Problem 1.6 – Acceleration of a rotating fluid.. A fluid in uniform rotation with angular velocity Ω is given by

$$\mathbf{u} = [-\Omega y, \Omega x, 0].$$

Create a vector plot of the fluid flow, and then calculate the acceleration of the fluid and show that it can be written as

$$-\Omega^2 \mathbf{r}.$$

Is this acceleration what you expect?

Solution by Michelle Denny

We can plot the fluid velocity in python, taking $\Omega = 1$ for simplicity. Note that as we move towards the centre, the fluid becomes stationary indicative of solid body rotation (i.e. Outer edges move faster than the middle).

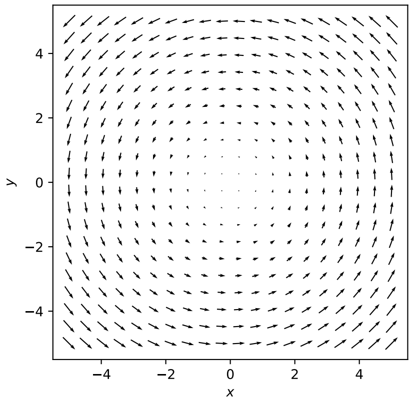

Figure 1: Vector plot of the fluid velocity, $\mathbf{u}$ for $\Omega = 1$.

To calculate the acceleration, we take the material derivative of $\mathbf{u}$, given by:

$$\boxed{\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}} \tag{1}$$

From the velocity equation we can ascertain that it is a 2D steady flow. Hence, $\partial\mathbf{u}/\partial t = 0$ and equation 1 becomes:

$$\frac{D\mathbf{u}}{Dt} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\mathbf{u}.$$

Solving the respective components gives us:

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\Omega^2 x,$$

and

$$\frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\Omega^2 y.$$

So the acceleration is

$$\frac{D\mathbf{u}}{Dt} = -\Omega^2 x\hat{\mathbf{x}} + -\Omega^2 y\hat{\mathbf{y}} = -\Omega^2[x, y] = -\Omega^2\mathbf{r}.$$

where,

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}.$$
