Analysis Problems

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Preface

The purpose of this book is to supply a collection of problems in analysis. Please submit your solution to one of th email addresses below.

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Prescribed books for problems.

1) Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume I: Introductory Level

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2) Problems and Solutions in Theoretical and Mathematical Physics, Third Edition, Volume II: Advanced Level

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Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\lvert z\rvert$	modulus of complex number z
	$\lvert x+iy\rvert = (x^2+y^2)^{1/2}, \;\; x,y \in \mathbb{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{v}$	column vector in $\mathbb{C}^n$
$\mathbf{v}^T$	transpose of $\mathbf{v}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\lVert\,.\,\rVert$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbb{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbb{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\mathrm{rank}(A)$	rank of matrix A
A^T	transpose of matrix A

$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices A and B
$[A,B] := AB - BA$	commutator for square matrices A and B
$[A,B]_+ := AB + BA$	anticommutator for square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

Chapter 1

Sums and Products

Problem 1. Let $m, n \geq 2$. Calculate the finite sum

$$S_{n,m} = \sum_{j=1}^{m} \sum_{k=1}^{n} |j - k|.$$

Problem 2. Let $n \in \mathbb{N}$. Calculate the finite sum

$$S_n = \sum_{j=-n}^{n} \sum_{k=-n}^{n} |j + k|.$$

Problem 3. Let $m \geq 1$ and $n \geq 1$. Find the sum

$$\sum_{k=0}^{\min(m,n)} (n + m - 2k + 1).$$

Problem 4. The *harmonic series* can be approximated by

$$\sum_{j=1}^{n} \frac{1}{j} \approx 0.5772 + \ln(n) + \frac{1}{2n}.$$

Calculate the left and rigt-hand side for $n = 1$ and $n = 10$.

Problem 5. The *Bernoulli numbers* B_0, B_1, B_2, ... are defined by the power series expansion

$$\frac{x}{e^x-1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j \equiv B_0 + \frac{B_1}{1!} x + \frac{B_2}{2!} x^2 + \cdots .$$

One finds $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$. One has $B_j = 0$ if $j \geq 3$ and j odd. The Bernoulli numbers are utilized in the *Euler summation formula*

$$\sum_{j=1}^{n} f(j) = \int_1^n f(t)dt + \frac{1}{2}(f(n)+f(1)) + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n) - f^{(2k-1)}(1)) + R_m(n)$$

where

$$|R_m(n)| \leq \frac{4}{(2\pi)^{2m}} \int_1^m |f^{(2m)}(t)| dt.$$

Calculate

$$\sum_{j=1}^{n} j^2$$

using $f(t) = t^2$.

Problem 6. Let $z \in \mathbb{C}$ and $|z| < 1$. Calculate the sum

$$(1-|z|^2)^{2s} \sum_{n=0}^{\infty} n \binom{2s+n-1}{n} |z|^{2n}.$$

Problem 7. Let $x \in \mathbb{R}$ and $r \in \mathbb{N}$ with $r \geq 1$. Find the sum

$$\sum_{k=0}^{r-1} \exp(-2\pi i k x / r).$$

Problem 8. Let $a_1, a_2, \ldots, a_n \in [0,1]$. Show that there exists a real number x in the unit interval such that the average of the (unsigned) distances from x to the a_j's is $1/2$.

Problem 9. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function. Show that there exist m differentiable functions g_1, g_2, ..., g_m defined on $\mathbb{R}^m$ with the properties

$$f(\mathbf{x}) = f(\mathbf{0}) + \sum_{j=1}^{m} x_j g_j(\mathbf{x})$$

and

$$g_j(\mathbf{0}) = \frac{\partial f}{\partial x_j}(\mathbf{0})$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$.

Problem 10. Let $x \in \mathbb{R}$. The sequence of functions $\{ f_k(x) \}$ is defined by $f_1(x) = \cos(x/2)$ and for $k > 1$ by

$$f_k(x) = f_{k-1}(x)\cos(x/2^k).$$

Thus

$$f_k(x) = \cos(x/2)\cos(x/2^2)\cdots\cos(x/2^k).$$

Obviously, we have $f_k(0) = 1$ for every k. Calculate $\lim_{k\to\infty} f_k(x)$ as a function of x for $x \neq 0$.

Problem 11. Show that

$$1 + 2\tanh^2\lambda + 3\tanh^4\lambda + \cdots + (n+1)\tanh^{2n}\lambda + \cdots \equiv \cosh^4\lambda.$$

Problem 12. Let n be a positive integer and $f(j) = j(j-1)(j-2)$ with $j = 1, 2, \ldots, n$. Let $a_j := f(j+1) - f(j)$. By calculating the sum $\sum_{j=1}^n a_j$ show that

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Problem 13. Show that

$$\sum_{j=2}^{\infty} \frac{1}{j(2j-1)} = 2\ln(2) - 1.$$

Problem 14. Show that

$$\left(\sum_{j=1}^N \frac{1}{u - v_j}\right)^2 - \sum_{j=1}^N \frac{1}{(u-v_j)^2} \equiv 2 \sum_{\substack{i<j\\ j=2\\ i=1}}^N \frac{1}{v_i - v_j}\left(\frac{1}{u - v_i} - \frac{1}{u - v_j}\right).$$

This identity plays a role in the *Bethe ansatz*.

Problem 15. Show that the series

$$f(\theta) = \sum_{j=0}^{\infty} \frac{\sin(3^j\theta)}{2^j}$$

is convergent. Is the series $df/d\theta$ convergent?

Problem 16. Find the radius of convergence of the power series

$$\sum_{j=0}^{\infty} \binom{j+k}{j} z^j, \qquad k > 0.$$

Problem 17. Find the radius of convergence of the power series

$$\sum_{j=1}^{\infty} \frac{j!}{j^j} z^{mj}, \qquad m = 1, 2, \ldots .$$

Problem 18. Let $(s_0, s_1, ..., s_{n-1})^T \in \mathbb{R}^n$, where $n = 2^k$. This vector in $\mathbb{R}^n$ can be associated with a piecewise constant function f defined on $[0, 1)$

$$f(x) = \sum_{j=0}^{2^k-1} s_j \Theta_{[j2^{-k},(j+1)2^{-k})}(x)$$

where $\Theta_{[j2^{-k},(j+1)2^{-k})}(x)$ is the step function

$$\Theta_{[j2^{-k},(j+1)2^{-k})}(x) := \begin{cases} 1 \; x \in [j2^{-k}, (j+1)2^{-k}) \\ 0 \; x \notin [j2^{-k}, (j+1)2^{-k}) \end{cases}$$

with the support $[j2^{-k}, (j+1)2^{-k})$. Let $x_{j+1} = 4x_j(1 - x_j)$ with $j = 0, 1, 2, \ldots$ and $x_0 = 1/3$. Then

$$x_0 = \frac{1}{3}, \quad x_1 = \frac{8}{9}, \quad x_2 = \frac{32}{81}, \quad x_3 = \frac{6272}{6561}.$$

Find $f(x)$ for this data set and then calculate

$$\int_0^1 f(x)dx.$$

Problem 19. Let $a_1, a_2, \ldots, a_n$ be a finite sequence of numbers. Its *Cesáro sum* is defined as

$$\frac{s_1 + s_2 + \cdots + s_n}{n}$$

where

$$s_k = a_1 + a_2 + \cdots + a_k$$

for each k, $1 \leq k \leq n$. Suppose that the Cesáro sum of the 99-term sequence $a_1, a_2, \ldots, a_{99}$ is 100. Find the Cesáro sum of the 100-term sequence 1, $a_1, a_2, \ldots, a_{99}$.

Problem 20. Each $x \in [0,1]$ can be written as

$$x = \sum_{j=1}^{\infty} \frac{\epsilon_j}{2^j}$$

with $\epsilon_j = 0$ or $\epsilon_j = 1$. Define the function $f : [0,1] \to [0,1)$ as

$$f(x) = \sum_{j=1}^{\infty} \frac{2\epsilon_j}{3^j}.$$

The function f is known as Cantor function. Let $x = 1/8$. Find $f(x)$.

Problem 21. Let $n, m \in \mathbb{N}$ and $a, b, c, d \in \mathbb{R}$. Show that

$$(a+b)^n (c+d)^m = \sum_{r=0}^{n} \sum_{s=0}^{m} \binom{n}{r} \binom{m}{s} a^r b^{n-r} c^s d^{m-s}.$$

Problem 22. Let $s \geq 2$. Simplify the sum

$$\sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \frac{1}{(j_1 + j_2)^s}.$$

Problem 23. Let $\ell \geq 1$. Study the *Gauss sum*

$$G(k, \ell) = \frac{1}{\sqrt{\ell}} \sum_{r=0}^{\ell-1} e^{2\pi i k r^2/\ell}.$$

Problem 24. Let n be an integer and $n > 1$. Show that

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Show that s_n cannot be integer for all $n > 1$.

Problem 25. The q-exponential function is defined by

$$e_q^x := \sum_{j=0}^{\infty} \frac{x^n}{[j]!}$$

where

$$[j] := \frac{q^j - q^{-j}}{q - q^{-1}}.$$

Find e_q^x for $x = 1$ and $q = 1/2$.

Problem 26. Consider

$$\frac{1+x}{1-x-x^2} = \sum_{j=0}^{\infty} c_j x^j.$$

Find c_j.

Problem 27. The *Cantor series approximation* is defined as follows. For arbitrary chosen integers $n_1, n_2, \ldots$ (equal or larger than 2), we can approximate any real number r_0 as follows

$$\begin{aligned} x_j &= \text{integer part}(r_j), \quad j = 0, 1, 2, \ldots \\ r_{j+1} &= (r_j - x_j) n_j \end{aligned}$$

and

$$r_0 \approx x_0 + \sum_{j=1}^{N} \frac{x_j}{n_1 n_2 \cdots n_j}.$$

The approximation error is

$$E_N = \frac{1}{n_1 n_2 \cdots n_N}.$$

Apply this approximation to $r_0 = 2/3$ and the golden mean number with $n_j = 2$ for all j and $N = 4$.

Problem 28. Calculate the sum

$$S = \sqrt{2+\sqrt{2}} - \sqrt{2}\sqrt{2-\sqrt{2}} - \sqrt{2-\sqrt{2}}.$$

Problem 29. Let $n_1, n_2, n_3 \in \mathbb{Z}$. Calculate

$$1 - e^{i\pi(n_1+n_2+n_3)}.$$

Problem 30. Let $a, b, c \in \mathbb{R}$. Factorize

$$(b-c)^3 + (c-a)^3 + (a-b)^3.$$

Problem 31. Show that $7 + 2\sqrt{10}$ has a square root the form $\sqrt{x} + \sqrt{y}$.

Problem 32. Let $n \geq 0$. Starting from

$$\sum_{j=0}^{n} c_j x^j = (1+x)^n$$

show that

$$\begin{aligned} c_0 + c_1 + \cdots + c_n &= 2^n \\ c_0 - c_1 + c_2 - c_3 + \cdots + (-1)^n c_n &= 0 \\ c_1 + 2c_2 + 3c_3 + \cdots n c_n &= n 2^{n-1} \\ c_0^2 + c_1^2 + \cdots + c_n^2 &= \frac{(2n)!}{(n!)^2}. \end{aligned}$$

Problem 33. Show that if $|x| < 1$ we have the expansion

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

Use this expansion to show that

$$\frac{1}{(a+x)^2} = \frac{1}{a^2}\left(1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \cdots\right).$$

Problem 34. Calculate

$$\sum_{j=0}^{\infty} \frac{j^3+1}{j!} x^j.$$

Hint. Write $j^3 + 1$ in the form

$$a + bj + cj(j-1) + dj(j-1)(j-2).$$

Problem 35. Let $n \in \mathbb{N}$. Find the sum

$$\sum_{k=-(n-1)/2}^{(n-1)/2} k^2$$

where k runs in steps of 1.

Problem 36. Let $x \in \mathbb{R}$. Show that the series

$$\frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \cdots$$

converges absolutely for all values of x, except for $+1$ and -1.

Problem 37. Assume that the series

$$1, \ 14, \ 51, \ 124, \ 245, \ 426, \ \ldots$$

is a the form

$$ak^3 + bk^2 + ck + d.$$

Find the integer coefficents a, b, c, d.

Problem 38. Apply mathematical induction to show that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2n-1) &= n^2 \\ 1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 &= \frac{1}{2}n(6n^2 - 3n - 1) \\ 1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 &= n^2(2n^2 - 1). \end{aligned}$$

Problem 39. Show that the series

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}, \quad \sum_{j=0}^{\infty} j z^j = \frac{z}{(1-z)^2}, \quad \sum_{j=0}^{\infty} j^2 z^j = \frac{z(1+z)}{(1-z)^3}$$

converge for all $|z| < 1$.

Problem 40. Find the sum

$$S_n = \frac{2}{1 \cdot 2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \cdots + \frac{2}{n \cdot (n+1) \cdot (n+2)}.$$

Calculate S_n for $n \to \infty$. Hint. Find a, b, c for

$$\frac{2}{n \cdot (n+1) \cdot (n+2)} = \frac{a}{n} + \frac{b}{n+1} + \frac{c}{n+2}.$$

Problem 41. Show that

$$\ln 2 = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots$$

Problem 42. Assuming that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=\ln(2).$$

Show that

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots=\frac{1}{2}\ln(2).$$

Note that the convergence of the first series is not absolute.

Problem 43. Let $x>0$. Shwo that a complicated way to calculate $1/x$ is given by

$$\frac{1}{x+1}+\frac{1!}{(x+1)(x+2)}+\frac{2!}{(x+1)(x+2)(x+3)}+\cdots$$

Problem 44. Show that

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}-\frac{1}{2}\ln(n)$$

tends to a finite value as $n\to\infty$. Find this value.

Problem 45. Let $a>0$. Show that

$$\sum_{n=-\infty}^{\infty}\frac{1}{z-na}=\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2a^2}.$$

Problem 46. Let $x\neq y$, $x\neq z$, $y\neq z$. Find the sum

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)}.$$

Problem 47. The sum

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{k_3=-\infty}^{\infty}{}'\frac{(-1)^{k_1+k_2+k_3}}{\sqrt{k_1^2+k_2^2+k_3^2}}$$

plays a role in solid state physics. Here $'$ indicates that the term $(k_1,k_2,k_3)=(0,0,0)$ is omitted. Given a positive integer N. Write a C++ program that implements the sum

$$\sum_{k_1=-N}^{N}\sum_{k_2=-N}^{N}\sum_{k_3=-N}^{N}{}'\frac{(-1)^{k_1+k_2+k_3}}{\sqrt{k_1^2+k_2^2+k_3^2}}.$$

Run the program for different N and compare with the exact value.

Problem 48. Let $\beta > 0$. Find the sum

$$\sum_{n=0}^{\infty} e^{-\beta n}.$$

Problem 49. Let n be a positive integer and $f(\theta_1, \ldots, \theta_N)$ be a periodic function, i.e. periodic 2π for each θ_j $(j = 1, \ldots, N)$. Show for large n we have

$$\left(\prod_{k=1}^{N} \int_0^{2\pi} d\theta_k\right) \left(\sum_{j=1}^{N} \cos\theta_j\right)^n f(\theta_1, \ldots, \theta_N) \approx N^n \left(\prod_{k=1}^{N} \int_{-\infty}^{\infty} d\theta_k e^{-n(\theta_k)^2/(2N)}\right) f(\theta_1, \ldots, \theta_N).$$

Problem 50. Let A, B be finite sets and $n(A)$, $n(B)$ the numbers of elements in A and B, respectively. Is

$$n(A) + n(B) = n(A \cup B) + n(A \cap B)\,?$$

Prove or disprove.

Problem 51. Find the sum

$$S_L = \sum_{n=1}^{L} \exp\left(i\pi \frac{2}{L} n^2\right)$$

for $L = 1, 2, 3$.

Problem 52. Given the expansion

$$\ln(1+x) = x - \frac{x^2}{+}\frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x \le 1.$$

Calculate $\ln(11)$.

Problem 53. Let $n \in \mathbb{N}_0$ and $p, a > 0$. Show that

$$\sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!} \exp(-2\pi kp/a) \equiv \frac{1}{(1 - \exp(-2\pi p/a))^{n+1}}.$$

Hint: Start with

$$f(n) := \sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!} \exp(-2\pi kp/a)$$

and find $f(n+1)$.

Problem 54. Let $(s_1, s_2) \in \mathbb{Z}^2$. Let $P(\phi_1, \phi_2)$ be a probability density and let $\theta_1, \theta_2 \in \mathbb{R}$. We can express the characteristic double sequence as

$$\chi(s_1, s_2) = \int_{\theta_1}^{\theta_1+2\pi} \int_{\theta_2}^{\theta_2+2\pi} \exp(i(s_1\phi_1 + s_2\phi_2))P(\phi_1, \phi_2)d\phi_1 d\phi_2.$$

Find $P(\phi_1, \phi_2)$ as a function of $\chi(s_1, s_2)$. Note that $\chi(0,0) = 1$.

Problem 55. Let $n \in \mathbb{N}$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}_0$. Consider the function

$$f(m_1, m_2, \ldots, m_n) := \begin{cases} \delta_{m_1,0} & \text{for } n = 1 \\ \delta_{m_2,0}\delta_{m_1,1} & \text{for } n = 2 \\ \delta_{m_n,0}\delta_{(m_1+\cdots+m_n),n-1} \prod_{j=1}^{n-2} H(j - \sum_{k=1}^{j} m_{n-k}) & \text{for } n \geq 3 \end{cases}$$

where H is the step function

$$H(x) := \begin{cases} 1 \text{ for } x \geq 0 \\ 0 \text{ for } x < 0 \end{cases}$$

Give an implementation of this function in SymbolicC++ for a given n using the `Verylong` class.

Problem 56. Let $f : [0,1] \to [0,1]$ be a differentiable function, for example the logistic map $f(x) = 4x(1-x)$. Let $f^{(k)}$ be the k-th iterate of the function f. Let

$$b_k := \sup_{0 \leq x \leq 1} \left| \frac{d}{dx} f^{(k)}(x) \right|.$$

Show that

$$\lim_{k \to \infty} (b_k)^{1/k}$$

exist. Apply the chain rule.

Problem 57. Let $n, m \geq 1$. Implement the sum

$$S_{n,m} := \frac{(2m)!}{2^{2m} m!} \sum_{k=0}^{m} 2^k \binom{n}{k} \binom{m}{k}$$

using the `Verylong` class and `Rational` class of SymbolicC++.

Problem 58. Let $n \geq 2$. Show that

$$\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}.$$

Problem 59. Let $y \in \mathbb{R}$ and $\lfloor y \rfloor$ be the integer part of y. Let $\omega = (1+\sqrt{5})/2$ be the golden mean number (which is an irrational number). We define

$$x_j := j + ((j+1)/\omega)(\lambda - 1), \qquad j = 0, 1, \ldots$$

where $\lambda > 0$ and

$$\mu_k = x_k - x_{k-1}, \qquad k = 1, 2, \ldots$$

Show that the sequence μ_1, μ_2, $\ldots$ is non-periodic.

Problem 60. Let $s > 0$ and

$$f(s) = \int_{t=0}^{\infty} \frac{\ln(1+st)}{1+t^2} dt.$$

Show that

$$f(s) = (1 - \ln(s))s + \frac{\pi s^2}{4} + \left(\frac{\ln(s)}{3} - \frac{1}{9}\right) s^3 + O(s^4).$$

Problem 61. Show that

$$\prod_{j=1}^{\infty} \cos\left(\frac{x}{2^j}\right) = \frac{\sin(x)}{x}$$

for $x \in \mathbb{R}$.

Problem 62. Is the series

$$S = \sum_{k=1}^{\infty} \frac{k}{1+2k^3}$$

convergent?

Problem 63. Show that

$$\frac{\pi}{4} = 4\arctan(1/5) - \arctan(1/239).$$

Hint. Let $\theta = \arctan(1/5)$. Thus $\tan(\theta) = 1/5$. Applying the double angle formula for the tangent we have $\tan(2\theta) = 5/12$ and $\tan(4\theta) = 120/119$. Finally apply that $\tan(\pi/4) = 1$.

Problem 64. Let z_j $(j = 1, \ldots, p)$ be fixed complex numbers. Is

$$\prod_{j=1}^{p} |z - z_j| > \prod_{j=1}^{p} (|z_j| - |z|)\,?$$

Problem 65. Let $n \geq 2$. Let S_n be the standard n-simplex embedded in $\mathbb{R}^n$

$$S_n := \left\{ \mathbf{x} \in \mathbb{R}^n \, : \, \sum_{j=1}^{n} x_j = 1, \text{ for } x_k \geq 0, \text{ for } k = 1, \ldots, n \right\}.$$

We denote by $S_{k,n-1}$ the kth face of the boundary of S_n. They are $(n-1)$-simplexes

$$S_{k,n-1} := \{ \mathbf{x} \in \mathbb{R}^n \, : \, x_k = 0, \ \mathbf{x} \in S_n \} \quad \text{for } k = 1, \ldots, n.$$

Show that the boundary ∂S_n of S_n is the union

$$\partial S_n = \cup_{j=1}^{n} S_{j,n-1}$$

of the faces.

Problem 66. A triple sum related to the Madelung constant is given by

$$\sum_{i,j,k=-\infty}^{\infty}{}' \frac{(-1)^{i+j+k}}{\sqrt{i^2+j^2+k^2}}$$

where the prime indicates that $(0,0,0)$ is excluded from the summation. Write a C++ program that finds

$$\sum_{i=-100}^{100} \sum_{j=-100}^{100} \sum_{k=-100}^{100} \frac{(-1)^{i+j+k}}{\sqrt{i^2+j^2+k^2}}.$$

Problem 67. Give a non-trival infinite sequence $(x_0, x_1, x_2, \ldots)$ such that

$$\sum_{j=0}^{\infty} \frac{|x_j|}{1+|x_j|}$$

is finite.

Problem 68. Let $n \in \mathbb{N}_0$. Consider an infinite number of time variable $\mathbf{t} = (t_1, t_2, t_3, \ldots)$. Consider the sum

$$p_n(x+t_1, t_2, t_3, \ldots) = \sum_{\substack{k_0+k_1+2k_2+3k_3+\cdots=n \\ k_0,k_1,k_2,k_3,\ldots \geq 0}} \frac{x^{k_0} t_1^{k_1} t_2^{k_2} \cdots}{k_0! k_1! k_2! \cdots}.$$

Find p_0, p_1, p_2, p_3.

Problem 69. Let n be an integer with $n \geq 1$. Simplify the series

$$S = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

Problem 70. Let $n \geq 1$. Show that

$$\sum_{j=1}^{n} \ln(1+1/j) = \ln(n+1).$$

Problem 71. Let $x \in (-1,1)$. Show that

$$\sum_{j=0}^{\infty} jx^j = \frac{x}{1-x)^2}.$$

Problem 72. Let M be an integer with $M \geq 2$ and $\Im(\phi) < 0$. Show that

$$S(k) = \sum_{m=1}^{M-1} \frac{e^{(2\pi imk)/M}}{\sin^2(\pi(m+\phi)/M)} = -\frac{4kMe^{-2\pi ik\phi/M}}{1-e^{-2\pi i\phi}} + \frac{M^2e^{-2\pi ik\phi/M}}{\sin^2(\pi\phi)}.$$

Note that

$$\frac{1}{\sin^2(\pi(m+\phi)/M)} = -4\sum_{p=0}^{\infty} pe^{-2\pi i(m+\phi)p/M}.$$

Problem 73. Let N be a positive integer and a, b be positive integers or positive half-integers. Find

$$f(N,a,b) = \frac{1}{2}(1+(-1)^{2a}+(-1)^{2b}+(-1)^{2a+2b+N}).$$

Problem 74. Let ℓ be a positive integer and $\phi \in \mathbb{R}$. Show that

$$\sum_{m=-\ell}^{+\ell} e^{im\phi} = \frac{\sin((\ell+1/2)\phi)}{\sin(\phi/2)}.$$

Problem 75. Let n be a positive integer. Show that

$$\frac{\sin(2^n\alpha)}{2^n\sin(\alpha)} = \cos(\alpha)\cos(2\alpha)\cos(2^2\alpha)\cdots\cos(2^{n-1}\alpha).$$

Problem 76. Let $-1 < x < +1$ and $n \geq 2$.
(i) Show that

$$S_n(x) = \sum_{j=0}^{n-1} jx^j = \frac{nx^n}{x-1} + \frac{x(1-x^n)}{(1-x)^2}.$$

(ii) Show that

$$\lim_{n\to\infty} S_n(x) = \frac{x}{(1-x)^2}.$$

(iii) Let $x = 1/2$. Show that $\lim_{n\to\infty} S_n(1/2) = 2$.

Problem 77. Let $n \geq 2$. Show that

$$\sum_{j=1}^{n} \ln(1+1/j) = \ln(n+1).$$

Problem 78. (i) Let $n \geq 2$. Show that

$$\sum_{j=0}^{n-1} \frac{1}{(j+1)(j+2)} = 1 - \frac{1}{n+1}$$

and thus show that

$$\sum_{j=0}^{\infty} \frac{1}{(j+1)(j+2)} = 1.$$

(ii) Calculate

$$\sum_{j=0}^{\infty} \frac{1}{(j+1)(j+2)(j+3)}.$$

Problem 79. Let $x, y, z \in \mathbb{R}$ and $x \neq y$, $x \neq z$, $y \neq z$. Show that

$$\frac{x}{(z-x)(x-y)} + \frac{y}{(x-y)(y-z)} + \frac{z}{(y-z)(z-x)} = 0.$$

Problem 80. Let

$$x \neq 0, \quad x \neq \pm 2\pi, \quad x \neq \pm 4\pi, \ldots$$

and $n \in \mathbb{N}$. Show that

$$\sin(x) + \sin(2x) + \cdots + \sin(nx) = \frac{\cos(x/2) - \cos((n+1/2)x)}{2\sin(x/2)}.$$

For the values $x = 0$, $x = \pm 2\pi$, $x = \pm 4\pi$ etc the sum is given by 0.

Problem 81. Let $n = 0, 1, 2, \ldots$. The function

$$K_n(\theta) := \frac{1}{n+1} \frac{\sin^2\left(\left(\frac{n+1}{2}\right)\theta\right)}{\sin^2\left(\frac{1}{2}\theta\right)}$$

is called the *Fejér kernel*.
(i) Show that

$$K_n(\theta) := \sum_{j=-n}^{j=+n} \left(1 - \frac{|j|}{n+1}\right) e^{ij\theta}.$$

(ii) Show that $K_n(\theta) \geq 0$.
(iii) Show that for any continuous 2π periodic function f on $\mathbb{R}$ one has

$$\begin{aligned} K_n * f(\theta) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta - \alpha) f(\alpha) d\alpha \\ &= \sum_{j=-n}^{j=+n} \left(1 - \frac{|j|}{n+1}\right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\alpha} f(\alpha) d\alpha\right) e^{ij\theta}. \end{aligned}$$

(iv) Show that $K_n \star f(\theta) \to f(\theta)$ uniformely in θ as $n \to \infty$.

Problem 82. The function

$$D_n(\theta) := \frac{\sin((n + \frac{1}{2})\theta)}{\sin(\frac{1}{2}\theta)}$$

is called the *Dirichlet kernel*. Show that

$$D_n(\theta) = \sum_{k=-n}^{k=+n} e^{ikx}.$$

Problem 83. Let $a > 0$. Show that

$$\sum_{j=-\infty}^{\infty} (-1)^j \frac{1}{j^2 + a^2} = \frac{\pi}{a \sinh(\pi a)}.$$

Problem 84. Let $\lambda > 0$.
(i) Calculate

$$S_1(\lambda) = \sum_{j=0}^{\infty} \frac{j e^{-\lambda} \lambda^j}{j!}.$$

(ii) Calculate

$$S_2(\lambda) = \sum_{j=0}^{\infty} \frac{j(j-1)e^{-\lambda}\lambda^j}{j!}.$$

Problem 85. (i) Let $\alpha > 0$. Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{k^2 + \alpha^2} = \frac{\pi}{2\alpha^2} \cdot \frac{\cosh(\alpha x)}{\sinh(\alpha x)} - \frac{1}{2\alpha^2}$$

where $-\pi \le x \le \pi$.
(ii) Let $\alpha > 0$. Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sin(kx)}{k^2 + \alpha^2} = -\frac{\pi}{2\alpha^2} \cdot \frac{\sinh(\alpha x)}{\sinh(\alpha \pi)}$$

for $-\pi < x < \pi$.

Chapter 2

Maps

Problem 1. Newton's sequence takes the form of a difference equation

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)}$$

where $t = 0, 1, 2, \ldots$ and x_0 is the initial value at $t = 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2 - 1$$

and $x_0 \neq 0$. Find the fixed points of f. Find the fixed points of the difference equation. Let $x_0 = 1/2$. Find x_1, x_2, x_3. Find Newton's sequence for this function. Obtain the exact solution of the difference equation.

Problem 2. (i) Solve the nonlinear recurrence relation

$$x_{n+1} = x_n^2, \qquad n = 0, 1, \ldots$$

where $x_0 = 2$.
(ii) Solve the linear recurrence relation

$$x_{n+1} = x_n + x_{n-1} + x_{n-2}, \qquad n = 2, 3, \ldots$$

and the initial values $x_0 = x_1 = x_2 = 1$.
(iii) Solve the linear recurrence relation

$$x_{n+1} = 1 + \sum_{j=0}^{n-1} x_j, \qquad x_0 = 1.$$

Problem 3. Let $x > 0$ and $p > 0$. Consider the map

$$f(x) = xe^{p-x}.$$

(i) Find the fixed points. Study the stability of the fixed points.
(ii) Show that f has a least one periodic point x^* with $x^* \neq 0$ or p.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a positive, continously differentiable function, defined for all real numbers and whose derivative is always negative. Show that for any real number x_0 (initial value) the sequence (x_k) obtained by *Newton's method*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

has always limit ∞.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuosly differentiable map. Let $f^{(n)}$ be the n-th iterate of f. Calculate the derivative of $f^{(n)}$ at x_0.

Problem 6. (i) Solve the second order linear difference equation

$$x_{t+2} = x_{t+1} + x_t \qquad t = 0, 1, 2, \ldots$$

where $x_0 = 0$ and $x_1 = 1$.
(ii) Give the definition of the golden mean number and derive this number.
(iii) Calculate

$$\lim_{t \to \infty} \frac{x_{t+1}}{x_t}.$$

Problem 7. The recursion relation

$$F_{n+2} = F_n + F_{n+1}$$

with the initial values $F_0 = F_1 = 1$ provides the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$. A generalization of the Fibonacci sequence is the q-analogue of the sequence defined by the recursion relation

$$F_{n+2}(q) = F_n(q) + qF_{n+1}(q)$$

and the initial condition $F_0(q) = 1$ and $F_1(q) = q$. Here, q is a real or complex number.
(i) Give the first five terms of the sequence.
(ii) Find a generating function of $F_n(q)$.

(iii) Find an explicit expression for $F_n(q)$.

Problem 8. (i) Find the linear map $f : \{0, 1\} \to \{-1, 1\}$ such that

$$f(0) = -1, \qquad f(1) = 1. \tag{1}$$

(ii) Find a linear map $g : \{-1, 1\} \to \{0, 1\}$ such that

$$g(-1) = 0, \qquad g(1) = 1. \tag{2}$$

This is obviously the inverse map of f.

Problem 9. Consider the differentiable function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1 - \cos(2\pi x)}{x}$$

where using L'Hospital $f(0) = 0$.
(i) Find the zeros of f.
(ii) Find the maxima and minima of f.

Problem 10. Given two manifolds M and N, a bijective map ϕ from M to N is called a *diffeomorphism* if both $\phi : M \to N$ and its inverse ϕ^{-1} are differentiable. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the analytic function

$$f(x) = 4x(1 - x)$$

and the analytic function $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = 1 - 2x^2.$$

(i) Can one find a diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$g = \phi \circ f \circ \phi^{-1}\,?$$

(ii) Consider the diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$

$$\psi(x) = \sinh(x).$$

Calculate $\psi \circ g \circ \psi^{-1}$.

Problem 11. Consider the polynomial $p(x) = x^3 - 3x + 3$. Show that for any positive integer N, there is an initial value x_0 such that the sequence x_0, x_1, x_2, ... obtained from *Newton's method*

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} = \frac{2x_n^3 - 3}{3(x_n^2 - 1)}, \qquad n = 0, 1, 2, \ldots$$

has period N.

Problem 12. Consider the linear recursion equation for $F(n)$ with $n = 0, 1, 2, \ldots$

$$(n+1)F(n+1) - nF(n-1) + F(n) = 0$$

where $F(0) = 0$, $F(1) = 1$. We define

$$f(z) := \sum_{n=1}^{\infty} F(n) z^n$$

where z is an undeterminate. Find the differential equation for f with the initial condition. Solve the differential equation.

Problem 13. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be an analytic function. Consider the map

$$\mathbf{x}_j = \mathbf{f}(\mathbf{x}_{j-1}) = \cdots = \mathbf{f}(\mathbf{x}_0).$$

To study the evolution of phase-space distributions, we can introduce the evolution operator $U(\mathbf{x}', \mathbf{x}, j)$ such that any initial phase-space distribution $\rho(\mathbf{x}, 0)$ evolves into

$$\rho(\mathbf{x}'; j) = \int_{\Omega} U(\mathbf{x}', \mathbf{x}; j) \rho(\mathbf{x}, 0) d\mathbf{x}$$

where Ω is the phase space area. Find $U(\mathbf{x}', \mathbf{x}; j)$.

Problem 14. Find the solution of the recursion relation

$$a_{j+2} = \frac{2}{3} a_{j+1} + \frac{1}{3} a_j, \qquad j = 1, 2, \ldots$$

with $a_1 = 5$, $a_2 = 1$. Calculate

$$\lim_{j \to \infty} a_j.$$

Problem 15. Solve the linear difference equation

$$x_{t+1} = 2x_t - t, \qquad t = 0, 1, 2, \ldots$$

with the initial value $x_0 = 1$.

Problem 16. Let

$$\mathbb{N}_0 := \{ 0, 1, 2, \ldots \}$$

Find an invertible map $f : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ with

$$f(0,0,0) = 0,\ f(0,0,1) = 1,\ f(0,1,0) = 2,\ f(0,1,1) = 3,$$

$$f(1,0,0) = 4,\ f(1,0,1) = 5,\ f(1,1,0) = 6,\ f(1,1,1) = 7$$

etc. Find the inverse function.

Problem 17. Consider the second order difference equation

$$x_{t+2} = x_{t+1} x_t$$

with the initial values $x_0 = a$, $x_1 = b$. Give the solution. What are the fixed points?

Problem 18. Let A be a set. Suppose that it is possible to define subsets A_1, A_2, ... of A which have the properties that
(i) the sets are pairwise disjoint; that is $A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, \ldots$ and $j \neq i$

(ii) $A_1 \cup A_2 \cup \cdots = A$.

Then the family of sets $\{ A_1, \ldots \}$ is called a *partition* of A.

Let $S = \{1, 2, 3, \ldots, 9\}$ and $A = \{1, 4, 7\}$. $B = \{2, 3, 5, 6\}$, $C = \{7, 8, 9\}$. Is $\{A, B, C\}$ a partition of S?

Problem 19. Let S be a finite set. Let $\mathcal{P}(S)$ be the power set of S. Then $\mathcal{P}(S)$ has $2^{|S|}$ elements.
(i) Show that the composition

$$A \circ B := (A \cup B) \cap (\complement A \cup \complement B)$$

defines a group, where $A, B \in \mathcal{P}(S)$. Here $\complement$ denotes the complement.
(ii) Show that the composition

$$A \circ B := (A \cap B) \cup (\complement A \cap \complement B)$$

defines a group, where $A, B \in \mathcal{P}(S)$. Here $\complement$ denotes the complement.

Problem 20. Find a function $f : [0,1] \to (0,1]$ that is one-to-one.

Problem 21. Let $n \in \mathbb{N}$. Consider the map

$$u_{t+1} = \frac{1}{2}\left(u_t + \frac{n}{u_t}\right), \qquad t = 0, 1, 2, \ldots$$

given the intial value u_0 with $u_0 > 0$. Show that

$$\frac{u_{t+1} - \sqrt{n}}{u_{t+1} + \sqrt{n}} = \left(\frac{u_t - \sqrt{n}}{u_t + \sqrt{n}} \right)^2 .$$

Show that $u_t \to \sqrt{n}$ as $t \to \infty$. Show that $\sqrt{n}$ is a fixed point.

Problem 22. Let $0 \le \alpha < \pi/4$. Consider the transformation

$$X(x,y,\alpha) = \frac{1}{\sqrt{\cos(2\alpha)}}(x\cos(\alpha) + iy\sin(\alpha))$$
$$Y(x,y,\alpha) = \frac{1}{\sqrt{\cos(2\alpha)}}(-ix\sin(\alpha) + y\cos(\alpha)).$$

(i) Show that $X^2 + Y^2 = x^2 + y^2$.
(ii) Do the matrices

$$\frac{1}{\sqrt{\cos(2\alpha)}} \begin{pmatrix} \cos(\alpha) & i\sin(\alpha) \\ -i\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

form a group under matrix multiplication?

Problem 23. A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is called homogeneousof degree r if

$$f(\epsilon x_1, \ldots, \epsilon x_n) = \epsilon^r f(x_1, \ldots, x_n). \tag{1}$$

Show that (Euler's identity)

$$\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} x_j = rf.$$

Problem 24. Find the invariance group of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sin(x).$$

Problem 25. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \cos(\sin(x)) - \sin(\cos(x)).$$

Show that f admits at least one critical point. By calculating the second order derivative find out whether this critical point refers to a maxima or minima.

Problem 26. The *Kustaanheimo-Stiefel transformation* is defined by the map from $\mathbb{R}^4$ (coordinates u_1, u_2, u_3, u_4) to $\mathbb{R}^3$ (coordinates x_1, x_2, x_3)

$$\begin{aligned} x_1(u_1,u_2,u_3,u_4) &= 2(u_1u_3 - u_2u_4) \\ x_2(u_1,u_2,u_3,u_4) &= 2(u_1u_4 + u_2u_3) \\ x_3(u_1,u_2,u_3,u_4) &= u_1^2 + u_2^2 - u_3^2 - u_4^2 \end{aligned}$$

together with the constraint

$$u_2du_1 - u_1du_2 - u_4du_3 + u_3du_4 = 0.$$

(i) Show that

$$r^2 = x_1^2 + x_2^2 + x_3^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2.$$

(ii) Show that

$$\Delta_3 = \frac{1}{4r}\Delta_4 - \frac{1}{4r^2}V^2$$

where

$$\Delta_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta_4 = \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}$$

and V is the vector field

$$V = u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_4}$$

(iii) Consider the differential one form

$$\alpha = u_2du_1 - u_1du_2 - u_4du_3 + u_3du_4.$$

Find $d\alpha$. Find $L_V\alpha$, where $L_V(.)$ denotes the Lie derivative.
(iv) Let $g(x_1(u_1,u_2,u_3,u_4), x_2(u_1,u_2,u_3,u_4), x_3(u_1,u_2,u_3,u_4))$ be a smooth function. Show that $L_Vg = 0$.

Problem 27. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$ be analytic functions. We define the *star product*

$$f(x_1,x_2)\star g(x_1,x_2) = \lim_{x_1'\to x_1, x_2'\to x_2} \exp\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2'} - \frac{\partial}{\partial x_1'}\frac{\partial}{\partial x_2}\right) f(x_1,x_2)g(x_1',x_2').$$

Let

$$f(x_1,x_2) = \sin(x_1+x_2), \quad g(x_1,x_2) = \sin(x_1-x_2).$$

Find the star product.

Problem 28. Let N_1, N_2 be given positive integers. Let $n_1 = 0, 1, \ldots, N_1 - 1$, $n_2 = 0, 1, \ldots, N_2 - 1$. There are $N_1 \cdot N_2$ points. The points (n_1, n_2) are a subset of $\mathbb{N}_0 \times \mathbb{N}_0$ and can be mapped one-to-one onto a subset of $\mathbb{N}_0$

$$j(n_1, n_2) = n_1 N_2 + n_2$$

where $j = 0, 1, \ldots, (N_1 - 1)(N_2 - 1)$. Find the inverse of this map.

Problem 29. Let $a \in \mathbb{R}$. Consider the transformation

$$\widetilde{t}(t, x) = \frac{1}{a} e^{ax} \sinh(at)$$
$$\widetilde{x}(t, x) = \frac{1}{a}(e^{ax} \cosh(at) - 1)$$

with $\lim_{a \to 0} \widetilde{t} = t$, $\lim_{a \to 0} \widetilde{x} = x$. Find the inverse of the transformation.

Problem 30. Show that the map $f : (0, 1) \to \mathbb{R}$

$$x \mapsto f(x) = \frac{x - 1/2}{x(x - 1)}$$

is bijective.

Problem 31. Consider the map $f : \mathbb{R} \to \{ (x_1, x_2) \, : \, x_1^2 + (x_2 - 1)^2 = 1 \wedge x_2 \neq 2 \}$ defined by

$$f(t) = \left(\frac{4t}{t^2 + 4}, \frac{2t^2}{t^2 + 4} \right).$$

Show that f is bijective. Show that f and f^{-1} are continuous.

Problem 32. Solve the initial value problem of the system of linear difference equations

$$x_{1,t+1} = -2x_{1,t}$$
$$x_{2,t+1} = \frac{8}{9} x_{1,t} - x_{2,t}$$

where $t = 0, 1, \ldots$.

Problem 33. Let $N \geq 2$ and $\delta := 1/(N + 1)$. Solve the linear one-dimensional linear difference equation

$$x_{j+1} - 2x_j + x_{j-1} = -12\delta^4 (j + 1)^2, \quad j = 0, 1, \ldots, N - 1$$

with the boundary conditions $x_{-1} = x_N = 0$.

Problem 34. (i) Consider the analytic map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x(1-x).$$

Find the fixed points of f. Are the fixed points stable? Prove or disprove.
(ii) Let $x_0 = 1/2$ and iterate

$$f(x_0), \quad f(f(x_0)), \quad f(f(f(x_0))), \ldots$$

Does this sequence tend to a fixed point? Prove or disprove.
(iii) Let $x_0 = 2$ and iterate

$$f(x_0), \quad f(f(x_0)), \quad f(f(f(x_0))), \ldots$$

Does this sequence tend to a fixed point? Prove or disprove.
(iv) Find the critical points of f. Then find the extrema of f.
(v) Find the roots of f, i.e. solve $f(x) = 0$.
(vi) Find the minima of the function $g(x) := |f(x)|$.

Problem 35. Let $c > 0$ and $0 \le v_1 < c$, $0 \le v_2 < c$. We define the composition

$$v_1 \star v_2 := \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

Is the composition associative?

Problem 36. Solve the initial value problem of the system of first order difference equations

$$\begin{aligned} x_{1,t+1} &= -2x_{1,t} \\ x_{2,t+1} &= \frac{8}{9}x_{1,t} - x_{2,t}. \end{aligned}$$

The first difference equation is independent of $x_{2,t}$ and we find the solution of the initial value problem

$$x_{1,t+1} = (-2)^t x_{1,0}.$$

Problem 37. Let m_A, m_B, $\mathbf{R}_A$, $\mathbf{R}_B$ be the masses and centre-of mass coordinates of mass A and B, respectively. We set $m = m_A + m_B$. Find the inverse of the transformation

$$\mathbf{r}(\mathbf{R}_A, \mathbf{R}_B) = \mathbf{R}_A - \mathbf{R}_B, \qquad \mathbf{R}(\mathbf{R}_A, \mathbf{R}_B) = \frac{1}{m}(m_A \mathbf{R}_A + m_B \mathbf{R}_B).$$

Problem 38. Let $x \in [0,1]$ and $\lfloor a \rfloor$ be the integer part of a. Show that the sequence x_t $(t = 0, 1, \ldots)$ given by

$$x_0 = 0, \quad \text{and} \quad x_{t+1} = x_t + \frac{1}{2^{t+1}} \lfloor 2^{t+1}(x - x_t) \rfloor$$

converges to x.

Problem 39. Consider the map $\mathbf{f} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$

$$f_1(n_1, n_2) = |n_1 - n_2|, \qquad f_2(n_1, n_2) = n_1 + n_2.$$

Is the map invertible?

Problem 40. The Kustaanheimo-Stiefel map $\mathbb{R}^4 \to \mathbb{R}^3$ $((u_1, u_2, u_3, u_4) \to (x_1, x_2, x_3))$ is given by

$$x_1 = 2(u_1 u_3 - u_2 u_4), \quad x_2 = 2(u_1 u_4 + u_2 u_3), \quad x_3 = u_1^2 + u_2^2 - u_3^2 - u_4^2$$

together with the constraint

$$\alpha \equiv u_2 du_1 - u_1 du_2 - u_4 du_3 + u_3 du_4 = 0.$$

Show that applying the Kustaanheimo-Stiefel map the Laplacian operator Δ_3 in $\mathbb{R}^3$ can be written as

$$\Delta_3 = \frac{1}{4r}\Delta_4 - \frac{1}{4r^2}V^2$$

where

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2} = u_1^2 + u_2^2 + u_3^2 + u_4^2, \quad V = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2} - u_4 \frac{\partial}{\partial u_3} + u_3 \frac{\partial}{\partial u_4}.$$

Problem 41. Let $n, m \geq 0$. Consider the differential operators

$$K_n = \sum_{j=0}^{n} u_j(x) \frac{d^j}{dx^j}, \qquad L_m = \sum_{j=0}^{m} v_j(x) \frac{d^j}{dx^j}$$

where the $u_j(x)$'s and $v_j(x)$'s are smooth functions. If the two differential operators K_n and L_m commute, then there is a nonzero polynomial $R(z, w)$ such that $R(K_n, L_m) = 0$. The curve Γ defined by $R(z, w) = 0$ is called the spectral curve. If we consider the eigenvalue problem

$$K_n \psi = z\psi, \qquad L_m \psi = w\psi$$

then $(z, w) \in \Gamma$.
(i) Let $(\alpha \in \mathbb{R})$

$$K = \left(\frac{d^2}{dx^2} + x^3 + \alpha\right)^2 + 2x, \quad L = \left(\frac{d^2}{dx^2} + x^3 + \alpha\right)^3 + 3x\frac{d^2}{dx^2} + 3\frac{d}{dx} + 3x(x^2 + \alpha).$$

Show that K and L commute with $w^2 = z^3 - \alpha$.
(ii) Show that

$$K = \left(\frac{d^3}{dx^3} + x^2 + \alpha\right)^2 + 2\frac{d}{dx}$$

and

$$L = \left(\frac{d^3}{dx^3} + x^2 + \alpha\right)^3 + 3\frac{d^4}{dx^4} + 3(x^2 + \alpha)\frac{d}{dx} + 3x$$

commute.

Problem 42. Let $F_{GG} : (\mathbb{N} \to \mathbb{N}) \to ((\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}))$ be defined by

$$[F_{GG}(f)](g) := f \circ g \circ g.$$

Let $Y : (\mathbb{N} \to \mathbb{N}) \to ((\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}))$ be defined by

$$Y(f) := F_{GG}(f) \circ F_{GG}(f).$$

Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$. Show that

$$[Y(f)](g) = f \circ [Y(f)](g).$$

Chapter 3

Functions

Problem 1. Let $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, \ldots, n$ be real-valued functions with continuous second-order partial derivatives everywhere on $\mathbb{R}^n$. Suppose that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leq i \leq n$. Prove that there is a function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f_i + \partial g / \partial x_i$ is linear for all $i = 1, 2, \ldots, n$. A linear function $p : \mathbb{R}^n \to \mathbb{R}$ is of the form

$$p(\mathbf{x}) = a_0 + a_1 x + a_2 x_2 + \cdots + a_n x_n.$$

Problem 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $\mathbf{x} \in \mathbb{R}^n$. Consider the invertible $n \times n$ matrix R and the transformation $\mathbf{x}' = R\mathbf{x}$. The transformation operator $\hat{P}_R$ associated with the invertible matrix R is defined by

$$\hat{P}_R f(\mathbf{x}) := f(R^{-1}\mathbf{x}).$$

Note that the operator acts upon the coordinates $\mathbf{x}$ and not on the argument of f. This means $\hat{P}_R \hat{P}_S f(\mathbf{x}) = f(S^{-1}R^{-1}\mathbf{x})$, where S is another invertible $n \times n$ matrix. Show that

$$\hat{P}_R \hat{P}_S f(\mathbf{x}) = \hat{P}_{RS} f(\mathbf{x}).$$

Problem 3. A ratio list is a finite list of positive numbers, $(r_1, r_2, \ldots, r_n)$. An iterated function system realizing a ratio list $(r_1, r_2, \ldots, r_n)$ in a metric space S is a list $(f_1, f_2, \ldots, f_n)$, where $f_j : S \to S$ is a similarity with ratio r_j. A nonempty compact set $K \subseteq S$ is an invariant set for the iterated function system $(f_1, f_2, \ldots, f_n)$ iff $K = f_1(K) \cup f_2(K) \cup \cdots \cup f_n(K)$. The triadic Cantor set is an invariant set for an iterated function system realizing the ratio list $(1/3, 1/3)$. The Sierpinski gasket is an invariant set for an iterated system realizing the ratio list $(1/2, 1/2, 1/2)$. The dimension associated with a ratio list $(r_1, r_2, \ldots, r_n)$ is the positive number s such that $r_1^s + r_2^s + \cdots + r_n^s = 1$. Let $(r_1, r_2, \ldots, r_n)$ be a ratio list. Suppose each $r_j < 1$. Show that there is a unique nonnegative number s satisfying

$$\sum_{j=1}^{n} r_j^s \equiv \sum_{j=1}^{n} e^{s \ln r_j} = 1.$$

The number s is 0 iff $n = 1$.

Problem 4. The arc length of the equilateral hyperbola

$$h(t) = \sqrt{t^2 - 1}, \qquad t \geq 1$$

starting at $t = 1$ is given by

$$L_h(x) = \int_1^x \sqrt{\frac{2t^2 - 1}{t^2 - 1}} dt$$

as a function of the terminal point $t = x$. The tangent line to the hyperbola at $t = x$ is

$$T_h(t) = \sqrt{x^2 - 1} + \frac{x}{\sqrt{x^2 - 1}}(t - x)$$

whose intersection with the t-axis is $t = 1/x$ ($t \in (0, 1)$). The line

$$N_h(t) = -\frac{\sqrt{x^2 - 1}}{x} t$$

is perpendicular to T_h passing through the origin.
(i) Find the point P_h where the lines T_h and N_h intersect.
(ii) Calculate the distance from $(x, h(x))$ to the common point P_h.

Problem 5. Given a differentiable function f. The logarithmic derivative of f is defined as

$$\frac{1}{f} \frac{df}{dx}.$$

When f is a real function of real variables and takes strictly positive values then the chain rule provides

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f} \frac{df}{dx}.$$

The logarithmic derivative has the following properties.
1. The logarithmic derivative of the product of functions is the sum of their logarithmic derivatives

$$\frac{1}{fg}\frac{d}{dx}(fg) = \frac{1}{f}\frac{df}{dx} + \frac{1}{g}\frac{dg}{dx}$$

2. The logarithmic derivative of the quotient of functions is the difference of their logarithmic derivatives

$$\frac{1}{f/g}\frac{d}{dx}(f/g) = \frac{1}{f}\frac{df}{dx} - \frac{1}{g}\frac{dg}{dx}$$

3. The logarithmic derivative of the α-th power of a function f is α times the logarithmic derivative of the function

$$\frac{1}{f^\alpha}\frac{d}{dx}(f^\alpha) = \alpha\frac{1}{f}\frac{df}{dx}$$

4. The logarithmic derivative of the exponential of a function equals the derivative of a function

$$\frac{1}{e^f}\frac{d}{dx}e^f = \frac{df}{dx}$$

(i) Find the logarithmic derivative of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cosh(x)$.
(ii) Let f be a meromorphic function in the open and conncted set $D \subseteq \mathbb{C}$. Let $G \subseteq D$ be a region such that its closure $\overline{G} \subseteq D$ and its boundary ∂G is a continuous curve not containing a zero or pole of f. Let N be the number of zeros of f lying inside G and P be the number of poles of f lying inside G. Then

$$N - P = \frac{1}{2\pi i}\int_{\partial G}\frac{1}{f(z)}\frac{df(z)}{dz}dz$$

where ∂G is an oriented boundary of G. Calculate the left and right-hand side for the function $f(z) = 1/z$ and $G = \{\,(x,y) \,:\, x^2 + y^2 \leq 1\,\}$.

Problem 6. Let f be an analytic function. Let $p \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Show that

$$\left(\frac{d}{dx} + \alpha x\right)^p f \equiv \exp\left(-\frac{\alpha x^2}{2}\right)\frac{d^p}{dx^p}\left(\exp\left(\frac{\alpha x^2}{2}\right) f\right).$$

Problem 7. Let $c_1, c_2 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Show that

$$\exp\left(c_1\frac{d}{dx}\right)\exp(c_2 x)f(x) \equiv \exp(c_2 x)\exp\left(c_1\frac{d}{dx}\right)\exp(c_1 c_2)f(x).$$

Problem 8. Consider

$$K(t,s) = \begin{cases} 1 \text{ if } 0 \le t \le s \\ 0 \text{ if } s \le t \le 1 \end{cases}.$$

Show that this *kernel* satisfies the functional equation

$$K(cu+a,t) = K\left(u, \frac{t-a}{c}\right)$$

where $c > 0$.

Problem 9. Find a continuous differentiable function $f : \mathbb{R} \to \mathbb{R}$ which has no fixed points and no critical points.

Problem 10. Let n be a non-negative integer. Let $k = 0, 1, \ldots, n$. Find the derivative

$$\frac{\partial^n}{\partial x^k \partial x^{n-k}}(x+y)^n.$$

Problem 11. Let $a, b, p, q, r \in \mathbb{R}$, $b > a$, $a < p$, $p < r$, $r < q$, $q < b$ and

$$p - a = r - p, \qquad q - r = b - q.$$

In fuzzy logic the following membership function plays an important role

$$f(x; a, b, r, p, q) = \begin{cases} 0 & x \le a \\ 2^{m-1}((x-a)/(r-a))^m & a < x \le p \\ 1 - 2^{m-1}((r-x)/(r-a))^m & p < x \le r \\ 1 - 2^{m-1}((x-r)/(b-r))^m & r < x \le q \\ 2^{m-1}((b-x)/(b-r))^m & q < x < b \\ 0 & x \ge b \end{cases}$$

where m is the fuzzifier. Im most cases $m = 2$. Where are the crossover points? What is the value at the centre?

Problem 12. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping of $(x, y) \to (u, v)$ given by

$$u(x,y) = e^x \cos(y), \qquad v(x,y) = e^x \sin(y).$$

What is the range of f? Show that the Jacobian determinant is not zero at any point of $\mathbb{R}^2$. Thus every pont of $\mathbb{R}^2$ has a neighbourhood in which f is one-to-one. Nevertheless f is not one-to-one on $\mathbb{R}^2$. What are the images under f of lines parallel to the coordinate axes?

Problem 13. The content (n-dimensional volume) bounded by a hypersphere of radius r is known to be

$$V_n = \frac{2r^n \pi^{n/2}}{n\Gamma(n/2)}$$

where Γ is the gamma function. Let $r = 1$. Find

$$\lim_{n\to\infty} V_n.$$

Problem 14. Let $x, y \in \mathbb{R}$. Calculate the square root

$$\sqrt{x^2 + y^2 - 2xy}.$$

Problem 15. A criterion for linearly independence of functions $f_0, f_1, \ldots, f_n \in C^n[a,b]$ is that the *Wronski determinant* is nonzero

$$\det \begin{pmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{pmatrix} \neq 0.$$

Here $'$ denotes derivative. Apply this criterion to the functions

$$f_0(x) = \cos(x), \qquad f_1(x) = \sin(x).$$

Problem 16. For a sphere of radius r and mass density ρ the mass that must be concentrated at its centre is $(\lambda \geq 0)$

$$M(\lambda) = \frac{4\pi r\rho}{\lambda^2}(\cosh(\lambda r) - \sinh(\lambda r)/(\lambda r)).$$

Find $\lim_{\lambda\to 0} M(\lambda)$.

Problem 17. Let $-1 < a < 1$. Find the inverse of the transformation

$$\lambda(z) = \frac{z-a}{1-az}.$$

Problem 18. Consider the Jacobi elliptic functions

$$\mathrm{sn}(x,k), \quad \mathrm{cn}(x,k), \quad \mathrm{dn}(x,k)$$

where $x \in \mathbb{R}$ and $k^2 \in [0,1]$. We have

$$\mathrm{sn}(x,0) = \sin(x), \quad \mathrm{cn}(x,0) = \cos(x), \quad \mathrm{dn}(x,0) = 1$$

and

$$\mathrm{sn}(x,1) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh(x), \quad \mathrm{cn}(x,1) = \mathrm{dn}(x,1) = \frac{2}{e^x + e^{-x}} \equiv \mathrm{sech}(x).$$

(i) Find an expression using Jacobi elliptic functions that interpolates between $\sin(x)$ for $k = 0$ and $\sinh(x)$ for $k = 1$.
(ii) Find an expression using Jacobi elliptic functions that interpolates between $\cos(x)$ for $k = 0$ and $\cosh(x)$ for $k = 1$.
(iii) Use this result to interpolate between the matrices

$$\begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}$$

and

$$\begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}.$$

Problem 19. Let $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable function. Find the determinant of the 3×3 matrix $A = (a_{jk})$

$$a_{jk} := \frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j}.$$

Problem 20. Let f_1, f_2 be differentiable functions and $f_1(x) > 0$, $f_2(x) > 0$ for all x. Let $f(x) = f_1(x) f_2(x)$. Find

$$\frac{d}{dx}(\ln(f(x))).$$

Problem 21. Let $x \in \mathbb{R}$. The sequence of functions $\{ f_k(x) \}$ is defined by $f_1(x) = \cos(x/2)$ and for $k > 1$ by

$$f_k(x) = f_{k-1}(x) \cos(x/2^k).$$

Thus

$$f_k(x) = \cos(x/2) \cos(x/2^2) \cdots \cos(x/2^k).$$

Obviously, we have $f_k(0) = 1$ for every k. Calculate $\lim_{k \to \infty} f_k(x)$ as a function of x for $x \neq 0$.

Problem 22. (i) Consider the transformation in $\mathbb{R}^3$

$$\begin{aligned} x_0(a,\theta_1) &= \cosh(a) \\ x_1(a,\theta_1) &= \sinh(a)\sin(\theta_1) \\ x_2(a,\theta_1) &= \sinh(a)\cos(\theta_1) \end{aligned}$$

where $a \geq 0$ and $0 \leq \theta_1 < 2\pi$. Find

$$x_0^2 - x_1^2 - x_2^2.$$

(ii) Consider the transformation in $\mathbb{R}^4$

$$\begin{aligned} x_0(a,\theta_1,\theta_2) &= \cosh(a) \\ x_1(a,\theta_1,\theta_2) &= \sinh(a)\sin(\theta_2)\sin(\theta_1) \\ x_2(a,\theta_1,\theta_2) &= \sinh(a)\sin(\theta_2)\cos(\theta_1) \\ x_3(a,\theta_1,\theta_2) &= \sinh(a)\cos(\theta_2) \end{aligned}$$

where $a \geq 0$, $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_2 \leq \pi$. Find

$$x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Extend the transformation to $\mathbb{R}^n$.

Problem 23. A fixed charge Q is located on the z-axis with coordinates $\mathbf{r}_a = (0,0,d/2)$, where d is interfocal distance of the *prolate spheroidal coordinates*

$$\begin{aligned} x(\eta,\xi,\phi) &= \frac{1}{2}d((1-\eta^2)(\xi^2-1))^{1/2}\cos\phi \\ y(\eta,\xi,\phi) &= \frac{1}{2}d((1-\eta^2)(\xi^2-1))^{1/2}\sin(\phi) \\ z(\eta,\xi,\phi) &= \frac{1}{2}d\eta\xi \end{aligned}$$

where $-1 \leq \eta \leq +1$, $1 \leq \xi \leq \infty$, $0 \leq \phi \leq 2\pi$. Express the *Coulomb potential*

$$V = \frac{Q}{|\mathbf{r} - \mathbf{r}_a|}$$

in prolate spheroidal coordinates.

Problem 24. *Toroidal coordinates* are defined by

$$\begin{aligned} x(\alpha,\beta,\phi) &= \frac{c\sinh(\alpha)\cos(\phi)}{\cosh(\alpha)-\cos(\beta)} \\ y(\alpha,\beta,\phi) &= \frac{c\sinh(\alpha)\sin(\phi)}{\cosh(\alpha)-\cos(\beta)} \\ z(\alpha,\beta,\phi) &= \frac{c\sin(\beta)}{\cosh(\alpha)-\cos(\beta)}. \end{aligned}$$

Use the L'Hosptial rule to find x, y, z for $\alpha \to 0$ and $\beta \to 0$.

Problem 25. Let $c, \theta \in \mathbb{R}$ and

$$f(\theta) = c(e^{i\theta} + e^{-i\theta}).$$

Calculate $\exp(f(\theta))$.

Problem 26. Let $n \geq 1$. Show that the vector space spanned by

$$x^n, yx^{n-1}, \ldots, y^{n-1}x, y^n$$

is $n+1$ dimensional.

Problem 27. A continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is called an *alternating function* if

$$f(x, y) = -f(y, x).$$

Give an example of an analytic alternating function. Find the minima and maxima of the function.

Problem 28. Let

$$f(\theta) = \frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}.$$

Find $f(\theta)$ for $\theta = n\pi$ $(n \in \mathbb{Z})$ using the L'Hospital rule. Plot $f(\theta)$ as a function of θ.

Problem 29. Show that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Problem 30. Consider the functions

$$j_1(x) = \frac{1}{x^2}(\sin(x) - x\cos(x))$$
$$j_2(x) = \frac{1}{x^3}((3 - x^2)\sin(x) - 3x\cos(x)).$$

Use the L'Hospital rule to find $j_1(0)$ and $j_2(0)$.

Problem 31. Find the invariance group of the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sin(x).$$

Problem 32. Let $a > 0$. Consider the transformation

$$u(x,y) = \frac{a\sin(2ax)}{2(\sin^2(ax)+\sinh^2(ay))}, \qquad v(x,y) = \frac{a\sinh(2ay)}{2(\sin^2(ax)+\sinh^2(ay))}.$$

Find the inverse transformation.

Problem 33. Let $\epsilon, x \in \mathbb{R}$. Find

$$\lim_{\epsilon\to 0}\frac{\sinh(\epsilon x)}{\sinh(\epsilon)}, \qquad \lim_{\epsilon\to 0}\frac{\sin(\epsilon x)}{\sin(\epsilon)}.$$

Problem 34. (i) Find an analytic function $f : \mathbb{R} \to \mathbb{R}$ that has no fixed point and no critical point. Draw the function.
(ii) Find an analytic function $f : \mathbb{R} \to \mathbb{R}$ that has no fixed point and exactly one critical point at $x = 0$. Draw the function.

Problem 35. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Calculate the commutator

$$\left[\cos(x)\frac{d}{dx}, \sin(x)\frac{d}{dx}\right] f.$$

Problem 36. (i) Consider the analytic function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1,x_2) = \sinh(x_2), \quad f_2(x_1,x_2) = \sinh(x_1).$$

Show that this function admits the (only) fixed point $(0,0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

(ii) Consider the analytic function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$

$$g_1(x_1,x_2) = \sinh(x_1), \quad g_2(x_1,x_2) = -\sinh(x_2).$$

Show that this function admits the (only) fixed point $(0,0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

(iii) Multiply the two matrices found in (i) and (ii).

(iv) Find the composite function $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{h}(\mathbf{x}) = (\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x})).$$

Show that this function also admits the fixed point $(0,0)$. Find the functional matrix at this fixed point

$$\left. \begin{pmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 \end{pmatrix} \right|_{(0,0)} .$$

Compare this matrix with the matrix found in (iii).

Problem 37. An approximation of e^{-x} $(x \geq 0)$ as a rational polynomial using a 3rd order Padé approximation is given by

$$e^{-x} \approx \frac{1 - x/2 + x^2/10 - x^3/120}{1 + x/2 + x^2/10 + x^3/120}.$$

Note that $e^{-x} \geq 0$ for all x. For which value of $x > 0$ does the right-hand side takes negative values?

Problem 38. Let $x \in \mathbb{R}$. Consider the function

$$f(x) = \frac{x - i}{x + i}.$$

Find $f(0)$, $f(1)$, $f(-1)$ and $f(x \to \infty)$. Is there an inverse?

Problem 39. Consider the function

$$f(x) = x^3 - x^2.$$

Find an approximation of the derivative of f by using

$$f'(x) \approx \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h} - \frac{1}{30} h^4 f^{(5)}(\xi)$$

$$f'(x) \approx \frac{-3f(x-h) - 10f(x) + 18f(x+h) - 6f(x+2h) + f(x+3h)}{12h} + \frac{1}{20} h^4 f^{(5)}(\xi)$$

$$f'(x) \approx \frac{-25f(x) + 48f(x+h) - 36f(x+2h) + 16f(x+3h) - 3f(x+4h)}{12h} - \frac{1}{5} h^4 f^{(5)}(\xi)$$

where $x \leq \xi \leq x + h$.

Problem 40. Consider the function

$$f(x) = \frac{3}{x^3}(\sin(x) - x\cos(x)).$$

Find $f(0)$ using the L'Hospital rule.

Problem 41. The *Hurwitz zeta-function* $\zeta_H(s,a)$ is defined by

$$\zeta_H(s,a) := \sum_{k=0}^{\infty}(n+a)^{-s}, \quad 0 < a \leq 1, \quad \Re(s) > 1.$$

(i) Show that the Hurwitz zeta-function can be written in the form

$$\zeta_H(s,a) = a^{-s} + \frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\int_0^{\infty} t^{s-1}e^{-(n+a)t}dt$$

where $\Gamma(s)$ is the Gamma function.
(ii) Show that

$$\frac{\partial}{\partial a}\zeta_H(s,a) = -s\zeta_H(s+1,a)$$
$$\frac{\partial}{\partial s}\zeta_H(s,a)\bigg|_{s=0} = \ln(\Gamma(a)) - \frac{1}{2}\ln(2\pi).$$

Problem 42. Let $z \in \mathbb{C}$. The *Airy functions* $Ai(z)$ and $Bi(z)$ are defined as sums of the power series

$$Ai(z) = c_1 f(z) - c_2 g(z)$$
$$Bi(z) = \sqrt{3}(c_1 f(z) + c_2 g(z))$$

where

$$f(z) = 1 + \frac{1}{3!}z^3 + \frac{1\cdot 4}{6!}z^6 + \frac{1\cdot 4\cdot 7}{9!}z^9 + \cdots$$
$$g(z) = z + \frac{2}{4!}z^4 + \frac{2\cdot 5}{7!}z^7 + \frac{2\cdot 5\cdot 8}{10!}z^{10} + \cdots$$

with

$$c_1 = \frac{1}{2\pi}\Gamma\left(\frac{1}{3}\right)3^{-1/6}, \qquad c_2 = \frac{1}{2\pi}\Gamma\left(\frac{2}{3}\right)3^{1/6}.$$

Show that the radius of convergence of these infinite series is infinite.

Problem 43. Find the inverse of the function $y = \tanh(x)$. Note that $y \in (-1,1)$.

Problem 44. Consider the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \sum_{j=0}^{\infty}\frac{x^j}{(j+1)!} \equiv \frac{e^x - 1}{x}.$$

Find the fixed points and critical points of the function. Note that $f(0) = 1$.

Problem 45. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Show that (differential identity)

$$\frac{d^n}{dx^n} f(x) = x^{-n} \frac{d^n}{d\epsilon^n} f(\epsilon x)\bigg|_{\epsilon=1} .$$

Problem 46. Find

$$\lim_{x \to 0} \frac{\sinh(x)}{x}, \qquad \lim_{x \to 0} \frac{x}{\sinh(x)} .$$

Problem 47. Find non-negative analytic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 0.$$

Problem 48. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{x\cosh(x) - \sinh(x)}{2x\sinh^2(x)} .$$

Find

$$\lim_{x \to 0} f(x), \qquad \lim_{x \to \infty} f(x).$$

Problem 49. Find an analytic function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\infty) = 1, \quad f(-\infty) = 0$$

and

$$f(x_1) \le f(x_2) \ \text{ whenever } \ x_1 \le x_2 .$$

Find the derivative of the function f and determine the maxima of the function df/dx.

Problem 50. The *Euler dilogarithm function* $\mathrm{Li}_2(x)$ is defined for $x \in (0,1)$ as

$$\mathrm{Li}_2(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^2} = -\int_0^x \frac{\ln(1-t)}{t} dt.$$

It can be analytically continued to the complex plane with the branch cut from 1 to ∞ along the real axis using the integral. Calculate $\mathrm{Li}_2(1/2)$.

Problem 51. Consider the functions

$$f(x) = \frac{x}{\sinh(x)}, \qquad g(x) = \frac{\sinh(x)}{x}.$$

Find

$$\lim_{x \to 0} f(x), \qquad \lim_{x \to 0} g(x).$$

Problem 52. Let Ai be the Airy function and J_n, I_n are the Bessel and modified Bessel functions of order n. Show that

$$\begin{aligned}
\frac{d^2 Ai}{dx^2} &= x Ai(x) \\
Ai(x) &= \frac{1}{3} x^{1/2} (I_{-1/3}(z) - I_{1/3}(z)) \\
Ai(-x) &= \frac{1}{3} x^{1/2} (J_{1/3}(z) + J_{-1/3}(z))
\end{aligned}$$

where $z := 2x^{3/2}/3$.

Problem 53. Let $k \in \mathbb{R}$, $x, y \in \{+1, -1\}$ such that $xy = \pm 1$. Show that

$$e^{kxy} = \cosh(k) + xy \sinh(k).$$

Problem 54. Let

$$f(x) = \frac{\sin(x)}{\sinh(x)}.$$

Find $f(0)$.

Problem 55. Let $\alpha \in \mathbb{R}$. Find

$$\lim_{\alpha \to \infty} \frac{1 - e^{-\alpha}}{\alpha}.$$

Problem 56. Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = f(-x), \quad f(x) = f(x + 2\pi), \quad f(0) = 0.$$

Problem 57. Let N be an integer and $N \geq 2$. The generalized hyperbolic function for a given N is defined as

$$f_j^{(N)}(x) := \sum_{k=0}^{\infty} \frac{x^{j+kN}}{(j+kN)!}, \qquad j = 0, 1, \ldots, N-1.$$

The functions $f_j^{(N)}$ are analytic.
(i) Show that

$$f_j^{(N)}(x) = f_{j+N}^{(N)}(x)$$

i.e. $f_j^{(N)}$ is periodic with respect to the index j.
(ii) Show that

$$\frac{d}{dx} f_j^{(N)}(x) = f_{j-1}^{(N)}(x).$$

(iii) Let $\omega^N = 1$, i.e. ω is the N-th primitive root of unity. Show that

$$f_j^{(N)}(x) = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-jk} \exp(\omega^k x).$$

Problem 58. Let $n = 2, 3, 4, \ldots$. Find

$$\frac{\Gamma(n/2)}{\Gamma(n)}$$

where Γ denotes the gamma function.

Problem 59. Consider the map $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^2 + x + 1.$$

Find the minima and maxima of

$$g(x) = f(f(x)) - f(x)f(x).$$

Problem 60. Consider the hyperspherical coordinates

$$\begin{aligned}
x_1(\theta, \phi, \psi) &= \cos(\theta) \\
x_2(\theta, \phi, \psi) &= \sin(\theta)\cos(\phi) \\
x_3(\theta, \phi, \psi) &= \sin(\theta)\sin(\phi)\cos(\psi) \\
x_4(\theta, \phi, \psi) &= \sin(\theta)\sin(\phi)\sin(\psi)
\end{aligned}$$

with $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. Show that the *angular distance* can be calculated as

$$\cos(d_{jk}) = \cos(\theta_j)\cos(\theta_k) + \sin(\theta_j)\sin(\theta_k)(\cos(\phi_j)\cos(\phi_k) + \sin(\phi_j)\sin(\phi_k)\cos(\psi_j - \psi_k)).$$

Problem 61. Let $-1 < x < 1$ and

$$\sum_{j=0}^{\infty} a_j x^j = \frac{1}{\sqrt{1-x}}.$$

Find the expansion coefficients a_j.

Problem 62. (i) Show that

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$$

(ii) Show that

$$\lim_{x\to 1} \frac{x-1}{\ln(x)} = 1.$$

Problem 63. Show that the function

$$f(x) = \cos(x)\cosh(x) + 1$$

has infinitely many roots, i.e. solutions of $f(x) = 0$. What happens for x large?

Problem 64. Let $\gamma > 0$. A random variable X is said to be Lorentzian with parameters (α, γ) if its probability density is given by

$$f_X(x) = \frac{1}{\pi} \frac{\gamma}{(x-\alpha)^2 + \gamma^2}, \quad x \in \mathbb{R}.$$

Let X, Y be two independent Lorentzian random variables with parameters (α, γ) and (β, δ), respectively. Let $\epsilon \geq 0$. Show that
(a) ϵX is distributed Lorentzian $(\epsilon\alpha, \epsilon\gamma)$
(b) $\epsilon + X$ is distributed Lorentzian $(\epsilon + \alpha, \gamma)$
(c) $-X$ is distributed Lorentzian $(-\alpha, \gamma)$
(d) $X + Y$ is distributed Lorentzian $(\alpha + \beta, \gamma + \delta)$
(e) X^{-1} is distributed Lorentzian $\left(\frac{\alpha}{\alpha^2+\gamma^2}, \frac{\gamma}{\alpha^2+\gamma^2}\right)$

Problem 65. The Jacobi elliptic functions $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ with $k \in [0,1]$ and $k^2 + k'^2 = 1$ have the properties

$$\mathrm{sn}(x,0) = \sin(x), \quad \mathrm{cn}(x,0) = \cos(x), \quad \mathrm{dn}(x,0) = 1$$

and

$$\mathrm{sn}(x,1) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \mathrm{cn}(x,1) = \frac{2}{e^x + e^{-x}} = \mathrm{dn}(x,1).$$

We define

$$\begin{aligned} u_1(x,y,k,k') &= \mathrm{sn}(x,k)\mathrm{dn}(y,k') \\ u_2(x,y,k,k') &= \mathrm{cn}(x,k)\mathrm{cn}(y,k') \\ u_3(x,y,k,k') &= \mathrm{dn}(x,k)\mathrm{sn}(y,k'). \end{aligned}$$

(i) Find $u_1(x,y,0,1)$, $u_2(x,y,0,1)$, $u_3(x,y,0,1)$ and calculate $u_1^2(x,y,0,1)+u_2^2(x,y,0,1)+u_3^2(x,y,0,1)$.
(ii) Find $u_1(x,y,1,0)$, $u_2(x,y,1,0)$, $u_3(x,y,1,0)$ and calculate $u_1^2(x,y,1,0)+u_2^2(x,y,1,0)+u_3^2(x,y,1,0)$.

Problem 66. Let m be an integer with $m \geq 1$. Then the gamma function Γ is given by

$$\Gamma(m) = (m-1)!.$$

Thus $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 6$, $\Gamma(5) = 24$. Furthermore

$$\Gamma(m+\frac{1}{2}) = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2m-1)}{2^m}\sqrt{\pi}.$$

Thus $\Gamma(3/2) = \sqrt{\pi}/2$, $\Gamma(5/2) = 3\sqrt{\pi}/4$, $\Gamma(7/2) = 15\sqrt{\pi}/8$, $\Gamma(9/2) = 105\sqrt{\pi}/16$. Calculate

$$\frac{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

for $m=5$, $m=10$, $m=20$.

Problem 67. Let $c_1 > 0$, $c_2 > 0$. Consider the function $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = x^2 + \frac{c_1 x^2}{1+c_1 x^2}.$$

Find the minima and maxima. Find the fixed points.

Problem 68. Let $c > 0$. Consider the function $f:\mathbb{R}\to\mathbb{R}$

$$f_c(x) = \frac{4}{x^2}\sin^2\left(\frac{cx}{2}\right).$$

Find $f_c(0)$. Show that f_c has a maximum at $x=0$.

Problem 69. (i) Find the minima and maxima of the analytic function $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = \cosh(x)\cos(x).$$

Find the fixed points.
(ii) Find the minima and maxima of the analytic function $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = \sinh(x)\sin(x).$$

Find the fixed points.

Problem 70. Find

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}.$$

Problem 71. Consider the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \frac{xe^x(e^y - 1) - ye^y(e^x - 1)}{xy(e^x - e^y)}.$$

(i) Show that $f(0,0) = \frac{1}{2}$ applying the L'Hospital rule.
(ii) Show that

$$f(x,x) = \frac{e^x - x - 1}{x^2}.$$

(iii) Show that

$$f(x,-x) = \frac{\tanh(x/2)}{x}.$$

Chapter 4

Polynomial

Problem 1. The *Chebyshev polynomials* are defined by

$$T_k(x) = \cos(k \arccos(x)), \quad k = 0, 1, \ldots \qquad x \in [-1, 1]$$

Thus the first six polynomials are

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x.
\end{aligned}$$

Find $1, x, x^2, x^3, x^4, x^5$ as functions of T_0, T_1, T_2, T_3, T_4, T_5.

Problem 2. Let $L_n(x)$, $H_n(x)$ be the *Laguerre polynomials* and *Hermite polynomials*, where $n = 0, 1, \ldots$. Let

$$L_n^{(\alpha)}(x) := \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha})$$

be the associated Laguerre polynomials with $\alpha > -1$ and $n = 0, 1, \ldots$. The Laguerre polynomials are recovered by setting $\alpha = 0$. We have

$$H_{2n}(x) = (-4)^n n! L_n^{(-1/2)}(x^2) \tag{1}$$

and the following addition formula for the associated Laguerre polynomials $L_n^\alpha(x)$

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_{n-k}^{(\alpha)}(x) L_k^{(\beta)}(y)\,. \qquad (2)$$

(i) Find a new sum rule by inserting (2) into (1).
(ii) Consider the sum rule

$$\frac{1}{n!2^n} H_n(\sqrt{2}x) H_n(\sqrt{2}y) = \sum_{k=1}^{n} (-1)^k L_{n-k}^{(-1/2)}((x+y)^2) L_k^{(-1/2)}((x-y)^2). \quad (3)$$

Insert (1) into (3) to find a sum rule for Hermite polynomials.

Problem 3. The *Hermite polynomial* of degree n can be written as

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}.$$

Express x^n using the Hermite polynomials.

Problem 4. Given the differentiable function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = 4x(1-x).$$

(i) Find the *fixed points* of $f(f(x))$.
(ii) Find the *critical points* of $f(f(x))$.

Problem 5. Let $P(z)$ be a polynomial of degree $n \geq 2$ with distinct zeros $\zeta_1, \dots, \zeta_n$. Show that

$$\sum_{j=1}^{n} \frac{1}{P'(\zeta_j)} = 0$$

where $'$ denotes the derivative, i.e. $P'(\zeta) \equiv dP(z=\zeta)/dz$.

Problem 6. Given a set of N real numbers $x_1, x_2, \dots, x_N$. It is often useful to express the sum of the j powers

$$s_j = x_1^j + x_2^j + \cdots + x_N^j, \qquad j = 0, 1, 2, \dots$$

in terms of the *elementary symmetric functions*

$$\sigma_1 = \sum_{i=1}^{N} x_i$$

$$\sigma_2 = \sum_{i<j}^{N} x_i x_j$$

$$\sigma_3 = \sum_{i<j<k}^{N} x_i x_j x_k$$

$$\vdots \quad \vdots$$

$$\sigma_N = x_1 x_2 \cdots x_N.$$

Consider the special case with three numbers x_1, x_2, x_3. Then the elementary symmetric functions are given by

$$\sigma_1 = x_1 + x_2 + x_3, \quad \sigma_2 = x_1 x_2 + x_1 x_3 + x_2 x_3, \quad \sigma_3 = x_1 x_2 x_3.$$

We know that the elementary symmetric functions are the coefficients (up to sign) of the polynomial with the roots x_1, x_2, x_3. In other words the values of x_1, x_2, x_3 each satisfy the polynomial equation

$$x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3 = 0. \tag{1}$$

Find a *recursion relation* for

$$s_j := x_1^j + x_2^j + x_3^j, \qquad j = 0, 1, 2, \ldots$$

and give the initial values s_0, s_1, s_2. Calculate s_3 and s_4.

Problem 7. The *dominant tidal potential* at position (r, ϕ, λ) due to the moon or sun is given by

$$U(\mathbf{r}) = \frac{GM^* r^2}{r^{*3}} P_2^0(\cos\psi)$$

where M^* is the mass of the moon or sun located at (r^*, ϕ^*, λ^*). Moreover, ψ is the angle between mass M^* and the observation point at (r, ϕ, λ), where ϕ is the latitude and λ is the longitude. By the *spherical cosine theorem* we have

$$\cos\psi = \sin(\phi)\sin(\phi^*) + \cos(\phi)\cos(\phi^*)\cos(\lambda - \lambda^*).$$

The *Legendre polynomials* are defined as

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

with $n = 0, 1, 2, \ldots$ and the *associated Legendre polynomials* are defined as

$$P_n^m(x) := (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) = \frac{(1 - x^2)^{m/2}}{2^n n!} \frac{d^{m+n}}{dx^{m+n}} (x^2 - 1)^n$$

with $P_n^0(x) = P_n(x)$ and $P_n^m = 0$ if $m > n$.
(i) Show that $U(\mathbf{r})$ can be written as

$$U(\mathbf{r}) = \frac{GM^* r^2}{r^{*3}} \left(P_2^0(\sin\phi) P_2^0(\sin\phi^*) + \frac{1}{3} P_2^1(\sin\phi) P_2^1(\sin\phi^*) \cos(\lambda - \lambda^*) \right.$$
$$\left. + \frac{1}{12} P_2^2(\sin\phi) P_2^2(\sin\phi^*) \cos(2(\lambda - \lambda^*)) \right).$$

(ii) Give an interpretation (maxima and nodes) of the terms in the parenthesis.

Problem 8. Let p_1 and p_2 two polynomials with $n = \text{degree}(p_1)$, $m = \text{degree}(p_2)$ and $n > m$. We expand the rational function

$$r(x) = \frac{p_2(x)}{p_1(x)}$$

with respect to powers of $1/x$, i.e.,

$$r(x) = d_1 x^{-1} + d_2 x^{-2} + \cdots$$

The coefficients $d_1, d_2, \ldots, d_{2n-1}$ are inserted into the $n \times n$ Hankel matrix

$$H_n = \begin{pmatrix} d_1 & d_2 & \ldots & d_n \\ d_2 & d_3 & \ldots & d_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ d_n & d_{n+1} & \ldots & d_{2n-1} \end{pmatrix}.$$

A *Hankel matrix* is a diagonal matrix in which all the elements are the same along any diagonal that slopes from northeast to southwest. If the determinant of this matrix is zero, then the two polynomials have a non-trival common divisor. Apply this algorithm to the polynomials

$$p_1(x) = x^3 + 6x^2 + 11x + 6, \qquad p_2(x) = x^2 + 6x + 8$$

to test whether they have a common non-trivial divisor.

Problem 9. Find the square root of $z = 4\exp(i\pi/4)$.

Problem 10. Show that the set

$$S = \{\, \omega_1 = -\frac{1}{2} + \frac{i}{2}\sqrt{3}, \quad \omega_2 = -\frac{1}{2} - \frac{i}{2}\sqrt{3}, \quad \omega_3 = 1 \,\}$$

of the cubic roots of 1, forms an abelian group with respect to multiplication on the set of complex numbers $\mathbb{C}$.

Problem 11. Consider the cubic equation

$$y^3 + py + q = 0, \qquad p, q \in \mathbb{R}, \quad pq \neq 0. \tag{1}$$

Show that applying the nonlinear transformation

$$y(z) := z - \frac{p}{3z}$$

equation (1) can be reduced to

$$z^6 + qz^3 - \frac{p^3}{27} = 0$$

and with $u = z^3$ to a quadratic equation.

Problem 12. Find all integers c for which the cubic equation

$$x^3 - x + c = 0$$

has three integer roots.

Problem 13. Let Φ be an endomorphism of the space $\mathbb{C}^n[X]$ of polynomials of degree n with complex coefficients, which maps a polynomial $p(X)$ to the polynomial $p(X+1)$. Let Ψ be an endomorphism of the space $\mathbb{C}^n[X]$ which maps a polynomial $p(X)$ to $(1-X)^n p\left(\frac{X}{1-X}\right)$, which of course is also a polynomial. Show that we have *braid-like relation*

$$\Phi \circ \Psi \circ \Phi = \Psi \circ \Phi \circ \Psi.$$

Problem 14. Let $a \in \mathbb{R}$. Let r and s be the roots of the quadratic equation

$$x^2 + ax + \frac{a^2 - 1}{2} = 0.$$

Find $r^3 + s^3$ in terms of a, and express it as a polynomial in a with rational coefficients.

Problem 15. Consider the polynomial $p(x) = x^3 - x^2 + x - 2$. Does there exist a nontrivial polynomial $q(x)$ with real coefficients such that the degree of every term of the product $p(x)q(x)$ is a multiple of 3? If so, find one. If not, show there is none.

Problem 16. (i) Let n be a positive integer. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The *Bernstein polynomials* of degree n associated with the continuous function f are given by

$$B_n(f,x) := \frac{1}{(b-a)^n} \sum_{j=0}^{n} \binom{n}{j} (x-a)^j (b-x)^{n-j} f(x_j)$$

where

$$x_j = a + j\frac{b-a}{n}, \qquad j = 0, 1, \ldots, n.$$

Consider the function $f : [0,1] \to \mathbb{R}$ given by $f(x) = \sin(4x)$. Show that $B_2(f,x)$ is not a "good approximation" for f. Consider $x = \pi/8$.
(ii) The *Bernstein basis polynomials* are defined as

$$B_{n,j}(x) := \binom{n}{j} x^j (1-x)^{n-j}, \quad j = 0, 1, 2, \ldots, n, \quad x \in [0,1].$$

Show that

$$\sum_{j=0}^{n} B_{n,j}(x) = 1.$$

Show that $B_{n,j}$ satisfies the recursion relations

$$\begin{aligned} B_{n,j}(x) &= (1-x)B_{n-1,j}(x) + xB_{n-1,j-1}(x), \qquad j = 1, 2, \ldots, n-1 \\ B_{n,0}(x) &= (1-x)B_{n-1,0}(x) \\ B_{n,n}(x) &= xB_{n-1,n-1}(x). \end{aligned}$$

Problem 17. A one-dimensional map f is called an *invariant* of a two-dimensional map g if

$$g(x, f(x)) = f(f(x)).$$

Let

$$f(x) = 2x^2 - 1.$$

Show that f is an invariant for

$$g(x,y) = y - 2x^2 + 2y^2 + d(1 + y - 2x^2).$$

Problem 18. Consider the functions $f(z) = z^3$ and $h(z) = z + 1/z$. Find a function p such that

$$h(f(z)) = p(h(z)). \tag{1}$$

Problem 19. Consider the map $f_c(z) = z^2 + c$, where $c \in \mathbb{C}$. Find all complex c-values where the map f_c has a fixed point z^* with $f_c'(z^*) = -1$.

Problem 20. Let $p(x,y)$ be a real polynomial. Show that if $p(x,y)=0$ for infinitely many (x,y) on the unit circle $x^2+y^2=1$, then $p(x,y)=0$ on the unit circle.

Problem 21. Show that if n is a positive integer then

$$(r(\sin\phi + i\sin\phi))^n \equiv r^n(\cos(n\phi)+i\sin(n\phi)). \tag{1}$$

Hint. Apply $\exp(i\phi)\equiv\cos\phi + i\sin\phi$.

Problem 22. Show that the equation

$$z^n = a \tag{1}$$

where n is a positive integer and a is any nonzero complex number, has exactly n roots. Hint. Set

$$a = \rho(\cos(\phi)+i\sin(\phi)). \tag{2}$$

Problem 23. Show that the cubic roots of unity $z^3=1$ are

$$1, \qquad w = -\frac{1}{2}+\frac{\sqrt{3}i}{2}, \qquad w^2 = -\frac{1}{2}-\frac{\sqrt{3}i}{2}. \tag{2}$$

Problem 24. Show that the n n-th roots of unity are

$$\rho = \cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right), \quad \rho^2, \quad \rho^3,\ldots,\rho^{n-1}, \qquad \rho^n\equiv 1. \tag{1}$$

Problem 25. Show that every polynomial $\alpha(x)\in C[x]$ of degree $m\geq 1$ has precisely m zeros over $\mathbb{C}$, where any zero of multiplicity is to be counted as n of the m zeros.

Problem 26. Show that if $r\in\mathbb{C}$ is a zero of any polynomial $\alpha(x)$ with real coefficients, then $\bar{r}$ is also a zero of $\alpha(x)$, where $\bar{r}$ denotes the complex conjugate of r.

Problem 27. Let A be a 2×2 matrix over the real numbers $\mathbb{R}$. The *trace* is defined as

$$\mathrm{tr}(A) = a_{11}+a_{22}.$$

It can be proved that the trace is the sum of the eigenvalues of A, i.e.

$$\mathrm{tr}(A) = \lambda_1+\lambda_2.$$

Thus we have

$$\mathrm{tr}A^2 = \lambda_1^2 + \lambda_2^2.$$

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find the eigenvalues using the two equations.

Problem 28. Find the zeros of the cubic polynomial

$$\alpha(x) := a_0 + a_1 x + a_2 x^2 + x^3 \tag{1}$$

over $\mathbb{C}$, where $a_0, a_1, a_2 \in \mathbb{R}$ and $a_0 \neq 0$.

Problem 29. Show that the zeros of $z^3 + 1 = 0$ are given by

$$-1, \qquad \frac{1}{2} + \frac{i}{2}\sqrt{3}, \qquad \frac{1}{2} - \frac{i}{2}\sqrt{3}\,. \tag{1}$$

Problem 30. Show that the zeros of

$$z^3 - 6z^2 + 11z - 6 = 0 \tag{1}$$

are given by 1, 2, and 3.

Problem 31. Find the zeros of the quartic polynomial

$$\alpha(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + x^4 \tag{1}$$

over $\mathbb{C}$ when $a_0 \neq 0$.

Problem 32. Find the zeros of

$$\alpha(x) = 35 - 16x - 4x^3 + x^4. \tag{1}$$

Problem 33. Let z be denote a root of the quadratic equation

$$z^2 + az + b = 0. \tag{1}$$

Show that the sequence of powers of z, $\{z^n\}$, $n \geq 2$, satisfies the linear difference equation

$$x_n + a x_{n-1} + b x_{n-2} = 0, \qquad n \geq 2. \tag{2}$$

Problem 34. The variational equation of the *Lorenz model*

$$\frac{dX}{dt} = -\sigma X + \sigma Y \tag{1a}$$

$$\frac{dY}{dt} = -XZ + \tau X - Y \tag{1b}$$

$$\frac{dX}{dt} = XY - bZ \tag{1c}$$

is given by

$$\begin{pmatrix} dx_0/dt \\ dy_0/dt \\ dz_0/dt \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ (\tau - Z) & -1 & -X \\ Y & X & -b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}. \tag{2}$$

(i) Show that Lorenz model possess the steady-state solution $X = Y = Z = 0$, representing the state of no convection.
(ii) Show that with this basic solution, the characteristic equation of the variational matrix is

$$(\lambda + b)(\lambda^2 + (\sigma + 1)\lambda + \sigma(1 - \tau)) = 0. \tag{3}$$

(iii) Show that this equation has three real roots when $\tau > 0$; all are negative when $\tau < 1$, but one is positive when $\tau > 1$. The criterion for the onset of convection is therefore $\tau = 1$.
(iv) Show that when $\tau > 1$, system (1) possess two additional steady state solutions

$$X = Y = \pm\sqrt{b(\tau - 1)}, \qquad Z = \tau - 1. \tag{4}$$

(v) Show that for either of these solutions, the characteristic equation of the matrix in (2) is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (\tau + \sigma)b\lambda + 2\sigma b(\tau - 1) = 0. \tag{5}$$

(vi) Show that this equation possesses one real negative root and two complex conjugate roots when $\tau > 1$. Show that the complex conjugate roots are pure imaginary if the product of the coefficients of λ^2 and λ equals the constant term, or

$$\tau = \sigma(\sigma + b + 3)(\sigma - b - 1)^{-1}. \tag{6}$$

Problem 35. The variational equation of the *Lotka Volterra model*

$$\frac{du_1}{dt} = u_1 - u_1 u_2, \qquad \frac{du_2}{dt} = -u_2 + u_1 u_2 \tag{1}$$

is given by

$$\begin{pmatrix} dv_1/dt \\ dv_2/dt \end{pmatrix} = \begin{pmatrix} 1-u_2 & -u_1 \\ u_2 & -1+u_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2}$$

where $u_1 > 0$ and $u_2 > 0$.
(i) Show that (1) possess the steady-state solution $u_1 = u_2 = 1$.
(ii) Show that with this basic solution, the characteristic equation of the matrix in (2) is

$$\lambda^2 + 1 = 0. \tag{3}$$

(iii) Find the solution of the characteristic equation and discuss.

Problem 36. Show the following. Let $P(z)$ be a polynomial. Then either

1. $P(z)$ has a fixed point q with $P'(q) = 1$,
2. $P(z)$ has a fixed point q with $|P'(q)| > 1$.

Problem 37. Let

$$P = \{f_i(x), i = 1, \ldots, k\}, \quad \mathbf{x} = (x_1 \ldots x_n) \tag{1}$$

be a set of multivariate polynomials in the ring $Q[x_1 \ldots x_n]$, with a solution set

$$S = S(f_i \ldots f_k) = \{x \,|\, f_i(x) = 0, \, \forall i = 1, \ldots, k\} \tag{2}$$

All polynomials

$$u(x) = \sum_j g_j(x) f_j(x) \tag{3}$$

for arbitrary polynomials g_j will vanish in all points of S. The set of all u establishes a polynomial ideal

$$I = I(f_1 \cdots f_k) = \left\{ \sum_j g_j(x) f_j(x) \right\} \tag{4}$$

and classical algebra tells that S is invariant if we replace the set $\{f_1, \cdots, f_k\}$ by any other basis for the ideal $I(f_1, \cdots f_k)$.

The *Buchberger algorithm* allows one to transform the set of polynomials into a canonical basis of the same ideal, the Gröbner basis $GB = GB(I)$. For the purpose of the equation solving especially Gröbner bases computed under lexicographical term ordering are important. They allow one to determine the set S directly. If I has dimension zero (S is a finite set of isolated points, GB has in most cases the form

$$g_1(x_1, x_k) = x_1 + c_{1,m-1} x_k^{m-1} + c_{1,m-2} x_k^{m-2} + \cdots + c_{1,0}$$

$$g_2(x_2, x_k) = x_2 + c_{2,m-1}x_k^{m-1} + c_{2,m-2}x_k^{m-2} + \cdots + c_{2,0}$$
$$\cdots$$
$$g_{k-1}(x_{k-1}, x_k) = x_{k-1} + c_{k-1,m-1}x_k^{m-1} + c_{k-1,m-2}x_k^{m-2} + \cdots + c_{k-1,0}$$
$$g_k(x_k) = x_k^m + c_{k,m-1}x_k^{m-1} + c_{k,m-2}x_k^{m-2} + \cdots + c_{k,0}\,.$$

A basis in this form has the elimination property: the variable dependency has been reduced to a triangular form, just as with a Gaussian elimination in the linear case. The last polynomial is univariate in x_k. It can be solved with usual algebraic or numeric techniques; its zero $\bar{x}_k$ then are propagated into the remaining polynomials, which then immediately allow one to determine the corresponding coordinates $(\bar{x}_1, \ldots, \bar{x}_{k-1})$.

Consider the system

$$\{y^2 - 6y, xy, \quad 2x^2 - 3y - 6x + 18, \quad 6z - y + 2x\}. \tag{4}$$

(i) Show that that this system has for $\{x, y, z\}$ the lexicographical Gröbner basis

$$\{g_1(x,z) = x - z^2 + 2z - 1, \quad g_2(y,z) = y - 2z^2 - 2z - 2, \quad g_3(z) = z^3 - 1\}. \tag{5}$$

(iii) Show that the roots of the third polynomial are given by

$$\left\{ z = 1, \quad z = \frac{1 - \sqrt{3}i}{2}, \quad z = \frac{\sqrt{3}i - 1}{2} \right\}. \tag{6}$$

If we propagate one of them into the basis, we generate univariate polynomials of degree one which can be solved immediately; e.g. Selecting $\frac{1-\sqrt{3}i}{2}$ for z the basis reduces to

$$\left\{ x = \frac{3\sqrt{3}i + 3}{2}, y, 0 \right\} \tag{7}$$

such that the final solution for this branch is

$$\left\{ x = \frac{3\sqrt{3}i + 3}{2}, \quad y = 0, \quad z = \frac{1 - \sqrt{3}i}{2} \right\}. \tag{8}$$

For zero dimensional problems the last polynomial will always be univariate. However, in degenerate cases the other polynomials can contain their leading variable in a higher degree, then containing more mixed terms with the following variables. And there can be additional polynomials with mixed leading terms (of lower degree) imposing some restrictions. But the varible dependency pattern will remain triangular (evenutally with more than k

rows). There is a special algorithm for decomposing such ideals using ideal quotients.

Problem 38. Let S_n be the *symmetric group*. The symmetric group S_n acts naturally on polynomials in n variables. For a polynomial p in n variables, define the symmetrized polynomial associated to p, $\mathrm{Sym}_n(p)$, by

$$\mathrm{Sym}_n(p) := \sum_{\sigma \in S_n} \sigma(p).$$

Let $n = 2$ and

$$p(x_1, x_2) = x_1^2 x_2 + 2x_2.$$

Find $\mathrm{Sym}_2(p)$.

Problem 39. Consider the system of equations

$$\begin{aligned} z_1 + z_2 + \cdots + z_{n-1} + z_n &= 0 \\ z_1 z_2 + z_2 z_3 + \cdots + z_{n-1} z_n + z_n z_1 &= 0 \\ &\vdots \\ z_1 z_2 \cdots z_{n-1} + z_2 z_3 \cdots z_n + \cdots + z_{n-1} z_n \cdots z_{n-3} + z_n z_1 \cdots z_{n-2} &= 0 \\ z_1 z_2 \cdots z_n &= 1. \end{aligned}$$

Find the solutions for the case $n = 2$ and $n = 3$. This system of equations arise as follows. Let p be a prime number. A vector $\mathbf{x} = (x_0, x_1, \ldots, x_{p-1}) \in \mathbb{C}^p$ viewed as a function $\mathbb{Z}/p \to \mathbb{C}$ has discrete Fourier transform $\hat{\mathbf{x}} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{p-1})$, where

$$\hat{x}_j = \sum_{k=0}^{p-1} \omega^{jk} x_k, \qquad \omega := e^{2\pi i/p}.$$

The vector $\mathbf{x}$ is called equimodular if all its coordinates have the same absolute value, and $\mathbf{x}$ is called bi-equimodular if both $\mathbf{x}$ and $\hat{\mathbf{x}}$ are equimodular. The question is: Which vectors are bi-equimodular?

Problem 40. Let $\alpha \in \mathbb{R}$. Find the roots of the characteristic equation

$$\lambda^6 - 2\cos(3\alpha)\lambda^3 + 1 = 0.$$

Problem 41. Show that the n-th order polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

which goes exactly through $n+1$ data points is unique.

Problem 42. Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial that satisfies

$$p(1-x) + 2p(x) = 3x$$

for all $x \in \mathbb{R}$. Find the values of $p(0)$ and $p(1)$. Give an example of a polynomial that satisfies this condition.

Problem 43. (i) Let $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$. Find the solutions of the system of equations

$$x_1 - \frac{1}{(x_1-x_2)^2} = 0, \qquad x_2 + \frac{1}{(x_1-x_2)^2} = 0.$$

(ii) Let $x_1, x_2, x_3 \in \mathbb{R}$ and $x_1 \neq x_2$, $x_1 \neq x_3$, $x_2 \neq x_3$. Find the solutions of the system of equations

$$\begin{aligned} x_1 - \frac{1}{(x_1-x_2)^2} - \frac{1}{(x_1-x_3)^2} &= 0 \\ x_2 + \frac{1}{(x_1-x_2)^2} - \frac{1}{(x_2-x_3)^2} &= 0 \\ x_3 + \frac{1}{(x_1-x_3)^2} + \frac{1}{(x_2-x_3)^2} &= 0. \end{aligned}$$

Problem 44. Let $z, w \in \mathbb{C}$. Find the solution of the system of equations

$$|z|^2 + |w|^2 = 1, \qquad z^2 + w^3 = 0.$$

Problem 45. Consider the two polynomials

$$p_1(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad p_2(x) = b_0 + b_1 x + \cdots + b_m x^m$$

where $n = \deg(p_1)$ and $m = \deg(p_2)$. Assume that $n > m$. Let $r(x) = p_2(x)/p_1(x)$. We expand $r(x)$ in powers of $1/x$, i.e.

$$r(x) = \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots$$

From the coefficients $c_1, c_2, \ldots, c_{2n-1}$ we can form an $n \times n$ *Hankel matrix*

$$H_n = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix}.$$

The determinant of this matrix is proportional to the *resultant* of the two polynomials. If the resultant vanishes, then the two polynomials have a non-trivial greates common divisor. Apply this theorem to the polynomials

$$p_1(x) = x^3 + 6x^2 + 11x + 6, \qquad p_2(x) = x^2 + 4x + 3.$$

Problem 46. Let p be a polynomial with coefficients in $\mathbb{R}$. If the equation $p(x) = 0$ has repeated roots, then $p(x)$ and $dp(x)/dx$ have a highest common factor. Apply this to the polynomial

$$p(x) = 32x^4 - 64x^3 + 24x^2 + 8x - 3.$$

Problem 47. Let p be a polynomial with real coefficients. The equation $p(x) = 0$ cannot have more positive roots than there are changes of sign from $+$ to $-$ and from $-$ to $+$ in the coefficients of the polynomial $p(x)$, or more negative roots than there are changes of sign in $p(-x)$ (Descartes rule of signs). Apply the rule to the polynomiol

$$p(x) = x^6 + 7x^3 + x - 2 = 0.$$

Problem 48. The Liouville-Riemann definition for the fractional integral operator D_x^{-q} is given by

$$D_x^{-q} f(x) := \frac{1}{\Gamma(q)} \int_0^x (x-y)^{q-1} f(y) dy, \quad q > 0.$$

The fractional differential operator D_x^{ν} for $\nu > 0$ is given by the definition

$$D_x^{\nu} f(x) := \frac{d^n}{dx^n}(D_x^{\nu-n} f(x)), \qquad \nu - n < 0.$$

Let $f(x) = x^2$. Find $D_x^{-q} f(x)$.

Problem 49. Let $n \in \mathbb{N}$. Show by induction that $x^n - y^n$ is divisible without remainder by $x - y$ for all values of n. We have

$$x^{n+1} - y^{n+1} \equiv x(x^n - y^n) + y^n(x - y).$$

Problem 50. Consider the polynomial

$$P_n(x) = \sum_{j=0}^{n} (-1)^j a_j x^{n-j}, \qquad a_0 = 1.$$

The homogeneous product sums symmetric functions $h_k(x_1, \ldots, x_n)$ of the zeros of this polynomial are defined as follows

$$\prod_{j=1}^{n}(1 - x_j x) = (1 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots)^{-1}.$$

Show that the first few sums are given explicitly by

$$h_1(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j$$
$$h_2(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^2 + \sum_{j<k}^{n} x_j x_k$$
$$h_3(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j^3 + \sum_{j\neq k}^{n} + \sum_{j<k<\ell}^{n} x_j x_k x_\ell.$$

Problem 51. Consider the quintic equation

$$p(x) = x^5 - 5x^3 + 5x - 5 = 0.$$

Is p irreducible over the rational numbers? What is the Galois group of p? Look for solutions of the form $r + 1/r$.

Problem 52. The *Bernoulli polynomials* $B_n(x)$ $(n = 0, 1, \ldots)$ can be defined recursively by

$$\frac{dB_n(x)}{dx} = nB_{n-1}, \quad n = 1, 2, \ldots$$

with $B_0(x) = 1$ and the condition

$$\int_0^1 B_n(x)dx = 0, \quad n \geq 1.$$

The *Bernoulli numbers* B_n are defined by $B_n := B_n(x = 0)$.
(i) Find the first four Bernoulli polynomials.
(ii) Show that

$$\begin{pmatrix} B_n(x) = \sum_{k=0}^{n} \\ \frac{n}{k} B_k x^{n-k}. \end{pmatrix}$$

(iii) Show that

$$B_n(1-x) = (-1)^n B_n(x).$$

Chapter 5

Equations

Problem 1. Let x_1, x_2 be positive real numbers. Consider the equation

$$x_2 \sin(\theta) = x_1 \cos(\theta).$$

Find $\sin\theta$.

Problem 2. Let L be a given positive real number. Solve the system of two coupled nonlinear equations

$$\begin{aligned} 1 &= x^2 L + 2x^2 y + Lx^2 y^2 \\ 0 &= -x^2 + 2x^2 y + Lx^2 y^2. \end{aligned}$$

Problem 3. Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z - 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Problem 4. Consider a triangle in the plane. Consider a point within the triangle. We draw lines from this point to the three vertices, thereby dividing the triangle into three triangles of area A_1, A_2, A_3. The sides of the triangle are designated by the same number as the opposite vertex. The areas are identified by the number of the adjacent side. The quantities L_j $(j = 1, 2, 3)$

$$L_1 = \frac{A_1}{A}, \qquad L_2 = \frac{A_2}{A}, \qquad L_3 = \frac{A_3}{A}$$

where A is the area of the original triangle, are defined to be the triangular coordinates. Show that

$$L_1 + L_2 + L_3 = 1.$$

The relationship between the Cartesian coordinates x, y which are the coordinates of the points in the elements, and the triangular coordinates L_1, L_2, L_3 are

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3, \qquad y = L_1 y_1 + L_2 y_2 + L_3 y_3$$

where x_j, y_j $(j = 1, 2, 3)$ are the coordinates of the nodes. The triangular coordinates can be expressed in terms of the known locations of the vertices, i.e.

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}.$$

Find L_1, L_2, L_3 as function of x_j, y_j, x, y.

Problem 5. Show that

$$\frac{1}{e^x - 1} \equiv -\frac{1}{e^{-x} - 1} - 1.$$

Problem 6. Let $\mathbf{q}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{x} \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}$. Simplify the expression

$$\|\mathbf{q} - \frac{\epsilon}{2}\mathbf{x} - \mathbf{q}_1\|^2 + \|\mathbf{q} + \frac{\epsilon}{2}\mathbf{x} - \mathbf{q}_2\|^2$$

where $\|\,.\,\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Apply the scalar product in $\mathbb{R}^n$.

Problem 7. Let $x, y \neq 0$. Find all solutions of the equation

$$\frac{4x^2y^2}{(x^2 + y^2)^2} = 1.$$

Problem 8. Let $\epsilon \in \mathbb{R}$. Consider the quadratic equation

$$x^2 - \epsilon x + \epsilon - 1 = 0.$$

Find the roots $x_1(\epsilon)$, $x_2(\epsilon)$. Then find the mimimum of $x_1^2+x_2^2$ with respect to ϵ.

A SymbolicC++ program to solve this problem is:

```
// quadratic.cpp

#include <iostream>
#include "symbolicc++.h"
using namespace std;

int main(void)
{
 Symbolic x("x"), eps("eps"), f = 0;
 Equations soln = solve((x^2)-eps*x+eps-1==0,x);
 cout << "Solutions: " << endl << soln << endl;
 Equations::iterator i;
 for(i=soln.begin();i!=soln.end();i++) f += (i->rhs^2);
 cout << "f(eps) = " << f << endl;
 Equations min = solve(df(f,eps)==0,eps);
 for(i=min.begin();i!=min.end();i++)
  if(double(df(f,eps,2)[*i]) > 0)
   cout << "Minimum at " <<  *i << endl;
 return 0;
}
/*
Solutions:
[ x == eps-1,
  x == 1 ]

f(eps) = eps^(2)-2*eps+2
Minimum at eps == 1
*/
```

Problem 9. Solve the equation

$$71x^2 = 133 \pmod{11}.$$

Problem 10. Let $x, y \in \mathbb{Z}$. Find all solutions of

$$16x + 7y = 601.$$

Problem 11. Let $x, y \in \mathbb{Z}$. Find all solutions of

$$18x + 12y = 4.$$

Problem 12. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \equiv (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Problem 13. Consider the two hyperplane ($n \geq 1$)

$$x_1 + x_2 + \cdots + x_n = 2, \qquad x_1 + x_2 + \cdots + x_n = -2.$$

The hyperplanes do no intersect. Find the shortest distance between the hyperplanes. First consider the cases $n = 1$ and $n = 2$ and then the general case. What happens if $n \to \infty$?

Problem 14. Solve the quadratic equation

$$x^2 - ix - (1 + i) = 0.$$

Problem 15. Solbe the equation

$$71x^2 = 133 \pmod{11}.$$

Problem 16. Let $z \in \mathbb{C}$. Solve

$$\left(\frac{z+1}{z-1}\right) = i.$$

Problem 17. Solve the system of nonlinear equations

$$\begin{aligned} x^2 - (y - z)^2 &= a^2 \\ y^2 - (z - x)^2 &= b^2 \\ z^2 - (x - y)^2 &= c^2. \end{aligned}$$

Problem 18. Let $x, y \in \mathbb{R}$. Find all solutions of

$$e^{-x-y} = e^{-x} + e^{-y}.$$

Problem 19. Show that the quartic equation

$$x^4 + qx^2 + rx + s = 0$$

can be written as

$$(x^2 - ex + f)(x^2 + ex + g) = 0$$

where e^2 is the root of the cubic equation

$$z^3 + 2qz^2 + (q^2 - 4s)z - r^2 = 0.$$

Problem 20. Solve the system of nonlinear equations

$$x_1 x_2 = 1, \quad x_2 x_3 = 2, \quad x_3 x_1 = 3.$$

Extend to the general case

$$x_1 x_2 = c_1, \quad x_2 x_3 = c_2, \ldots, x_{n-1} x_n = c_{n-1}, \quad x_n x_1 = c_n$$

where c_j $(j = 1, \ldots, n)$ are positive constants.

Problem 21. Let

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

be elements of $\mathbb{C}^2$. Solve the equation $\mathbf{z}^* \mathbf{w} = \mathbf{w}^* \mathbf{z}$.

Problem 22. A special set of coordinates on S^n called spheroconical (or elliptic spherical) coordinates are defined as follows: For a given set of real numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_{n+1}$ and nonzero x_1, …, x_{n+1} the coordinates λ_j $(j = 1, \ldots, n)$ are the solutions of the equation

$$\sum_{j=1}^{n+1} \frac{x_j^2}{\lambda - \alpha_j}.$$

Find the solutions for $n = 2$.

Problem 23. Find all solutions of the system of equations

$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha)\cos(\beta) \\ \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\beta) \\ \sin(\alpha)\sin(\beta) \end{pmatrix}.$$

Problem 24. Let $\phi \in [0, 2\pi)$. Solve the cubic equation

$$4x^3 - 3x - \cos\phi = 0$$

over the real numbers.

Problem 25. Are there solutions of the equation ($z \in \mathbb{C}$)

$$\sin(z) = e^z$$

and

$$\cos(z) = e^z\,?$$

Problem 26. Let $\epsilon > 0$. Find the solution of the coupled two non-linear equations

$$\epsilon x^2 + x - y - 1 = 0, \qquad \epsilon y^2 + x - y - 1 = 0.$$

Study $\epsilon \to 0$ for these solutions.

Problem 27. Consider the six 3×3 matrices

$$X_{12} = \begin{pmatrix} a_{12} & b_{12} & 0 \\ c_{12} & d_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X'_{12} = \begin{pmatrix} a'_{12} & b'_{12} & 0 \\ c'_{12} & d'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X_{13} = \begin{pmatrix} a_{13} & 0 & b_{13} \\ 0 & 1 & 0 \\ c_{13} & 0 & d_{13} \end{pmatrix}, \quad X'_{13} = \begin{pmatrix} a'_{13} & 0 & b'_{13} \\ 0 & 1 & 0 \\ c'_{13} & 0 & d'_{13} \end{pmatrix},$$

$$X_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{23} & b_{23} \\ 0 & c_{23} & d_{23} \end{pmatrix}, \quad X'_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a'_{23} & b'_{23} \\ 0 & c'_{23} & d'_{23} \end{pmatrix}.$$

Find the 9 conditions on the entries such that (local Yang-Baxter equation)

$$X_{12}X_{13}X_{23} = X'_{23}X'_{13}X'_{12}.$$

Find solutions of these 9 equations for the 24 unkowns $a_{12}, \ldots, d'_{23}$.

Problem 28. Find the solution of the equation

$$e^{-x} - 1 = 2(\sqrt{2} + 1).$$

Problem 29. Solve the equations

$$2\arcsin(x) + \arcsin(2x) - \pi/2 = 0$$

$$2\arcsin(x) - \arccos(3x) = 0$$

$$\arccos(x) - \arctan(x) = 0$$
$$\arccos(2x^2 - 4x - 2) - 2\arcsin(x) = 0$$
$$2\arctan(x) - \arctan\left(\frac{2x}{1-x^2}\right) = 0.$$

Problem 30. Let $x \in \mathbb{R}$. Find all values of x that satisfy

$$|5x + 7| = 3.$$

Then find the smallest and largest values.

Problem 31. Let $\theta \in [0, 2\pi)$. Can one $x \in \mathbb{R}$ such that

$$\sin(\theta) = \frac{e^{-x/2}}{\sqrt{1+e^{-x}}}\,?$$

Problem 32. Let $x < 1$. Find a solution of the equation

$$f^2(x) - 2f(x) + x = 0.$$

Problem 33. What are the conditions on $c_1 > 0$ and $c_2 > 0$ such that the system of equations

$$x_1 + x_2 = c_1, \qquad x_1 x_2 = c_2$$

has real solutions?

Problem 34. Let m, n be positive integers. Find all solution of the system of equations

$$2mn + (m^2 + n^2) = (m^2 - n^2)^2, \quad m - n = 1.$$

Problem 35. Let $x \in \mathbb{R}$.
(i) Find all solutions of $2x = |x| + 1$.
(ii) Find all solutions of $2x = -|x| + 1$.

Problem 36. Let $x \in \mathbb{Z}$. Solve the equation

$$x^2 - 2x + 2 = 0 \bmod 5.$$

Problem 37. (i) Show that $\cos(\pi/7)$ is a real root of

$$8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0.$$

(ii) Show that

$$\cos(\pi/7)\cos(2\pi/7)\cos(4\pi/7) = -1/8.$$

Chapter 6

Normed Spaces

Problem 1. (i) Let $a, b \in \mathbb{R}$. Show that

$$d(a,b) := |\arctan(a) - \arctan(b)|$$

defines a distance in $\mathbb{R}$.
(ii) Show that $x_n = \arctan(n)$, $(n \in \mathbb{N})$ is a *Cauchy sequence*. Is the metric space $\{\mathbb{R}\,,\, d\}$ complete?

Problem 2. Let $n \geq 1$, $0 \leq a < b$ and $p \in \mathbb{R}^n$. Show that there exists a map $k : C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $k(\mathbf{x}) = 0$ for $\|\mathbf{x} - \mathbf{p}\| \geq b$, $k(\mathbf{x}) = 1$ for $\|\mathbf{x} - \mathbf{p}\| \leq a$, and $0 < k(\mathbf{x}) \leq 1$ for $\|\mathbf{x} - \mathbf{p}\| \leq b$.

Problem 3. Let $\mathbf{x} \in \mathbb{R}^2$. Is

$$\|\mathbf{x}\| := |x_1 x_2|^{1/2}$$

a norm on $\mathbb{R}^2$?

Problem 4. Show that if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq M(|x - y|)^a$$

for some fixed $M > 0$ and $a > 1$, then f is a constant function, i.e., f is identically equal to some real number b for all $x \in \mathbb{R}$.

Problem 5. Let f, g be continuously differentiable functions on the interval $[0,1]$. One defines

$$\langle f, g\rangle = \int_0^1 \left(f(x)\overline{g(x)} + \frac{df}{dx}\overline{\frac{dg}{dx}} \right) dx.$$

Show that this satisfies the properties of an inner product. Calculate $\langle f, g\rangle$ for $f(x) = \sin(x)$, $g(x) = \cos(x)$. Extend it to

$$\langle f, g\rangle = \int_0^1 \left(\sum_{j=0}^{n} \frac{d^j f(x)}{dx^j} \overline{\frac{d^j g(x)}{dx^j}} \right) dx.$$

Problem 6. The p-norm of a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}$$

where $p \in \mathbb{R}^+$. Find the norm for $p \to \infty$.

Problem 7. Let $\mathbf{x} \in \mathbb{R}^n$ and $\|\mathbf{x}\|$ be the Euclidean norm of $\mathbf{x}$. If $B \subset \mathbb{R}^m$ is a nonempty compect set and $\mathbf{x} \in \mathbb{R}^n$, then we define the distance

$$\text{dist}(\mathbf{x}, B) := \min\{ \|\mathbf{x} - \mathbf{b}\| \, : \, \mathbf{b} \in B \}.$$

If $A, B \subset \mathbb{R}^n$ are nonempty compact sets then we define the distance

$$\text{dist}(A, B) := \max\{ \text{dist}(\mathbf{a}, B) \, : \, \mathbf{a} \in A \}.$$

Show that

$$\text{dist}(A, B) = 0 \;\Leftrightarrow\; A \subset B$$

and $\text{dist}(A, B) < \epsilon$ means that $A \subset N_\epsilon(B)$ where

$$N_\epsilon(B) := \{ \mathbf{x} \, : \, \text{dist}(\mathbf{x}, B) < \epsilon \}$$

is the ϵ-neighborhood of B.

Problem 8. Let $x, y \in \mathbb{R}$. Is

$$d(x, y) = \frac{|x - y|}{2 + |x - y|}$$

a metric on $\mathbb{R}$?

Problem 9. Let $\mathbf{v}$ and $\mathbf{w}$ be two normalized column vectors in $\mathbb{C}^n$. Does

$$D(\mathbf{v},\mathbf{w}) := 2\arccos\left(\sqrt{\frac{(\mathbf{v}^*\mathbf{w})(\mathbf{w}^*\mathbf{v})}{(\mathbf{v}^*\mathbf{v})(\mathbf{w}^*\mathbf{w})}}\right)$$

provide a distance measure between $\mathbf{v}$ and $\mathbf{w}$.

Chapter 7

Complex Numbers and Complex Functions

Problem 1. Let $z \in \mathbb{C}$. Solve the nonlinear equation

$$z^3 = z|z|^2.$$

Problem 2. (i) Find the complex numbers z satisfying $z^2 = \bar{z}$.
(ii) Find the complex numbers z satisfying $z^3 = \bar{z}$.

Problem 3. Solve the cubic equation $z^3 = -1$. Do the solutions form a group under multiplication? If not, what numbers have to be added to form a group. Find them by multiplication of the solutions of the cubic equation.

Problem 4. Let $\phi \in [0, 2\pi)$. Consider the complex number $z = 1 - e^{i\phi}$. Find the condition on ϕ such that $|z| = 1$.

Problem 5. Let $\phi \in [0, 2\pi)$ and $\theta \in (-\pi, \pi)$. Can any complex number be represented by

$$z = e^{i\phi} \tan\left(\frac{1}{2}\theta\right) ?$$

Find

$$(zd\bar{z} - \bar{z}dz, \quad (zd\bar{z} - \bar{z}dz) \otimes (zd\bar{z} - \bar{z}dz).$$

Problem 6. Show that

$$\frac{1}{5}(-1+2i) = \frac{1}{\sqrt{5}} e^{i(\pi - \arctan(2))}.$$

Problem 7. Let $x, y \in \mathbb{R}$. Solve

$$\frac{x-i}{x+i}\frac{y-i}{y+i} = \frac{xy-i}{xy+i}.$$

Problem 8. Let $\epsilon \in \mathbb{R}$ and $z \in \mathbb{C}$. Consider the product

$$f(z) = \prod_{k=1}^{\infty} \frac{1-e^{-\epsilon k}}{1-e^{-\epsilon(k+z)}}.$$

Find $f(0)$. Show that

$$f(z) = (1-e^{-\epsilon z}) f(z-1).$$

Show that

$$f(m) = \prod_{k=1}^{m} (1-e^{-\epsilon k}), \qquad m = 1, 2, \ldots$$

Show that f is periodic with period $2\pi i/\epsilon$. Show that f has simple poles at $z = -m$ ($m = 1, 2, \ldots$). Show that the residues are given by

$$\lim_{z \to -m} (z+m) f(z) = \frac{(-1)^{m+1} \exp(-\frac{1}{2}\epsilon m^2 + \frac{1}{2}\epsilon m)}{\epsilon f(m-1)}.$$

Problem 9. (i) Calculate

$$p(\alpha, \beta, \theta, \phi) = |\cos(\alpha)\cos(\beta)\sin(\theta)e^{i\phi} + \sin(\alpha)\sin(\beta)\cos(\theta)|^2.$$

(ii) Show that

$$p(\alpha, \beta, \theta, \phi) \leq 1.$$

(iii) Simplify the result from (i) for $\theta = \pi/4$ and $\phi = 0$.

Problem 10. Let $f(z_1)$ and $g(z_2)$ be a pair of analytic functions of z_1 and z_2, respectively. We define

$$f(z_1) \circ g(z_2) := \frac{1}{2\pi} \int_0^{2\pi} f(z_1 e^{i\theta}) g(z_2 e^{-i\theta}) d\theta.$$

Let

$$f(z_1) = \sum_{k=0}^{\infty} a_k z_1^k, \qquad g(z_2) = \sum_{k=0}^{\infty} b_k z_2^k$$

for $|z_j| < R$, $j = 1, 2$. Find $f(z_1) \circ g(z_2)$.

Problem 11. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Find

$$\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}.$$

Problem 12. Study the behaviour (stability) of the fixed points of the complex map $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = z^2.$$

Problem 13. Let $z = x + iy$ ($x, y \in \mathbb{R}$) be a nonzero complex number. We define the *principal argument* by $z = |z| \exp(i\arg(z))$, where $\arg(z) \in (-\pi, \pi]$ and we define the imaginary remainder $\mathrm{Imr}(z)$ and the imaginary quotient $\mathrm{Imq}(z)$ by

$$\Im(z) = \mathrm{Imr}(z) + 2\pi \mathrm{Imq}(z)$$

where $\mathrm{Imr}(z) \in (-\pi, \pi]$ and $\mathrm{Imq}(z) \in \mathbb{Z}$. Show that

$$\ln(e^z) = \Re(z) + i\mathrm{Imr}(z)$$

and in particular

$$\ln(e^z) = z \quad \text{iff} \quad \Im(z) \in (-\pi, \pi].$$

Problem 14. Let $z \in \mathbb{C}$ and consider the analytic map

$$f(z) = \exp(z).$$

Find the solutions (fixed points) of the equation

$$z = f(z).$$

We set $z = x + iy$ ($x, y \in \mathbb{R}$. Then

$$x + iy = \exp(x + iy) \equiv e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

Thus we have to solve

$$e^x \cos(y) - x = 0, \qquad e^x \sin(y) - y = 0.$$

Problem 15. Let A be an $n\times n$ matrix. Suppose f is an analytic function inside on a closed contour Γ which encircles $\lambda(A)$, where $\lambda(A)$ denotes the eigenvalues of A. We define $f(A)$ to be the $n \times n$ matrix

$$f(A) = \frac{1}{2\pi i} \oint_\Gamma f(z)(zI_n - A)^{-1} dz.$$

This is a matrix version of the *Cauchy integral theorem*. The integral is defined on an element-by-element basis $f(A) = (f_{jk})$, where

$$f_{jk} = \frac{1}{2\pi i} \oint_\Gamma f(z) \mathbf{e}_j^T (zI_n - A)^{-1} \mathbf{e}_d z.$$

Let $f(z) = z^2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate $f(A)$.

Problem 16. Show that

$$(1+i)^{1/2} = \pm 2^{1/4} \left(\cos(\pi/8) + i \sin(\pi/8)\right).$$

Problem 17. Let a, b, ϕ be fixed real numbers. Consider the function

$$w(z) = z + az^2 + be^{i\phi} z^3.$$

Write it as real and imginary part, with $w = u + iv$ and $z = x + iy$.

Problem 18. Let $n = 0, 1, \dots$ and $x \in \mathbb{R}$. Find the real and imaginary part of the functions

$$f_n(x) = \frac{1}{\sqrt{\pi}} \frac{(ix-1)^n}{(ix+1)^{n+1}}.$$

Problem 19. Let $x, c \in \mathbb{R}$ and $c \neq 0$. Find the real and imaginary part of the function

$$f_c(x) = \frac{x - ic}{x + ic}.$$

Problem 20. Consider the z-transform

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \qquad x(n) = \frac{1}{2\pi i} \oint x(z) z^{n-1} dz.$$

Let

$$S(N) := \sum_{n=1}^{N} x(n).$$

Then

$$S(N) = \sum_{n=1}^{N} x(n) = \frac{1}{2\pi i} \oint x(z) \sum_{n=1}^{N} z^{n-1} dz.$$

It follows that (geometric series)

$$S(N) = \frac{1}{2\pi i} \oint \frac{x(z)(z^N - 1)}{z-1} dz.$$

Apply this expression and the *residue theorem* to calculate

$$S(N) = \sum_{n=1}^{N} n^3.$$

Problem 21. (i) Consider the complex number $z = e^{i\phi}$. Let $n \in \mathbb{N}$. Find $\sqrt[n]{z}$.
(ii) Let $z = re^{i\phi}$ and $w = x + iy$ $(x, y \in \mathbb{R})$. Find z^w.

Problem 22. Consider the complex numbers

$$z_1 = 0.4 + 0.3i, \qquad z_2 = 5 + 2i.$$

Calculate $z_1^{z_2}$. Hint: Set $z_1 = r_1 e^{i\phi_1}$.

Problem 23. Consider the complex numbers

$$z_1 = x_1 + iy_1 = 1 + 4i, \qquad z_2 = x_2 + iy_2 = 3 - 2i.$$

Calculate $\log_{z_2} z_1$.

Problem 24. Let $A > 0$ and $B \geq 0$. Consider the quadratic conformal map in the complex w-plane of the unit disc in the complex z-plane

$$w(z) = Az + Bz^2.$$

Let θ be the polar angle in the complex z-plane.
(i) Show that with the notation $w := u + iv$, $z := x + iy$ (with u, v, x, y real) one has the parametric equation of the boundary

$$u(\theta) = A\cos(\theta) + B\cos(2\theta), \qquad v(\theta) = A\sin(\theta) + B\sin(2\theta).$$

(ii) Let $C := B/A$. Show that for $C = 0$ (i.e. $B = 0$) one obtains a circular disc. Show that for $C = 1/4$ the curvature vanishes at $\theta = \pi$. Show that for $C = 1/2$ the derivative dw/dz vanishes at the boundary, i.e. at $\theta = \pi$.

Problem 25. Show that

$$\frac{1}{(n-k)!} = \frac{1}{2\pi i}\oint dt \frac{e^t}{t^{n-k+1}}$$

where the integration contour is a small circle around the origin in the complex plane.

Problem 26. (i) Solve the equation

$$\left(\frac{z+\frac{i}{2}}{z-\frac{i}{2}}\right)^4 = 1.$$

(ii) Solve the system of equations

$$\left(\frac{z_1+\frac{i}{2}}{z_1-\frac{i}{2}}\right)^4 = \frac{z_1-z_2+i}{z_1-z_2-i}, \quad \left(\frac{z_2+\frac{i}{2}}{z_2-\frac{i}{2}}\right)^4 = \frac{z_2-z_1+i}{z_2-z_1-i}.$$

These equations play a role for the Bethe ansatz for spin systems.

Problem 27. Let z be a complex number such that $|z| < \infty$ and $\Re(z) > 0$. Consider

$$Z(z) = \int_0^\infty \exp(-zx - x^2)dx.$$

Show that

$$Z(z) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(\frac{1}{2}+\frac{1}{2}k)}{k!} z^k.$$

Problem 28. Let $z \neq 0$. Show that the function

$$f(z) = \frac{\ln(z)}{z-1}$$

is analytic near $z = 1$ and admits the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(z-1)^n \quad \text{for} \quad |z-1| < 1.$$

Problem 29. Is the function $f : \mathbb{C} \to \mathbb{C}$

$$f(z) = z + |z|$$

continuous? Find the fixed points of f.

Problem 30. We know that $0 \leq |\sin(x)| \leq 1$ and $0 \leq |\cos(x)| \leq 1$ for $x \in \mathbb{R}$. This is no longer true for $\sin(z)$ and $\cos(z)$ with $z \in \mathbb{C}$.
(i) Let $a > 0$. Show that

$$|\sin(az)| = \sqrt{\sinh^2(ay) + \sin^2(ax)}, \quad |\cos(az)| = \sqrt{\sinh^2(ay) + \cos^2(ax)}.$$

(ii) Find all solutions of $|\sin(z)| = 2$ and $|\cos(z)| = 2$.
(iii) Find all solutions of $\cos(z) = i$.
(iv) Let $n \in \mathbb{N}$. Show that $(x, y \in \mathbb{R})$

$$|\sin(n(x+iy))| = \sqrt{\sin^2(nx)\cosh({}^2(ny) + \cos^2(nx)\sinh^2(ny)} > |\sinh^2(ny)| \to \infty \text{ as } n \to \infty$$

for any $y \neq 0$.

Problem 31. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ with $z_1\bar{z}_1 \neq 0$, $z_2\bar{z}_2 \neq 0$. Find the conditions such that

$$\begin{pmatrix} \bar{z}_1 & \bar{z}_2 \end{pmatrix} \begin{pmatrix} z_2 \\ -z_1 \end{pmatrix} = 0.$$

Chapter 8

Integration

Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $f(x_1, x_2) = f(x_2, x_1)$ for all $x_1, x_2 \in \mathbb{R}$. Let $b, a \in \mathbb{R}$ and $b > a$. Calculate

$$\int_a^b \int_a^b f(x_1, x_2) \sin(x_1 - x_2) dx_1 dx_2.$$

Problem 2. The time-average of a continuous function f is

$$\langle f \rangle := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) dt.$$

Find the time-average of the functions

$$f_1(t) = \cos(\omega t) \sin(\omega t), \quad f_2(t) = \cos^2(\omega t), \quad f_3(t) = \sin^2(\omega t).$$

Problem 3. Let $j, k = 1, 2, \ldots$. Consider the function

$$f(j, k) = \int_0^1 x^{j-1}(1-x)^{k-1} dx, \qquad j, k = 1, 2, \ldots.$$

Thus $f(1, 1) = 1$. Is

$$f(j-1, k+1) = \frac{k}{j-1} f(j, k), \qquad j \geq 2$$

and

$$f(j+1,k) = f(j,k) - f(j,k+1)\,?$$

Prove or disprove.

Problem 4. Draw the functions

$$\begin{aligned} f_1(x) &= \cos(2\pi x) \\ f_2(x) &= \cos(2\pi(\cos(2\pi x))) \\ f_3(x) &= \cos(2\pi(\cos(2\pi(\cos(2\pi x))))). \end{aligned}$$

Extend it to $f_n(x)$. Find the integral

$$\int_0^{2\pi} f_n(x)dx.$$

Problem 5. Let

$$x(\tau) = \begin{cases} 1 \quad \text{for} \quad 0 \le \tau \le 1 \\ 0 \,\text{otherwise} \end{cases}$$

and

$$h(\tau) = \begin{cases} 1 \quad \text{for} \quad 0 \le \tau \le 1 \\ 0 \,\text{otherwise} \end{cases}.$$

Find the *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Problem 6. Let f be a continuous function.
(i) Show that the double integral

$$\int_0^x d\xi \int_0^\xi f(t)dt$$

can be expressed by a single integral.
(ii) Show that $(n \ge 2)$

$$\int_0^x d\xi_1 \int_0^{\xi_1} d\xi_2 \cdots \int_0^{\xi_{n-1}} f(\xi_n)d\xi_n$$

can be expressed by a single integral.

Problem 7. Calculate the integral

$$I = \int_0^{\pi/2} \sin^{2n}(\theta)\cos^{2n+1}(\theta)d\theta$$

by considering the double integral

$$\int\int_D (r\sin(\theta))^{2n}(r\cos(\theta)^{2n+1}e^{-r^2}rdrd\theta$$

where D is the first quadrant.

Problem 8. Let $T > 0$ and $\omega = 2\pi/T$. Let $m, n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$. Calculate

$$I(\alpha,\beta) = \frac{1}{T}\int_0^T c_m \sin(m\omega t + \phi_m - \alpha)c_n \sin(n\omega t + \phi_n - \beta)dt.$$

Problem 9. Find the integral

$$I(\ell) = \int_0^{2\pi}\left(\int_0^{|\cos(\theta)|(\ell/2)} dp\right)d\theta.$$

Problem 10. A *cubic B-spline* with uniform knot spacing, centered at the origin, is given by

$$B(x) = \begin{cases} \frac{1}{6}(2-|x|)^3 & \text{if} \quad 1 \le |x| < 2 \\ \frac{1}{6}((2-|x|)^3 - 4(1-|x|)^3) & \text{if} \quad 0 \le |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the integral

$$\int_{-\infty}^{\infty} B(x)dx.$$

Problem 11. Evaluate the quadruple integral

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 dx' \int_0^1 dy' \left(\frac{1}{((x-x')^2+(y-y')^2)^{1/2}} - \frac{1}{((x-x')^2+(y-y')^2+1)^{1/2}}\right).$$

Problem 12. Let $c \in \mathbb{R}$. Calculate the integral

$$f(c) = \int_0^{2\pi} \exp(ce^{i\phi})d\phi$$

by finding an ordinary differential equation for f together with the initial conditions. Obviously $f(0) = 2\pi$.

Problem 13. Let $\alpha > 0$. Find the integral

$$\int_{-\infty}^{\infty} \operatorname{sech}(\alpha t)dt.$$

Problem 14. Let $z \in (0,1]$ and $x \in (0,1]$. Find the integral

$$f(z) = 1 - \int_z^1 dx \int_{z/x}^1 dy.$$

Problem 15. Let $b > a$. Find the mean and variance of random variable x with uniform probabilty density function p

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Problem 16. (i) Let $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that the integral

$$\int \int_{r \ge 1} \cdots \int \frac{dx_1 \cdots dx_n}{r^\alpha}$$

is finite for $\alpha > n$ and infinite for $\alpha \le n$.
(ii) Let $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that the integral

$$\int \int_{r \le 1} \cdots \int \frac{dx_1 \cdots dx_n}{r^\alpha}$$

is finite for $\alpha < n$ and infinite for $\alpha \ge n$.

Problem 17. Consider a one-dimensional lattice (chain) with lattice constant a. Let k be the sum over the first Brioullin zone we have

$$\frac{1}{N} \sum_{k \in 1.BZ} F(\epsilon(k)) \to \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} F(\epsilon(k))dk = G$$

where

$$\epsilon(k) = \epsilon_0 - 2\epsilon_1 \cos(ka).$$

Using the identity

$$\int_{-\infty}^{\infty} \delta(E - \epsilon(k))F(E)dE \equiv F(\epsilon(k))$$

we can write

$$G = \frac{a}{2\pi} \int_{-\infty}^{\infty} F(E) \left(\int_{-\pi/a}^{\pi/a} \delta(E - \epsilon(k))dk \right) dE.$$

Calculate

$$g(E) = \int_{-\pi/a}^{\pi/a} \delta(E - \epsilon(k))dk$$

where $g(E)$ is called the density of states.

Problem 18. Consider a three-dimensional probability distribution $f(x_1, x_2, x_3)$ such that for all j

$$f_x(x_j) = \frac{1}{2\sqrt{\pi}} \left(1 + 2x_j^2 - \frac{-1 + 2x_j^2}{\sqrt{2}} \right) e^{-x_j^2}$$

where $f_x(x)$ is the probability density associated with an individual variable. This means

$$f_x(x_1) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_1, x_2, x_3)dx_2dx_3$$

etc. Is it possible that the probabilty density associated with the sum of these variables $s = x_1 + x_2 + x_3$ is given by

$$f_s(s) = \frac{1}{2\sqrt{\pi}} \left(1 + 2s^2 + \frac{-1 + 2s^2}{\sqrt{2}} \right) e^{-s^2}$$

provided that $f(x_1, x_2, x_3)$ is non-negative?

Problem 19. Let

$$i_1(t) = I^2 \sin^2(\omega t), \qquad i_2(t) = I^2 \sin^2(\omega t + \phi).$$

Calculate

$$\langle i_1(t)i_2(t) \rangle := \lim_{T \to \infty} \frac{1}{T} \int_0^T i_1(t)i_2(t)dt$$

and

$$\langle i_1(t)i_2(t)\rangle - \langle i_1(t)\rangle\langle i_2(t)\rangle.$$

Problem 20. Let $f(t)$ be a continuous function. Show that

$$\int_0^x d\zeta \int_0^\zeta f(t)dt = \int_0^x (x-t)f(t)dt.$$

Problem 21. (i) Calculate

$$\int_{-\infty}^{\infty} \mathrm{sech}(t)\tanh(t)\cos(t+t_0)dt.$$

(ii) Calculate

$$\int_{-\infty}^{\infty} \mathrm{sech}^2(t)\tanh^2(t)dt.$$

Problem 22. Let $\lambda > 0$. Calculate

$$\int_{-\infty}^{\infty} \frac{\sin(\lambda t)}{\lambda\sinh(t)}dt.$$

Problem 23. Calculate

$$\int_{-\infty}^{\infty} \exp(-x^2+2ixy)dx.$$

Problem 24. Calculate the integral

$$I(\lambda) = \int_{-\infty}^{\infty} \frac{e^t}{(1+e^t)^2}\cos(t+\lambda)dt$$

using the residue technique.

Problem 25. Let m be an non-negative integer. Find

$$\int_0^\pi d\theta|\cos^{2m+1}(\theta)|.$$

Problem 26. Let $\epsilon > 0$. Calculate

$$\int_0^\infty \frac{k^2dk}{e^{\epsilon k^2}-1}.$$

Problem 27. The mother *Haar wavelet* is given by

$$f(t) = \begin{cases} -1 & \text{for} \quad 0 \le t < 1/2 \\ +1 & \text{for} \quad 1/2 \le t < 1 \\ \ 0 & \text{otherwise} \end{cases}$$

Find the Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt.$$

Problem 28. The *Poisson wavelet* is given by

$$f(t) = \left(t\frac{d}{dt} + 1 \right) P(t)$$

where

$$P(t) = \frac{1}{\pi} \frac{1}{1+t^2}.$$

Find the Fourier transform of f.

Problem 29. Let $m \in \mathbb{Z}$. Calculate

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} e^{im\phi} 2\cos(\phi) d\phi.$$

Problem 30. Show that (Fresnel's integral)

$$\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} sin(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Problem 31. Let $0 < \alpha < 1$. Find the integral

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx.$$

Problem 32. Show that

$$2\int_0^{\infty} \frac{\sin(2px)\sin(qx)}{x} dx = \ln \frac{|2p+q|}{|2p-q|}.$$

Problem 33. The linear one-dimensional *diffusion equation* is given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad t \ge 0, \quad -\infty < x < \infty$$

where $u(x,t)$ denotes the concentration at time t and position $x \in \mathbb{R}$. D is the diffusion constant which is assumed to be independent of x and t. Given the initial condition $c(x,0) = f(x)$, $x \in \mathbb{R}$ the solution of the one-dimensional diffusion equation is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t|x',0)f(x')dx'$$

where

$$G(x,t|x',t') = \frac{1}{\sqrt{4\pi D(t-t')}} \exp\left(-\frac{(x-x')^2}{4D(t-t')}\right).$$

Here $G(x,t|x',t')$ is called the fundamental solution of the diffusion equation obtained for the initial data $\delta(x-x')$ at $t=t'$, where δ denotes the Dirac delta function.
(i) Let $u(x,0) = f(x) = \exp(-x^2/(2\sigma))$. Find $u(x,t)$.
(ii) Let $u(x,0) = f(x) = \exp(-|x|/\sigma)$. Find $u(x,t)$.

Problem 34. The sum

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \exp\left(2\cos\left(\frac{k\pi}{n+1}\right)\right)$$

can be cast into the integral

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n \exp(2\cos(\pi x))dx. \tag{1}$$

Calculate this integral.

Problem 35. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. The *Dirichlet integral identity* is given by

$$\int_0^u \int_0^{u-w_2} f(u-w_1-w_2)w_1^{\mu_1-1}w_2^{\mu_2-1}dw_1dw_2 =$$

$$\frac{\Gamma(\mu_1)\Gamma(\mu_2)}{\Gamma(\mu_1+\mu_2)}\int_0^u f(u-w)w^{\mu_1+\mu_2-1}dw.$$

Let $f(x) = e^{-x}$. Calculate the left and right-hand side of this identity.

Problem 36. For a $\lambda/2$ antenna we obtain the expression

$$E(r,\theta) = -\frac{\omega I_0 \sin\theta}{4\pi\epsilon_0 c^2 r}\int_{-\lambda/4}^{\lambda/4} \cos(k\ell)\sin(\omega(t - c^{-1}(r - \ell\cos\theta)))d\ell.$$

Calculate $E(r,\theta)$.

Problem 37. Consider a one-dimensional chain of length N with open end boundary conditions. The counting is from left to right starting at 0. The canonical partition function $Z(\beta)$ ($\beta > 0$) is given by the multiple integral

$$Z_N(\beta) = \int_{-1}^{1} ds_0 \int_{-1}^{1} ds_1 \cdots \int_{-1}^{1} ds_{N-1} e^{\beta|s_0 - s_1|} e^{\beta|s_1 - s_2|} \cdots e^{\beta|s_{N-2} - s_{N-1}|}.$$

Show that there is a coordinate transformation which decouples the sites. Find $Z_2(\beta)$ and $Z_3(\beta)$.

Problem 38. Let $r_1 > 0, r_2 > 0$. Find the integral

$$I(r_1, r_2) = \int_{-1}^{1} \frac{dx}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 x}}.$$

Problem 39. Let

$$S := \{\, (x,y) \in \mathbb{R}^2 \; : \; x, y \ge 0, \;\; 0 \le x_1^2 + x_2^2 \le 1 \,\}.$$

Let m, n be nonnegative integers. Find the integral

$$\int_S x^{2m+1} y^{2n+1} dx dy.$$

Problem 40. Calculate

$$\int\int_A dx dy$$

where

$$A = \{(x,y) \; : \; x, y \ge 0, \; x + y \le 1, \; x \ge 1/3\,\}.$$

Problem 41. Show that

$$e^{a^2} = \int_{\mathbb{R}} dx \exp\left(-\frac{x^2}{2} + \sqrt{2}ax\right).$$

Problem 42. Calculate the integral

$$\int_0^1 |\cos(2\pi x)| dx$$

using the random numnber generator described in problem 7, chapter 10, page 250, Problems and Solutions in Scientific Computing. Compare to the exact result by solving the integral.

Problem 43. Consider the function $\phi : \mathbb{R} \to \mathbb{R}$

$$\phi(x) := \begin{cases} 1 & \text{for} \quad x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find

$$\psi(x) := \phi(2x) - \phi(2x-1).$$

Draw the function. Calculate

$$\int_{-\infty}^{+\infty} \psi(x) dx.$$

Problem 44. Calculate the integral

$$I = \int_1^\infty dx \int_1^\infty dy \frac{\sqrt{x^2-1}\sqrt{y^2-1}}{(x+y)^6}$$

using the substitution $x = \cosh(\alpha)$, $y = \cosh(\beta)$.

Problem 45. Let $u \geq 0$ and

$$\rho(u) = \frac{1}{2}\exp(-\sqrt{u}).$$

Find

$$\rho_n = \int_0^{-\infty} u^n \rho(u) du.$$

Problem 46. A random variable X is said to be Lorentzian if its probability density p_X is a the form

$$p_X(x) = \frac{1}{\pi}\frac{\gamma}{(x-\alpha)^2+\gamma^2}$$

where $\gamma > 0$. We say that X is (α, γ) to indicate that the random variable is Lorentzian with the probability density given above. Let X, Y be two independent random variables with (α, γ) and (β, δ). Let ϵ be a nonnegative real number. Show that

$$\begin{aligned} \epsilon X &\quad \text{is} \quad (\epsilon\alpha, \epsilon\gamma) \\ \epsilon + X &\quad \text{is} \quad (\epsilon + \alpha, \gamma) \\ -X &\quad \text{is} \quad (-\alpha, \gamma) \\ X + Y &\quad \text{is} \quad (\alpha + \beta, \gamma + \delta) \\ X^{-1} &\quad \text{is} \quad \left(\frac{\alpha}{\alpha^2 + \gamma * 2}, \frac{\gamma}{\alpha^2 + \gamma^2} \right). \end{aligned}$$

Problem 47. Calculate the integral

$$I = \int\int_D \sqrt{x^2 + 4y^2} dydx$$

where D is the domain bounded by the positive x-axis, the positive y-axis and the parabola $y^2 = 1 - x$.

Problem 48. Find the integral

$$I = \int_1^\infty dx \int_1^\infty dy \frac{\sqrt{x^2-1}\sqrt{y^2-1}}{(x+y)^6}.$$

Hint. Use the substitutions $x = \cos(\alpha)$, $y = \cosh(\beta)$ and show that the integral can be written as

$$I = \int_0^\infty d\alpha \int_0^\infty d\beta \frac{\sinh^2(\alpha)\sinh^2(\beta)}{(\cosh(\alpha) + \cosh(\beta))^6}.$$

Problem 49. Show that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{sech}(\alpha - \beta) e^{i\omega\beta} d\beta = \operatorname{sech}(\frac{1}{2}\pi\omega) e^{i\omega\alpha}.$$

Problem 50. Calculate the definite integral

$$\int_0^1 \sin(x^2) dx.$$

Problem 51. (i) Consider the wavelet $(\omega_0 > 0)$

$$\psi_0(t) = (e^{i\omega_0 t} - e^{-\omega_0^2/2}) e^{-t^2/2}.$$

Show that

$$\int_{-\infty}^{\infty} \psi_0(t)dt = 0.$$

Hint: Use $(a > 0)$

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{a}}e^{(b^2-4ac)/4a}.$$

(ii) We define

$$\psi_n(t) := \frac{d}{dt}\psi_{n-1}(t), \quad n = 1, 2, \ldots$$

Show that

$$\int_{-\infty}^{\infty} t^k \psi_n(t)dt = 0, \qquad 0 \le k \le n.$$

Problem 52. Let $b > a$. Consider the integral

$$\int_{y=a}^{y=b} f(y)dy$$

where f is a continuous function in $[a, b]$. Apply the transformation

$$y(x) = \frac{1}{2}((b-a)x + a + b)$$

so that the integration range is between -1 and $+1$. Then the Gauss quadrature can be applied which extends over the interval $[-1, +1]$.

Problem 53. Let $\epsilon > 0$. Find f of the equation

$$\exp(-\epsilon t) = 1 - \int_0^t f(s)ds.$$

Problem 54. Calculate the integral

$$\frac{1}{2\pi^2}\int_0^{\pi}\int_0^{\pi} d\alpha d\alpha' \ln(2 - \cos(\alpha) - \cos(\alpha'))$$

utilizing the identity

$$\cos(\alpha) + \cos(\alpha') \equiv 2\cos\left(\frac{\alpha+\alpha'}{2}\right)\cos\left(\frac{\alpha-\alpha'}{2}\right)$$

the transformation $x = (\alpha + \alpha')/2$, $y = (\alpha - \alpha')/2$ and the integral

$$\int_0^\pi \ln(1 + \sin(x))dx = -\pi \ln(2) + G$$

where G is the Catalan constant.

Problem 55. (i) Find the area of the set

$$S_2 := \{ (x_1, x_2) \, : \, 1 \geq x_1 \geq x_2 \geq 0 \}.$$

(ii) Find the volume of the set

$$S_3 := \{ (x_1, x_2, x_3) \, : \, 1 \geq x_1 \geq x_2 \geq x_3 \geq 0 \}.$$

Extend the n-dimensions.

Problem 56. Let ω_1, ω_2 be real and positive. Find

$$J(\omega_1, \omega_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}x^2 + i\omega_1 x + i\omega_2 x^2)dx.$$

Problem 57. Let e be the eccentricity of an ellipse, i.e. $\sqrt{1-e^2} = b/a$. Let $n \geq 1$. Calculate the integral

$$\int_0^{2\pi} \frac{dx}{\left(\frac{1}{e} - \cos(x)\right)^n}$$

by making the substitution $z = \exp(ix)$. Apply the residue theorem.
Hint. We have

$$\int_0^{2\pi} \frac{dx}{\left(\frac{1}{e} - \cos(x)\right)^n} = \int_{|z|=1} \frac{dz}{\left(\frac{1}{e} - \frac{1}{2}\left(z + \frac{1}{z}\right)\right)^n iz} = \int_{|z|=1} \frac{i(-1)^{n+1}2^n z^{n-1}dz}{\left(z^2 - \frac{2}{e}z + 1\right)^n}.$$

Problem 58. Let $a > 0$ and $R \geq 0$. Find

$$I_a(R) = \int_0^R \int_0^R \frac{dxdy}{(x-y)^2 + a^2} \equiv \int_0^R \int_0^R \frac{dxdy}{x^2 + y^2 - 2xy + a^2}.$$

Problem 59. Let $q^2 > 0$ and $(p-q)^2 > 0$. Find

$$\int_0^\infty \exp(-xq^2)dx, \qquad \int_0^\infty \exp(-x(p-q)^2)dx.$$

Problem 60. Find $\alpha > 0$ such that

$$\int_{-\infty}^{\infty} \exp(-\alpha|x|)dx = 1.$$

Afterwards calculate

$$\int_{-\infty}^{\infty} x\exp(-\alpha|x|)dx, \qquad \int_{-\infty}^{\infty} x^2\exp(-\alpha|x|)dx.$$

Problem 61. Let $c > 0$. Show that

$$\exp\left(\frac{1}{2}cy^2\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}x^2 + \sqrt{c}yx\right).$$

Problem 62. The Gauss invariant for two given closed loops C_α and C_β in $\mathbb{R}^3$ parametrized by $\mathbf{r}_\alpha(s)$, $\mathbf{r}_\beta$ is defined by

$$G(C_\alpha, C_\beta) := \frac{1}{4\pi}\oint_{C_\alpha} ds \oint_{C_\beta} ds' \frac{d\mathbf{r}_\alpha(s)}{ds} \times \frac{d\mathbf{r}_\beta(s')}{ds'} \frac{\mathbf{r}_\alpha(s) - \mathbf{r}_\beta(s')}{|\mathbf{r}_\alpha(s) - \mathbf{r}_\beta(s')|^3}$$

where $\times$ denotes the vector product. Find $G(C_\alpha, C_\beta)$ for the two curves

$$C_\alpha \,:\, x_1^2 + x_3^2 = 1, \qquad C_\beta \,:\, (x_1 - 1)^2 + x_2^2 = 1.$$

Problem 63. Let $x \in \mathbb{R}$. We define $[x]$ as the integer part of x and $\{x\} := x - [x]$. Calculate the integrals

$$\int_0^{7/2} [x]dx, \qquad \int_0^{7/2} \{x\}dx.$$

Problem 64. Let $x \in \mathbb{R}$. Consider the integral

$$f(x) = \int_x^{\infty} \frac{e^{iy}}{y} dy.$$

(i) Show that $f(-|x|) = f(|x|) - i\pi$.
(ii) Show that for large x

$$f(x) = e^{ix}\left(\frac{i}{x} + \frac{1}{x^2} + \cdots\right).$$

Problem 65. Let $\sigma > 0$. Show that

$$\frac{1}{4\sigma^2}\int_0^\infty x^{5/2}\exp\left(-\frac{x^2}{8\sigma^2}\right)dx = \left(\frac{32}{10}\right)^{1/4}\Gamma(3/4)\sigma^{3/2}.$$

Problem 66. Let $n = 0, 1, 2, \ldots$. We define

$$a_n := \int_0^{\pi/2}\cos^{2n}(x)dx, \qquad b_n := \int_0^{\pi/2} x^2\cos^{2n}(x)dx.$$

Then $a_0 = \pi/2$ and $b_0 = \pi^3/24$. Show that using integration by parts

$$a_n = (2n-1)(a_{n-1} - a_n).$$

Show that for $n \geq 1$ we have

$$a_n = (2n-1)nb_{n-1} - 2n^2 b_n.$$

Show that ($n \geq 1$)

$$0 \leq \frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2} = 2\frac{b_n}{a_n} \leq \frac{\pi^2}{4(n+1)}.$$

Problem 67. Calculate

$$I = \int_0^{10}(x - \mathrm{int}(x))dx$$

where $\mathrm{int}(x)$ defines the largest integer less than x.

Problem 68. Show that

$$e^{-x^2/2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}y^2 + ixy\right)dy.$$

Problem 69. Let $0 < a < b < 1$. Find

$$\int_a^b \frac{\ln(1-x)}{x}dx.$$

Problem 70. Show that

$$\int_0^{\pi/2}\cos^{10}(x)\cos^8(2x)\cos^6(4x)\cos^4(6x)\cos^2(8x)dx =$$

$$\frac{5166673\pi}{536870912} + \frac{2966549762512816}{98120709987525225}.$$

Problem 71. Let $z \in \mathbb{C}$. Let $\ln(1+z)$ be the branch of the logarithm defined on $\mathbb{C} \setminus (-\infty, -1]$. Calculate

$$I_n(r) = Pv \int_{|z|=r} z^{n-1} \ln(1+z) dz, \quad r > 0, \; n \in \mathbb{Z}$$

where Pv is the principal value.

Problem 72. Let $b > a$ and a, b be finite. Consider the integral

$$\int_a^b f(x) dx.$$

(i) Apply the transformation

$$x = \frac{1}{(}a + b + (b-a)\tanh(y) \Leftrightarrow y = \tanh^{-1}\left(\frac{2x-a-b}{b-a}\right)$$

to the integral.
(ii) Apply the transformation

$$x = \frac{1}{2}\left(a + b + (b-a)\tanh\left(\frac{\pi}{2}\sinh(y)\right)\right)$$

with

$$\frac{dx}{dy} = \frac{(b-a)\pi\cosh(y)/4}{\cosh^2(\pi\sinh(y)/2}$$

to the integral.

Problem 73. Calculate the integral

$$\int_0^\pi \frac{1+\sin(x)}{1+\cos(x)} dx$$

utilizing the transformation $t = \tan(x/2)$ with $-\pi < x < \pi$. From this transformation it follows that

$$x = 2\arctan(t), \qquad dx = \frac{2}{1+t^2} dt$$

and $x = 0 \to t = 0$, $x = \pi/2 \to t = 1$.

Problem 74. Let $a > 0$. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be continuous functions with $f(-x) = f(x)$ and $g(-x) = -g(x)$. Show that

$$\int_{-a}^{a} f(x)g(x)dx = 0.$$

Problem 75. Calculate the definite integral

$$\int_0^1 \sin(x^2)dx.$$

Problem 76. Let $\Re(a) > 0$ and $n = 0, 1, \ldots$. Show that

$$\begin{aligned}\int_{-\infty}^{\infty} x^n e^{-ax^2+px}dx &= \frac{\partial^n}{\partial p^n}\int_{-\infty}^{\infty} e^{-ax^2+px}dx \\ &= \frac{\partial^n}{\partial p^n}\sqrt{\frac{\pi}{a}}e^{p^2/(4a)}.\end{aligned}$$

Problem 77. Calculate the integral

$$\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{dx_1dx_2dx_3dx_4dx_5dx_6}{1+x_1+x_2+x_3+x_4+x_5+x_6}.$$

Problem 78. Show that

$$\int_0^{\infty} e^{-x}x^2dx = 2!, \qquad \int_0^{\infty} e^{-x}x^3dx = 3!.$$

Let $n = 4, 5, \ldots$. Show that

$$\int_0^{\infty} e^{-x}x^n dx = n!.$$

Problem 79. Find the integral

$$\int_0^{r_0} \frac{r^3}{\sqrt{1+r^2/\ell^2}}dr.$$

Problem 80. Let $k_1 > 0$, $k_2 > 0$, $k_3 > 0$. Find the integral

$$\int_0^{\infty} \sin(k_1 r)\sin(k_2 r)\sin(k_3 r)\frac{dr}{r}.$$

Problem 81. Let $\alpha > 0$. Show that

$$\int_0^\infty x e^{-\alpha x^2} J_0(\beta x) dx = \frac{1}{2\alpha} e^{-\beta^2/(4\alpha)}.$$

Problem 82. Calculate

$$F(s,t) = \int_{\mathbb{R}} e^{-ixt-|x-s|} dx.$$

Problem 83. Let $x \in (0,1)$. Calculate

$$P \int_0^1 \frac{y^m}{(y-x)} dy$$

for $m = 0$, $m = 1$, $m = 2$, $m = 3$.

Problem 84. Let $a, b > 0$ and $b > |a|$. Show that

$$\int_{\mathbb{R}} \frac{dx}{x^2 + 2ax + b^2} = \frac{\pi}{\sqrt{b^2 - a^2}}.$$

Problem 85. Let $a, b > 0$. Show that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = \ln\left(\frac{b}{a}\right).$$

Problem 86. Let $x > 0$. Show that

$$\int_0^\infty e^{-tx} dt = \frac{1}{x}.$$

Problem 87. Let $n_1, n_2 \in \mathbb{Z}$. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ with a period 2π for both x_1 and x_2

$$f(x_1, x_2) = \sum_{n_1, n_2 = -\infty}^{\infty} c_{n_1, n_2} e^{i(n_1 x_1 + n_2 x_2)}$$

and

$$\int_0^{2\pi} \int_0^{2\pi} f(x_1, x_2) dx_1 dx_2 = 0.$$

Then

$$c_{n_1,n_2} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1,x_2)e^{-i(n_1x_1+n_2x_2)}dx_1dx_2$$

(i) Let

$$f(x_1,x_2) = \sin(x_1) + \cos(x_2).$$

Find c_{n_1,n_2}.
(ii) Let

$$f(x_1,x_2) = \sin(x_1)\cos(x_2).$$

Find c_{n_1,n_2}.

Problem 88. $a, b \in \mathbb{R}$. Let

$$f(x) = a\exp(bx).$$

Find the conditions on a and b such that

$$\int_0^1 f(x)dx = 1, \qquad \int_0^1 xf(x)dx = \frac{1}{2}.$$

Problem 89. *Dawson's integral* is given by

$$f(x) = \int_0^x e^{t^2-x^2}dt, \quad x \geq 0.$$

(i) Show that for all complex values z the function f satisfies the linear differential equation

$$\frac{df(z)}{dz} + zf(z) = 1.$$

(ii) Let $j = 1, 2, \ldots$. Show that

$$f^{(j+1)}(z) + 2zf^{(j)} + 2jf^{(j-1)}(z) = 0, \quad j = 1, 2, \ldots$$

where $f^{(j)}$ indicates the jth derivative.

Problem 90. Find a non-negative analytic function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)dxdy = 1$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} x^2dxdy = \frac{1}{2}, \qquad \int_{\mathbb{R}}\int_{\mathbb{R}} y^2dxdy = \frac{1}{2}.$$

Problem 91. Let $a, b, c \in \mathbb{R}$ and $a + b\cos(\theta) + c\sin(\theta) \neq 0$. Show that

$$\int_{-\pi}^{+\pi} \frac{b\sin(\theta) - c\cos(\theta)}{a + b\cos(\theta) + c\sin(\theta)} d\theta = 0.$$

Problem 92. Consider the compact set

$$S := \{ (x, y) \, : \, y \geq x^2, \, y \leq \sqrt{x}, \, x, y \geq 0 \}.$$

Thus $S \subset [0, 1] \times [0, 1]$. Find

$$I(S) = \int_S d\mu = \int_S dxdy.$$

Problem 93. Let $a > 0$. Show that

$$\int_0^\infty \frac{\cos(ax)}{1 + x^2} dx = \frac{\pi}{2} e^{-a}.$$

Problem 94. Show that

$$\int_0^\infty \frac{dx}{\cosh^3(x)} = \frac{\pi}{4}.$$

Problem 95. Show that

$$\int_0^\infty \frac{\ln(\cosh(x))}{\cosh^3(x)} dx = \frac{\pi}{4}(\ln(2) - 1/2).$$

Problem 96. Let $m, n = 0, 1, 2, \ldots$. Find the integral

$$f_{mn}(t) = \int_0^t (t - \tau)^m \tau^n d\tau.$$

Problem 97. Let $a, b > 0$. Find the integral

$$\int_0^\infty \frac{\cos(at) - \cos(bt)}{t} dt.$$

Problem 98. Let $q^2 > 0$. Calculate the integral

$$\int_0^\infty \exp(-q^2 x)dx.$$

Problem 99. Let $x > 0$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is integrable over $[0, x]$ for all $x > 0$ and $\lim_{x\to\infty} f(x) = a$. Show that

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(s)ds = a.$$

Utilize

$$\left|\frac{1}{x}\int_0^x f(s)ds - a\right| = \left|\frac{1}{x}\int_0^x (f(s) - a)ds\right| \le \left|\frac{1}{x}\right| \int_0^x |f(s) - a|ds.$$

Problem 100. Let $a > 0$. Consider the compact set (*lemniscate*)

$$S := \{\, (x, y) \,:\, (x^2 + y^2)^2 \le 2a^2 xy \,\}.$$

Find

$$I(S) = \int_S d\mu = \int_S dxdy.$$

Introduce polar coordinates $x(r, \phi) = r\cos(\phi)$, $y(r, \phi) = r\sin(\phi)$. Thus we have

$$x^2 + y^2 = r^2, \qquad xy = r^2 \cos(\phi)\sin(\phi) \equiv \frac{1}{2}\sin(2phi).$$

Hilbert Transform

Problem 101. The *Hilbert transform* H and its inverse is given by

$$g(y) = H(f(x)) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{f(x)}{x-y} dx$$
$$f(x) = H^{-1}(g(y)) = \frac{1}{\pi} P \int_{\mathbb{R}} \frac{g(y)}{y-x} dy$$

where the *Cauchy principal value* is defined by

$$P \int_{\mathbb{R}} f(x) dx := \lim_{R \to \infty} \int_{-R}^{+R} f(x) dx.$$

The Hilbert transform relates parts of the function in the same domain. Let $k > 0$. Find the Hilbert transform of $f(x) = \cos(kx)$.

Problem 102. Consider the Hilbert transform

$$H[f] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx'.$$

Find H^2.

Problem 103. The *Hilbert transform* acts on a one-dimensional function $g(s)$ by a convolution with the kernel $1/(\pi s)$. The singularity at $s = 0$ is handled in the Cauchy principal value sense. The Fourier transform of the Hilbert kernel is $-i \operatorname{sgn}\sigma$. Thus the Hilbert transform of g is

$$Hg(s) = \int_{-\infty}^{\infty} \frac{g(s-s')}{\pi s'} ds' = \int_{-\infty}^{\infty} (-i \operatorname{sgn}\sigma) G(\sigma) e^{i2\pi s\sigma} d\sigma$$

where $G(\sigma)$ is the Fourier transform of g, i.e.

$$G(\sigma) = \int_{-\infty}^{\infty} g(s) e^{-i2\pi\sigma s} ds.$$

Suppose g is a function whose support is strictly less than radius R, i.e. $g(s) = 0$ for all $|s| > R - \epsilon$ for some small positive ϵ. Find an inversion formula.

Problem 104. The *Hilbert transform* of a function $f \in L_2(\mathbb{R})$ is defined as

$$H(f)(y) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(x)}{x-y} dx$$

where PV stands for *principal value*. Calculate the Hilbert transform of $f(x) = 1/(1+x^4)$.

Problem 105. The *Radon transform* for a function $f(x,y)$ is given by the integral transform

$$P(r,\theta) = Rf(x,y) = \int_{-\infty}^{+\infty} f(r\cos(\theta) - s\sin(\theta), r\sin(\theta) + s\cos(\theta))ds.$$

The function $P(r,\theta)$ desribes the values of points on projections. Show that the inverse Radon transform can be given by

$$f(x,y) = R^{-1}P(r,\theta) = -\frac{1}{2\pi}BHDP(r,\theta)$$

where D is the partial differential operator $Dg(r,\theta) = \partial g/\partial r$ with respect to r, H is the Hilbert transform operator

$$Hg(r,\theta) = -\frac{1}{\pi}\int \frac{g(u,\theta)}{r-u}du$$

and B is the backprojection operator

$$Bg(r,\theta) = \int_0^{\pi} g(x\cos(\theta) + y\sin(\theta), \theta)d\theta.$$

Problem 106. The *Hilbert transform* of a function $f \in L_2(\mathbb{R})$ is defined as

$$H(f)(y) = \frac{1}{\pi}PV\int_{-\infty}^{\infty} \frac{f(x)}{x-y}dx$$

where PV stands for *principal value*. Calculate the Hilbert transform of

$$f(x) = \exp(-x^2/2).$$

Problem 107. Consider the function $f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} 1/x - \text{int}(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where $\text{int}(y)$ denotes the integer part of y. The function is integrable.
(i) Let $k \geq 1$ be a positive integer. Find

$$\int_{1/(k+1)}^{1/k} f(x)dx.$$

(ii) Find

$$\int_{1/k}^{1} f(x)dx.$$

(iii) Find

$$\int_{0}^{1} f(x)dx.$$

Chapter 9

Functional Equations

Problem 1. To solve a number of nonlinear functional equation it is helpful to have the solution of the linear functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

where we assume that the function f is continouos.

Problem 2. Find the solution of the functional equations

$$f(x+y) = f(x)f(y) \tag{1}$$

where we assume that f is continuous.

Problem 3. Find the solution of the functional equation

$$f(xy) = f(x) + f(y) \tag{1}$$

where we assume that the function f is continuous.

Problem 4. Find the solution of the functional equation

$$f(xy) = f(x)f(y). \tag{1}$$

We assume that the function f is continuous.

Problem 5. Find the solution of the *Jensen equation*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \tag{1}$$

where we assume that f is continuous.

Problem 6. Show that the functional equation

$$f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$$

admits the solutions

$$f(x) = \tan(cx).$$

Problem 7. Show that the functional equation

$$f(x+y) = \frac{f(x)+f(y)}{1+\frac{f(x)f(y)}{C^2}} \tag{1}$$

admits the solution

$$f(x) = C\tanh(cx). \tag{2}$$

Problem 8. Show that the functional equation

$$f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)}$$

admits the solution

$$f(x) = \frac{c}{x}.$$

Problem 9. Show that the functional equation

$$f(x+y) = \frac{f(x)+f(y)-1}{2f(x)+2f(y)-2f(x)f(y)-1}$$

admits the solution

$$f(x) = \frac{1}{1+\tan cx}.$$

Problem 10. Show that the functional equation

$$f(x+y) = \frac{f(x)+f(y)+2f(x)f(y)}{1-f(x)f(y)}$$

admits the solution

$$f(x) = \frac{cx}{1-cx}.$$

Problem 11. Show that the functional equation

$$f(x+y) = f(x)f(y) + \sqrt{f(x)^2}\sqrt{f(y)^2 - 1}$$

admits the solution

$$f(x) = \cosh(cx).$$

Problem 12. Show that the functional equation

$$f(x+y+axy) = f(x)f(y)$$

admits the solution

$$f(x) = (1+ax)^c.$$

Problem 13. Show that the functional equation

$$f(x+y) - f(x-y) = 4\sqrt{f(x)f(y)}$$

admits the solution

$$f(x) = cx^2.$$

Problem 14. Solve the functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

assuming that f is a continuous function.

Problem 15. Show that the trigonometric functions $f(x) = \cos(x)$ and $g(x) = \sin(x)$ satisfy the system of functional equations

$$\begin{aligned} g(x+y) &= g(x)f(y) + f(x)g(y) \\ f(x+y) &= f(x)f(y) - g(x)g(y) \\ g(x-y) &= g(x)f(y) - g(y)f(x) \\ f(x-y) &= f(x)f(y) + g(x)g(y). \end{aligned}$$

Problem 16. Show that the Jacobi elliptic functions satisfy the system of functional equations

$$f(x \pm y) = \frac{f(x)g(y)h(y) \pm f(y)g(x)h(x)}{1 - k^2 f(x)^2 f(y)^2}$$

$$g(x \pm y) = \frac{g(x)g(y) \mp f(x)f(y)h(x)h(y)}{1 - k^2 f(x)^2 f(y)^2}$$

$$h(x \pm y) = \frac{h(x)h(y) \mp k^2 f(x)f(y)g(x)g(y)}{1 - k^2 f(x)^2 f(y)^2}.$$

Problem 17. Let a be a positive integer with $a \geq 2$. Let $1 \leq x \leq a$. Consider the equation

$$g(x-1) - 2g(x) + g(x+1) = -\lambda g(x).$$

Show that

$$g_j(x) = \sin(j\pi x/(a+1)), \qquad \lambda_j(x) = 2(1 - \cos(j\pi/(a+1)))$$

satisfy this equation.

Problem 18. Let a, c, ϵ be positive constants. Solve the functional equation

$$g((\theta + c)(\operatorname{mod} 1)) = ag(\theta) + \epsilon \sin(2\pi\theta).$$

Chapter 10

Convex Functions

Problem 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^3.$$

Show that f is not convex. Show that f is convex if we restrict to the domain $x \geq 0$.

Problem 2. Consider the convex functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that $f + g$ is convex. Show that $\max\{ f, g \}$ is convex.

Problem 3. **Definition (Convex Set)**. A subset C of $\mathbb{R}^n$ is said to be *convex* if for any $\mathbf{a}$ and $\mathbf{b}$ in C and any θ in $\mathbb{R}$, $0 \leq \theta \leq 1$, the n-tuple $\theta\mathbf{a} + (1-\theta)\mathbf{b}$ also belongs to C. In other words, if $\mathbf{a}$ and $\mathbf{b}$ are in C then

$$\{ \theta\mathbf{a} + (1-\theta)\mathbf{b} \, : \, 0 \leq \theta \leq 1 \} \subset C.$$

Definition (Convex Polyhedron). Let $\mathbf{a}_1, \ldots, a_p$ be p points in $\mathbb{R}^n$. The n-tuple

$$\sum_{j=1}^{p} \theta_j a_j, \quad \theta_j \geq 0, \quad j = 1, \ldots, p, \qquad \sum_{j=1}^{p} \theta_j = 1$$

is called a convex combination (or a convex sum) of $\mathbf{a}_1, \ldots, \mathbf{a}_p$. If $X \subset \mathbb{R}^n$ then the set of all (finite) convex combination of points of X is called the convex hull of X and is denoted by $H(X)$. If X is finite, $X = \{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$, then $H(X)$ is called the convex polyhedron spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_p$ and is also denoted by $H(\mathbf{a}_1, \ldots, \mathbf{a}_p)$.

Defintion. Let S be a nonempty convex set in $\mathbb{R}^n$, where $\mathbb{R}^n$ is the n-dimensional Euclidean space. The function $f : S \to \mathbb{R}$ is said to be *convex* if

$$f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)$$

for each $\mathbf{x}_1, \mathbf{x}_2 \in S$ and for each $\lambda \in [0,1]$. The function is said to be strictly convex if the above inequality holds as a strict inequality for each distinct $\mathbf{x}_1, \mathbf{x}_2 \in S$ and for each $\lambda \in (0,1)$.

Let

$$\mathbf{a}_1 = (1,-1), \qquad \mathbf{a}_2 = (2,2), \qquad \mathbf{a}_3 = (3,1)$$

be points of $\mathbb{R}^2$. Find the convex polyhedron.

Problem 4. Show that the function $\phi : [0,\infty) \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \ln x & \text{if } x \neq 0 \end{cases} \tag{1}$$

is strictly convex, i.e.,

$$\phi(\alpha x + \beta y) \leq \alpha\phi(x) + \beta\phi(y) \tag{2}$$

if $x, y \in [0,\infty), \alpha\beta \geq 0,\ \alpha + \beta = 1$ with equality only when $x = y$ or $\alpha = 0$ or $\beta = 0$. By induction we get

$$\phi\left(\sum_{i=1}^{k} \alpha_i x_i\right) \leq \sum_{i=1}^{k} \alpha_i \phi(x_i) \tag{3}$$

if

$$x_i \in [0,\infty), \quad \alpha_i \geq 0, \quad \sum_{i=1}^{k} \alpha_i = 1. \tag{4}$$

Equality holds only when all the x_i, corresponding to non-zero α_i, are equal.

Problem 5. Let $x \geq 0$ and

$$f(x) = x\ln(x) - x + 1. \tag{1}$$

(i) Show that

$$f(x) \geq 0. \tag{2}$$

(ii) Show that from L'Hospital rule we find that $f(0) = 1$.
(iii) Show that the function is convex.

Problem 6. Show that $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |x| \tag{1}$$

is convex.

Problem 7. Let $\{ f_1, f_2, \ldots, f_n \}$ be a set of convex functions from $\mathbb{R}^n \to \mathbb{R}$. Show that the nonnegative linear combination

$$f(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x}) + \cdots + \alpha_n f_n(\mathbf{x}), \qquad \alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$$

is convex.

Problem 8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = ax^2 + by^2 + 2cxy + d$$

where $a, b, c, d \in \mathbb{R}$. For what values of a, b, c, d is f concave?

Chapter 11

Inequalities

Problem 1. Let $a, b \in \mathbb{R}$. Show that

$$2ab \leq a^2 + b^2.$$

Problem 2. Let $a, b \in \mathbb{R}$. Show that

$$|a + b| \leq |a| + |b|.$$

Problem 3. Let $x \geq 0$ and $0 < p < 1$. Show that

$$\frac{1}{p}(1 - x^p) \geq 1 - x.$$

Problem 4. Let $x \in (0, 1)$. Show that

$$x(1 - x) < x.$$

Problem 5. Let $n \in \mathbb{N}$.
(i) Show that $n < 2^n$.
(ii) Show that $n^2 < 4^n$.
(iii) Show that if $n \geq 4$ then $2^n < n!$.

Problem 6. Let x, y be two nonnegative real numbers. Show that

$$xy \le \left(\frac{x+y}{2}\right)^2.$$

Problem 7. Let a, b, c, d be nonnegative real numbers. Show that

$$(abcd)^{1/4} \le \frac{1}{4}(a+b+c+d).$$

Problem 8. Let x_j $(j = 1, \ldots, n)$ be nonegative real numbers. Show that

$$x_1 x_2 \cdots x_n \le \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n.$$

Problem 9. Let a, b, c, d be nonnegative real numbers. Show that

$$a^4 + b^4 + c^4 + d^4 \ge 4abcd.$$

Problem 10. Let $x \ge 0$. Show that

$$1 - e^{-x} \ge \frac{x}{1+x}.$$

Problem 11. Let

$$e_n := 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

where $n = 1, 2, \ldots$. Let $m > n$. Show that

$$|e_m - e_n| = e_m - e_n \le \frac{2}{(n+1)!}.$$

Problem 12. Does the inequality

$$1 + 2\cos(\theta) - \cos(2\theta) \le 2$$

hold for all $\theta \in [0, 2\pi)$.

Problem 13. Let $x > 0$. Show that

$$1 - \frac{1}{x} \le \ln(x) \le x - 1$$

with equality iff $x = 1$.

Problem 14. Show that if a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a second order derivative which is non-negative (positive) everywhere, then the function is convex (strictly convex).

Problem 15. Let $a_1, a_2, \ldots, a_n$ be positive numbers and $b_1, b_2, \ldots, b_n$ be nonnegative numbers such that

$$\sum_{j=1}^{n} b_j > 0.$$

Show that (*log-sum inequality*)

$$\sum_{j=1}^{n} \left(a_j \log \frac{a_j}{b_j} \right) \geq \left(\sum_{j=1}^{n} a_j \right) \log \frac{\left(\sum_{j=1}^{n} a_j \right)}{\left(\sum_{j=1}^{n} b_j \right)}$$

with the conventions based on continuity arguments

$$0 \cdot \log 0 = 0, \qquad 0 \cdot \log \frac{p}{0} = \infty, \quad p > 0.$$

Show that equality holds if and only if $a_j/b_j = \text{constant}$ for all $j = 1, 2, \ldots, n$.

Problem 16. Let x_j $(j = 1, 2, \ldots, n)$ be positive real numbers. Show that

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{\sum_{k=1}^{n} x_k}{n}.$$

This is the arithmetic-geometric mean inequality.

Problem 17. Let n be a positive integer and $a, b \geq c/2 > 0$. Show that

$$|a^{-n} - b^{-n}| \leq 4nc^{-n-1}|a - b|.$$

Problem 18. Let $X \subset \mathbb{R}$ be an interval. A function $\psi : X \to \mathbb{R}$ is *convex* if for all $x_1, x_2 \in X$ and numbers $\alpha_1, \alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$,

$$\psi(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 \psi(x_1) + \alpha_2 \psi(x_2). \tag{1}$$

This means that every chord of the graph of ψ lies above the graph. Let $\psi : X \to \mathbb{R}$ be convex, let $x_1, x_2, \ldots, x_n \in X$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$

satisfy $\sum_{j=1}^{n} \alpha_j = 1$. Show that (Jensen's inequality)

$$\psi\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \psi(x_i). \tag{2}$$

Problem 19. Consider the differentiable function $f : [0,\infty) \to \mathbb{R}$

$$f(x) = \frac{x}{1+x}.$$

Let $a, b \in \mathbb{R}^+$. Show that

$$f(|a+b|) \leq f(|a|+|b|). \tag{1}$$

Problem 20. Let $a, b \in \mathbb{R}$. Show that

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

Problem 21. Show that $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |x|$$

is convex.

Problem 22. Let $n \in \mathbb{N}$ and $n \geq 2$. Show that

$$e^n \ln(n+1) < e^{n+1} \ln(n).$$

Problem 23. Show that the function $f : (0,\infty) \to \mathbb{R}$

$$f(x) = x \ln(x)$$

is convex.

Problem 24. Let $x, y \in \mathbb{R}$. Show that

$$\frac{x+y}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Problem 25. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$\|AB\| \leq \|A\| \cdot \|B\|.$$

Problem 26. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$\|A + B\| \leq \|A\| + \|B\|.$$

Problem 27. Let $\mathcal{H}$ be a Hilbert space. Let $\psi, \phi \in \mathcal{H}$. Assume that $\psi \neq 0, \phi \neq 0$. Show that

$$|\langle \phi, \psi \rangle| \leq \sqrt{\langle \phi, \phi \rangle} \sqrt{\langle \psi, \psi \rangle} \tag{1}$$

where $\langle \, , \, \rangle$ denotes the scalar product in the Hilbert space.

Problem 28. Let $n \geq 2$. Let $x_1, x_2, \ldots, x_n$ be given positive real number with

$$x_1 < x_2 < \cdots < x_n.$$

Let $\lambda_1, \ldots, \lambda_n \geq 0$ and $\sum_{j=1}^{n} \lambda_j = 1$. Show that

$$\left(\sum_{j=1}^{n} \lambda_j x_j \right) \left(\sum_{j=1}^{n} \lambda_j x_j^{-1} \right) \leq A^2 G^{-2}$$

where

$$A = \frac{1}{2}(x_1 + x_n), \qquad G = (x_1 x_n)^{1/2}.$$

Problem 29. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ (column vectors) and $\epsilon > 0$. Show that

$$2\mathbf{x}^T \mathbf{y} \leq \epsilon \mathbf{x}^T \mathbf{x} + \frac{1}{\epsilon} \mathbf{y}^T \mathbf{y}.$$

Problem 30. Let $a, b \in \mathbb{R}$ and $\epsilon > 0$. Show that

$$2ab \leq \epsilon a^2 + \epsilon^{-1} b^2.$$

Problem 31. Show that the analytic function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(\alpha, \beta) = \sin(\alpha) \sin(\beta) \cos(\alpha - \beta)$$

is bounded between $-1/8$ and 1.

Problem 32. Let n be an integer and $n \geq 2$. Show that

$$e^n \ln(n+1) < e^{n+1} \ln(n).$$

Problem 33. Let $n \in \mathbb{N}$ and $n \geq 2$. Show that

$$\frac{1}{\sqrt{n-1}} > 2\sqrt{n} - 2\sqrt{n-1} > \frac{1}{\sqrt{n}}.$$

Use this reult to show that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

diverges.

Problem 34. Let $x \in (0,1)$. Show that

$$x > \ln(1+x) > x - \frac{x^2}{2}.$$

Problem 35. Let a, b, c be positive numbers. Assume that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Show that

$$x + y + z \leq \sqrt{a^2 + b^2 + c^2}.$$

Use the Lagrange multiplier method.

Problem 36. Let a, b, c, d be positive numbers and $s := a + b + c + d$. Show that

$$\frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c} + \frac{s}{s-d} \geq \frac{16}{3}.$$

Problem 37. Let $x \geq 0$ and $0 < p < 1$. Show that

$$\frac{1}{p}(1 - x^p) \geq 1 - x.$$

Problem 38. Let b be a real number such that the elements of the infinite sequence $(a_k)_{k=1}^{\infty}$ satisfy

$$a_{k+m} \leq a_k + a_m + b$$

for all $k, m = 1, 2, \ldots$. Show that

$$a := \lim_{k \to \infty} \frac{a_k}{k}$$

exists and $a_k \geq ka - b$ for all k.

Problem 39. Let $x \geq 0$. Show that

$$1 - e^{-x} \geq \frac{x}{1+x}.$$

Problem 40. Let $n \in \mathbb{N}$ and $n \geq 2$. Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Problem 41. Let $a, b \geq 0$. Show that

$$\frac{a+b}{1+a+b} \leq \frac{a+b+ab}{1+a+b+ab} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Problem 42. Show that

$$\frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin^2(x-y)}{(x-y)^2} dx dy < \frac{1}{4}.$$

Problem 43. Let $x_1, x_2, x_3 \in \mathbb{R}$. Show that

$$x_1^2 + x_2^2 + x_3^2 \geq x_1 x_2 + x_2 x_3 + x_3 x_1.$$

Why could it be useful to consider the function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 - x_3 x_1\,?$$

Problem 44. Let x_1, x_2, $\dots$, x_n be positive numbers. Assume that the numbers satisfy

$$\sum_{j=1}^{n} x_j = 1, \qquad \sum_{j=1}^{n} x_j^2 = b^2.$$

Show that

$$\max\{\, x_j \,:\, 1 \leq j \leq n\} \leq \frac{1}{n}(1 + \sqrt{n-1}\sqrt{nb^2 - 1}).$$

Problem 45. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Show that

$$|f(x_2) - f(x_1)| \leq 4|x_2 - x_1|$$

for all x_1 and x_2 in $[-2, 2]$. Hint. Apply

$$x_1^2 - x_2^2 \equiv (x_1 + x_2)(x_1 - x_2).$$

Problem 46. Let $n \in \mathbb{N}$ and $n \geq 2$. Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Problem 47. Let $n \in \mathbb{N}$. Show that

$$\frac{n}{2} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1} \leq n.$$

Problem 48. Let $a, x, y \in \mathbb{R}$. Show that

$$|\sqrt{a^2 + x^2} - \sqrt{a^2 + y^2}| \leq |x - y|.$$

Matrices

Problem 49. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$|\mathrm{tr}(AB^*)|^2 \leq \mathrm{tr}(AA^*)\mathrm{tr}(BB^*).$$

Problem 50. Let A and B be two $n \times n$ matrices over $\mathbb{R}$. It can be shown that

$$\mathrm{tr}e^{A+B} \leq \mathrm{tr}(e^A e^B) \leq \frac{1}{2}\mathrm{tr}(e^{2A} + e^{2B})$$
$$\mathrm{tr}e^{A+B} \leq \mathrm{tr}(e^A e^B) \leq (\mathrm{tr}e^{pA})^{1/p}(\mathrm{tr}e^{qB})^{1/q}$$

where $p > 1$, $q > 1$ with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Is

$$(\mathrm{tr}e^{pA})^{1/p}(\mathrm{tr}e^{qB})^{1/q} \leq \frac{1}{2}\mathrm{tr}(e^{2A} + e^{2B})\,?$$

Prove or disprove.

Problem 51. Let A, B be hermitian matrices. Then

$$\mathrm{tr}(e^{A+B}) \leq \mathrm{tr}(e^A e^B).$$

Assume that

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate the left and right-hand side of the inequality. Does equality hold?

Problem 52. Let A, B be positive definite matrices. Show that

$$\mathrm{tr}(AB)^{2^{p+1}} \leq \mathrm{tr}(A^2B^2)^{2^p}, \qquad p \text{ a non-negative integer}$$

Problem 53. Let A be an $n \times n$ matrix with $\|A\| < 1$.
(i) Show that $(I_n + A)^{-1}$ exists.
(ii) Show that

$$\|(I_n + A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Problem 54. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $a_{jj} \geq 1$ for all j and

$$\sum_{j \neq k}^{n} a_{jk}^2 < 1.$$

Show that A is inververtible.

Problem 55. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let I_n be the $n \times n$ identity matrix. Assume that

$$\sum_{k=1}^{n} |a_{jk}| < 1$$

for each j. Show that $I_n - A$ is invertible.

Problem 56. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$. Let B be an $n \times n$ positive semidefinite matrix over $\mathbb{R}$. Show that

$$\det(A + B) \geq \det(A).$$

Problem 57. Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that there exists nonnull vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbb{R}^n$ such that

$$\frac{\mathbf{x}_1^T A \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \leq \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \leq \frac{\mathbf{x}_2^T A \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2}$$

for every nonnull vector $\mathbf{x}$ in $\mathbb{R}^n$.

Problem 58. Let A be an $n \times n$ skew-symmetric matrix over $\mathbb{R}$. Show that

$$\det(I_n + A) \geq 1$$

with equality holding if and only if $A = 0$.

Problem 59. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $a_{jj} \geq 1$ for $j = 1, 2, \ldots, n$ and

$$\sum_{j \neq k}^{n} a_{jk}^2 < 1.$$

Show that A is invertible.

Problem 60. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that

$$|a_{jj}| > \sum_{k \neq j}^{n} |a_{jk}|$$

for all $j = 1, 2, \ldots, n$. Show that A is invertible.

Problem 61. Let A, B be $n \times n$ positive definite matries. Show that

$$\mathrm{tr}(A \ln A) - \mathrm{tr}(A \ln B) \geq \mathrm{tr}(A - B).$$

Problem 62. Let $\mathbf{v}$ be a normalized (column) vector in $\mathbb{C}^n$ and let A be an $n \times n$ hermitian matrix. Is

$$\mathbf{v}^* e^A \mathbf{v} \geq e^{\mathbf{v}^* A \mathbf{v}}$$

for all normalized $\mathbf{v}$? Prove or disprove.

Problem 63. Consider the two manifolds

$$x_1^2 + x_2^2 = 1, \qquad y_1^2 + y_2^2 = 1.$$

Show that

$$|x_1 y_1 + x_2 y_2| \leq 1.$$

Hint. Set

$$x_1(t) = \cos(t), \quad x_2(t) = \sin(t), \quad y_1(t) = \cos(\tau), \quad y_2(t) = \sin(\tau).$$

Problem 64. Let $n \in \mathbb{N}$ and $x > 0$. Show that

$$x^{n+1} + \frac{1}{x^{n+1}} > x^n + \frac{1}{x^n}.$$

Problem 65. Find all $x \in \mathbb{R}$ and $x > 0$ such that

$$|\sqrt{x} - \sqrt{2}| < 1.$$

Problem 66. Let j, k be positive intgers. Find all pairs (j, k) such that the following four inequalities are satisfied

$$j + k < 10, \quad j + k \geq 6, \quad \frac{j}{k} > 1, \quad \frac{j}{k} < 2.$$

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