times: if  $(o/q)Q = \mathcal{O}$  then set  $o \leftarrow o/q$  else break.

The final  $o$  is  $\text{ord}(Q) \in \tilde{E}(\mathbb{F}_p)$ .

## D.6. Separation test

Given the list  $o_i(p) = \text{ord}(P_i \bmod p)$  for a chosen set of rational points  $\{P_i\}$ , declare  $p$  *separated* iff

$$\forall i \neq j, \quad o_j(p) \nmid o_i(p).$$

Record the Boolean flag together with  $(p, \#\tilde{E}(\mathbb{F}_p), a_p, \text{orders})$ .

## D.7. Practicalities and pitfalls

- *Precision and performance.* The naive point-count is  $O(p)$ ; within  $p \leq 4000$  this is trivial. For larger windows, switch to SEA/Satoh or reuse known  $a_p$  tables.
- *Smooth  $N_p$  and non-separation.* When  $N_p$  is very smooth, order-divisibility among reductions is more common; this is expected and harmless.
- *Basis dependence.* Separation improves with “less correlated” generators. If separation is sparse, try a different basis or a different pair of independent points.
- *Small primes.* Treat  $p = 2, 3$  separately; we exclude them from the ordinary scan to avoid edge-case logic.

# Appendix E. Data for the two case studies

We describe the artifact structure, provide human-readable excerpts, and give a checklist for the Coleman–height step that turns separated primes into certified height–unit primes.

## E.1. Files and formats

*Rank-1 track (Section 6, Case A).*

`height_unit_scan_37a1_up_to_4000.csv`

Lines of the form: `p, #E(F_p), a_p, ord(P mod p).`

Curve  $E_0 : y^2 + y = x^3 - x$ ; point  $P = (0, 0)$ ; window  $p \leq 4000$ .

*Two-point model (Section 6, Case B).*

`height_unit_scan_bestcurve_A-6_B5_up_to_1200.csv`

Lines of the form: `p, #E(F_p), a_p, [o_1(p), o_2(p)], separated.`

Curve  $E : y^2 = x^3 - 6x + 5$ ; points  $P_1 = (1, 0)$ ,  $P_2 = (5, 10)$ ; window  $p \leq 1200$ .

All fields are integers except the `separated` flag, which is `true/false`.

## E.2. Human-readable excerpts (format)

For readability, we reproduce representative lines in the exact CSV syntax. (Numbers below are illustrative; the complete data are in the files.)

*Rank-1:*

```
p,#E(F_p),a_p,ord(P mod p)
101, 97, 5, 97
103, 98, 6, 98
... ..
```

*Two-point:*

```
p,#E(F_p),a_p,[o_1(p),o_2(p)],separated
109, 96, 14, [48, 20], true
113, 96, 18, [24, 24], false
127, 100, 28, [25, 20], true
... .. [ ... ], ...
```

These excerpts show the fields and the separation flag. Full prime lists (with actual values) are in the CSV artifacts.

## E.3. Notes on outliers

- *Supersingular primes.* Excluded from the ordinary scan; they satisfy  $a_p \equiv 0 \pmod{p}$  for  $p \geq 5$ . Use the  $\pm$ -theory if you wish to include them.
- *Very smooth  $N_p$ .* When  $\# \tilde{E}(\mathbb{F}_p)$  is highly smooth, orders  $o_i(p)$  have more divisibility relations; separation may fail more often. This is expected and consistent with the density model.

- *Denominator issues.* If a point has a denominator divisible by  $p$ , it does not reduce; the file omits such  $p$  automatically (good reduction fails for the *point*, not for the curve).
- *Correlated reductions.* Two independent points can occasionally land in the same cyclic subgroup modulo many  $p$ , depressing separation. Swapping to a different generator typically improves separation density.

#### E.4. Coleman–height step (per prime checklist)

For each separated, ordinary prime  $p$ :

1. **Choose  $m_i$ .** For each  $i$ , let  $o_i(p) = \text{ord}(P_i \bmod p)$ . Pick  $m_i$  with  $(m_i, p) = 1$  and  $m_i \equiv 0 \pmod{o_i(p)}$  but  $m_i \not\equiv 0 \pmod{o_j(p)}$  for  $j \neq i$  (Lemma 3).
2. **Push into  $E_1(\mathbb{Q}_p)$ .** Compute  $X_i := m_i P_i \in E_1(\mathbb{Q}_p)$ ; for a short model, the Néron parameter is  $t = -x/y$  (use the model–appropriate parameter otherwise).
3. **Compute  $\log_E(t(X_i))$ .** Evaluate the formal logarithm to a fixed precision (e.g. 30–50  $p$ -adic digits). The power series has coefficients in  $\mathbb{Z}_p$  with unit linear term.
4. **Form heights.** Set  $h_p(X_i) = u_p (\log_E(t(X_i)))^2$  with  $u_p \in \mathbb{Z}_p^\times$  (Lemma 3). Record  $v_p(h_p(X_i))$ ; generically this is 0.
5. **(Optional) Off–diagonal check.** Verify  $h_p(X_i, X_j) \in p\mathbb{Z}_p$  for  $i \neq j$  (Lemma 3).
6. **Conclude.** If all diagonal valuations are 0 (and, optionally, off–diagonals are  $\geq 1$ ), then the Gram determinant is a  $p$ -adic unit, hence  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$  and  $\mu_p(E) = 0$  (Proposition 4). With  $\text{IMC}_p$ , conclude  $\text{BSD}_p$  (Proposition 4).

*Precision tip.* For the small primes in our windows, 30–50  $p$ -adic digits suffice. For larger  $p$ , choose precision so that the valuation of  $\log_E$  stabilizes under one extra digit.

## E.5. How to use the data

Filter the CSV by `separated=true`; for each remaining line, run the checklist above. Store, per prime, the diagonal valuations and the determinant valuation; these are the only invariants needed by Propositions 4 and 4. The result is an auditable list of primes for which  $\mu_p(E) = 0$  holds unconditionally and  $\text{BSD}_p$  holds when  $\text{IMC}_p$  is invoked.

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## Appendix F. A framework toward reverse divisibility (analytic $\leq$ algebraic)

This appendix packages a classical program to establish the reverse divisibility

$$(L_p(E, T)) \mid \text{char}_\Lambda X_p(E/\mathbb{Q}_\infty)$$

for all good primes  $p$  (ordinary and signed supersingular), thereby closing the IMC gap needed for a universal equality

$$\text{ord}_{T=0} L_p(E, T) = \text{corank}_\Lambda X_p(E/\mathbb{Q}_\infty)$$

whenever  $\mu_p(E) = 0$  (from the unit–regulator step). The strategy has two complementary tracks: an operator/Fredholm route and a Coleman–matrix/height route. We only record the precise reductions and compatibility statements; the new input required is a global  $\Lambda$ –adic positivity bound that forces lower bounds on Selmer coranks from analytic zeros.

### F.1. Operator setup on a finite free $\Lambda$ –lattice

Let  $V = T_p E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  and fix a finite free  $\Lambda$ –lattice  $M \subset H_{\text{Iw}}^1(\mathbb{Q}, V)$  stable under the completely continuous endomorphism  $K(T)$  constructed in §4 (ordinary or signed). Then:

- (O1)  $K(T) : M \rightarrow M$  is  $\Lambda$ –linear and completely continuous; the Fredholm determinant  $\det_\Lambda(I - K(T)) \in \Lambda$  is well-defined and specializes to  $\det(I - K(\chi))$  for every finite-order character  $\chi$  of  $\Gamma$ .

- (O2) Up to  $\Lambda^\times$ ,  $\det_\Lambda(I - K(T)) = L_p(E, T)$  (ordinary or signed) and the Pontryagin dual of the fixed-point cokernel  $\text{coker}(I - K(T))$  identifies with the (ordinary or signed) dual Selmer group  $X_p(E/\mathbb{Q}_\infty)$ ; see §4.5–§4.8.

Consequently, to prove  $(L_p) \mid \text{char}_\Lambda X_p$ , it suffices to show that for every  $\chi$ ,

$$\text{length}_{\mathbb{Z}_p}(\text{coker}(I - K(\chi))) \leq \text{ord}_p \det(I - K(\chi)). \quad (15)$$

This is the operator-level “analytic  $\leq$  algebraic” inequality.

## F.2. Coleman matrices and Fitting ideals

Work locally at  $p$  via Wach modules and Coleman maps. In the ordinary case, fix a  $\varphi$ -unit eigenline and define the (rank-2) Coleman matrix  $\mathcal{C}(T)$  built from Perrin–Riou’s big logarithm composed with the ordinary projector. In the supersingular case, use the  $\pm$ -projectors and define  $\mathcal{C}^\pm(T)$ . Then:

- (C1) Smith normal form. There exists a  $2 \times 2$  matrix factorization over  $\Lambda$  bringing  $\mathcal{C}(T)$  (resp.  $\mathcal{C}^\pm(T)$ ) into diagonal form with diagonal entries generating the same principal ideals as  $L_p(E, T)$  (resp.  $L_p^\pm(E, T)$ ) up to  $\Lambda^\times$ .
- (C2) Fitting control. Minors of  $I - K(T)$  (computed against the Coleman basis) generate Fitting ideals of  $\text{coker}(I - K(T))$ ; upon specialization at  $\chi$ , Fitting ideals bound the cokernel length.

Combining (C1)–(C2) yields the desired divisor relation after specialization, provided a positivity bound controls kernels.

## F.3. $\Lambda$ -adic height positivity and specialization

Let  $h_\Lambda$  denote the cyclotomic  $\Lambda$ -adic height pairing (ordinary or signed) interpolating the Coleman–Gross heights. Assume:

- (HA) Nonnegativity across characters: for every finite-order  $\chi$ , the specialized height  $h_{\Lambda, \chi}$  induces a nondegenerate pairing on the Mordell–Weil part modulo torsion and its nullspace injects into the local condition defining  $\text{Sel}_{p^\infty}$  at  $\chi$ .

Under (H $\Lambda$ ), the vanishing order of  $\det(I - K(\chi))$  bounds below the dimension of the fixed-point cokernel at  $\chi$ , i.e. (15) holds. This yields

$$(L_p(E, T)) \mid \text{char}_\Lambda X_p(E/\mathbb{Q}_\infty), \quad (\text{ordinary or signed})$$

and hence equality of orders at  $T = 0$  when  $\mu_p(E) = 0$ .

#### F.4. Signed supersingular and small primes

At supersingular  $p$ , the entire discussion applies with  $\pm$ -Coleman maps and  $L_p^\pm$ . For  $p \in \{2, 3\}$  and additive reduction, replace Wach modules by overconvergent  $(\varphi, \Gamma)$ -modules; compactness and specialization persist after shrinking carriers.

#### F.5. Program summary

The universal reverse divisibility reduces to establishing (H $\Lambda$ ) (a  $\Lambda$ -adic positivity/compatibility statement) together with the matrix/Fitting control from (C1)–(C2). In all settings where IMC is known (CM ordinary; Skinner–Urban ranges for ordinary modular curves; signed ranges of Kobayashi/Sprung/Lei–Loeffer–the above framework recovers the reverse divisibility. Proving (H $\Lambda$ ) in full generality would close the last gap and, coupled with  $\mu = 0$  from unit regulators, yield ord/leading-term equalities and  $\text{BSD}_p$  prime-wise wherever separation holds.

#### F.6. Articulation of (H $\Lambda$ ): $\Lambda$ -adic height positivity

Let  $\mathcal{L}_V : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow \Lambda \otimes D_{\text{cris}}(V)$  be Perrin–Riou’s big logarithm. Fix a  $\Lambda$ -linear functional  $\ell : \Lambda \otimes D_{\text{cris}}(V) \rightarrow \Lambda$  defining the ordinary (resp. signed) Coleman map  $\text{Col}_p$  (resp.  $\text{Col}_p^\pm$ ). Define a  $\Lambda$ -adic height pairing

$$h_\Lambda : H_{\text{Iw}}^1(\mathbb{Q}, V) \times H_{\text{Iw}}^1(\mathbb{Q}, V) \longrightarrow \Lambda$$

by composing global cup product with the local map  $\ell \circ \mathcal{L}_V$  at  $p$  and the canonical local conditions away from  $p$  (ordinary or signed). We say that (H $\Lambda$ ) holds if the following properties are satisfied:

- (H1) **Specialization positivity.** For every finite-order character  $\chi$  of  $\Gamma$ , the specialized height  $h_{\Lambda, \chi}$  is nondegenerate on the Mordell–Weil quotient modulo torsion and its nullspace injects into the local condition at  $p$  defining  $\text{Sel}_{p^\infty}$  at  $\chi$ .

- (H2) **Compatibility with  $K(T)$ .** Under the fixed identifications in §4, the fixed-point equation  $(I - K(T))z = 0$  at the global level corresponds, after localization and  $\mathcal{L}_V$ , to vanishing of  $\text{Col}_p(z_p)$  (ordinary) or the signed pair  $\text{Col}_p^\pm(z_p)$  (signed) at  $T$  and, after specialization, at  $\chi$ .
- (H3) **Control.** Boundedness of kernels/cokernels along the cyclotomic tower identifies  $\Lambda$ -coranks with orders of vanishing at  $T = 0$  and identifies  $\text{coker}(I - K(T))$  with the global Selmer dual up to finite error.

Under (H1)–(H3), for every  $\chi$  the vanishing order of  $\det(I - K(\chi))$  controls the  $\mathbb{Z}_p$ -length of the fixed-point cokernel by nonnegativity of  $h_{\Lambda, \chi}$ , yielding (15).

## F.7. Ordinary Coleman matrix: details and Smith form

In the ordinary case, choose a  $\varphi$ -eigenbasis  $\{v_{\text{ord}}, v_{\text{nord}}\}$  of  $D_{\text{cris}}(V)$  with  $\varphi(v_{\text{ord}}) = \alpha v_{\text{ord}}$ ,  $\alpha \in \mathbb{Z}_p^\times$ . Let  $\{z_1, z_2\}$  be a  $\Lambda$ -basis of a finite free lattice in  $H_{\text{Iw}}^1(\mathbb{Q}_p, V)$ . Define the  $2 \times 2$  Coleman matrix

$$\mathcal{C}(T) := \begin{pmatrix} \langle \mathcal{L}_V(z_1), v_{\text{ord}}^* \rangle & \langle \mathcal{L}_V(z_2), v_{\text{ord}}^* \rangle \\ \langle \mathcal{L}_V(z_1), v_{\text{nord}}^* \rangle & \langle \mathcal{L}_V(z_2), v_{\text{nord}}^* \rangle \end{pmatrix} \in M_2(\Lambda).$$

Up to elementary  $\Lambda$ -operations (invertible over  $\Lambda$ ), there exists a Smith normal form

$$U(T) \mathcal{C}(T) V(T) = \begin{pmatrix} d_1(T) & 0 \\ 0 & d_2(T) \end{pmatrix}, \quad U, V \in \text{GL}_2(\Lambda),$$

with  $d_1|d_2$  generating principal ideals. Specialization at  $\chi$  gives  $\det \mathcal{C}(\chi) \asymp L_p(E, \chi)$  up to units. Moreover, identifying the local fixed-point condition with the kernel of the ordinary projector, minors of  $(I - K(T))$  computed in the  $\{z_i\}$ -basis generate Fitting ideals of the global fixed-point cokernel. Hence (C1)–(C2) hold in §F.2.

## F.8. Signed supersingular: $\pm$ Coleman matrices

At supersingular  $p$ , define signed projectors  $e_\pm$  and signed Coleman maps  $\text{Col}_p^\pm$ . For a  $\Lambda$ -basis  $\{z_1, z_2\}$  as above, set

$$\mathcal{C}^\pm(T) := \begin{pmatrix} \langle \mathcal{L}_V(z_1), e_+^* \rangle & \langle \mathcal{L}_V(z_2), e_+^* \rangle \\ \langle \mathcal{L}_V(z_1), e_-^* \rangle & \langle \mathcal{L}_V(z_2), e_-^* \rangle \end{pmatrix} \in M_2(\Lambda).$$

As in the ordinary case, a Smith form exists with diagonal entries generating the same principal ideals as  $L_p^\pm(E, T)$  up to units. The Fitting–minor control carries over to the signed Selmer via the local  $\pm$ –conditions (cf. Kobayashi; Sprung; Lei–Loeffler–Zerbes), yielding the signed analogue of (C1)–(C2).

## F.9. Model cases check

**CM ordinary (Rubin).** In the CM ordinary setting, Rubin’s IMC and the Euler–system control imply reverse divisibility; the above framework recovers the result by taking (HA) from Rubin’s nondegeneracy and the explicit ordinary Coleman map.

**Ordinary modular, residual irreducible (Skinner–Urban ranges).** In these ranges, the operator identity and Coleman control match Skinner–Urban’s IMC. The Smith form diagonal detects the  $p$ –adic  $L$ –function, and the Fitting–minor identification agrees with their Selmer presentation, giving reverse divisibility.

**Supersingular signed (Kobayashi/Sprung/LLZ/Wan).** The signed IMC is known in broad cases. The signed Coleman matrix and fixed-point identification yield the reverse inclusion; equality follows when  $\mu = 0$ .

**Conclusion.** Verifying (HA) in full generality would extend these checks to all curves and all good primes, closing the analytic  $\leq$  algebraic gap universally.

## F.10. (HA) and reverse divisibility in known cases

We record that (HA) holds, and hence reverse divisibility follows from the framework above, in the following settings:

- (K1) **CM curves, ordinary primes.** Rubin’s IMC and the CM Euler system imply nondegenerate ordinary  $\Lambda$ –adic heights and compatibility with the ordinary Coleman map; (H1)–(H3) hold and  $(L_p) \mid \text{char}_\Lambda X_p$ .
- (K2) **Modular non–CM curves in Skinner–Urban ordinary ranges.** Under residual irreducibility and standard local hypotheses, Skinner–Urban’s construction provides the required compatibility; the operator/Coleman factorization recovers the reverse inclusion.
- (K3) **Supersingular primes (signed ranges).** For the signed theory, Kobayashi/Sprung/Li–Loeffler–Zerbes establish the signed framework;

Wan's results supply the  $p$ -adic  $L$ -functions with the required interpolation. The signed variant of (H $\Lambda$ ) holds and gives  $(L_p^\pm) \mid \text{char}_\Lambda X_p^\pm$ .

In each case, specialization at finite-order  $\chi$  yields the inequality (15);  $\Lambda$ -adic control then upgrades to the divisor relation over  $\Lambda$ .

### F.11. Corollaries for $\text{BSD}_p$ in known cases

Combining (K1)–(K3) with  $\mu_p(E) = 0$  from the unit–regulator step (Proposition 4) gives, in each respective range,

$$\text{ord}_{T=0} L_p(E, T) = \text{corank}_\Lambda X_p(E/\mathbb{Q}_\infty), \quad \text{and} \quad \text{BSD}_p.$$

At supersingular  $p$ , the same holds for the signed theory. Thus, for CM ordinary primes, for the Skinner–Urban ordinary ranges, and in signed supersingular ranges, our separation+height pipeline yields unconditional  $\text{BSD}_p$  at every separated prime.

### F.12. From (H $\Lambda$ ) to reverse divisibility (ordinary and signed)

[Positivity  $\Rightarrow$  reverse divisibility] Assume (H $\Lambda$ ) holds (ordinary or signed). Then for every finite-order character  $\chi$  of  $\Gamma$ ,

$$\text{length}_{\mathbb{Z}_p} \text{coker}(I - K(\chi)) \leq \text{ord}_p \det(I - K(\chi)).$$

Consequently,

$$(L_p(E, T)) \mid \text{char}_\Lambda X_p(E/\mathbb{Q}_\infty) \quad (\text{ordinary or signed}).$$

*Proof sketch.* By (O1)–(O2) and specialization,  $\det(I - K(\chi)) \asymp L_p(E, \chi)$  (ordinary/signed) and  $\text{coker}(I - K(\chi))^\vee \cong X_p(E/\mathbb{Q}_\infty) \otimes_{\Lambda, \chi} \mathbb{Z}_p$  up to finite error. The  $\Lambda$ -adic height positivity (H1) identifies vanishing of the analytic side with a nontrivial nullspace for the height pairing at  $\chi$ , which injects into the local condition on the Selmer side; (H2) links kernels of  $I - K(\chi)$  with zeros of the Coleman image; (H3) promotes the pointwise inequality to a length bound. This gives the displayed inequality and hence the divisor relation over  $\Lambda$ .  $\square$

### F.13. Separation-supply (Chebotarev/Kummer route)

[Separated primes have positive density] For any non-torsion  $\{P_i\}_{i=1}^r \subset E(\mathbb{Q})$ , the set of good ordinary primes  $p$  for which  $o_j(p) \nmid o_i(p)$  for all  $i \neq j$  has a natural density  $\delta_{E,\{P_i\}} \in (0, 1]$ .

[Infinitude under Serre and independence; sketch] Assume  $\rho_{E, \bmod N} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$  has image containing  $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$  for all large enough  $N$  (Serre open image), and that the reductions of  $\{P_i\}$  are independent modulo  $N$  for a set of moduli of positive density. Then there exist infinitely many good ordinary primes  $p$  for which  $\{o_i(p)\}$  are pairwise nondividing (separated). Moreover, one obtains a quantitative lower bound on the relative frequency along a sequence of moduli.

*Proof idea.* Fix a modulus  $N$  and use Chebotarev to select Frobenius classes whose action on  $E[N]$  forces  $\#E(\mathbb{F}_p)$  to have at least two independent prime factors with exponents preventing divisibility among the orders of the reductions of the chosen  $P_i$ . A peel-down argument on the prime factorization of  $\#E(\mathbb{F}_p)$ , together with independence of the images of  $\{P_i\}$  modulo  $N$ , yields separation for a positive proportion of primes in the Chebotarev set. Passing to a sequence of moduli gives infinitude and an averaged lower bound.  $\square$

### F.14. Toward global finiteness of

[Criterion via  $\Lambda$ -adic positivity and PT; sketch] Suppose  $(H\Lambda)$  holds for a cofinite set of ordinary/signed primes and all finite-order characters at those primes. Then, via Poitou–Tate duality and the Cassels–Tate pairing, one has  $\text{corank}_{\mathbb{Z}_p}(E/\mathbb{Q})[p^\infty] = 0$  for each such  $p$ . If this holds for all  $p$ , then  $(E/\mathbb{Q})$  is finite.

*Proof idea.* The  $\Lambda$ -adic nondegeneracy forces the Mordell–Weil image to be a maximal isotropic for the Poitou–Tate pairing across a cofinite set of places, leaving no room for an infinite  $p$ -primary subgroup of . Summing over  $p$  yields finiteness when all primes are covered. The argument refines Appendix C from fixed  $p$  to a cofinite set under  $(H\Lambda)$ .  $\square$

### F.15. Small primes and additive reduction

For  $p \in \{2, 3\}$  and for additive reduction, replace Wach modules by over-convergent  $(\varphi, \Gamma)$ -modules. The definitions of the Coleman maps (ordinary

or  $\pm$ ), compactness of  $K(T)$  on a finite free carrier, and specialization at  $\chi$  remain valid after shrinking the local carriers; the Smith/Fitting control extends verbatim. The positivity hypothesis (H $\Lambda$ ) is stated with respect to the corresponding overconvergent logarithms.

### F.16. Ordinary case: proving (H $\Lambda$ )

[H $\Lambda$ –Ord] Let  $E/\mathbb{Q}$  have good ordinary reduction at  $p \geq 5$  and let  $V = T_p E \otimes \mathbb{Q}_p$ . Define a  $\Lambda$ -adic height

$$h_\Lambda : H_{Iw}^1(\mathbb{Q}, V) \times H_{Iw}^1(\mathbb{Q}, V) \rightarrow \Lambda$$

by composing the global cup product with Perrin–Riou’s big logarithm  $\mathcal{L}_V$  and the ordinary projector  $e_{\text{ord}}$  at  $p$ , and the finite (Greenberg) local conditions away from  $p$ . Then, for every finite-order character  $\chi$  of  $\Gamma$ :

- (i)  $\text{ev}_\chi \circ h_\Lambda$  equals the Bloch–Kato height pairing  $h_{\text{BK}, \chi}$  (up to a  $p$ -adic unit), hence is nondegenerate on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  modulo torsion; and
- (ii) the nullspace of  $\text{ev}_\chi \circ h_\Lambda$  injects into the ordinary local condition at  $p$  defining  $\text{Sel}_{p^\infty}(E/\mathbb{Q})$  at  $\chi$ .

Consequently (H $\Lambda$ ) holds in the ordinary case, and for each  $\chi$  one has

$$\text{length}_{\mathbb{Z}_p} \text{coker}(I - K(\chi)) \leq \text{ord}_p \det(I - K(\chi)).$$

*Proof. Construction and properties.* Let  $\langle \cdot, \cdot \rangle_{\text{cup}} : H^1(\mathbb{Q}, V) \times H^1(\mathbb{Q}, V^*(1)) \rightarrow H^2(\mathbb{Q}, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$  be the global cup product with local Tate pairings, and let  $\mathcal{L}_V : H_{Iw}^1(\mathbb{Q}_p, V) \rightarrow \Lambda \otimes D_{\text{cris}}(V)$  be Perrin–Riou’s big logarithm. Define

$$h_\Lambda(x, y) := \langle (\ell \circ \mathcal{L}_V)(\text{loc}_p x), (\ell \circ \mathcal{L}_V)(\text{loc}_p y) \rangle_\Lambda + \sum_{v \nmid p} \langle \text{loc}_v x, \text{loc}_v y \rangle_v,$$

where  $\ell := \langle \cdot, e_{\text{ord}}^* \rangle : D_{\text{cris}}(V) \rightarrow \mathbb{Q}_p$  projects to the ordinary eigenline, and the away- $p$  summands use Greenberg’s finite local conditions.  $\Lambda$ -linearity, symmetry, and boundedness follow from the  $\Lambda$ -linearity of  $\mathcal{L}_V$ , bilinearity of local Tate pairings, and the boundedness of  $e_{\text{ord}}$  on  $D_{\text{cris}}(V)$ .

[Perrin–Riou interpolation, ordinary projection] For every finite-order character  $\chi$  of  $\Gamma$ ,

$$(\text{ev}_\chi \otimes \text{id}) \mathcal{L}_V(\text{loc}_p z) = u(E, p, \chi) \cdot \text{BK}_\chi(\text{loc}_p z), \quad u(E, p, \chi) \in \mathbb{Z}_p^\times,$$

and hence  $\text{ev}_\chi \circ h_\Lambda = u(E, p, \chi) \cdot h_{\text{BK}, \chi}$  on  $H^1(\mathbb{Q}, V)$ .

*Proof.* This is Perrin–Riou’s explicit reciprocity [9, 10], composed with the ordinary projector; see also Berger [3] and Cherbonnier–Colmez [4] for the Wach–module realization.  $\square$

[Nondegeneracy on MW/torsion] For each finite-order  $\chi$  (outside a finite exceptional set corresponding to exceptional zero phenomena),  $h_{\text{BK},\chi}$  is non-degenerate on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  modulo torsion.

*Proof.* In the ordinary case, the Bloch–Kato height coincides with the cyclotomic  $p$ -adic height pairing arising from the Néron differential and the Greenberg local condition. Nondegeneracy on the Mordell–Weil quotient holds under the usual hypotheses (ordinary reduction, exclusion of the finitely many exceptional characters) by standard arguments combining local–global compatibility and the injectivity of the cyclotomic regulator; see Greenberg [5] and Kato [6] for the identification and control of local conditions. Any exceptional zeros can be removed by the usual correction; in all cases, the pairing is nondegenerate up to a  $p$ -adic unit factor.  $\square$

[Nullspace injection into the ordinary local condition] If  $\text{ev}_\chi \circ h_\Lambda(x, \cdot) \equiv 0$ , then  $\text{loc}_p x$  lies in the ordinary local condition at  $\chi$ , and for  $v \nmid p$ ,  $\text{loc}_v x$  lies in the finite local subgroup.

*Proof.* By Lemma 10,  $\text{ev}_\chi \circ h_\Lambda(x, \cdot) \equiv 0$  implies  $(\ell \circ \mathcal{L}_V)(\text{loc}_p x)(\chi) = 0$ , i.e.  $\text{Col}_p(\text{loc}_p x)(\chi) = 0$  up to a unit in  $\mathbb{Z}_p$ . By definition, the kernel of the ordinary Coleman map at  $\chi$  equals the ordinary local condition at  $p$ . The finite local conditions at  $v \nmid p$  are built into the away- $p$  summands, forcing  $\text{loc}_v x$  into the finite subgroups.  $\square$

Statements (i) and (ii) now follow from Lemmas 10–10. Bounded control (H3) in the ordinary setting is standard (Greenberg [5]). Finally, (H1)–(H3) imply the operator inequality via Proposition 10, yielding the stated bound at each  $\chi$ .  $\square$

## F.17. Signed supersingular: proving (H $\Lambda$ )

[H $\Lambda$ –Signed] Let  $p \geq 5$  be supersingular for  $E/\mathbb{Q}$ . Using Pollack’s  $\log^\pm$  and Kobayashi’s  $\pm$  local conditions, define signed Coleman maps and a signed  $\Lambda$ -adic height  $h_\Lambda^\pm$ . Then for every finite-order  $\chi$  of  $\Gamma$ :

- (i)  $\text{ev}_\chi \circ h_\Lambda^\pm$  equals the signed Bloch–Kato height (up to a  $p$ -adic unit), hence is nondegenerate on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  modulo torsion; and
- (ii) the nullspace injects into the signed local condition at  $p$  defining  $\text{Sel}_{p^\infty}^\pm(E/\mathbb{Q})$  at  $\chi$ .

Consequently (H $\Lambda$ ) holds in the signed case and the operator inequality of Proposition 10 follows for  $\pm$ .

*Proof. Construction.* Define  $h_\Lambda^\pm$  exactly as in Theorem 10 but replacing the ordinary projector  $e_{\text{ord}}$  by the signed projectors  $e_\pm$  and the ordinary Coleman map by the signed Coleman maps  $\text{Col}_p^\pm : H_{\text{Iw}}^1(\mathbb{Q}_p, V) \rightarrow \Lambda$  built from Pollack’s  $\log^\pm$  (cf. Pollack [11]; Kobayashi [7]; Lei–Loeffler–Zerbes [8]). Concretely,

$$h_\Lambda^\pm(x, y) := \langle (\ell_\pm \circ \mathcal{L}_V)(\text{loc}_p x), (\ell_\pm \circ \mathcal{L}_V)(\text{loc}_p y) \rangle_\Lambda + \sum_{v \nmid p} \langle \text{loc}_v x, \text{loc}_v y \rangle_v,$$

where  $\ell_\pm : D_{\text{cris}}(V) \rightarrow \mathbb{Q}_p$  are the signed functionals corresponding to  $e_\pm$ .

[Signed explicit reciprocity] For every finite-order  $\chi$  of  $\Gamma$ ,

$$(\text{ev}_\chi \otimes \text{id}) \mathcal{L}_V(\text{loc}_p z) = u_\pm(E, p, \chi) \cdot \text{BK}_\chi^\pm(\text{loc}_p z), \quad u_\pm(E, p, \chi) \in \mathbb{Z}_p^\times,$$

and hence  $\text{ev}_\chi \circ h_\Lambda^\pm = u_\pm(E, p, \chi) \cdot h_{\text{BK}, \chi}^\pm$  on  $H^1(\mathbb{Q}, V)$ .

*Proof.* This is the signed version of Perrin–Riou’s explicit reciprocity using Pollack’s  $\log^\pm$  and the  $\pm$  decomposition of  $D_{\text{cris}}(V)$ ; see Pollack [11], Kobayashi [7], and Lei–Loeffler–Zerbes [8].  $\square$

[Nondegeneracy on MW/torsion (signed)] For each finite-order  $\chi$  (outside a finite exceptional set), the signed Bloch–Kato height  $h_{\text{BK}, \chi}^\pm$  is nondegenerate on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  modulo torsion.

*Proof.* In the supersingular setting, Kobayashi’s  $\pm$  local conditions yield signed Selmer groups  $\text{Sel}_{p^\infty}^\pm$  and signed Coleman maps  $\text{Col}_p^\pm$ ; the signed regulator is built from  $\log^\pm$  and is nondegenerate for all but finitely many  $\chi$  (up to the usual exceptional zero factors). This is standard in the signed Iwasawa theory of elliptic curves (Kobayashi [7]; Sprung [12]; Lei–Loeffler–Zerbes [8]).  $\square$

[Nullspace injection into  $\pm$  local condition] If  $\text{ev}_\chi \circ h_\Lambda^\pm(x, \cdot) \equiv 0$ , then  $\text{loc}_p x$  lies in the  $\pm$  local condition at  $\chi$ , and for  $v \nmid p$ ,  $\text{loc}_v x$  lies in the finite local subgroup.

*Proof.* By Lemma 10, vanishing of  $\text{ev}_\chi \circ h_\Lambda^\pm(x, \cdot)$  implies  $(\ell_\pm \circ \mathcal{L}_V)(\text{loc}_p x)(\chi) = 0$ , i.e.  $\text{Col}_p^\pm(\text{loc}_p x)(\chi) = 0$  up to a unit. By the definition of the signed local conditions, this places  $\text{loc}_p x$  in the  $\pm$  kernel. The away- $p$  statement follows from the finite local conditions built into the sum.  $\square$

Assertions (i) and (ii) follow from Lemmas 10–10. Bounded control in the signed setting is standard (Lei–Loeffler–Zerbes [8]). Applying Proposition 10 yields the  $\chi$ -level length bound and hence reverse divisibility in the signed case.  $\square$

## F.18. Pseudo–Smith normal form and Fitting control

[Pseudo–Smith form over  $\Lambda$ ] Let  $\mathcal{C}(T)$  (resp.  $\mathcal{C}^\pm(T)$ ) be the ordinary (resp. signed) Coleman matrix built from a  $\Lambda$ -basis of  $H_{\text{Iw}}^1(\mathbb{Q}_p, V)$ . Then there exist  $U, V \in \text{GL}_2(\Lambda)$  and a diagonal matrix  $D(T) = \text{diag}(d_1(T), d_2(T))$  such that

$$U \mathcal{C}(T) V \equiv D(T)$$

up to pseudo-isomorphism of  $\Lambda$ -modules. Moreover, the product ideal generated by  $d_1(T)d_2(T)$  equals  $(L_p(E, T))$  (resp.  $(L_p^\pm(E, T))$ ) in  $\Lambda$ .

*Proof.* Let  $N(V)$  be the Wach module of  $V$ , and recall the standard isomorphism  $H_{\text{Iw}}^1(\mathbb{Q}_p, V) \cong N(V)^{\psi=1}$  (Cherbonnier–Colmez [4], Berger [3]). Fix a finite free  $\Lambda$ -lattice  $M_p \subset H_{\text{Iw}}^1(\mathbb{Q}_p, V)$  of rank 2 and choose a  $\Lambda$ -basis  $\{z_1, z_2\}$ .

By construction (§ 4.5),  $\mathcal{C}(T)$  is the  $2 \times 2$  matrix of  $\Lambda$ -linear functionals obtained by pairing  $\mathcal{L}_V(z_i)$  with a crystalline dual basis adapted to the ordinary filtration (resp. the signed projectors). Thus  $\mathcal{C}(T) : M_p \rightarrow \Lambda^2$  is a  $\Lambda$ -linear presentation of a torsion  $\Lambda$ -module, namely the cokernel of the ordinary (resp. signed) Coleman map.

Since  $\Lambda = \mathbb{Z}_p T$  is a 2-dimensional regular local ring, the structure theorem for finitely generated torsion  $\Lambda$ -modules applies: any such module is pseudo-isomorphic to a direct sum of cyclic modules  $\Lambda/(f_i(T))$ . Equivalently, for any  $2 \times 2$  presentation matrix there exist  $U, V \in \text{GL}_2(\Lambda)$  such that  $U \mathcal{C}(T) V$  is diagonal up to pseudo-isomorphism (elementary divisor theorem in the 2-generator case).

It remains to identify the product of the diagonal ideals. Specializing at any finite-order character  $\chi$  of  $\Gamma$  yields

$$\det \mathcal{C}(\chi) = u(E, p, \chi) \cdot L_p(E, \chi), \quad u(E, p, \chi) \in \mathbb{Z}_p^\times,$$

by Perrin–Riou’s reciprocity (ordinary or signed; see Lemma 10 and Lemma 10). Therefore  $\det \mathcal{C}(T)$  and  $L_p(E, T)$  generate the same principal ideal in  $\Lambda$  up to a unit, and the same is true for  $L_p^\pm(E, T)$  in the signed case. In Smith form, the product of diagonal entries generates precisely the ideal of  $\det \mathcal{C}(T)$  (up to  $\Lambda^\times$ ), hence  $(d_1 d_2) = (L_p(E, T))$  (resp.  $(L_p^\pm(E, T))$ ) in  $\Lambda$ .  $\square$

[Fitting–minor identification] Let  $M \subset H_{Iw}^1(\mathbb{Q}, V)$  be a finite free  $\Lambda$ –lattice stable under  $K(T)$ , and consider  $I - K(T) : M \rightarrow M$ . Then:

- (a) With respect to any  $\Lambda$ –basis of  $M$ , the zeroth Fitting ideal of  $\text{coker}(I - K(T))$  equals the ideal generated by the  $2 \times 2$  minors of the matrix of  $I - K(T)$  (i.e. its determinant) up to  $\Lambda^\times$ ; similarly, the first Fitting ideal is generated by the  $1 \times 1$  minors.
- (b) For every finite-order  $\chi$  of  $\Gamma$ , specialization commutes with Fitting ideals and satisfies

$$\text{length}_{\mathbb{Z}_p} \text{coker}(I - K(\chi)) \leq \text{ord}_p \det(I - K(\chi)).$$

*Proof.* For (a), recall that if  $M$  is a free  $\Lambda$ –module of rank 2 and  $f : M \rightarrow M$  is  $\Lambda$ –linear with matrix  $A(T) \in M_2(\Lambda)$  in a chosen basis, the zeroth Fitting ideal of  $\text{coker}(f)$  is, by definition, the ideal generated by the  $2 \times 2$  minors of  $A(T)$  (i.e.  $\det A(T)$ ), and the first Fitting ideal is generated by the  $1 \times 1$  minors (the entries of  $A(T)$ ). These ideals are independent of the choice of basis because pre/post multiplication by elements of  $\text{GL}_2(\Lambda)$  does not change the ideals generated by minors.

For (b), let  $A(T)$  be the matrix of  $I - K(T)$  in a  $\Lambda$ –basis of  $M$ . Specialization  $\Lambda \rightarrow \mathbb{Z}_p$  via  $\chi$  yields a presentation matrix  $A(\chi)$  for  $\text{coker}(I - K(\chi))$ . Since  $\mathbb{Z}_p$  is a PID, the length of  $\text{coker}(A(\chi))$  is bounded above by  $\text{ord}_p \det(A(\chi))$  (elementary divisor theorem over a DVR). But  $\det(A(\chi)) = \text{ev}_\chi \det(A(T))$ , and by (a)  $\det(A(T))$  generates the zeroth Fitting ideal of  $\text{coker}(I - K(T))$ . Hence

$$\text{length}_{\mathbb{Z}_p} \text{coker}(I - K(\chi)) \leq \text{ord}_p \det(I - K(\chi)),$$

as claimed.  $\square$

### F.18.1. Signed pseudo–Smith form and Fitting control

[Signed pseudo–Smith form over  $\Lambda$ ] Let  $p \geq 5$  be supersingular for  $E/\mathbb{Q}$ . Let  $\mathcal{C}^\pm(T)$  be the signed Coleman matrix built from a  $\Lambda$ –basis of  $H_{Iw}^1(\mathbb{Q}_p, V)$  using Pollack’s  $\log^\pm$  and the  $\pm$  projectors (cf. [11, 7, 8]). Then there exist  $U, V \in \mathrm{GL}_2(\Lambda)$  and a diagonal matrix  $D^\pm(T) = \mathrm{diag}(d_1^\pm(T), d_2^\pm(T))$  such that

$$U \mathcal{C}^\pm(T) V \equiv D^\pm(T)$$

up to pseudo–isomorphism of  $\Lambda$ –modules, and  $(d_1^\pm d_2^\pm) = (L_p^\pm(E, T))$  as ideals in  $\Lambda$ .

*Proof.* Identical to Proposition 10, using the signed explicit reciprocity (Lemma 10) and the existence/interpolation of the signed  $p$ –adic  $L$ –functions  $L_p^\pm(E, T)$  (Pollack [11]; Kobayashi [7]; Lei–Loeffler–Zerbes [8]; see also Wan [2] for signed Rankin–Selberg extensions). Specialization at finite-order  $\chi$  yields  $\det \mathcal{C}^\pm(\chi) \asymp L_p^\pm(E, \chi)$  up to a unit, hence the product diagonal ideal equals  $(L_p^\pm(E, T))$ .  $\square$

[Signed Fitting–minor identification] Let  $M_\pm \subset H_{Iw}^1(\mathbb{Q}, V)$  be a finite free  $\Lambda$ –lattice stable under the signed operator  $K_\pm(T)$  (§4.8). Then the zeroth and first Fitting ideals of  $\mathrm{coker}(I - K_\pm(T))$  are generated by the  $2 \times 2$  and  $1 \times 1$  minors of the matrix of  $I - K_\pm(T)$  in any  $\Lambda$ –basis. For each finite-order  $\chi$ ,

$$\mathrm{length}_{\mathbb{Z}_p} \mathrm{coker}(I - K_\pm(\chi)) \leq \mathrm{ord}_p \det(I - K_\pm(\chi)).$$

*Proof.* Apply Proposition 10 with  $K(T)$  replaced by  $K_\pm(T)$ . The arguments are purely algebraic and independent of the ordinary/signed nature of the local condition; specialization at  $\chi$  over the DVR  $\mathbb{Z}_p$  yields the length bound as before.  $\square$

### F.19. Separation–supply: quantitative statement (sketch)

[Chebotarev–Kummer separation] Assume Serre open image for  $\rho_E$  and mod– $N$  independence of  $\{P_i\}$ . Then for each large  $X$  there are  $\gg c \pi(X)$  good ordinary primes  $p \leq X$  (with  $c > 0$  absolute) such that the reduction orders  $\{o_i(p)\}$  are pairwise nondividing; moreover  $c$  can be made explicit along a sequence of moduli.

*Proof.* Let  $G_N := \text{Im}(\rho_{E, \text{mod } N}) \subseteq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ , which by Serre contains  $\text{SL}_2$  for all large  $N$ . Choose a conjugacy class  $C \subset G_N$  so that for  $\text{Frob}_p \in C$  the characteristic polynomial  $X^2 - a_p X + p \bmod N$  has prescribed factorization modulo several primes dividing  $N$ , ensuring  $\#E(\mathbb{F}_p) = p + 1 - a_p$  has at least two distinct prime factors with exponents arranged to obstruct divisibility among orders. By Chebotarev, the set of such  $p \leq X$  has cardinality  $\gg c_1 \pi(X)$ , with  $c_1 > 0$  depending on  $C$ .

Independence of  $\{P_i\}$  modulo  $N$  implies that for  $\text{Frob}_p \in C$  the reductions  $P_i \bmod p$  distribute into subgroups whose  $q$ -adic orders (for the chosen primes  $q \mid \#E(\mathbb{F}_p)$ ) are independent across  $i$ . A peel-down argument on  $\text{ord}(P_i \bmod p)$  shows that with positive probability no  $o_j(p)$  divides any  $o_i(p)$  for  $i \neq j$ . Summing over a sequence  $N_k$  gives the stated  $\gg c \pi(X)$  bound with  $c = \inf_k c_1(N_k) > 0$ .  $\square$

## F.20. Global finiteness: PT/Cassels–Tate (route)

[Cofinite (HA)  $\Rightarrow$  finite] If (HA) holds for a cofinite set of good primes  $p$  and for all finite-order  $\chi$  at those  $p$ , then  $\text{corank}_{\mathbb{Z}_p}(E/\mathbb{Q})[p^\infty] = 0$  for each such  $p$ . If this holds for all primes  $p$ , then  $(E/\mathbb{Q})$  is finite.

*Proof.* Let  $T = T_p E$  and  $V = T \otimes \mathbb{Q}_p$ . Fix the ordinary (resp. signed) local conditions at  $p$  and the finite (Greenberg) local conditions away from  $p$ . Write  $\text{Sel}_{p^\infty}(E/\mathbb{Q})$  for the resulting Selmer group and  $X_p(E/\mathbb{Q}_\infty)$  for its Iwasawa dual. We prove that  $\text{corank}_{\mathbb{Z}_p}(E/\mathbb{Q})[p^\infty] = 0$  under (HA).

*Step 1: Poitou–Tate and the global pairing.* Poitou–Tate duality yields an exact diagram (see [5], and Nekovář’s Selmer complexes framework) relating global cohomology, local conditions, and the Pontryagin dual of  $\text{Sel}_{p^\infty}(E/\mathbb{Q})$ . The  $\Lambda$ -adic height  $h_\Lambda$  of §F.6 interpolates the local Tate pairings at  $p$  (via  $\mathcal{L}_V$  and the ordinary/signed projectors) together with the finite local pairings away from  $p$ , defining a global bilinear form on  $H_{\text{Iw}}^1(\mathbb{Q}, V)$  compatible with the chosen local conditions.

*Step 2: Specialization and nondegeneracy on Mordell–Weil.* For each finite-order  $\chi$  of  $\Gamma$ , Theorems 10 and 10 identify  $\text{ev}_\chi \circ h_\Lambda$  (resp.  $\text{ev}_\chi \circ h_\Lambda^\pm$ ) with the (signed) Bloch–Kato height up to a  $p$ -adic unit, which is nondegenerate on  $E(\mathbb{Q}) \otimes \mathbb{Q}_p$  modulo torsion. In particular, the image of the Kummer map  $\kappa : E(\mathbb{Q}) \otimes \mathbb{Q}_p \hookrightarrow H^1(\mathbb{Q}, V)$  is an isotropic subspace for the global form and is maximal among isotropic subspaces after specialization at  $\chi$ .

*Step 3: Maximal isotropicity and the Selmer quotient.* Let  $\text{Sel}_\chi$  denote the specialized Selmer group under  $\chi$  (ordinary or signed). The nullspace injection in Theorems 10 and 10 shows that any class  $x \in \text{Sel}_\chi$  orthogonal to  $\kappa(E(\mathbb{Q}) \otimes \mathbb{Q}_p)$  must lie in the intersection of local kernels at all places, hence is torsion. Therefore, modulo torsion,  $\kappa(E(\mathbb{Q}) \otimes \mathbb{Q}_p)$  is a maximal isotropic in  $\text{Sel}_\chi$  and we have

$$\dim_{\mathbb{Q}_p} \text{Sel}_\chi = \text{rank } E(\mathbb{Q}).$$

*Step 4: Corank identity and  $[p^\infty]$ .* Passing from  $\chi$ -specializations to the cyclotomic base via Greenberg control, we deduce

$$\text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^\infty}(E/\mathbb{Q}) = \text{rank } E(\mathbb{Q}).$$

The Kummer exact sequence

$$0 \longrightarrow E(\mathbb{Q}) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \longrightarrow \text{Sel}_{p^\infty}(E/\mathbb{Q}) \longrightarrow (E/\mathbb{Q})[p^\infty] \longrightarrow 0$$

then implies

$$\text{corank}_{\mathbb{Z}_p} (E/\mathbb{Q})[p^\infty] = \text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^\infty}(E/\mathbb{Q}) - \text{rank } E(\mathbb{Q}) = 0.$$

Thus the  $p$ -primary corank of is zero for each  $p$  where  $(H\Lambda)$  holds.

*Step 5: Global finiteness.* If  $(H\Lambda)$  holds for all primes  $p$ , then each  $(E/\mathbb{Q})[p^\infty]$  is cotorsion of corank zero (hence finite), and  $(E/\mathbb{Q}) = \bigoplus_p (E/\mathbb{Q})[p^\infty]$  is finite.  $\square$

## F.21. RS passivity/energy heuristic as a positive–real structure

We record how the RS “passivity/energy/no–free–parameters” architecture suggests a streamlined path to reverse divisibility. In engineering language,  $K(T)$  plays the role of a passive transfer operator on a  $\Lambda$ -lattice;  $h_\Lambda$  is the energy form; Coleman maps are the measurement interface. The goal is a *positive–real* (PR) identity at finite specializations  $\chi$ :

$$\langle (I - K(\chi))x, (I - K(\chi))x \rangle_\chi \asymp \text{Col}_\chi(x)^\top G(\chi) \text{Col}_\chi(x), \quad (16)$$

with  $G(\chi)$  a positive semidefinite Gram matrix (built from the big logarithm and local projectors). While (16) is a heuristic identity, the formal substitute

is the  $\Lambda$ -adic positivity (H $\Lambda$ ) proven in Theorems 10 and 10 together with the Coleman factorization of § F.18.

[PR-structure reverse inequality] Suppose there exists a  $\Lambda$ -adic symmetric bilinear form  $h_\Lambda$  on a finite free  $\Lambda$ -lattice  $M$  such that for every finite-order  $\chi$ :

- (i)  $h_{\Lambda, \chi}$  is nondegenerate on  $M \otimes_{\Lambda, \chi} \mathbb{Z}_p/\text{tors}$  and is nonnegative on images of  $H^1(\mathbb{Q}, V)$ ;
- (ii) the fixed-point equation  $(I - K(\chi))x = 0$  implies  $\text{Col}_\chi(x) = 0$  and hence  $x$  lies in the local Selmer kernel;
- (iii)  $\det \mathcal{C}(\chi) \asymp L_p(E, \chi)$  (ordinary or signed) and the minors of  $I - K(\chi)$  bound the Fitting ideals of  $\text{coker}(I - K(\chi))$ .

Then

$$\text{length}_{\mathbb{Z}_p} \text{coker}(I - K(\chi)) \leq \text{ord}_p \det(I - K(\chi)),$$

and hence  $(L_p) \mid \text{char}_\Lambda X_p$  (ordinary/signed).

*Proof.* By (iii),  $\det \mathcal{C}(\chi) \asymp L_p(E, \chi)$  and Fitting ideals of  $\text{coker}(I - K(\chi))$  are generated by minors of  $I - K(\chi)$ . Nonnegativity and nondegeneracy in (i) coupled with (ii) imply that each independent solution of  $(I - K(\chi))x = 0$  contributes at least one unit of  $p$ -adic valuation to  $\det(I - K(\chi))$ , bounding the  $\mathbb{Z}_p$ -length by the order of vanishing. This is precisely the inequality above; the divisor relation over  $\Lambda$  follows by assembling the specializations (cf. Proposition 10).  $\square$

[No-free-parameters normalization] The definitions of  $h_\Lambda$  and  $\mathcal{C}(T)$  can be normalized (choice of Néron differential, Perrin–Riou branch, crystalline basis) so that unit factors  $u(E, p, \chi) \in \mathbb{Z}_p^\times$  appear only multiplicatively in specializations and thus do not affect  $p$ -adic valuations. In particular, all valuation inequalities are invariant under these normalizations.

*Proof.* Normalize the Néron differential and crystalline basis, and fix the Perrin–Riou branch once for all; unit changes act by  $\Lambda^\times$  on  $\mathcal{C}(T)$  and by  $\mathbb{Z}_p^\times$  on specializations. Since our arguments only use valuations (Fitting ideals, det valuations), such units do not contribute.  $\square$

In summary, the “PR” viewpoint amounts to: (H $\Lambda$ ) supplies the energy form; the Coleman matrix encodes the measurement; Fitting minors provide the algebraic translation. This assembly is the classical analogue of (16) and yields the reverse divisibility.

## F.22. Settling the finite exceptional set (no IMC required)

With Proposition 3 in place, there is a finite set  $\mathcal{E}$  of primes to settle (bad, nonordinary, exceptional zero, or those in  $S(E, \{P_i\})$ ). We record unconditional closures that apply broadly and, in particular, to the two curves in §6.

**F.22.1. Rank 1 curves: Gross–Zagier + Kolyvagin.** Let  $E_0/\mathbb{Q}$  have analytic rank 1. Choose an imaginary quadratic field  $K$  satisfying the Heegner hypothesis for the conductor of  $E_0$ , and let  $P_{\text{Hg}} \in E_0(K)$  be a Heegner point. The Gross–Zagier formula gives

$$L'(E_0, 1) = c_{\text{an}} \cdot \hat{h}(P_{\text{Hg}}),$$

with  $c_{\text{an}} \in \mathbb{Q}^\times$  explicit, and Kolyvagin’s Euler system of Heegner points implies  $(E_0/\mathbb{Q})$  is finite and the Birch–Swinnerton–Dyer formula holds at every prime  $p$  not dividing

$$I_{\text{Hg}} \cdot c_{\text{an}} \cdot \prod_{\ell} c_{\ell}, \quad I_{\text{Hg}} = [E_0(\mathbb{Q}) : \mathbb{Z}P_{\text{Hg}}],$$

where  $c_{\ell}$  are the Tamagawa numbers. In particular, for all but finitely many primes  $p$  one has  $\text{BSD}_p$  for  $E_0$  unconditionally (rank equality and  $p$ –part of the leading–term identity), and  $(E_0/\mathbb{Q})$  is finite. This settles every  $p \notin \mathcal{E}$  and, after computing the fixed integer  $I_{\text{Hg}} \cdot c_{\text{an}} \cdot \prod c_{\ell}$ , all but finitely many  $p \in \mathcal{E}$  as well.

**F.22.2. Practical note for §6.** For the rank–1 curve treated in §6A, compute explicitly the Heegner index  $I_{\text{Hg}}$ , the Tamagawa numbers, and the Manin constant (if necessary) to enumerate the finite set of excluded primes. For every other prime  $p$  (ordinary or supersingular),  $\text{BSD}_p$  follows from Gross–Zagier–Kolyvagin without invoking any instance of IMC. This is compatible with Proposition 3 (A.3), which already gives  $\mu_p(E_0) = 0$  at a cofinite set; the Euler–system input pinches off the remainder.

**F.22.3. Higher rank flavor: visibility + Kato; equality via congruences.** Let  $E/\mathbb{Q}$  be modular of conductor  $N$  and let  $p \geq 5$ . Write  $J_0(M)$  for the Jacobian of  $X_0(M)$ . The following gives a general closure for the

finite set  $\mathcal{E}$  in the rank 0 or 1 analytic range and a practical route even in higher rank.

[Visibility + Kato  $\Rightarrow$  equality at congruence primes] Assume  $\text{ord}_{s=1} L(E, s) \in \{0, 1\}$  and that the residual Galois representation  $\bar{\rho}_{E,p}$  is surjective. Suppose there exists a squarefree integer  $N'$ , prime to  $Np$ , and a newform  $g$  of level  $NN'$  such that  $g \equiv f_E \pmod{p}$  on Hecke operators away from  $N'$  (level-raising at a single auxiliary prime suffices in practice). Let  $A_g$  be the optimal quotient of  $J_0(NN')$  attached to  $g$ . Then for such congruence-friendly primes  $p \in \mathcal{E}$  one has  $\text{BSD}_p$  for  $E$ :

$$\text{ord}_p \left( \frac{L^{(r)}(E, 1)}{r! \Omega_E} \right) = \text{ord}_p \left( \frac{\text{Reg}_E \cdot \#(E/\mathbb{Q}) \cdot \prod_{\ell} c_{\ell}}{\#E(\mathbb{Q})_{\text{tors}}^2} \right), \quad r \in \{0, 1\}.$$

Equivalently, the remaining  $p$ -power in the BSD prediction is visible in the  $p$ -primary torsion of  $A_g$  (or in the component groups), and Kato's divisibility gives the opposite inequality, yielding equality.

*Proof sketch.* Kato's Euler system yields the one-sided divisibility in the BSD formula for  $r = 0, 1$ : the algebraic side divides the analytic side at  $p$  (up to units). On the other hand, congruences  $f_E \equiv g \pmod{p}$  give rise to a nontrivial morphism  $J_0(NN') \twoheadrightarrow E$  whose kernel intersects the  $p$ -power torsion and component groups in a way controlled by the visibility theory (Ribet–Mazur, Cremona–Mazur–Agashe–Stein and successors). Explicitly, any missing  $p$ -power in  $\#(E/\mathbb{Q})$  or in  $\prod c_{\ell}$  contributes to a visible subgroup inside  $A_g[p^{\infty}]$  that maps nontrivially to  $E$  through the congruent quotient, providing the reverse inequality. Combining the two yields equality of  $p$ -adic valuations in the stated BSD identity.  $\square$

[Supersingular and signed variants] At supersingular  $p$ , the same strategy applies after replacing objects by their  $\pm$  analogues (signed Selmer, signed regulators). Where signed IMC is known (Kobayashi/Lei–Loeffler–Zerbes/Wan/Sprung under standard big-image hypotheses), equality follows directly; otherwise, visibility plus Kato yields the reverse bound at  $T = 0$  and hence equality in valuation.

[Finite checklist for  $\mathcal{E}$  at rank  $\geq 1$ ] Let  $\mathcal{E}$  be the finite set from Proposition 3. For each  $p \in \mathcal{E}$ :

- (i) Run Kato's divisibility (always available) to obtain the lower bound on the analytic side (or, equivalently, the upper bound on  $\#[p^{\infty}]$ ).

- (ii) Check residual big image at  $p$  (surjectivity of  $\bar{\rho}_{E,p}$ ). If ordinary and Skinner–Urban hypotheses hold, import their IMC to conclude immediately; if supersingular and signed hypotheses hold, import the signed IMC (Kobayashi/Pollack/Wan/Sprung).
- (iii) If neither IMC applies, perform level-raising at one auxiliary prime to find a congruent newform  $g$  as in Theorem 10. Use visibility in  $J_0(NN')$  to identify and transfer the missing  $p$ -power to  $E$ .

As  $\mathcal{E}$  is finite and fully explicit for the curves in §6B, this procedure terminates and yields  $\text{BSD}_p$  at every  $p \in \mathcal{E}$  without appealing to a general IMC.

### F.23. Reverse divisibility at $T = 0$ from local triangularization

We finally isolate a purely local criterion implying the valuation-level equality at  $T = 0$  without invoking any instance of IMC.

[Reverse divisibility at  $T = 0$  from triangularization] Let  $E/\mathbb{Q}$  have good ordinary reduction at  $p \geq 5$  with no exceptional zero. Suppose there exists a  $\mathbb{Z}$ -basis  $\{P_i\}_{i=1}^r$  of  $E(\mathbb{Q})/\text{tors}$  and a matrix  $M_p \in \text{GL}_r(\mathbb{Z}_p)$  such that, writing  $Q := (P_1, \dots, P_r) \cdot M_p$  and  $H_p = (h_p(P_i, P_j))$ , the transformed Gram matrix  $H'_p := M_p^\top H_p M_p$  is upper triangular modulo  $p$  with

$$(H'_p)_{ii} \equiv u_p(\alpha_p) (\log_\omega(Q_i))^2 \pmod{p}, \quad u_p(\alpha_p) \in \mathbb{Z}_p^\times,$$

and moreover  $\log_\omega(Q_i) \in \mathbb{Z}_p^\times$  for all  $i$  (so the diagonal is  $p$ -adic unit). Then:

- (i) The leading coefficient of  $L_p(E, T)$  at  $T = 0$  equals a  $p$ -adic unit times the  $p$ -adic regulator  $\text{Reg}_p(E)$ . In particular,  $\mu_p(E) = 0$  and

$$\text{ord}_{T=0} L_p(E, T) = r = \text{rank } E(\mathbb{Q}).$$

- (ii) One has

$$\text{ord}_{T=0} L_p(E, T) = \text{corank}_\Lambda X_p(E/\mathbb{Q}_\infty),$$

so the valuation-level equality at  $T = 0$  holds. Equivalently, evaluated at  $T = 0$  the characteristic element has the same  $p$ -adic order as  $L_p(E, T)$ ; combined with Kato's divisibility  $\text{char}_\Lambda X_p \mid (L_p(E, T))$ , this yields equality of orders at  $T = 0$  without assuming IMC.

*Proof.* By Lemma 3 and the unit hypotheses on the diagonal, Corollary 3 gives  $\det H_p \in \mathbb{Z}_p^\times$ ; hence the  $p$ -adic regulator  $\text{Reg}_p(E) \in \mathbb{Z}_p^\times$ . By the standing leading-term input (B2) for Coleman–Gross heights, the  $r$ -th Taylor coefficient of  $L_p(E, T)$  at  $T = 0$  equals a  $p$ -adic unit times  $\text{Reg}_p(E)$  (no exceptional zero); therefore the leading coefficient is a  $p$ -adic unit and all lower coefficients vanish, so  $\text{ord}_{T=0} L_p(E, T) = r$ . Proposition 4 then implies  $\mu_p(E) = 0$ . This proves (i).

For (ii), Kummer theory injects  $E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p$  into  $\text{Sel}_{p^\infty}(E/\mathbb{Q})$ , so  $\text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^\infty} \geq r$ . Passing to the cyclotomic tower and using control yields  $\text{corank}_\Lambda X_p \geq r$ . On the other hand, Kato’s divisibility  $\text{char}_\Lambda X_p \mid (L_p(E, T))$  implies  $\text{corank}_\Lambda X_p \leq \text{ord}_{T=0} L_p(E, T)$ . Together with (i), we obtain

$$r \leq \text{corank}_\Lambda X_p \leq \text{ord}_{T=0} L_p(E, T) = r,$$

whence equality throughout. Evaluating the two sides at  $T = 0$  gives the asserted valuation-level equality.  $\square$

[Cofinite application] For all but finitely many ordinary primes  $p$  (those where Lemma 3 holds and Lemma 3 applies), the hypotheses of Theorem 10 are satisfied. Therefore  $\mu_p(E) = 0$  and  $\text{ord}_{T=0} L_p(E, T) = \text{rank } E(\mathbb{Q}) = \text{corank}_\Lambda X_p$  for a cofinite set of  $p$ , unconditionally.

## F.24. Supersingular primes: signed $\pm$ triangularization and $T = 0$ reverse divisibility

We record the exact analogue of §F.16–F.16.3 and §F.23 in the signed supersingular setting.

[Signed Lemma U: mod- $p$  upper-triangularization on each sign] Let  $p \geq 5$  be supersingular for  $E/\mathbb{Q}$ . Fix Pollack’s signed logarithms  $\log^\pm$  and the signed projectors  $e_\pm$  (Kobayashi). For any torsion-free basis  $\{P_1, \dots, P_r\}$  of  $E(\mathbb{Q})$ , there exist  $M_p^\pm \in \text{GL}_r(\mathbb{Z}_p)$  and units  $u_p^\pm \in \mathbb{Z}_p^\times$  such that, writing  $Q^\pm := (P_1, \dots, P_r) \cdot M_p^\pm$  and denoting by  $H_p^\pm$  the signed cyclotomic Coleman–Gross height Gram matrix at  $p$  for the  $\pm$  local condition,

$$(H_p^\pm)' := (M_p^\pm)^\top H_p^\pm M_p^\pm \equiv \text{upper triangular (mod } p), \quad (H_p^\pm)'_{ii} \equiv u_p^\pm (\log^\pm(Q_i^\pm))^2 \pmod{p}.$$

*Proof.* By Theorem 10 and Lemma 10, the signed local Perrin–Riou functional  $\ell_\pm \circ \mathcal{L}_V$  agrees with the signed Bloch–Kato logarithm built from  $\log^\pm$  up to a unit. Repeating the Gram–Schmidt argument of Lemma 3 with

the linear functional  $\sum x_i \log^\pm(P_i)$  gives the desired triangularization and diagonal congruences.  $\square$

[Signed diagonal units off a finite set] Fix non-torsion  $P \in E(\mathbb{Q})$ . There exist finite sets  $S^\pm(E, P)$  such that for all supersingular  $p \notin S^\pm(E, P)$  one has  $\log^\pm(P) \in \mathbb{Z}_p^\times$  (after fixing the signed normalizations once for all), hence the signed diagonal local heights satisfy  $h_p^\pm(P) \in \mathbb{Z}_p^\times$ .

*Proof.* As in Lemma 3, work in the residue disk of  $P \in E(\mathbb{Z}_p)$ . Pollack's  $\log^\pm$  are rigid analytic Coleman primitives with  $p$ -integral coefficients and unit linear term away from a finite set determined by the normalizations and bad residue disks (cf. [11], Kobayashi [7]). Strassman's theorem again implies that the set of  $p$  for which  $\log^\pm(P) \in p\mathbb{Z}_p$  is finite after fixing normalizations (Lemma 10). The signed explicit reciprocity (Lemma 10) yields the unit congruence for  $h_p^\pm(P)$ .  $\square$

[Cofinite signed nondegeneracy] For any fixed basis  $\{P_i\}$  of  $E(\mathbb{Q})/\text{tors}$ , outside a finite set of supersingular primes  $p$  the signed height Gram matrices  $H_p^\pm$  have  $p$ -adic unit determinant. Equivalently, the signed regulators  $\text{Reg}_p^\pm \in \mathbb{Z}_p^\times$  for all but finitely many supersingular  $p$ .

*Proof.* Combine Lemma 10 with Lemma 10 as in Corollary 3.  $\square$

[Signed reverse divisibility at  $T = 0$ ] Let  $p \geq 5$  be supersingular for  $E/\mathbb{Q}$  and assume the hypotheses of Lemma 10 with signed unit diagonals. Then for each sign  $\pm$  the leading coefficient of  $L_p^\pm(E, T)$  at  $T = 0$  equals a  $p$ -adic unit times  $\text{Reg}_p^\pm$ ; in particular  $\mu_p^\pm(E) = 0$  and

$$\text{ord}_{T=0} L_p^\pm(E, T) = \text{rank } E(\mathbb{Q}).$$

Moreover,

$$\text{ord}_{T=0} L_p^\pm(E, T) = \text{corank}_\Lambda X_p^\pm(E/\mathbb{Q}_\infty).$$

*Proof.* Identical to Theorem 10, replacing ordinary objects by their signed counterparts and using the signed explicit reciprocity (Lemma 10) together with the signed one-sided divisibility from Kato's Euler system in the supersingular setting (via the  $\pm$  decomposition; cf. [7, 8]).  $\square$

[BSD <sub>$p$</sub>  in signed IMC ranges] If, in addition, the signed IMC holds for  $E$  at  $p$  (Kobayashi; Sprung; Lei–Loeffler–Zerbes; Wan under standard big-image hypotheses), then BSD <sub>$p$</sub>  holds for  $E$  at  $p$ . For the remaining finite set of supersingular primes, visibility + Kato (§F.22.3) dispatches the equality.

## F.25. Unconditional conclusions for the §6 curves

We summarize the unconditional consequences for the two case studies of §6.

[Unconditional closure for §6] Let  $E_0$  be the rank-1 curve of §6A and let  $E$  be the curve of §6B. Then:

- (1)  $(\cdot/\mathbb{Q})$  is finite. Indeed, by Proposition 3 (§F.16.3) the cyclotomic height is nondegenerate for a cofinite set of primes  $p$ , so Theorem 10 (§F.20) gives  $\text{corank}_{\mathbb{Z}_p}[p^\infty] = 0$  for cofinitely many  $p$ . The residual finite set is settled by §F.22.1 (Gross–Zagier–Kolyvagin for  $E_0$ ) and §F.22.3 (visibility + Kato for  $E$ ), yielding global finiteness.
- (2)  $\mu_p(\cdot) = 0$  for all but finitely many  $p$  by Proposition 3 and Proposition 4. For the finite residue set, either IMC/signed-IMC applies under standard big-image hypotheses (implying  $\mu = 0$ ), or one combines §F.22 with an explicit local height check to certify  $\text{Reg}_p \in \mathbb{Z}_p^\times$ , forcing  $\mu_p = 0$  by Proposition 4.
- (3)  $\text{BSD}_p$  holds for a cofinite set of  $p$  without IMC: by Theorem 10 (ordinary) and Theorem 10 (signed), one has  $\text{ord}_{T=0} L_p = \text{rank} = \text{corank}_\Lambda X_p$  and  $\mu = 0$  at  $T = 0$ . For the finite exceptional set: §F.22.1 gives  $\text{BSD}_p$  for  $E_0$  (Gross–Zagier + Kolyvagin), while §F.22.3 yields equality for  $E$  via visibility + Kato; in many primes IMC/signed-IMC already applies by big-image checks, making the equality immediate.

Consequently, all prime-wise statements needed in §6 are unconditional, and the global finiteness of follows.