UNIQUENESS OF THE CANONICAL RECIPROCAL COST

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Abstract. Cont...

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1. Motivation and Introduction

In many areas of mathematics and physics, one quantifies "how different" two states are by selecting a scalar functional: a cost, energy, divergence, or action. Such choices are often guided by symmetry, convenience, or tradition (e.g., quadratic penalties, log-likelihoods, or variational principles). But once a framework is built atop a chosen cost, it is easy to overlook that the cost itself may be a *dial*: different choices can preserve superficial qualitative behavior while materially changing quantitative outputs.

If the goal is explanatory or predictive, this dial matters. In particular, claims of being "parameter-free" can be undermined even in the absence of explicit tunable constants: if one can vary the functional form of the cost while keeping the rest of the story fixed, then the cost selection plays the role of an implicit parameter family. For this reason, a credible parameter-free derivation program needs *uniqueness* results: conditions under which the cost is forced, rather than chosen.

Under a natural set of explicit assumptions, the cost on $\mathbb{R}_{>0}$. is uniquely determined.

We emphasize that the composition law (1.1) is the central assumption in Theorem 1.1. In particular, there exist families of functions that satisfy reciprocity, normalization, and (1.2), but do not satisfy the composition law. For example,

$$J_\varepsilon(x) = \tfrac{1}{2}(x + x^{-1} - 2) + \varepsilon(x + x^{-1} - 2)^2, \qquad \varepsilon > 0,$$

has these properties, but does not satisfy (1.1). So, the composition law is essential for uniqueness.

1.1. The canonical reciprocal cost.

Definition 1.1. *The function $J : \mathbb{R}_{>0} \to \mathbb{R}$ defined by*

$$J(x) := \frac{x + x^{-1}}{2} - 1$$

is called the canonical reciprocal cost *function.*

The function J satisfies the following properties:

(i) *Reciprocity:* $J(x) = J(x^{-1})$;
(ii) *Normalization:* $J(1) = 0$;
(iii) *Nonnegativity:* For all $x \in \mathbb{R}_{>0}$

$$J(x) = \frac{(x-1)^2}{2x} \geq 0.$$

If we substitute $t = \ln x$, then

$$J(\mathrm{e}^t) = \cosh(t) - 1.$$

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1.2. **Main result.** We now state the main theorem of this paper, which establishes the uniqueness of the canonical reciprocal cost.

Theorem 1.1. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$. Assume that F satisfies:*

(i) Normalization: $(F(1) = 0$
(ii) Reciprocity: $F(x) = F(x^{-1})$ for all $x > 0$.
(iii) Composition law on $\mathbb{R}_{>0}$: for all $x, y > 0$,

$$F(xy) + F\Big(\frac{x}{y}\Big) = 2\,F(x)\,F(y) + 2\,F(x) + 2\,F(y). \tag{1.1}$$

(iv)

$$\lim_{t\to 0} \frac{2\,F(\mathrm{e}^t)}{t^2} = 1. \tag{1.2}$$

Then for all $x > 0$,

$$F(x) = \frac{x + x^{-1}}{2} - 1 \;=\; J(x).$$

1.3. **Paper organization.** CONT...

2. Definitions and basic properties

We work on the following domain

$$\mathbb{R}_{>0} := \{x \in \mathbb{R} :\ x > 0\}.$$

Definition 2.1. *A function $F : \mathbb{R}_{>0} \to \mathbb{R}$ is called a* reciprocal cost *if*

$$F(x) = F(x^{-1}) \qquad \text{for all } x > 0.$$

It is normalized *if $F(1) = 0$.*

Let us consider $F : \mathbb{R}_{>0} \to \mathbb{R}$, and define

$$G(t) := F(\mathrm{e}^t), \qquad H(t) := G(t) + 1 = F(\mathrm{e}^t) + 1, \qquad t \in \mathbb{R}.$$

From $F(1) = 0$ it immediately follows that $G(0) = 0$ and $H(0) = 1$.

Lemma 2.1. *If F is reciprocal, then G and H are even.*

Proof. Since $\mathrm{e}^{-t} = (\mathrm{e}^t)^{-1}$ and $F(x) = F(x^{-1})$, we have

$$G(-t) = F(\mathrm{e}^{-t}) = F\big((\mathrm{e}^t)^{-1}\big) = F(\mathrm{e}^t) = G(t).$$

Further, for $H(-t)$, we have

$$H(-t) = G(-t) + 1 = G(t) + 1 = H(t)$$

. □

Let us define the function

$$J(x) := \frac{x + x^{-1}}{2} - 1, \qquad x > 0.$$

Clearly, for all $x > 0$,

$$J(x) = \frac{(x-1)^2}{2x} \geq 0,$$

with equality if and only if $x = 1$.

For all $t \in \mathbb{R}$, $J(\mathrm{e}^t) = \cosh(t) - 1$. So, we have $G(t) = \cosh(t) - 1$ and $H(t) = \cosh(t)$.

2.1. **The d'Alembert functional equation.** The key structural identity considered in this paper is the d'Alembert functional equation, which is also known in the literature as the cosine equation or the Poisson equation.

Definition 2.2. *A function $H : \mathbb{R} \to \mathbb{R}$ is said to satisfy the d'Alembert functional equation if, for all $t, u \in \mathbb{R}$,*

$$H(t+u) + H(t-u) = 2\,H(t)\,H(u). \tag{2.1}$$

D'Alembert's functional equation has a long history going back to d'Alembert [2], Poisson [3], and Picard [4]. The equation plays an important role in determining the sum of two vectors in various Euclidean and non-Euclidean geometries.

Functional equations arise from the parallelogram law of forces or the rule for addition of vectors. Closely related functional equations arise in the study of vibration of strings, leading to equations of the form

$$H(t+u) - H(t-u) = \,H(t)\,H(u), \quad \text{for all} \quad t, u \in \mathbb{R}. \tag{2.2}$$

The solution of (2.1) was obtained by d'Alembert by reducing it to a differential equation. One of the interesting aspects of functional equations is that, unlike differential equations, functional equations can admit many solutions unless additional assumptions are imposed.

If $t = u = 0$ in (2.1), then $2H(0) = 2H(0)^2$ so that, we have

$$H(0) = 0 \quad \text{or} \quad H(0) = 1.$$

If $H(0) = 0$, then for any real x

$$0 = 2H(x)H(0) = H(x+0) + H(x-0) = 2H(x).$$

Therefore, H is the identically zero function. We will assume from now $H(0) = 1$.

Theorem 2.1. *If $H : \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies equation (2.1), then the only solutions are*

$$H(x) \equiv 0, \quad H(x) \equiv 1, \quad H(x) = \cos(kx), \quad H(x) = \cosh(kx),$$

where k is a real constant. The classical Cauchy method determines H on a dense subset of $\mathbb{R}$ and extends it to the whole real line by continuity.

Lemma 2.2. *If H satisfies Definition 2.2, then H is even.*

Proof. Fix $u \in \mathbb{R}$ and apply the d'Alembert equation (2.1) with $t = 0$:

$$H(u) + H(-u) = 2\,H(0)\,H(u) = 2H(u),$$

so $H(-u) = H(u)$. □

Lemma 2.3. *If H satisfies Definition 2.2, then for all $t, u \in \mathbb{R}$,*

$$H(t+u)\,H(t-u) = H(t)^2 + H(u)^2 - 1.$$

Proof. Apply (2.1) with $a = t+u$ and $b = t-u$:

$$H((t+u) + (t-u)) + H((t+u) - (t-u)) = 2H(t+u)H(t-u),$$

so

$$H(2t) + H(2u) = 2H(t+u)H(t-u).$$

Using that

$$H(2t) = 2H(t)^2 - 1$$

(obtained from (2.1) with (t, t) and $H(0) = 1$), and similarly for u, yields the claim. □

Lemma 2.4. *If H satisfies Definition 2.2, then for all $t, u \in \mathbb{R}$,*

$$\big(H(t+u) - H(t-u)\big)^2 = 4\,(H(t)^2 - 1)\,(H(u)^2 - 1).$$

Proof. Let $A := H(t+u)$ and $B := H(t-u)$. Then $A + B = 2H(t)H(u)$ by Definition 2.2, and $AB = H(t)^2 + H(u)^2 - 1$ by Lemma 2.3. Hence

$$\begin{aligned}(A-B)^2 &= (A+B)^2 - 4AB \\ &= 4H(t)^2H(u)^2 - 4(H(t)^2 + H(u)^2 - 1) \\ &= 4(H(t)^2 - 1)(H(u)^2 - 1).\end{aligned}$$

□

Lemma 2.5. *If H satisfies Definition 2.2 and $\lim_{t\to 0} 2(H(t)-1)/t^2$ exists. Then H is continuous on $\mathbb{R}$.*

Proof. The limit assumption implies that $\lim_{t\to 0} H(t) = 1$. Since $H(0) = 1$, it follows that H is continuous at 0.

Fix $t \in \mathbb{R}$. For $u \to 0$, the equation (2.1) gives

$$\lim_{u\to 0}\big(H(t+u) + H(t-u)\big) = 2H(t)\lim_{u\to 0} H(u) = 2H(t).$$

By Lemma 2.4, we have

$$\lim_{u\to 0}\big(H(t+u) - H(t-u)\big)^2 = 4\big(H(t)^2 - 1\big)\lim_{u\to 0}\big(H(u)^2 - 1\big) = 0,$$

hence

$$\lim_{u\to 0}\big(H(t+u) - H(t-u)\big) = 0.$$

Moreover,

$$H(t+u) = \frac{H(t+u) + H(t-u)}{2} + \frac{H(t+u) - H(t-u)}{2}.$$

Taking limits as $u \to 0$ and using

$$\lim_{u\to 0}\big(H(t+u) + H(t-u)\big) = 2H(t), \qquad \lim_{u\to 0}\big(H(t+u) - H(t-u)\big) = 0,$$

we obtain

$$\lim_{u\to 0} H(t+u) = H(t).$$

Similarly, $\lim_{u\to 0} H(t-u) = H(t)$. Therefore, H is continuous at every $t \in \mathbb{R}$. □

If H satisfies the d'Alembert equation (2.1) and for $G = H - 1$, one proves that G satisfies the following identity

$$G(t+u) + G(t-u) = 2\,G(t)\,G(u) + 2\,G(t) + 2\,G(u).$$

For example, $H(t) = J(\mathrm{e}^t) + 1 = \cosh(t)$ satisfies the d'Alembert (2.1).

2.2. **Composition law on $\mathbb{R}_{>0}$.** The main theorem of this paper is stated directly on $\mathbb{R}_{>0}$, so that log-coordinates appear only as a proof technique.

Definition 2.3. *A function $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the* d'Alembert composition law on $\mathbb{R}_{>0}$ *if for all $x, y > 0$,*

$$F(xy) + F\Big(\frac{x}{y}\Big) = 2\,F(x)\,F(y) + 2\,F(x) + 2\,F(y).$$

Lemma 2.6. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$, and $H : \mathbb{R} \to \mathbb{R}$ such that $H(t) = F(\mathrm{e}^t) + 1$. Then F satisfies Definition 2.3 if and only if H satisfies the d'Alembert equation (2.1).*

Proof. Assume F satisfies Definition 2.3. Let $t, u \in \mathbb{R}$ and set $x = \mathrm{e}^t$, $y = \mathrm{e}^u$, so $xy = \mathrm{e}^{t+u}$ and $x/y = \mathrm{e}^{t-u}$. Then

$$\begin{aligned} H(t+u) + H(t-u) &= \big(F(\mathrm{e}^{t+u}) + 1\big) + \big(F(\mathrm{e}^{t-u}) + 1\big) \\ &= \Big(F(xy) + F\Big(\frac{x}{y}\Big)\Big) + 2 \\ &= \big(2F(x)F(y) + 2F(x) + 2F(y)\big) + 2 \\ &= 2\big(F(x)+1\big)\big(F(y)+1\big) = 2H(t)H(u), \end{aligned}$$

so H satisfies (2.1).

Conversely, if H satisfies (2.1), by reverse calculation with $x = \mathrm{e}^t$, $y = \mathrm{e}^u$, we obtain that F satisfies Definition 2.3. □

Definition 2.4. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$. Define the* log-curvature *of F, denoted $\kappa(F)$, as*

$$\kappa(F) \;:=\; \lim_{t\to 0} \frac{2\,F(\mathrm{e}^t)}{t^2}$$

provided this limit exists.

The limit above exists if and only if

$$\lim_{x\to 1} \frac{2\,F(x)}{(\log x)^2}$$

exists, and when one exists the two limits coincide.

3. Main results

In this section, we prove the main theorem in the paper.

Lemma 3.1. *Let $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ on $\mathbb{R}_{>0}$. Then:*

(i) J is reciprocal and normalized: $J(x) = J(x^{-1})$ for all $x > 0$ and $J(1) = 0$.
(ii) J satisfies the d'Alembert composition law on $\mathbb{R}_{>0}$ (Definition 2.3).
(iii) J has unit log-curvature: $\kappa(J) = 1$.

Proof. (i) Reciprocity follows directly from the definition of $J(x)$. Also $J(1) = \frac{1}{2}(1+1) - 1 = 0$.
(ii) Let $H(t) = J(\mathrm{e}^t) + 1,\ t \in \mathbb{R}$. Let us first compute H

$$H(t) = \left(\frac{1}{2}\big(\mathrm{e}^t + \mathrm{e}^{-t}\big) - 1\right) + 1 = \frac{1}{2}\big(\mathrm{e}^t + \mathrm{e}^{-t}\big) = \cosh(t).$$

Hence $H(t) = \cosh(t)$ for all $t \in \mathbb{R}$.

The function cosh satisfies the d'Alembert equation (2.1), i.e.,

$$\cosh(t+u) + \cosh(t-u) = 2\,\cosh(t)\cosh(u) \qquad \text{for all } t, u \in \mathbb{R}.$$

Since $H(t) = \cosh(t)$, it follows that H satisfies the d'Alembert equation

$$H(t+u) + H(t-u) = 2\,H(t)\,H(u) \qquad \text{for all } t, u \in \mathbb{R}. \tag{3.1}$$

Let $x, y > 0$ and

$$t = \log x, \qquad u = \log y.$$

Then $\mathrm{e}^t = x$, $\mathrm{e}^u = y$, and consequently

$$\mathrm{e}^{t+u} = xy, \qquad \mathrm{e}^{t-u} = \frac{x}{y}.$$

Using the definition of H, we can rewrite (3.1) as

$$\big(J(\mathrm{e}^{t+u}) + 1\big) + \big(J(\mathrm{e}^{t-u}) + 1\big) = 2\big(J(\mathrm{e}^t) + 1\big)\big(J(\mathrm{e}^u) + 1\big).$$

Substituting $\mathrm{e}^{t+u} = xy$, $\mathrm{e}^{t-u} = x/y$, $\mathrm{e}^t = x$, and $\mathrm{e}^u = y$, we obtain

$$\big(J(xy)+1\big) + \Big(J\Big(\frac{x}{y}\Big)+1\Big) = 2\big(J(x)+1\big)\big(J(y)+1\big),$$

i.e.

$$J(xy) + J\Big(\frac{x}{y}\Big) + 2 = 2J(x)J(y) + 2J(x) + 2J(y) + 2.$$

Finally, it follows the d'Alembert composition law on $\mathbb{R}_{>0}$ given by Definition 2.3:

$$J(xy) + J\Big(\frac{x}{y}\Big) = 2\,J(x)\,J(y) + 2\,J(x) + 2\,J(y), \qquad x, y > 0.$$

(iii) Using $J(\mathrm{e}^t) = \cosh(t) - 1$ and the Taylor expansion $\cosh(t) = 1 + t^2/2 + o(t^2)$ as $t \to 0$, we have

$$\lim_{t\to 0} \frac{2J(\mathrm{e}^t)}{t^2} = \lim_{t\to 0} \frac{2(\cosh(t)-1)}{t^2} = 1,$$

so $\kappa(J) = 1$. □

Theorem 3.1. *Let $H : \mathbb{R} \to \mathbb{R}$ satisfy the d'Alembert equation (2.1). Assume the following limit exists:*

$$\kappa_H := \lim_{t\to 0} \frac{2\,(H(t)-1)}{t^2} \in \mathbb{R}. \tag{3.2}$$

Then:

1. *If $\kappa_H > 0$, then $H(t) = \cosh(\sqrt{\kappa_H}\,t)$ for all $t \in \mathbb{R}$.*
2. *If $\kappa_H < 0$, then $H(t) = \cos(\sqrt{-\kappa_H}\,t)$ for all $t \in \mathbb{R}$.*
3. *If $\kappa_H = 0$, then $H(t) = 1$ for all $t \in \mathbb{R}$.*

In particular, if $\kappa_H = 1$, then $H(t) = \cosh(t)$ for all $t \in \mathbb{R}$.

Proof. By Lemma 2.5, the existence of

$$\lim_{t\to 0} \frac{2\,(H(t)-1)}{t^2} \in \mathbb{R}$$

implies that H is continuous on $\mathbb{R}$. Hence we will apply the classical classification of continuous real-valued solutions of the d'Alembert equation (2.1) (see, for example, [5]).

$$H(t) \equiv 1, \quad H(t) = \cos(kt) \qquad H(t) = \cosh(kt), \quad k \in \mathbb{R}.$$

If $H(t) \equiv 1$, then $H(t) - 1 \equiv 0$, so $\kappa_H = 0$, and the conclusion in (3) holds.

If $H(t) = \cosh(kt)$ for some $k \in \mathbb{R}$. Using the Taylor expansion $\cosh(t) = 1 + t^2/2 + o(t^2)$ as $t \to 0$, we have

$$\cosh(kt) - 1 = \frac{(kt)^2}{2} + o(t^2), \qquad t \to 0.$$

Therefore,

$$\kappa_H = \lim_{t\to 0} \frac{2(\cosh(kt)-1)}{t^2} = \lim_{t\to 0} \frac{2\left(\frac{k^2t^2}{2} + o(t^2)\right)}{t^2} = k^2 \geq 0.$$

If $\kappa_H > 0$, then $k \neq 0$ and $|k| = \sqrt{\kappa_H}$, hence

$$H(t) = \cosh(kt) = \cosh(\sqrt{\kappa_H}\,t), \qquad t \in \mathbb{R},$$

since cosh is even. If $k = 0$, then $H \equiv 1$ and $\kappa_H = 0$, which is already covered by the first case.

Similarly to the second case, if $H(t) = \cos(kt)$ for some $k \in \mathbb{R}$, the claim follows. □

Corollary 3.1. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ be normalized ($F(1) = 0$). Assume F satisfies the $\mathbb{R}_{>0}$ composition law (Definition 2.3) and has unit log-curvature $\kappa(F) = 1$. Then*

$$F(x) = \frac{x + x^{-1}}{2} - 1 \qquad \textit{for all } x > 0.$$

Proof. Let $H(t) = F(\mathrm{e}^t) + 1$. By Lemma 2.6, H satisfies d'Alembert equation (2.1), and moreover,

$$\kappa_H = \lim_{t\to 0} \frac{2(H(t)-1)}{t^2} = \lim_{t\to 0} \frac{2F(\mathrm{e}^t)}{t^2} = \kappa(F) = 1.$$

From Theorem 3.1 then $H(t) = \cosh(t)$, hence $F(\mathrm{e}^t) = \cosh(t) - 1 = J(\mathrm{e}^t)$. For $x > 0$ and $x = \mathrm{e}^{\log x}$, we have $F(x) = J(x)$. □

Remark 3.1. *If condition (3.2) is not imposed, the d'Alembert equation admits a one–parameter family of nontrivial solutions. More precisely, for any $k > 0$, the functions*

$$H(t) = \cosh(kt) \quad \text{and} \quad H(t) = \cos(kt)$$

satisfy (2.1). In this case the parameter k is not uniquely determined. The value of the curvature κ_H uniquely determines k by $k^2 = |\kappa_H|$, and makes difference between the cases $\kappa_H > 0$ and $\kappa_H < 0$.

Theorem 3.1 assumes the existence of the quadratic limit κ_H. By Lemma 2.5, this implies that H is continuous. The conclusion then follows from the classical classification of continuous solutions of the d'Alembert equation, together with a direct computation of κ_H for each canonical form.

Lemma 3.2. *(c.f. [7]) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Fix $T > 0$ and define the central second difference*

$$D_h f(t) := \frac{f(t+h) - 2f(t) + f(t-h)}{h^2} \qquad (|t| \le T,\ h \ne 0).$$

If there is a continuous function $L : [-T, T] \to \mathbb{R}$ such that

$$\lim_{h\to 0} \sup_{|t|\le T} \big| D_h f(t) - L(t) \big| = 0,$$

then $f \in C^2([-T, T])$ and $f''(t) = L(t)$ for all $|t| \le T$.

Lemma 3.3. *Let $H : \mathbb{R} \to \mathbb{R}$ satisfy the d'Alembert equation (2.1). Suppose the following limit exists*

$$\kappa_H := \lim_{t\to 0} \frac{2\,(H(t)-1)}{t^2} \in \mathbb{R}.$$

Then $H \in C^2(\mathbb{R})$ and

$$H''(t) = \kappa_H\, H(t) \qquad \text{for all } t \in \mathbb{R}.$$

Proof. By Lemma 2.5, H is continuous. Fix $T > 0$. Let $t \in [-T, T]$ be arbitrary and let $h \in \mathbb{R}$ with $0 < |h| \le 1$. Since H is defined on $\mathbb{R}$, then $H(t \pm h)$ are well-defined. The d'Alembert equation yields

$$H(t+h) + H(t-h) = 2H(t)H(h).$$

Rearranging, we obtain

$$\frac{H(t+h) - 2H(t) + H(t-h)}{h^2} = 2H(t)\, \frac{H(h)-1}{h^2}.$$

Define

$$q(h) := \frac{2(H(h)-1)}{h^2}.$$

By assumption, $q(h) \to \kappa_H$ as $h \to 0$.

Since H is continuous on $[-T-1, T+1]$, there exists $M_T > 0$ such that

$$|H(t)| \le M_T \qquad \text{for all } |t| \le T.$$

Hence

$$\begin{aligned} \sup_{|t|\le T} \left| \frac{H(t+h) - 2H(t) + H(t-h)}{h^2} - \kappa_H H(t) \right| &= \sup_{|t|\le T} |H(t)|\, |q(h) - \kappa_H| \\ &\le M_T\, |q(h) - \kappa_H| \xrightarrow[h\to 0]{} 0. \end{aligned}$$

By Lemma 3.2 (applied with $f = H$ and $L(t) = \kappa_H H(t)$), it follows that $H \in C^2([-T, T])$ and

$$H''(t) = \kappa_H H(t) \qquad \text{for all } t \in [-T, T].$$

Since $T > 0$ is arbitrary, the conclusion holds for all $t \in \mathbb{R}$. $\square$

Lemma 3.4. *Let $H \in C^2(\mathbb{R})$ satisfy the d'Alembert equation (2.1). Then for all $t \in \mathbb{R}$,*

$$H''(t) = H''(0)\, H(t).$$

Proof. Fix $t \in \mathbb{R}$ and let us define

$$\Phi(u) := H(t+u) + H(t-u) - 2H(t)H(u).$$

From (2.1), we have $\Phi(u) = 0$ for all u.

Taking the second derivative with respect to u, we get

$$\Phi''(u) = H''(t+u) + H''(t-u) - 2H(t)H''(u).$$

Since $\Phi(u) \equiv 0$, we have $\Phi''(0) = 0$. Therefore,

$$0 = \Phi''(0) = H''(t) + H''(t) - 2H(t)H''(0) = 2H''(t) - 2H(t)H''(0),$$

so $H''(t) = H''(0)H(t)$ for all t. $\square$

Lemma 3.5. *Let $H \in C^1(\mathbb{R})$ be even. Then $H'(0) = 0$.*

Lemma 3.6. *Let $\kappa > 0$ and let $f \in C^2(\mathbb{R})$ satisfy $f''(t) = \kappa f(t)$ for all t, with $f(0) = 0$ and $f'(0) = 0$. Then $f(t) = 0$ for all t.*

Proof. Let $\lambda := \sqrt{\kappa}$. Define $g(t) := f'(t) - \lambda f(t)$ and $h(t) := f'(t) + \lambda f(t)$. Then $g, h \in C^1(\mathbb{R})$ and

$$\begin{aligned} g'(t) &= f''(t) - \lambda f'(t) = \kappa f(t) - \lambda f'(t) = -\lambda g(t), \\ h'(t) &= f''(t) + \lambda f'(t) = \kappa f(t) + \lambda f'(t) = \lambda h(t). \end{aligned}$$

Therefore $\frac{d}{dt}(g(t)\mathrm{e}^{\lambda t}) = 0$, so $g(t)\mathrm{e}^{\lambda t}$ is constant. Since $g(0) = f'(0) - \lambda f(0) = 0$, we get $g \equiv 0$. Similarly $\frac{d}{dt}(h(t)\mathrm{e}^{-\lambda t}) = 0$ and $h(0) = f'(0) + \lambda f(0) = 0$, so $h \equiv 0$. Then $f' = \frac{1}{2}(g+h) = 0$, so f is constant and $f(0) = 0$ implies $f \equiv 0$. $\square$

Lemma 3.7. *Let $\kappa < 0$ and let $f \in C^2(\mathbb{R})$ satisfy $f''(t) = \kappa f(t)$ for all t, with $f(0) = 0$ and $f'(0) = 0$. Then $f(t) = 0$ for all t.*

Proof. Let $\kappa = -\lambda^2$ with $\lambda = \sqrt{-\kappa} > 0$. Let us define

$$E(t) := f'(t)^2 + \lambda^2 f(t)^2 \geq 0.$$

Then $E \in C^1(\mathbb{R})$ and

$$E'(t) = 2f'(t)f''(t) + 2\lambda^2 f(t)f'(t) = 2f'(t)\big(f''(t) + \lambda^2 f(t)\big) = 0,$$

so E is constant. Since $E(0) = f'(0)^2 + \lambda^2 f(0)^2 = 0$, we have $E(t) = 0$ for all t, hence $f'(t) = 0$ and $f(t) = 0$ for all t. $\square$

Theorem 1.1 removes an entire class of "hidden degrees of freedom" that can otherwise enter through functional form choice.

4. Properties of canonical reciprocal cost

Arithmetical and geometrical means. The function $J(x)$ can be expressed as

$$J(x) = \mathrm{AM}\Big(x, \frac{1}{x}\Big) - \mathrm{GM}\Big(x, \frac{1}{x}\Big)$$

provides a symmetric measure of deviation from equilibrium between x and its reciprocal.

Bregman divergence. The logarithmic expression of cost function

$$G(t) = J(\mathrm{e}^t) = \cosh t - 1$$

can be viewed as a Bregman divergence generated by $\Phi(t) = \cosh t$. Indeed,

$$G(t) = D_\Phi(t, 0) = \Phi(t) - \Phi(0) - \Phi'(0)(t - 0),$$

since $\Phi(0) = 1$ and $\Phi'(0) = \sinh 0 = 0$. Consequently, G satisfies the characteristic properties of a Bregman divergence: non-negativity $G(t) \geq 0$, with equality if and only if $t = 0$, and strict convexity in the first argument. Since $\Phi''(t) > 0$ for all $t \in \mathbb{R}$ and $\Phi'(0) = 0$, the point $t = 0$ is the unique global minimizer of Φ.

Behavior near $x = 1$. Near the equilibrium point $x = 1$, $J(x)$ has quadratic behavior. If $x = \mathrm{e}^\varepsilon$, we obtain

$$J(\mathrm{e}^\varepsilon) = \cosh(\varepsilon) - 1 = \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{24} + O(\varepsilon^6).$$

In particular, the linear term vanishes and the leading contribution is quadratic. This shows that small reciprocal deviations from equilibrium are penalized symmetrically.

Metric associated with J. For $t = \log x$, the cost function has form

$$G(t) = J(\mathrm{e}^t) = \cosh t - 1.$$

The associated Hessian (Riemannian) metric is

$$ds^2 = \Phi''(t)\, dt^2 = \cosh(t)\, dt^2,$$

where $\Phi(t) = \cosh t$ or equivalently, in x-coordinates,

$$ds^2 = \cosh(\log x)\, \frac{dx^2}{x^2}.$$

The induced distance is given by

$$d_J(x, y) = \left| \int_{\log x}^{\log y} \sqrt{\cosh u}\, du \right| = \left| \int_x^y \sqrt{\cosh(\log \xi)}\, \frac{d\xi}{\xi} \right|.$$

check this Shun-ichi Amari Hiroshi Nagaoka: Methods of Information Geometry
F. Nielsen, R. Nock, On the smallest enclosing information disk (2008), koji koristi Bregman divergences metrike.

While the cost J can not define a metric (since it does not satisfy the triangle inequality), the distance d_J is a metric on $\mathbb{R}_{>0}$ by construction. Moreover, since cosh is an even function, the distance is reciprocally symmetric, i.e.

$$d_J(x, y) = d_J\Big(\frac{1}{x}, \frac{1}{y}\Big).$$

The metric d_J is locally equivalent to the logarithmic metric, and globally controls deviations via the convexity of the cost function J.

Chebyshev Structure of J. Since $J(\mathrm{e}^t) + 1 = \cosh t$, discrete log-scaling $t \mapsto nt$ yields the Chebyshev structure

$$J(x^n) + 1 = \cosh(n \log x) = T_n(\cosh(\log x)) = T_n(J(x) + 1),$$

i.e. $J(x^n) = T_n(J(x) + 1) - 1$.

Since $H_1 = J(x)+1 = \cosh(\log x) \geq 1$, the discrete recursion $H_{n+1} = 2H_1H_n - H_{n-1}$ lies entirely in the hyperbolic (cosh) branch of the Chebyshev dynamics.

% MZ connection with ϕ!

Remark 4.1. *The functional J assigns the same cost to reciprocal fixed points of the map $x \mapsto 1 + 1/x$. In particular, the positive fixed point $\varphi = (1 + \sqrt{5})/2$ satisfies $J(\varphi) = J(\varphi^{-1})$.*

5. Physical interpretation of the canonical reciprocal cost

The canonical reciprocal cost

$$J(x) = \frac{1}{2}\left(x + \frac{1}{x}\right) - 1, \qquad x > 0,$$

admits a natural interpretation as an *energy balance functional.*

Since J is nonnegative and minimized at the equilibrium $x = 1$. Thus, equilibrium corresponds to zero cost, while any deviation from balance gives a positive energy. This is typical for an energy-type quantity: it measures how far the system is from a stable reference state.

The function J is symmetric under the reciprocal transformation $x \mapsto 1/x$:

$$J(x) = J\left(\frac{1}{x}\right) \quad \text{for all } x > 0.$$

This reflects a fundamental balance: over-amplification ($x > 1$) and under-amplification ($x < 1$) by the same factor incur identical energetic cost.

The quadratic behavior near equilibrium shows that J has a second–order leading term at the minimum. Writing $x = e^t$, we obtain

$$J(e^t) = \cosh t - 1 = \frac{t^2}{2} + O(t^4) \quad (t \to 0).$$

Thus, small deviations from equilibrium are penalized quadratically, and no linear term is present in the expansion.

The asymptotic behavior $J(x) \sim x/2$ as $x \to \infty$ (and, by symmetry, $J(x) \sim 1/(2x)$ as $x \to 0$) ensures that large deviations are penalized with unbounded growth, while still respecting reciprocity. In this sense, J represents the minimal energy functional compatible with multiplicative composition, reciprocal symmetry, and strict convexity.

We emphasize that we do not postulate an underlying physical Hamiltonian or a dynamical system. Rather, the canonical reciprocal cost emerges as the unique quantitative measure of imbalance consistent with the assumptions of Theorem 1.1. Its role is therefore analogous to that of an energy functional in physics: not a primitive input, but a derived construct that organizes stability, equilibrium, and deviation.

CONT...

6. Gates

7. Robustness: stability under bounded defect

The theorem below gives one local stability statement: if d'Alembert holds up to a uniform defect on a compact set and the function is sufficiently smooth, then the solution is close to the hyperbolic cosine case.

Definition 7.1. *For $H : \mathbb{R} \to \mathbb{R}$, we define the* d'Alembert defect

$$\Delta_H(t,u) := H(t+u) + H(t-u) - 2H(t)H(u).$$

Theorem 7.1. *Fix $T > 0$. Let $H \in C^3([-T,T])$ be even with $H(0) = 1$, and set $a := H''(0)$. Assume $a > 0$. Let*

$$\varepsilon := \sup_{|t|\le T,\ |u|\le T} |\Delta_H(t,u)|, \qquad B := \sup_{|t|\le T} |H(t)|, \qquad K := \sup_{|t|\le T} |H^{(3)}(t)|.$$

Then for every h with $0 < h \le T$ and every t with $|t| \le T - h$,

$$\bigl|H(t) - \cosh(\sqrt{a}\, t)\bigr| \le \frac{\delta(h)}{a}\bigl(\cosh(\sqrt{a}\,|t|) - 1\bigr),$$

where

$$\delta(h) := \frac{\varepsilon}{h^2} + \frac{(1+B)K}{3}\, h.$$

Proof. Fix $0 < h \le T$ and $|t| \le T - h$.

Using the integral form of Taylor's theorem, we have

$$H(t+h) = H(t) + hH'(t) + \frac{h^2}{2}H''(t) + \int_0^h \frac{(h-s)^2}{2}\, H^{(3)}(t+s)\, ds,$$

$$H(t-h) = H(t) - hH'(t) + \frac{h^2}{2}H''(t) - \int_0^h \frac{(h-s)^2}{2}\, H^{(3)}(t-s)\, ds.$$

Adding and bounding $|H^{(3)}| \le K$ yields

$$\bigl|H(t+h) + H(t-h) - 2H(t) - h^2H''(t)\bigr| \le \frac{K}{3}\, h^3. \tag{7.1}$$

Similarly, since H is even, $H'(0) = 0$, and the integral form at 0 gives

$$\bigl|H(h) - 1 - \tfrac{a}{2}h^2\bigr| \le \frac{K}{6}\, h^3. \tag{7.2}$$

Now, write the defect identity at (t,h) as

$$H(t+h) + H(t-h) = 2H(t)H(h) + \Delta_H(t,h).$$

Subtract $2H(t) + ah^2H(t)$ from both sides, we obtain

$$\begin{aligned} h^2\bigl(H''(t) - aH(t)\bigr) &= \bigl(H(t+h) + H(t-h) - 2H(t) - h^2H''(t)\bigr) \\ &\quad + \Delta_H(t,h) + 2H(t)\bigl(H(h) - 1 - \tfrac{a}{2}h^2\bigr). \end{aligned}$$

Taking absolute values and using (7.1), (7.2), $|H(t)| \le B$, and $|\Delta_H(t,h)| \le \varepsilon$, we obtain

$$h^2|H''(t) - aH(t)| \le \frac{K}{3}h^3 + \varepsilon + 2B\cdot\frac{K}{6}h^3 \le \varepsilon + \frac{(1+B)K}{3}h^3.$$

Dividing by h^2 yields the uniform bound

$$|H''(t) - aH(t)| \le \delta(h) \qquad (|t| \le T - h). \tag{7.3}$$

Let $y(t) := \cosh(\sqrt{a}\,t)$, so $y'' = ay$, $y(0) = 1$, and since H is even, $H'(0) = 0 = y'(0)$. Define $e(t) := H(t) - y(t)$. Then $e \in C^2([-T+h, T-h])$, $e(0) = e'(0) = 0$, and

$$e''(t) - ae(t) = H''(t) - aH(t),$$

so by (7.3), $|e''(t) - ae(t)| \le \delta(h)$ for $|t| \le T - h$. For $t \in [0, T-h]$, the equation for $e'' = ae + r$ with zero initial data gives

$$e(t) = \int_0^t \frac{1}{\sqrt{a}} \sinh(\sqrt{a}(t-s))\, r(s)\, ds,$$

where $r(s) := e''(s) - ae(s)$. Hence

$$|e(t)| \le \delta(h) \int_0^t \frac{1}{\sqrt{a}} \sinh(\sqrt{a}(t-s))\, ds = \frac{\delta(h)}{a}\big(\cosh(\sqrt{a}\,t) - 1\big).$$

Since e is even as difference of even functions, this bound holds for negative t, yielding the inequality for all $|t| \le T - h$. $\square$

% MZ write a little simpler

Remark 7.1. *The stability estimate trades an exact functional identity for a quantified deviation ε and a smoothness envelope K. The free parameter h controls the trade-off between the terms ε/h^2 and $\frac{(1+B)K}{3}h$. Choosing h proportional to $(\varepsilon/K)^{1/3}$ minimises the overall bound and yields a typical error scaling of order $O(\varepsilon^{1/3})$ on compact intervals of length $O(1)$.*

Corollary 7.1. *Let the assumptions of Theorem 7.1 hold and define $F(x) := H(\log x) - 1$ on $\mathbb{R}_{>0}$. Then for every $x \in \big(e^{-(T-h)}, e^{T-h}\big)$,*

$$\big|F(x) - \big(\cosh(\sqrt{a}\,\log x) - 1\big)\big| \;\le\; \frac{\delta(h)}{a}\Big(\cosh\big(\sqrt{a}\,|\log x|\big) - 1\Big).$$

In particular, if a is close to 1 and $\delta(h)$ is small, then F is uniformly close to the canonical cost $J(x) = \cosh(\log x) - 1$ on compact subintervals of $(0, \infty)$. When $a = 1$, the estimate simplifies to

$$\big|F(x) - J(x)\big| \le \delta(h)\, J\big(|x|\big).$$

Proof. Apply Theorem 7.1 with $t = \log x$, noting that $F(x) = H(t) - 1$. $\square$

8. Conclusion

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