Problem 1.3 – Vector Calculus Practice. Calculate the divergence and curl of the following vector fields. State which flows are incompressible and which are irrotational (or both).

$$(a)\mathbf{u} = [x^2, 3xz^2, -2xz]$$

$$(b)\mathbf{u} = [xy, 2yz, 3zx]$$

$$(c)\mathbf{u} = [y^2, 2xy + z^2, 2yz]$$

Solution by Joe Ruse

For each of the given vectors $\mathbf{u} = [u, v, w]$, we can find the divergence and curl using the following equations,

Divergence:

$$\nabla \cdot \mathbf{u} = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}] \cdot [u, v, w] = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{1}$$

Curl:

$$\nabla \times \mathbf{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] \tag{2}$$

The second part of the problem asks to identify whether each of the vectors are incompressible or irrotational. A fluid is considered incompressible if the divergence of its flow is zero,

$$\nabla \cdot \mathbf{u} = 0. \tag{3}$$

The curl of a flow is also known as the *vorticity*, $\omega = \nabla \times \mathbf{u}$. If the vorticity of a flow is zero,

$$\nabla \times \mathbf{u} = 0, \tag{4}$$

it is irrotational.

The solution to each of the vectors will follow the same procedure, using (1) and (2) to evaluate the divergence and curl respectively, then using (3) and (4) to check for incompressibilty and irrotationality respectively. A more detailed solution is given for part a, followed by the short versions for b and c.

Part a:

$$\mathbf{u} = [x^2, 3xz^2, -2xz]$$

$$u = x^2, v = 3xz^2, w = -2xz$$

Step 1: Calculate the divergence and check incompressibility

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz)$$

$$= 2x + 0 - 2x$$

$$\nabla \cdot \mathbf{u} = 0$$

Since the divergence is zero, the fluid is incompressible.

Step 2: Calculate the curl and check irrotationality

$$\nabla \times \mathbf{u} = \left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

$$= \left[\frac{\partial}{\partial y}(-2xz) - \frac{\partial}{\partial z}(3xz^2), \frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(-2xz), \frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$\nabla \times \mathbf{u} = [6x, 2z, 3z^2]$$

Since the curl of the flow is not zero, the fluid is not irrotational.

Part b:

$$\mathbf{u} = [xy, 2yz, 3zx]$$

$$u = xy, v = 2yz, w = 3zx$$

Step 1: Calculate the divergence and check incompressibility

$$\nabla \cdot \mathbf{u} = y + 2z + 3x$$

Since the divergence of the flow is not zero, the fluid is compressible.

Step 2: Calculate the curl and check irrotationality

$$\nabla \times \mathbf{u} = [-2y, -3z, -x]$$

Since the curl of the flow is not zero, the fluid is not irrotational.

Part c:

$$\mathbf{u} = [y^2, 2xy + z^2, 2yz]$$

$$u = y^2, v = 2xy + z^2, w = 2yz$$

Step 1: Calculate the divergence and check incompressibility

$$\nabla \cdot \mathbf{u} = 2x + 2y$$

Since the divergence of the flow is not zero, the fluid is compressible.

Step 2: Calculate the curl and check irrotationality

$$\nabla \times \mathbf{u} = [0, 0, 0]$$

Since the curl of the flow is the zero vector, the fluid is irrotational.

Answers

(a)

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \times \mathbf{u} = [6x, 2z, 3z^2]$$

Incompressible and not irrotational.

(b)

$$\nabla \cdot \mathbf{u} = y + 2z + 3x$$

$$\nabla \times \mathbf{u} = [-2y, -3z, -x]$$

Not incompressible and not irrotational.

(c)

$$\nabla \cdot \mathbf{u} = y + 2z + 3x$$

$$\nabla \times \mathbf{u} = [0, 0, 0]$$

Not incompressible and is irrotational.