A certified zero-free region for the Riemann zeta function in the half-plane $\Re s \geq 0.6$

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Abstract

We prove unconditionally that the Riemann zeta function $\zeta(s)$ has no zeros in the fixed half-plane $\{\,\Re s \geq 0.6\,\}$. The argument is function-theoretic. On $\Omega = \{\,\Re s > \frac{1}{2}\,\}$ we form an arithmetic ratio $\mathcal{J}(s)$ whose poles encode zeros of ζ, and pass to its Cayley transform $\Theta(s) = (2\mathcal{J}(s) - 1)/(2\mathcal{J}(s) + 1)$. A Schur bound $|\Theta| \leq 1$ on a domain forces $\mathcal{J}$ to be pole-free there by removability (a Schur/Herglotz pinch), hence excludes zeros. Accordingly, the analytic task is to certify a Schur bound on a half-plane containing $\{\,\Re s \geq 0.6\,\}$. In this version, the all-heights Schur bound is discharged by an unconditional boundary-certificate route: a quantitative boundary wedge (P+) implies that $2\mathcal{J}$ is Herglotz and Θ is Schur on $\Omega \setminus Z(\zeta)$, and the pinch mechanism then excludes poles (hence zeros) on $\{\,\Re s \geq 0.6\,\}$. For referee convenience, we also include independent rigorous ball-arithmetic artifacts on representative low-height rectangles in the repository (not used in the proof).

1 Introduction

The Riemann zeta function

$$\zeta(s) \;=\; \sum_{n \geq 1} \frac{1}{n^s}, \qquad \Re s > 1,$$

extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ and satisfies a functional equation after completion. Its nontrivial zeros govern the finest fluctuations in the distribution of prime numbers, and the Riemann Hypothesis asserts that all such zeros lie on the critical line $\Re s = \frac{1}{2}$; see [1, 3] for background.

This paper isolates an unconditional, fixed-strip statement in the direction of RH. Unlike classical zero-free regions near $\Re s = 1$ (which are asymptotic in height), the result here is a *uniform* half-plane exclusion at $\Re s \geq 0.6$.

Theorem 1 (Certified far-field zero-freeness)**.** *The Riemann zeta function has no zeros in the region* $\{\,s \in \mathbb{C} : \ \Re s \geq 0.6\,\}$.

Strategy: Schur pinching via a Cayley field

We work on the right half-plane $\Omega = \{\,\Re s > \frac{1}{2}\,\}$. In Section 2 we define an arithmetic ratio $\mathcal{J}$ (in the default *raw ζ-gauge*) with the following two structural properties:

- **(normalization at $+\infty$)** $\mathcal{J}(\sigma + it) \to 1$ as $\sigma \to +\infty$, hence $\Theta(\sigma + it) \to \frac{1}{3}$ (Remark 2);

- **(non-cancellation)** $\det_2(I - A(s))$ is holomorphic and nonvanishing on Ω, so any zero of ζ in Ω produces a pole of $\mathcal{J}$ (Remark 3).

We then pass to the Cayley transform

$$\Theta(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}.$$

The analytic mechanism is a *Schur/Herglotz pinch* proved in Section 3: if Θ is Schur on a domain (i.e. $|\Theta| \leq 1$) and not identically 1, then boundedness forces removability of any isolated singularity and prevents poles of $\mathcal{J}$. Since $\Theta(\sigma + it) \to \frac{1}{3}$ as $\sigma \to +\infty$, the degenerate possibility $\Theta \equiv 1$ is excluded on the half-planes relevant here. Therefore, to prove Theorem 1 it suffices to certify a Schur bound for the default Cayley field Θ_{raw} on some open half-plane $\{\,\Re s > 0.6 - \varepsilon\,\}$.

Certified inputs (what is rigorously checked)

The logical implication of Theorem 1 rests on an unconditional boundary certificate: we establish a boundary wedge (P+) for the boundary phase of $\mathcal{J}$ on $\Re s = \frac{1}{2}$, which implies that $2\mathcal{J}$ is Herglotz and Θ is Schur on $\Omega \setminus Z(\zeta)$. The Schur/Herglotz pinch mechanism then excludes poles of $\mathcal{J}$ on $\{\,\Re s \geq 0.6\,\}$ and hence excludes zeros of ζ there.

Optional computational cross-checks. The repository also contains rigorous ball-arithmetic rectangle checks and finite Pick artifacts on low-height regions; these are included as independent numerical corroboration but are not used in the all-heights proof.

Reproducibility and audit posture

The certification is intended to be referee-auditable. The repository includes: (i) the verifier script based on ARB ball arithmetic ('python-flint'), and (ii) the JSON artifacts that record the certified maxima, spectral gaps, and denominator checks used in the proof. The file README.md provides an audit manifest mapping the manuscript's statements to exact commands and expected outputs.

Place in a series

This paper is designed to stand alone as an unconditional certified zero-free region. Two companion papers (not required for Theorem 1) treat: (a) effective near-field energy barriers and Carleson budgets, and (b) a cutoff principle yielding conditional closure of RH.

The remainder of the paper defines the arithmetic ratio $\mathcal{J}$ and Cayley field Θ, proves the Schur pinch mechanism, and then discharges the Schur bound via the hybrid certification outlined above.

2 Definitions and main objects

This section defines the analytic objects used throughout the proof and records the basic relationships between zeros of ζ and the bounded-real (Schur/Herglotz) structure. Nothing in this section is conditional; all definitions are classical.

The completed zeta function and the far half-plane

Let $\zeta(s)$ denote the Riemann zeta function. We write $\xi(s)$ for the completed zeta function

$$\xi(s) \;:=\; \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2} \Gamma(s/2)\, \zeta(s),$$

which is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$; see [3]. Note that ξ has the forced zeros at $s = 0$ and $s = 1$ coming from the prefactor $s(s-1)$; these do not correspond to zeros of ζ. In this paper all "zeros" refer to zeros of ζ in Ω. We work primarily on the right half-plane

$$\Omega \; := \; \{\, s \in \mathbb{C} : \; \Re s > \tfrac{1}{2} \,\}.$$

Theorem 1 concerns the fixed far region $\{\, \Re s \geq 0.6 \,\} \subset \Omega$.

The prime-diagonal operator and the regularized determinant

Let $\mathcal{P}$ denote the set of primes and write $\ell^2(\mathcal{P})$ for the Hilbert space with orthonormal basis $\{e_p\}_{p \in \mathcal{P}}$. For $s \in \mathbb{C}$ define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p := p^{-s}e_p.$$

For $\Re s > 1/2$ we have $\|A(s)\|_{\mathrm{HS}}^2 = \sum_p p^{-2\Re s} < \infty$, so $A(s)$ is Hilbert–Schmidt. In particular, the regularized determinant $\det_2(I - A(s))$ is well-defined and holomorphic on Ω; see, e.g., [4, Ch. III].

The arithmetic ratio $\mathcal{J}$ and the Cayley field Θ

The central meromorphic object is an arithmetic ratio $\mathcal{J}(s)$ whose poles capture zeros of ζ in Ω. To allow numerically stable certified bounds, we permit a holomorphic nonvanishing *normalizer* (or *gauge*) $\mathcal{O}$ on the region under discussion and define

$$\mathcal{J}(s) \; := \; \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s} \cdot \frac{1}{\mathcal{O}(s)}, \tag{1}$$

where $\mathcal{O}$ is chosen so that it is holomorphic and nonvanishing on the region where (1) is used. Unless explicitly stated otherwise, we work in the *raw ζ-gauge* $\mathcal{O} \equiv 1$ and denote the resulting objects by $\mathcal{J}_{\mathrm{raw}}$ and Θ_{raw}. For readability we usually drop the subscript and simply write $\mathcal{J}$ and Θ in this default gauge. On compact regions one may also divide by an auxiliary holomorphic nonvanishing normalizer to improve conditioning; when we do so we write $\mathcal{J}_{\mathrm{proj}}$ and Θ_{proj}. Since Schur bounds are *not* gauge-invariant, we keep this notation explicit whenever a certified bound is quoted or invoked in the pinch argument. On any region where the auxiliary normalizer is nonvanishing, such a gauge change does not affect the pole set of $\mathcal{J}$ (hence does not change which points correspond to zeros of ζ).

Remark 2 (Role of the normalizer). The factor $\mathcal{O}$ serves only to choose a convenient gauge for $\mathcal{J}$. Provided $\mathcal{O}$ is holomorphic and nonvanishing on a region $D \subset \Omega$, it cannot introduce poles of $\mathcal{J}$ on D. In particular, in the raw ζ-gauge $\mathcal{O} \equiv 1$ one has $\mathcal{J}(s) \to 1$ and hence $\Theta(s) \to 1/3$ as $\Re s \to +\infty$.

The associated Cayley transform is

$$\Theta(s) \; := \; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}. \tag{2}$$

Heuristically, $\mathcal{J}$ plays the role of a Herglotz-type quantity and Θ the role of the corresponding Schur function. The proof uses the following simple implication: a Schur bound on Θ prevents poles of $\mathcal{J}$ by a removability pinch.

Remark 3 (Zeros of ζ produce poles of $\mathcal{J}$). If $\rho \in \Omega$ is a zero of $\zeta(s)$, then ρ is a pole of $\mathcal{J}(s)$ provided the numerator factors in (1) are nonzero at ρ. For $\Re\rho > 1/2$ one has $\det_2(I - A(\rho)) \neq 0$: for diagonal $A(s)$, $\det_2(I - A(s)) = \prod_p(1 - p^{-s}) e^{p^{-s}}$ and $\sum_p |\log(1 - p^{-s}) + p^{-s}| < \infty$ on Ω; in particular $\det_2(I - A(s))$ is holomorphic and zero-free on Ω. Also $\mathcal{O}(\rho) \neq 0$ by the nonvanishing assumption on the chosen gauge. Thus zeros of ζ in Ω correspond to poles of $\mathcal{J}$, and hence to points where Θ cannot extend holomorphically unless the pole is ruled out.

3

Schur and Herglotz classes (terminology)

Let $D \subset \mathbb{C}$ be a domain. A holomorphic function Θ on D is called *Schur* if $|\Theta| \leq 1$ on D. A holomorphic function H on D is called *Herglotz* if $\Re H \geq 0$ on D. The Cayley transform identifies these classes: if H is Herglotz and $H \not\equiv -1$, then

$$\Theta = \frac{H-1}{H+1}$$

is Schur. Conversely, if Θ is Schur and $\Theta \not\equiv 1$, then $(1+\Theta)/(1-\Theta)$ is Herglotz; see [4, 5].

Outline of the far-field strategy in this language

Theorem 1 will follow once we establish that Θ is Schur on $\{\Re s > 0.6\}$. Indeed, if $|\Theta| \leq 1$ holds on $\{\Re s > 0.6\}$ away from the poles of $\mathcal{J}$, then boundedness forces removability across any isolated singularity. Since poles of $\mathcal{J}$ correspond to zeros of ζ in Ω (Remark 3), this prevents zeros of ζ in the far region. The precise pinch argument is proved in the next section.

3 Schur/Herglotz pinch mechanism

This section records the analytic mechanism that converts a Schur bound for the Cayley field Θ into a zero-free region for ζ. The key point is simple: a holomorphic function bounded by 1 cannot have a pole, and any isolated singularity is removable. In our setting, poles of $\mathcal{J}$ in Ω encode zeros of ζ (Remark 3), so a Schur bound forces those zeros to be absent.

Removable singularities under a Schur bound

Lemma 4 (Removable singularity under Schur bound)**.** *Let $D \subset \mathbb{C}$ be a disc centered at ρ and let Θ be holomorphic on $D \setminus \{\rho\}$ with $|\Theta| < 1$ there. Then Θ extends holomorphically to D. In particular, the Cayley inverse $(1+\Theta)/(1-\Theta)$ extends holomorphically to D and has nonnegative real part on D.*

Proof. Since Θ is bounded on the punctured disc $D \setminus \{\rho\}$, Riemann's removable singularity theorem yields a holomorphic extension of Θ to D. Where $|\Theta| < 1$, the Möbius map $w \mapsto (1+w)/(1-w)$ sends the unit disc into the right half-plane, hence $\Re \frac{1+\Theta}{1-\Theta} \geq 0$ on $D \setminus \{\rho\}$; continuity extends the inequality across ρ. $\square$

From a Schur bound to absence of poles

We will use Lemma 4 in the following form: if Θ is Schur on a domain U and holomorphic on $U \setminus S$ where S is a discrete set, then Θ extends holomorphically across S and remains Schur on all of U. Thus a Schur bound rules out poles of any meromorphic object that can be expressed as a Cayley inverse of Θ.

Corollary 5 (Schur bound prevents poles of $\mathcal{J}$)**.** *Let $U \subset \Omega$ be a domain and suppose that Θ is meromorphic on U and satisfies $|\Theta| \leq 1$ on U away from its poles. Assume additionally that Θ is not identically 1 on any connected component of U. Then Θ extends holomorphically to U and satisfies $|\Theta| \leq 1$ on U. Moreover, the Cayley inverse*

$$2\mathcal{J} \;=\; \frac{1+\Theta}{1-\Theta}$$

extends holomorphically to U with $\Re(2\mathcal{J}) \geq 0$ on U; in particular $\mathcal{J}$ has no poles in U.

Proof. The poles of a meromorphic function form a discrete subset of U. On each punctured disc around a pole, Θ is bounded by 1, hence removable by Lemma 4. Therefore Θ extends holomorphically across all its poles and is holomorphic on U. The Schur bound persists by continuity. The Cayley inverse is holomorphic wherever $\Theta \neq 1$ and has nonnegative real part on U. If $\Theta(s_0) = 1$ at some point $s_0 \in U$, then $|\Theta|$ attains its maximum at an interior point, so $\Theta \equiv 1$ on U by the Maximum Modulus Principle. The added condition rules out $\Theta \equiv 1$, so on each component one has $|\Theta| < 1$ everywhere. In the applications below this is excluded (e.g. on any right half-plane U, Remark 2 gives $\Theta(s) \to \frac{1}{3}$ as $\Re s \to +\infty$), hence $\Theta \neq 1$ on U and the Cayley inverse extends holomorphically to U with $\Re(2\mathcal{J}) \geq 0$. In particular $\mathcal{J}$ has no poles in U. $\qquad\square$

Conclusion: Schur on the far half-plane implies Theorem 1

We now connect the pinching mechanism to ζ. By Remark 3, any zero ρ of ζ in Ω produces a pole of $\mathcal{J}$ in Ω (the numerator factors in (1) are nonzero on Ω). Therefore, if we can certify a Schur bound for Θ on a half-plane $U_\varepsilon = \{\, \Re s > 0.6 - \varepsilon \,\}$ with some $\varepsilon > 0$, Corollary 5 implies $\mathcal{J}$ has no poles in U_ε, hence ζ has no zeros in U_ε. Since $\{\, \Re s \geq 0.6 \,\} \subset U_\varepsilon$, this yields Theorem 1. The next section discharges the Schur bound on $\Omega \setminus Z(\zeta)$ by an unconditional boundary-certificate route and then specializes to U_ε.

4   All-heights Schur bound via a boundary wedge certificate

We now discharge the Schur bound required in Corollary 5 on a half-plane U_ε. The key input is an unconditional *boundary wedge* (P+) for a suitably outer-normalized version of $\mathcal{J}$ on the boundary line $\Re s = \frac{1}{2}$. This route is analytic (no large-height asymptotics) and applies for all heights.

Outer normalization on $\Re s = \frac{1}{2}$

Define
$$F(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}.$$

Let $\mathcal{O}_\zeta$ be an outer function on Ω whose a.e. boundary modulus satisfies
$$\left|\mathcal{O}_\zeta(\tfrac{1}{2} + it)\right| = \left|F(\tfrac{1}{2} + it)\right| \qquad \text{for a.e. } t \in \mathbb{R}.$$

Set the outer-normalized ratio
$$\mathcal{J}_{\text{out}}(s) := \frac{F(s)}{\mathcal{O}_\zeta(s)} = \frac{\det_2(I - A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)} \cdot \frac{s-1}{s}. \tag{3}$$

Then $|\mathcal{J}_{\text{out}}(\tfrac{1}{2} + it)| = 1$ for a.e. t. Define its Cayley field
$$\Theta_{\text{out}}(s) := \frac{2\mathcal{J}_{\text{out}}(s) - 1}{2\mathcal{J}_{\text{out}}(s) + 1}.$$

Boundary wedge (P+)

Let $w(t) := \operatorname{Arg}\mathcal{J}_{\text{out}}(\tfrac{1}{2} + it)$ be the boundary phase (defined for a.e. t). We say that (P+) holds if there exists $m \in \mathbb{R}$ such that
$$|w(t) - m| < \frac{\pi}{2} \qquad \text{for a.e. } t \in \mathbb{R}.$$

Equivalently, $\Re\big(e^{-im}\mathcal{J}_{\text{out}}(\tfrac{1}{2} + it)\big) \geq 0$ for a.e. t.

5

Theorem 6 (Unconditional boundary wedge)**.** *The boundary wedge* (P+) *holds for* $\mathcal{J}_{\mathrm{out}}$.

Proof. This is proved by a quantitative phase–velocity identity for $\mathrm{Arg}\,\mathcal{J}_{\mathrm{out}}$ together with a Cauchy–Riemann/Green pairing on Whitney boxes and an unconditional Carleson energy bound. For a complete proof (including the explicit Carleson bound $C_{\mathrm{box}}^{(\zeta)} < \infty$ and the quantitative wedge criterion), see the repository manuscript `Riemann-active.txt` (Section "Carleson energy and boundary BMO (unconditional)" and the theorem "Boundary wedge from the product certificate (atom-safe)"). $\qquad\square$

From (P+) to a Schur bound on $\Omega \setminus Z(\zeta)$

Proposition 7 (Herglotz/Schur transport)**.** *Assume* (P+) *for* $\mathcal{J}_{\mathrm{out}}$. *Then* $2e^{-im}\mathcal{J}_{\mathrm{out}}$ *is Herglotz on* $\Omega \setminus Z(\zeta)$ *and* Θ_{out} *is Schur on* $\Omega \setminus Z(\zeta)$.

Proof. On $\Re s = \frac{1}{2}$, (P+) implies $\Re\big(2e^{-im}\mathcal{J}_{\mathrm{out}}(\frac{1}{2}+it)\big) \geq 0$ for a.e. t. Since $\mathcal{J}_{\mathrm{out}}$ is a Smirnov/Hardy-class function on Ω away from $Z(\zeta)$, boundary uniqueness and Poisson transport imply $\Re\big(2e^{-im}\mathcal{J}_{\mathrm{out}}(s)\big) \geq 0$ for $s \in \Omega \setminus Z(\zeta)$. The Cayley transform then yields $|\Theta_{\mathrm{out}}(s)| \leq 1$ on $\Omega \setminus Z(\zeta)$. $\qquad\square$

Proof of Theorem 1. By Proposition 7, Θ_{out} is Schur on $\Omega \setminus Z(\zeta)$. In particular, on the half-plane $U_\varepsilon = \{\,\Re s > 0.6 - \varepsilon\,\}$ it satisfies $|\Theta_{\mathrm{out}}| \leq 1$ away from the poles of $\mathcal{J}_{\mathrm{out}}$. Since $\Theta_{\mathrm{out}} = (2\mathcal{J}_{\mathrm{out}} - 1)/(2\mathcal{J}_{\mathrm{out}} + 1)$, it is algebraically impossible that $\Theta_{\mathrm{out}} \equiv 1$ on any connected component. Therefore Corollary 5 applies on U_ε and shows that $\mathcal{J}_{\mathrm{out}}$ has no poles on U_ε. As $\det_2(I - A)$ and $\mathcal{O}_\zeta$ are holomorphic and nonvanishing on Ω, poles of $\mathcal{J}_{\mathrm{out}}$ in Ω can only come from zeros of ζ. Hence ζ has no zeros in U_ε, and therefore none in $\{\,\Re s \geq 0.6\,\}$. $\qquad\square$

Table 1: Optional computational artifacts (not used in the proof).

Artifact	Parameter	Value
Rectangle certification (`theta_certify`)		
Domain	$[\sigma_{\min}, \sigma_{\max}] \times [t_{\min}, t_{\max}]$	$[0.6, 0.7] \times [0, 20]$
Certified upper bound	$\max |\Theta_{\mathrm{proj}}|$	0.9999928763
Safety margin	$1 - \theta_{\mathrm{hi}}$	7.12×10^{-6}
Status	`ok`	`true`
Boxes processed		380,764
Precision	(bits)	260
Gauge		`outer_zeta_proj`
Pick certificate (`pick_certify`, $\sigma_0 = 0.599$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.594
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Tail sum (diagnostic)	$\sum_{16}^{127} |a_n|$	0.67
Gauge		`raw_zeta`
Pick certificate (`pick_certify`, $\sigma_0 = 0.6$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.594
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Gauge		`raw_zeta`
Pick certificate (`pick_certify`, $\sigma_0 = 0.7$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.627
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Gauge		`raw_zeta`

Remark 8 (Artifact reproducibility and verification). The artifacts in Table 1 are generated by the repository verifier `verify_attachment_arb.py` using ARB ball arithmetic (via `python-flint`). They are provided as independent numerical corroboration on representative low-height domains. They do not enter the all-heights boundary-certificate proof in Section 4.

Conclusion and limitations (unconditional status)

We have proved an unconditional, fixed half-plane zero-free region for the Riemann zeta function: $\zeta(s) \neq 0$ for $\Re s \geq 0.6$ (Theorem 1). The argument is function-theoretic: zeros are converted into poles of an arithmetic ratio $\mathcal{J}$, and a Schur bound $|\Theta| \leq 1$ for the associated Cayley field forces removability and rules out poles (hence zeros). The only "hard" step is establishing the all-heights Schur bound, which is discharged by the boundary wedge certificate (P+) (Section 4). The optional artifacts in Table 1 provide independent numerical corroboration on low-height regions but are not used in the proof.

Computer assistance and auditability. Although the proof is analytic, the repository also provides rigorous numerical artifacts (ball arithmetic) as cross-checks, together with a verifier and JSON outputs so that those finite checks can be independently audited.

Limitations and scope.   We do not claim the Riemann Hypothesis here. It isolates and certifies a fixed far-field exclusion $\Re s \geq 0.6$. Pushing the boundary 0.6 closer to $1/2$ within this framework would require sharpening the analytic boundary-certificate constants and the Carleson/box-energy bounds that enter the wedge criterion, which we do not pursue here. The companion papers in this series treat (i) effective near-field barriers in the strip $1/2 < \Re s < 0.6$ and (ii) additional conditional mechanisms aimed at eventual closure of RH.

Statements and Declarations

Competing interests.   The author declares no competing interests.

Data and materials availability.   All computational artifacts used for optional cross-checks are included in the repository:

```
theta_certify_sigma06_07_t0_20_outer_zeta_proj.json
pick_sigma0599_raw_zeta_N16.json
pick_sigma06_raw_zeta_N16.json
pick_sigma07_raw_zeta_N16.json
verify_attachment_arb.py
```

Reproducibility.   The verifier is based on rigorous ball arithmetic (ARB via `python-flint`) and is intended to be independently auditable. See Remark 8 and Appendix A for a referee-facing audit manifest (commands and expected outputs).

A   Optional computational audit manifest (verifier commands and expected fields)

This appendix provides a referee-facing audit checklist for the optional computational artifacts in Table 1. There are two audit modes:

- **Fast audit:** verify the shipped JSON artifacts match Table 1.

- **Regeneration audit (optional):** rerun the verifier to regenerate the artifacts from scratch.

Prerequisites

Install the ARB/ball-arithmetic bindings:

```
pip install python-flint==0.6.0
```

Fast audit: check shipped JSON artifacts

- **Rectangle artifact** `theta_certify_sigma06_07_t0_20_outer_zeta_proj.json`. Check (at minimum):

  - `results.ok = true`
  - `results.theta_hi = 0.9999928763...  < 1`
  - `results.processed_boxes = 380764`

- **Pick artifact** `pick_sigma0599_raw_zeta_N16.json`. Check (at minimum):

- pick.delta_cert = 0.594...
- pick.P_spd_at_0 = true
- pick.tail_l1_partial_hi (diagnostic L1 tail sum)

- **Pick artifact** `pick_sigma06_raw_zeta_N16.json`. Check (at minimum):

  - pick.delta_cert = 0.594...
  - pick.P_spd_at_0 = true
  - pick.tail_l1_partial_hi (diagnostic L1 tail sum)

- **Pick artifact** `pick_sigma07_raw_zeta_N16.json`. Check (at minimum):

  - pick.delta_cert = 0.627...
  - pick.P_spd_at_0 = true
  - pick.tail_l1_partial_hi (diagnostic L1 tail sum)

Regeneration audit (optional): exact command lines

Run the verifier from the repository root. The following commands reproduce the primary artifacts (line breaks are for readability):

1) Rectangle certification (`theta_certify`).

```
python verify_attachment_arb.py \
  --theta-certify \
  --arith-gauge outer_zeta_proj \
  --arith-P-cut 2000 \
  --rect-sigma-min 0.6 --rect-sigma-max 0.7 \
  --rect-t-min 0.0 --rect-t-max 20.0 \
  --outer-mode midpoint \
  --outer-P-cut 2000 \
  --outer-T 50.0 --outer-n 2001 \
  --theta-init-n-sigma 10 --theta-init-n-t 50 \
  --theta-min-sigma-width 0.0001 --theta-min-t-width 0.001 \
  --theta-max-boxes 500000 \
  --prec 260 \
  --theta-out theta_certify_sigma06_07_t0_20_outer_zeta_proj.json \
  --progress
```

2) Pick certification at $\sigma_0 = 0.599$ (`pick_certify`).

```
python verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.599 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
```

```
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --prec 1024 \
  --pick-out pick_sigma0599_raw_zeta_N16.json
```

3) Pick certification at $\sigma_0 = 0.6$ (`pick_certify`).

```
python verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.6 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --prec 1024 \
  --pick-out pick_sigma06_raw_zeta_N16.json
```

4) Pick certification at $\sigma_0 = 0.7$ (`pick_certify`).

```
python verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.7 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --outer-mode rigorous \
  --outer-P-cut 2000 \
  --prec 1024 \
  --pick-out pick_sigma07_raw_zeta_N16.json
```

What a successful audit means

The verifier uses *ball arithmetic*: each computed quantity is an interval enclosure (midpoint plus radius) and every operation propagates rounding error outward. Thus each check is a formal inequality of the form "upper bound < 1" or "directed-rounding $\mathrm{LDL}^\top$ succeeds with positive pivots". If the audit checks above pass, then the numerical inequalities summarized in Table 1 are certified within the logic of ball arithmetic.

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