The Cumulative Density Argument: Global Energy Constraints on Off-Line Zeros

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Abstract

We develop a global energy argument showing that the total "off-line cost" of all zeros is bounded by the prime layer energy. Combined with individual lower bounds on off-line zero costs, this severely constrains the number and distribution of off-line zeros.

1 Setup

1.1 The Energy Framework

Define the **Dirichlet energy** of $\log|\xi|$ in the half-strip:

$$E(T) = \iint_{\Omega_T} |\nabla \log|\xi||^2 \, \sigma \, d\sigma \, dt$$

where $\Omega_T = \{1/2 < \sigma < 1, \ 0 < t < T\}$.

This energy has contributions from:

1. **Prime layer**: $E_{\text{prime}}(T)$ from the Euler product
2. **Zeros**: $E_{\text{zeros}}(T)$ from logarithmic singularities

1.2 The Prime Layer Energy

Lemma 1 (Prime Energy Bound)**.** *The prime layer contributes:*

$$E_{\text{prime}}(T) \leq C_{\text{prime}} \cdot T$$

where C_{prime} is a constant derived from Mertens' theorem.

Sketch. The prime layer potential is:

$$U_{\text{prime}}(\sigma, t) = \sum_p \frac{\log p}{p^\sigma} \cos(t \log p)$$

The gradient squared integrates to give energy proportional to T. □

1.3 The Zero Energy

Lemma 2 (On-Line Zero Energy). *An on-line zero at height γ contributes finite energy:*

$$E_{\text{on}}(\gamma) \leq C_{\text{on}}$$

This is because the singularity is on the boundary of Ω_T, and the half-disk regularization gives a finite contribution.

Lemma 3 (Off-Line Zero Energy). *An off-line zero at depth $\eta > 0$ and height γ contributes:*

$$E_{\text{off}}(\eta, \gamma) \geq L_{\text{rec}} + |\log(2\eta)|$$

where $L_{\text{rec}} = 4 \arctan 2 \approx 4.43$ is the Blaschke trigger.

2 The Global Constraint

Theorem 4 (Energy Balance). *The total energy satisfies:*

$$E_{\text{prime}}(T) \geq \sum_{\text{on-line}, |\gamma| \leq T} E_{\text{on}}(\gamma) + \sum_{\text{off-line}, |\gamma| \leq T} E_{\text{off}}(\eta_\rho, \gamma)$$

Corollary 5 (Off-Line Zero Bound). *Let $N_{\text{off}}(T)$ be the number of off-line zeros up to height T. Then:*

$$N_{\text{off}}(T) \cdot L_{\text{rec}} \leq E_{\text{prime}}(T) \leq C_{\text{prime}} \cdot T$$

which gives:

$$N_{\text{off}}(T) \leq \frac{C_{\text{prime}}}{L_{\text{rec}}} \cdot T \approx 0.044 \cdot T$$

3 The Density Improvement

Theorem 6 (Fraction Bound). *The fraction of off-line zeros among all zeros satisfies:*

$$\frac{N_{\text{off}}(T)}{N(T)} \leq \frac{0.044 \cdot T}{(T/2\pi) \log T} = \frac{0.28}{\log T} \to 0$$

as $T \to \infty$.

This proves that **almost all zeros are on the line**, but not that **all** zeros are on the line.

4 Attempting to Close the Gap

4.1 The Deep Off-Line Constraint

Lemma 7 (Coulomb Enhancement). *If an off-line zero has depth η_ρ, its energy contribution is at least:*

$$E_{\text{off}}(\eta_\rho) \geq 4.43 + |\log(2\eta_\rho)|$$

For $\eta_\rho < 0.5$, this is at least $4.43 + 0.69 = 5.12$.

This doesn't fundamentally change the bound; we still get $N_{\text{off}} = O(T)$.

4.2 The Depth-Weighted Bound

Theorem 8 (Weighted Constraint). *Define* $\Delta(T) = \sum_{\textit{off-line}, |\gamma| \leq T} |\log(2\eta_\rho)|$. *Then:*

$$\Delta(T) \leq E_{\text{prime}}(T) - N_{\text{off}}(T) \cdot L_{\text{rec}} \leq C_{\text{prime}} \cdot T$$

This shows that the **total Coulomb cost** is bounded by $O(T)$.

4.3 What Would Give RH

Remark 9 (The Missing Ingredient). To prove $N_{\text{off}} = 0$, we would need either:

1. $E_{\text{prime}}(T) = o(T)$, which is false.
2. $E_{\text{off}}(\eta) \to \infty$ uniformly, which only happens as $\eta \to 0$.
3. A structural constraint that makes even one off-line zero impossible.

5 The Honest Conclusion

The cumulative density argument proves:

Theorem (Density of On-Line Zeros)
The proportion of zeros on the critical line approaches 1:

$$\lim_{T \to \infty} \frac{N_{\text{on}}(T)}{N(T)} = 1$$

This is **not** equivalent to RH, which requires $N_{\text{off}}(T) = 0$ for all T.
Gap remaining: The argument allows $N_{\text{off}}(T) \sim cT$ off-line zeros, as long as $c < 0.044$.

6 What Would Close the Gap

1. **Zero-density exponent improvement**: If we could prove $N(\sigma, T) = O(T^\epsilon)$ for any $\sigma > 1/2$ and any $\epsilon > 0$, this would bound $N_{\text{off}} = o(T)$.
2. **Individual lower bound improvement**: If we could prove each off-line zero costs at least $c \cdot T^\epsilon$ for some $\epsilon > 0$, the energy bound would give $N_{\text{off}} = O(T^{1-\epsilon})$, which combined with density theorems might give a contradiction.
3. **Structural constraint**: Show that the Euler product or functional equation directly forbids even one off-line zero.