The hydraulic jump ("shocks" and viscous flow in the kitchen sink)

Robert P. Godwin
Los Alamos National Laboratory, P. O. Box 1663, Los Alamos, New Mexico 87545

(Received 17 August 1992; accepted 27 January 1993)

The radial location and flow discontinuity of the hydraulic jump in a sink are discussed. The importance of viscosity in determining the location of the jump is emphasized. A model, which highlights the importance of viscous flow near the sink surface, allows us to predict the jump radius and the flow depth on the upstream and downstream sides of the jump as functions of viscosity, gravity, and flow velocity. In our approximate treatment, the jump location is independent of gravity and inversely proportional to the cube root of the viscosity.

I. INTRODUCTION

The fascinating hydraulic jump can easily be observed in a sink.[1,2] A casual survey indicates that few physicists have observed or can satisfactorily explain the jump. The existence of the hydraulic jump and the conditions which must be met at the jump are discussed in fluid dynamics texts.[3–6] The radial position of the jump, which I have not found discussed in textbooks, will be emphasized in this report. The jump position is determined by the viscosity of the fluid[2] and is direct, readily observed, evidence for the existence of a viscous boundary layer. It illustrates the important difference between flow for which the viscosity is identically zero and flow for which the Reynolds number is very large.[7,8] It is also direct evidence for the validity of the "no-slip" wall boundary condition, the existence of which was debated for the last half of the 19th century.[9]

The jump appears as a distinct discontinuity in radial flow away from the point of impact of a vertical water stream impinging on a flat sink bottom. At the discontinuity, the flow velocity abruptly decreases and the water depth increases. The jump is related to, but importantly different from,[10] the tidal bore and is the incompressible flow analog of a shock in compressible flows. Across a shock front, there is a velocity discontinuity and a density (rather than depth) discontinuity. Since most shock experiments are performed with supersonic wind tunnels, sophisticated gas guns, or high explosives, it is useful to have a commonly available experimental shock-front analogy. While hydraulic jumps and bores are legitimate shock analogs, they are only qualitative analogs.[11,12] Figure 1 schematically illustrates the jump and introduces the notation that we will use in describing it; r_1 is the jump radius, a the radius of the vertical water stream, $v_1(v_2)$ the flow velocity upstream (downstream) of the jump, and $h_1(h_2)$ the water depth just upstream (downstream) of the jump. An observed jump does not have as sharp a discontinuity as indicated in Fig. 1, but our analysis assumes a sharp jump. (In the case of shocks, the discontinuity is very sharp.) Table I summarizes the characteristics of two hydraulic jumps observed in a sink.

Assuming that only nonviscous flow and gravity are important, we use dimensional analysis to examine the jump location. With Q the volume flow rate,

$$r_1 = f(Q,g,\rho,a,h_2) \tag{1}$$

and ignoring h_1 since $h_1 \ll h_2$. Try

$$r_1 \propto Q^\alpha g^\beta \rho^\gamma a^\delta h_2^\epsilon \tag{2}$$

or

$$r_1 \propto \left(\frac{L^3}{T}\right)^\alpha \left(\frac{L}{T^2}\right)^\beta \left(\frac{M}{L^3}\right)^\gamma L^\delta L^\epsilon.$$

This yields

$$M: \gamma = 0,$$

$$L: 1 = 3\alpha + \beta + \delta + \epsilon,$$

and

$$T: \beta = -\alpha/2, \tag{3}$$

giving

$$\delta = 1 - \tfrac{5}{2}\alpha - \epsilon.$$

Therefore

$$r_1 \propto Q^\alpha g^{-\alpha/2} a^{[1-(5/2)\alpha-\epsilon]} h_2^\epsilon, \tag{4}$$

indicating a solution requiring only nonviscous flow and gravity is likely. (We will see later that this conclusion is misleading and related to the difference between truly nonviscous flow and flow for which the Reynolds number is large.) The jump appears to be independent of the fluid density. (Our further analysis will show this density independence is the result of a balance between the kinetic pressure $\sim \rho v_1^2$ and the hydrostatic pressure $\sim \rho g h_2$ at the jump.) Dimensional analysis does not allow us to isolate r_1, the observable of most interest to us, from h_2. We will show that considering the physical origins of r_1 resolves this dilemma.

II. INVISCID FLOW

Temporarily ignoring the existence of the step, the collision of the vertical stream with the sink surface gives radially spreading flow (via Bernoulli's equation) which conserves mass and velocity at distances well away from the stagnation point at $r=z=0$. Except at radii $\sim a$, we have, to a very good approximation,

$$Q = \pi a^2 v_1 = 2\pi r h v_1, \tag{5}$$

where r is the radius and h the depth at an arbitrary location in the flow; v_1 is the velocity the water would have acquired after gravitational acceleration at the $z=0$ level, if no sink bottom existed. To conserve mass, we have from Eq. (5)

$$h = a^2/2r. \tag{6}$$

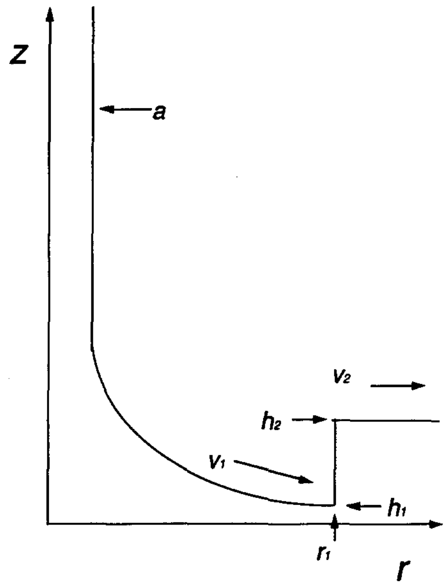

Fig. 1. The hydraulic jump; r_1 is the jump radius, a the radius of the vertical water stream, $v_1(v_2)$ the flow velocity upstream (downstream) of the jump, and $h_1(h_2)$ the water depth upstream (downstream) of the jump.

III. THE JUMP CONDITION

The key to understanding the discontinuity is knowing what condition to apply at the jump. The insight lies in the conservation of momentum or, equivalently, the concept of thrust.[3,13] (Energy is *not* conserved at the jump. More properly stated, at the jump kinetic energy is converted into potential energy and into increasing the internal energy of the fluid.) For steady-state flow, the net thrust must be zero at the jump. The momentum through a surface at the step is canceled by the hydraulic pressure head on the downstream side of the jump; $h_1 \ll h_2$ and $v_2 \ll v_1$. Indeed, we set $v_2=0$ for an approximate analysis. Doing so, we have as the momentum flux upstream of the step

$$\frac{\Delta(mv_1)}{\Delta t}=(\rho 2\pi r\Delta rh)\frac{v_1}{\Delta t}=2\rho\pi rhv_1^2, \tag{7}$$

and, since the average hydrostatic pressure of the water column downstream of the step is $\rho gh_2/2$, the effective force downstream of the step

$$F=\tfrac{1}{2}\rho gh_2\cdot 2\pi rh_2=\rho\pi rgh_2^2. \tag{8}$$

In steady state, Eqs. (7) and (8) are equal, giving at the step

Table I. Measured characteristics of two sink hydraulic jumps.

Parameter	A	B
r_1(cm)	5	1
a(cm)	~0.5	~0.1
h_2(cm)	~0.4	~0.2
h_1(cm)	$< h_2$	$< h_2$
Q(cm^3/s)	62	1.5
v_1(cm/s)[a]	79	48

[a] $v_1=Q/\pi a^2$. The volume flow rate Q was determined from the filling time of a measuring cup.

$$2h_1v_1^2=gh_2^2 \text{ or } h_2^2=h_1h_0, \tag{9}$$

where $h_0 \equiv 2v_1^2/g$. [Equation (9) is the simplification of a more general jump condition[4] for the special case of $h_1 \ll h_2 \ll h_0$.] Using Eqs. (5) and (6), this can be written

$$r_1=Q^2/\pi^2ga^2h_2^2, \tag{10}$$

which is consistent with Eq. (4) if $\alpha=-\epsilon=2$. A proper derivation has done no better than dimensional analysis at isolating r_1 from h_2.

In a more detailed treatment[2] the velocity v_2 downstream from the step is not neglected, giving a solution accurate to $O(h_1/h_2)^2$. The more exact solution is

$$\frac{r_1h_2^2ga^2}{Q^2}=\frac{1}{\pi^2}-\frac{gh_2a^4}{2Q^2}. \tag{11}$$

For our sink experiment, the terms on the rhs are $\sim 10^{-1}$ and $<10^{-2}$, respectively, and Eq. (10) is an excellent approximation.

IV. BOUNDARY LAYER FLOW

Return to the unresolved issue of separating r_1, the jump radius, from h_2, the depth of the flow downstream of the jump. Assume the jump location may be influenced by viscosity ν and by surface tension s, but not directly by gravity which acts vertically. Dimensional analysis assuming

$$r_1 \propto a^\alpha v_1^\beta \nu^\delta s^\gamma \tag{12}$$

gives $\gamma=0$ (indicating surface tension is unlikely to be important), $\delta=-\beta$ and $\alpha=1+\beta$. Thus dimensional analysis leads us to guess

$$r_1 \propto a(av_1/\nu)^\beta, \tag{13}$$

implying that a viscous model for the jump location is worth pursuing.

We assume h_1, the flow depth just upstream of the jump, is associated with the viscous laminar boundary layer thickness[2,8,14,15]

$$\Delta=k(\nu t)^{1/2}, \tag{14}$$

where k is a dimensionless constant of order 1 and t is time. For $h \gg \Delta$, far upstream of the jump, deviations from inviscid flow will be negligible. As $h \to \Delta$ the no-slip boundary condition at the interface will become important and eventually dominate the flow behavior. A rigorous analysis of the jump assuming laminar viscous flow has been attempted in Ref. 2. We use a heuristic approach to predict the scaling and magnitude of the jump location. We postulate that the hydraulic jump, with its associated energy dissipation and turbulence, will occur when the laminar boundary layer thickness approaches the depth predicted from nonviscous flow theory. Equivalently, we assume that the jump occurs where $h_1 \approx \Delta$ and $t \approx r_1/v_1$.

The Reynolds number in the region of the jump is

$$R_e \equiv UL/\nu \approx v_1r_1/\nu. \tag{15}$$

We have assumed the characteristic fluid velocity U and length L to be v_1 and r_1, respectively. For our water jump, $R_e \approx 4\times 10^4$. The Reynolds number of our flow is below the $\sim 3\times 10^5$ where the transition from viscous laminar to turbulent flow occurs.[14,15] Thus our assumption of viscous

Table II. Hydraulic jumps for liquids with a wide range of viscosities.

Liquid	$\nu(\text{cm}^2/\text{s})$	Predicted $r_1(\text{cm})$[a]	Measured $r_1(\text{cm})$
Water	10^{-2}	5	5
Cooking oil	1	1	1
Glycerin[b]	10	0.5	0.5
Syrup	10^3	0.1[c]	piles up[c]

[a]Assuming our viscous model is correct and all other flow parameters are held constant, i.e., assuming r_1 scales as $\nu^{-1/3}$.
[b]Glycerin is particularly interesting for "kitchen" hydrodynamics experiments. With typical scale lengths of 1 to 10 cm and flow velocities of 10 to 100 cm/s, $R_e \sim 10$ and the ratio $\Delta/L \approx R_e^{1/2} \sim 1$.
[c]Obviously the prediction of a jump radius smaller than the vertical stream radius is unphysical. The observed oscillating "buckling" pile up of the syrup represents a *low* R_e instability worthy of study in its own right![16,17] By diluting thick syrup with increasing amounts of water, we were able to create arbitrary jump radii between the vertical column radius a and the jump radius for pure water.

laminar boundary layer flow is consistent with the conditions of our experiment.

Assuming $h_1 \approx \Delta$ and combining Eqs. (14) and (6) yields a prediction for the jump radius of

$$r_1 \approx \frac{a^{4/3}}{(2k)^{2/3}} \left(\frac{v_1}{\nu}\right)^{1/3} = 0.63 \frac{a^{4/3}}{k^{2/3}} \left(\frac{v_1}{\nu}\right)^{1/3} \tag{16}$$

in agreement with the dimensional analysis and $\beta = 1/3$. We also have

$$h_1 = \frac{a^2}{2r_1} \approx \frac{(ka)^{2/3}}{2^{1/3}} \left(\frac{\nu}{v_1}\right)^{1/3} \tag{17}$$

and

$$h_2 = \sqrt{h_0 h_1} \approx (2ka)^{1/3} \nu^{1/6} v_1^{5/6} g^{-1/2}. \tag{18}$$

Using this h_2 and Q from Eq. (5) in the r_1 of Eq. (10) gives the jump radius of Eq. (16). Our viscous model is equivalent to the analysis which led to Eq. (10). It has, however, allowed us to remove the ambiguity of Eq. (10) with respect to the values of r_1 and h_2.

To estimate the values of r_1, h_1, and h_2 we require a numerical value for the dimensionless constant k. For apparently rather arbitrary historical reasons, the constant is usually chosen so that Δ is that thickness at which the flow velocity has reached 99% of its bulk flow value.[14,15] In that case, $k \approx 4$. We, in a more typical modeling approach, choose k so that Δ is that distance from the wall at which the flow velocity has reached e^{-1} of its bulk value. With this choice, $k \approx 1.1$ is appropriate.[15] For our sink experiment with $k = 1.1$, a water viscosity of $\nu = 10^{-2}$ cm^2/s, $v_1 = 80$ cm/s, and $a = 0.5$ cm the viscous boundary layer model predicts $r_1 = 5$ cm, $h_1 = 3 \times 10^{-2}$ cm, and $h_2 = 0.6$ cm in satisfactory agreement with observations.

We can test the model by performing experiments with fluids having quite different viscosities, while keeping the other flow parameters as constant as possible. In Table II we tabulate the properties of several common liquids, the implications of the model for those liquids, and measured jump radii. The model correctly predicts the behavior of the jump as the viscosity is dramatically changed.

V. ANOTHER APPROACH TO BOUNDARY LAYER FLOW

Tani[18] has estimated the jump radius directly from the Navier-Stokes equations by applying the no-slip boundary condition at the $z=0$ surface, assuming the downward flowing fluid column can be replaced by a point source of radial flow rate Q at the $r=z=0$ stagnation point and assuming the simplest velocity profile which is consistent with the conditions at both the $z=0$ and the fluid surfaces. Using Tani's approach, but keeping only the zeroth order in h/r one finds

$$\frac{dh}{dr} = \frac{5\pi\nu}{Q} r. \tag{19}$$

Integration gives

$$h = \frac{5\pi\nu}{Q} \frac{r^2}{2}. \tag{20}$$

Defining Q as that of the descending columnar flow provides a prediction of the radius where the viscous boundary layer engulfs the entire flow:

$$h = \frac{5\nu r^2}{2a^2 v_1}. \tag{21}$$

This viscous laminar flow depth will coincide with the "Bernoulli inviscid flow depth" of Eq. (6) when

$$r_1 \approx \frac{a^{4/3}}{5^{1/3}} \left(\frac{v_1}{\nu}\right)^{1/3} = 0.59 a^{4/3} \left(\frac{v_1}{\nu}\right)^{1/3}. \tag{22}$$

Assuming this gives the location of the hydraulic jump and comparing it to Eq. (16), we find the same scaling and, with our choice of $k = 1.1$, agreement in magnitude.

The agreement in the predictions of our two superficially different approaches for estimating the jump radius suggests the robustness of the hydraulic jump and related phenomena. The jump and its physical analogs are important features of a variety of flows.[19]

VI. SUMMARY

We have shown that the readily observed hydraulic jump can be understood by applying the laws of mass and momentum conservation, which are presented in hydrodynamics texts but not widely appreciated by physicists. We have also shown, both analytically and empirically, that the radial location of the hydraulic jump is determined by the fluid viscosity. The jump occurs at a flow depth essentially equal to the viscous laminar boundary layer. Thus the hydraulic jump dramatically highlights the important fact that, while many flows can be effectively modeled by nonviscous hydrodynamics, there is always a boundary layer where the no-slip boundary dominates the flow. Sometimes the boundary layer, which is often microscopic, has dramatic macroscopic effects. Commonly used hydrodynamics textbooks do not point out the effect of viscosity in determining the radial location of the hydrodynamic jump.

ACKNOWLEDGMENTS

I thank P. Blewett and W. Harvey for discussions of analytical and numerical hydrodynamics related to the hy-

draulic jump, M. Klein for bringing Ref. 11 to my attention, and G. H. McCall for suggesting that this analysis be published. I also thank a referee for emphasizing the work of Yeh[10] on the differences between a static radial jump and a moving bore.

[1] J. Walker, "Sink hydraulic jump," in *The Flying Circus of Physics* (Wiley, New York, 1975), Item 4.58, p. 93.
[2] E. J. Watson, "The radial spread of a liquid jet over a horizontal plane," J. Fluid Mech. **20**, 481–499 (1964).
[3] D. J. Acheson, "Hydraulic jumps and shock waves," in *Elementary Fluid Dynamics* (Oxford, New York, 1990), pp. 100–104.
[4] S. Whitaker, "Hydraulic jump," in *Introduction to Fluid Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1968), pp. 382–385.
[5] H. Lamb, *Hydrodynamics* (Dover, New York, 1945), p. 280.
[6] F. Harlow and A. A. Amsden, "The hydraulic jump," in *Fluid Dynamics*, Los Alamos National Laboratory Monograph, LA-4700 (Los Alamos Natl. Lab., New Mexico, 1971), pp. 77–78.
[7] R. Feynman, R. B. Leighton, and M. Sands, "The Flow of Wet Water," in *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), Vol. II, Chap. 41.
[8] D. J. Acheson, "Elementary viscous flow," in *Elementary Fluid Dynamics* (Oxford, New York, 1990), Chap. 2.
[9] D. J. Acheson, "Boundary layers," in *Elementary Fluid Dynamics* (Oxford, New York, 1990), Chap. 8.
[10] H. H. Yeh, "Vorticity-generation mechanisms in bores," Proc. R. Soc. London Ser. A, **432**, 215–231 (1991).
[11] P. Krehl and M. van der Geest, "The discovery of the Mach reflection effect and its demonstration in an auditorium," Shock Waves **1**, 3–15 (1991).
[12] F. R. Gilmore, M. S. Plesset, and H. E. Crossley, "The analogy between hydraulic jumps in liquids and shock waves in gases," J. Appl. Phys. **21**, 243–249 (1950).
[13] R. Courant and R. O. Friedricks, "Thrust," in *Supersonic Flow and Shock Waves* (Springer, New York, 1948), pp. 392–394.
[14] S. Whitaker, "Flow around immersed bodies," in *Introduction to Fluid Mechanics* (Prentice-Hall, Englewood Cliffs, NJ, 1968), Chap. 11.
[15] H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill, New York, 1979), Chaps. II and VII.
[16] J. Walker, "Liquid rope coils," in *The Flying Circus of Physics* (Wiley, New York, 1975), Item 4.125, p. 111.
[17] D. J. Acheson, "The buckling of viscous jets," in *Elementary Fluid Dynamics* (Oxford, New York, 1990), p. 342.
[18] I. Tani, "Water jump in the boundary layer," J. Phys. Soc. Jpn. **4**, 212–215 (1949).
[19] G. Wallerstein and S. Elgar, "Shock waves in stellar atmospheres and breaking waves on an ocean beach," Science **256**, 1531–1536 (1992).

Critical current density of $YBa_2Cu_3O_{7-x}$

K. E. Oldenburg and W. A. Morrison
Chemistry Department, University of Evansville, Evansville, Indiana 47722

G. C. Brown
Physics Department, University of Evansville, Evansville, Indiana 47722

(Received 8 September 1992; accepted 27 January 1993)

An experiment to determine the critical current density of superconducting $YBa_2Cu_3O_{7-x}$ is described. The experiment has been used in a solid state physics course and would also be suitable for modern physics or advanced laboratory courses as well as for courses in physical or inorganic chemistry.

I. INTRODUCTION

Experiments using high temperature superconductors provide an interesting opportunity to bring some of the excitement of modern physics into the undergraduate curriculum. Since several possible practical applications of high temperature superconductors depend on their ability to carry large currents, we report on an experiment for determining critical current density J_c. The critical current density is the maximum or critical current I_c per area at which the material remains superconducting.

II. SAMPLES

$YBa_2Cu_3O_{7-x}$ samples were prepared by both mortar and pestle[1] and wet milling[2] techniques. In both cases yttrium oxide (Y_2O_3), barium carbonate ($BaCO_3$), and copper oxide (CuO) were used in a 1:3.50:2.11 mass ratio. Mixing was accomplished in the first case by grinding with mortar and pestle. In the wet milling method, the starting reagents were placed in a polyethylene jar with corundum grinding media and a liquid, usually acetone or an alcohol. After mixing on a ball mill for 12 to 24 h, the liquid was evaporated by heating the mixture on a hot plate. The remainder of the sample preparation process was similar for both the mortar and pestle and wet milling produced powders. The powders were calcined in a muffle furnace at temperatures between 900 and 940 °C for 12 h. The samples were reground with mortar and pestle and pressed into 0.5 in. diam. pellets with a hydraulic press at about 40 000 psi. Finally, the pellets were sintered at temperatures between 900 and 940 °C for 12 h in the muffle furnace. For the critical current density measurements, thin strips of rectangular cross section were cut from the middle of the pellets by hand with a glass scoring saw. Most of the rectangular samples had cross-sectional areas between 0.01 and 0.02 cm^2. Leads were attached with indium solder.

III. EXPERIMENTAL METHOD AND RESULTS

The sample is cooled to its critical temperature and below using a closed cycle cryosystem. A simpler version of