Problem 1.5 - Concentration of a dye. Consider the flow about a stagnation point,

$$\mathbf{u} = [\alpha x, -\alpha y],$$

where α is a positive constant. Suppose a coloured dye (food colouring, for example) is introduced into the fluid, and its concentration is given by:

$$c(x, y, t) = \beta x^2 y e^{-\alpha t},$$

for $y > 0$, and where β is a constant. Does the dye concentration for any particular fluid element change with time?

Solution by Simon Theriault:
To find how the concentration changes for any particular fluid element, we must assume that our fluid element moves with the flow. Thus, we must use the **total derivative**:

$$\frac{Df}{Dt} = (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \frac{\partial f}{\partial t}$$

Where our function f is our concentration of die in our fluid element, c. Expanding out the dot product, we get:

$$\frac{Df}{Dt} = u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} + \frac{\partial c}{\partial t}$$

Subbing in our function $c(x, y, t)$ and taking the derivatives, we get:

$$\frac{Df}{Dt} = (\alpha x)(2\beta x y e^{-\alpha t}) + (-\alpha y)(\beta x^2 e^{-\alpha t}) + 0 + (-\alpha\beta x^2 y e^{-\alpha t})$$

$$\frac{Df}{Dt} = 2\alpha\beta x^2 y e^{-\alpha t} - 2\alpha\beta x^2 y e^{-\alpha t} = 0$$

Therefore, our concentration of dye doesn't change as our fluid element flows with the fluid.