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Y. Brechet and Z. Néda

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Y. Brechet
Laboratoire de Thermodynamique et Physico-Chimie Metallurgique ENSEEG/INPG, Domaine Universitaire de Grenoble, BP75 38402 Saint Martin d'Heres Cedex, France

Z. Néda
Laboratoire de Thermodynamique et Physico-Chimie Metallurgique ENSEEG/INPG, Domaine Universitaire de Grenoble, BP75 38402 Saint Martin d'Heres Cedex, France and Babeş-Bolyai University, Department of Physics, str. Kogalniceanu 1, RO-3400 Cluj-Napoca, Romania

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We study both experimentally and theoretically the classical problem of the circular hydraulic jump. By means of elementary hydrodynamics we investigate the scaling laws governing the position of the hydraulic jump and compare our predictions with experimental data. The results of our simple model are in good agreement with the experiments and with more elaborate approaches. The problem can be effectively used for educational purposes, being appropriate both for experimental investigations and for theoretical application of many fluid mechanics concepts.

I. INTRODUCTION

The hydraulic jump (HJ) following the impact of a liquid jet on a plate is a very common example of a free boundary problem in fluid mechanics: It is observable in everyday life as one opens a water tap into a sink (see Fig. 1). When the vertical jet hits the horizontal plate, it first spreads out radially into a thin layer. At some distance from the jet, however, the height of liquid suddenly jumps to a higher value. The position of this jump is dependent mainly on the rate of water flow from the tap. The existence of the hydraulic jump and the well-defined conditions which must be met at the jump are discussed in many fluid dynamics textbooks.[1,2] On the other hand, the physics underlying the radial location of the HJ is less well known and is seldom, if ever, discussed in textbooks. The complex details of flow in the neighborhood of the jump remain a topic of active study.

The phenomenon presents interest not only from a purely scientific viewpoint, but also for technical applications. The turbulence accompanying the hydraulic jump can be used effectively for mixing fluids or for oxygenating water. The energy dissipation in the neighborhood of the jump may be useful for reducing the kinetic energy of the flow.

The phenomenon presents many interesting aspects one can investigate both experimentally and theoretically:

- the flow profile in the laminar flow region, before the circular HJ occurs
- the flow in the turbulent region of the HJ
- the mechanism which causes the jump
- the energy dissipation in the region of the HJ
- the scaling of the radius R of the circular HJ, as a function of:

(i) the impact speed of the jet
(ii) volume flow rate of the jet
(iii) density and viscosity of the liquid
(iv) the boundary conditions governed by the size and geometry of the plate

- the standing ripples of the free surface around the HJ, and their dependence on the surface tension of the liquid.

It seems that Lord Rayleigh[3] was the first to attempt a solution of these problems in his theory of shallow water flows. Since then, a considerable amount of work has been devoted to this question both from experimental and theoretical viewpoints.

Many experimental studies of the HJ have been conducted.[4–13] Olsson *et al.*[4] investigated the flow profile in the laminar flow regime, the speed at the free surface of the fluid as a function of the distance from the incidence point of the jet, and the radius of the HJ as a function of the characteristic Reynolds number of the flow. Ishigai *et al.*[5] considered the problem from a thermodynamic viewpoint, studying the dissipated heat in the region of the HJ. Khalifa *et al.*[6] studied the flow in the turbulent regime. Craik *et al.*[7] used a light absorption technique, and investigated the flow in the region of the HJ. They also analyzed the scaling laws of R as a function of the flow rate and drop height of the incident jet. Liu *et al.*[8] investigated the wavelike instabilities in the region of the HJ as a function of the liquid's surface tension, and Siwon[9] studied the flow in the impact and laminar flow region of the jet. Very recently two new experimental studies were reported: Ellegaard *et al.*[10–12] observed an interesting transition of the HJ in a high viscosity fluid when the depth of the fluid far away from the jet is controlled; Hansen *et al.*[13] measured the power spectra of surface waves generated by an unstable circular jump.

Pure theoretical studies were performed in Refs. 14–22. Kurihara[14] was the first to derive a theoretical scaling law for R as a function of the relevant physical parameters, but his results do not agree with experiments. Khalifa *et al.*[15] and Bowles *et al.*[16] used numerical methods to solve the differential equations describing the phenomenon. Buevich *et al.*[17,18] proposed an analytic approximation, which provides a good qualitative description of the phenomenon. Bohr *et al.*[19] performed analytic calculations, using complex mathematical apparatus, and obtained scaling results for R in very good agreement with experiments. They also elaborated a simple viscous theory[20] of free-surface flows in boundary layers which yields the structure of the stationary HJ. Higuera[21] used a boundary layer approximation and numerical computations to study the flow profile, and obtained results in good agreement with the experimental ones.

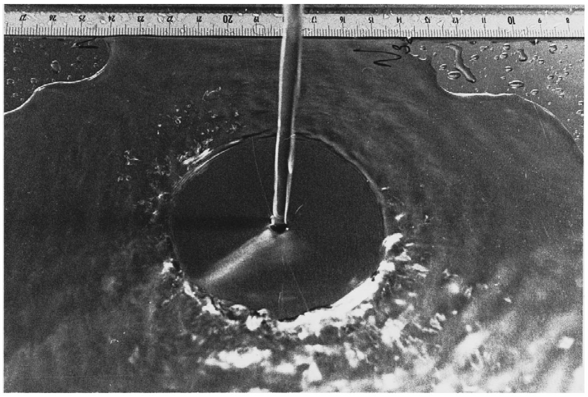

Fig. 1. The circular hydraulic jump.

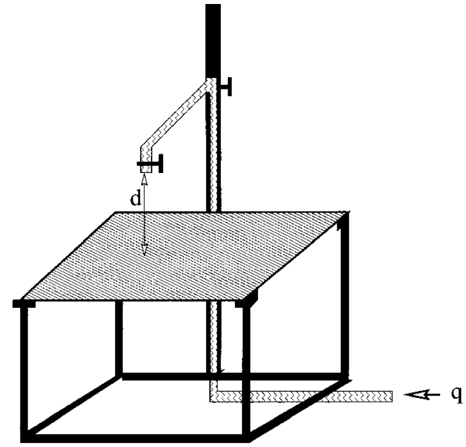

Fig. 2. Schematic experimental setup for studying the influence of the boundary conditions, flow rate (q), and drop height (d) on the radius of the circular hydraulic jump.

Recently[22] he published an asymptotic order-of-magnitude description for the structure of a circular laminar HJ.

There are also some studies which consider the problem both from experimental and theoretical viewpoints.[23–27] Tani[23] determines the scaling of R as a function of the imposed flow rate, but his experimental and theoretical results are not in agreement. Watson[24] approximates analytically all the interesting aspects of the problem, and compares the analytic results with his own experimental data. His approximation describes the flow profile well, but fails to match the scaling of R as a function of flow rate. Nakaryakov[25] studies energy dissipation in the region of the hydraulic jump, getting theoretical results in good agreement with his own experimental data. Godwin[26] studied the scaling of R as a function of the liquid's viscosity, and his theoretical results are in good agreement with the experiments and the one given by Bohr *et al.* In this journal, Blackford[27] recently published a new model for the HJ. The depth profile of the flow was solved by computer simulation. Suitably adjusting two free parameters the theoretical predictions were found to be in good agreement with experimental results. However, the important problem of the scaling law for the radius of the jump as a function of volume flow rate and viscosity was not addressed. We feel that the readers of this journal would benefit from a discussion of this aspect of the problem. Thus the main question we will address in this contribution concerns the scaling laws which govern the position (radius R) of the HJ. Our purpose is not to further refine the technical approaches discussed above, but rather to present an elementary fluid mechanics description, accessible to undergraduate students, which captures the main results of the more complicated methods. We will also consider the problem experimentally using a very simple and universally accessible experimental device. Indeed, this problem provides an excellent introduction for students to fluid-dynamics research. The experiment, though quite simple, has to be done with care. The mathematics of the modeling is straightforward. Moreover, the problem allows one to introduce and test the applicability of many basic principles of fluid mechanics.

II. EXPERIMENTAL SETUP

The experimental setups are very simple and can easily be used as classroom experiments. Qualitative features of the circular HJ can be determined from experiments in any home or laboratory sink in which there is a relatively smooth horizontal basin bottom and in which the faucet can be rotated so the flow column impacts the bottom. For quantitative measurements our basic assumption is in accordance with all previous studies considering valid the

$$R \sim q^{\alpha} d^{\beta} \nu^{\gamma} \tag{1}$$

scaling relation for a given tap with fixed nozzle diameter. Formula (1) means that for a given liquid the volume flow rate, q, and the drop height, d, of the incident jet are the most important parameters governing the R radius of the HJ. On the other hand, the most important liquid parameter is the kinematic viscosity, ν, while the density seems to play no important role. We will also check this hypothesis.

We used two experimental setups. The *first setup* (Fig. 2) was designed to study R as a function of the geometry of the plate, volume flow rate, and the height of the incident jet. Water comes out from a tap into a plastic tube and falls on a Plexiglas™ plate whose height d with respect to the exit of the tube can be varied. The flow rate of water, q, is measured simply as the amount of liquid falling on the plate. It is shown to be constant and can be regulated from the tap. In order not to disturb the hydraulic jump, the plates were large (at least three times the radius of the hydraulic jump) and had free boundaries. The Plexiglas plate was firmly attached to a support in order to prevent any vibrations. Under the plate and protected by a glass plate, a sheet of millimetric paper was inserted, allowing us to measure the radius of the circular hydraulic jump. Measurements for various flow rates were done, repeatedly increasing or decreasing the volume flow rate, without any sign of hysteresis. From this reproducibility we could deduce that the surface state of the plate was not changing during the experiments. The *second experimental setup* (Fig. 3) was used to study the scaling of R as a function of the density ρ and kinematic viscosity ν of the liquid. The liquid in question (methyl alcohol, ammonia, oil, and double-distilled water) is placed in a reservoir and runs through a vertical tube with a tap, to fall on the horizontal Plexiglas plate. The liquid is recollected under the plate. The measurements are made using a convenient level of the liquid in the reservoir and a convenient drop height. The flow rate is previously determined for the chosen configuration. We use the scaling from the first experiment to rescale the

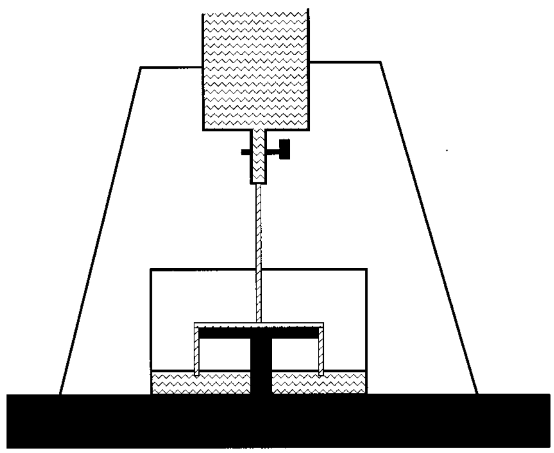

Fig. 3. Schematic experimental setup for studying the influence of the liquid's density and viscosity on the radius of the circular hydraulic jump.

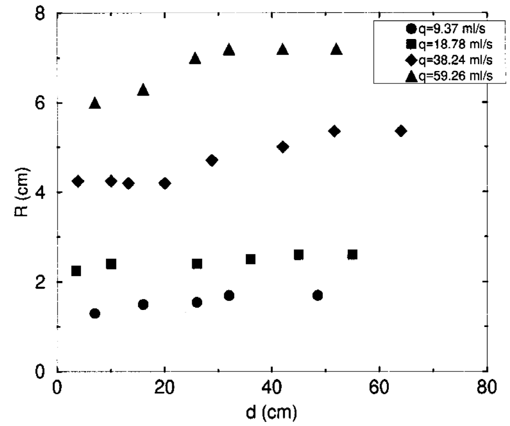

Fig. 4. Experimental results for the variation of the radius R of the HJ as a function of the height of the fall (d). Results for different volume flow rates (q).

jump on the same q and d parameters; thus we can study the effects of viscosity and density. The viscosities and densities of the liquids are taken from physical tables.

III. EXPERIMENTAL RESULTS

The precision of our measurements is influenced mostly by the width of the jump region and is estimated roughly to be 4 mm.

A. Influence of the geometry and size of the plate

The flow rate was fixed at $q=38.24$ ml/s, and the drop height at $d=42$ cm. We considered five different plates. Their characteristic sizes and geometries, together with the obtained radii R of the HJ, are presented in Table I.

The horizontality of the plates, and the constancy of the volume flow rate, was verified for each measurement.

These results suggest that the effect of the size and geometry of the Plexiglas plate is small, when we use open boundaries.

If we used dish-like geometry, however, the height of the fluid layer after the jump would be increased, and the radius of the HJ would be appreciably reduced. As revealed by recent experiments,[10–13] for high viscosity liquids, stationary polygonal patterns form, breaking the axial symmetry. We conclude thus, that the boundary conditions on a perfectly flat plate do not influence the radius of the HJ, but vertical obstacles imposed on the downstream flow will reduce the value of R and in some conditions also the circular aspect of the HJ can be destroyed.

Table I. Geometry and sizes of the considered Plexiglas plates.

Geometry of the plate	Characteristic size (cm)	R (cm)
Square	15	5.1
Square	20	5.0
Square	30	5.2
Disc	10 (radius)	5.1
Equilateral triangle	30	4.9

B. Influence of the volume flow rate and height of the fall

Figures 4 and 5 show the influence of the volume flow rate, q, and the drop height, d, on the position of the hydraulic jump.

From Figs. 4 and 5 we see that the radius of the hydraulic jump is an increasing function of q. It seems also to increase slightly with increasing drop height, but no definite conclusion concerning this latter dependence can be made due to the experimental scatter. Neglecting the dependence on d, and fitting together all experimental results as a function of q (Fig. 5), we obtain the scaling law:

$$R \sim q^{0.703}. \tag{2}$$

C. Influence of liquid density and viscosity

In order to study this dependence we used four different liquids: double-distilled water, methyl alcohol, automobile engine oil (Shell), and ammonia. The relative dynamic viscosity (μ_r), relative density (ρ_r), and relative kinematic viscosity (ν_r), of these liquids, at room temperature and with respect to the water, are presented in Table II.

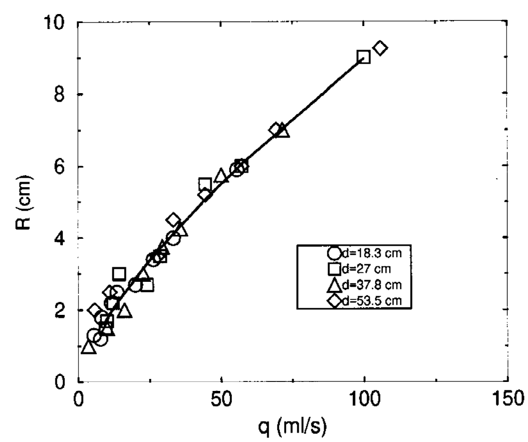

Fig. 5. Experimental results for the variation of the radius R of the HJ as a function of the volume flow rate (q). Results for different falling heights (d). The best fit scaling index [considering all the experimental points for different (d) values] is 0.703. The fit is indicated by the continuous curve.

Table II. The liquids used, their relative dynamic viscosity (μ_r), relative density (ρ_r), and relative kinematic viscosity (ν_r).

Liquid	μ_r	ρ_r	ν_r
Oil	880	0.945	931
Water	1	1	1
Alcohol	0.6	0.8	0.75
Ammonia	0.155	0.617	0.2512

We fixed the drop height at $d=34$ cm, and measured both the flow rate, q, and the radius, R, of the HJ for a given level of the liquid in the reservoir. The results are shown in Table III.

By using Eq. (1), we rescaled all the results to a flow rate of $q_0=60$ ml/s. The dependence of R on ν_r is plotted in Fig. 6.

As can be seen from Fig. 6, the dependence of R on the viscosity ν_r can be nicely fitted by the power law

$$R \sim \nu_r^{-0.295}. \tag{3}$$

The dependence on density does not show a monotonic behavior and no scaling law could be determined.

IV. THEORETICAL APPROACH

Many attempts have been made in the literature to derive the scaling laws for the radius R of the circular hydraulic jump as a function of the physical parameters governing the phenomenon.[14–19,21–26]

Due to the fact that most current theoretical works are very technical, and thus inaccessible to nonspecialists, one aim of our paper is to approach the phenomenon on the level of elementary fluid dynamics.

In the following we will present some simple theoretical approaches to the phenomenon. First we will present the theory of the circular hydraulic jump for ideal fluids, and then consider the real problem of viscous fluids.

Although our derivation will remain mostly on the level of elementary hydrodynamics, our results are in good agreement with all the accepted ones.

A. Ideal fluids

Our notation is outlined in Fig. 7: q is the volume flow rate, u the speed, and a the diameter of the jet at the contact point with the horizontal plate, h, v, and H, V the height and speed of the liquid layer just before and after the hydraulic jump, respectively. We denote the density of the liquid by ρ, the gravitational acceleration by g, and the radius of the hydraulic jump by R.

With ideal (nonviscous) fluids, we will have no velocity gradient inside the liquid layer in the vertical direction. In the neighborhood of R, if we take two cylindrical sections of the flow (one before and the other right after the jump), the condition for conservation of momentum can be written as

$$\frac{dp}{dt}=2\pi R\rho H V^2-2\pi R\rho h v^2=F_1-F_2, \tag{4}$$

where

$$F_1=2\pi R\rho g\int_0^h x\,dx=\rho g\pi R h^2,$$
$$F_2=2\pi R\rho g\int_0^H x\,dx=\rho g\pi R H^2. \tag{5}$$

We make the simplifying assumption

$$h \ll H, \tag{6}$$

which by the continuity equation

$$2\pi R v h=2\pi R V H, \tag{7}$$

leads us to

$$V \ll v. \tag{8}$$

Accepting (6) and (8), momentum conservation (4) gives

$$\rho v^2 h \approx \tfrac{1}{2}\rho g H^2. \tag{9}$$

Table III. Measured volume flow rate q and radius R of the HJ for the considered liquids. ($d=34$ cm).

Liquid	q (ml/s)	R (cm)
Oil	57	1
Water	60	7.4
Alcohol	48	7.1
Ammonia	37	8.8

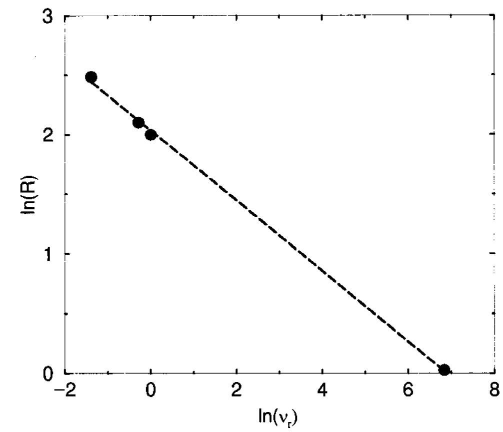

Fig. 6. Experimental results for the variation of the radius R of the HJ as a function of viscosity shown on a log–log plot. The best fit line indicates the scaling index -0.295.

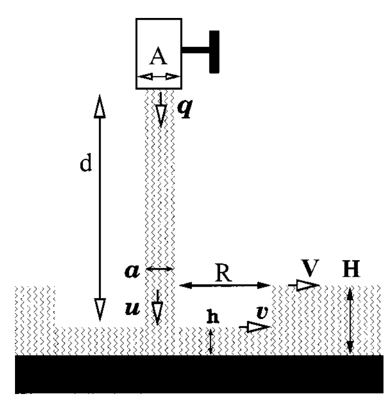

Fig. 7. Notation for the ideal fluid approximation.

An elementary application of Bernoulli's theorem shows that the radial velocity of the liquid before the hydraulic jump is uniformly equal to the velocity $u=\sqrt{2gd}$ of the incident jet:

$$v=u. \tag{10}$$

From (10) and the mass conservation equations,

$$2\pi R v h=\frac{\pi}{4}a^2u=q, \tag{11}$$

we get

$$v=\frac{4q}{\pi a^2}, \tag{12}$$

$$h=\frac{q}{2\pi R v}=\frac{a^2}{8R}. \tag{13}$$

Inserting these in the approximate relation (9), we get the radius of the hydraulic jump:

$$R=\frac{4q^2}{\pi^2 a^2 g H^2}. \tag{14}$$

The diameter a is determined by the volume flow rate, q, the nozzle diameter, A, of the tap, and the drop height, d. By using the continuity equation inside the vertical jet and considering the acceleration of the liquid in the jet as g, we get

$$a=\left(\frac{\pi^2 g d}{8q^2}+\frac{1}{A^4}\right)^{-1/4}. \tag{15}$$

We should mention here that for real viscous fluids the above formula has proved to be inadequate,[7] but it is justified in our nonviscous fluid approximation. For small heights and high volume flow rates ($gd/q^2 \ll 1/A^4$), the $a \approx A$ approximation is sufficient; however, this limit is not always applicable in typical experimental conditions. Accepting formula (15) with the above remarks, finally we get

$$R=\frac{4q^2\sqrt{\dfrac{\pi^2 g d}{8q^2}+\dfrac{1}{A^4}}}{\pi^2 g H^2}. \tag{16}$$

This very first theoretical description already gives interesting results. We see that the radius of the jump is strongly dependent on the jet's volume flow rate, q, and the imposed downstream boundary conditions which influence H. The radius depends slightly on the height d and it is independent of the density of the liquid.

Neglecting the viscosity is of course a rough approximation. In the following, by means of elementary fluid dynamics we will try to approach also the realistic case of viscous liquids.

B. Viscous fluids

We will present three approaches to the problem, of different levels of complexity, all leading to the same scaling law of R as a function of the relevant physical parameters. The physical picture underlying our approximation is the one proposed by Godwin,[26] and it is based on the important role played by the viscous boundary layer.[28] What happens at small r is that the jet spreads sideways with a velocity:

$$v_0=\sqrt{2gd} \tag{17}$$

in a laminar fashion as though it were nonviscous, except for a boundary layer near the surface of the plate. The thickness of this boundary layer can be shown to be approximately

$$\delta=\sqrt{\frac{\nu r}{v_0}}=\sqrt{\frac{\nu r}{\sqrt{2gd}}}. \tag{18}$$

The flow inside the boundary layer can be described as a viscous laminar flow, and the flow above it as a laminar flow with velocity v_0. Godwin[26] conjectured that the HJ occurs when the boundary layer reaches the total height of the fluid film and obtained good agreement with experiments for the viscosity dependence of the jump radius.

Our first two approaches are based on Godwin's conjecture. We will show that not only the right viscosity dependence, but also a good approximation of the dependence on volume flow rate, can be achieved. In our last approach we will go even further and show that one can derive the same scaling law by analyzing the stability of the flow after the boundary layer is fully developed (i.e., reaches the fluid film surface). Using an analogy with flow in a diverging channel, we identify the HJ with the instability that can appear in such cases.

1. First approach

This theory is a very simple one and it is a first generalization of formula (14) to viscous flows. Making the approximation that the boundary layer is a very small part of the liquid film thickness right upto the HJ, we use the nonviscous result and formula (9), writing

$$R=\frac{4q^2}{2\pi^2 a^2 \bar{v} h}, \tag{19}$$

where $\bar{v}$ is the average fluid velocity at the given radius. We have $\bar{v} \approx v_0$ close to the impact point of the jet, and $\bar{v}$ is always smaller than v_0 as the boundary layer develops. In particular, at the site of the jump we will always have $\bar{v} < v_0$, and replacing $\bar{v}$ by v_0, (19) will give us a lower bound for R:

$$R>\frac{4q^2}{2\pi^2 v_0^2 h}. \tag{20}$$

This is an inequality, but one can still hope that the scaling relationship we are looking for will be preserved. Using Eq. (12) we can now write:

$$R>\frac{q^2}{\dfrac{\pi r^2}{4}2\pi v h}=\frac{q}{2\pi v h}. \tag{21}$$

Following Godwin's conjecture about the position of the HJ,

$$h=\sqrt{\frac{\nu R}{v_0}}, \tag{22}$$

we get immediately

$$R>\left(\frac{g^{-1/4}}{2^{5/4}\pi}\right)^{2/3} q^{2/3} d^{-1/6} \nu^{-1/3}. \tag{23}$$

Assuming that this inequality has the same scaling as the original equality, we have

$$R \sim q^{2/3} d^{-1/6} \nu^{-1/3}. \tag{24}$$

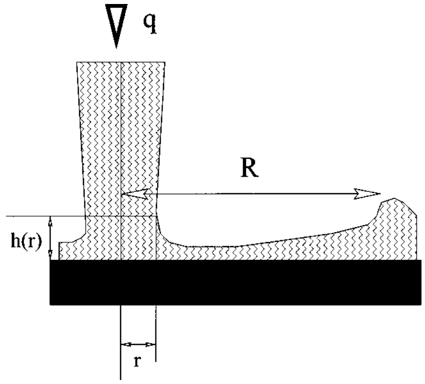

Fig. 8. Notation for the viscous fluid approximation.

2. Second approach

This is a more accurate approximation, although the hydrodynamics involved is still elementary. It is also based on Godwin's conjecture and gives the same scaling law for the position of the jump as our previous approximation. In contrast with the first approach considered, we now get an equality for R, which is more useful than the lower bond (23).

The outline of our method is the following:

- We determine the velocity profile $v(x,h)$, and flow profile $h(r)$, in a viscous laminar flow where the boundary layer is fully developed. (The velocity profile will be needed in order to determine the flow profile).
- We use Godwin's conjecture and identify R from the $h(R)=\delta$ equality, presuming that the shape of the boundary layer evolution is continuous at R.

a. Velocity profile in a viscous free surface laminar flow with cylindrical symmetry. The problem has cylindrical symmetry and thus the current position in the plane of impact will be given by the distance r to the impact point. The notation used in the model is shown in Fig. 8. We will denote the velocity profile by $v(x)$, the height of liquid at a distance r from the impact position by $h(r)$, and the volume flow rate by q.

For a real fluid the velocity profile of a free-surface flow is generally self-similar, i.e., satisfying the condition:

$$v(x)=v(h)f\left(\frac{x}{h}\right). \tag{25}$$

In the above equation the value of h can be r dependent, but the form of the function f is independent of r. The velocity profile must also satisfy the boundary conditions:

$$v(0)=0, \tag{26}$$

$$\left.\frac{dv}{dx}\right|_h=0. \tag{27}$$

There are of course many functions satisfying the conditions (25)–(27), and we will consider in the following the simplest possible one, which is a second-order polynomial in the variable x/h:

$$v(x)=v(h)\left[2\frac{x}{h}-\left(\frac{x}{h}\right)^2\right]. \tag{28}$$

The flow speed at the free surface $v(h)$ can be found by inserting (28) in the continuity equation

$$q=2\pi r\int_0^h v(x)dx, \tag{29}$$

to yield:

$$v(x)=\frac{3q}{4\pi r h^3}(2hx-x^2). \tag{30}$$

b. Flow profile in a viscous free-surface laminar flow with cylindrical symmetry. In order to get the flow profile $h(r)$, we now study the energy balance. The energy dissipation in a volume element situated between r and $r+dr$ with height between x and $x+dx$ can be expressed both as the dissipation associated with viscosity and as the divergence of the kinetic energy flux J. The energy dissipation rate in our volume element can be written as

$$d^2W_d=(2\pi\mu)r\left(\frac{\partial v}{\partial x}\right)^2 dx\,dr. \tag{31}$$

The kinetic energy flux flowing into our volume element is given by

$$dJ=\pi\rho r v(x)^2 v(x)dx. \tag{32}$$

Integrating the quantities in Eqs. (31) and (32) gives the dissipation rate dW_d between r and $r+dr$, and the kinetic energy flux $J(r)$ through a cylindrical surface of height $h(r)$ and radius r, respectively,

$$dW_d=(2\pi\mu)r\,dr\int_0^h\left(\frac{dv}{dx}\right)^2dx=\frac{3}{2\pi}\frac{\mu q^2}{h^3 r}dr, \tag{33}$$

$$J=\int_0^h dJ=\frac{27}{140\pi^2}\frac{\rho q^3}{h^2 r^2}. \tag{34}$$

The energy balance relation is $J(r+dr)-J(r)+dW_d=0$, which leads to

$$\frac{dJ}{dr}+\frac{dW_d}{dr}=0. \tag{35}$$

Performing the differentiation in (35) leads to the analytically solvable differential equation

$$\frac{h}{r}\left(\frac{h'}{h}+\frac{1}{r}\right)=b, \tag{36}$$

where

$$b=\frac{35\pi\mu}{9\rho q}. \tag{37}$$

The solution of the above equation is

$$h(r)=\frac{b}{3}r^2+\frac{C}{r}, \tag{38}$$

with C an integration constant.

Let us underline two important facts:

- As we already mentioned, the quadratic form of the velocity profile was chosen somewhat arbitrarily. There are many possible functions satisfying the imposed boundary conditions and Eq. (25). One can just presume that the chosen form approximates the reality. If we use the same (26)–(27) assumption for f and now make use of the momentum conservation principle to determine the flow profile we would get the same differential equation (35), but the coefficient b would differ by a factor of 54/35 from the one obtained by our energetic arguments. This difference is a sign that the function f is not really parabolic; however we think it is an acceptable first approximation.

- In writing the energy balance (35), we have neglected both the gravitational potential energy and the surface tension contributions. Neglecting the surface tension in the laminar flow regime is acceptable due to the fact that the height of the fluid layer is much smaller than the average radius of curvature of the free surface. Were we to consider details in the free surface as the standing ripples around the hydraulic jump, we would have certainly to take into account those capillarity effects. As far as gravitational effects are concerned, they are certainly negligible in the laminar flow regime since the height of the liquid layer varies slowly. However, those effects are likely to be important both close to the impact point, and to determine the height and shape of the hydraulic jump.

c. The radius of the HJ. As we already emphasized, the jump radius will be obtained from the $h(R)=\delta(R)$ equality. A problem which arises here is the form (38) of $h(R)$ which contains an undermined integration constant: C. However, in the jump region the second term in (38) is supposed to be smaller than the first one, and as a first approximation it might be neglected. One thus gets

$$\frac{b}{3}R^2=\sqrt{\frac{R\nu}{\sqrt{2gd}}}, \tag{39}$$

which leads to

$$R=\left(\frac{27g^{-1/4}}{2^{1/4}35\pi}\right)^{2/3}q^{2/3}d^{-1/6}\nu^{-1/3}. \tag{40}$$

The scaling is in agreement with the previous approach, and we obtain a good numerical estimate for the radius of the jump as well (see the discussion part).

3. Third approach

The drawbacks of the previous two approaches were that we used an unproved conjecture regarding the condition for the jump. With this third approach, we intend to determine the scaling of R as a function of d, g, ν, and q from basic principles of hydrodynamics, without using Godwin's conjecture. We will only presume that the HJ occurs after the boundary layer reaches the free surface. We use the flow profile given by Eq. (38). Because we assumed that the HJ is after the point (r_0) where the boundary layer is fully developed, the constant C in (38) can be determined by imposing smooth evolution of the boundary layer at r_0. This means that both the $h(r)$ and $\delta(r)$ functions and their derivatives must match at r_0:

$$\delta(r_0)=\sqrt{\frac{\nu r_0}{\sqrt{2gd}}}=\frac{b}{3}r_0^2+\frac{C}{r_0}=h(r_0), \tag{41}$$

$$\frac{d\delta(r_0)}{dr}=\frac{1}{2}\sqrt{\frac{\nu}{r_0\sqrt{2gd}}}=\frac{2b}{3}r_0-\frac{C}{r_0^2}=\frac{dh(r_0)}{dr}. \tag{42}$$

It is straightforward to determine the values of r_0 and C:

$$C=\frac{27}{140\pi\sqrt{2}}qg^{-1/2}d^{-1/2}, \tag{43}$$

$$r_0=\left(\frac{27g^{-1/4}}{2^{5/4}35\pi}\right)^{2/3}q^{2/3}d^{-1/6}\nu^{-1/3}. \tag{44}$$

Knowing C, we have the flow profile (38), which appears to be like the one in a convergent, then divergent, channel. When the channel is convergent, the flow is stable. When the channel is diverging, instabilities can appear. Given that in the region where we would expect the hydraulic jump the flow profile is divergent, it seems reasonable to think that the appearance of these instabilities corresponds to the hydraulic jump.

It is known from the literature[29] that for a flow in a diverging channel instabilities might appear when the Reynolds number, Re, exceeds a critical value, $\mathrm{Re}_{\max}(\theta)$, which is dependent on the θ angle of the diverging channel. In the limit $\theta\rightarrow 0$, $\mathrm{Re}_{\max}=K/\theta$ (K a real number), and the condition for getting instabilities becomes:

$$\mathrm{Re}\,\theta>K. \tag{45}$$

The K number is known for a flow between two diverging plates,[29] but this value should certainly not be used in our free surface flow (which looks like half of the flow between the two plates) with cylindrical symmetry. We will thus treat K as an undetermined number.

The use of the $\theta\rightarrow 0$ limit is totally justified since the flow profile is very slowly varying in the laminar flow regime.

Adapting this criterion to our situation:

$$\mathrm{Re}=\frac{h\langle v\rangle}{\nu}=\frac{q}{2\pi r\nu}, \tag{46}$$

$$\theta\approx tg(\theta)=\frac{dh}{dr}=\frac{2br}{3}-\frac{C}{r^2}, \tag{47}$$

leads to the condition:

$$\frac{q}{2\pi r\nu}\left(\frac{2br}{3}-\frac{C}{r^2}\right)>K. \tag{48}$$

The radius of the hydraulic jump can now be identified as the smallest value of r satisfying the condition (48), and we get the equation

$$\frac{35}{27}-\frac{qC}{2\pi\nu R^3}=K \tag{49}$$

for the radius, R, of the hydraulic jump. In the above equation, K being the only undetermined constant, using the value (43) for C we will get the scaling law

$$R\sim q^{2/3}d^{-1/6}\nu^{-1/3}. \tag{50}$$

This is exactly the scaling law obtained in our previous approximations. One can also realize that the r_0 radius (44) has the same scaling properties, which explains why Godwin's conjecture works well.

V. DISCUSSION

We now compare our experimental and theoretical results, and discuss them in connection with earlier results in the literature.

Our experimental results are in good agreement with those in the literature. The scaling law for R as a function of the flow rate (2), the weak dependence as a function of the drop height, d, and the independence of R on the geometry of the horizontal plate, confirm the results of Watson[24] and Craik

Table IV. The scaling exponents for the radius of the HJ obtained by our approximations, the ones accepted in the literature, and the experimentally obtained ones.

Physical quantity	Exponent	Our theory	Bohr *et al.*	Godwin	Our experiment
Flow rate (q)	α	2/3	5/8	$\cdots$	0.703
Height (d)	β	$-1/6$	0	$\cdots$	≈ 0
Viscosity (ν)	γ	$-1/3$	$-3/8$	$-1/3$	-0.295

et al.[7] The scaling of the jump radius as a function of the viscosity is in agreement with the results obtained by Godwin.[26]

The theoretical results (16) obtained using the ideal fluid approximation are only qualitatively confirmed by experiments. The strong dependence of R on the flow rate, q, and the weak variation as a function of d are in qualitative agreement with our observations. However, the observed scaling of R as a function of q (1) is different from the one indicated in (16), which would suggest roughly a quadratic law. The dependence of R on H, which is governed by the boundary conditions imposed on the downstream flow, are partially confirmed by the experiments. The size and geometry of the plate do not influence the radius of the HJ, but vertical obstacles imposed on the downstream flow (which can increase H considerably) will reduce the value of R.

Theoretical results obtained by considering real, viscous liquids lead to the same scaling law for R (50) in all of the three approaches we used. These results are in good agreement with both our experimental results and the results of more refined approximations (Bohr *et al.*, Ref. 19). In Table IV we summarize the scaling exponents obtained with our approximations, the ones given in the literature, and the experimentally obtained ones.

Formula (40), which gives the value of R explicitly, is an acceptable approximation. For example with volume flow rate $q = 50\,\text{ml/s}$ and drop height $d = 37.8\,\text{cm}$, (40) gives $R \approx 4$ cm, a fairly reasonable estimate of the measured value of $R \approx 5.5\,\text{cm}$.

VI. CONCLUSIONS

The circular hydraulic jump, a long-standing problem in fluid mechanics and an everyday life experiment, can be approached with simple, undergraduate-level fluid mechanics. Of course, in order to do so, some heuristic principles have to be used. The results obtained are in good agreement with the latest and more refined theories. These theoretical results can be verified experimentally with a very simple setup.

It appears that this problem is well suited to inspire the taste for research in students. Further theoretical or experimental questions, that could be used as laboratory or seminar exercises, include determining the:

- shape and height of the jump
- thickness evolution of the fluid film in the laminar flow (e.g., using laser interferometry)
- the influence of substrate (e.g., Teflon™ or paraffin coating of the plate)
- influence of the angle of impact (e.g., inclining the plate)

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A NEW ANGLE ON RELATIVITY THEORY

Homework question. Charlie has just caught a lake trout 20 inches long. Zipping by in her motor boat, the game warden sees the fish as 12 inches long. Uh-oh! The minimum legal length is 16 inches.
(a) How fast was the game warden going?
(b) Will Charlie have to pay a fine? Briefly, why?

Catherine's answers.
(a) $v=0.8c$.
(b) Charlie will not pay a fine because as soon as he is done measuring the fish, he will proudly hold it perpendicular for his wife to take a photograph, where upon the warden will see she was mistaken.

Wesleyan University, Spring 1998.