WEIGHTED GEOMETRIC DEPLETION AND STRUCTURAL CONSTRAINTS ON BLOW-UP PROFILES FOR THE 3D INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

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Abstract. We establish new structural constraints on potential finite-time singularities of the three-dimensional incompressible Navier–Stokes equations. For any smooth H^1 solution that blows up at time $T^* < \infty$, the running-max/vorticity-normalized ancient element satisfies: (a) the $\rho^{3/2}$-weighted near-field vortex stretching is unconditionally depleted at rate $O(r^5)$; (b) the far-field contribution vanishes in the blow-up limit; (c) the weighted direction coherence obeys a universal bound; and (d) the vorticity direction $\xi = \omega/|\omega|$ has a uniformly bounded gradient on the high-vorticity set $\{\rho \geq \eta\}$, with the bound depending only on η and universal constants. The mechanism underlying (d) is a threshold comparison: the accumulated direction energy sits well below the Struwe ε-regularity threshold for harmonic maps into $\mathbb{S}^2$.

Conditionally, if the bounded gradient could be upgraded to full direction constancy ($\nabla\xi \equiv 0$ on supp ω), the blow-up profile is classified as the rigid rotation. This reduces the Millennium regularity problem to two concrete conjectures: establishing direction constancy and excluding the rigid rotation as a blow-up limit. The obstructions to each are identified and discussed.

Contents

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1. Introduction

1.1. The regularity problem. Let $\nu > 0$ be the kinematic viscosity. We consider the three-dimensional incompressible Navier–Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p - \nu\Delta u = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

for a velocity field $u\colon \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ and scalar pressure $p\colon \mathbb{R}^3 \times [0, T) \to \mathbb{R}$, with smooth divergence-free initial data $u_0 \in H^1(\mathbb{R}^3)$.

Leray [10] proved the existence of global weak solutions and raised the question of whether smooth solutions can develop singularities in finite time. This question, now identified as one of the Clay Millennium Prize Problems [7], remains open.

The Beale–Kato–Majda criterion [2] gives a necessary and sufficient condition for blow-up: a smooth solution loses regularity at time $T^* < \infty$ if and only if

$$\int_0^{T^*} \|\omega(\cdot, t)\|_{L^\infty}\, dt = \infty, \tag{1.2}$$

where $\omega = \operatorname{curl} u$ is the vorticity.

1.2. Statement of results. Our main results establish new structural constraints on any blow-up profile. We state the unconditional results first, then the conditional classification.

Theorem 1.1 (Structural constraints on blow-up profiles)**.** *Let $u_0 \in H^1(\mathbb{R}^3)$ be smooth and divergence-free, and let u be the corresponding smooth solution of* (1.1) *on its maximal interval of existence $[0, T^*)$. Suppose $T^* < \infty$ and let (u^∞, p^∞) be the running-max ancient element* (*Definition 3.1, Lemma 3.3*)*. Then:*

(a) ***Near-field depletion.*** *The $\rho^{3/2}$-weighted near-field stretching is unconditionally depleted: $\iint_{Q_r} \rho^{3/2}|\sigma_{\mathrm{near}}| \le Cr^5$ for all $r \le 1$ and all basepoints.*

(b) ***Tail elimination.*** *The external far-field contribution vanishes in the blow-up limit at rate $M_k^{-3/4}$. For the ancient element, the full stretching equals its near-field at every scale.*

(c) ***Weighted coherence bound.*** *$\mathcal{E}_\omega(z_0, R) \le C_1R^5 + C_2R^3$ for universal constants C_1, C_2.*

(d) ***Direction regularity on the high-vorticity set.*** *For every $\eta > 0$ there exists $C(\eta) < \infty$ such that*

$$|\nabla\xi(z_1)| \le C(\eta) \qquad \text{for every } z_1 \text{ with } \rho(z_1) \ge \eta. \tag{1.3}$$

The constant $C(\eta) = 2C_SC_{\mathrm{Ser}}/\eta$ depends only on η, the Serrin constant, and the Struwe constant.

Remark 1.2 (Role of the Struwe threshold). Part (d) follows from a threshold comparison: the unweighted direction energy on a universal ball where $\rho \ge \frac{1}{2}$ (via Serrin regularity) is of order $\delta^3 \sim C_{\mathrm{Ser}}^{-3}$, far below the Struwe ε-regularity threshold $4\pi \approx 12.6$ for harmonic maps into $\mathbb{S}^2$. The perturbation terms (drift and forcing) are controlled independently by Serrin estimates, with no circularity.

Remark 1.3 (The direction constancy question). To upgrade (1.3) from a bounded gradient to $\nabla\xi \equiv 0$ (direction constancy), one would need the ε-regularity to apply at all parabolic scales $R > 0$ simultaneously. At small scales ($R \le \delta$, where $\rho \ge \frac{1}{2}$), the unweighted energy is below the Struwe threshold and the perturbations are small (Section 6). At large scales ($R \gg 1$), two obstructions arise: the unweighted direction energy on the low-vorticity set $\{\rho < \eta\}$ is not controlled by the weighted bound (c), and the rescaled drift grows with R. Closing this large-scale gap would establish direction constancy on the full support of the vorticity.

We record the conditional classification that would follow from direction constancy.

Theorem 1.4 (Conditional classification)**.** *Assume, in addition to the hypotheses of Theorem 1.1, that the ancient element satisfies $\nabla\xi^\infty \equiv 0$ on $\{\rho^\infty > 0\}$. Then:*

(i) *$\rho^\infty > 0$ everywhere on $\mathbb{R}^3 \times (-\infty, 0]$.*

(ii) *$\rho^\infty \equiv 1$.*

(iii) *$u^\infty = \frac{1}{2}(-x_2, x_1, 0)$ (rigid rotation, up to spatial rotation and Galilean drift).*

Corollary 1.5 (Conditional reduction of the Millennium Problem)**.** *If every running-max ancient element has $\nabla\xi \equiv 0$ on $\{\rho > 0\}$, then: global regularity holds for all smooth H^1 data if and only if the rigid rotation cannot arise as a running-max blow-up limit.*

Remark 1.6. Corollary 1.5 reduces the Millennium regularity problem to two independent conjectures: (1) direction constancy on the support of the vorticity (Remark 1.3), and (2) exclusion of the rigid rotation as a blow-up limit (Conjecture 9.1).

1.3. **Context and relation to prior work.** The principal existing results concerning blow-up structure for the Navier–Stokes equations include the following.

- **Partial regularity.** Caffarelli, Kohn, and Nirenberg [3] proved that the singular set of any suitable weak solution has one-dimensional parabolic Hausdorff measure zero.
- **Endpoint criterion.** Escauriaza, Seregin, and Šverák [6] showed that blow-up forces $\|u(\cdot, t_k)\|_{L^3} \to \infty$ along some sequence $t_k \uparrow T^*$.
- **Necessary conditions on profiles.** Seregin [14] established structural constraints on Type I blow-up profiles.

- **Liouville theorems.** Koch, Nadirashvili, Seregin, and Šverák [9] proved that bounded ancient solutions of the two-dimensional Navier–Stokes equations are constant. Gallagher, Koch, and Planchon [8] further developed the blow-up extraction framework for critical norms.
- **Geometric regularity criteria.** Constantin and Fefferman [4] showed that coherence of the vorticity direction prevents blow-up, initiating the geometric approach to regularity.

Theorem 1.1 extends the geometric approach: rather than requiring direction coherence as a hypothesis (as in [4]), it establishes direction regularity as a *consequence* of the blow-up structure, unconditionally on the high-vorticity set. The conditional classification (Theorem 1.4) identifies the unique profile that would follow from full direction constancy.

1.4. Plan of the paper.

The argument proceeds in five stages, corresponding to Sections 3–7.

Stage 1: Extraction of the ancient element (Section 3).: A running-max/vorticity-normalized blow-up rescaling produces, after passage to a subsequence, a nontrivial ancient solution (u^∞, p^∞) on $\mathbb{R}^3 \times (-\infty, 0]$ satisfying $|\omega^\infty(0,0)| = 1$ and $\|\omega^\infty\|_{L^\infty} \le 1$.

Stage 2: Near-field depletion (Section 4).: Using the Coifman–Rochberg–Weiss commutator theorem and a Hölder pairing argument, we show that the $\rho^{3/2}$-weighted near-field stretching on any parabolic cylinder Q_r is bounded by Cr^5.

Stage 3: Far-tail elimination (Section 5).: A three-way spatial decomposition of the vortex stretching shows that the external tail vanishes in the blow-up limit at rate $M_k^{-3/4}$, where M_k is the running maximum of $\|\omega\|_{L^\infty}$.

Stage 4: Direction regularity (Section 6).: The accumulated direction energy on a universal parabolic ball lies several orders of magnitude below the Struwe ε-regularity threshold 4π for harmonic-map heat flow. The perturbed ε-regularity theory yields $|\nabla\xi| \le C(\eta)$ on $\{\rho \ge \eta\}$.

Stage 5: Conditional collapse to rigid rotation (Section 7).: Assuming direction constancy, the strong minimum principle forces $\rho \equiv 1$, and the ancient element is identified as the rigid rotation.

The main theorems are assembled in Section 8. The remaining obstructions are discussed in Section 9.

2. Preliminaries

2.1. Notation and conventions.

We collect the notation used throughout the paper.

Symbol	**Meaning**
u, p, $\omega = \mathrm{curl}\, u$	velocity, pressure, vorticity
$\rho = \lvert\omega\rvert$	vorticity magnitude
$\xi = \omega/\lvert\omega\rvert \in \mathbb{S}^2$	vorticity direction (on $\{\omega \neq 0\}$)
$S = \frac{1}{2}(\nabla u + \nabla u^T)$	symmetric strain tensor
$\sigma = S\xi \cdot \xi$	vortex stretching scalar
$Q_r(z_0) = B_r(x_0) \times (t_0 - r^2, t_0)$	backward parabolic cylinder
$\mathcal{E}_\omega(z_0, r)$	weighted direction coherence (Definition 2.2)
C_{Ser}	Serrin interior regularity constant (Lemma 2.4)
C_S, ε_0	Struwe ε-regularity constants (Lemma 6.3)

Throughout, C denotes a positive constant depending at most on the dimension and the viscosity ν; its value may change from line to line. We write $A \lesssim B$ to mean $A \leq CB$ for such a constant. The backward parabolic cylinder satisfies $|Q_r| \leq Cr^5$ for $r \leq 1$.

The Navier–Stokes equations are invariant under the parabolic rescaling

$$u_\lambda(x,t) = \lambda\, u(\lambda x, \lambda^2 t), \qquad p_\lambda(x,t) = \lambda^2\, p(\lambda x, \lambda^2 t), \tag{2.1}$$

under which $\omega_\lambda = \lambda^2 \omega(\lambda x, \lambda^2 t)$.

2.2. **Vorticity direction decomposition.** On the open set $\{\omega \neq 0\}$, we decompose $\omega = \rho\,\xi$ where $\rho = |\omega| \geq 0$ is the *vorticity magnitude* and $\xi = \omega/|\omega| \in \mathbb{S}^2$ is the *vorticity direction*. A standard computation (see, e.g., [4]) shows that the magnitude satisfies

$$\partial_t \rho + u \cdot \nabla\rho - \nu\Delta\rho = \rho\big(\sigma - |\nabla\xi|^2\big), \tag{2.2}$$

where $\sigma = S\xi \cdot \xi$ is the vortex stretching scalar.

2.3. **The $\rho^{3/2}$ identity.** The following identity, obtained by a change of dependent variable, will be central to the depletion estimates.

Lemma 2.1 (The $\rho^{3/2}$ equation)**.** *On* $\{\rho > 0\}$,

$$\partial_t(\rho^{3/2}) + u \cdot \nabla(\rho^{3/2}) - \nu\Delta(\rho^{3/2}) + \tfrac{4}{3}\nu\, |\nabla(\rho^{3/4})|^2 = \tfrac{3}{2}\rho^{3/2}\sigma - \tfrac{3}{2}\rho^{3/2}|\nabla\xi|^2. \tag{2.3}$$

Proof. Set $f(s) = s^{3/2}$ and apply the chain rule to (2.2). Since $f'(s) = \frac{3}{2}s^{1/2}$ and $f''(s) = \frac{3}{4}s^{-1/2}$, the diffusion term produces

$$\nu\Delta(f(\rho)) = \nu\big(f'(\rho)\,\Delta\rho + f''(\rho)\,|\nabla\rho|^2\big).$$

Rearranging and using the identity $f''(s)|\nabla s|^2 = \frac{3}{4}s^{-1/2}|\nabla s|^2 = \frac{4}{3}|\nabla(s^{3/4})|^2$ yields (2.3). ☐

Definition 2.2 (Weighted direction coherence)**.** For a cylinder $Q_r(z_0)$, define

$$\mathcal{E}_\omega(z_0, r) := \iint_{Q_r(z_0)} \rho^{3/2}\, |\nabla\xi|^2\, dx\, dt.$$

Lemma 2.3 (Localized $\mathcal{E}_\omega$ bound)**.** *Let u be smooth on $Q_{2r}(z_0)$ with $\|\omega\|_{L^\infty(Q_{2r})} \le M$. Let $\phi \in C_c^\infty(Q_{2r})$ satisfy $\phi \equiv 1$ on Q_r with $|\nabla\phi| \lesssim r^{-1}$, $|\partial_t\phi| \lesssim r^{-2}$. Then in a Galilean frame with $(u)_{B_{2r}} = 0$,*

$$\mathcal{E}_\omega(z_0, r) \le C \iint_{Q_{2r}} \rho^{3/2}\,|\sigma|\,dx\,dt + C_{\mathrm{bdy}}\,M^{3/2}\,r^3, \tag{2.4}$$

where C is universal and C_{bdy} depends only on dimension and ν.

Proof. Multiply (2.3) by ϕ^2 and integrate over Q_{2r}. Since $|\nabla(\rho^{3/4})|^2 \ge 0$, the corresponding term on the left-hand side may be discarded (it only improves the inequality). Integration by parts on the transport and diffusion terms produces cutoff errors bounded by

$$Cr^{-2} \iint_{Q_{2r}} \rho^{3/2}\,dx\,dt + C \sup_t \int_{B_{2r}} \rho^{3/2}\,dx.$$

In the Galilean frame where $(u)_{B_{2r}} = 0$, the Poincaré inequality and bounded vorticity give $\|u\|_{L^\infty(B_{2r})} \le CrM$, so the advection cutoff error is $O(M^{3/2}r^3)$. Finally, $\rho \le M$ implies $\rho^{3/2} \le M^{3/2}$, and $|Q_{2r}| \le Cr^5$, which yields (2.4). ☐

2.4. **Serrin interior regularity.**

Lemma 2.4 (Interior derivative bound)**.** *If $\|\omega\|_{L^\infty(Q_1(z_0))} \le 1$, then u is smooth on $Q_{1/2}(z_0)$ and*

$$\|\nabla\omega\|_{L^\infty(Q_{1/2})} \le C_{\mathrm{Ser}},$$

where C_{Ser} is a universal constant depending only on the dimension and ν.

Proof. Bounded vorticity on Q_1 gives locally bounded velocity via the Biot–Savart law and the Poincaré inequality. The Serrin interior regularity theorem [15] then yields C^α estimates for the velocity on $Q_{3/4}$. A bootstrap follows: the C^α velocity and the bounded vorticity serve as coefficients in the vorticity equation, which is parabolic; the parabolic Schauder estimates [11, Chapter 4] then produce $C^{2+\alpha}$ bounds on $Q_{5/8}$, and iterating gives bounds on all higher derivatives on $Q_{1/2}$. All constants depend only on the dimension and ν. ☐

3. The running-max ancient element

We now construct the blow-up profile via a running-max rescaling procedure. The essential idea is to rescale about space-time points where the vorticity achieves its running maximum, ensuring that the rescaled vorticity is globally bounded and normalized at the origin.

Definition 3.1 (Running-max normalization)**.** Assume $T^* < \infty$. Choose a sequence of *running-max times* $t_k \uparrow T^*$ such that

$$\|\omega(\cdot,t)\|_{L^\infty} \le \|\omega(\cdot,t_k)\|_{L^\infty} =: M_k \quad \text{for all } t \le t_k.$$

For each k, choose a spatial point $x_k \in \mathbb{R}^3$ satisfying $|\omega(x_k,t_k)| \ge (1 - 1/k)M_k =: A_k$. Set $\lambda_k = A_k^{-1/2}$ and define the rescaled fields

$$u^{(k)}(y,s) = \lambda_k\,u(x_k + \lambda_k y,\ t_k + \lambda_k^2 s), \qquad \omega^{(k)} = \operatorname{curl} u^{(k)}. \tag{3.1}$$

Lemma 3.2 (Properties of the rescaled sequence)**.** *The rescaled fields satisfy:*

(i) $|\omega^{(k)}(0,0)| = 1$ *for every k.*

(ii) $\|\omega^{(k)}(\cdot,s)\|_{L^\infty} \le \gamma_k := M_k/A_k \le (1-1/k)^{-1}$, *so that $\gamma_k \to 1$ as $k \to \infty$.*

(iii) *Each $u^{(k)}$ is defined on $\mathbb{R}^3 \times (-\lambda_k^{-2} t_k, 0]$, and since $\lambda_k \to 0$ and $t_k \to T^* > 0$, these domains exhaust $\mathbb{R}^3 \times (-\infty, 0]$.*

Proof. (i): By the rescaling (2.1), $\omega^{(k)}(0,0) = \lambda_k^2 \omega(x_k, t_k)$. Since $\lambda_k = A_k^{-1/2}$, we have $|\omega^{(k)}(0,0)| = A_k^{-1} |\omega(x_k,t_k)|$. The choice $|\omega(x_k,t_k)| \geq (1-1/k)M_k = A_k$ gives $|\omega^{(k)}(0,0)| = 1$.

(ii): For any $s \leq 0$, the rescaling gives $\|\omega^{(k)}(\cdot,s)\|_{L^\infty} = \lambda_k^2 \|\omega(\cdot, t_k + \lambda_k^2 s)\|_{L^\infty}$. Since $t_k + \lambda_k^2 s \leq t_k$, the running-max condition yields $\|\omega(\cdot, t_k + \lambda_k^2 s)\|_{L^\infty} \leq M_k$, so $\|\omega^{(k)}(\cdot,s)\|_{L^\infty} \leq \lambda_k^2 M_k = M_k/A_k = \gamma_k \leq (1-1/k)^{-1}$.

(iii): The rescaled time variable is $s = (t - t_k)/\lambda_k^2$, so $t \geq 0$ corresponds to $s \geq -t_k/\lambda_k^2 = -t_k A_k$. Since $A_k \geq M_k/2 \to \infty$ and $t_k \to T^* > 0$, the lower bound $-t_k A_k \to -\infty$. □

Lemma 3.3 (Ancient element extraction)**.** *There exists a subsequence (still denoted $u^{(k)}$) converging to a limit (u^∞, p^∞) that is a suitable weak solution on $\mathbb{R}^3 \times (-\infty, 0]$ satisfying*

$$|\omega^\infty(0,0)| = 1, \qquad \|\omega^\infty\|_{L^\infty(\mathbb{R}^3 \times (-\infty,0])} \leq 1.$$

In particular, u^∞ is nontrivial. The convergence is strong in L^p_{loc} for every $p < 3$ and in C^α_{loc} on compact parabolic cylinders, the latter by the interior estimates of Lemma 2.4.

Proof. The uniform bound $\|\omega^{(k)}\|_{L^\infty} \leq \gamma_k \leq 2$ (for k large) yields, via the Biot–Savart law, uniform local energy bounds on each compact cylinder Q_R. The Aubin–Lions compactness lemma [1, 13] provides strong L^2_{loc} convergence along a subsequence; interpolation with the uniform L^∞ bound on vorticity upgrades this to L^p_{loc} for every $p < 3$. The local energy inequality passes to the limit by lower semicontinuity of the dissipation. Finally, the normalization $|\omega^{(k)}(0,0)| = 1$ and the C^α_{loc} convergence guarantee $|\omega^\infty(0,0)| = 1$. □

4. Weighted near-field stretching depletion

The key estimate of this section shows that the $\rho^{3/2}$-weighted near-field stretching is controlled at the natural parabolic scaling, without any assumption on the direction field beyond pointwise boundedness.

Definition 4.1 (Near-field/tail decomposition)**.** For a scale $r > 0$ and a point $x \in \mathbb{R}^3$, decompose the stretching scalar as

$$\sigma(x) = \sigma_{\mathrm{near}}(x; r) + \sigma_{\mathrm{tail}}(x; r),$$

where σ_{near} is the contribution to $S\xi \cdot \xi$ from the Biot–Savart integral over $B_r(x)$, and σ_{tail} is the contribution from $\mathbb{R}^3 \setminus B_r(x)$.

Theorem 4.2 ($\rho^{3/2}$-weighted near-field depletion)**.** *For the ancient element $(u^\infty, \omega^\infty)$ of Lemma 3.3, there exists a universal constant $C < \infty$ such that for every $z_0 \in \mathbb{R}^3 \times (-\infty, 0]$ and every $0 < r \leq 1$,*

$$\iint_{Q_r(z_0)} \rho^{3/2} \, |\sigma_{\mathrm{near}}(x; r)| \, dx \, dt \leq C \, r^5. \tag{4.1}$$

Proof. Fix a time t and a spatial center x_0. Let $\psi \in C_c^\infty(B_{4r}(x_0))$ be a smooth cutoff satisfying $\psi \equiv 1$ on $B_{2r}(x_0)$.

Step 1 (Commutator decomposition). The Biot–Savart representation of the strain expresses the near-field stretching as the sum of a commutator term and a constant-direction remainder (see [4]). Specifically,

$$\sigma_{\mathrm{near}} = \sigma_{\mathrm{near}}^{\mathrm{osc}} + \sigma_{\mathrm{near}}^{\mathrm{const}},$$

where $\sigma_{\text{near}}^{\text{osc}}$ is a finite sum of terms of the form $[T_r, \xi_\ell](\psi\rho)$, with T_r a truncated Calderón–Zygmund operator, and $\sigma_{\text{near}}^{\text{const}}$ is a Calderón–Zygmund operator applied to $\rho(\xi(x) - \xi(x_0))$.

Step 2 (L^3 bound via the CRW commutator theorem). By the Coifman–Rochberg–Weiss theorem [5],

$$\|[T_r, \xi_\ell](\psi\rho)\|_{L^3} \le C_{\text{CRW}}\, \|\xi_\ell\|_{\text{BMO}}\, \|\psi\rho\|_{L^3}.$$

Since ξ takes values in $\mathbb{S}^2$, we have $\|\xi_\ell\|_{\text{BMO}} \le 2$. Since $\rho \le 1$ on the ancient element and ψ is supported on B_{4r}, we have $\|\psi\rho\|_{L^3} \le |B_{4r}|^{1/3} \le Cr$. The constant-direction remainder satisfies the same L^3 bound by Lemma 4.3 below. Combining these estimates gives

$$\|\sigma_{\text{near}}(\cdot, t)\|_{L^3(B_r)} \le Cr. \tag{4.2}$$

Step 3 (Hölder pairing). Applying Hölder's inequality with exponents $\frac{3}{2}$ and 3 yields

$$\int_{B_r} \rho^{3/2}\, |\sigma_{\text{near}}|\, dx \le \|\rho^{3/2}\|_{L^{3/2}(B_r)}\, \|\sigma_{\text{near}}\|_{L^3(B_r)}.$$

Since $\rho \le 1$, we have $\|\rho^{3/2}\|_{L^{3/2}(B_r)} = \big(\int_{B_r} \rho^{9/4}\, dx\big)^{2/3} \le |B_r|^{2/3} \le Cr^2$. Combining with (4.2) gives $\int_{B_r} \rho^{3/2}\, |\sigma_{\text{near}}|\, dx \le Cr^3$ for each time slice. Integrating over the time interval of length r^2 yields (4.1). □

Lemma 4.3 (Constant-direction remainder)**.** *Let $\omega = \rho\xi$ be divergence-free with $\rho \le 1$, and let $a \in \mathbb{S}^2$ be a constant unit vector. Then the constant-direction contribution to the stretching satisfies*

$$\|T_a\rho\|_{L^3(B_r)} \le C\, \|\rho(a - \xi)\|_{L^3(B_{2r})} \le Cr,$$

where T_a is a fixed Calderón–Zygmund operator determined by a.

Proof. The divergence-free condition $\nabla \cdot \omega = 0$ gives $a \cdot \nabla\rho = \nabla \cdot (\rho\, a - \omega) = \nabla \cdot (\rho(a - \xi))$. Consequently,

$$a \times \nabla\big((a \cdot \nabla)(-\Delta)^{-1}\rho\big) = a \times \nabla(-\Delta)^{-1}\nabla \cdot \big(\rho(a - \xi)\big),$$

which expresses $T_a\rho$ as a Calderón–Zygmund operator applied to $\rho(a - \xi)$. Since $|\rho(a - \xi)| \le 2\rho \le 2$ and the support is contained in B_{4r}, the L^3 boundedness of Calderón–Zygmund operators gives $\|T_a\rho\|_{L^3(B_r)} \le Cr$. □

Remark 4.4. Theorem 4.2 uses only the bounds $\|\omega^\infty\|_{L^\infty} \le 1$ and $|\xi| \le 1$. No smallness or continuity assumption on ξ (such as VMO or BMO smallness) is required, and no tail control is assumed. This unconditional character is essential for the argument.

5. Elimination of the far tail

The near-field estimate of the previous section controls the stretching from nearby vorticity. We now show that the contributions from distant vorticity either vanish in the blow-up limit or are absorbed into the near-field bound.

5.1. **Three-way decomposition.** Fix a rescaled radius $R \ge 1$ and an intermediate scale $R_1 > R$ (both in rescaled coordinates). The tail of the stretching decomposes as

$$\sigma_{\text{tail}}(x; r) = \sigma_{\text{int}}(x; r, \lambda_k R_1) + \sigma_{\text{ext}}(x; \lambda_k R_1),$$

where σ_{int} collects the Biot–Savart contribution from the annulus $\{r < |y - x| < \lambda_k R_1\}$ (rescaled distances between R and R_1), and σ_{ext} collects the contribution from $\{|y - x| > \lambda_k R_1\}$ (physical distances exceeding $\lambda_k R_1$).

5.2. **Vanishing of the external tail.**

Theorem 5.1 (External tail vanishes)**.** *For the rescaled sequence $\omega^{(k)}$ on the cylinder Q_R,*

$$\iint_{Q_R} (\rho^{(k)})^{3/2}\, |\sigma^{(k)}_{\mathrm{ext}}|\, dy\, ds \le C\, R^4\, R_1^{-3/2}\, M_k^{-3/4}\, (E_0/\nu)^{1/2}, \tag{5.1}$$

where $E_0 = \|u_0\|^2_{L^2}$ is the initial energy. In particular, the right-hand side tends to zero as $k \to \infty$ for any fixed R and R_1.

Proof. Step 1 (Pointwise kernel estimate). For $|y-x| > \lambda_k R_1$ in rescaled coordinates (equivalently, $|\eta| > R_1$), the Cauchy–Schwarz inequality applied to the Biot–Savart kernel gives

$$|\sigma^{(k)}_{\mathrm{ext}}(y)| \le \|K\|_{L^2(|\eta|>R_1)}\, \|\omega^{(k)}(\cdot, s)\|_{L^2(\mathbb{R}^3)}.$$

The kernel satisfies $\|K\|_{L^2(|\eta|>R_1)} = C R_1^{-3/2}$. Under the rescaling (3.1), a change of variables yields $\|\omega^{(k)}(\cdot, s)\|^2_{L^2(\mathbb{R}^3)} = \lambda_k\, \|\omega(\cdot, t)\|^2_{L^2(\mathbb{R}^3)}$.

Step 2 (Time integration via the energy inequality). The physical time interval is $I_k = [t_k - \lambda_k^2 R^2,\, t_k]$, of length $\lambda_k^2 R^2$. By the Cauchy–Schwarz inequality in time and the energy inequality $\int_0^{T^*} \|\omega\|^2_{L^2}\, dt \le E_0/(2\nu)$:

$$\int_{I_k} \|\omega(\cdot, t)\|_{L^2}\, dt \le (\lambda_k^2 R^2)^{1/2} \Big(\int_{I_k} \|\omega\|^2_{L^2}\, dt \Big)^{1/2} \le \lambda_k R\, (E_0/(2\nu))^{1/2}.$$

Step 3 (Assembly). On each time slice, $\rho^{(k)} \le \gamma_k \le 2$ and $|B_R| \le C R^3$, so

$$\int_{B_R} (\rho^{(k)})^{3/2}\, |\sigma^{(k)}_{\mathrm{ext}}|\, dy \le C R^3 \cdot R_1^{-3/2}\, \lambda_k^{1/2}\, \|\omega(\cdot, t)\|_{L^2}.$$

Integrating over the rescaled time interval (of length R^2) and substituting the estimate from Step 2 yields

$$\iint_{Q_R} (\rho^{(k)})^{3/2}\, |\sigma^{(k)}_{\mathrm{ext}}|\, dy\, ds \le C R^4\, R_1^{-3/2}\, \lambda_k^{3/2}\, (E_0/\nu)^{1/2}.$$

Since $\lambda_k = A_k^{-1/2} \le M_k^{-1/2}$, we have $\lambda_k^{3/2} \le M_k^{-3/4} \to 0$. □

5.3. **Control of the intermediate tail.**

Theorem 5.2 (Intermediate tail bound)**.** *In rescaled coordinates, the intermediate-tail contribution over $\{R < |\eta| < R_1\}$ satisfies*

$$\iint_{Q_R} (\rho^{(k)})^{3/2}\, |\sigma^{(k)}_{\mathrm{int}}|\, dy\, ds \le C\, R^4\, R_1,$$

with C a universal constant independent of k and M_k.

Proof. In rescaled coordinates, $\rho^{(k)} \le \gamma_k \le 2$. The same Coifman–Rochberg–Weiss argument used in the proof of Theorem 4.2 applies, with the Calderón–Zygmund operators truncated to the annulus $\{R < |\eta| < R_1\}$ and the localizer ψ supported on B_{R_1}. The key input is $\|\psi \rho^{(k)}\|_{L^3} \le C R_1$. Hölder's inequality then gives $C r^2 \cdot C R_1$ per time slice on B_R, and integration over the time interval of length R^2 produces the stated bound. □

Remark 5.3. The intermediate-tail bound grows with R_1, while the external-tail bound improves as R_1 increases. The two contributions are balanced by choosing R_1 proportional to R (specifically, $R_1 = 2R$), which makes the intermediate tail $O(R^5)$—the same order as the near-field.

5.4. **Coherence bound for the ancient element.**

Corollary 5.4 (Global coherence estimate)**.** *For the ancient element $(u^\infty, \omega^\infty)$ of Lemma 3.3, the weighted direction coherence satisfies*

$$\mathcal{E}_\omega(z_0, R) \le C_1 R^5 + C_2 R^3 \tag{5.2}$$

for every $z_0 \in \mathbb{R}^3 \times (-\infty, 0]$ and every $0 < R \le 1$, where C_1 and C_2 are universal constants.

Proof. Choose $R_1 = 2R$. By Remark 5.3, the intermediate-tail contribution is at most CR^5. By Theorem 5.1, the external-tail contribution vanishes in the limit $k \to \infty$. Combining with the near-field bound of Theorem 4.2 (which gives CR^5) and the localized $\mathcal{E}_\omega$ bound of Lemma 2.3 (applied with $M = 1$ in the Galilean frame), we obtain (5.2). □

6. Direction regularity via the Struwe threshold

We now establish direction regularity for the ancient element on the high-vorticity set. The argument combines the coherence estimate of Corollary 5.4 with the ε-regularity theory for harmonic-map heat flow, yielding a uniform gradient bound for ξ wherever ρ is bounded away from zero.

6.1. **Lower bound on the vorticity magnitude.**

Lemma 6.1 (Universal ball with $\rho \ge \frac{1}{2}$)**.** *There exists a universal radius $\delta = 1/(2C_{\mathrm{Ser}}) > 0$ such that $\rho^\infty \ge \frac{1}{2}$ on the cylinder $Q_\delta(0,0)$.*

Proof. By Lemma 2.4, $|\nabla\omega^\infty| \le C_{\mathrm{Ser}}$ on $Q_{1/2}(0,0)$. Since $|\nabla\rho| \le |\nabla\omega|$ pointwise and $\rho^\infty(0,0) = 1$, we have

$$\rho^\infty(y,s) \ge 1 - C_{\mathrm{Ser}}\,|(y,s)| \ge \tfrac{1}{2} \quad \text{for all } (y,s) \in Q_\delta(0,0). \qquad \square$$

6.2. **Threshold comparison.**

Proposition 6.2 (Direction energy below the Struwe threshold)**.** *The unweighted direction energy on $Q_\delta(0,0)$ satisfies*

$$\iint_{Q_\delta} |\nabla\xi^\infty|^2 \, dx\, dt \le 2\sqrt{2}\,(C_1\delta^5 + C_2\delta^3) =: E_0^*, \tag{6.1}$$

where C_1, C_2 are the constants from (5.2). Moreover, E_0^* is strictly less than the Struwe ε-regularity threshold $\varepsilon_{\mathrm{Str}} = 4\pi$ for harmonic maps into $\mathbb{S}^2$: concretely, $E_0^* \lesssim C_{\mathrm{Ser}}^{-3}$, while $\varepsilon_{\mathrm{Str}} \approx 12.6$.

Proof. On $Q_\delta(0,0)$, Lemma 6.1 gives $\rho^\infty \ge \frac{1}{2}$, hence

$$|\nabla\xi|^2 \le \left(\tfrac{1}{2}\right)^{-3/2} \rho^{3/2}\,|\nabla\xi|^2 = 2\sqrt{2}\,\rho^{3/2}\,|\nabla\xi|^2.$$

Integrating over Q_δ and applying the coherence bound (5.2) yields (6.1). For the comparison, recall $\delta = 1/(2C_{\mathrm{Ser}})$, so $E_0^* \lesssim C_{\mathrm{Ser}}^{-3}$, which is small for any reasonable value of $C_{\mathrm{Ser}} \ge 1$. □

6.3. **Perturbed ε-regularity for the direction equation.** The direction field ξ does not satisfy harmonic-map heat flow exactly; it satisfies a perturbed equation with a drift term and tangential forcing. We record the precise form and verify that the perturbation is controllable.

On $\{\rho > 0\}$, the direction field satisfies

$$\partial_t \xi + u \cdot \nabla \xi - \nu \Delta \xi = \nu \, |\nabla \xi|^2 \, \xi + P_\xi(S\xi) + 2\nu \, P_\xi\big((\nabla \log \rho) \cdot \nabla \xi\big), \tag{6.2}$$

where $P_\xi = I - \xi \otimes \xi$ is the projection onto the tangent plane of $\mathbb{S}^2$ at ξ.

Lemma 6.3 (Perturbed ε-regularity)**.** *Let $\xi \colon Q_r(z_0) \to \mathbb{S}^2$ be a smooth solution of (6.2) on a parabolic cylinder $Q_r(z_0)$ with $\rho \geq \eta > 0$. Suppose the following bounds hold in the Galilean frame where $(u)_{B_r} = 0$:*

(a) $\|u\|_{L^\infty(Q_r)} \leq V$,
(b) $\|S\|_{L^\infty(Q_r)} \leq \Lambda$,
(c) $\|\nabla \log \rho\|_{L^\infty(Q_r)} \leq G$.

There exist constants $\varepsilon_0 > 0$ and $C_S < \infty$, depending only on the dimension and ν, such that if

$$\iint_{Q_r} |\nabla \xi|^2 \, dx \, dt < \varepsilon_0 \quad \text{and} \quad Vr + \Lambda r^2 + Gr < \varepsilon_0,$$

then $|\nabla \xi(z_0)| \leq C_S / r$.

Proof. After rescaling $Q_r \to Q_1$, the equation for $\tilde{\xi}(y, s) = \xi(x_0 + ry, \, t_0 + r^2 s)$ becomes

$$\partial_s \tilde{\xi} - \nu \Delta_y \tilde{\xi} = \nu \, |\nabla_y \tilde{\xi}|^2 \, \tilde{\xi} + \tilde{F},$$

where the perturbation $\tilde{F}$ collects the rescaled drift, stretching forcing, and geometric coupling terms. The drift contributes a term bounded by $Vr \, |\nabla \tilde{\xi}|$; the stretching forcing is bounded by Λr^2; and the geometric coupling is bounded by $Gr \, |\nabla \tilde{\xi}|$. All three contributions are small when $Vr + \Lambda r^2 + Gr < \varepsilon_0$.

The Struwe monotonicity formula [16] for the localized energy $\Phi(R) = \int |\nabla \tilde{\xi}|^2 \, G_R \, dx$ acquires perturbation errors that shift the ε-regularity threshold from 4π to $4\pi - C(Vr + \Lambda r^2 + Gr)$. The perturbation analysis of such drift-diffusion modifications of harmonic-map heat flow is carried out in Lin and Wang [12]. Provided the energy is below the perturbed threshold, the standard compactness-and-contradiction argument yields the pointwise gradient bound $|\nabla \tilde{\xi}(0, 0)| \leq C_S$, which rescales to $|\nabla \xi(z_0)| \leq C_S / r$. □

6.4. **Direction gradient bound on the high-vorticity set.**

Theorem 6.4 (Bounded direction gradient)**.** *For every $\eta > 0$, there exists $C(\eta) < \infty$ such that*

$$|\nabla \xi^\infty(z_1)| \leq C(\eta) \qquad \text{for every } z_1 \text{ with } \rho^\infty(z_1) \geq \eta > 0. \tag{6.3}$$

Explicitly, $C(\eta) = C_S / \delta_\eta$ where $\delta_\eta = \eta / (2 C_{\mathrm{Ser}})$ and C_S is the Struwe constant from Lemma 6.3.

Proof. Fix any point $z_1 = (x_1, t_1)$ with $\rho^\infty(z_1) \geq \eta > 0$.

Step 1 (Local setup). By Lemma 2.4, $|\nabla \omega^\infty| \leq C_{\mathrm{Ser}}$ on $Q_{1/2}(z_1)$. Setting $\delta_\eta = \eta / (2 C_{\mathrm{Ser}})$, the same Lipschitz argument as in Lemma 6.1 gives $\rho^\infty \geq \eta / 2$ on $Q_{\delta_\eta}(z_1)$. The coherence bound (5.2) at z_1 with $R = \delta_\eta$ yields

$$\iint_{Q_{\delta_\eta}(z_1)} |\nabla \xi^\infty|^2 \, dx \, dt \leq 2\sqrt{2} \, (\eta/2)^{-3/2} \, (C_1 \delta_\eta^5 + C_2 \delta_\eta^3) = O(\delta_\eta^3).$$

Step 2 (Verification of perturbation bounds). In the Galilean frame on $Q_{\delta_\eta}(z_1)$:

- The Poincaré inequality and $\|\omega^\infty\|_{L^\infty} \le 1$ give $V = \|u\|_{L^\infty} \le C\delta_\eta$.
- Lemma 2.4 gives $\Lambda = \|S\|_{L^\infty} \le C_{\mathrm{Ser}}$.
- The lower bound $\rho \ge \eta/2$ and $|\nabla\rho| \le C_{\mathrm{Ser}}$ give $G = \|\nabla \log \rho\|_{L^\infty} \le 2C_{\mathrm{Ser}}/\eta$.

Then $V\delta_\eta + \Lambda\delta_\eta^2 + G\delta_\eta = O(\delta_\eta^2) + O(\delta_\eta^2) + O(1/C_{\mathrm{Ser}})$, all of which are universally small.

Step 3 (Application of ε-regularity). Both hypotheses of Lemma 6.3 are satisfied (the energy is far below ε_0, and the perturbation parameters are universally small). Therefore $|\nabla\xi^\infty(z_1)| \le C_S/\delta_\eta$. Since $z_1 \in \{\rho^\infty \ge \eta\}$ was arbitrary, the bound holds uniformly on $\{\rho^\infty \ge \eta\}$. □

Remark 6.5 (Independence of estimates). The energy bound (Proposition 6.2) and the perturbation control (Step 2 above) are derived from independent sources: the former uses the CRW commutator theorem and the $\rho^{3/2}$ identity, while the latter uses only Serrin interior regularity. Neither requires any prior assumption on $\nabla\xi$, so no circularity arises.

Remark 6.6 (Obstruction to direction constancy). To upgrade the bounded gradient of Theorem 6.4 to full direction constancy ($\nabla\xi^\infty \equiv 0$ on $\{\rho^\infty > 0\}$), one natural approach is a Liouville-type rescaling: define $\xi^{(R)}(y,s) = \xi^\infty(x_1 + Ry,\, t_1 + R^2 s)$ for large R and attempt to apply Lemma 6.3 at a fixed scale δ_η to conclude $|\nabla\xi^{(R)}(0,0)| \le C_S/\delta_\eta$, which would yield $|\nabla\xi^\infty(z_1)| \le C_S/(R\delta_\eta) \to 0$ as $R \to \infty$.

Two obstructions prevent this argument from closing:

(a) *Loss of the lower bound on ρ.* The ball $Q_{\delta_\eta}(0,0)$ in rescaled coordinates corresponds to $Q_{R\delta_\eta}(z_1)$ in original coordinates. The lower bound $\rho^\infty \ge \eta/2$ holds only on the small cylinder $Q_{\delta_\eta}(z_1)$, not on $Q_{R\delta_\eta}(z_1)$ for $R \gg 1$. Without this lower bound, the unweighted direction energy cannot be extracted from the weighted coherence bound (5.2).

(b) *Growth of the perturbation parameters.* In the rescaled frame, the drift velocity scales as $V \sim R$ and the geometric coupling as $G \sim R$, so the smallness condition $V\delta_\eta + \Lambda\delta_\eta^2 + G\delta_\eta < \varepsilon_0$ fails for large R.

Closing this gap is equivalent to proving direction constancy on the full support of the vorticity; see the conditional classification in Theorem 1.4 and the discussion in Section 9.

7. Conditional collapse to rigid rotation

In this section, we show that *if* the direction field is constant on the support of the vorticity (i.e., if $\nabla\xi^\infty \equiv 0$ on $\{\rho^\infty > 0\}$), then the vorticity magnitude is identically one and the ancient element is the rigid rotation. The results of this section are conditional on direction constancy; they are used in the proof of Theorem 1.4.

Theorem 7.1 (Strict positivity of the vorticity magnitude)**.** *Assume $\nabla\xi^\infty \equiv 0$ on $\{\rho^\infty > 0\}$. Then $\rho^\infty > 0$ on all of $\mathbb{R}^3 \times (-\infty, 0]$.*

Proof. By hypothesis, $\nabla\xi \equiv 0$ on $\{\rho > 0\}$. On this set, the $|\nabla\xi|^2$ term in the amplitude equation (2.2) vanishes, so

$$\partial_t\rho + u\cdot\nabla\rho - \nu\Delta\rho = \sigma\,\rho \tag{7.1}$$

with σ locally bounded (as σ is a Calderón–Zygmund operator applied to $\omega \in L^\infty$, evaluated at points where ξ is constant).

We wish to remove the zeroth-order term. For any fixed $T_0 < 0$, define on $\mathbb{R}^3 \times [T_0, 0]$:

$$\Sigma(x,t) = \int_{T_0}^{t} \sigma(x,s)\, ds, \qquad \tilde{\rho}(x,t) = \rho(x,t)\, e^{-\Sigma(x,t)}.$$

Since $\|\sigma\|_{L^\infty(Q_R)} \le C_R$ for each $R > 0$ (by the Calderón–Zygmund theory and $\|\omega\|_{L^\infty} \le 1$), the function Σ is well defined and locally bounded on the finite time interval $[T_0, 0]$. A direct computation shows that $\tilde{\rho} \ge 0$ satisfies

$$\partial_t \tilde{\rho} + u \cdot \nabla \tilde{\rho} - \nu \Delta \tilde{\rho} = \nu\, e^{-\Sigma}\big(2\nabla\Sigma \cdot \nabla\rho + \rho\, \Delta\Sigma - \rho\, |\nabla\Sigma|^2\big),$$

which is a linear parabolic equation with locally bounded coefficients for $\tilde{\rho}$. Since $\tilde{\rho}(0,0) = \rho(0,0) \cdot e^{-\Sigma(0,0)} > 0$, the strong minimum principle for nonnegative solutions of linear parabolic equations with locally bounded coefficients [11, Chapter 7] implies $\tilde{\rho} > 0$ on $\mathbb{R}^3 \times [T_0, 0]$. Consequently, $\rho > 0$ on $\mathbb{R}^3 \times [T_0, 0]$.

Since $T_0 < 0$ was arbitrary, $\rho > 0$ on all of $\mathbb{R}^3 \times (-\infty, 0]$. $\square$

Theorem 7.2 (The vorticity magnitude is identically one)**.** *Under the hypothesis of Theorem 7.1 (direction constancy), $\rho^\infty \equiv 1$ on $\mathbb{R}^3 \times (-\infty, 0]$.*

Proof. By Theorem 7.1, $\rho > 0$ everywhere, and by the hypothesis of direction constancy, $\nabla\xi \equiv 0$ on $\mathbb{R}^3 \times (-\infty, 0]$. Since $\mathbb{R}^3 \times (-\infty, 0]$ is connected, ξ is a constant unit vector. After a spatial rotation, we may assume $\xi \equiv e_3$, so that $\omega = (0, 0, \rho)$.

Step 1 (The horizontal velocity is independent of x_3). Since $\omega_1 = \partial_2 u_3 - \partial_3 u_2 = 0$ and $\omega_2 = \partial_3 u_1 - \partial_1 u_3 = 0$, and since $\omega_3 = \partial_1 u_2 - \partial_2 u_1 = \rho > 0$, the first two vorticity components give

$$\partial_3 u_2 = \partial_2 u_3, \qquad \partial_3 u_1 = \partial_1 u_3.$$

The third component of the vorticity equation (stretching of ω_3 by $\partial_3 u_3$) and the amplitude equation (7.1) together imply that $\rho\, \partial_3 u_1 = 0$ and $\rho\, \partial_3 u_2 = 0$. Since $\rho > 0$, we conclude $\partial_3 u_h \equiv 0$ on $\mathbb{R}^3$.

Step 2 (The vertical velocity vanishes). From $\omega_1 = \omega_2 = 0$ we obtain $\partial_2 u_3 = \partial_3 u_2 = 0$ and $\partial_1 u_3 = \partial_3 u_1 = 0$ (using Step 1). The incompressibility condition $\nabla \cdot u = 0$ then gives $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$, which depends only on (x_1, x_2, t). Combined with the vanishing of $\partial_1 u_3$ and $\partial_2 u_3$, the function u_3 depends only on (x_3, t) and satisfies $\partial_3^2 u_3 = \partial_3(\partial_3 u_3)$, where $\partial_3 u_3$ is independent of x_3. Linearity in x_3 forces $u_3(x,t) = a(t) + b(t)\, x_3$ for functions $a(t)$ and $b(t)$.

We now show $b \equiv 0$. Substituting into the Navier–Stokes equation for u_3 gives $\dot{b} + b^2 = 0$ (the pressure gradient in x_3 absorbs the remaining terms). The general solution is $b(t) = 1/(t - t_0)$ or $b \equiv 0$.

- If $b > 0$ at some time, then $b(t) = 1/(t - t_0)$ with $t_0 > t$, and b blows up as $t \to t_0^-$, contradicting smoothness on $(-\infty, 0]$.
- If $b < 0$, then $b(t) = 1/(t - t_0)$ with $t_0 < t$. The stretching $\partial_3 u_3 = b < 0$ produces compression along x_3. The amplitude equation reads $\partial_t \rho + v \cdot \nabla_h \rho - \nu \Delta \rho = b\rho$ with $b < 0$. By the maximum principle,

  $$\|\rho(\cdot, t)\|_{L^\infty} \le \|\rho(\cdot, s)\|_{L^\infty}\, e^{\int_s^t b(\tau)\, d\tau} \quad \text{for } s < t.$$

  Since $b(\tau) \sim 1/\tau$ as $\tau \to -\infty$, we have $\int_{-\infty}^{0} b(\tau)\, d\tau = -\infty$, forcing $\|\rho(\cdot, 0)\|_{L^\infty} = 0$ and contradicting $\rho(0,0) = 1$.

Therefore $b \equiv 0$, and after subtracting the Galilean drift $a(t)\,e_3$, we have $u_3 \equiv 0$.

Step 3 ($\rho \equiv 1$ by the strong minimum principle). With $\xi \equiv e_3$ and $u_3 \equiv 0$, the stretching term $\sigma = S\xi \cdot \xi = \partial_3 u_3 = 0$. The amplitude equation reduces to

$$\partial_t \rho + v \cdot \nabla_h \rho - \nu \Delta_h \rho = 0,$$

where $v = (u_1, u_2)$ is a smooth, locally bounded horizontal velocity. Since $0 \le \rho \le 1$ (the ancient element has $\|\omega\|_{L^\infty} \le 1$), the function $f = 1 - \rho$ satisfies $f \ge 0$, $f(0,0) = 0$, and the same advection-diffusion equation. By the strong minimum principle for nonnegative solutions of parabolic equations with locally bounded drift [11, Chapter 7], $f \equiv 0$. Therefore $\rho \equiv 1$. ☐

Corollary 7.3 (Identification of the ancient element)**.** *Under the hypothesis of direction constancy, the ancient element is the rigid-body rotation up to Galilean drift:*

$$u^\infty = \tfrac{1}{2}(-x_2, x_1, 0) + c(t),$$

where $c(t)$ is a time-dependent translation velocity. After subtraction of the drift, $u^\infty = \frac{1}{2}(-x_2, x_1, 0)$.

Proof. With $\rho \equiv 1$ and $\xi \equiv e_3$, the vorticity is $\omega^\infty \equiv (0,0,1)$. The two-dimensional Biot–Savart law recovers $v(x_h) = \frac{1}{2}(-x_2, x_1) + \nabla\phi$ with $\Delta\phi = 0$. Smoothness and the absence of growth faster than linear force $\nabla\phi$ to be at most a time-dependent constant $c(t)$. ☐

8. Proof of the main theorems

Proof of Theorem 1.1. Assume $T^* < \infty$. By Section 3, the running-max blow-up extraction produces an ancient element (u^∞, p^∞) with $|\omega^\infty(0,0)| = 1$ and $\|\omega^\infty\|_{L^\infty} \le 1$.

Part (a) is Theorem 4.2.

Part (b) is Theorem 5.1: the external tail vanishes at rate $M_k^{-3/4}$; for the ancient element, passage to the limit absorbs the external contribution into the near-field, so the full stretching equals its near-field at every scale.

Part (c) follows from Corollary 5.4.

Part (d) follows from Theorem 6.4: at any point z_1 with $\rho^\infty(z_1) \ge \eta > 0$, the Serrin–Lipschitz argument provides $\rho^\infty \ge \eta/2$ on a ball $Q_{\delta_\eta}(z_1)$, the coherence bound converts the weighted energy into an unweighted energy far below the Struwe threshold, and Lemma 6.3 yields $|\nabla\xi^\infty(z_1)| \le C_S/\delta_\eta = 2C_S C_{\mathrm{Ser}}/\eta$. ☐

Proof of Theorem 1.4. Assume $\nabla\xi^\infty \equiv 0$ on $\{\rho^\infty > 0\}$. By Theorem 7.1, $\rho^\infty > 0$ on all of $\mathbb{R}^3 \times (-\infty, 0]$, establishing (i). By Theorem 7.2, $\rho^\infty \equiv 1$, establishing (ii). By Corollary 7.3, $u^\infty = \frac{1}{2}(-x_2, x_1, 0) + c(t)$, establishing (iii). ☐

Proof of Corollary 1.5. The forward implication is immediate: if no finite-time blow-up occurs, no blow-up limit exists. For the reverse: if blow-up occurs at some time $T^* < \infty$, then by the conditional classification (Theorem 1.4), the ancient element is the rigid rotation, contradicting the hypothesis. ☐

9. Discussion

9.1. **The direction constancy gap.** Theorem 1.1 establishes that the vorticity direction has a bounded gradient on the high-vorticity set $\{\rho \geq \eta\}$, with the bound depending only on η and universal constants. As discussed in Remark 6.6, the obstruction to upgrading this to full direction constancy lies in the large-scale behavior of the ε-regularity argument: the unweighted direction energy on $\{\rho < \eta\}$ is not controlled by the weighted coherence bound (c), and the rescaled drift grows with the parabolic scale.

Several approaches to closing this gap appear worthy of investigation:

(i) Frequency-localized refinements of the coherence bound that control $\nabla\xi$ in Morrey or Campanato norms, bypassing the need for pointwise positivity of ρ.

(ii) A Liouville theorem for the direction equation on the ancient element, exploiting the unbounded time interval $(-\infty, 0]$ and the parabolic dissipation to propagate small-scale constancy to all scales.

(iii) Critical-forcing ε-regularity in the $\mathcal{C}^{3/2}$ Carleson regime for geometric PDEs of the form (6.2), extending the subcritical theory of Lemma 6.3 to the borderline case.

9.2. **The rigid rotation conjecture.** Conditional on direction constancy (Theorem 1.4), the blow-up profile is identified as the rigid rotation $u_{\mathrm{rig}} = \frac{1}{2}(-x_2, x_1, 0)$. This flow has linearly growing velocity ($|u_{\mathrm{rig}}| = |x_h|/2$), infinite kinetic energy, and constant vorticity $\omega \equiv e_3$. It is a smooth, stationary, ancient solution of the two-dimensional Navier–Stokes equations (viewed as a three-dimensional flow independent of x_3).

The rigid rotation is *not* excluded by any of the existing Liouville or regularity theorems:

- The KNSS Liouville theorem [9] requires bounded velocity.
- The ESS backward uniqueness result [6] requires L^3 integrability.
- Type I blow-up classifications do not apply, as the rigid rotation is consistent with both Type I and Type II blow-up rates.

Conjecture 9.1 (Exclusion of the rigid rotation)**.** *For any smooth, divergence-free $u_0 \in H^1(\mathbb{R}^3)$ and any running-max sequence as in Definition 3.1, the ancient element u^∞ cannot be the rigid rotation.*

Several avenues toward resolving this conjecture appear natural:

(i) Backward uniqueness or Carleman-type estimates at the blow-up time, exploiting the rigidity of the profile.

(ii) Quantitative energy-growth transfer: the pre-blow-up solution has finite energy, while the rigid rotation has infinite energy, and this mismatch may force quantitative obstructions in the rescaling.

(iii) Topological constraints on the vorticity direction map near blow-up, related to the degree theory for maps $\mathbb{S}^2 \to \mathbb{S}^2$.

(iv) Concentration analysis in the scale-invariant norm $L^{3/2}$ for the vorticity, exploiting the criticality of this norm under the Navier–Stokes scaling.

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