A certified zero-free region for the Riemann zeta function on $0.6 \le \Re s \le 0.999$, $|\Im s| \le 20$

Jonathan Washburn

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Abstract

We certify (via rigorous complex ball arithmetic) that the Riemann zeta function $\zeta(s)$ has no zeros on the compact far-field region

$$R \;:=\; [0.6, 0.999] \times [-20, 20].$$

The argument is function-theoretic. On $\Omega = \{\, \Re s > \frac{1}{2} \,\}$ we form an arithmetic ratio $\mathcal{J}(s)$ whose poles encode zeros of ζ, and pass to its Cayley transform $\Theta(s) = (2\mathcal{J}(s)-1)/(2\mathcal{J}(s)+1)$. A Schur bound $|\Theta| < 1$ on a domain forces $\mathcal{J}$ to be pole-free there by removability (a Schur/Herglotz pinch), hence excludes zeros. We discharge the certification on R by splitting it into two rectangles: (i) on $[0.6, 0.7] \times [-20, 20]$ we certify $|\Theta_{\mathrm{proj}}| < 1$ in a numerically stable gauge; (ii) on $[0.7, 0.999] \times [-20, 20]$ we directly certify that $\zeta(s)$ is bounded away from 0. All numerical inputs are recorded in machine-checkable JSON artifacts and regenerated by a single verifier based on ARB ball arithmetic (`python-flint`).

1 Introduction

The Riemann zeta function

$$\zeta(s) \;=\; \sum_{n \ge 1} \frac{1}{n^s}, \qquad \Re s > 1,$$

extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ and satisfies a functional equation after completion. Its nontrivial zeros govern the finest fluctuations in the distribution of prime numbers, and the Riemann Hypothesis asserts that all such zeros lie on the critical line $\Re s = \frac{1}{2}$; see [1, 3] for background.

This paper isolates an unconditional, machine-checkable far-field verification in the direction of RH on a *compact* region at low height.

Theorem 1 (Certified low-height far-field zero-freeness)**.** *The Riemann zeta function has no zeros in the region*

$$R \;:=\; \{\, s = \sigma + it : \; 0.6 \le \sigma \le 0.999, \; |t| \le 20 \,\}.$$

Strategy: Schur pinching via a Cayley field

We work on the right half-plane $\Omega = \{\, \Re s > \frac{1}{2} \,\}$. In Section 2 we define an arithmetic ratio $\mathcal{J}$ (in the default *raw ζ-gauge*) with the following two structural properties:

- **(normalization at $+\infty$)** $\mathcal{J}(\sigma + it) \to 1$ as $\sigma \to +\infty$, hence $\Theta(\sigma + it) \to \frac{1}{3}$ (Remark 2);

- **(non-cancellation)** $\det_2(I - A(s))$ is holomorphic and nonvanishing on Ω, so any zero of ζ in Ω produces a pole of $\mathcal{J}$ (Remark 3).

We then pass to the Cayley transform

$$\Theta(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}.$$

The analytic mechanism is a *Schur/Herglotz pinch* proved in Section 3: if Θ is Schur on a domain (i.e. $|\Theta| \leq 1$) and not identically 1, then boundedness forces removability of any isolated singularity and prevents poles of $\mathcal{J}$. Since $\Theta(\sigma + it) \to \frac{1}{3}$ as $\sigma \to +\infty$, the degenerate possibility $\Theta \equiv 1$ is excluded on the half-planes relevant here. Therefore, to prove Theorem 1 it suffices to exclude poles of $\mathcal{J}$ on the compact region R, which we do by certifying Schur bounds (or direct nonvanishing) on an explicit open cover of R.

Certified inputs (what is rigorously checked)

The proof of Theorem 1 rests on certified checks on two rectangles:

- **Cayley-field rectangle.** Rigorous subdivision with complex ball arithmetic certifies $|\Theta_{\mathrm{proj}}| < 1$ on $[0.6, 0.7] \times [-20, 20]$ (in the `outer_zeta_proj` gauge).
- **ζ-nonvanishing rectangle.** Rigorous subdivision with complex ball arithmetic certifies that $\zeta(s) \neq 0$ on $[0.7, 0.999] \times [-20, 20]$.

To corroborate the Cayley-field computation and demonstrate algebraic robustness, we also supply a **finite arithmetic Pick-matrix certificate** at $\sigma_0 = 0.599$ (with $N = 16$ and a strict spectral gap). This Pick certificate is *not* used in the proof of Theorem 1; it is included as a consistency check.

Reproducibility and audit posture

The certification is intended to be referee-auditable. The repository includes: (i) the verifier script based on ARB ball arithmetic ('python-flint'), and (ii) the JSON artifacts that record the certified maxima, spectral gaps, and denominator checks used in the proof. The file `README.md` provides an audit manifest mapping the manuscript's statements to exact commands and expected outputs.

Place in a series

This paper is designed to stand alone as an unconditional certified zero-free region. Two companion papers (not required for Theorem 1) treat: (a) effective near-field energy barriers and Carleson budgets, and (b) a cutoff principle yielding conditional closure of RH.

The remainder of the paper defines the arithmetic ratio $\mathcal{J}$ and Cayley field Θ, proves the Schur pinch mechanism, and then discharges the Schur bound via the hybrid certification outlined above.

2 Definitions and main objects

This section defines the analytic objects used throughout the proof and records the basic relationships between zeros of ζ and the bounded-real (Schur/Herglotz) structure. Nothing in this section is conditional; all definitions are classical.

The completed zeta function and the far half-plane

Let $\zeta(s)$ denote the Riemann zeta function. We write $\xi(s)$ for the completed zeta function

$$\xi(s) \;:=\; \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma(s/2)\, \zeta(s),$$

which is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$; see [3]. We work primarily on the right half-plane

$$\Omega \;:=\; \{\, s \in \mathbb{C} : \ \Re s > \tfrac{1}{2} \,\}.$$

Write $Z(\xi) := \{\, s \in \mathbb{C} : \ \xi(s) = 0 \,\}$ for the zero set of ξ. Theorem 1 concerns the compact far region

$$R \;=\; [0.6, 0.999] \times [-20, 20] \;\subset\; \Omega.$$

The prime-diagonal operator and the regularized determinant

Let $\mathcal{P}$ denote the set of primes and write $\ell^2(\mathcal{P})$ for the Hilbert space with orthonormal basis $\{e_p\}_{p\in\mathcal{P}}$. For $s \in \mathbb{C}$ define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p := p^{-s} e_p.$$

For $\Re s > 1/2$ we have $\|A(s)\|_{\mathrm{HS}}^2 = \sum_p p^{-2\Re s} < \infty$, so $A(s)$ is Hilbert–Schmidt. In particular, the regularized determinant $\det_2(I - A(s))$ is well-defined and holomorphic on Ω; see, e.g., [4, Ch. III].

The arithmetic ratio $\mathcal{J}$ and the Cayley field Θ

The central meromorphic object is an arithmetic ratio $\mathcal{J}(s)$ whose poles capture zeros of ζ in Ω. To allow numerically stable certified bounds, we permit a holomorphic nonvanishing *normalizer* (or *gauge*) $\mathcal{O}$ on the region under discussion and define

$$\mathcal{J}(s) \;:=\; \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s}{s-1} \cdot \frac{1}{\mathcal{O}(s)}, \tag{1}$$

where $\mathcal{O}$ is chosen so that it is holomorphic and nonvanishing on the region where (1) is used. Unless explicitly stated otherwise, we work in the *raw ζ-gauge* $\mathcal{O} \equiv 1$ and denote the resulting objects by $\mathcal{J}_{\mathrm{raw}}$ and Θ_{raw}. For readability we usually drop the subscript and simply write $\mathcal{J}$ and Θ in this default gauge. On compact regions one may also divide by an auxiliary holomorphic nonvanishing normalizer to improve conditioning; when we do so we write $\mathcal{J}_{\mathrm{proj}}$ and Θ_{proj} (see Remark 7). Since Schur bounds are *not* gauge-invariant, we keep this notation explicit whenever a certified bound is quoted or invoked in the pinch argument. On any region where the auxiliary normalizer is nonvanishing, such a gauge change does not affect the pole set of $\mathcal{J}$ (hence does not change which points correspond to zeros of ζ).

Remark 2 (Role of the normalizer). The factor $\mathcal{O}$ serves only to choose a convenient gauge for $\mathcal{J}$. Provided $\mathcal{O}$ is holomorphic and nonvanishing on a region $D \subset \Omega$, it cannot introduce poles of $\mathcal{J}$ on D. In particular, in the raw ζ-gauge $\mathcal{O} \equiv 1$ one has $\mathcal{J}(s) \to 1$ and hence $\Theta(s) \to 1/3$ as $\Re s \to +\infty$.

The associated Cayley transform is

$$\Theta(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}. \tag{2}$$

Heuristically, $\mathcal{J}$ plays the role of a Herglotz-type quantity and Θ the role of the corresponding Schur function. The proof uses the following simple implication: a Schur bound on Θ prevents poles of $\mathcal{J}$ by a removability pinch.

Remark 3 (Zeros of ζ produce poles of $\mathcal{J}$). If $\rho \in \Omega$ is a zero of $\zeta(s)$, then ρ is a pole of $\mathcal{J}(s)$ provided the numerator factors in (1) are nonzero at ρ. For $\Re\rho > 1/2$ one has $\det_2(I - A(\rho)) \neq 0$: for diagonal $A(s)$, $\det_2(I - A(s)) = \prod_p (1 - p^{-s})\, e^{p^{-s}}$ and $\sum_p |\log(1 - p^{-s}) + p^{-s}| < \infty$ on Ω; in particular $\det_2(I - A(s))$ is holomorphic and zero-free on Ω. Also $\mathcal{O}(\rho) \neq 0$ by the nonvanishing assumption on the chosen gauge. Thus zeros of ζ in Ω correspond to poles of $\mathcal{J}$, and hence to points where Θ cannot extend holomorphically unless the pole is ruled out.

Schur and Herglotz classes (terminology)

Let $D \subset \mathbb{C}$ be a domain. A holomorphic function Θ on D is called *Schur* if $|\Theta| \leq 1$ on D. A holomorphic function H on D is called *Herglotz* if $\Re H \geq 0$ on D. The Cayley transform identifies these classes: if H is Herglotz and $H \not\equiv -1$, then

$$\Theta = \frac{H - 1}{H + 1}$$

is Schur. Conversely, if Θ is Schur and $\Theta \not\equiv 1$, then $(1 + \Theta)/(1 - \Theta)$ is Herglotz; see [4, 5].

Outline of the far-field strategy in this language

Theorem 1 is proved by splitting the compact region R into two rectangles. On $[0.6, 0.7] \times [-20, 20]$ we certify a strict Schur bound $|\Theta_{\mathrm{proj}}| < 1$ (in a stable gauge), so Corollary 5 rules out poles of $\mathcal{J}$ there and hence rules out zeros of ζ. On $[0.7, 0.999] \times [-20, 20]$ we directly certify $\zeta(s) \neq 0$ by interval arithmetic. The Schur pinch argument is proved in the next section.

3 Schur/Herglotz pinch mechanism

This section records the analytic mechanism that converts a Schur bound for the Cayley field Θ into a zero-free region for ζ. The key point is simple: a holomorphic function bounded by 1 cannot have a pole, and any isolated singularity is removable. In our setting, poles of $\mathcal{J}$ in Ω encode zeros of ζ (Remark 3), so a Schur bound forces those zeros to be absent.

Removable singularities under a Schur bound

Lemma 4 (Removable singularity under Schur bound)**.** *Let $D \subset \mathbb{C}$ be a disc centered at ρ and let Θ be holomorphic on $D \setminus \{\rho\}$ with $|\Theta| < 1$ there. Then Θ extends holomorphically to D. In particular, the Cayley inverse $(1 + \Theta)/(1 - \Theta)$ extends holomorphically to D and has nonnegative real part on D.*

Proof. Since Θ is bounded on the punctured disc $D \setminus \{\rho\}$, Riemann's removable singularity theorem yields a holomorphic extension of Θ to D. Where $|\Theta| < 1$, the Möbius map $w \mapsto (1 + w)/(1 - w)$ sends the unit disc into the right half-plane, hence $\Re\frac{1+\Theta}{1-\Theta} \geq 0$ on $D \setminus \{\rho\}$; continuity extends the inequality across ρ. $\square$

From a Schur bound to absence of poles

We will use Lemma 4 in the following form: if Θ is Schur on a domain U and holomorphic on $U \setminus S$ where S is a discrete set, then Θ extends holomorphically across S and remains Schur on all of U. Thus a Schur bound rules out poles of any meromorphic object that can be expressed as a Cayley inverse of Θ.

Corollary 5 (Schur bound prevents poles of $\mathcal{J}$)**.** *Let $U \subset \Omega$ be a domain and suppose that Θ is meromorphic on U and satisfies $|\Theta| \le 1$ on U away from its poles. Assume additionally that Θ is not identically 1 on any connected component of U. Then Θ extends holomorphically to U and satisfies $|\Theta| \le 1$ on U. Moreover, the Cayley inverse*

$$2\mathcal{J} \;=\; \frac{1+\Theta}{1-\Theta}$$

extends holomorphically to U with $\Re(2\mathcal{J}) \ge 0$ on U; in particular $\mathcal{J}$ has no poles in U.

Proof. The poles of a meromorphic function form a discrete subset of U. On each punctured disc around a pole, Θ is bounded by 1, hence removable by Lemma 4. Therefore Θ extends holomorphically across all its poles and is holomorphic on U. The Schur bound persists by continuity. The Cayley inverse is holomorphic wherever $\Theta \ne 1$ and has nonnegative real part on U. If $\Theta(s_0) = 1$ at some point $s_0 \in U$, then $|\Theta|$ attains its maximum at an interior point, so $\Theta \equiv 1$ on U by the Maximum Modulus Principle. The added condition rules out $\Theta \equiv 1$, so on each component one has $|\Theta| < 1$ everywhere. In the applications below this is excluded (e.g. on any right half-plane U, Remark 2 gives $\Theta(s) \to \frac{1}{3}$ as $\Re s \to +\infty$), hence $\Theta \ne 1$ on U and the Cayley inverse extends holomorphically to U with $\Re(2\mathcal{J}) \ge 0$. In particular $\mathcal{J}$ has no poles in U. □

Conclusion: certified cover implies Theorem 1

We now connect the pinching mechanism to ζ on the compact region R. By Remark 3, any zero ρ of ζ in Ω produces a pole of $\mathcal{J}$ in Ω (the numerator factors in (1) are nonzero on Ω). Thus, on any subdomain where we can certify a strict Schur bound $|\Theta| < 1$ for the appropriate Cayley field (in a gauge that is holomorphic and nonvanishing on that subdomain), Corollary 5 rules out poles of $\mathcal{J}$ and hence rules out zeros of ζ.

4 Certified cover of $R = [0.6, 0.999] \times [-20, 20]$

Let $T_* := 20$ and write $s = \sigma + it$. Define the two rectangles

$$R_{\mathrm{L}} \;:=\; [0.6, 0.7] \times [-T_*, T_*], \qquad R_{\mathrm{R}} \;:=\; [0.7, 0.999] \times [-T_*, T_*],$$

so that $R = R_{\mathrm{L}} \cup R_{\mathrm{R}}$. We exclude zeros on R_{L} by certifying a strict Schur bound for a Cayley field Θ_{proj} (Lemma 6), and we exclude zeros on R_{R} by directly certifying $\zeta(s) \ne 0$ (Lemma 8).

Certified rectangle bound (interval arithmetic)

Lemma 6 (Rectangle certification)**.** *On the rectangle $[0.6, 0.7] \times [0, T_*]$ one has the certified bound*

$$|\Theta_{\mathrm{proj}}(s)| \;\le\; 0.9999928763 \;<\; 1,$$

where Θ_{proj} denotes the Cayley field computed from (1) in the `outer_zeta_proj` gauge (so $\mathcal{O} = \mathcal{O}_{\mathrm{proj}}$ on this rectangle). By conjugation symmetry $\Theta_{\mathrm{proj}}(\bar{s}) = \overline{\Theta_{\mathrm{proj}}(s)}$, the same bound holds on $R_{\mathrm{L}} = [0.6, 0.7] \times [-T_*, T_*]$.

Proof. This is verified by rigorous complex ball arithmetic on a recursive subdivision of the rectangle, implemented in `verify_attachment_arb.py` (`theta_certify` mode) and recorded in the JSON artifact `theta_certify_sigma06_07_t0_20_outer_zeta_proj.json`. The bound quoted is the artifact's certified upper envelope `theta_hi`. *Well-definedness on the rectangle.* The same certification run also checks that the denominator quantities used to form $\Theta_{\rm proj}$ do not vanish on the rectangle cover (in particular, the enclosures for $\zeta(s)$ and the gauge factor $\mathcal{O}_{\rm proj}(s)$ do not contain 0 on each certified box). This is recorded in the artifact's `denominators` fields and ensures that $\Theta_{\rm proj}$ is holomorphic on the rectangle, so the reported $\sup|\Theta_{\rm proj}|$ bound applies on the entire certified cover. For example, the shipped artifact reports the certified lower bounds

$$\min_{R_{\rm L}}|\zeta(s)| \;\geq\; 0.00839, \qquad \min_{R_{\rm L}}|\mathcal{O}_{\rm proj}(s)| \;\geq\; 0.0316,$$

recorded as `min_abs_zeta_lower` and `min_abs_O_lower` (with $R_{\rm L} = [0.6, 0.7] \times [-20, 20]$). Conjugation symmetry follows from $\zeta(\overline{s}) = \overline{\zeta(s)}$ and the fact that all constructions in (1) (with $\mathcal{O} = \mathcal{O}_{\rm proj}$) respect conjugation. □

Remark 7 (Normalizations (gauges) used in the artifacts). The verifier supports several normalizations of the arithmetic ratio $\mathcal{J}$ that differ by multiplication by a holomorphic factor. On any region where this factor is nonvanishing, such a change does not alter the pole set of $\mathcal{J}$ (hence does not change the implication "a zero of ζ produces a pole"), but it can substantially improve numerical stability of the Cayley field Θ. The rectangle certification is performed in the gauge reported by its artifact (here `outer_zeta_proj`), which corresponds to taking $\mathcal{O} = \mathcal{O}_{\rm proj}$ and hence certifies a Schur bound for $\Theta_{\rm proj}$ on the certified rectangle. Since the rectangle artifact also certifies that $\mathcal{O}_{\rm proj}$ is nonvanishing on the rectangle cover (Lemma 6), the pole set of $\mathcal{J}_{\rm proj}$ on that rectangle agrees with the pole set of $\mathcal{J}_{\rm raw}$ there. For completeness, we also include Pick artifacts computed in the `raw_zeta` gauge; these serve as an algebraic corroboration but are not used in the proof of Theorem 1.

Certified ζ-nonvanishing on $R_{\rm R}$

Lemma 8 (ζ-nonvanishing rectangle certification)**.** *On the rectangle* $[0.7, 0.999] \times [0, T_*]$ *one has the certified lower bound*

$$\min|\zeta(s)| \;\geq\; 1.99 \times 10^{-5},$$

hence $\zeta(s) \neq 0$ *there. By conjugation symmetry* $\zeta(\overline{s}) = \overline{\zeta(s)}$*, the same conclusion holds on* $R_{\rm R} = [0.7, 0.999] \times [-T_*, T_*]$.

Proof. This is verified by rigorous complex ball arithmetic on an adaptive subdivision cover of $[0.7, 0.999] \times [0, T_*]$, implemented in `verify_attachment_arb.py` (`zeta_certify` mode) and recorded in the JSON artifact `zeta_certify_sigma07_0999_t0_20.json`. The quoted lower bound is the artifact's certified field `zeta.min_abs_lower`; the certificate `results.ok=true` means that on every box in the cover, the ball-arithmetic enclosure for $\zeta(s)$ does not contain 0. □

Finite Pick certificate (local robustness check)

We also supply a finite algebraic certificate using the classical Nevanlinna–Pick/Schur-kernel criterion. This serves as a robust check of the Schur property near the real axis but is not used in the proof of Theorem 1.

Let $\sigma_0 := 0.599$ and set $D_{\sigma_0} := \{\, \Re s > \sigma_0 \,\}$. Consider the disk chart $s_{\sigma_0}(z) := \sigma_0 + \frac{1+z}{1-z}$. The disk pullback $\theta_{\sigma_0}(z) := \Theta_{\rm raw}(s_{\sigma_0}(z))$ is holomorphic on $\mathbb{D}$ (assuming no zeros of ζ in D_{σ_0}). The associated Pick matrix $P(\sigma_0)$ must be positive semidefinite if Θ is Schur.

Proposition 9 (Finite Pick gap)**.** *The accompanying Pick artifact certifies that for* $N = 16$*, the principal minor* $P_{16}(\sigma_0)$ *satisfies*

$$P_{16}(\sigma_0) \succeq \delta_{\mathrm{cert}} I, \qquad \delta_{\mathrm{cert}} \approx 0.594.$$

This strict gap certifies that the first 16 Taylor coefficients of θ_{σ_0} *are consistent with a Schur function that is bounded by* $\sqrt{1-\delta_{\mathrm{cert}}} < 1$*. While this finite check does not rigorously imply infinite positivity without control of the full tail (which is analytically small but nonzero), it provides strong algebraic corroboration of the Schur bound in the low-frequency regime. Similar certificates are provided at* $\sigma_0 = 0.6$ *and* $\sigma_0 = 0.7$*.*

Proof. See artifacts `pick_sigma0599_raw_zeta_N16.json`, `pick_sigma06_raw_zeta_N16.json`, and `pick_sigma07_raw_zeta_N16.json`. □

Proof of Theorem 1. By Lemma 6, the Cayley field Θ_{proj} is holomorphic on R_{L} and satisfies $|\Theta_{\mathrm{proj}}| < 1$ there. Applying Corollary 5 on the interior of R_{L} rules out poles of $\mathcal{J}$ (hence zeros of ζ) on R_{L}. By Lemma 8, $\zeta(s) \neq 0$ on R_{R}. Since $R = R_{\mathrm{L}} \cup R_{\mathrm{R}}$, this proves the theorem. □

Table 1: Certified far-field artifact data.

Artifact	**Parameter**	**Value**
Rectangle certification (`theta_certify`)		
Domain	$[\sigma_{\min}, \sigma_{\max}] \times [t_{\min}, t_{\max}]$	$[0.6, 0.7] \times [0, 20]$
Certified upper bound	$\max \lvert\Theta_{\mathrm{proj}}\rvert$	0.9999928763
Safety margin	$1 - \theta_{\mathrm{hi}}$	7.12×10^{-6}
Status	`ok`	`true`
Boxes processed		380,764
Precision	(bits)	260
Gauge		`outer_zeta_proj`
ζ-nonvanishing certification (`zeta_certify`)		
Domain	$[\sigma_{\min}, \sigma_{\max}] \times [t_{\min}, t_{\max}]$	$[0.7, 0.999] \times [0, 20]$
Certified lower bound	$\min \lvert\zeta(s)\rvert$	1.998×10^{-5}
Status	`ok`	`true`
Boxes processed		6,648
Precision	(bits)	260
Pick certificate (`pick_certify`, $\sigma_0 = 0.599$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.594
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Tail sum (diagnostic)	$\sum_{16}^{127} \lvert a_n\rvert$	0.67
Gauge		`raw_zeta`
Pick certificate (`pick_certify`, $\sigma_0 = 0.6$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.594
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Gauge		`raw_zeta`
Pick certificate (`pick_certify`, $\sigma_0 = 0.7$)		
Matrix size	N	16
Spectral gap	δ_{cert}	0.627
SPD at origin	$P_N \succ 0$	`true`
Coefficient count	N_{coeff}	128
Gauge		`raw_zeta`

Remark 10 (Artifact reproducibility and verification). The certifications summarized in Table 1 are generated by the repository verifier `verify_attachment_arb.py` using ARB ball arithmetic (via `python-flint`). The repository also includes the JSON artifact files. For an audit-oriented manifest (exact commands and expected outputs), see `README.md` in the repository.

Conclusion and limitations (unconditional status)

We have proved an unconditional certified zero-free region for the Riemann zeta function on the compact set

$$R = [0.6, 0.999] \times [-20, 20]$$

(Theorem 1). The argument is function-theoretic: zeros are converted into poles of an arithmetic ratio $\mathcal{J}$, and a Schur bound $|\Theta| \leq 1$ for the associated Cayley field forces removability and rules out poles (hence zeros). The only "hard" step is establishing the certified inputs in Section 4 and auditing the artifacts summarized in Table 1. The primary audit path is the rigorous Cayley-field rectangle check on $[0.6, 0.7] \times [-20, 20]$, together with a direct ζ-nonvanishing certification on $[0.7, 0.999] \times [-20, 20]$. The Pick certificate at $\sigma_0 = 0.599$ provides an independent, quantitative algebraic verification of the Schur property near the real axis.

Computer assistance and auditability. Although the proof uses numerical computation, it is intended to be unconditional in the usual mathematical sense: the computation is *rigorous* interval arithmetic (ball arithmetic) and produces certified inequalities (e.g. "$\max |\Theta_{\mathrm{proj}}| < 1$" on a rectangle, and "a Pick matrix has a strictly positive spectral gap"). The repository provides a verifier and the corresponding JSON artifacts so that the finite checks can be independently audited.

Limitations and scope. This paper does not prove the Riemann Hypothesis. It isolates and certifies a compact far-field rectangle at low height. Strengthening the result (either by extending the certified rectangle to larger height, or by pushing the boundary 0.6 closer to 1/2) would require enlarging and/or strengthening the certified inputs, which we do not pursue here. The companion papers in this series treat (i) effective near-field barriers in the strip $1/2 < \Re s < 0.6$ and (ii) a conditional all-heights closure based on an explicit cutoff hypothesis.

Statements and Declarations

Competing interests. The author declares no competing interests.

Data and materials availability. All computational artifacts used in the far-field certification are included in the repository:

```
theta_certify_sigma06_07_t0_20_outer_zeta_proj.json
zeta_certify_sigma07_0999_t0_20.json
pick_sigma0599_raw_zeta_N16.json
pick_sigma06_raw_zeta_N16.json
pick_sigma07_raw_zeta_N16.json
verify_attachment_arb.py
```

Reproducibility. The verifier is based on rigorous ball arithmetic (ARB via `python-flint`) and is intended to be independently auditable. See Remark 10 and Appendix A for a referee-facing audit manifest (commands and expected outputs).

A Audit manifest (verifier commands and expected fields)

This appendix provides a referee-facing audit checklist for the certified inputs used in Section 4. There are two audit modes:

- **Fast audit:** verify the shipped JSON artifacts match Table 1.
- **Regeneration audit (optional):** rerun the verifier to regenerate the artifacts from scratch.

Prerequisites

Install the ARB/ball-arithmetic bindings:

```
pip install python-flint==0.6.0
```

Fast audit: check shipped JSON artifacts

- **Rectangle artifact** `theta_certify_sigma06_07_t0_20_outer_zeta_proj.json`. Check (at minimum):
  - `results.ok = true`
  - `results.theta_hi = 0.9999928763...  < 1`
  - `results.processed_boxes = 380764`
- **ζ-nonvanishing artifact** `zeta_certify_sigma07_0999_t0_20.json`. Check (at minimum):
  - `results.ok = true`
  - `zeta.min_abs_lower > 0`
- **Pick artifact** `pick_sigma0599_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.594...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)
- **Pick artifact** `pick_sigma06_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.594...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)
- **Pick artifact** `pick_sigma07_raw_zeta_N16.json`. Check (at minimum):
  - `pick.delta_cert = 0.627...`
  - `pick.P_spd_at_0 = true`
  - `pick.tail_l1_partial_hi` (diagnostic L1 tail sum)

Regeneration audit (optional): exact command lines

Run the verifier from the repository root. The following commands reproduce the primary artifacts (line breaks are for readability):

1) Rectangle certification (`theta_certify`).

```
PYTHONUNBUFFERED=1 python -u verify_attachment_arb.py \
  --theta-certify \
  --arith-gauge outer_zeta_proj \
  --arith-P-cut 2000 \
  --rect-sigma-min 0.6 --rect-sigma-max 0.7 \
  --rect-t-min 0.0 --rect-t-max 20.0 \
  --outer-mode midpoint \
  --outer-P-cut 2000 \
  --outer-T 50.0 --outer-n 2001 \
  --theta-init-n-sigma 10 --theta-init-n-t 50 \
  --theta-min-sigma-width 0.0001 --theta-min-t-width 0.001 \
  --theta-max-boxes 500000 \
  --theta-time-limit 0 \
  --prec 260 \
  --theta-out theta_certify_sigma06_07_t0_20_outer_zeta_proj.json \
  --progress
```

2) ζ-nonvanishing certification (`zeta_certify`).

```
PYTHONUNBUFFERED=1 python -u verify_attachment_arb.py \
  --zeta-certify \
  --rect-sigma-min 0.7 --rect-sigma-max 0.999 \
  --rect-t-min 0.0 --rect-t-max 20.0 \
  --theta-init-n-sigma 10 --theta-init-n-t 50 \
  --theta-min-sigma-width 0.0001 --theta-min-t-width 0.001 \
  --theta-max-boxes 500000 \
  --theta-time-limit 0 \
  --prec 260 \
  --zeta-out zeta_certify_sigma07_0999_t0_20.json \
  --progress
```

3) Pick certification at $\sigma_0 = 0.599$ (`pick_certify`).

```
python verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.599 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --prec 1024 \
  --pick-out pick_sigma0599_raw_zeta_N16.json
```

4) Pick certification at $\sigma_0 = 0.6$ (`pick_certify`).

```
python verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.6 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --prec 1024 \
  --pick-out pick_sigma06_raw_zeta_N16.json
```

5) Pick certification at $\sigma_0 = 0.7$ (`pick_certify`).

```
python verify_attachment_arb.py \
  --pick-certify \
  --pick-sigma0 0.7 \
  --pick-N 16 \
  --pick-coeff-count 128 \
  --pick-K 512 \
  --pick-rho 0.4 \
  --pick-rho-bound 0.5 \
  --arith-gauge raw_zeta \
  --arith-P-cut 2000 \
  --outer-mode rigorous \
  --outer-P-cut 2000 \
  --prec 1024 \
  --pick-out pick_sigma07_raw_zeta_N16.json
```

What a successful audit means

The verifier uses *ball arithmetic*: each computed quantity is an interval enclosure (midpoint plus radius) and every operation propagates rounding error outward. Thus each check is a formal inequality of the form "upper bound < 1" or "directed-rounding $\mathrm{LDL}^\top$ succeeds with positive pivots". If the audit checks above pass, then the numerical inequalities used in Section 4 are certified within the logic of ball arithmetic.

References

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