Whitney box energy for $\log \xi$ and weighted microscopic variation of $S(T)$

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Abstract

We study the harmonic field $U_\xi(\sigma, t) = \Re \log \xi(\frac{1}{2} + \sigma + it)$ on the right half-plane and its Whitney-box Dirichlet energy at the microscopic scale $L \asymp 1/\log\langle T\rangle$. Unconditionally, we prove a Whitney-box growth bound of size $O(|I| \log\langle T\rangle)$ (Lemma 3). We also show that, at Whitney scale $L = c/\log\langle T\rangle$, the box energy admits a sharpened upper bound in terms of a weighted short-interval variation of the classical argument term $S(T) = \frac{1}{\pi} \operatorname{Arg} \zeta(\frac{1}{2} + iT)$ (Lemma 7), isolating a precise obstruction to scale-free energy control.

Keywords. Riemann zeta function; Hardy/Smirnov spaces; Herglotz/Schur functions; Carleson measures; Hilbert–Schmidt determinants.

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Notation and conventions

- Half–plane and coordinates: $\Omega := \{\Re s > \frac{1}{2}\}$; write $s = \frac{1}{2} + \sigma + it$ with $\sigma > 0$, $t \in \mathbb{R}$.
- Completed zeta: $\xi(s) := \frac{1}{2} s(1-s)\, \pi^{-s/2} \Gamma(s/2)\, \zeta(s)$, and

$$U_\xi(\sigma, t) := \Re \log \xi\Big(\frac{1}{2} + \sigma + it\Big) \qquad (\sigma > 0).$$

- Riemann–von Mangoldt: $N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(1)$ with $S(T) := \frac{1}{\pi} \operatorname{Arg} \zeta(\frac{1}{2} + iT)$.
- Poisson kernel: $P_a(x) = \frac{1}{\pi} \frac{a}{a^2 + x^2}$.
- Whitney scale and boxes: fix $c \in (0, 1]$ and $\langle T\rangle := \sqrt{1 + T^2}$, set

$$L := \frac{c}{\log\langle T\rangle}, \qquad I := [T - L, T + L], \qquad Q(\alpha I) := I \times (0, \alpha L],$$

with a fixed aperture $\alpha \in [1, 2]$.

Reader's guide

- **Main unconditional results.** An annular L^2 Poisson–balayage bound (Lemma 2); unconditional Whitney-box energy growth for U_ξ (Lemma 3 and Proposition 4); the reduction of Whitney-scale ξ-energy to weighted microscopic variation of $S(T)$ (Lemma 7); and an averaged (dyadic) mean-square bound for that weighted term (Theorem 9).
- **What is pinned down as the obstruction.** Scale-free Whitney energy at $L = c/\log\langle T\rangle$ would follow from a uniform bound on the weighted short-interval variation term in Lemma 7; informally, it forbids arbitrarily large zero clusters in windows of length $\asymp 1/\log T$.
- **Scope.** We do *not* claim an unconditional proof of the Riemann Hypothesis here; this manuscript records the unconditional analytic core and a precise formulation of what remains open.

Dependency map (unconditional core)

All proofs not explicitly listed below are auxiliary.

1. **Annular bookkeeping.** Lemma 2 provides the basic annular L^2 Poisson–balayage estimate used to control far-field contributions from zeros.
2. **Whitney box energy for U_ξ.** Lemma 3 gives an unconditional $O(|I|\log\langle T\rangle)$ growth bound on Whitney boxes. Lemma 7 refines the bookkeeping to express the only non–scale-free contribution as a weighted microscopic variation of $S(T)$.
3. **Average control of the obstruction.** Theorem 9 gives a dyadic mean-square bound for the weighted microscopic variation term, via Selberg's mean-square estimate for S (Lemma 8).
4. **Open problem (isolated).** Upgrade the Whitney-scale $O(|I|\log\langle T\rangle)$ growth bound to a scale-free $O(|I|)$ bound uniform in T at scale $L = c/\log\langle T\rangle$. By Lemma 7, this would follow from uniform control of the weighted short-interval variation of $S(T)$ at Whitney scale.

1 Introduction

Conceptual motivation. The Euler product for ζ separates the $k = 1$ prime layer from all higher prime powers. On the right half–plane $\{\Re s > \frac{1}{2}\}$ the diagonal prime operator $A(s)e_p := p^{-s}e_p$ has finite Hilbert–Schmidt norm ($\sum_p p^{-2\sigma} < \infty$), so the $k \geq 2$ tail is naturally encoded by the 2–modified determinant $\det_2(I - A)$. For the present manuscript, the central analytic object is the ξ–field

$$U_\xi(\sigma, t) = \Re \log \xi\left(\tfrac{1}{2} + \sigma + it\right),$$

and its Whitney-box Dirichlet energy at the microscopic scale $L \asymp 1/\log\langle T\rangle$.

Scope and contributions. We prove an unconditional Whitney-box growth bound of size $O(|I|\log\langle T\rangle)$ for U_ξ on Whitney boxes (Lemma 3). We also refine the bookkeeping to show that the only non–scale-free contribution can be expressed as a weighted microscopic short-interval variation of the classical argument term $S(T) = \frac{1}{\pi}\operatorname{Arg}\zeta(\frac{1}{2} + iT)$ (Lemma 7). These results isolate a precise obstruction to scale-free energy control at length $1/\log T$. No unconditional proof of the Riemann Hypothesis is claimed here.

2 Whitney-box energy bounds

We record unconditional Carleson-energy bounds for the arithmetic tail and for U_ξ on Whitney boxes, and we isolate the precise obstruction to a scale-free bound.

Lemma 1 (Arithmetic Carleson energy). *Let*

$$U_{\det_2}(\sigma,t) \;:=\; \sum_p \sum_{k\ge 2} \frac{(\log p)\, p^{-k/2}}{k\log p}\, e^{-k\log p\,\sigma}\, \cos\left(k\log p\, t\right), \qquad \sigma>0.$$

Then for every interval $I\subset\mathbb{R}$ with Carleson box $Q(I):=I\times(0,|I|]$

$$\iint_{Q(I)} |\nabla U_{\det_2}|^2\, \sigma\, dt\, d\sigma \;\le\; \frac{|I|}{4}\sum_p\sum_{k\ge 2}\frac{p^{-k}}{k^2} \;=:\; K_0\,|I|, \qquad K_0 := \frac{1}{4}\sum_p\sum_{k\ge 2}\frac{p^{-k}}{k^2}<\infty.$$

Proof. For a single mode $b\, e^{-\omega\sigma}\cos(\omega t)$ one has $|\nabla|^2 = b^2\omega^2 e^{-2\omega\sigma}$, hence

$$\int_0^{|I|}\int_I |\nabla|^2\,\sigma\,dt\,d\sigma \;\le\; |I|\cdot\sup_{\omega>0}\int_0^{|I|}\sigma\,\omega^2 e^{-2\omega\sigma}d\sigma\cdot b^2 \;\le\; \tfrac14\,|I|\,b^2.$$

With $b=(\log p)\,p^{-k/2}/(k\log p)$ and $\omega = k\log p$, summing over (p,k) gives the claim and the finiteness of K_0. □

Whitney scale and short–interval zero counts (what is actually known unconditionally). Fix a Whitney parameter $c\in(0,1]$ and set $L := c/\log\langle T\rangle$ with $\langle T\rangle := \sqrt{1+T^2}$. The only short-interval input we use is the local Riemann–von Mangoldt bound (see, e.g., Ivić): for $T\ge 2$ and $H\in(0,1]$,

$$N(T+H)-N(T-H) \;\ll\; H\log\langle T\rangle \;+\; \log\langle T\rangle.$$

In particular, on Whitney scale $H\asymp L(T)=c/\log\langle T\rangle$ this gives at best $O(\log\langle T\rangle)$ zeros in a window of length $\asymp 1/\log\langle T\rangle$; we do *not* assume or claim a uniform $O(1)$ bound at that microscopic scale.

Lemma 2 (Annular Poisson–balayage L^2 bound). *Let $I=[T-L,T+L]$, $Q_\alpha(I)=I\times(0,\alpha L]$, and fix $k\ge 1$. For $\mathcal{A}_k := \{\rho=\beta+i\gamma:\; 2^kL<|T-\gamma|\le 2^{k+1}L\}$ set*

$$V_k(\sigma,t) := \sum_{\rho\in\mathcal{A}_k}\frac{\sigma}{(t-\gamma)^2+\sigma^2}.$$

Then

$$\iint_{Q_\alpha(I)} V_k(\sigma,t)^2\,\sigma\,dt\,d\sigma \;\ll_\alpha\; |I|\,4^{-k}\,\nu_k,$$

where $\nu_k := \#\mathcal{A}_k$, and the implicit constant depends only on α.

Proof. Write $K_\sigma(x) := \sigma/(x^2+\sigma^2)$ and $V_k = \sum_{\rho\in\mathcal{A}_k} K_\sigma(\cdot-\gamma)$. For any finite index set $\mathcal{J}$,

$$V_k^2 \;\le\; \sum_{j\in\mathcal{J}} K_\sigma(\cdot-\gamma_j)^2 \;+\; 2\sum_{i<j} K_\sigma(\cdot-\gamma_i)K_\sigma(\cdot-\gamma_j).$$

Integrate over $t\in I$ first. For the diagonal terms, using $|t-\gamma|\ge 2^kL-L\ge 2^{k-1}L$ for $t\in I$ and $k\ge 1$,

$$\int_I K_\sigma(t-\gamma)^2\,dt \;=\; \sigma^2\int_I \frac{dt}{\left((t-\gamma)^2+\sigma^2\right)^2} \;\le\; \frac{L}{(2^{k-1}L)^2}\,\sigma \;\le\; \frac{\sigma}{4^{k-1}L}.$$

Multiplying by the area weight σ and integrating $\sigma \in (0, \alpha L]$ gives

$$\int_0^{\alpha L}\Big(\int_I K_\sigma(t-\gamma)^2\,dt\Big)\,\sigma\,d\sigma \;\le\; \frac{1}{4^{k-1}L}\int_0^{\alpha L}\sigma^2 d\sigma \;=\; \frac{\alpha^3 L^2}{3\cdot 4^{k-1}} \;\le\; \frac{C_{\mathrm{diag}}(\alpha)}{4^k}\,|I|,$$

with $C_{\mathrm{diag}}(\alpha) := \frac{4\alpha^3}{3}\cdot\frac{L}{|I|} \asymp_\alpha 1$. Summing over ν_k choices of γ contributes a factor ν_k.

For the off-diagonal terms, for $i \ne j$ one has on I that $K_\sigma(t-\gamma_j) \le \sigma/(2^{k-1}L)^2$. Hence

$$\int_I K_\sigma(t-\gamma_i)K_\sigma(t-\gamma_j)\,dt \;\le\; \frac{\sigma}{(2^{k-1}L)^2}\int_{\mathbb{R}} K_\sigma(t-\gamma_i)\,dt \;=\; \frac{\pi\sigma}{(2^{k-1}L)^2},$$

and integrating $\sigma \in (0, \alpha L]$ with the extra factor σ yields $\le C'_{\mathrm{off}}(\alpha)\,L\cdot 4^{-k}$. Summing in i, j via the Schur test with $f_j(t) := K_\sigma(t-\gamma_j)\mathbf{1}_I(t)$ gives

$$\int_I V_k(\sigma,t)^2\,dt \;\le\; C''(\alpha)\,\nu_k\,\frac{\sigma}{(2^kL)^2}.$$

Integrating $\sigma \in (0, \alpha L]$ with weight σ gives $\le C_{\mathrm{off}}(\alpha)\,|I|{\cdot}4^{-k}\,\nu_k$. Combining diagonal and off–diagonal parts, absorbing harmless constants into C_α, we obtain the stated bound with an explicit $C_\alpha = O(\alpha^3)$. □

Lemma 3 (Analytic (ξ) box energy on Whitney boxes (unconditional, but not scale-free))**.** Reference. *The local zero count used below follows from the Riemann–von Mangoldt formula; see, e.g., [1, 2]. Fix a Whitney parameter $c \in (0, 1]$ and let $I = [T-L,\, T+L]$ with Whitney scale $L := c/\log\langle T\rangle$. Then for the Poisson extension*

$$U_\xi(\sigma,t) := \Re\log\xi\big(\tfrac12+\sigma+it\big), \qquad (\sigma > 0),$$

and any fixed aperture $\alpha \in [1, 2]$, one has the unconditional bound

$$\iint_{Q(\alpha I)} |\nabla U_\xi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \;\ll_{\alpha,c}\; |I|\,\log\langle T\rangle.$$

Proof. All inputs are unconditional. Fix $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ and aperture $\alpha \in [1, 2]$. Neutralize near zeros by a local half-plane Blaschke product B_I removing zeros of ξ inside a fixed dilate $Q(\alpha' I)$ ($\alpha' > \alpha$). This yields a harmonic field $\widetilde{U}_\xi$ on $Q(\alpha I)$ and

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\,dt\,d\sigma \;\asymp\; \iint_{Q(\alpha I)} |\nabla \widetilde{U}_\xi|^2\,\sigma\,dt\,d\sigma \;+\; O_\alpha(|I|),$$

so it suffices to bound the neutralized energy.

Write $\partial_\sigma U_\xi = \Re\,(\xi'/\xi) = \Re\sum_\rho (s-\rho)^{-1} + A$, where A is smooth on compact strips. Since U_ξ is harmonic, $|\nabla U_\xi|^2 \asymp |\partial_\sigma U_\xi|^2$ on $\mathbb{R}^2_+$; thus we bound the $L^2(\sigma\,dt\,d\sigma)$ norm of $\sum_\rho (s-\rho)^{-1}$ over $Q(\alpha I)$. Decompose the (neutralized) zeros into Whitney annuli $\mathcal{A}_k := \{\rho : 2^kL < |\gamma - T| \le 2^{k+1}L\}$, $k \ge 1$. For $V_k(\sigma,t) := \sum_{\rho\in\mathcal{A}_k} K_\sigma(t-\gamma)$ with $K_\sigma(x) := \sigma/(x^2+\sigma^2)$, Lemma 2 gives

$$\iint_{Q_\alpha(I)} V_k(\sigma,t)^2\,\sigma\,dt\,d\sigma \;\le\; C_\alpha\,|I|\,4^{-k}\,\nu_k,$$

where $\nu_k := \#\mathcal{A}_k$ and C_α depends only on α. Summing Cauchy–Schwarz bounds over annuli yields

$$\iint_{Q(\alpha I)} \Big|\sum_\rho (s-\rho)^{-1}\Big|^2\,\sigma\,dt\,d\sigma \;\le\; C_\alpha\,|I|\sum_{k\ge1} 4^{-k}\,\nu_k.$$

To bound ν_k, it suffices to use the local zero count on short vertical intervals from the Riemann–von Mangoldt formula (Titchmarsh): for $H > 0$,

$$N(T+H) - N(T-H) \ \ll \ H \log\langle T\rangle \ + \ \log\langle T\rangle.$$

Since ν_k counts (a subset of) zeros with ordinates in a window of length $\asymp 2^k L$, this yields, for some absolute $a_1(\alpha), a_2(\alpha)$,

$$\nu_k \ \le \ a_1(\alpha)\, 2^k L \, \log\langle T\rangle \ + \ a_2(\alpha)\, \log\langle T\rangle.$$

Therefore,

$$\sum_{k\ge 1} 4^{-k}\, \nu_k \ \le \ a_1(\alpha)\, L\, \log\langle T\rangle \sum_{k\ge 1} 2^{-k} \ + \ a_2(\alpha)\, \log\langle T\rangle \sum_{k\ge 1} 4^{-k} \ \ll \ L\, \log\langle T\rangle \ + \ \log\langle T\rangle.$$

On Whitney scale $L = c/\log\langle T\rangle$ this is $\ll \log\langle T\rangle$. Adding the neutralized near-field $O(|I|)$ and the smooth A contribution, we obtain

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \, dt\, d\sigma \ \le \ C_\xi\, |I|,$$

with C_ξ depending on α and growing at most like $O(\log\langle T\rangle)$ on Whitney scale. This proves the lemma. □

Proposition 4 (Whitney box energy growth for U_ξ (unconditional))**.** *Fix $\alpha \in [1,2]$ and a Whitney parameter $c \in (0,1]$. For each Whitney base interval $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$ one has*

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \, dt\, d\sigma \ \ll_{\alpha,c} \ |I| \, \log\langle T\rangle.$$

Proof. This is exactly Lemma 3. □

Definition 5 (Scale-free Whitney box-energy bound for U_ξ)**.** Fix $\alpha \in [1,2]$ and a Whitney parameter $c \in (0,1]$. We say that U_ξ satisfies a *scale-free Whitney box-energy bound* (at (α, c)) if there exists a finite constant $K_\xi = K_\xi(\alpha, c) < \infty$ such that for every Whitney base interval $I = [T-L, T+L]$ with $L = c/\log\langle T\rangle$,

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2 \, \sigma \, dt\, d\sigma \ \le \ K_\xi\, |I|.$$

Remark 6 (Number-theoretic interpretation)*.* The scale-free Whitney box-energy bound in Definition 5 is not merely a technical Carleson constant: it is a scale-free prohibition on extreme microscopic zero clustering at scale $H \asymp 1/\log\langle T\rangle$. One clean way to pin this to a classical object is via the $S(T)$ term in the Riemann–von Mangoldt formula: Lemma 7 shows that a uniform weighted short-interval variation bound for S at Whitney scale implies Definition 5.

Lemma 7 (Whitney ξ-energy reduces to weighted short-interval variation of S)**.** *Fix $\alpha \in [1,2]$ and $c \in (0,1]$. Let $T \ge 3$, set $L := c/\log\langle T\rangle$, and let $I := [T-L, T+L]$. Define*

$$U_\xi(\sigma, t) := \Re \log \xi\Big(\tfrac{1}{2} + \sigma + it\Big) \qquad (\sigma > 0),$$

and write the Riemann–von Mangoldt decomposition

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(1/T), \qquad S(T) := \frac{1}{\pi} \operatorname{Arg} \zeta\Big(\tfrac{1}{2} + iT\Big).$$

Then there exists a constant $C_{\alpha,c} < \infty$ such that

$$\iint_{Q(\alpha I)} |\nabla U_\xi(\sigma,t)|^2\,\sigma\,dt\,d\sigma \;\le\; C_{\alpha,c}\,|I|\Bigg(1+\sum_{k\ge 1} 4^{-k}\,\big|S(T+2^kL)-S(T-2^kL)\big|\Bigg).$$

In particular, a uniform bound

$$\sup_{T\ge 3}\;\sum_{k\ge 1} 4^{-k}\,\big|S(T+2^kL)-S(T-2^kL)\big| \;<\; \infty \qquad \Big(L=\frac{c}{\log\langle T\rangle}\Big)$$

implies the scale-free Whitney box-energy bound of Definition 5 for some finite $K_\xi = K_\xi(\alpha,c)$ independent of T.

Proof. This is a sharpened version of the annular bookkeeping in Lemma 3. As there, neutralize zeros of ξ in a fixed dilate $Q(\alpha' I)$ by a local half-plane Blaschke product and reduce to bounding the neutralized energy on $Q(\alpha I)$; this changes the energy by at most $O_\alpha(|I|)$.

Decompose the (neutralized) zeros by dyadic annuli in ordinate,

$$\mathcal{A}_k := \{\rho=\beta+i\gamma:\; 2^kL < |\gamma-T| \le 2^{k+1}L\}, \qquad \nu_k := \#\mathcal{A}_k,$$

and let $V_k(\sigma,t) := \sum_{\rho\in\mathcal{A}_k} K_\sigma(t-\gamma)$ with $K_\sigma(x) := \sigma/(x^2+\sigma^2)$. Arguing exactly as in the proof of Lemma 3 and using Lemma 2,

$$\iint_{Q(\alpha I)} |\nabla U_\xi|^2\,\sigma\,dt\,d\sigma \;\ll_\alpha\; |I|\Big(1+\sum_{k\ge 1} 4^{-k}\,\nu_k\Big).$$

To control ν_k, note that trivially

$$\nu_k \;\le\; N(T+2^{k+1}L) - N(T-2^{k+1}L).$$

Using the displayed form of $N(T)$ and Taylor expansion of the main term on $[T-2^{k+1}L,\, T+2^{k+1}L]$, one has

$$N(T+H)-N(T-H) \;=\; \frac{H}{\pi}\log\frac{T}{2\pi} \;+\; O(1+H^2/T) \;+\; \big(S(T+H)-S(T-H)\big)$$

for $T\ge 3$ and $0<H\le T/2$. Applying this with $H=2^{k+1}L$ (and absorbing the finitely many k with $2^{k+1}L > T/2$ into the harmless $O(1)$ term, since their 4^{-k} weights are summable) yields

$$\nu_k \;\ll_c\; 2^kL\log\langle T\rangle \;+\; 1 \;+\; \big|S(T+2^{k+1}L)-S(T-2^{k+1}L)\big|.$$

Insert this into the weighted sum:

$$\sum_{k\ge 1} 4^{-k}\,\nu_k \;\ll_c\; (L\log\langle T\rangle)\sum_{k\ge 1}2^{-k} \;+\; \sum_{k\ge 1}4^{-k} \;+\; \sum_{k\ge 1}4^{-k}\,\big|S(T+2^{k+1}L)-S(T-2^{k+1}L)\big|.$$

Since $L\log\langle T\rangle = c$ and the geometric series converge, the first two terms are $O_c(1)$. Reindexing the last term ($j=k+1$) and absorbing the factor 4 into the constant gives

$$\sum_{k\ge 1}4^{-k}\,\nu_k \;\ll_c\; 1+\sum_{j\ge 1}4^{-j}\,\big|S(T+2^jL)-S(T-2^jL)\big|.$$

Substituting back into the energy bound yields the stated estimate (with constants depending only on (α,c)). $\square$

3 Averaged control of the weighted microscopic S-variation

The scale-free bound of Definition 5 remains open. Nevertheless, one can control the weighted short-interval variation term from Lemma 7 on average using Selberg's mean-square bound for $S(t)$.

Lemma 8 (Selberg mean-square bound for S)**.** *For every $X \geq 3$ one has*

$$\int_0^X S(t)^2 \, dt \ \ll \ X \log\log X.$$

Proof. This is classical (Selberg); see, e.g., [2]. □

Theorem 9 (Dyadic mean-square bound for the weighted S-variation)**.** *Fix $c \in (0, 1]$ and let $T \geq 3$. Set $L := c/\log\langle T\rangle$ and define, for $t \in [T, 2T]$,*

$$V_T(t) \ := \ \sum_{k\geq 1} 4^{-k} \left| S(t + 2^k L) - S(t - 2^k L) \right|.$$

Then

$$\frac{1}{T} \int_T^{2T} V_T(t)^2 \, dt \ \ll \ \log\log T.$$

In particular,

$$\frac{1}{T} \int_T^{2T} V_T(t) \, dt \ \ll \ \sqrt{\log\log T},$$

and for any $A > 0$,

$$\left| \{ t \in [T, 2T] : \ V_T(t) > A\sqrt{\log\log T} \} \right| \ \ll \ A^{-2}\, T.$$

Proof. Let $\Delta_k(t) := S(t + 2^k L) - S(t - 2^k L)$. By Cauchy–Schwarz with weights $w_k := 4^{-k}$,

$$V_T(t)^2 \ \leq \ \Big(\sum_{k\geq 1} w_k \Big) \sum_{k\geq 1} w_k \, |\Delta_k(t)|^2.$$

Using $|a - b|^2 \leq 2(a^2 + b^2)$, we have

$$|\Delta_k(t)|^2 \ \leq \ 2\Big(S(t + 2^k L)^2 + S(t - 2^k L)^2 \Big).$$

Integrating $t \in [T, 2T]$ and changing variables yields

$$\int_T^{2T} |\Delta_k(t)|^2 \, dt \ \leq \ 2 \int_{T-2^k L}^{2T+2^k L} S(u)^2 \, du \ + \ 2 \int_{T-2^k L}^{2T+2^k L} S(u)^2 \, du \ \leq \ 4 \int_0^{4T} S(u)^2 \, du.$$

Therefore

$$\int_T^{2T} V_T(t)^2 \, dt \ \ll \ \Big(\sum_{k\geq 1} 4^{-k} \Big) \sum_{k\geq 1} 4^{-k} \int_0^{4T} S(u)^2 \, du \ \ll \ \int_0^{4T} S(u)^2 \, du \ \ll \ T \log\log T$$

by Lemma 8. The L^1 bound follows from Cauchy–Schwarz, and the large-deviation bound from Chebyshev. □

Remark 10. Theorem 9 gives a density-one control at the scale $\sqrt{\log\log T}$, but it does not imply the scale-free bound of Definition 5; it is compatible with rare microscopic windows where $V_T(t)$ is large.

Remark 11 (Bottleneck / open problem). An open problem is to upgrade the Whitney-scale estimate of Proposition 4 to a *scale-free* bound uniform in T, i.e. to prove that U_ξ satisfies Definition 5. By Lemma 7, this would follow from a uniform (in T) bound on the weighted short-interval variation term

$$\sum_{k\geq 1} 4^{-k} \left|S(T+2^k L) - S(T-2^k L)\right| \qquad \Big(L = \frac{c}{\log\langle T\rangle}\Big),$$

or by any equivalent analytic input strong enough to replace Proposition 4 by a scale-free $O(|I|)$ estimate at Whitney scale.

References

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