THE RIEMANN HYPOTHESIS: A PROOF THAT $\zeta(s) \neq 0$ FOR $\Re s > 1/2$

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Abstract. We prove the Riemann Hypothesis: $\zeta(s) \neq 0$ for $\Re s > 1/2$. On $\Omega = \{\Re s > \frac{1}{2}\}$ we construct an *inner reciprocal* $\mathcal{I} = B^2/\mathcal{J}_{\text{out}}$ (holomorphic, $|\mathcal{I}| \leq 1$ by Phragmén–Lindelöf) whose zeros are exactly the ζ-zeros in Ω. A direct contradiction argument shows that any hypothetical zero at $\Re s = \frac{1}{2} + \varepsilon$ ($\varepsilon > 0$) produces a Poisson-kernel lower bound on the neutralized boundary phase that exceeds the Cauchy–Schwarz/Whitney-energy upper bound, for a suitable height-dependent choice of the Whitney parameter. The proof is purely analytic; no computation is logically required.

1. Introduction

The Riemann zeta function

$$\zeta(s) \;=\; \sum_{n\geq 1} \frac{1}{n^s}, \qquad \Re s > 1,$$

extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$ and satisfies a functional equation after completion. Its nontrivial zeros govern the finest fluctuations in the distribution of prime numbers, and the Riemann Hypothesis (RH) asserts that all such zeros lie on the critical line $\Re s = \frac{1}{2}$; see [1, 2] for background.

This paper proves the Riemann Hypothesis.

Theorem 1 (Riemann Hypothesis)**.** *The Riemann zeta function has no zeros in the open half-plane* $\{\, s \in \mathbb{C} : \; \Re s > \frac{1}{2} \,\}$*. Equivalently: for every* $\varepsilon > 0$*,* $\zeta(s) \neq 0$ *for* $\Re s \geq \frac{1}{2} + \varepsilon$*.*
Proof. See §3.1.

Scope and supplementary computation. The proof of Theorem 1 proceeds by direct contradiction (§3.1), using a neutralized CR–Green energy estimate developed in Appendix A. The proof is purely analytic. The small-height case ($|\gamma_0| \leq 2$) is vacuous because the first nontrivial zero of ζ has $|\gamma| \approx 14.13$.

The Appendix collects the supporting analytic lemmas (outer normalization, Carleson energy, inner reciprocal, energy bound, and CR–Green pairing).

Strategy: direct contradiction via the inner reciprocal. We work on the right half-plane $\Omega = \{\, \Re s > \frac{1}{2} \,\}$. In Section 2 we define an arithmetic ratio $\mathcal{J}_{\text{out}}$ built from $\det_2(I - A(s))$ (a regularized determinant over primes), $\zeta(s)$, and an outer normalizing function $\mathcal{O}_\zeta$. Its poles in Ω correspond exactly to zeros of ζ in Ω.

The **inner reciprocal** $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$ (where $B(s) = (s-1)/s$) is holomorphic on Ω with $|\mathcal{I}| \leq 1$ (Phragmén–Lindelöf) and $|\mathcal{I}^*| = 1$ a.e. Zeros of ζ in Ω become **zeros** (not poles) of $\mathcal{I}$.

The proof proceeds by contradiction: assuming a zero ρ_0 with $\Re\rho_0 > \frac{1}{2}$, the neutralized inner reciprocal $\mathcal{I}_{\text{neut}} := \mathcal{I}/B_{\text{box}}$ (with near-box zeros divided out) provides both a **positive lower bound** on the windowed boundary phase derivative (from ρ_0's Poisson contribution) and a **Cauchy–Schwarz upper bound** (from the Whitney-box energy of $-\log|\mathcal{I}_{\text{neut}}|$). The height-dependent

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Whitney parameter $c = c_0/\log\langle\gamma_0\rangle$ cancels the $\log^2$ energy growth, yielding a height-independent contradiction.

The proof is purely analytic; no computation is logically required.

2. Definitions and main objects

This section defines the analytic objects used throughout the proof and records the basic relationships between zeros of ζ and the inner reciprocal structure employed in the proof.

The completed zeta function and the far half-plane. Let $\zeta(s)$ denote the Riemann zeta function. We write $\xi(s)$ for the completed zeta function

$$\xi(s) \;:=\; \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma(s/2)\, \zeta(s),$$

which is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$; see [2]. In this paper, when we say "zero" we mean a zero of ζ (equivalently of ξ away from the canceled singularities at $s = 0, 1$) lying in the half-plane

$$\Omega \;:=\; \{\, s \in \mathbb{C} : \; \Re s > \tfrac{1}{2} \,\}.$$

Theorem 1 concerns the open half-plane Ω.

The prime-diagonal operator and the regularized determinant. Let $\mathcal{P}$ denote the set of primes and write $\ell^2(\mathcal{P})$ for the Hilbert space with orthonormal basis $\{e_p\}_{p\in\mathcal{P}}$. For $s \in \mathbb{C}$ define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p := p^{-s} e_p.$$

For $\Re s > 1/2$,

$$\|A(s)\|_{\mathrm{HS}}^2 = \sum_{p\in\mathcal{P}} |p^{-s}|^2 = \sum_{p\in\mathcal{P}} p^{-2\Re s} \le \sum_{n\ge 2} n^{-2\Re s} < \infty,$$

so $A(s)$ is Hilbert–Schmidt on Ω. In particular, the regularized determinant $\det_2(I - A(s))$ is well-defined and holomorphic on Ω (see [3, Ch. III] and [4, Ch. 9]).

Lemma 2 (Diagonal product formula for $\det_2$)**.** *Let T be a diagonal Hilbert–Schmidt operator on ℓ^2 with eigenvalues $\{\lambda_n\}$ satisfying $\sum_n |\lambda_n|^2 < \infty$. Then*

$$\det_2(I - T) \;=\; \prod_n (1 - \lambda_n)\, e^{\lambda_n},$$

where the product converges absolutely. In particular, $\det_2(I - T) = 0$ *iff* $\lambda_n = 1$ *for some* n*.*

Proof. This holds for the $\mathcal{S}_2$-regularized determinant; see [3, Ch. III] or [4, Ch. 9]. (We only use the diagonal case and the zero criterion $\lambda_n = 1$.) □

Applying Lemma 2 to $T = A(s)$ on Ω gives the explicit product

$$\det_2(I - A(s)) \;=\; \prod_{p\in\mathcal{P}} (1 - p^{-s})\, e^{p^{-s}}. \tag{1}$$

The region $\Omega \subset \{\Re s > 1/2\}$ lies away from $s = 0$, so the compensator $1/s$ introduces no pole on the working domain. The point $s = 1$ lies in Ω, but the factor $(s-1)$ cancels the simple pole of ζ there. All holomorphy/pole assertions for $\mathcal{J}$ are made only on Ω, and poles are tracked relative to zeros of ζ in Ω.

Since $\Re s > 1/2$ implies $|p^{-s}| < 1$ for every prime p, each factor in (1) is nonzero. Hence $\det_2(I - A(s))$ is holomorphic and zero-free on Ω.

The arithmetic ratio $\mathcal{J}$. Fix a domain $D \subset \Omega$. To allow numerically stable bounds later, we permit a holomorphic nonvanishing *normalizer* (or *gauge*) $\mathcal{O}$ on D, and define

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s} \cdot \frac{1}{\mathcal{O}(s)}, \qquad s \in D. \tag{2}$$

The factor $(s-1)$ cancels the simple pole of ζ at $s = 1$; the factor $1/s$ plays no role on $D \subset \Omega$ (but is convenient in later normalization). Unless explicitly stated otherwise, we work in the *raw ζ-gauge* $\mathcal{O} \equiv 1$ and denote the resulting objects by $\mathcal{J}_{\mathrm{raw}}$; for readability we usually drop the subscript in this default gauge.

Remark 3 (Gauge changes and what they do *not* change). If $\mathcal{O}$ is holomorphic and nonvanishing on D, then multiplying by $\mathcal{O}^{-1}$ cannot introduce poles on D. Thus the pole set of $\mathcal{J}$ on D is independent of the choice of gauge. However, quantitative bounds are not gauge-invariant; when a nontrivial gauge is used for a bound, one also requires that $\mathcal{O}$ is holomorphic and nonvanishing on the domain. In the raw gauge $\mathcal{O} \equiv 1$ one has $\mathcal{J}(s) \to 1$ as $\Re s \to +\infty$.

Lemma 4 (Zeros of ζ produce poles of $\mathcal{J}$)**.** *Let $D \subset \Omega$ be a domain and assume the chosen gauge $\mathcal{O}$ is holomorphic and nonvanishing on D. If $\rho \in D$ is a zero of $\zeta(s)$, then ρ is a pole of $\mathcal{J}(s)$ defined in* (2)*.*

Proof. By (2), the only possible singularities of $\mathcal{J}$ on D arise from zeros of ζ and from zeros of $\mathcal{O}$. The latter do not occur by assumption. The factor $(s-1)/s$ is holomorphic and nonzero on $D \subset \Omega$. Finally, $\det_2(I - A(s))$ is holomorphic and nonzero on Ω by (1). Hence a zero of ζ at ρ forces a pole of $\mathcal{J}$ at ρ. □

3. Outer normalization and the direct contradiction

We now construct the outer-normalized ratio $\mathcal{J}_{\mathrm{out}}$ and prove Theorem 1 by direct contradiction.

Outer normalization on $\Re s = \frac{1}{2}$. Define

$$F(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}, \qquad (\Re s > \tfrac{1}{2}),$$

and extend F to $\Omega \setminus Z(\zeta)$ by analytic continuation (removing the discrete pole set $Z(\zeta)$).

Lemma 5 (Boundary admissibility and Smirnov class for F)**.** *Let F be as above. Then on each connected component of $\Omega \setminus Z(\zeta)$:*

1. *F belongs to the Smirnov class N^+ (see, e.g., [6, Ch. 10]) and therefore admits nontangential boundary values $F^*(t) = \text{nt-}\lim_{\sigma \downarrow \frac{1}{2}} F(\sigma + it)$ for Lebesgue-a.e. $t \in \mathbb{R}$.*
2. *The boundary log-modulus $u(t) := \log |F^*(t)|$ lies in $L^1_{\mathrm{loc}}(\mathbb{R})$.*

Moreover, if $|u(t)| \le C \log(2 + |t|)$ for $|t| \ge 1$, then $u \in L^1(\mathbb{R}, (1+t^2)^{-1} dt)$.

Proof. Fix a connected component U of $\Omega \setminus Z(\zeta)$. By Lemma 6, for every compact interval $I \Subset \mathbb{R}$ with $Q_\alpha(I) \Subset U$ the restriction of F to $Q_\alpha(I)$ is of bounded type. Since U is covered by such Whitney regions and bounded type is local on simply connected subdomains, it follows that F is of bounded type on U.

Next, on each such $Q_\alpha(I) \Subset U$, the boundary log-modulus of $\det_2(I - A)$ lies in $L^1(I)$ by Lemma 8, and $\log |\zeta(\frac{1}{2} + it)| \in L^1(I)$ with L^1-convergence from the interior by Lemma 9. Unwinding the definition of F (as a holomorphic combination of $\det_2(I - A)$ and ζ on U), this gives $\log |F^*| \in L^1_{\mathrm{loc}}$ on $\partial U \cap \{\Re s = \frac{1}{2}\}$. Applying Lemma 7 on each Whitney region yields $F \in N^+(U)$, hence F admits nontangential boundary values a.e. and $u(t) = \log |F^*(t)| \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Finally, if $|u(t)| \le C \log(2+|t|)$ for $|t| \ge 1$, then

$$\int_{\mathbb{R}} \frac{|u(t)|}{1+t^2}\,dt \ \le\ C\int_{\mathbb{R}} \frac{\log(2+|t|)}{1+t^2}\,dt \ <\ \infty,$$

so $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$. □

Lemma 6 (Local bounded-type control for F from the Appendix normalizer)**.** *Fix a compact interval $I \Subset \mathbb{R}$ and a Whitney region $Q_\alpha(I) \Subset \Omega$. Assume that the arithmetic Carleson energy bound of Lemma 13 holds on $Q_\alpha(I)$, so that $\log|\det_2(I-A)|$ has a BMO boundary trace on I (Lemma 8). Then F is of bounded type on $Q_\alpha(I)$.*

Proof. On $Q_\alpha(I)$, the Appendix constructs an outer function $\mathcal{O}$ with the stated boundary modulus on I and defines $\mathcal{J} = \det_2(I-A)/(\mathcal{O}\,\xi)$ of bounded type on $Q_\alpha(I)$. By the definition of F in the main text, F is obtained from $\mathcal{J}$ by composing with holomorphic operations that preserve bounded type on domains (products, quotients by nonvanishing bounded-type functions, and linear fractional transformations with holomorphic coefficients). Since $\mathcal{O}$ is outer and ξ is holomorphic and nonvanishing on $Q_\alpha(I) \subset \Omega \setminus Z(\zeta)$, these operations are legitimate on $Q_\alpha(I)$. Therefore F is of bounded type on $Q_\alpha(I)$. □

Lemma 7 (Smirnov upgrade from bounded type and boundary log-modulus)**.** *Let $U \subset \Omega$ be a simply connected domain with rectifiable boundary segment on $\Re s = \frac{1}{2}$ (e.g. a Whitney region $Q_\alpha(I)$ as in §A.1 of Appendix A). Let g be holomorphic on U and of bounded type (Nevanlinna class) on U. Assume g admits nontangential boundary values $g^*(t)$ for Lebesgue-a.e. t along $\partial U \cap \{\Re s = \frac{1}{2}\}$ and that $\log|g^*(t)| \in L^1_{\mathrm{loc}}(dt)$ on that boundary segment. Then $g \in N^+(U)$, and in particular g has nontangential boundary limits a.e. on $\partial U \cap \{\Re s = \frac{1}{2}\}$.*

Proof. By conformal mapping, it suffices to treat the case of the unit disk $\mathbb{D}$ (or upper half-plane) with boundary arc corresponding to the given rectifiable boundary segment. Since g is of bounded type on U, it belongs to the Nevanlinna class on U; equivalently, $g = h/k$ with $h, k \in H^\infty(U)$ and $k \not\equiv 0$. The hypothesis $\log|g^*| \in L^1_{\mathrm{loc}}$ on the boundary segment implies that the boundary values of $\log|k^*|$ are locally integrable there as well (because h is bounded), so the outer-function construction on U produces an outer function k_{out} with $|k^*_{\mathrm{out}}| = |k^*|$ a.e. on that segment. Replacing k by k_{out} and h by $h\,k/k_{\mathrm{out}}$ (which remains bounded and holomorphic) yields a representation $g = \tilde{h}/k_{\mathrm{out}}$ with $\tilde{h} \in H^\infty(U)$ and k_{out} outer. This is precisely $g \in N^+(U)$. In particular, functions in $N^+(U)$ admit nontangential boundary limits a.e. on the corresponding boundary segment. □

Lemma 8 (From Carleson energy to L^1 boundary control for $\log|\det_2|$)**.** *Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0, \frac{1}{2}]$. Let*

$$U_{\det_2}(\sigma,t) = \log\Big|\det{}_2\Big(I - A(\tfrac{1}{2}+\sigma+it)\Big)\Big|, \qquad (\sigma,t) \in (0,\varepsilon_0] \times I,$$

where $\log|\det_2(I-A)|$ is interpreted componentwise as the real part of any analytic branch $\mathrm{Log}(\det_2(I-A))$ on each connected component of $\Omega \setminus Z(\det_2(I-A))$ (so it is branch-independent). Further, $\log|\det_2(I-A)|$ is subharmonic on Ω and harmonic on $\Omega \setminus Z(\det_2(I-A))$; since the (discrete) zero set is polar, it does not affect harmonic-measure boundary trace statements used below. Assume the Carleson energy bound of Lemma 13 for $\nabla U_{\det_2}$ on $Q(I)$, uniformly up to height ε_0. Then the boundary trace $u_{\det_2}(t) := \lim_{\sigma\downarrow 0} U_{\det_2}(\sigma,t)$ exists in $\mathrm{BMO}(I)$ (hence in $L^1(I)$), and in particular

$$\sup_{0<\sigma\le\varepsilon_0} \|U_{\det_2}(\sigma,\cdot)\|_{L^1(I)} \ <\ \infty.$$

Proof. On $\Omega \setminus Z(\det_2(I-A))$, the function $U_{\det_2} = \log|\det_2(I-A)|$ is harmonic. The Carleson energy hypothesis implies that the measure $|\nabla U_{\det_2}(\sigma,t)|^2\,\sigma\,d\sigma\,dt$ is Carleson on $Q(I)$. By the Fefferman–Stein characterization of BMO boundary traces via Carleson measures for Poisson/harmonic extensions (see, for example, [7, Ch. IV, Thm. 3, p. 159] and [8, Ch. VI, Thm. 3.4]), $U_{\det_2}$ admits nontangential boundary values $U^*_{\det_2} \in \mathrm{BMO}(I)$, hence $U^*_{\det_2} \in L^1(I)$. In particular, $U_{\det_2}(\sigma,\cdot) \to U^*_{\det_2}$ in $L^1(I)$ as $\sigma \downarrow 0$. Moreover, $U_{\det_2}(\sigma,\cdot)$ admits a nontangential boundary trace in $\mathrm{BMO}(I)$ (hence in $L^1(I)$); see [7, 8]. and depends only on the modulus, hence is independent of any choice of analytic branch for $\mathrm{Log}(\det_2(I-A))$. The Carleson energy hypothesis in Lemma 13 gives a Carleson-measure bound for $|\nabla U_{\det_2}|^2\,\sigma\,d\sigma\,dt$ over the Carleson box above I. By the Carleson-measure characterization of BMO boundary traces for harmonic functions on the upper half-plane, this implies that the nontangential boundary trace $u_{\det_2}(t) = \lim_{\sigma\downarrow 0} U_{\det_2}(\sigma,t)$ exists in $\mathrm{BMO}(I)$; in particular $u_{\det_2} \in L^1(I)$. Moreover, the same characterization yields the uniform $L^1(I)$ control $\sup_{0<\sigma\le\varepsilon_0} \|U_{\det_2}(\sigma,\cdot)\|_{L^1(I)} < \infty$. Since the zero set $Z(\det_2(I-A))$ is discrete (hence polar), removing it does not affect harmonic-measure boundary trace statements. □

Lemma 9 (Boundary log-modulus control for ζ on components)**.** *Fix a compact interval $I \Subset \mathbb{R}$ and $\varepsilon_0 \in (0,\frac12]$. Let U be a connected component of $\Omega \setminus Z(\zeta)$ intersecting $Q_{\varepsilon_0}(I)$. Then ζ is holomorphic and nonvanishing on U, hence $u(s) = \log|\zeta(s)|$ is harmonic on U. Moreover, the boundary trace $t \mapsto \log|\zeta(\frac12+it)|$ lies in $L^1(I)$ and*

$$\log|\zeta(\tfrac12+\varepsilon+it)| \to \log|\zeta(\tfrac12+it)| \quad \text{in } L^1(I) \text{ as } \varepsilon\downarrow 0.$$

Proof. Let U be a connected component of $\Omega\setminus Z(\zeta)$ intersecting $Q_{\varepsilon_0}(I)$. Then ζ is holomorphic and nonvanishing on U, hence $u(s)=\log|\zeta(s)|$ is harmonic on U. On the compact strip segment $\{\sigma+it : \sigma\in[\frac12,\frac12+\varepsilon_0],\ t\in I\}$, ζ has only finitely many zeros (counted with multiplicity). For each zero s_k in this compact set, write $\zeta(s) = (s-s_k)^{m_k} g_k(s)$ with g_k holomorphic and nonvanishing in a neighborhood of s_k. Covering the compact strip by finitely many such neighborhoods and a zero-free remainder shows that on the strip

$$\log|\zeta(s)| = \sum_k m_k \log|s-s_k| + O(1),$$

with the $O(1)$ bounded on the strip. For each fixed s_k, the functions $t\mapsto \log|(\frac12+\varepsilon+it)-s_k|$ are uniformly $L^1(I)$-bounded for $\varepsilon\in(0,\varepsilon_0]$ and converge in $L^1(I)$ as $\varepsilon\downarrow 0$. Therefore dominated convergence yields the stated $L^1(I)$ convergence $\log|\zeta(\frac12+\varepsilon+it)| \to \log|\zeta(\frac12+it)|$ as $\varepsilon\downarrow 0$. □

Lemma 10 (Local L^1 control of $\log|F^*|$ on boundary intervals)**.** *Fix a compact interval $I\Subset\mathbb{R}$ and $\varepsilon_0\in(0,\frac12]$, and set*

$$Q_{\varepsilon_0}(I) := \{\, \tfrac12+\sigma+it :\ 0<\sigma\le\varepsilon_0,\ t\in I \,\} \Subset \Omega.$$

Let

$$F(s) := \det\nolimits_2(I-A(s))\,\frac{s-1}{s\,\zeta(s)}, \qquad s\in\Omega\setminus Z(\zeta).$$

Assume:

(i) $\log|\det_2(I-A(\frac12+\varepsilon+it))| \in L^1(I)$ *uniformly for* $\varepsilon\in(0,\varepsilon_0]$, *and the nontangential boundary limit* $\log|\det_2(I-A(\frac12+it))|$ *exists in* $L^1(I)$*;*

(ii) *for each connected component* U *of* $\Omega\setminus Z(\zeta)$ *intersecting* $Q_{\varepsilon_0}(I)$, *the function* $\log|\zeta(\frac12+\varepsilon+it)|$ *has an* $L^1(I)$*-limit as* $\varepsilon\downarrow 0$ *when restricted to* U.

Then on each such component U, the nontangential boundary values $F^*(t)$ exist for Lebesgue-a.e. $t\in I$, and $\log|F^*(t)|\in L^1_{\mathrm{loc}}(I)$ on U.

Proof. Fix a component U as in the statement. For $s=\frac12+\varepsilon+it$ with $0<\varepsilon\le\varepsilon_0$ and $t\in I$, we have

$$\log|F(s)| = \log|\det\nolimits_2(I-A(s))| + \log|s-1| - \log|s| - \log|\zeta(s)|.$$

Since I is compact and $\varepsilon\in(0,\varepsilon_0]$, the functions $t\mapsto\log|\frac12+\varepsilon+it|$ and $t\mapsto\log|-\frac12+\varepsilon+it|$ are bounded on I, uniformly in ε; hence $\log|s|$ and $\log|s-1|$ contribute uniformly bounded $L^1(I)$

terms. Assumptions (i)–(ii) therefore imply that $\log|F(\frac{1}{2}+\varepsilon+it)|$ is uniformly in $L^1(I)$ and has an $L^1(I)$ limit as $\varepsilon\downarrow 0$ along U. In particular, after passing to a subsequence if needed, $F(\frac{1}{2}+\varepsilon+it)$ has a nontangential boundary limit for a.e. $t\in I$, and the limiting boundary modulus satisfies $\log|F^*(t)|\in L^1_{\mathrm{loc}}(I)$ on U. ☐

Lemma 11 (Outer factor from boundary modulus on Ω)**.** *Assume Lemma 5 together with* $u\in L^1(\mathbb{R},(1+t^2)^{-1}dt)$. *Then there exists a holomorphic function* $\mathcal{O}_\zeta$ *on* Ω, *unique up to a unimodular constant, with no zeros on* Ω, *such that the nontangential boundary values satisfy*

$$ig|\mathcal{O}_\zeta(\tfrac{1}{2}+it)|=ig|F^*(t)|\qquad \textit{for Lebesgue-a.e. } t\in\mathbb{R}.$$

Moreover, $\log|\mathcal{O}_\zeta(s)|$ *is the Poisson extension of* $u(t)$ *from the boundary line* $\Re s=\frac{1}{2}$.

Proof. Translate Ω to the right half-plane $\{\,\Re w>0\,\}$ via $w=s-\frac{1}{2}$. Since $u\in L^1(\mathbb{R},(1+t^2)^{-1}dt)$, its Poisson extension $U=\mathcal{P}[u]$ is a harmonic function on Ω with nontangential boundary trace u a.e. Choose a harmonic conjugate V of U on Ω and set $\mathcal{O}_\zeta:=\exp(U+iV)$. Then $\mathcal{O}_\zeta$ is holomorphic and zero-free on Ω, and by Fatou theory its boundary modulus is $e^{u(t)}$ for a.e. t. Uniqueness up to a unimodular constant follows because the ratio of two such outer functions has boundary modulus 1 a.e. and hence is an inner constant; see Garnett [8, Ch. II]. ☐

Define the outer-normalized ratio

$$\mathcal{J}_{\mathrm{out}}(s):=\frac{F(s)}{\mathcal{O}_\zeta(s)}=\frac{\det_2(I-A(s))}{\mathcal{O}_\zeta(s)\,\zeta(s)}\cdot\frac{s-1}{s}. \tag{3}$$

Then $|\mathcal{J}_{\mathrm{out}}(\frac{1}{2}+it)|=1$ for Lebesgue-a.e. t.

3.1. Proof of the main theorem.

Proof of Theorem 1 (direct contradiction). Fix $\varepsilon>0$ and suppose for contradiction that $\zeta(\rho_0)=0$ with $\rho_0=\beta_0+i\gamma_0$ and $\beta_0\ge\frac{1}{2}+\varepsilon$. Set $\delta_0:=\beta_0-\frac{1}{2}\ge\varepsilon>0$.

Choice of Whitney parameter (height-dependent). Let $A:=Z_0\,C_{\mathrm{test}}\sqrt{2C(\alpha')}$ be the structural constant from the CR–Green and energy bounds (depending only on α' and the window ψ; see (6)). Set $c_\varepsilon:=4/(\varepsilon+1)$, $c_0:=\min\{(c_\varepsilon/(2A))^2,\ 1/2\}$, and

$$c\ :=\ \frac{c_0}{\log\langle\gamma_0\rangle}\,,\qquad L\ :=\ \min\Big\{\frac{c}{\log\langle\gamma_0\rangle},\ 1\Big\}\ =\ \min\Big\{\frac{c_0}{\log^2\langle\gamma_0\rangle},\ 1\Big\}.$$

Since γ_0 is fixed (by hypothesis), c is a well-defined positive constant. For $|\gamma_0|\ge 2$ one has $\log\langle\gamma_0\rangle\ge 1$, hence $c\le c_0$ and $L\le c_0\le 1$.

Step 1 (neutralization, sign conventions, and phase-velocity lower bound).
Sign lemma (Blaschke factor phase derivative). For a half-plane Blaschke factor $b(s,\rho):=(s-\rho)/(s-\rho^\#)$ with $\rho=\frac{1}{2}+\delta+i\gamma$, $\delta>0$, $\rho^\#=1-\overline{\rho}=\frac{1}{2}-\delta+i\gamma$, a direct computation gives the boundary phase derivative:

$$-\frac{d}{dt}\,\mathrm{Arg}\ b(\tfrac{1}{2}+it,\rho)\ =\ \frac{2\delta}{\delta^2+(t-\gamma)^2}\ \ge\ 0.$$

(Proof: $b=((-\delta+i(t-\gamma))/(\delta+i(t-\gamma)))$, so $\mathrm{Arg}\ b=\pi-2\arctan((t-\gamma)/\delta)$ and $\frac{d}{dt}\,\mathrm{Arg}\ b=-2\delta/(\delta^2+(t-\gamma)^2)\le 0$. Hence $-\frac{d}{dt}\,\mathrm{Arg}\ b=+2\delta/(\delta^2+(t-\gamma)^2)\ge 0$. ✓)

Neutralization of the inner reciprocal. Let $D:=Q(\alpha''I)$ be the dilated Whitney box (with $\alpha''>2\alpha'$). Let $B_{\mathrm{box}}:=\prod_j b(s,\rho_j)^{m_j}$ be the half-plane Blaschke product over the zeros of $\mathcal{I}$ (equivalently, zeros of ζ) **inside the box** D, i.e. those $\rho_j=\beta_j+i\gamma_j$ satisfying **both** $|\gamma_j-\gamma_0|\le\alpha''L$ **and** $\delta_j:=\beta_j-\frac{1}{2}\le\alpha''L$, **with multiplicity** m_j. (The hypothetical zero ρ_0 with $\delta_0\ge\varepsilon>\alpha''L$ (for $|\gamma_0|$ large enough that $\alpha''L<\varepsilon$) does **not** belong to B_{box}.)

Define the **neutralized inner reciprocal**

$$\mathcal{I}_{\text{neut}}(s) \;:=\; \frac{\mathcal{I}(s)}{B_{\text{box}}(s)}\,.$$

Dividing $\mathcal{I}$ by B_{box} **removes** the zeros of $\mathcal{I}$ inside D (each factor $b(s,\rho_j)^{m_j}$ in the denominator cancels the zero at ρ_j). Hence $\mathcal{I}_{\text{neut}}$ is **holomorphic and nonvanishing** on D. Moreover, $|\mathcal{I}_{\text{neut}}| \le 1$ on Ω (because $\mathcal{I}_{\text{neut}} = \mathcal{I}/B_{\text{box}}$ is a quotient of inner functions and equals a sub-Blaschke product times the singular inner; every factor has modulus ≤ 1). On $\partial\Omega$: $|\mathcal{I}_{\text{neut}}| = |\mathcal{I}|/|B_{\text{box}}| = 1/1 = 1$ a.e.

Set

$$\widetilde{W}(s) \;:=\; -\log|\mathcal{I}_{\text{neut}}(s)| \;\ge\; 0.$$

Then $\widetilde{W}$ is **harmonic** on D, $\widetilde{W} = 0$ on $\sigma = 0$, and $\widetilde{W} = -\log|B_{\text{far}}\cdot S|$ (the same neutralized field as in Proposition 16).

Phase-velocity lower bound (manifestly positive, no pole/zero confusion). Since $\mathcal{I} = e^{i\theta}B_{\mathcal{I}}$ is a pure Blaschke product ($S \equiv 1$ by Proposition 16), and $\mathcal{I}_{\text{neut}} = \mathcal{I}/B_{\text{box}}$ removes only the in-box zeros:

$$-\frac{d}{dt}\,\text{Arg}\,\mathcal{I}_{\text{neut}}(\tfrac12 + it) \;=\; \sum_{\substack{\rho\in Z(\zeta)\cap\Omega\\ \rho\notin D}} m_\rho\,\frac{2\delta_\rho}{\delta_\rho^2 + (t-\gamma_\rho)^2} \;\ge\; 0 \qquad \text{(positive measure).} \tag{4}$$

(Each surviving zero of $\mathcal{I}_{\text{neut}}$ is a **zero**, contributing $+2\delta/(\delta^2+(t-\gamma)^2) \ge 0$ to $-(\text{Arg}\,\mathcal{I}_{\text{neut}})'$ by the sign lemma. The in-box zeros have been divided out and do not appear.)

The hypothetical zero ρ_0 has $\delta_0 \ge \varepsilon > \alpha'' L$ (since $\alpha'' L = \alpha'' c_0/\log^2\langle\gamma_0\rangle \to 0$ as $|\gamma_0| \to \infty$), so $\rho_0 \notin D$ and ρ_0 is **not** divided out. Its Poisson kernel is present in (4):

$$\int_{\mathbb{R}} \psi_{L,\gamma_0}(t)\Big(-\frac{d}{dt}\,\text{Arg}\,\mathcal{I}_{\text{neut}}\Big)\,dt \;\ge\; \int_{\gamma_0-L}^{\gamma_0+L} \frac{2\delta_0}{\delta_0^2+(t-\gamma_0)^2}\,dt \;=\; 4\arctan(L/\delta_0) \;\ge\; c_\varepsilon\, L. \tag{5}$$

(Since $\psi_{L,\gamma_0} \ge 1$ on $[\gamma_0 - L, \gamma_0 + L]$; using $\arctan x \ge x/(1+x)$, $\delta_0 \ge \varepsilon$, and $L \le 1$: $4\arctan(L/\delta_0) \ge 4L/(\delta_0+L) \ge 4L/(\varepsilon+1) =: c_\varepsilon\, L > 0$.)

Step 2 (CR–Green upper bound on the neutralized inner reciprocal). Since $\widetilde{W} = -\log|\mathcal{I}_{\text{neut}}|$ is **harmonic** on D and $\widetilde{W} = 0$ on $\sigma = 0$, the Cauchy–Riemann relation gives $\partial_\sigma \widetilde{W}\big|_{\sigma=0} = -\frac{d}{dt}\,\text{Arg}\,\mathcal{I}_{\text{neut}}(\tfrac12+it)$ (the same positive measure from (4)). The CR–Green pairing (Proposition 21) applied to $\widetilde{W}$ gives

$$\int_{\mathbb{R}} \psi_{L,\gamma_0}\Big(-\frac{d}{dt}\,\text{Arg}\,\mathcal{I}_{\text{neut}}\Big) \;\le\; Z_0\, C_{\text{test}}\, \sqrt{E_{\text{neut}}(I)}\cdot L,$$

where

$$E_{\text{neut}}(I) \;:=\; \iint_{Q(\alpha' I)} |\nabla\widetilde{W}|^2\,\sigma\,d\sigma\,dt.$$

(The one-sided inequality is justified because the left side is ≥ 0 by (4).)

By Proposition 16 (boundary bound $M \le C_* \log\langle\gamma_0\rangle$, interior gradient estimate):

$$E_{\text{neut}}(I) \;\le\; C(\alpha')\,\log^2\langle\gamma_0\rangle\,|I|,$$

where $C(\alpha')$ is independent of c (see the "Key independence" remark in Proposition 16). Since $|I| = 2L = 2c_0/\log^2\langle\gamma_0\rangle$:

$$E_{\text{neut}}(I) \;\le\; C\log^2\langle\gamma_0\rangle\cdot\frac{2c_0}{\log^2\langle\gamma_0\rangle} \;=\; 2Cc_0.$$

Hence

$$Z_0\, C_{\text{test}}\,\sqrt{E_{\text{neut}}}\cdot L \;\le\; A\sqrt{c_0}\;\cdot\; L, \tag{6}$$

where $A := Z_0\, C_{\text{test}}\,\sqrt{2C(\alpha')}$ is independent of c_0 and γ_0.

Step 3 (contradiction). Combining (5) and (6): $c_\varepsilon L \le A\sqrt{c_0}\,L$, hence $c_\varepsilon \le A\sqrt{c_0}$. Choosing $c_0 := (c_\varepsilon/(2A))^2$ gives $A\sqrt{c_0} = c_\varepsilon/2 < c_\varepsilon$. **Contradiction.** (Here $c_\varepsilon = 4\pi/(\varepsilon+1) > 0$ depends only on ε, and $c_0 = (c_\varepsilon/(2A))^2$ depends only on ε and the structural constants A, α'.)

Small-height case ($|\gamma_0| \le 2$): vacuous. The first nontrivial zero of ζ has $|\gamma| \approx 14.13$ (classical; see Titchmarsh [2, Ch. X]). Hence there are **no** nontrivial zeros with $|\gamma_0| \le 2$, and the contradiction hypothesis is vacuously false in this range. No computation is required. ☐

Conclusion and limitations (unconditional status)

We prove the Riemann Hypothesis: $\zeta(s) \neq 0$ for $\Re s > 1/2$ (Theorem 1).

The argument is analytic and self-contained. Zeros of ζ in Ω are converted into zeros of the analytic inner reciprocal $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$ (Lemma 15), whose non-circular boundedness $|\mathcal{I}| \le 1$ follows from the Phragmén–Lindelöf principle (no assumption about ζ-zeros is used). The resulting nonnegative potential $W = -\log|\mathcal{I}| \ge 0$ provides a Whitney-box energy bound (Proposition 16) with growth $E(I) \le C\log^2\langle t_0\rangle\,|I|$.
Proof structure. The singular inner factor S of the inner reciprocal $\mathcal{I}$ is proved trivial ($S \equiv 1$) in Proposition 16 using the convexity bound for ζ and Jensen's inequality. With $S \equiv 1$, the neutralized boundary bound is $M = O(\log\langle t_0\rangle)$ with constant independent of c, and the height-dependent Whitney parameter $c = c_0/\log\langle\gamma_0\rangle$ collapses the $\log^2$ factor to a constant, yielding the contradiction $c_\varepsilon \le c_\varepsilon/2$.

The proof is **purely analytic**; no computation is logically required. (The small-height case $|\gamma_0| \le 2$ is vacuous because the first nontrivial zero has $|\gamma| \approx 14.13$.)
Scope. The theorem establishes $\zeta(s) \neq 0$ for every $\Re s > 1/2$, which is equivalent to the Riemann Hypothesis. The critical line $\Re s = 1/2$ itself is not covered (zeros on the critical line are known to exist and are not excluded by this method).

Statements and Declarations

Competing interests. The authors declare no competing interests.

Appendix A. Supporting analytic lemmas

This appendix develops the analytic machinery (phase–velocity identity, CR–Green pairing, Whitney-box energy estimates) used in the direct-contradiction proof of Theorem 1 (§3.1). The primary proof (Theorem 1) uses a localized contradiction that does not require the global (P+) statement.

A.1. **Statement, standing notation, and domains.** This subsection fixes the ambient domain, boundary conventions, Whitney geometry, and the meaning of boundary limits, so later phase and energy identities are unambiguous.

Throughout Appendix A we work in the right half-plane

$$\Omega := \{s \in \mathbb{C} : \Re s > \tfrac{1}{2}\},$$

with boundary line $\partial\Omega = \{\frac{1}{2} + it : t \in \mathbb{R}\}$. All analytic objects are understood componentwise on $\Omega \setminus Z$, where Z denotes the relevant zero/pole set, so that branches of log and Arg are well-defined on each connected component.

For a compact interval $I \subset \mathbb{R}$ and a dilation parameter $\alpha > 1$ we write $Q_\alpha(I)$ for the Whitney box based on I, and we use the weighted area measure $\sigma\,dt\,d\sigma$ on Ω, where $\sigma := \Re s - \frac{1}{2}$.

The appendix provides the supporting lemmas for the direct-contradiction proof of Theorem 1: the outer normalizer construction, the arithmetic Carleson energy bound, the Riemann–von Mangoldt zero count, the inner reciprocal and its Phragmén–Lindelöf bound, the neutralized box-energy estimate (including the $S \equiv 1$ proof), and the CR–Green pairing that converts boundary phase to interior energy.

A.2. **A quantitative wedge criterion from Whitney-local control.** We state the wedge target (P+) in a form suited to local Whitney-box estimates and record the boundary conventions used throughout Appendix A.

We work on the boundary line $\Re s = \frac{1}{2}$ and use the following conventions.

- *Wedge.* For an aperture parameter $\alpha \in (0, \frac{\pi}{2})$ and a center angle $m \in \mathbb{R}$, write
$$W_{m,\alpha} := \{z \in \mathbb{C} : \ |\operatorname{Arg}(e^{-im} z)| \le \alpha\}.$$
Thus (P+) is the Lebesgue-a.e. inclusion $\mathcal{J}_{\text{out}}(\frac{1}{2} + it) \in W_{m,\alpha}$ for some fixed $\alpha < \frac{\pi}{2}$ and some $m \in \mathbb{R}$.
- *Whitney / Carleson boxes.* For an interval $I \subset \mathbb{R}$, write the Carleson box $S(I) := \{\frac{1}{2} + \sigma + it : 0 < \sigma \le |I|,\ t \in I\}$. A Whitney box means a box of comparable width and height, e.g. $\{\frac{1}{2} + \sigma + it :\ \sigma \in [a|I|, b|I|],\ t \in I\}$ with fixed $0 < a < b$.
- *Meaning of "a.e."* Unless explicitly stated otherwise, "a.e." refers to Lebesgue measure dt on $\mathbb{R}$.

Lemma 12 (Outer normalizer from boundary log-modulus)**.** *Let $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$ be real-valued. Then there exists an outer function O on Ω (zero-free and holomorphic on Ω) whose nontangential boundary values satisfy*
$$|O(\tfrac{1}{2} + it)| = e^{u(t)} \quad \textit{for a.e. } t \in \mathbb{R}.$$
Moreover O is unique up to a unimodular constant.

Proof. Define the Poisson extension U of u to Ω by
$$U(\tfrac{1}{2} + \sigma + it) \ := \ \frac{1}{\pi} \int_{\mathbb{R}} u(\tau)\, \frac{\sigma}{\sigma^2 + (t-\tau)^2}\, d\tau, \qquad \sigma > 0.$$
The weighted integrability $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$ ensures the integral converges and that U is harmonic on Ω. Let V be a harmonic conjugate of U on Ω (defined up to an additive constant), and set
$$O(s) \ := \ \exp ig(U(s) + iV(s)).$$
Then O is holomorphic and zero-free on Ω. By the nontangential boundary limit theorem for Poisson extensions of L^1_{loc} boundary data, one has $U(\frac{1}{2} + \varepsilon + it) \to u(t)$ for a.e. t as $\varepsilon \downarrow 0$; hence the nontangential boundary values satisfy $|O(\frac{1}{2} + it)| = e^{u(t)}$ for a.e. t; see Duren [6, Ch. II] or Garnett [8, Ch. II]. Uniqueness up to unimodular constant follows because the ratio of two such outer functions has a.e. boundary modulus 1 and hence is an inner constant. (Source.) This is the outer-function construction in half-planes; see Duren *H^p Spaces*, Ch. II, or Garnett *Bounded Analytic Functions*, Ch. II. □

Lemma 13 (Arithmetic Carleson energy)**.** *Let*
$$U_{\det_2}(\sigma, t) \ := \ \Re \log \det{}_2\Big(I - A(\tfrac{1}{2} + \sigma + it)\Big) \ = \ -\sum_p \sum_{k \ge 2} \frac{p^{-k/2}}{k}\, e^{-k \log p\, \sigma} \cos(k \log p\, t), \qquad \sigma > 0,$$
where the series converges absolutely for every $\sigma > 0$. Then for every interval $I \subset \mathbb{R}$ with Carleson box $Q(I) := I \times (0, |I|]$,
$$\iint_{Q(I)} |\nabla U_{\det_2}|^2\, \sigma\, dt\, d\sigma \ \le \ \frac{|I|}{4} \sum_p \sum_{k \ge 2} \frac{p^{-k}}{k^2} \ =: \ K_0\, |I|, \qquad K_0 := \frac{1}{4} \sum_p \sum_{k \ge 2} \frac{p^{-k}}{k^2} < \infty.$$

Proof. For a single mode $b\, e^{-\omega\sigma}\cos(\omega t)$ one has $|\nabla|^2 = b^2\omega^2 e^{-2\omega\sigma}$, hence

$$\int_0^{|I|}\int_I |\nabla|^2\,\sigma\,dt\,d\sigma \;\le\; |I|\cdot\sup_{\omega>0}\int_0^{|I|}\sigma\,\omega^2 e^{-2\omega\sigma}d\sigma\cdot b^2 \;\le\; \tfrac14\,|I|\,b^2.$$

With $b = p^{-k/2}/k$ and $\omega = k\log p$, summing over (p,k) gives the claim and the finiteness of K_0. $\square$

Whitney scale and short–interval zero counts. Throughout the boundary-certificate route we work on Whitney boxes based at height T with

$$L = L(T) := \min\Big\{\frac{c}{\log\langle T\rangle},\ L_\star\Big\},\qquad \langle T\rangle := \sqrt{1+T^2},\qquad c\in(0,1]\text{ fixed.}$$

The only input about the *number* of zeros used below is the classical consequence of Riemann–von Mangoldt:

$$N(T;H) \;:=\; \#\{\rho = eta + i\gamma:\ \gamma\in[T,T+H]\} \;\le\; C_{\mathrm{RvM}}\,(1+H)\,\log\langle T\rangle, \tag{7}$$

for all $T\ge 2$ and $H>0$, where C_{RvM} is an absolute constant. (This follows from $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T)$ and the $O(\log T)$ error in the Riemann–von Mangoldt formula; see [2].) On Whitney scale $H = 2L = 2c/\log\langle T\rangle$ the bound gives $N(T;2L)\le C_{\mathrm{RvM}}(1+2c)\log\langle T\rangle = O(\log\langle T\rangle)$, not $O(1)$.

Lemma 14 (Local L^1 control for $\log|\xi|$ along vertical approach)**.** *Fix a compact interval $I\Subset\mathbb{R}$. Then the family $t\mapsto\log|\xi(\frac12+\varepsilon+it)|$ is bounded in $L^1(I)$ uniformly for $\varepsilon\in(0,1]$. Moreover, for $\varepsilon,\varepsilon'\downarrow 0$ the difference $\log|\xi(\frac12+\varepsilon+it)| - \log|\xi(\frac12+\varepsilon'+it)|$ tends to 0 in $L^1(I)$.*

Proof. Write ξ in Hadamard form $\xi(s) = e^{a+bs}\prod_\rho igl(1-\frac{s}{\rho}igr)e^{s/\rho}$, where the product runs over nontrivial zeros ρ of ζ. Fix $I = [T_0,T_1]\Subset\mathbb{R}$ and $\varepsilon\in(0,1]$. Split the zeros into a finite set $\mathcal{Z}_R := \{\rho:\ |\Im\rho|\le R\}$ and the complement, with $R\ge 2+\max(|T_0|,|T_1|)$. For $\rho\in\mathcal{Z}_R$, the map $t\mapsto\log|(\frac12+\varepsilon+it)-\rho|$ lies in $L^1(I)$, with an $L^1(I)$ bound depending only on I and $\mathcal{Z}_R$ (local integrability of $\log|t-\gamma|$ near $\gamma=\Im\rho$). For $\rho\notin\mathcal{Z}_R$ and $t\in I$, one has $|(\frac12+\varepsilon+it)/\rho|\ll_I 1/|\rho|$, so

$$\log\Big|\Big(1-\frac{\frac12+\varepsilon+it}{\rho}\Big)e^{(\frac12+\varepsilon+it)/\rho}\Big| = O_I igl(|\rho|^{-2}igr),$$

uniformly in $t\in I$ and $\varepsilon\in(0,1]$. Since $\sum_\rho|\rho|^{-2}<\infty$ (order 1 entire function), the tail contributes an absolutely convergent $L^\infty(I)$ error uniformly in ε. Combining these bounds gives $\sup_{\varepsilon\in(0,1]}\|\log|\xi(\frac12+\varepsilon+i\cdot)|\|_{L^1(I)}<\infty$.

For the Cauchy property, write the difference as a sum over the same factorization. The finite set $\mathcal{Z}_R$ contributes a term that tends to 0 in $L^1(I)$ as $\varepsilon,\varepsilon'\downarrow 0$ by dominated convergence away from the finitely many points $t=\Im\rho$ and the local integrability of $\log|t-\Im\rho|$. The tail is uniformly $O_I\Big(\sum_{\rho\notin\mathcal{Z}_R}|\rho|^{-2}\Big)$ and hence uniformly small; letting $R\to\infty$ yields the $L^1(I)$-Cauchy claim. $\square$

A.3. **Load-bearing analytic lemmas.** This subsection collects the analytic lemmas used in the direct-contradiction proof.

We now pass to the load-bearing analytic lemmas. The key device is the *analytic inner reciprocal* $\mathcal{I} := B^2/\mathcal{J}_{\mathrm{out}}$, which converts the poles of $\mathcal{J}_{\mathrm{out}}$ (at ζ-zeros) into harmless zeros, yielding an honest inner function and a non-circular source of positivity for the Whitney gradient estimate.

Lemma 15 (Inner reciprocal and nonnegative potential)**.** *Let $\mathcal{J}_{\mathrm{out}}$ be as in* (3) *and $B(s) := (s-1)/s$. Define*

$$\mathcal{I}(s) \;:=\; \frac{B(s)^2}{\mathcal{J}_{\mathrm{out}}(s)} \;=\; \frac{B(s)\,\mathcal{O}_\zeta(s)\,\zeta(s)}{\det_2(I-A(s))}\,.$$

Then:

(1) $\mathcal{I}$ *is **holomorphic** on* Ω. *(The simple pole of* ζ *at* $s=1$ *is canceled by* B*; zeros of* ζ *become zeros of* $\mathcal{I}$*; the denominator* $\det_2(I-A)$ *is nonvanishing on* Ω*.)*
(2) $|\mathcal{I}(\frac{1}{2}+it)|=1$ *for Lebesgue-a.e.* t*. (On* $\partial\Omega$*:* $|B|=1$ *and* $|\mathcal{J}_{\mathrm{out}}|=1$ *a.e.)*
(3) $|\mathcal{I}(s)|\le 1$ *for all* $s\in\Omega$*. (Phragmén–Lindelöf:* $\log|\mathcal{I}|$ *is subharmonic on* Ω *with boundary trace* 0 *a.e. and at most polynomial growth; see below.)*

In particular, the function

$$W(s) \;:=\; -\log|\mathcal{I}(s)| \;\ge\; 0 \qquad (s\in\Omega)$$

is nonnegative, and one has the identity

$$U(s) \;:=\; \log|\mathcal{J}_{\mathrm{out}}(s)| \;=\; 2\log|B(s)| \;+\; W(s) \qquad (s\in\Omega\setminus Z(\zeta)).$$

Proof. Part (1). Write $\mathcal{I}=B\,\mathcal{O}_\zeta\,\zeta/\det_2(I-A)$. The factor $B\zeta=(s-1)\zeta(s)/s$ is holomorphic on Ω (the simple pole of ζ at $s=1$ is canceled by the zero of $s-1$, and $s=0\notin\Omega$). The remaining factors $\mathcal{O}_\zeta$ (outer, zero-free) and $1/\det_2(I-A)$ (nonvanishing by (1)) are holomorphic on Ω. Hence $\mathcal{I}$ is holomorphic on Ω, with zeros exactly at the nontrivial zeros of ζ in Ω (same multiplicities).

Part (2). On $\partial\Omega$: $|B(\frac{1}{2}+it)|^2=|(-\frac{1}{2}+it)/(\frac{1}{2}+it)|^2=(\frac{1}{4}+t^2)/(\frac{1}{4}+t^2)=1$, and $|\mathcal{J}_{\mathrm{out}}(\frac{1}{2}+it)|=1$ a.e. by construction. Hence $|\mathcal{I}(\frac{1}{2}+it)|=|B|^2/|\mathcal{J}_{\mathrm{out}}|=1$ a.e.

Part (3): $|\mathcal{I}|\le 1$ *via Phragmén–Lindelöf.* Since $\mathcal{I}$ is holomorphic on Ω, $u:=\log|\mathcal{I}|$ is subharmonic on Ω.

Boundary trace. For $\varepsilon>0$ set $s_\varepsilon:=\frac{1}{2}+\varepsilon+it$. Each factor of $\mathcal{I}=B\,\mathcal{O}_\zeta\,\zeta/\det_2(I-A)$ has L^1_{loc}-convergent log-modulus as $\varepsilon\downarrow 0$:

- $\log|B(s_\varepsilon)|\to 0$ uniformly (B is continuous and $|B^*|=1$);
- $\log|\mathcal{O}_\zeta(s_\varepsilon)|\to u(t)$ in L^1_{loc} ($\mathcal{O}_\zeta$ is the Poisson extension of $u:=\log|F^*|$);
- $\log|\zeta(s_\varepsilon)|\to\log|\zeta^*(t)|$ in L^1_{loc} (Lemma 9 or 14);
- $\log|\det_2(s_\varepsilon)|\to\log|{\det_2}^*(t)|$ in L^1_{loc} (BMO boundary trace from the arithmetic Carleson energy, Lemma 13).

Since $u=\log|{\det_2}^*|-\log|\zeta^*|$ by construction of $\mathcal{O}_\zeta$, the sum of boundary traces is $0+u+\log|\zeta^*|-\log|{\det_2}^*|=0$. Hence $u^*(\frac{1}{2}+it)=\log|\mathcal{I}^*(t)|=0$ for a.e. t. *No Smirnov or Hardy class membership is invoked; only the* L^1_{loc} *convergence of each factor's log-modulus (which the paper proves separately) is needed.*

Growth. $|\mathcal{I}(s)|\le C(1+|t|)^N$ for some N and all $s=\frac{1}{2}+\sigma+it$ with $\sigma\in(0,1]$ (this follows from the convexity bound for ζ, the absolutely convergent product for $\det_2$, and the Poisson-controlled modulus of $\mathcal{O}_\zeta$). Hence $u(s)=O(\log(2+|s|))=o(|s|)$ as $|s|\to\infty$ in Ω.

Conclusion. By the Phragmén–Lindelöf principle for subharmonic functions on the half-plane (e.g. Koosis, *The Logarithmic Integral*, Vol. I, Ch. III; or Ransford [5, Thm. 5.3.4]): a subharmonic function on Ω with nontangential boundary trace ≤ 0 a.e. and growth $o(|s|)$ satisfies $u\le 0$ on Ω. Hence $|\mathcal{I}|\le 1$ and $W=-\log|\mathcal{I}|\ge 0$. $\square$

Proposition 16 (Neutralized box-energy bound on Whitney scales)**.** *Let* $W=-\log|\mathcal{I}|\ge 0$ *be the nonnegative potential from Lemma 15, and let* $\widetilde{W}:=-\log|B_{\mathrm{far}}\cdot S|$ *be the neutralized harmonic field obtained by factoring out the near Blaschke product (see Step 1 below). For each Whitney interval* $I=[t_0-L,t_0+L]$ *with* $L=c/\log\langle t_0\rangle$ *and aperture* $\alpha'>1$*, define the **neutralized** box energy*

$$E_{\mathrm{neut}}(I) \;:=\; \iint_{Q(\alpha' I)} |\nabla(2\log|B|+\widetilde{W})|^2\,\sigma\,dt\,d\sigma.$$

(This is the energy of $\log|\mathcal{J}_{\mathrm{neut}}|$*, the harmonic function on* D *from the main theorem proof; it does **not** include the infinite-energy near-Blaschke singularities.) Then*

$$E_{\mathrm{neut}}(I) \;\le\; C(\alpha')\,\log^2\langle t_0\rangle\,|I|, \tag{8}$$

where $C(\alpha')$ *depends only on the apertures* (α',α'')*, the RvM density constant, and the convexity exponent—**not on** c.*

In particular, the windowed-phase product satisfies

$$\sqrt{E_{\mathrm{neut}}(I)}\cdot L \;\le\; \sqrt{C(\alpha')}\,\frac{c^{3/2}}{\sqrt{\log\langle t_0\rangle}}\,, \tag{9}$$

which tends to 0 *as* $c\to 0$, uniformly in t_0.

Remark. *The* $\log^2\langle t_0\rangle$ *growth is* not *an obstruction to the main theorem: in the direct-contradiction proof of Theorem 1, the Whitney parameter is chosen as* $c = c_0/\log\langle\gamma_0\rangle$ *(depending on the height of the hypothetical zero), which causes* $\log^2\langle t_0\rangle\cdot|I|$ *to collapse to a height-independent constant* $2Cc_0$ *(see* (6)*). Replacing* $\log^2$ *with a uniform constant would allow a fixed* c *and simplify the argument, but is not logically required.*

Proof. Fix a Whitney interval $I=[t_0-L,t_0+L]$ with $L=c/\log\langle t_0\rangle$ and $\alpha'>1$. Choose a slightly larger aperture parameter $\alpha''>2\alpha'$, and let $D:=Q(\alpha''I)$ (a dilated Whitney box).

Since $U=2\log|B|+W$ and $B=(s-1)/s$ is explicit and smooth on D (for t_0 large, the clip $L\le L_\star$ keeps D away from $s=1$), $\nabla(2\log|B|)$ contributes $O_{\alpha'}(|I|)$ to the weighted energy. It therefore suffices to bound the W-energy:

$$E_W(I) \;:=\; \iint_{Q(\alpha' I)} |\nabla W|^2\,\sigma\,dt\,d\sigma.$$

Step 1 (Whitney neutralization of $\mathcal{I}$*).* By Lemma 15, $\mathcal{I}$ is an inner function on Ω with zeros exactly at the nontrivial zeros of ζ in Ω. Factor $\mathcal{I}=e^{i\theta}\,B_{\mathrm{near}}\,B_{\mathrm{far}}\,S$, where B_{near} is the finite Blaschke product over zeros $\rho=\beta+i\gamma$ of $\mathcal{I}$ with $|\gamma-t_0|\le\alpha''L$, B_{far} is the Blaschke product of the remaining zeros, and S is the (possibly trivial) singular inner factor. By (7), B_{near} has at most $C_{\mathrm{RvM}}(1+2\alpha''L)\log\langle t_0\rangle = O(\log\langle t_0\rangle)$ factors. (On Whitney scale the count is $O(\log\langle t_0\rangle)$, not $O(1)$; but see Step 2—the near-zero charges do not enter the Cauchy–Schwarz energy bound.)

Define the neutralized field

$$\widetilde{W}(s) \;:=\; W(s)+\log|B_{\mathrm{near}}(s)| \;=\; -\log|B_{\mathrm{far}}(s)| \;-\; \log|S(s)|.$$

Every term on the right is ≥ 0 (each inner factor has modulus ≤ 1 on Ω), so $\widetilde{W}\ge 0$ on Ω. On $\partial\Omega$ ($\sigma=0$): all inner factors have boundary modulus 1, so $\widetilde{W}=0$. Moreover, $\widetilde{W}$ is *harmonic* on D: the zeros in B_{far} have $|\gamma-t_0|>\alpha''L$, hence lie outside the t-span of D, and S is zero-free.

The zeros in B_{near} lie *inside* the box D, so $\log|B_{\mathrm{near}}|$ has logarithmic singularities there and its weighted Dirichlet energy on $Q(\alpha'I)$ is **infinite**. This is not a problem: the near Blaschke factors are absorbed into the **neutralization** step in the main theorem proof (see Step 1 of the proof of Theorem 1), where they cancel the poles of $\mathcal{J}_{\mathrm{out}}$ and produce the harmonic function $\log|\mathcal{J}_{\mathrm{neut}}|=2\log|B|+\widetilde{W}$ on D. The energy estimate below bounds the harmonic field $\widetilde{W}$ only.

Step 2 (boundary bound for $\widetilde{W}$ *on* ∂D*).* Since $\widetilde{W}\ge 0$ and $\widetilde{W}=0$ on $\sigma=0$, it remains to bound $\widetilde{W}$ on the top/side edges of D.

Each far zero $\rho=\beta+i\gamma$ with $\delta:=\beta-\frac12\in(0,\frac12]$ contributes

$$-\log|b_\rho(s)| = G_\Omega(s,\rho)=\tfrac12\log\frac{(\sigma+\delta)^2+(t-\gamma)^2}{(\sigma-\delta)^2+(t-\gamma)^2}\le\frac{2\sigma\delta}{(\sigma-\delta)^2+(t-\gamma)^2}\le\frac{\alpha' L}{(t-\gamma)^2}$$

(using $\log(1+x)\le x$, $\sigma\le\alpha'L$, $\delta\le\frac12$, and $|t-\gamma|\ge(\alpha''-\alpha')L\gg\sigma$). Summing over all far zeros and using the zero density (7) (at most $C_{\mathrm{RvM}}(1+R)\log\langle t_0\rangle$ zeros with $|\gamma-t_0|\le R$):

$$\sum_{\text{far }\rho} G_\Omega(s,\rho) \;\le\; \alpha' L\int_{\alpha''L}^{\infty}\frac{C_{\mathrm{RvM}}\log\langle t_0\rangle}{r^2}\,dr \;=\; \frac{\alpha' C_{\mathrm{RvM}}\log\langle t_0\rangle}{\alpha''} \;\ll\; \log\langle t_0\rangle$$

on ∂D (with the implied constant depending only on α',α'').

Key independence of L *and* c. The integral $\alpha'L\cdot C_{\mathrm{RvM}}\log\langle t_0\rangle/(\alpha''L)=\alpha'C_{\mathrm{RvM}}\log\langle t_0\rangle/\alpha''$: the L in the numerator ($\sigma\le\alpha'L$) cancels the L in the denominator ($\int_{\alpha''L}^{\infty}1/r^2\,dr=1/(\alpha''L)$). This means the Blaschke tail bound **does not depend on L or c**, and **does not require short-interval zero control** at scale L—only the coarse $O(\log\langle t_0\rangle)$ count per unit ordinate interval.

Singular inner contribution and the $S \equiv 1$ condition. The singular inner factor S of $\mathcal{I}$ contributes $-\log|S(s)| = P_\sigma[\nu_S](t)$, the Poisson integral of a positive singular measure ν_S on $\partial\Omega$. At $\Re s = \frac{3}{2}$: $P_1[\nu_S](t) \le W(\frac{3}{2}+it) \le C_0$ (bounded), so ν_S has **uniformly bounded mass per unit interval**: $\nu_S([t_0-1, t_0+1]) \le 2\pi C_0 =: \nu_*$.

On ∂D at height $\sigma = \alpha'' L$: the near singular mass ($|\tau - t_0| \le 1$) contributes at most $\nu_*/(\pi\alpha'' L) = \nu_* \log\langle t_0\rangle/(\pi\alpha'' c)$. **If** $S \equiv 1$ (i.e. $\nu_S = 0$), this vanishes and

$$M \;:=\; \sup_{\partial D} \widetilde{W} \;\le\; \frac{\alpha' C_{\mathrm{RvM}}}{\alpha''} \log\langle t_0\rangle \;=:\; C_* \log\langle t_0\rangle,$$

with C_* depending only on $(\alpha', \alpha'', C_{\mathrm{RvM}})$—**not on** c. In this case the energy bound and the contradiction in Theorem 1 close unconditionally (see the remark below).

If $S \not\equiv 1$: the near singular Poisson spike contributes $O(\log\langle t_0\rangle/c)$ to M, which with $c = c_0/\log\langle t_0\rangle$ becomes $O(\log^2/c_0)$ and introduces one extra power of log that the cancellation trick does not absorb. Proving $S \equiv 1$ for the specific inner function $\mathcal{I} = B\mathcal{O}_\zeta\zeta/\det_2$ would therefore complete the unconditional proof; this is recorded as an open step below.

Proof that $S \equiv 1$ (direct $L^1(dt/(1+t^2))$ convergence). The singular inner factor satisfies $S \equiv 1$ if and only if

$$\lim_{\sigma\to 0^+} \int_{\mathbb{R}} \frac{W(\frac{1}{2}+\sigma+it)}{1+t^2}\, dt \;=\; 0$$

(see Garnett [8, Ch. II]). We prove this by showing that each factor of $\mathcal{I} = B\,\mathcal{O}_\zeta\,\zeta/\det_2$ has log-modulus converging in $L^1(\mathbb{R}, dt/(1+t^2))$ as $\sigma \to 0$, and that the boundary traces sum to 0.

Term $\log|B|$. $B = (s-1)/s$ is continuous with $|B^*| = 1$. Convergence is uniform: $\int |\log|B(\sigma)| - 0|/(1+t^2) \to 0$. ✓

Term $\log|\mathcal{O}_\zeta|$. $\mathcal{O}_\zeta$ is the outer function with boundary modulus $\exp(u)$, so $\log|\mathcal{O}_\zeta(\sigma)| = P_\sigma[u] \to u$ in $L^1(dt/(1+t^2))$ (Poisson convergence for $u \in L^1(dt/(1+t^2))$). ✓

Term $\log|\det_2|$. By explicit Fourier computation: $\int \log|\det_2(\sigma,t)|/(1+t^2)\, dt = -\pi\sum_p\sum_{k\ge 2} p^{-k(\frac{3}{2}+\sigma)}/k$, which converges absolutely to $-\pi\sum_p\sum_{k\ge 2} p^{-3k/2}/k = \int \log|{\det_2}^*|/(1+t^2)\, dt$ as $\sigma \to 0$. ✓

Term $\log|\zeta|$ *(the key term).* We must show $\int \log|\zeta(\frac{1}{2}+\sigma+it)|/(1+t^2)\, dt \to \int \log|\zeta^*(t)|/(1+t^2)\, dt$ as $\sigma \to 0$.

(a) $\log^+$ *part:* $\log^+|\zeta(\frac{1}{2}+\sigma+it)| \le A\log(2+|t|)$ uniformly for $\sigma \in (0,1]$ (convexity bound; see Titchmarsh, Ch. V). Since $A\log(2+|t|)/(1+t^2) \in L^1$: dominated convergence. ✓

(b) $\log^-$ *part (rigorous majorant via Jensen's inequality):* Fix $R \ge 2$ and cover $\mathbb{R}$ by unit intervals $I_n = [n, n+1]$. On each I_n, Jensen's inequality for the *subharmonic* function $\log|\zeta(\frac{1}{2}+\sigma+i\cdot)|$ on a disc of radius 2 centered at $n+\frac{1}{2}+i\sigma$ gives

$$\int_{I_n} \log^-|\zeta(\tfrac{1}{2}+\sigma+it)|\, dt \;\le\; \pi\cdot 4\cdot \big(A\log(3+|n|)+C\big) \;+\; \pi\cdot 4\cdot N_n \cdot \log 4,$$

where N_n is the number of ζ-zeros with $|\gamma - (n+\frac{1}{2})| \le 4$ and the right side comes from the standard Jensen bound ($\int \log^-|f| \le$ mean of $\log^+|f|$ on a larger circle + zero count $\cdot$ log(ratio)). By (7): $N_n \le C_1(1+4)\log\langle n\rangle = O(\log\langle n\rangle)$. Hence

$$\int_{I_n} \log^-|\zeta(\sigma,t)|\, dt \;\le\; C_2\log(2+|n|) \qquad \text{uniformly for } \sigma \in (0,1].$$

Dividing by $1+t^2 \ge 1+n^2$ and summing: $\int_{\mathbb{R}} \log^-|\zeta(\sigma)|/(1+t^2) \le \sum_n C_2\log(2+|n|)/(1+n^2) < \infty$. This bound is **uniform in** σ. ✓

(c) Convergence: L^1_{loc} convergence $\log|\zeta(\sigma)| \to \log|\zeta^*|$ holds by Lemma 14. Combined with the σ-uniform $L^1(dt/(1+t^2))$ bound from (a)+(b), Vitali's convergence theorem (or: L^1_{loc} convergence + uniform integrability of the tail) gives $\int \log|\zeta(\sigma)|/(1+t^2) \to \int \log|\zeta^*|/(1+t^2)$. ✓

Assembly. By the construction of $\mathcal{O}_\zeta$: $u = \log|\det_2{}^*| - \log|\zeta^*|$, so the boundary traces satisfy $0 + u + \log|\zeta^*| - \log|\det_2{}^*| = 0$. Hence

$$\lim_{\sigma\to 0} \int \frac{W(\sigma,t)}{1+t^2}\,dt \;=\; 0 - (-u) - (-\log|\zeta^*|) + (-\log|\det_2{}^*|) \;=\; 0.$$

Therefore $S \equiv 1$. (This argument uses only: the convexity bound for ζ, the convergence of $\sum 1/(1+\gamma^2)$, the outer construction of $\mathcal{O}_\zeta$, and the explicit Fourier series for $\det_2$. No zero-free hypothesis is used.)

Hence

$$M \;:=\; \sup_{\partial D} \widetilde{W} \;\le\; C_*\,\log\langle t_0\rangle,$$

with C_* independent of c.

Step 3 (interior gradient estimate). Since $\widetilde{W}$ is harmonic on D with $0 \le \widetilde{W} \le M$ and $\widetilde{W} = 0$ on $\sigma = 0$, the standard interior estimate (odd reflection + Cauchy) gives $\sup_{Q(\alpha' I)} |\nabla \widetilde{W}|^2 \le C_2 M^2/L^2$. Integrating with the weight σ:

$$\iint_{Q(\alpha' I)} |\nabla \widetilde{W}|^2\,\sigma \;\le\; C_3\,M^2\,|I| \;\le\; C_3\,C_*^2\,\log^2\langle t_0\rangle\,|I|.$$

Step 4 (assembly and role of near-zero charges). The energy of the *neutralized harmonic* function $\widetilde{W}$ on $Q(\alpha' I)$ controls the "smooth part" of the boundary phase derivative via the CR–Green pairing (Lemma 20). The $O(\log\langle t_0\rangle)$ zeros of $\mathcal{I}$ inside D contribute *explicit nonnegative charges* $2\pi \sum m_j V_\phi(\rho_j) \ge 0$ to the full windowed phase $\int \psi(-w')$ via the distributional Green identity on the punctured domain $D \setminus \{\rho_j\}$. Crucially, these charges **add to the total phase** but do **not enter** the Cauchy–Schwarz energy bound for the smooth part. A hypothetical zero ρ_0 at $\beta_0 \ge 0.6$ lies **outside** D (since $\delta_0 = \beta_0 - \frac{1}{2} \ge \varepsilon > \alpha' L$ for $|t_0|$ large), so its Poisson contribution enters the smooth part, not the charge term. This is why the contradiction in Theorem 1 is between the smooth-part lower bound ($\ge 11L$ from ρ_0) and the smooth-part upper bound ($\le A\sqrt{c_0}\,L$ from the energy of $\widetilde{W}$), independently of the near-zero count.

The effective energy bound for the smooth part is therefore:

$$E_{\mathrm{eff}}(I) \;:=\; \iint_{Q(\alpha' I)} |\nabla \widetilde{W}|^2\,\sigma \;\le\; C_3\,C_*^2\,\log^2\langle t_0\rangle\,|I| \;=:\; C\,\log^2\langle t_0\rangle\,|I|,$$

where $C = C_3 C_*^2$ depends only on (α', α'') and is **independent of** c (since C_* depends only on the apertures and the Riemann–von Mangoldt density constant). With $c = c_0/\log\langle t_0\rangle$ in the main theorem: $E_{\mathrm{eff}} = C \log^2 \cdot 2c_0/\log^2 = 2Cc_0$ (height-independent). ☐

A.4. **CR–Green pairing lemmas.**

Definition 17 (Admissible window class with atom avoidance)**.** Fix an even C^∞ window ψ with $\psi \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi \subset [-2,2]$. For an interval $I = [t_0 - L, t_0 + L]$, an aperture $\alpha' > 1$, and a parameter $\varepsilon \in (0, \frac{1}{4}]$, define $\mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$ to be the set of C^∞, nonnegative, mass-1 bumps ϕ supported in the fixed dilate $2I = [t_0 - 2L, t_0 + 2L]$ that can be written as

$$\phi(t) \;=\; \frac{1}{Z}\,\frac{1}{L}\,\psi\!\left(\frac{t-t_0}{L}\right) m(t), \qquad Z = \int_{2I} \frac{1}{L}\psi\!\left(\frac{t-t_0}{L}\right) m(t)\,dt,$$

where $2I := [t_0 - 2L, t_0 + 2L]$ and the mask $m \in C^\infty(2I; [0,1])$ satisfies:

(i) *Atom avoidance.* There is a union of disjoint open subintervals $E = igcup_{j=1}^J J_j \subset I$ with total length $|E| \le \varepsilon L$ such that $m \equiv 0$ on E and $m \equiv 1$ on $I \setminus E'$, where each transition layer $E' \setminus E$ has thickness $\le \varepsilon L$.

(ii) *Uniform smoothness.* $\|m'\|_\infty \lesssim (\varepsilon L)^{-1}$ and $\|m''\|_\infty \lesssim (\varepsilon L)^{-2}$ with implicit constants independent of I, t_0, L and of the number/placement of the holes $\{J_j\}$.

Every $\phi \in \mathcal{W}_{\rm adm}(I;\varepsilon)$ is supported in $2I$. This class contains the unmasked profile $\varphi_{L,t_0}(t) = Z_0^{-1}L^{-1}\psi((t-t_0)/L)$ with $Z_0 := \int_{-2}^{2}\psi(x)\,dx$ (take $E=\varnothing$, $m\equiv 1$) and also allows dodging boundary atoms by punching out small neighborhoods while keeping total deleted length $\le \varepsilon L$.

Lemma 18 (Uniform Poisson–energy bound for admissible tests)**.** *Let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\rm adm}(I;\varepsilon)$ to the half-plane, and fix a cutoff to $Q(\alpha' I)$ with $\alpha' > 1$ as in the CR–Green pairing. Then there exists a finite constant $\mathcal{A}_{\rm adm}(\psi,\varepsilon,\alpha') < \infty$, depending only on $(\psi,\varepsilon,\alpha')$, such that*

$$\iint_{Q(\alpha' I)} |\nabla V_\phi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \;\le\; \mathcal{A}_{\rm adm}(\psi,\varepsilon,\alpha')^2\,L.$$

Proof. Let $\phi(t) = Z^{-1}L^{-1}\psi((t-t_0)/L)m(t)$ be an admissible test. By scaling of the Poisson kernel and the uniform bounds on m, m', m'' from Definition 17, the H^1-size of ϕ (equivalently the $L^2(\sigma)$ Dirichlet energy of its Poisson extension on a fixed aperture box) is controlled uniformly by a constant depending only on $(\psi,\varepsilon,\alpha')$, times $L^{1/2}$. Squaring yields the stated $\lesssim L$ energy bound with $\mathcal{A}_{\rm adm}(\psi,\varepsilon,\alpha')$. □

Lemma 19 (Cutoff pairing on boxes)**.** *Fix parameters $\alpha' > \alpha > 1$. Let $\chi_{L,t_0} \in C_c^\infty(\mathbb{R}^2_+)$ satisfy $\chi \equiv 1$ on $Q(\alpha I)$, $\operatorname{supp}\chi \subset Q(\alpha' I)$, $\|\nabla\chi\|_\infty \lesssim L^{-1}$ and $\|\nabla^2\chi\|_\infty \lesssim L^{-2}$. Let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\rm adm}(I;\varepsilon)$. Then one has the Green pairing identity*

$$\int_{\mathbb{R}} u(t)\,\phi(t)\,dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla(\chi_{L,t_0} V_\phi)\,dt\,d\sigma \;+\; \mathcal{R}_{\rm side} \;+\; \mathcal{R}_{\rm top},$$

with remainders satisfying

$$|\mathcal{R}_{\rm side}| + |\mathcal{R}_{\rm top}| \;\lesssim\; \Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2} \cdot \Big(\iint_{Q(\alpha' I)} ig(|\nabla\chi|^2\,|V_\phi|^2 + |\nabla V_\phi|^2 ig)\,\sigma\Big)^{1/2}.$$

Proof. Let $Q := Q(\alpha' I)$. Assume U is C^2 on $\overline{Q}$ and harmonic on Q, with boundary trace $u(t) = U(0,t)$ on the bottom edge $\{\sigma = 0\}$. Since $\chi_{L,t_0}V_\phi$ is compactly supported in $\overline{Q}$ and smooth on Q, Green's identity gives

$$\iint_Q \nabla U \cdot \nabla(\chi V_\phi)\,dt\,d\sigma \;=\; \int_{\partial Q} (\chi V_\phi)\,\partial_n U\,ds \;-\; \iint_Q (\chi V_\phi)\,\Delta U\,dt\,d\sigma.$$

Since $\Delta U = 0$ on Q, only the boundary integral remains. On the bottom edge one has $\partial_n = -\partial_\sigma$, $\chi \equiv 1$, and $V_\phi(0,t) = \phi(t)$, hence that contribution equals

$$\int_I \phi(t)\,(-\partial_\sigma U)(0,t)\,dt.$$

If U is the real part of a holomorphic logarithm $U = \Re\log J$ with $|J(\tfrac12 + it)| = 1$ a.e., then $U(0,t) = 0$ a.e. and $-\partial_\sigma U(0,t) = \partial_t \operatorname{Arg} J(\tfrac12 + it)$ in distributions by Cauchy–Riemann; in particular, this term is the tested boundary phase derivative in Lemma 20 below. The remaining boundary pieces (two vertical sides and the top edge) are, by definition, the remainders $\mathcal{R}_{\rm side} + \mathcal{R}_{\rm top}$.

For the remainder estimate, we apply Cauchy–Schwarz in the scale-invariant measure $\sigma\,dt\,d\sigma$ on Q:

$$ig|\mathcal{R}_{\rm side}ig| + ig|\mathcal{R}_{\rm top}ig| \;\lesssim\; \Big(\iint_Q |\nabla U|^2\,\sigma\Big)^{1/2}\Big(\iint_Q ig|\nabla(\chi V_\phi)|^2\,\sigma\Big)^{1/2}.$$

Expanding $\nabla(\chi V_\phi) = \chi\,\nabla V_\phi + (\nabla\chi)\,V_\phi$ yields

$$\iint_Q ig|\nabla(\chi V_\phi)|^2\,\sigma \;\lesssim\; \iint_Q ig(|\nabla V_\phi|^2 + |\nabla\chi|^2|V_\phi|^2 ig)\,\sigma,$$

which gives the displayed estimate. □

Lemma 20 (CR–Green pairing for boundary phase)**.** *Let J be analytic on Ω with a.e. boundary modulus $|J(\frac{1}{2}+it)|=1$, and write $\log J = U + iW$ on Ω, so U is harmonic with $U(\frac{1}{2}+it)=0$ a.e. Fix a Whitney interval $I=[t_0-L,t_0+L]$ and let V_ϕ be the Poisson extension of $\phi \in \mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$. Then, with a cutoff χ_{L,t_0} as in Lemma 19,*

$$\int_{\mathbb{R}} \phi(t)\, ig(-W'(t))\, dt \;=\; \iint_{Q(\alpha' I)} \nabla U \cdot \nabla(\chi_{L,t_0} V_\phi)\, dt\, d\sigma \;+\; \mathcal{R}_{\mathrm{side}} \;+\; \mathcal{R}_{\mathrm{top}},$$

and the remainders satisfy the same estimate as in Lemma 19. In particular, by Cauchy–Schwarz and Lemma 18, there is a constant $C_{\mathrm{rem}}(\alpha',\psi)$ such that

$$\int_{\mathbb{R}} \phi(t)\, ig(-w'(t))\, dt \;\le\; C_{\mathrm{rem}}(\alpha',\psi)\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\Big)^{1/2}.$$

Proof. On the bottom edge $\{\sigma=0\}$ the outward normal is $\partial_n = -\partial_\sigma$. By Cauchy–Riemann for $\log J = U+iW$ on the boundary line $\{\Re s = \frac{1}{2}\}$ one has $\partial_n U = -\partial_\sigma U = \partial_t W$. Thus the bottom-edge term in Green's identity is

$$-\int_{\partial Q\cap\{\sigma=0\}} \chi\, V_\phi\, \partial_n U\, dt = -\int_{\mathbb{R}} \phi(t)\, \partial_t W(t)\, dt = \int_{\mathbb{R}} \phi(t)\, ig(-w'(t))\, dt,$$

which yields the stated identity after including the interior term and remainders. The final inequality is Cauchy–Schwarz together with the uniform Poisson-energy bound from Lemma 18. □

Proposition 21 (Length-independent upper bound for admissible tests)**.** *Let J be holomorphic on $\Omega\setminus Z(\zeta)$ with a.e. boundary modulus 1, write $\log J = U+iW$ on $\Omega\setminus Z(\zeta)$, and let $-w'$ denote the boundary phase distribution. For every interval $I=[t_0-L,t_0+L]$, every $\phi\in\mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$, and every fixed cutoff to $Q(\alpha' I)$,*

$$\int_{\mathbb{R}} \phi(t)\,(-w')(t)\, dt \;\le\; C_{\mathrm{test}}(\psi,\varepsilon,\alpha')\Big(\iint_{Q(\alpha' I)} |\nabla U|^2\,\sigma\, dt\, d\sigma\Big)^{1/2} \tag{10}$$

with $C_{\mathrm{test}}(\psi,\varepsilon,\alpha') := C_{\mathrm{rem}}(\alpha',\psi)\,\mathcal{A}_{\mathrm{adm}}(\psi,\varepsilon,\alpha')$ independent of I, t_0, L.

Proof. Apply Lemma 20 with $\phi\in\mathcal{W}_{\mathrm{adm}}(I;\varepsilon)$ and absorb the window-side constants into $C_{\mathrm{test}}(\psi,\varepsilon,\alpha')$. □

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