

# THE RIEMANN HYPOTHESIS VIA INNER FUNCTIONS AND SCHUR CERTIFICATION

JONATHAN WASHBURN AND AMIR RAHNAMAI BARGHI

**ABSTRACT.** Starting from the Euler product and the regularized determinant  $\det_2(I - A(s))$  over primes, we construct an inner function  $\mathcal{I}$  on  $\{\Re s > \frac{1}{2}\}$  whose zero set coincides with that of  $\zeta$ , and prove unconditionally that  $\mathcal{I}$  is a *pure Blaschke product* (the singular inner factor is trivial). The Riemann Hypothesis is equivalent to the statement that this Blaschke product has no zeros. We establish this zero-free property via the *Schur/Nevanlinna–Pick pathway*: the Cayley transform  $\Xi := (2\mathcal{J} - 1)/(2\mathcal{J} + 1)$  of the arithmetic ratio converts any would-be pole of  $\mathcal{J}$  (i.e. any zero of  $\zeta$ ) into a boundary hit  $\Xi \rightarrow 1$ ; a global Schur bound  $|\Xi| \leq 1$  then forces the singularity to be removable, excluding the zero. Under the Nyquist bandwidth hypothesis (T7-Hyp)—a Recognition Science prediction that prime-frequency observables are bandlimited by  $\Omega_{\max} = 1/(2\tau_0)$ —the windowed prime sum in the explicit formula becomes a *finite* sum, the Pick spectral gap persists uniformly as  $\sigma_0 \rightarrow (\frac{1}{2})^+$ , and the Schur bound closes on all of  $\{\Re s > \frac{1}{2}\}$ , yielding the Riemann Hypothesis.

## 1. INTRODUCTION

The Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad \Re s > 1,$$

extends meromorphically to  $\mathbb{C}$  with a simple pole at  $s = 1$  and satisfies a functional equation after completion. Its nontrivial zeros govern the distribution of prime numbers, and the Riemann Hypothesis asserts that all such zeros lie on the critical line  $\Re s = \frac{1}{2}$ ; see [2, 4, 6, 14] for background.

**Theorem 1** (Inner-function encoding of the zeros of  $\zeta$ ). *Let  $\Omega := \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$ . There exists a function  $\mathcal{I}$ , constructed explicitly from  $\zeta$ , the regularized determinant  $\det_2(I - A(s))$ , and an outer normalizer  $\mathcal{O}_\zeta$  (§§2–3, Lemma 25), with the following properties:*

- (a)  *$\mathcal{I}$  is holomorphic on  $\Omega$  with  $|\mathcal{I}(s)| \leq 1$  for all  $s \in \Omega$ .*
- (b)  *$|\mathcal{I}(\frac{1}{2} + it)| = 1$  for Lebesgue-a.e.  $t \in \mathbb{R}$ .*
- (c) *The zeros of  $\mathcal{I}$  in  $\Omega$  are exactly the nontrivial zeros of  $\zeta$  in  $\Omega$ , with the same multiplicities.*
- (d)  *$\mathcal{I}$  is a pure Blaschke product: the singular inner factor is trivial,  $S \equiv 1$ .*

**Corollary 2** (Equivalence with the Riemann Hypothesis). *The Riemann Hypothesis is equivalent to the statement  $\mathcal{I} \equiv e^{i\theta}$  for some  $\theta \in \mathbb{R}$ , i.e., the Blaschke product is empty.*

*Proof.* If RH holds,  $\mathcal{I}$  has no zeros and is inner, hence a unimodular constant. Conversely, if  $\mathcal{I} \equiv e^{i\theta}$ , part (c) of Theorem 1 implies  $\zeta$  has no zeros in  $\Omega$ .  $\square$

**Theorem 3** (Riemann Hypothesis under T7-Hyp). *Assume the Nyquist bandwidth hypothesis (T7-Hyp, Hypothesis 19). Then  $\zeta(s) \neq 0$  for all  $s \in \Omega$ .*

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Theorem 1 and Corollary 2 are proved unconditionally in §§2–3 and Appendix A. Section 4 establishes Theorem 3 via the Schur/Nevanlinna–Pick pathway: the Cayley transform of  $\mathcal{J}$  is shown to be a Schur function on  $\Omega$  under T7-Hyp, which excludes all poles of  $\mathcal{J}$  and hence all zeros of  $\zeta$ .

**Notation.** Throughout we use the following conventions.

- $\Omega := \{s \in \mathbb{C} : \Re s > \frac{1}{2}\}$  denotes the open half-plane to the right of the critical line, with boundary  $\partial\Omega = \{\frac{1}{2} + it : t \in \mathbb{R}\}$ .
- $\sigma := \Re s - \frac{1}{2}$  is the distance from the critical line.
- $\langle T \rangle := (1 + T^2)^{1/2}$  is the Japanese bracket.
- For a compact interval  $I \subset \mathbb{R}$ ,  $|I|$  denotes its length and

$$Q_\alpha(I) := \left\{ \frac{1}{2} + \sigma + it : 0 < \sigma \leq \alpha |I|, t \in I \right\}$$

is the Whitney box with aperture  $\alpha > 0$ .

- “A.e.” refers to Lebesgue measure on  $\mathbb{R}$  unless stated otherwise.

**Strategy.** On  $\Omega$  we construct an *inner reciprocal*  $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$ , where  $B(s) = (s - 1)/s$ , from the Riemann zeta function, the regularized determinant  $\det_2(I - A(s))$  over primes, and an outer normalizer  $\mathcal{O}_\zeta$ ; the construction is carried out in §2–§3. Lemma 25 shows that  $\mathcal{I}$  is holomorphic on  $\Omega$  with  $|\mathcal{I}| \leq 1$  (via the Phragmén–Lindelöf principle) and boundary modulus 1 a.e. Crucially, zeros of  $\zeta$  in  $\Omega$  become *zeros* (not poles) of  $\mathcal{I}$ . The unconditional proof that  $S \equiv 1$  (Proposition 26) then identifies  $\mathcal{I}$  as a pure Blaschke product, yielding Theorem 1.

To prove Theorem 3, we use the *Schur/Nevanlinna–Pick pathway* (§4), which avoids Cauchy–Schwarz entirely. Define the *Cayley field*  $\Xi(s) := (2\mathcal{J}(s) - 1)/(2\mathcal{J}(s) + 1)$ . If  $\zeta(\rho) = 0$  then  $\mathcal{J}$  has a pole at  $\rho$ , forcing  $\Xi(\rho) \rightarrow 1$ . A global Schur bound  $|\Xi| \leq 1$  makes this singularity removable (by Riemann’s theorem), so  $\mathcal{J}$  has no poles and  $\zeta$  has no zeros. The Schur property is certified via the Nevanlinna–Pick criterion: a finite Pick matrix with positive spectral gap, plus a quantitative Taylor tail bound, implies  $|\Xi| \leq 1$  globally. Under T7-Hyp the windowed prime sum is finite, the tail bound is uniform, and the Schur certificate closes on all of  $\Omega$ .

## 2. DEFINITIONS AND MAIN OBJECTS

This section introduces the principal objects of the proof: the prime-diagonal operator  $A(s)$  and its regularized determinant  $\det_2(I - A(s))$ , and the arithmetic ratio  $\mathcal{J}$  formed from  $\det_2$  and  $\zeta$ .

**The completed zeta function.** Let  $\zeta(s)$  denote the Riemann zeta function. We write  $\xi(s)$  for the completed zeta function

$$\xi(s) := \frac{1}{2}s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

which is entire and satisfies the functional equation  $\xi(s) = \xi(1 - s)$ ; see [14]. Throughout, by a *zero* we mean a zero of  $\zeta$  (equivalently of  $\xi$ , away from the canceled singularities at  $s = 0, 1$ ) lying in the half-plane  $\Omega$ .

**The prime-diagonal operator and the regularized determinant.** Let  $\mathcal{P}$  denote the set of primes and write  $\ell^2(\mathcal{P})$  for the Hilbert space with orthonormal basis  $\{e_p\}_{p \in \mathcal{P}}$ . For  $s \in \mathbb{C}$  define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \rightarrow \ell^2(\mathcal{P}), \quad A(s)e_p := p^{-s}e_p.$$

For  $\Re s > 1/2$ ,

$$\|A(s)\|_{\text{HS}}^2 = \sum_{p \in \mathcal{P}} |p^{-s}|^2 = \sum_{p \in \mathcal{P}} p^{-2\Re s} \leq \sum_{n \geq 2} n^{-2\Re s} < \infty,$$

so  $A(s)$  is Hilbert–Schmidt on  $\Omega$ . In particular, the regularized determinant  $\det_2(I - A(s))$  is well-defined and holomorphic on  $\Omega$  (see [10, Ch. III] and [12, Ch. 9]).

**Lemma 4** (Diagonal product formula for  $\det_2$ ). *Let  $T$  be a diagonal Hilbert–Schmidt operator on  $\ell^2$  with eigenvalues  $\{\lambda_n\}$  satisfying  $\sum_n |\lambda_n|^2 < \infty$ . Then*

$$\det_2(I - T) = \prod_n (1 - \lambda_n) e^{\lambda_n},$$

where the product converges absolutely. In particular,  $\det_2(I - T) = 0$  iff  $\lambda_n = 1$  for some  $n$ .

*Proof.* This holds for the  $\mathcal{S}_2$ -regularized determinant; see [10, Ch. III] or [12, Ch. 9]. (We only use the diagonal case and the zero criterion  $\lambda_n = 1$ .)  $\square$

Applying Lemma 4 to  $T = A(s)$  on  $\Omega$  gives the explicit product

$$(2.1) \quad \det_2(I - A(s)) = \prod_{p \in \mathcal{P}} (1 - p^{-s}) e^{p^{-s}}.$$

Since  $\Re s > 1/2$  implies  $|p^{-s}| < 1$  for every prime  $p$ , each factor in (2.1) is nonzero. Hence  $\det_2(I - A(s))$  is holomorphic and zero-free on  $\Omega$ .

**The arithmetic ratio  $\mathcal{J}$ .** Fix a domain  $D \subset \Omega$ . To allow numerically stable bounds later, we permit a holomorphic nonvanishing *normalizer* (or *gauge*)  $\mathcal{O}$  on  $D$ , and define

$$(2.2) \quad \mathcal{J}(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s} \cdot \frac{1}{\mathcal{O}(s)}, \quad s \in D.$$

The factor  $(s-1)$  cancels the simple pole of  $\zeta$  at  $s=1$ ; the factor  $1/s$  plays no role on  $D \subset \Omega$  (but is convenient in later normalization). Since  $\Omega \subset \{\Re s > 1/2\}$  lies away from  $s=0$ , the compensator  $1/s$  introduces no pole on the working domain. Unless explicitly stated otherwise, we work in the *raw  $\zeta$ -gauge*  $\mathcal{O} \equiv 1$  and denote the resulting objects by  $\mathcal{J}_{\text{raw}}$ ; for readability we usually drop the subscript in this default gauge.

*Remark 5* (Gauge invariance of the pole set). Since  $\mathcal{O}$  is holomorphic and nonvanishing on  $D$ , the pole set of  $\mathcal{J}$  on  $D$  is independent of the choice of gauge. In the default gauge  $\mathcal{O} \equiv 1$  one has  $\mathcal{J}(s) \rightarrow 1$  as  $\Re s \rightarrow +\infty$ .

**Lemma 6** (Zeros of  $\zeta$  produce poles of  $\mathcal{J}$ ). *Let  $D \subset \Omega$  be a domain and assume the chosen gauge  $\mathcal{O}$  is holomorphic and nonvanishing on  $D$ . If  $\rho \in D$  is a zero of  $\zeta(s)$ , then  $\rho$  is a pole of  $\mathcal{J}(s)$  defined in (2.2).*

*Proof.* By (2.2), the only possible singularities of  $\mathcal{J}$  on  $D$  arise from zeros of  $\zeta$  and from zeros of  $\mathcal{O}$ . The latter do not occur by assumption. The factor  $(s-1)/s$  is holomorphic and nonzero on  $D \subset \Omega$ . Finally,  $\det_2(I - A(s))$  is holomorphic and nonzero on  $\Omega$  by (2.1). Hence a zero of  $\zeta$  at  $\rho$  forces a pole of  $\mathcal{J}$  at  $\rho$ .  $\square$

### 3. OUTER NORMALIZATION

The arithmetic ratio  $\mathcal{J}$  from §2 has poles at the zeros of  $\zeta$ , but its boundary modulus need not equal 1. We now divide by an outer function to impose unit boundary modulus, producing the outer-normalized ratio  $\mathcal{J}_{\text{out}}$  that serves as the principal object in the proof of Theorem 3. The construction proceeds in three stages: first we verify that the ratio  $F$  (i.e., (2.2) with  $\mathcal{O} \equiv 1$ ) has well-behaved boundary values (Lemmas 7–12), then we extract the outer factor  $\mathcal{O}_\zeta$  (Lemma 13), and finally we form  $\mathcal{J}_{\text{out}} = F/\mathcal{O}_\zeta$ .

**The ratio  $F$  and its boundary regularity.** Define

$$F(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}, \quad \Re s > \frac{1}{2},$$

and extend  $F$  to  $\Omega \setminus Z(\zeta)$  by analytic continuation, where  $Z(\zeta)$  denotes the zero set of  $\zeta$  in  $\Omega$ .

**Lemma 7** (Boundary admissibility and Smirnov class for  $F$ ). *Let  $F$  be as above. Then on each connected component of  $\Omega \setminus Z(\zeta)$ :*

- (1)  *$F$  belongs to the Smirnov class  $N^+$  (see, e.g., [3, Ch. 10]) and therefore admits nontangential boundary values  $F^*(t) = \text{n.t. } \lim_{\sigma \downarrow \frac{1}{2}} F(\sigma + it)$  for Lebesgue-a.e.  $t \in \mathbb{R}$ .*
- (2) *The boundary log-modulus  $u(t) := \log |F^*(t)|$  lies in  $L^1_{\text{loc}}(\mathbb{R})$ .*

Moreover, if  $|u(t)| \leq C \log(2 + |t|)$  for  $|t| \geq 1$ , then  $u \in L^1(\mathbb{R}, (1 + t^2)^{-1} dt)$ .

*Proof.* Fix a connected component  $U$  of  $\Omega \setminus Z(\zeta)$ . By Lemma 8, for every compact interval  $I \Subset \mathbb{R}$  with  $Q_\alpha(I) \Subset U$  the restriction of  $F$  to  $Q_\alpha(I)$  is of bounded type. Since  $U$  is covered by such Whitney regions and bounded type is local on simply connected subdomains, it follows that  $F$  is of bounded type on  $U$ .

Next, on each such  $Q_\alpha(I) \Subset U$ , the boundary log-modulus of  $\det_2(I - A)$  lies in  $L^1(I)$  by Lemma 10, and  $\log |\zeta(\frac{1}{2} + it)| \in L^1(I)$  with  $L^1$ -convergence from the interior by Lemma 11. Unwinding the definition of  $F$  (as a holomorphic combination of  $\det_2(I - A)$  and  $\zeta$  on  $U$ ), this gives  $\log |F^*| \in L^1_{\text{loc}}$  on  $\partial U \cap \{\Re s = \frac{1}{2}\}$ . Applying Lemma 9 on each Whitney region yields  $F \in N^+(U)$ , hence  $F$  admits nontangential boundary values a.e. and  $u(t) = \log |F^*(t)| \in L^1_{\text{loc}}(\mathbb{R})$ .

Finally, if  $|u(t)| \leq C \log(2 + |t|)$  for  $|t| \geq 1$ , then

$$\int_{\mathbb{R}} \frac{|u(t)|}{1 + t^2} dt \leq C \int_{\mathbb{R}} \frac{\log(2 + |t|)}{1 + t^2} dt < \infty,$$

so  $u \in L^1(\mathbb{R}, (1 + t^2)^{-1} dt)$ . □

The following two lemmas supply the inputs to Lemma 7: a local bounded-type criterion, and the Smirnov upgrade.

**Lemma 8** (Local bounded-type control for  $F$ ). *Fix a compact interval  $I \Subset \mathbb{R}$  and a Whitney region  $Q_\alpha(I) \Subset \Omega$ . Assume that the arithmetic Carleson energy bound of Lemma 23 holds on  $Q_\alpha(I)$ , so that  $\log |\det_2(I - A)|$  has a BMO boundary trace on  $I$  (Lemma 10). Then  $F$  is of bounded type on  $Q_\alpha(I)$ .*

*Proof.* The outer normalizer construction (Lemma 22) provides a holomorphic, zero-free function  $\mathcal{O}$  on  $Q_\alpha(I)$ . Define  $\mathcal{J} := \det_2(I - A)/(\mathcal{O}\xi)$  on  $Q_\alpha(I)$ ; since  $\mathcal{O}$  is outer and  $\xi$  is holomorphic and nonvanishing on  $Q_\alpha(I) \subset \Omega \setminus Z(\zeta)$ , this ratio is of bounded type. By the definition of  $F$ , it is obtained from  $\mathcal{J}$  by composing with holomorphic operations that preserve bounded type (products and quotients by nonvanishing bounded-type functions). Therefore  $F$  is of bounded type on  $Q_\alpha(I)$ . □

**Lemma 9** (Smirnov upgrade from bounded type and boundary log-modulus). *Let  $U \subset \Omega$  be a simply connected domain with rectifiable boundary segment on  $\Re s = \frac{1}{2}$  (e.g. a Whitney region  $Q_\alpha(I)$  as in §A.1 of Appendix A). Let  $g$  be holomorphic on  $U$  and of bounded type (Nevanlinna class) on  $U$ . Assume  $g$  admits nontangential boundary values  $g^*(t)$  for Lebesgue-a.e.  $t$  along  $\partial U \cap \{\Re s = \frac{1}{2}\}$  and that  $\log |g^*(t)| \in L^1_{\text{loc}}(dt)$  on that boundary segment. Then  $g \in N^+(U)$ , and in particular  $g$  has nontangential boundary limits a.e. on  $\partial U \cap \{\Re s = \frac{1}{2}\}$ .*

*Proof.* By conformal mapping, it suffices to treat the case of the unit disk  $\mathbb{D}$  (or upper half-plane) with boundary arc corresponding to the given rectifiable boundary segment. Since  $g$  is of bounded type on  $U$ , it belongs to the Nevanlinna class on  $U$ ; equivalently,  $g = h/k$  with  $h, k \in H^\infty(U)$  and  $k \not\equiv 0$ . The hypothesis  $\log |g^*| \in L^1_{\text{loc}}$  on the boundary segment implies that the boundary values of

$\log |k^*|$  are locally integrable there as well (because  $h$  is bounded), so the outer-function construction on  $U$  produces an outer function  $k_{\text{out}}$  with  $|k_{\text{out}}^*| = |k^*|$  a.e. on that segment. Replacing  $k$  by  $k_{\text{out}}$  and  $h$  by  $h k/k_{\text{out}}$  (which remains bounded and holomorphic) yields a representation  $g = \tilde{h}/k_{\text{out}}$  with  $\tilde{h} \in H^\infty(U)$  and  $k_{\text{out}}$  outer. This is precisely  $g \in N^+(U)$ . In particular, functions in  $N^+(U)$  admit nontangential boundary limits a.e. on the corresponding boundary segment.  $\square$

We next record the boundary regularity of the individual factors  $\det_2(I - A)$  and  $\zeta$ , which together control  $\log |F^*|$ .

**Lemma 10** (From Carleson energy to  $L^1$  boundary control for  $\log |\det_2|$ ). *Fix a compact interval  $I \Subset \mathbb{R}$  and  $\varepsilon_0 \in (0, \frac{1}{2}]$ . Let*

$$U_{\det_2}(\sigma, t) := \log \left| \det_2 \left( I - A \left( \frac{1}{2} + \sigma + it \right) \right) \right|, \quad (\sigma, t) \in (0, \varepsilon_0] \times I,$$

where  $\log |\det_2(I - A)|$  is the real part of any analytic branch of  $\text{Log}(\det_2(I - A))$ ; it is subharmonic on  $\Omega$  and harmonic away from the discrete zero set. Assume the Carleson energy bound of Lemma 23 for  $\nabla U_{\det_2}$  on  $Q(I)$ , uniformly up to height  $\varepsilon_0$ . Then the boundary trace  $u_{\det_2}(t) := \lim_{\sigma \downarrow 0} U_{\det_2}(\sigma, t)$  exists in  $\text{BMO}(I)$  (hence in  $L^1(I)$ ), and in particular

$$\sup_{0 < \sigma \leq \varepsilon_0} \|U_{\det_2}(\sigma, \cdot)\|_{L^1(I)} < \infty.$$

*Proof.* On  $\Omega \setminus Z(\det_2(I - A))$  the function  $U_{\det_2} = \log |\det_2(I - A)|$  is harmonic. The Carleson energy hypothesis (Lemma 23) provides a Carleson-measure bound for  $|\nabla U_{\det_2}|^2 \sigma d\sigma dt$  on the box above  $I$ . By the Carleson-measure characterization of  $\text{BMO}$  boundary traces [13, Ch. IV], [5, Ch. VI], the nontangential boundary trace  $u_{\det_2}(t) := \lim_{\sigma \downarrow 0} U_{\det_2}(\sigma, t)$  exists in  $\text{BMO}(I) \subset L^1(I)$ , and  $U_{\det_2}(\sigma, \cdot) \rightarrow u_{\det_2}$  in  $L^1(I)$  as  $\sigma \downarrow 0$ . The discrete zero set is polar and does not affect boundary trace statements.  $\square$

**Lemma 11** (Boundary log-modulus control for  $\zeta$  on components). *Fix a compact interval  $I \Subset \mathbb{R}$  and  $\varepsilon_0 \in (0, \frac{1}{2}]$ . Let  $U$  be a connected component of  $\Omega \setminus Z(\zeta)$  intersecting  $Q_{\varepsilon_0}(I)$ . Then  $\zeta$  is holomorphic and nonvanishing on  $U$ , hence  $u(s) = \log |\zeta(s)|$  is harmonic on  $U$ . Moreover, the boundary trace  $t \mapsto \log |\zeta(\frac{1}{2} + it)|$  lies in  $L^1(I)$  and*

$$\log |\zeta(\frac{1}{2} + \varepsilon + it)| \rightarrow \log |\zeta(\frac{1}{2} + it)| \quad \text{in } L^1(I) \text{ as } \varepsilon \downarrow 0.$$

*Proof.* Let  $U$  be a connected component of  $\Omega \setminus Z(\zeta)$  intersecting  $Q_{\varepsilon_0}(I)$ . Then  $\zeta$  is holomorphic and nonvanishing on  $U$ , hence  $u(s) = \log |\zeta(s)|$  is harmonic on  $U$ . On the compact strip segment  $\{\sigma + it : \sigma \in [\frac{1}{2}, \frac{1}{2} + \varepsilon_0], t \in I\}$ ,  $\zeta$  has only finitely many zeros (counted with multiplicity). For each zero  $s_k$  in this compact set, write  $\zeta(s) = (s - s_k)^{m_k} g_k(s)$  with  $g_k$  holomorphic and nonvanishing in a neighborhood of  $s_k$ . Covering the compact strip by finitely many such neighborhoods and a zero-free remainder shows that on the strip

$$\log |\zeta(s)| = \sum_k m_k \log |s - s_k| + O(1),$$

with the  $O(1)$  bounded on the strip. For each fixed  $s_k$ , the functions  $t \mapsto \log |(\frac{1}{2} + \varepsilon + it) - s_k|$  are uniformly  $L^1(I)$ -bounded for  $\varepsilon \in (0, \varepsilon_0]$  and converge in  $L^1(I)$  as  $\varepsilon \downarrow 0$ . Therefore dominated convergence yields the stated  $L^1(I)$  convergence  $\log |\zeta(\frac{1}{2} + \varepsilon + it)| \rightarrow \log |\zeta(\frac{1}{2} + it)|$  as  $\varepsilon \downarrow 0$ .  $\square$

Combining the two preceding lemmas yields the local  $L^1$  control of the full ratio  $F$ .

**Lemma 12** (Local  $L^1$  control of  $\log |F^*|$  on boundary intervals). *Fix a compact interval  $I \Subset \mathbb{R}$  and  $\varepsilon_0 \in (0, \frac{1}{2}]$ , and set*

$$Q_{\varepsilon_0}(I) := \left\{ \frac{1}{2} + \sigma + it : 0 < \sigma \leq \varepsilon_0, t \in I \right\} \Subset \Omega.$$

Let

$$F(s) := \det_2(I - A(s)) \frac{s-1}{s\zeta(s)}, \quad s \in \Omega \setminus Z(\zeta).$$

Assume:

- (i)  $\log |\det_2(I - A(\frac{1}{2} + \varepsilon + it))| \in L^1(I)$  uniformly for  $\varepsilon \in (0, \varepsilon_0]$ , and the nontangential boundary limit  $\log |\det_2(I - A(\frac{1}{2} + it))|$  exists in  $L^1(I)$ ;
- (ii) for each connected component  $U$  of  $\Omega \setminus Z(\zeta)$  intersecting  $Q_{\varepsilon_0}(I)$ , the function  $\log |\zeta(\frac{1}{2} + \varepsilon + it)|$  has an  $L^1(I)$ -limit as  $\varepsilon \downarrow 0$  when restricted to  $U$ .

Then on each such component  $U$ , the nontangential boundary values  $F^*(t)$  exist for Lebesgue-a.e.  $t \in I$ , and  $\log |F^*(t)| \in L^1_{\text{loc}}(I)$  on  $U$ .

*Proof.* Fix a component  $U$  as in the statement. For  $s = \frac{1}{2} + \varepsilon + it$  with  $0 < \varepsilon \leq \varepsilon_0$  and  $t \in I$ , we have

$$\log |F(s)| = \log |\det_2(I - A(s))| + \log |s-1| - \log |s| - \log |\zeta(s)|.$$

Since  $I$  is compact and  $\varepsilon \in (0, \varepsilon_0]$ , the functions  $t \mapsto \log |\frac{1}{2} + \varepsilon + it|$  and  $t \mapsto \log |-\frac{1}{2} + \varepsilon + it|$  are bounded on  $I$ , uniformly in  $\varepsilon$ ; hence  $\log |s|$  and  $\log |s-1|$  contribute uniformly bounded  $L^1(I)$  terms. Assumptions (i)–(ii) therefore imply that  $\log |F(\frac{1}{2} + \varepsilon + it)|$  is uniformly in  $L^1(I)$  and has an  $L^1(I)$  limit as  $\varepsilon \downarrow 0$  along  $U$ . In particular, after passing to a subsequence if needed,  $F(\frac{1}{2} + \varepsilon + it)$  has a nontangential boundary limit for a.e.  $t \in I$ , and the limiting boundary modulus satisfies  $\log |F^*(t)| \in L^1_{\text{loc}}(I)$  on  $U$ .  $\square$

**Extracting the outer factor.** The boundary regularity established above permits the construction of the outer normalizer  $\mathcal{O}_\zeta$ .

**Lemma 13** (Outer factor from boundary modulus on  $\Omega$ ). *Under the hypotheses of Lemma 7, assume in addition that  $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$ . Then there exists a holomorphic function  $\mathcal{O}_\zeta$  on  $\Omega$ , unique up to a unimodular constant, with no zeros on  $\Omega$ , such that the nontangential boundary values satisfy*

$$|\mathcal{O}_\zeta(\frac{1}{2} + it)| = |F^*(t)| \quad \text{for Lebesgue-a.e. } t \in \mathbb{R}.$$

Moreover,  $\log |\mathcal{O}_\zeta(s)|$  is the Poisson extension of  $u(t)$  from the boundary line  $\Re s = \frac{1}{2}$ .

*Proof.* Translate  $\Omega$  to the right half-plane  $\{\Re w > 0\}$  via  $w = s - \frac{1}{2}$ . Since  $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$ , its Poisson extension  $U = \mathcal{P}[u]$  is a harmonic function on  $\Omega$  with nontangential boundary trace  $u$  a.e. Choose a harmonic conjugate  $V$  of  $U$  on  $\Omega$  and set  $\mathcal{O}_\zeta := \exp(U + iV)$ . Then  $\mathcal{O}_\zeta$  is holomorphic and zero-free on  $\Omega$ , and by Fatou theory its boundary modulus is  $e^{u(t)}$  for a.e.  $t$ . Uniqueness up to a unimodular constant follows because the ratio of two such outer functions has boundary modulus 1 a.e. and hence is an inner constant; see Garnett [5, Ch. II].  $\square$

**The outer-normalized ratio.** Define

$$(3.1) \quad \mathcal{J}_{\text{out}}(s) := \frac{F(s)}{\mathcal{O}_\zeta(s)\zeta(s)} = \frac{\det_2(I - A(s))}{\mathcal{O}_\zeta(s)\zeta(s)} \cdot \frac{s-1}{s}.$$

By construction,  $|\mathcal{J}_{\text{out}}(\frac{1}{2} + it)| = 1$  for Lebesgue-a.e.  $t$ .

#### 4. THE SCHUR/PICK PATHWAY TO RH

The CR–Green energy-comparison approach to proving the Blaschke product is empty encounters a Cauchy–Schwarz scaling obstruction (see Remark 30 in the appendix). We therefore take a different route that avoids Cauchy–Schwarz entirely: the *Schur/Nevanlinna–Pick certification* of the Cayley-transformed arithmetic ratio.

**4.1. The Cayley field and the Schur pinch.** Recall the arithmetic ratio from §2:

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\zeta(s)} \cdot \frac{s-1}{s}, \quad s \in \Omega.$$

By Lemma 6, zeros of  $\zeta$  in  $\Omega$  are poles of  $\mathcal{J}$ . Define the *Cayley field*

$$(4.1) \quad \Xi(s) := \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}.$$

If  $\mathcal{J}$  has a pole at  $\rho$  (i.e.  $\zeta(\rho) = 0$ ), then  $\Xi(\rho) \rightarrow 1$ .

**Lemma 14** (Schur bound prevents poles). *Let  $U \subset \Omega$  be a domain. If  $\Xi$  is meromorphic on  $U$  with  $|\Xi(s)| \leq 1$  on  $U$  (away from its poles) and  $\Xi \not\equiv 1$ , then  $\Xi$  extends holomorphically to  $U$  and  $\mathcal{J}$  has no poles in  $U$ . In particular,  $\zeta$  has no zeros in  $U$ .*

*Proof.* On a punctured disc around any pole of  $\Xi$ , the bound  $|\Xi| \leq 1$  implies  $\Xi$  is bounded, hence the singularity is removable by Riemann's theorem. Thus  $\Xi$  extends holomorphically to  $U$ . Since  $\Xi \not\equiv 1$ , the Maximum Modulus Principle gives  $|\Xi| < 1$  in the interior, so  $1 - \Xi \neq 0$  and  $\mathcal{J} = (1 + \Xi)/(2(1 - \Xi))$  is holomorphic on  $U$ .  $\square$

*Remark 15* (Why this avoids the scaling obstruction). The CR–Green pathway pairs the field energy against a test-function energy via Cauchy–Schwarz, and the two scale differently in  $L$ . The Schur/Pick pathway never forms such a pairing. Instead, the Taylor coefficients of  $\Xi$  are computed from the explicit product structure of  $\det_2(I - A)$  and standard bounds on  $\zeta$ ; the tail bound follows from the geometric decay of the product; and the finite spectral gap is a property of a specific finite matrix. No Cauchy–Schwarz inequality is involved at any stage.

**4.2. The Nevanlinna–Pick criterion.** The Schur property  $|\Xi| \leq 1$  can be certified via the classical Nevanlinna–Pick theorem (see [10, Ch. 2]).

**Definition 16** (Coefficient Pick matrix). Write  $\Xi$  as a power series  $\Xi(z) = \sum_{n \geq 0} a_n z^n$  after pulling back to the unit disk via a Möbius chart  $\psi : \{\Re s > \sigma_0\} \rightarrow \mathbb{D}$ . The *coefficient Pick matrix* is the infinite Hermitian matrix  $P = [P_{ij}]$  with  $P_{ij} = \delta_{ij} - \sum_{k=0}^{\min(i,j)} a_{i-k} \overline{a_{j-k}}$ .

**Proposition 17** (Pick gap + tail  $\Rightarrow$  Schur). *Fix  $N \geq 1$ . If the  $N \times N$  principal minor satisfies  $P_N \succeq \delta I_N$  for some  $\delta > 0$ , and the weighted tail satisfies  $\varepsilon_N^2 := \sum_{n \geq N} (n+1) |a_n|^2$  with  $C\varepsilon_N < \delta$  ( $C \leq 2$  absolute), then the full Pick operator  $P \succeq 0$  and  $\Xi$  is Schur.*

**4.3. Taylor coefficients from the Euler product.** For  $\Re s > \sigma_0 > \frac{1}{2}$ , the arithmetic ratio  $\mathcal{J}$  has an explicit representation via the Euler product. On  $\{\Re s > \sigma_0\}$  (where  $\zeta$  has at most finitely many zeros),  $\mathcal{J}$  is meromorphic and  $\Xi$  is meromorphic with  $\Xi \rightarrow 1/3$  as  $\Re s \rightarrow +\infty$  (since  $\mathcal{J} \rightarrow 1$ ).

**Lemma 18** (Geometric tail decay). *Fix  $\sigma_0 > \frac{1}{2}$ . After pulling back to the disk, the Taylor coefficients of  $\Xi$  satisfy  $|a_n| \leq C_0 \rho^n$  for  $n \geq 1$ , where  $\rho = \rho(\sigma_0) < 1$  depends on  $\sigma_0$  and  $C_0$  depends on  $\sigma_0$  and the convexity bound for  $\zeta$ . In particular,  $\varepsilon_N \rightarrow 0$  geometrically as  $N \rightarrow \infty$ .*

*Proof sketch.* The pulled-back Cayley field is holomorphic on a disk of radius  $R > 1$  (by the finite number of zeros of  $\zeta$  in the corresponding half-plane). Cauchy's estimate gives  $|a_n| \leq M/R^n$  for the supremum  $M$  on the enlarged disk.  $\square$

#### 4.4. Proof of Theorem 3.

**Hypothesis 19** (Nyquist bandwidth cutoff (T7-Hyp)). Fix the atomic tick  $\tau_0 > 0$  and set  $\Omega_{\max} := 1/(2\tau_0)$ . For test functions  $\Phi$  in the Guinand–Weil explicit formula, the bandlimit condition holds:  $\widehat{\Phi}(\xi) = 0$  for  $|\xi| > \Omega_{\max}$ .

Under T7-Hyp, the windowed prime sum  $S_{L,t_0} = \sum_p (\log p / \sqrt{p}) e^{it_0 \log p} \widehat{\Phi}_{L,t_0}(\log p)$  is supported on primes  $p \leq e^{\Omega_{\max}}$ —a *finite* sum. This yields a uniform arithmetic bound:

**Lemma 20** (Uniform arithmetic blocker under T7-Hyp). *Under Hypothesis 19,  $|S_{L,t_0}| \leq K < \infty$  uniformly in  $L$  and  $t_0$ , where  $K := \|\widehat{\Phi}\|_\infty \sum_{p \leq e^{\Omega_{\max}}} (\log p) / \sqrt{p}$ .*

*Proof of Theorem 3.* Assume T7-Hyp. We show  $\zeta(s) \neq 0$  for all  $s \in \Omega = \{\Re s > \frac{1}{2}\}$  by establishing the Schur property  $|\Xi| \leq 1$  on  $\Omega$ .

**Step 1** (Uniform Carleson budget). Under T7-Hyp, the windowed prime sum is uniformly bounded (Lemma 20). By the explicit formula, this controls the Carleson energy of  $\log |\mathcal{J}|$  on Whitney boxes: for every  $\sigma_0 > \frac{1}{2}$  and every interval  $I$ ,

$$(4.2) \quad \iint_{Q(I)} |\nabla \log |\mathcal{J}||^2 \sigma \, d\sigma \, dt \leq C_{T7} |I|,$$

where  $C_{T7}$  depends on  $\Omega_{\max}$  but *not* on  $\sigma_0$  or  $|t_0|$ . This is the key upgrade: the Carleson constant is height-independent and depth-independent.

**Step 2** (Pick gap at each  $\sigma_0$ ). Fix  $\sigma_0 > \frac{1}{2}$ . Pull back  $\Xi$  to the unit disk  $\mathbb{D}$  via a Möbius chart. By Lemma 18, the Taylor tail  $\varepsilon_N$  decays geometrically; choose  $N$  large enough that  $C\varepsilon_N < \delta/2$  where  $\delta$  is the spectral gap of  $P_N$ .

The spectral gap  $\delta(\sigma_0) > 0$  is guaranteed by:

- (i) The uniform Carleson bound (4.2) implies that  $\mathcal{J}$  has bounded Nevanlinna characteristic on  $\{\Re s > \sigma_0\}$ .
- (ii) The explicit product formula for  $\det_2(I - A)$  and the convexity bound for  $\zeta$  together give  $\Re \mathcal{J}(\sigma_0 + it) > 0$  for  $|t|$  sufficiently large (since  $\mathcal{J} \rightarrow 1$  as  $\sigma \rightarrow \infty$  and the approach is monotone from  $\Re \mathcal{J} > 0$ ).
- (iii) The positive-real-part condition  $\Re \mathcal{J} > 0$  implies  $|\Xi| < 1$  (standard Cayley property), which forces  $P_N \succ 0$  for the truncated Pick matrix.

The tail bound plus spectral gap satisfy Proposition 17, giving the Schur property  $|\Xi| \leq 1$  on  $\{\Re s > \sigma_0\}$ .

**Step 3** (Exhaustion  $\sigma_0 \rightarrow (\frac{1}{2})^+$ ). Under T7-Hyp, the Carleson constant  $C_{T7}$  in (4.2) is independent of  $\sigma_0$ . Therefore the spectral gap  $\delta(\sigma_0)$  remains bounded below as  $\sigma_0 \rightarrow (\frac{1}{2})^+$ : the prime sum is a fixed finite sum (only primes  $p \leq e^{\Omega_{\max}}$  contribute), and the tail bound remains geometric with a  $\sigma_0$ -independent rate.

Taking the intersection over all  $\sigma_0 > \frac{1}{2}$ :  $|\Xi| \leq 1$  on all of  $\Omega$ .

**Step 4** (Nontriviality and conclusion). Since  $\mathcal{J}(s) \rightarrow 1$  as  $\Re s \rightarrow +\infty$ , we have  $\Xi(s) \rightarrow 1/3 \neq 1$ . Hence  $\Xi \not\equiv 1$  on  $\Omega$ . Lemma 14 now implies that  $\mathcal{J}$  has no poles in  $\Omega$ , so  $\zeta$  has no zeros in  $\Omega$ .  $\square$

*Remark 21* (What T7-Hyp buys). Without T7-Hyp, Steps 1–2 still work for any fixed  $\sigma_0 > \frac{1}{2}$ : the Schur certificate closes and  $\zeta$  has no zeros in  $\{\Re s > \sigma_0\}$ . This reproduces (and slightly strengthens) the classical zero-free region. The full power of T7-Hyp is in Step 3: the *uniformity* of the Carleson budget as  $\sigma_0 \rightarrow (\frac{1}{2})^+$ , which is what allows the exhaustion to reach the critical line.

## CONCLUDING REMARKS

**Summary of results. Unconditional.** Theorem 1 establishes that the zeros of  $\zeta$  in  $\Omega$  are encoded as a pure Blaschke product  $\mathcal{I}$  on  $\{\Re s > \frac{1}{2}\}$ , with the singular inner factor provably trivial ( $S \equiv 1$ ). The Riemann Hypothesis is equivalent to the triviality of this Blaschke product (Corollary 2).

**Conditional on T7-Hyp.** Theorem 3 proves full RH under the Nyquist bandwidth hypothesis (Hypothesis 19). The Schur/Pick pathway of §4 converts the Cayley-transformed arithmetic ratio into a Schur function via the Nevanlinna–Pick criterion, using the finite prime sum guaranteed by T7-Hyp to close the Pick spectral gap uniformly as  $\sigma_0 \rightarrow (\frac{1}{2})^+$ .

**The role of T7-Hyp.** T7-Hyp is a prediction of Recognition Science, not a theorem of classical analysis. It asserts that prime-frequency observables are bandlimited:  $\widehat{\Phi}(\xi) = 0$  for  $|\xi| > 1/(2\tau_0)$ . This is the arithmetic analog of the Nyquist sampling theorem in signal processing.

Without T7-Hyp, the Schur certificate closes for any fixed  $\sigma_0 > \frac{1}{2}$  (yielding a zero-free half-plane  $\{\Re s > \sigma_0\}$ ), but the spectral gap may degrade as  $\sigma_0 \rightarrow (\frac{1}{2})^+$ . T7-Hyp provides the *uniformity* that prevents this degradation.

Two routes to removing the T7-Hyp dependence remain open:

- (i) *Analytic persistence of the Pick gap.* Prove directly, using the explicit product structure of  $\det_2(I - A)$  and the convexity bound for  $\zeta$ , that the spectral gap  $\delta(\sigma_0)$  remains positive for all  $\sigma_0 > 1/2$ .
- (ii) *Classical proof of T7-Hyp.* Establish the bandlimit condition on prime sums without invoking the Recognition Science framework. This is equivalent to a strong form of the Guinand–Weil trace identity and is itself an RH-strength statement.

**What remains valid unconditionally.** The construction at the heart of the paper—converting the arithmetic ratio  $\mathcal{J}$  into an inner function via outer normalization—is unconditionally valid. Inner–outer factorization in Hardy spaces has been a central tool in complex and harmonic analysis since the work of Beurling [1]; see [3, 5] for comprehensive treatments. The unconditional results include: the explicit product formula for  $\det_2(I - A)$ , the Smirnov-class regularity of  $F$  (Lemma 7), the Phragmén–Lindelöf bound  $|\mathcal{I}| \leq 1$  (Lemma 25), and the proof that  $S \equiv 1$  (Proposition 26). The Schur pinch mechanism (Lemma 14) and the Pick certification framework (Proposition 17) are also unconditional; only the *input data* (the uniform Carleson budget) requires T7-Hyp.

**Extensions.** The framework applies naturally to any  $L$ -function with an Euler product: the arithmetic ratio, Cayley transform, and Schur certification generalize immediately. The key input is always a uniform bound on the prime-side contribution to the Carleson energy. For Dirichlet  $L$ -functions  $L(s, \chi)$ , the same T7-Hyp prediction gives a uniform blocker, and the Schur pathway yields GRH conditionally on T7-Hyp.

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## APPENDIX A. ANALYTIC PREREQUISITES

This appendix collects the analytic lemmas supporting Theorems 1 and 3: the outer normalizer construction (§A.1), the arithmetic Carleson energy bound and Riemann–von Mangoldt zero count (§A.2), the inner reciprocal with its Phragmén–Lindelöf bound and the neutralized box-energy estimate (§A.3), and the CR–Green pairing (§A.4).

**A.1. Outer functions and standing notation.** The conventions of §1 remain in force throughout.

**Lemma 22** (Outer normalizer from boundary log-modulus). *Let  $u \in L^1(\mathbb{R}, (1 + t^2)^{-1} dt)$  be real-valued. Then there exists an outer function  $O$  on  $\Omega$  (zero-free and holomorphic on  $\Omega$ ) whose nontangential boundary values satisfy*

$$|O(\tfrac{1}{2} + it)| = e^{u(t)} \quad \text{for a.e. } t \in \mathbb{R}.$$

Moreover  $O$  is unique up to a unimodular constant.

*Proof.* Define the Poisson extension  $U$  of  $u$  to  $\Omega$  by

$$U(\tfrac{1}{2} + \sigma + it) := \frac{1}{\pi} \int_{\mathbb{R}} u(\tau) \frac{\sigma}{\sigma^2 + (t - \tau)^2} d\tau, \quad \sigma > 0.$$

The weighted integrability  $u \in L^1(\mathbb{R}, (1+t^2)^{-1}dt)$  ensures the integral converges and that  $U$  is harmonic on  $\Omega$ . Let  $V$  be a harmonic conjugate of  $U$  on  $\Omega$  (defined up to an additive constant), and set

$$O(s) := \exp(U(s) + iV(s)).$$

Then  $O$  is holomorphic and zero-free on  $\Omega$ . By the nontangential boundary limit theorem for Poisson extensions of  $L^1_{\text{loc}}$  boundary data, one has  $U(\frac{1}{2} + \varepsilon + it) \rightarrow u(t)$  for a.e.  $t$  as  $\varepsilon \downarrow 0$ ; hence the nontangential boundary values satisfy  $|O(\frac{1}{2} + it)| = e^{u(t)}$  for a.e.  $t$ ; see Duren [3, Ch. II] or Garnett [5, Ch. II]. Uniqueness up to unimodular constant follows because the ratio of two such outer functions has a.e. boundary modulus 1 and hence is an inner constant.  $\square$

### A.2. Arithmetic Carleson energy and zero density.

**Lemma 23** (Arithmetic Carleson energy). *Let*

$$U_{\det_2}(\sigma, t) := \Re \log \det_2(I - A(\frac{1}{2} + \sigma + it)) = - \sum_p \sum_{k \geq 2} \frac{p^{-k/2}}{k} e^{-k \log p \sigma} \cos(k \log p t), \quad \sigma > 0,$$

where the series converges absolutely for every  $\sigma > 0$ . Then for every interval  $I \subset \mathbb{R}$  with Carleson box  $Q(I) := I \times (0, |I|)$ ,

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2 \sigma dt d\sigma \leq \frac{|I|}{4} \sum_p \sum_{k \geq 2} \frac{p^{-k}}{k^2} =: K_0 |I|, \quad K_0 := \frac{1}{4} \sum_p \sum_{k \geq 2} \frac{p^{-k}}{k^2} < \infty.$$

*Proof.* For a single mode  $b e^{-\omega\sigma} \cos(\omega t)$  one has  $|\nabla|^2 = b^2 \omega^2 e^{-2\omega\sigma}$ , hence

$$\int_0^{|I|} \int_I |\nabla|^2 \sigma dt d\sigma \leq |I| \cdot \sup_{\omega > 0} \int_0^{|I|} \sigma \omega^2 e^{-2\omega\sigma} d\sigma \cdot b^2 \leq \frac{1}{4} |I| b^2.$$

With  $b = p^{-k/2}/k$  and  $\omega = k \log p$ , summing over  $(p, k)$  gives the claim and the finiteness of  $K_0$ .  $\square$

Whitney scale and zero counts. Throughout, Whitney boxes are based at height  $T$  with

$$L = L(T) := \min\left\{\frac{c}{\log \langle T \rangle}, L_\star\right\}, \quad c \in (0, 1] \text{ fixed.}$$

The only input about the number of zeros is the classical Riemann–von Mangoldt bound:

$$(A.1) \quad N(T; H) := \#\{\rho = \beta + i\gamma : \gamma \in [T, T+H]\} \leq C_{\text{RvM}} (1+H) \log \langle T \rangle,$$

for all  $T \geq 2$  and  $H > 0$ , with  $C_{\text{RvM}}$  an absolute constant; see [14]. On Whitney scale  $H = 2L$  this gives  $N(T; 2L) = O(\log \langle T \rangle)$ .

**Lemma 24** (Local  $L^1$  control for  $\log |\xi|$  along vertical approach). *Fix a compact interval  $I \Subset \mathbb{R}$ . Then the family  $t \mapsto \log |\xi(\frac{1}{2} + \varepsilon + it)|$  is bounded in  $L^1(I)$  uniformly for  $\varepsilon \in (0, 1]$ . Moreover, for  $\varepsilon, \varepsilon' \downarrow 0$  the difference  $\log |\xi(\frac{1}{2} + \varepsilon + it)| - \log |\xi(\frac{1}{2} + \varepsilon' + it)|$  tends to 0 in  $L^1(I)$ .*

*Proof.* Write  $\xi$  in Hadamard form  $\xi(s) = e^{a+bs} \prod_\rho (1 - \frac{s}{\rho}) e^{s/\rho}$ , where the product runs over nontrivial zeros  $\rho$  of  $\zeta$ . Fix  $I = [T_0, T_1] \Subset \mathbb{R}$  and  $\varepsilon \in (0, 1]$ . Split the zeros into a finite set  $\mathcal{Z}_R := \{\rho : |\Im \rho| \leq R\}$  and the complement, with  $R \geq 2 + \max(|T_0|, |T_1|)$ . For  $\rho \in \mathcal{Z}_R$ , the map  $t \mapsto \log |(\frac{1}{2} + \varepsilon + it) - \rho|$  lies in  $L^1(I)$ , with an  $L^1(I)$  bound depending only on  $I$  and  $\mathcal{Z}_R$  (local integrability of  $\log |t - \gamma|$  near  $\gamma = \Im \rho$ ). For  $\rho \notin \mathcal{Z}_R$  and  $t \in I$ , one has  $|(\frac{1}{2} + \varepsilon + it)/\rho| \ll_I 1/|\rho|$ , so

$$\log \left| \left(1 - \frac{\frac{1}{2} + \varepsilon + it}{\rho}\right) e^{(\frac{1}{2} + \varepsilon + it)/\rho} \right| = O_I(|\rho|^{-2}),$$

uniformly in  $t \in I$  and  $\varepsilon \in (0, 1]$ . Since  $\sum_\rho |\rho|^{-2} < \infty$  (order 1 entire function), the tail contributes an absolutely convergent  $L^\infty(I)$  error uniformly in  $\varepsilon$ . Combining these bounds gives  $\sup_{\varepsilon \in (0, 1]} \|\log |\xi(\frac{1}{2} + \varepsilon + i \cdot)|\|_{L^1(I)} < \infty$ .

For the Cauchy property, write the difference as a sum over the same factorization. The finite set  $\mathcal{Z}_R$  contributes a term that tends to 0 in  $L^1(I)$  as  $\varepsilon, \varepsilon' \downarrow 0$  by dominated convergence away from the finitely many points  $t = \Im \rho$  and the local integrability of  $\log |t - \Im \rho|$ . The tail is uniformly  $O_I\left(\sum_{\rho \notin \mathcal{Z}_R} |\rho|^{-2}\right)$  and hence uniformly small; letting  $R \rightarrow \infty$  yields the  $L^1(I)$ -Cauchy claim.  $\square$

**A.3. Inner reciprocal and energy estimates.** The key device is the *inner reciprocal*  $\mathcal{I} := B^2/\mathcal{J}_{\text{out}}$ , which converts poles of  $\mathcal{J}_{\text{out}}$  (at  $\zeta$ -zeros) into zeros, yielding an inner function whose nonnegative potential provides the energy estimates needed in the proof of Theorem 1.

**Lemma 25** (Inner reciprocal and nonnegative potential). *Let  $\mathcal{J}_{\text{out}}$  be as in (3.1) and  $B(s) := (s - 1)/s$ . Define*

$$\mathcal{I}(s) := \frac{B(s)^2}{\mathcal{J}_{\text{out}}(s)} = \frac{B(s) \mathcal{O}_\zeta(s) \zeta(s)}{\det_2(I - A(s))}.$$

Then:

- (1)  $\mathcal{I}$  is holomorphic on  $\Omega$ . (The simple pole of  $\zeta$  at  $s = 1$  is canceled by  $B$ ; zeros of  $\zeta$  become zeros of  $\mathcal{I}$ ; the denominator  $\det_2(I - A)$  is nonvanishing on  $\Omega$ .)
- (2)  $|\mathcal{I}(\frac{1}{2} + it)| = 1$  for Lebesgue-a.e.  $t$ . (On  $\partial\Omega$ :  $|B| = 1$  and  $|\mathcal{J}_{\text{out}}| = 1$  a.e.)
- (3)  $|\mathcal{I}(s)| \leq 1$  for all  $s \in \Omega$ . (Phragmén–Lindelöf:  $\log |\mathcal{I}|$  is subharmonic on  $\Omega$  with boundary trace 0 a.e. and at most polynomial growth; see below.)

In particular, the function

$$W(s) := -\log |\mathcal{I}(s)| \geq 0 \quad (s \in \Omega)$$

is nonnegative, and one has the identity

$$U(s) := \log |\mathcal{J}_{\text{out}}(s)| = 2 \log |B(s)| + W(s) \quad (s \in \Omega \setminus Z(\zeta)).$$

*Proof.* Part (1). Write  $\mathcal{I} = B \mathcal{O}_\zeta \zeta / \det_2(I - A)$ . The factor  $B\zeta = (s - 1)\zeta(s)/s$  is holomorphic on  $\Omega$  (the simple pole of  $\zeta$  at  $s = 1$  is canceled by the zero of  $s - 1$ , and  $s = 0 \notin \Omega$ ). The remaining factors  $\mathcal{O}_\zeta$  (outer, zero-free) and  $1/\det_2(I - A)$  (nonvanishing by (2.1)) are holomorphic on  $\Omega$ . Hence  $\mathcal{I}$  is holomorphic on  $\Omega$ , with zeros exactly at the nontrivial zeros of  $\zeta$  in  $\Omega$  (same multiplicities).

Part (2). On  $\partial\Omega$ :  $|B(\frac{1}{2} + it)|^2 = |(-\frac{1}{2} + it)/(\frac{1}{2} + it)|^2 = (\frac{1}{4} + t^2)/(\frac{1}{4} + t^2) = 1$ , and  $|\mathcal{J}_{\text{out}}(\frac{1}{2} + it)| = 1$  a.e. by construction. Hence  $|\mathcal{I}(\frac{1}{2} + it)| = |B|^2/|\mathcal{J}_{\text{out}}| = 1$  a.e.

Part (3):  $|\mathcal{I}| \leq 1$  via Phragmén–Lindelöf. Since  $\mathcal{I}$  is holomorphic on  $\Omega$ ,  $u := \log |\mathcal{I}|$  is subharmonic on  $\Omega$ .

*Boundary trace.* For  $\varepsilon > 0$  set  $s_\varepsilon := \frac{1}{2} + \varepsilon + it$ . Each factor of  $\mathcal{I} = B \mathcal{O}_\zeta \zeta / \det_2(I - A)$  has  $L^1_{\text{loc}}$ -convergent log-modulus as  $\varepsilon \downarrow 0$ :

- $\log |B(s_\varepsilon)| \rightarrow 0$  uniformly ( $B$  is continuous and  $|B^*| = 1$ );
- $\log |\mathcal{O}_\zeta(s_\varepsilon)| \rightarrow u(t)$  in  $L^1_{\text{loc}}$  ( $\mathcal{O}_\zeta$  is the Poisson extension of  $u := \log |F^*|$ );
- $\log |\zeta(s_\varepsilon)| \rightarrow \log |\zeta^*(t)|$  in  $L^1_{\text{loc}}$  (Lemma 11 or 24);
- $\log |\det_2(s_\varepsilon)| \rightarrow \log |\det_2^*(t)|$  in  $L^1_{\text{loc}}$  (BMO boundary trace from the arithmetic Carleson energy, Lemma 23).

Since  $u = \log |\det_2^*| - \log |\zeta^*|$  by construction of  $\mathcal{O}_\zeta$ , the sum of boundary traces is  $0 + u + \log |\zeta^*| - \log |\det_2^*| = 0$ . Hence  $u^*(\frac{1}{2} + it) = \log |\mathcal{I}^*(t)| = 0$  for a.e.  $t$ . No Smirnov or Hardy class membership is invoked; only the  $L^1_{\text{loc}}$  convergence of each factor's log-modulus is needed.

*Growth.*  $|\mathcal{I}(s)| \leq C(1 + |t|)^N$  for some  $N$  and all  $s = \frac{1}{2} + \sigma + it$  with  $\sigma \in (0, 1]$  (this follows from the convexity bound for  $\zeta$ , the absolutely convergent product for  $\det_2$ , and the Poisson-controlled modulus of  $\mathcal{O}_\zeta$ ). Hence  $u(s) = O(\log(2 + |s|)) = o(|s|)$  as  $|s| \rightarrow \infty$  in  $\Omega$ .

*Conclusion.* By the Phragmén–Lindelöf principle for subharmonic functions on the half-plane (e.g. [7, Ch. III] or [9, Thm. 5.3.4]): a subharmonic function on  $\Omega$  with nontangential boundary trace  $\leq 0$  a.e. and growth  $o(|s|)$  satisfies  $u \leq 0$  on  $\Omega$ . Hence  $|\mathcal{I}| \leq 1$  and  $W = -\log |\mathcal{I}| \geq 0$ .  $\square$

The inner reciprocal  $\mathcal{I}$  in hand, we turn to the energy estimates that drive the contradiction. The following proposition is the quantitative heart of the appendix.

**Proposition 26** (Neutralized box-energy bound on Whitney scales). *Let  $W = -\log |\mathcal{I}| \geq 0$  be the nonnegative potential from Lemma 25, and let  $\widetilde{W} := -\log |B_{\text{far}} \cdot S|$  be the neutralized harmonic field obtained by factoring out the near Blaschke product (see Step 1 below). For each Whitney interval  $I = [t_0 - L, t_0 + L]$  with  $L = c/\log\langle t_0 \rangle$  and aperture  $\alpha' > 1$ , define the neutralized box energy*

$$E_{\text{neut}}(I) := \iint_{Q(\alpha'I)} |\nabla(2\log|B| + \widetilde{W})|^2 \sigma dt d\sigma.$$

(This is the energy of  $\log|\mathcal{J}_{\text{neut}}|$ , the harmonic function on  $D$  from the main theorem proof; it does not include the infinite-energy near-Blaschke singularities.) Then

$$(A.2) \quad E_{\text{neut}}(I) \leq C(\alpha') \log^2\langle t_0 \rangle |I|,$$

where  $C(\alpha')$  depends only on the apertures  $(\alpha', \alpha'')$ , the RvM density constant, and the convexity exponent—not on  $c$ .

In particular, the windowed-phase product satisfies

$$(A.3) \quad \sqrt{E_{\text{neut}}(I)} \cdot L \leq \sqrt{C(\alpha')} \frac{c^{3/2}}{\sqrt{\log\langle t_0 \rangle}},$$

which tends to 0 as  $c \rightarrow 0$ , uniformly in  $t_0$ .

*Remark 27.* The  $\log^2\langle t_0 \rangle$  growth is not an obstruction to the main theorem: in the proof of Theorem 3, the Whitney parameter is chosen as  $c = c_0/\log\langle \gamma_0 \rangle$  (depending on the height of the hypothetical zero), which causes  $\log^2\langle t_0 \rangle \cdot |I|$  to collapse to a height-independent constant  $2Cc_0$  (see (??)). Replacing  $\log^2$  with a uniform constant would allow a fixed  $c$  and simplify the argument, but is not logically required.

*Proof.* Fix a Whitney interval  $I = [t_0 - L, t_0 + L]$  with  $L = c/\log\langle t_0 \rangle$  and  $\alpha' > 1$ . Choose a slightly larger aperture parameter  $\alpha'' > 2\alpha'$ , and let  $D := Q(\alpha''I)$  (a dilated Whitney box).

Since  $U = 2\log|B| + W$  and  $B = (s-1)/s$  is explicit and smooth on  $D$  (for  $t_0$  large, the clip  $L \leq L_\star$  keeps  $D$  away from  $s = 1$ ),  $\nabla(2\log|B|)$  contributes  $O_{\alpha'}(|I|)$  to the weighted energy. It therefore suffices to bound the  $W$ -energy:

$$E_W(I) := \iint_{Q(\alpha'I)} |\nabla W|^2 \sigma dt d\sigma.$$

*Step 1 (Whitney neutralization).* By Lemma 25,  $\mathcal{I}$  is an inner function on  $\Omega$  with zeros exactly at the nontrivial zeros of  $\zeta$  in  $\Omega$ . Factor  $\mathcal{I} = e^{i\theta} B_{\text{near}} B_{\text{far}} S$ , where  $B_{\text{near}}$  is the finite Blaschke product over zeros  $\rho = \beta + i\gamma$  of  $\mathcal{I}$  with  $|\gamma - t_0| \leq \alpha''L$ ,  $B_{\text{far}}$  is the Blaschke product of the remaining zeros, and  $S$  is the (possibly trivial) singular inner factor. By (A.1),  $B_{\text{near}}$  has at most  $C_{\text{RvM}}(1 + 2\alpha''L) \log\langle t_0 \rangle = O(\log\langle t_0 \rangle)$  factors. (On Whitney scale the count is  $O(\log\langle t_0 \rangle)$ , not  $O(1)$ ; but see Step 2—the near-zero charges do not enter the Cauchy–Schwarz energy bound.)

Define the neutralized field

$$\widetilde{W}(s) := W(s) + \log|B_{\text{near}}(s)| = -\log|B_{\text{far}}(s)| - \log|S(s)|.$$

Every term on the right is  $\geq 0$  (each inner factor has modulus  $\leq 1$  on  $\Omega$ ), so  $\widetilde{W} \geq 0$  on  $\Omega$ . On  $\partial\Omega$  ( $\sigma = 0$ ): all inner factors have boundary modulus 1, so  $\widetilde{W} = 0$ . Moreover,  $\widetilde{W}$  is *harmonic* on  $D$ : the zeros in  $B_{\text{far}}$  have  $|\gamma - t_0| > \alpha''L$ , hence lie outside the  $t$ -span of  $D$ , and  $S$  is zero-free.

The zeros in  $B_{\text{near}}$  lie *inside* the box  $D$ , so  $\log|B_{\text{near}}|$  has logarithmic singularities there and its weighted Dirichlet energy on  $Q(\alpha'I)$  is infinite. This is not a problem: the near Blaschke factors are absorbed into the neutralization step in the main theorem proof (see the neutralization step in §4),

where they cancel the poles of  $\mathcal{J}_{\text{out}}$  and produce the harmonic function  $\log |\mathcal{J}_{\text{neut}}| = 2 \log |B| + \widetilde{W}$  on  $D$ . The energy estimate below bounds the harmonic field  $\widetilde{W}$  only.

*Step 2 (boundary bound on  $\partial D$ ).* Since  $\widetilde{W} \geq 0$  and  $\widetilde{W} = 0$  on  $\sigma = 0$ , it remains to bound  $\widetilde{W}$  on the top/side edges of  $D$ .

Each far zero  $\rho = \beta + i\gamma$  with  $\delta := \beta - \frac{1}{2} \in (0, \frac{1}{2}]$  contributes

$$-\log |b_\rho(s)| = G_\Omega(s, \rho) = \frac{1}{2} \log \frac{(\sigma + \delta)^2 + (t - \gamma)^2}{(\sigma - \delta)^2 + (t - \gamma)^2} \leq \frac{2\sigma\delta}{(\sigma - \delta)^2 + (t - \gamma)^2} \leq \frac{\alpha'L}{(t - \gamma)^2}$$

(using  $\log(1 + x) \leq x$ ,  $\sigma \leq \alpha'L$ ,  $\delta \leq \frac{1}{2}$ , and  $|t - \gamma| \geq (\alpha'' - \alpha')L \gg \sigma$ ). Summing over all far zeros and using the zero density (A.1) (at most  $C_{\text{RvM}}(1 + R) \log \langle t_0 \rangle$  zeros with  $|\gamma - t_0| \leq R$ ):

$$\sum_{\text{far } \rho} G_\Omega(s, \rho) \leq \alpha'L \int_{\alpha''L}^{\infty} \frac{C_{\text{RvM}} \log \langle t_0 \rangle}{r^2} dr = \frac{\alpha' C_{\text{RvM}} \log \langle t_0 \rangle}{\alpha''} \ll \log \langle t_0 \rangle$$

on  $\partial D$  (with the implied constant depending only on  $\alpha', \alpha''$ ).

*Key independence of  $L$  and  $c$ .* The integral  $\alpha'L \cdot C_{\text{RvM}} \log \langle t_0 \rangle / (\alpha''L) = \alpha' C_{\text{RvM}} \log \langle t_0 \rangle / \alpha''$ : the  $L$  in the numerator ( $\sigma \leq \alpha'L$ ) cancels the  $L$  in the denominator ( $\int_{\alpha''L}^{\infty} 1/r^2 dr = 1/(\alpha''L)$ ). The Blaschke tail bound does not depend on  $L$  or  $c$ , and does not require short-interval zero control at scale  $L$ —only the coarse  $O(\log \langle t_0 \rangle)$  count per unit ordinate interval.

*Singular inner contribution and the  $S \equiv 1$  condition.* The singular inner factor  $S$  of  $\mathcal{I}$  contributes  $-\log |S(s)| = P_\sigma[\nu_S](t)$ , the Poisson integral of a positive singular measure  $\nu_S$  on  $\partial\Omega$ . At  $\Re s = \frac{3}{2}$ :  $P_1[\nu_S](t) \leq W(\frac{3}{2} + it) \leq C_0$  (bounded), so  $\nu_S$  has uniformly bounded mass per unit interval:  $\nu_S([t_0 - 1, t_0 + 1]) \leq 2\pi C_0 =: \nu_*$ .

On  $\partial D$  at height  $\sigma = \alpha''L$ : the near singular mass ( $|\tau - t_0| \leq 1$ ) contributes at most  $\nu_* / (\pi \alpha''L) = \nu_* \log \langle t_0 \rangle / (\pi \alpha''c)$ . If  $S \equiv 1$  (i.e.  $\nu_S = 0$ ), this vanishes and

$$M := \sup_{\partial D} \widetilde{W} \leq \frac{\alpha' C_{\text{RvM}}}{\alpha''} \log \langle t_0 \rangle =: C_* \log \langle t_0 \rangle,$$

with  $C_*$  depending only on  $(\alpha', \alpha'', C_{\text{RvM}})$ —not on  $c$ . In this case the energy bound closes unconditionally (see the remark below).

If  $S \not\equiv 1$ : the near singular Poisson spike contributes  $O(\log \langle t_0 \rangle / c)$  to  $M$ , which with  $c = c_0 / \log \langle t_0 \rangle$  becomes  $O(\log^2/c_0)$  and introduces one extra power of  $\log$  that the cancellation trick does not absorb. Proving  $S \equiv 1$  for the specific inner function  $\mathcal{I} = B\mathcal{O}_\zeta\zeta / \det_2$  would therefore complete the unconditional proof; this is recorded as an open step below.

*Proof that  $S \equiv 1$ .* The singular inner factor satisfies  $S \equiv 1$  if and only if

$$\lim_{\sigma \rightarrow 0^+} \int_{\mathbb{R}} \frac{W(\frac{1}{2} + \sigma + it)}{1 + t^2} dt = 0$$

(see Garnett [5, Ch. II]). We prove this by showing that each factor of  $\mathcal{I} = B\mathcal{O}_\zeta\zeta / \det_2$  has log-modulus converging in  $L^1(\mathbb{R}, dt/(1 + t^2))$  as  $\sigma \rightarrow 0$ , and that the boundary traces sum to 0.

*Term  $\log |B|$ .*  $B = (s-1)/s$  is continuous with  $|B^*| = 1$ ; convergence is uniform.

*Term  $\log |\mathcal{O}_\zeta|$ .*  $\mathcal{O}_\zeta$  is the outer function with boundary modulus  $\exp(u)$ , so  $\log |\mathcal{O}_\zeta(\sigma)| = P_\sigma[u] \rightarrow u$  in  $L^1(dt/(1 + t^2))$  by Poisson convergence.

*Term  $\log |\det_2|$ .* By explicit Fourier computation,

$$\int_{\mathbb{R}} \frac{\log |\det_2(\sigma, t)|}{1 + t^2} dt = -\pi \sum_p \sum_{k \geq 2} \frac{p^{-k(\frac{3}{2} + \sigma)}}{k},$$

which converges absolutely to  $-\pi \sum_p \sum_{k \geq 2} p^{-3k/2}/k$  as  $\sigma \rightarrow 0$ .

*Term  $\log |\zeta|$  (the key term).* We must show  $\int \log |\zeta(\frac{1}{2} + \sigma + it)| / (1 + t^2) dt \rightarrow \int \log |\zeta^*(t)| / (1 + t^2) dt$  as  $\sigma \rightarrow 0$ .

(a) *The  $\log^+$  part.*  $\log^+ |\zeta(\frac{1}{2} + \sigma + it)| \leq A \log(2 + |t|)$  uniformly for  $\sigma \in (0, 1]$  (convexity bound; Titchmarsh [14, Ch. V]). Since  $A \log(2 + |t|)/(1 + t^2) \in L^1$ , dominated convergence applies.

(b) *The  $\log^-$  part.* Cover  $\mathbb{R}$  by unit intervals  $I_n = [n, n + 1]$ . On each  $I_n$ , Jensen's inequality for the subharmonic function  $\log |\zeta(\frac{1}{2} + \sigma + i\cdot)|$  on a disc of radius 2 centered at  $n + \frac{1}{2} + i\sigma$  gives

$$\int_{I_n} \log^- |\zeta(\frac{1}{2} + \sigma + it)| dt \leq \pi \cdot 4 \cdot (A \log(3 + |n|) + C) + \pi \cdot 4 \cdot N_n \cdot \log 4,$$

where  $N_n$  is the number of  $\zeta$ -zeros with  $|\gamma - (n + \frac{1}{2})| \leq 4$  and the right side comes from the standard Jensen bound ( $\int \log^- |f| \leq \text{mean of } \log^+ |f| \text{ on a larger circle} + \text{zero count} \cdot \log(\text{ratio})$ ). By (A.1):  $N_n \leq C_1(1 + 4) \log \langle n \rangle = O(\log \langle n \rangle)$ . Hence

$$\int_{I_n} \log^- |\zeta(\sigma, t)| dt \leq C_2 \log(2 + |n|) \quad \text{uniformly for } \sigma \in (0, 1].$$

Dividing by  $1 + t^2 \geq 1 + n^2$  and summing:  $\int_{\mathbb{R}} \log^- |\zeta(\sigma)|/(1 + t^2) \leq \sum_n C_2 \log(2 + |n|)/(1 + n^2) < \infty$ . This bound is uniform in  $\sigma$ .

(c) *Convergence.*  $L^1_{\text{loc}}$  convergence  $\log |\zeta(\sigma)| \rightarrow \log |\zeta^*|$  holds by Lemma 24. Combined with the  $\sigma$ -uniform  $L^1(dt/(1+t^2))$  bound from (a) and (b), Vitali's convergence theorem gives  $\int \log |\zeta(\sigma)|/(1+t^2) \rightarrow \int \log |\zeta^*|/(1+t^2)$ .

*Assembly.* By the construction of  $\mathcal{O}_\zeta$ :  $u = \log |\det_2^*| - \log |\zeta^*|$ , so the boundary traces satisfy  $0 + u + \log |\zeta^*| - \log |\det_2^*| = 0$ . Hence

$$\lim_{\sigma \rightarrow 0} \int \frac{W(\sigma, t)}{1+t^2} dt = 0 - (-u) - (-\log |\zeta^*|) + (-\log |\det_2^*|) = 0.$$

Therefore  $S \equiv 1$ . (This argument uses only: the convexity bound for  $\zeta$ , the convergence of  $\sum 1/(1+\gamma^2)$ , the outer construction of  $\mathcal{O}_\zeta$ , and the explicit Fourier series for  $\det_2$ . No zero-free hypothesis is used.)

Hence

$$M := \sup_{\partial D} \widetilde{W} \leq C_* \log \langle t_0 \rangle,$$

with  $C_*$  independent of  $c$ .

*Step 3 (interior gradient estimate).* Since  $\widetilde{W}$  is harmonic on  $D$  with  $0 \leq \widetilde{W} \leq M$  and  $\widetilde{W} = 0$  on  $\sigma = 0$ , the standard interior estimate (odd reflection + Cauchy) gives  $\sup_{Q(\alpha'I)} |\nabla \widetilde{W}|^2 \leq C_2 M^2 / L^2$ . Integrating with the weight  $\sigma$ :

$$\iint_{Q(\alpha'I)} |\nabla \widetilde{W}|^2 \sigma \leq C_3 M^2 |I| \leq C_3 C_*^2 \log^2 \langle t_0 \rangle |I|.$$

*Step 4 (assembly).* The energy of the neutralized harmonic function  $\widetilde{W}$  on  $Q(\alpha'I)$  controls the smooth part of the boundary phase derivative via the CR–Green pairing (Lemma 32). The  $O(\log \langle t_0 \rangle)$  zeros of  $\mathcal{I}$  inside  $D$  contribute nonnegative charges to the total windowed phase via the distributional Green identity, but they do not enter the Cauchy–Schwarz energy bound. A hypothetical zero  $\rho_0$  with  $\delta_0 \geq \varepsilon > \alpha'L$  lies outside  $D$ , so its Poisson contribution enters the smooth part and produces the lower bound in Theorem 1.

The effective energy bound is therefore

$$E_{\text{eff}}(I) := \iint_{Q(\alpha'I)} |\nabla \widetilde{W}|^2 \sigma \leq C \log^2 \langle t_0 \rangle |I|,$$

where  $C = C_3 C_*^2$  depends only on  $(\alpha', \alpha'')$  and is independent of  $c$ . With  $c = c_0 / \log \langle t_0 \rangle$  in the main theorem,  $E_{\text{eff}} = 2C c_0$ , independent of height.  $\square$

**A.4. CR–Green pairing lemmas.** The final set of tools converts boundary phase integrals into Dirichlet-energy estimates via Green’s theorem on Whitney boxes.

**Definition 28** (Admissible window class with atom avoidance). Fix an even  $C^\infty$  window  $\psi$  with  $\psi \equiv 1$  on  $[-1, 1]$  and  $\text{supp } \psi \subset [-2, 2]$ . For an interval  $I = [t_0 - L, t_0 + L]$ , an aperture  $\alpha' > 1$ , and a parameter  $\varepsilon \in (0, \frac{1}{4}]$ , define  $\mathcal{W}_{\text{adm}}(I; \varepsilon)$  to be the set of  $C^\infty$ , nonnegative, mass-1 bumps  $\phi$  supported in the fixed dilate  $2I = [t_0 - 2L, t_0 + 2L]$  that can be written as

$$\phi(t) = \frac{1}{Z} \frac{1}{L} \psi\left(\frac{t - t_0}{L}\right) m(t), \quad Z = \int_{2I} \frac{1}{L} \psi\left(\frac{t - t_0}{L}\right) m(t) dt,$$

where  $2I := [t_0 - 2L, t_0 + 2L]$  and the mask  $m \in C^\infty(2I; [0, 1])$  satisfies:

- (i) *Atom avoidance.* There is a union of disjoint open subintervals  $E = \bigcup_{j=1}^J J_j \subset I$  with total length  $|E| \leq \varepsilon L$  such that  $m \equiv 0$  on  $E$  and  $m \equiv 1$  on  $I \setminus E'$ , where each transition layer  $E' \setminus E$  has thickness  $\leq \varepsilon L$ .
- (ii) *Uniform smoothness.*  $\|m'\|_\infty \lesssim (\varepsilon L)^{-1}$  and  $\|m''\|_\infty \lesssim (\varepsilon L)^{-2}$  with implicit constants independent of  $I, t_0, L$  and of the number/placement of the holes  $\{J_j\}$ .

Every  $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$  is supported in  $2I$ . This class contains the unmasked profile  $\varphi_{L, t_0}(t) = Z_0^{-1} L^{-1} \psi((t - t_0)/L)$  with  $Z_0 := \int_{-2}^2 \psi(x) dx$  (take  $E = \emptyset$ ,  $m \equiv 1$ ) and also allows dodging boundary atoms by punching out small neighborhoods while keeping total deleted length  $\leq \varepsilon L$ .

**Lemma 29** (Poisson–energy bound for admissible tests). *Let  $V_\phi$  be the Poisson extension of  $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$  to the half-plane, and fix a cutoff to  $Q(\alpha'I)$  with  $\alpha' > 1$  as in the CR–Green pairing. Then*

$$(A.4) \quad \iint_{Q(\alpha'I)} |\nabla V_\phi(\sigma, t)|^2 \sigma dt d\sigma \leq \frac{1}{2} \|\phi\|_{L^2}^2 = \frac{\mathcal{A}_{\text{adm}}(\psi, \varepsilon)^2}{L},$$

where  $\mathcal{A}_{\text{adm}}(\psi, \varepsilon)^2 := \|\psi m\|_{L^2}^2 / (2Z^2)$  depends only on  $(\psi, \varepsilon)$ , not on  $L$ . In particular, the energy scales as  $1/L$ , not  $L$ .

*Proof.* On the full half-plane the  $\sigma$ -weighted Dirichlet energy of a Poisson extension satisfies the identity

$$\iint_{\mathbb{R}_+^2} |\nabla V_\phi|^2 \sigma d\sigma dt = \frac{1}{2} \|\phi\|_{L^2(\mathbb{R})}^2,$$

which follows from Plancherel and the explicit formula  $\int_0^\infty \sigma \omega^2 e^{-2\omega\sigma} d\sigma = \frac{1}{4}$  applied mode-by-mode. Since the box  $Q(\alpha'I) \subset \mathbb{R}_+^2$ , restriction to  $Q(\alpha'I)$  can only decrease the integral. For the unmasked profile  $\phi(t) = (Z_0 L)^{-1} \psi((t - t_0)/L)$ :

$$\|\phi\|_{L^2}^2 = \frac{1}{Z_0^2 L^2} \int_{\mathbb{R}} |\psi((t - t_0)/L)|^2 dt = \frac{\|\psi\|_{L^2}^2}{Z_0^2 L}.$$

Hence the energy is  $\|\psi\|_{L^2}^2 / (2Z_0^2 L) = O(1/L)$ .  $\square$

**Remark 30** (Scaling error and its impact). An earlier version of this paper claimed the energy in (A.4) was  $\lesssim L$ , which would make the constant  $C_{\text{test}}$  in Proposition 33 independent of  $L$ . The correct scaling  $1/L$  causes  $C_{\text{test}}$  to blow up like  $1/\sqrt{L}$ . When this corrected bound is propagated through the CR–Green pathway (Appendix A.4) (see Remark ?? below), the upper bound on the windowed phase becomes proportional to  $\sqrt{L}$  rather than to  $L$ , and the contradiction mechanism in Step 3 fails. See §4 for details.

The next two lemmas implement the Cauchy–Riemann/Green pairing that converts the boundary phase integral into a box energy.

**Lemma 31** (Cutoff pairing on boxes). *Fix parameters  $\alpha' > \alpha > 1$ . Let  $\chi_{L,t_0} \in C_c^\infty(\mathbb{R}_+^2)$  satisfy  $\chi \equiv 1$  on  $Q(\alpha I)$ ,  $\text{supp } \chi \subset Q(\alpha' I)$ ,  $\|\nabla \chi\|_\infty \lesssim L^{-1}$  and  $\|\nabla^2 \chi\|_\infty \lesssim L^{-2}$ . Let  $V_\phi$  be the Poisson extension of  $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$ . Then one has the Green pairing identity*

$$\int_{\mathbb{R}} u(t) \phi(t) dt = \iint_{Q(\alpha' I)} \nabla U \cdot \nabla (\chi_{L,t_0} V_\phi) dt d\sigma + \mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}},$$

with remainders satisfying

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \lesssim \left( \iint_{Q(\alpha' I)} |\nabla U|^2 \sigma \right)^{1/2} \cdot \left( \iint_{Q(\alpha' I)} (|\nabla \chi|^2 |V_\phi|^2 + |\nabla V_\phi|^2) \sigma \right)^{1/2}.$$

*Proof.* Let  $Q := Q(\alpha' I)$ . Assume  $U$  is  $C^2$  on  $\overline{Q}$  and harmonic on  $Q$ , with boundary trace  $u(t) = U(0, t)$  on the bottom edge  $\{\sigma = 0\}$ . Since  $\chi_{L,t_0} V_\phi$  is compactly supported in  $\overline{Q}$  and smooth on  $Q$ , Green's identity gives

$$\iint_Q \nabla U \cdot \nabla (\chi V_\phi) dt d\sigma = \int_{\partial Q} (\chi V_\phi) \partial_n U ds - \iint_Q (\chi V_\phi) \Delta U dt d\sigma.$$

Since  $\Delta U = 0$  on  $Q$ , only the boundary integral remains. On the bottom edge one has  $\partial_n = -\partial_\sigma$ ,  $\chi \equiv 1$ , and  $V_\phi(0, t) = \phi(t)$ , hence that contribution equals

$$\int_I \phi(t) (-\partial_\sigma U)(0, t) dt.$$

If  $U$  is the real part of a holomorphic logarithm  $U = \Re \log J$  with  $|J(\frac{1}{2} + it)| = 1$  a.e., then  $U(0, t) = 0$  a.e. and  $-\partial_\sigma U(0, t) = \partial_t \text{Arg } J(\frac{1}{2} + it)$  in distributions by Cauchy–Riemann; in particular, this term is the tested boundary phase derivative in Lemma 32 below. The remaining boundary pieces (two vertical sides and the top edge) are, by definition, the remainders  $\mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}}$ .

For the remainder estimate, we apply Cauchy–Schwarz in the scale-invariant measure  $\sigma dt d\sigma$  on  $Q$ :

$$|\mathcal{R}_{\text{side}}| + |\mathcal{R}_{\text{top}}| \lesssim \left( \iint_Q |\nabla U|^2 \sigma \right)^{1/2} \left( \iint_Q |\nabla (\chi V_\phi)|^2 \sigma \right)^{1/2}.$$

Expanding  $\nabla(\chi V_\phi) = \chi \nabla V_\phi + (\nabla \chi) V_\phi$  yields

$$\iint_Q |\nabla(\chi V_\phi)|^2 \sigma \lesssim \iint_Q (|\nabla V_\phi|^2 + |\nabla \chi|^2 |V_\phi|^2) \sigma,$$

which gives the displayed estimate.  $\square$

**Lemma 32** (CR–Green pairing for boundary phase). *Let  $J$  be analytic on  $\Omega$  with a.e. boundary modulus  $|J(\frac{1}{2} + it)| = 1$ , and write  $\log J = U + iW$  on  $\Omega$ , so  $U$  is harmonic with  $U(\frac{1}{2} + it) = 0$  a.e. Fix a Whitney interval  $I = [t_0 - L, t_0 + L]$  and let  $V_\phi$  be the Poisson extension of  $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$ . Then, with a cutoff  $\chi_{L,t_0}$  as in Lemma 31,*

$$\int_{\mathbb{R}} \phi(t) (-W'(t)) dt = \iint_{Q(\alpha' I)} \nabla U \cdot \nabla (\chi_{L,t_0} V_\phi) dt d\sigma + \mathcal{R}_{\text{side}} + \mathcal{R}_{\text{top}},$$

and the remainders satisfy the same estimate as in Lemma 31. In particular, by Cauchy–Schwarz and Lemma 29,

$$\int_{\mathbb{R}} \phi(t) (-w'(t)) dt \leq \frac{C_{\text{rem}}(\alpha', \psi)}{\sqrt{L}} \left( \iint_{Q(\alpha' I)} |\nabla U|^2 \sigma \right)^{1/2},$$

where  $C_{\text{rem}}$  depends only on  $(\alpha', \psi)$ . (The factor  $1/\sqrt{L}$  arises from  $\mathcal{A}_{\text{adm}} \sim 1/\sqrt{L}$  in Lemma 29; see Remark 30.)

*Proof.* On the bottom edge  $\{\sigma = 0\}$  the outward normal is  $\partial_n = -\partial_\sigma$ . By Cauchy–Riemann for  $\log J = U + iW$  on the boundary line  $\{\Re s = \frac{1}{2}\}$  one has  $\partial_n U = -\partial_\sigma U = \partial_t W$ . Thus the bottom-edge term in Green’s identity is

$$-\int_{\partial Q \cap \{\sigma=0\}} \chi V_\phi \partial_n U dt = -\int_{\mathbb{R}} \phi(t) \partial_t W(t) dt = \int_{\mathbb{R}} \phi(t) (-w'(t)) dt,$$

which yields the stated identity after including the interior term and remainders. The final inequality is Cauchy–Schwarz together with the uniform Poisson-energy bound from Lemma 29.  $\square$

**Proposition 33** (Upper bound for admissible tests). *Let  $J$  be holomorphic on  $\Omega \setminus Z(\zeta)$  with a.e. boundary modulus 1, write  $\log J = U + iW$  on  $\Omega \setminus Z(\zeta)$ , and let  $-w'$  denote the boundary phase distribution. For every interval  $I = [t_0 - L, t_0 + L]$ , every  $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$ , and every fixed cutoff to  $Q(\alpha' I)$ ,*

$$(A.5) \quad \int_{\mathbb{R}} \phi(t) (-w')(t) dt \leq \frac{C_{\text{test}}(\psi, \varepsilon, \alpha')}{\sqrt{L}} \left( \iint_{Q(\alpha' I)} |\nabla U|^2 \sigma dt d\sigma \right)^{1/2}$$

with  $C_{\text{test}}(\psi, \varepsilon, \alpha') := C_{\text{rem}}(\alpha', \psi) \mathcal{A}_{\text{adm}}(\psi, \varepsilon)$  independent of  $I, t_0, L$ . The factor  $1/\sqrt{L}$  is forced by the correct Poisson-energy scaling (Lemma 29, Remark 30).

*Proof.* Apply Lemma 32 with  $\phi \in \mathcal{W}_{\text{adm}}(I; \varepsilon)$ . The window-side Cauchy–Schwarz factor is  $(\iint |\nabla(\chi V_\phi)|^2 \sigma)^{1/2} \lesssim \mathcal{A}_{\text{adm}}/\sqrt{L}$  by Lemma 29.  $\square$

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RECOGNITION PHYSICS RESEARCH INSTITUTE, AUSTIN, TX, USA  
*Email address:* `jon@recognitionphysics.org`

RECOGNITION PHYSICS RESEARCH INSTITUTE, AUSTIN, TX, USA  
*Email address:* `arahnamab@gmail.com`