Problem 1.8 – Vorticity of a line vortex. Show that the vorticity of a line vortex,

$$\mathbf{u} = \frac{k}{s}\hat{\boldsymbol{\phi}},$$

is zero everywhere except at the origin, where it blows up.

Solution by Adam Callanan

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$$

The velocity being in the $\hat{\phi}$ direction indicates cylindrical coordinates. Thus the curl must be calculated in cylindrical coordinates too as follows

$$\boldsymbol{\omega} = \frac{1}{s}(\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z})\hat{\mathbf{s}} - (\frac{\partial u_z}{\partial s} - \frac{\partial u_s}{\partial z})\hat{\boldsymbol{\phi}} + \frac{1}{s}(\frac{\partial (s u_\phi)}{\partial s} - \frac{\partial u_s}{\partial \phi})\hat{\mathbf{z}}.$$

It can be seen at this stage that $\boldsymbol{\omega}$ will blow up as s approaches 0. Specifically,

$$\lim_{s \to 0} \boldsymbol{\omega} = \infty\hat{\boldsymbol{s}} - 0\hat{\boldsymbol{\phi}} + \infty\hat{\boldsymbol{z}}.$$

Continuing the curl calculation for the case $s \neq 0$, by subbing in $\mathbf{u}$, we're left with

$$\boldsymbol{\omega} = \frac{1}{s}(\frac{\partial (s[\frac{k}{s}])}{\partial s})\hat{\mathbf{z}}.$$

Which reduces to

$$\boldsymbol{\omega} = \frac{1}{s}(\frac{\partial k}{\partial s})\hat{\mathbf{z}}$$

but since k is a constant:

$$\boldsymbol{\omega} = 0.$$

$\therefore$ the vorticity of $\mathbf{u}$ is zero everywhere except at the origin, where it blows up.