Problems and Solutions in Hilbert space theory, wavelets and generalized functions

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Preface

The purpose of this book is to supply a collection of problems in Hilbert space theory, wavelets and generalized functions.

Prescribed books for problems.

1) Hilbert Spaces, Wavelets, Generalized Functions and Modern Quantum Mechanics

by Willi-Hans Steeb
Kluwer Academic Publishers, 1998
ISBN 0-7923-5231-9

2) Classical and Quantum Computing with C++ and Java Simulations

by Yorick Hardy and Willi-Hans Steeb
Birkhauser Verlag, Boston, 2002
ISBN 376-436-610-0

3) Problems and Solutions in Quantum Computing and Quantum Information, second edition

by Willi-Hans Steeb and Yorick Hardy
World Scientific, Singapore, 2006
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6077.html`

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Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\|z\|$	modulus of complex number z
	$\|x+iy\| = (x^2+y^2)^{1/2}, \ \ x, y \in \mathbf{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbb{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\\| . \\|$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbb{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbb{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\mathrm{rank}(A)$	rank of matrix A
A^T	transpose of matrix A

$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices A and B
$[A, B] := AB - BA$	commutator for square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator for square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
δ	delta function
Θ	Heaviside's function
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

Chapter 1

General

Problem 1. Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\, ,\, \rangle$. Let $u, v \in \mathcal{H}$.
(i) Show that

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|.$$

(ii) Show that

$$\|u+v\| \leq \|u\| + \|v\|.$$

Problem 2. Consider a Hilbert space $\mathcal{H}$ with scalar product $\langle\, ,\, \rangle$. The scalar product implies a norm via $\|f\|^2 := \langle f, f\rangle$, where $f \in \mathcal{H}$.
(i) Show that

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

(ii) Assume that $\langle f, g\rangle = 0$, where $f, g \in \mathcal{H}$. Show that

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2.$$

Problem 3. Let $f, g \in \mathcal{H}$. Use the Schwarz inequality

$$|\langle f, g\rangle|^2 \leq \langle f, f\rangle\langle g, g\rangle = \|f\|^2\, \|g\|^2$$

to prove the triangle inequality

$$\|f+g\| \leq \|f\| + \|g\|.$$

Problem 4. Consider a complex Hilbert space $\mathcal{H}$ and $|\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}$. Let $c_1, c_2 \in \mathbb{C}$. An *antilinear operator* K in this Hilbert space $\mathcal{H}$ is characterized by

$$K(c_1|\phi_1\rangle + c_2|\phi_2\rangle) = c_1^* K|\phi_1\rangle + c_2^* K|\phi_2\rangle.$$

A *comb* is an antilinear operator K with zero expectation value for all states $|\psi\rangle$ of a certain complex Hilbert space $\mathcal{H}$. This means

$$\langle\psi|K|\psi\rangle = \langle\psi|LC|\psi\rangle = \langle\psi|L|\psi^*\rangle = 0$$

for all states $|\psi\rangle \in \mathcal{H}$, where L is a linear operator and C is the complex conjugation.
(i) Consider the two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$. Find a unitary 2×2 matrix such that

$$\langle\psi|UC|\psi\rangle = 0.$$

(ii) Consider the Pauli spin matrices with $\sigma_0 = I_2$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$. Find

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\langle\psi|\sigma_\mu C|\psi\rangle g^{\mu,\nu}\langle\psi|\sigma_\nu C|\psi\rangle$$

where $g^{\mu,\nu} = \mathrm{diag}(-1, 1, 0, 1)$.

Problem 5. Let P be a nonzero projection operator in a Hilbert space $\mathcal{H}$. Show that $\|P\| = 1$.

Problem 6. A family, $\{\psi_j\}_{j\in J}$ of vectors in the Hilbert space, $\mathcal{H}$, is called a *frame* if for any $f \in \mathcal{H}$ there exist two constants $A > 0$ and $0 < B < \infty$, such that

$$A\|f\|^2 \le \sum_{j\in J}|\langle\psi_j|f\rangle|^2 \le B\|f\|^2.$$

Consider the Hilbert space $\mathcal{H} = \mathbb{R}^2$ and the family of vectors

$$\left\{ \psi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Show that we have a tight frame.

Problem 7. Let $T : X \to Y$ be a linear map between linear spaces (vector spaces) X, Y. The *null space* or *kernel* of the linear map T, denoted by $\ker T$, is the subset of X defined by

$$\ker T := \{\, x \in X \,:\, Tx = 0 \,\}.$$

The *range* of T, denoted by ran$T$, is the subset of Y defined by

$$\mathrm{ran}T := \{\, y \in Y \,:\, \text{there exists } x \in X \text{ such that } Tx = y \,\}.$$

Let P be a projection operator in a Hilbert space $\mathcal{H}$. Show that ran$P$ is closed and

$$\mathcal{H} = \mathrm{ran}P \oplus \ker P$$

is the orthogonal direct sum of ran$P$ and ker$P$.

Problem 8. Let $\mathcal{H}$ be an arbitrary Hilbert space with scalar product $\langle\, ,\, \rangle$. Show that if φ is a bounded linear functional on the Hilbert space $\mathcal{H}$, then there is a unique vector $u \in \mathcal{H}$ such that

$$\varphi(x) = \langle u, x \rangle \qquad \text{for all } \; x \in \mathcal{H}.$$

Problem 9. Let $\mathcal{H}$ be an arbitrary Hilbert space. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ satisfies the *Fredholm alternative* if one of the following two alternatives holds:

(i) either $Ax = 0$, $A^*x = 0$ have only the zero solution, and the linear equations $Ax = y$, $A^*x = y$ have a unique solution $x \in \mathcal{H}$ for every $y \in \mathcal{H}$;
(ii) or $Ax = 0$, $A^*x = 0$ have nontrivial, finite-dimensional solution spaces of the same dimension, $Ax = y$ has a (nonunique) solution if and only if $y \perp u$ for every solution u of $A^*u = 0$, and $A^*x = y$ has a (nonunique) solution if and only if $y \perp u$ for every solution u of $Au = 0$.

Give an example of a bounded linear operator that satisfies the Fredholm alternative.

Problem 10. Let (M, d) be a complete metric space (for example a Hilbert space) and let $f : M \to M$ be a mapping such that

$$d(f^{(m)}(x), f^{(m)}(y)) \le k d(x, y), \qquad \forall x, y \in M$$

for some $m \ge 1$, where $0 \le k < 1$ is a constant. Show that the map f has a unique fixed point in M.

Problem 11. Let $\mathcal{H}$ be a Hilbert space and let $f : \mathcal{H} \to \mathcal{H}$ be a monotone mapping such that for some constant $\beta > 0$

$$\|f(u) - f(v)\| \le \beta \|u - v\| \qquad \forall\, u, v \in \mathcal{H}.$$

Show that for any $w \in \mathcal{H}$, the equation

$$u + f(u) = w$$

has a unique solution u.

Problem 12. Let $f, g \in \mathcal{H}$. Find all solutions to the equations

$$\langle f, g\rangle\langle g, f\rangle = i.$$

Problem 13. Let $f, g \in \mathcal{H}$. Show that

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$$

where the norm is implied by the scalar product of the Hilbert space.

Problem 14. Show that

$$\langle f, g\rangle = \frac{1}{4}\|f+g\|^2 - \frac{1}{4}\|f-g\|^2$$

or

$$\langle f, g\rangle = \frac{1}{4}\|f+g\|^2 - \frac{1}{4}\|f-g\|^2 + \frac{i}{4}\|f+ig\|^2 - \frac{i}{4}\|f-ig\|^2$$

depending on whether we are dealing with a real and complex Hilbert space.

Problem 15. Given a Hilbert space $\mathcal{H}$ and a Hilbert subspace $\mathcal{G}$ of $\mathcal{H}$. The Hilbert space *projection theorem* states that for every $f \in \mathcal{H}$, there exists a unique $g \in G$ such that

$$(i) \quad f - g \in G^{\perp}$$

$$(ii) \quad \|f-g\| = \inf_{h\in\mathcal{G}} \|f-h\|$$

where the space $\mathcal{G}^{\perp}$ is defined by

$$\mathcal{G}^{\perp} := \{\, k \in \mathcal{H} \,:\, \langle k|u\rangle = 0 \text{ for all } u \in \mathcal{G}\,\}.$$

Show that if g is the minimizer of $\|f-h\|$ over all $h \in \mathcal{G}$, then it is true that $f - g \in \mathcal{G}^{\perp}$.

Problem 16. Let $\{\,\phi_n\,\}_{n\in\mathbb{Z}}$ be an orthonormal basis in a Hilbert space $\mathcal{H}$. Then any vector $f \in \mathcal{H}$ can be written as

$$f = \sum_{n\in\mathbb{Z}} \langle f, \phi_n\rangle \phi_n\,.$$

Now suppose that $\{\,\psi_n\,\}_{n\in\mathbb{Z}}$ is also a basis for $\mathcal{H}$, but it is not orthonormal. Show that if we can find a so-called dual basis $\{\,\chi_n\,\}_{n\in\mathbb{Z}}$ satisfying

$$\langle \psi_n|\chi_m\rangle = \delta(n-m)$$

then for any vector $f \in \mathcal{H}$, we have

$$f = \sum_{n \in \mathbb{Z}} \langle f | \chi_n \rangle \psi_n.$$

Here $\delta(n-m)$ denotes the Kronecker delta with $\delta(n-m) = 0$ if $n = m$ and 1 otherwise.

Problem 17. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two normed spaces. Show that the product vector spaces $X = X_1 \times X_2$ is also a normed vector space if we define

$$\|x\| := \max(\|x_1\|_1, \|x_2\|_2)$$

with $x = (x_1, x_2)$.

Problem 18. Let A be a linear bounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Let $u, v \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Consider the equation

$$Au - \lambda u = v.$$

(i) Show that for λ nonreal (i.e. it has an imaginary part) v cannot vanish unless u vanishes.
(ii) Show that for λ nonreal we have

$$\|(A - \lambda I)^{-1} v\| \leq \frac{1}{|\Im \lambda|} \|v\|.$$

Problem 19. Let $\mathbb{E}$ be the exterior of the unit disc

$$\{ z \in \mathbb{C} : |z| > 1 \}$$

and $\mathbb{T}$ the unit circle

$$\{ z \in \mathbb{C} : |z| = 1 \}.$$

Let $\mathcal{H}_2(\mathbb{E})$ be the *Hardy spacee* of square integrable functions on $\mathbb{T}$, analytic in the region $\mathbb{E}$. The inner product for $f(z), g(z) \in \mathcal{H}_2(\mathbb{E})$ is defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\omega})^* g(e^{i\omega}) d\omega = \frac{1}{2\pi i} \oint_{\mathbb{T}} f^*(1/z^*) g(z) \frac{dz}{z}.$$

Let $f(z) = z^2$ and $g(z) = z + 1$. Find the scalar product $\langle f, g \rangle$.

Problem 20. Let $\mathcal{O} = \{ u_1 . u_2, \dots \}$ be an orthonormal set in a infinite dimensional Hilbert space. Show that if

$$x = \sum_{j=1}^{\infty} c_j u_j$$

then

$$\|x\|^2 = \sum_{j=1}^{\infty} |c_j|^2.$$

Problem 21. Two Cauchy sequences $\{ x_k \}$ and $\{ y_k \}$ are said to be equivalent if for all $\epsilon > 0$, there is a $k(epsilon)$ such that for all $j \geq k(\epsilon)$ we have $d(x_j, y_j) < \epsilon$. One writes $\{ x_k \} \sim \{ y_k \}$. Obviously, $\sim$ is an equivalence relationship. Show that equivalent Cauchy sequences have the same limit.

Problem 22. Consider the sequence $\{ x_k \}$, $k = 1, 2, \ldots$ in $\mathbb{R}$ defined by $x_k = 1/k^2$ for all $k = 1, 2, \ldots$. Show that this sequence is a Cauchy sequence.

Problem 23. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S}$ be a sub Hilbert space. Show that any element u of $\mathcal{H}$ can be decomposed uniquely

$$u = v + w$$

where v is in $\mathcal{S}$ and w is in $\mathcal{S}^{\perp}$.

Problem 24. Let u, v_1, v_2 be elements of a Hilbert space. Show that

$$2\|u - v_1\|^2 + 2\|u - v_2\|^2 = \|2\left(u - \frac{v_1 + v_2}{2}\right)\|^2 + \|v_1 - v_2\|^2.$$

Problem 25. Let P be the set of prime numbers. We define the set

$$S := \{(p, q) \, : \, p, q \in P \; p \leq q \}.$$

Show that

$$d((p_1, q_1), (p_2, q_2)) := |p_1 q_1 - p_2 q_2|$$

defines a metric.

Problem 26. Consider the vector space of all continuous functions defined on $[a, b]$. We define a metric

$$d(f, g) := \max_{a \leq x \leq b} |f(x) - g(x)|.$$

Let $a = \pi$, $b = \pi$, $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Find $d(f, g)$.

Problem 27. The $n \times n$ matrices over $\mathbb{R}$ form a vector space. Show that

$$d(A, B) := \sum_{j=1}^{n} \sum_{k=1}^{n} |a_{jk} - b_{jk}|$$

defines a metric.

Problem 28. Let $n \geq 1$. Consider the continuous function

$$f_n(t) = \begin{cases} 0 & 0 \leq t < 1/2 - 1/n \\ 1/2 + \frac{n}{2}(t - 1/2) & 1/2 - 1/n \leq t \leq 1/2 + 1/n \\ 1 & 1/2 + 1/n \leq t \leq 1 \end{cases}$$

Show that the sequence $\{ f_n(t) \}$ is not a Cauchy sequence for the uniform norm, but with any of the L^p norms $(1 \leq p < \infty)$ it is a Cauchy sequence.

Problem 29. The sequence space consists of the set of all (bounded or unbounded) sequences of complex

$$x = (\chi_1, \chi_2, \ldots)$$

Thus we have a vector space. Can we define a metric in this vector space which is implied by a norm?

Chapter 2

Finite Dimensional Hilbert Spaces

Problem 1. Consider the Hilbert space $\mathbb{R}^4$ and the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

(i) Show that the vectors are linearly independent.
(ii) Use the *Gram-Schmidt orthogonalization process* to find mutually orthogonal vectors.

Problem 2. Consider the Hilbert space $\mathbb{R}^4$. Show that the vectors (*Bell basis*)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

are linearly independent. Show that they form a orthonormal basis in the Hilbert space $\mathbb{R}^4$.

Problem 3. Consider the Hilbert space $\mathbb{R}^4$. Find all pairwise orthogonal vectors (column vectors) $\mathbf{x}_1, \dots, \mathbf{x}_p$, where the entries of the column vectors can only be $+1$ or -1. Calculate the matrix

$$\sum_{i=1}^{p} \mathbf{x}_i \mathbf{x}_i^T$$

and find the eigenvalues and eigenvectors of this matrix

Problem 4. A sequence $\{ f_n \}$ $(n \in \mathbb{N})$ of elements in a normed space E is called a *Cauchy sequence* if, for every $\epsilon > 0$, there exists a number M_ϵ, such that $\|f_p - f_q\| < \epsilon$ for $p, q > M_\epsilon$. Consider the Hilbert space $\mathbb{R}$. Show that

$$s_n = \sum_{j=1}^{n} \frac{1}{(j-1)!}, \qquad n \geq 1$$

is a Cauchy sequence.

Problem 5. Two Cauchy sequences $\{ x_k \}$ and $\{ y_k \}$ are said to be equivalent if for all $\epsilon > 0$, there is a $k(\epsilon)$ such that for all $j \geq k(\epsilon)$ we have $d(x_j, y_j) < \epsilon$. One writes $\{ x_k \} \sim \{ y_k \}$. Obviously, $\sim$ is an equivalence relationship. Show that equivalent Cauchy sequences have the same limit.

Problem 6. Consider the sequence $\{ x_k \}$, $k = 1, 2, \ldots$ in $\mathbb{R}$ defined by $x_k = 1/k^2$ for all $k = 1, 2, \ldots$. Show that this sequence is a Cauchy sequence.

Problem 7. Consider the Hilbert space $\mathbb{C}^2$ and the vectors

$$|0\rangle = \begin{pmatrix} i \\ i \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Normalize these vectors and then calculate the probability $|\langle 0|1\rangle|^2$.

Problem 8. Consider the Hilbert space $\mathbb{R}^n$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 \equiv 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2).$$

Note that

$$\|\mathbf{x}\|^2 := \langle \mathbf{x}, \mathbf{y} \rangle.$$

Problem 9. Let $|0\rangle$, $|1\rangle$ be an orthonormal basis in the Hilbert space $\mathbb{C}^2$. Let

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

where $\theta, \phi \in \mathbb{R}$.

(i) Find $\langle\psi|\psi\rangle$.
(ii) Find the probability $|\langle 0|\psi\rangle|^2$. Discuss $|\langle 0|\psi\rangle|^2$ as a function of θ.
(iii) Assume that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the 2×2 matrix $|\psi\rangle\langle\psi|$ and calculate the eigenvalues.

Problem 10. Consider the Hilbert space $\mathbb{R}^2$. Show that the vectors

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

are linearly independent. Find

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Show that these four vectors form a basis in $\mathbb{R}^4$. Consider the 4×4 matrix Q which is constructed from the four vectors given above, i.e. the columns of the 4×4 matrix are the four vectors. Find Q^T. Is Q invertible? If so find the inverse Q^{-1}. What is the use of the matrix Q?

Problem 11. Consider the Hilbert space $\mathbb{R}^4$. Let A be a symmetric 4×4 matrix over $\mathbb{R}$. Assume that the eigenvalues are given by $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$ and $\lambda_4 = 3$ with the corresponding normalized eigenfunctions

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Find the matrix A by means of the *spectral theorem.*

Problem 12. Show that the 2×2 matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ & -1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $M_2(\mathbb{C})$.

Problem 13. Consider the Hilbert space $\mathcal{H}$ of the 2×2 matrices over the complex numbers with the scalar product

$$\langle A, B \rangle := \mathrm{tr}(AB^*), \qquad A, B \in \mathcal{H}.$$

Show that the rescaled Pauli matrices $\mu_j := \frac{1}{\sqrt{2}}\sigma_j$, $j = 1, 2, 3$

$$\mu_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus the rescaled 2×2 identity matrix

$$\mu_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $\mathcal{H}$.

Problem 14. Let A, B be two $n \times n$ matrices over $\mathbb{C}$. We introduce the scalar product

$$\langle A, B \rangle := \frac{\mathrm{tr}(AB^*)}{\mathrm{tr} I_n} = \frac{1}{n}\mathrm{tr}(AB^*).$$

This provides us with a Hilbert space.
The Lie group $SU(N)$ is defined by the complex $n \times n$ matrices U

$$SU(N) := \{\, U \,:\, U^*U = UU^* = I_n \,,\, \det(U) = 1 \,\}.$$

The dimension is $N^2 - 1$. The Lie algebra $su(N)$ is defined by the $n \times n$ matrices X

$$su(N) := \{\, X \,:\, X^* = -X \,,\, \mathrm{tr} X = 0 \,\}.$$

(i) Let $U \in SU(N)$. Calculate $\langle U, U \rangle$.
(ii) Let A be an arbitrary complex $n \times n$ matrix. Let $U \in SU(N)$. Calculate $\langle UA, UA \rangle$.
(iii) Consider the Lie algebra $su(2)$. Provide a basis. The elements of the basis should be orthogonal to each other with respect to the scalar product given above. Calculate the commutators of these matrices.

Problem 15. Let $\hat{H} = \omega S_1$ be a Hamilton operator, where

$$S_1 := \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and ω is the frequency.
(i) Find $\exp(-i\hat{H}t/\hbar)\psi(0)$, where $\psi(0) = (1, 1, 1)^T/\sqrt{3}$.

(ii) Calculate the time evolution of

$$S_3 := \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

using the Heisenberg equation of motion. The matrices S_x, S_y, S_z are the spin-1 matrices, where

$$S_2 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

Problem 16. Consider the linear operator

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in the Hilbert space $\mathbb{R}^3$. Find

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

using the method of the Lagrange multiplier.

Problem 17. Consider the Hilbert space $\mathbb{R}^4$. Show that the *Bell basis*

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

forms an orthonormal basis in this Hilbert space.

Problem 18. Consider the Hilbert space $\mathbb{R}^3$. Let $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{x}$ is considered as a column vector. Find the matrix $\mathbf{x}\mathbf{x}^T$. Show that at least one eigenvalue is equal to 0.

Problem 19. (i) Consider the Hilbert space $\mathbb{C}^4$. Show that the matrices

$$\Pi_1 = \frac{1}{2}(I_2 \otimes I_2 + \sigma_1 \otimes \sigma_1), \qquad \Pi_2 = \frac{1}{2}(I_2 \otimes I_2 - \sigma_1 \otimes \sigma_1)$$

are projection matrices in $\mathbb{C}^4$.

(ii) Find $\Pi_1\Pi_2$. Discuss.
(iii) Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ be the standard basis in $\mathbb{C}^4$. Calculate

$$\Pi_1\mathbf{e}_j, \qquad \Pi_2\mathbf{e}_j, \qquad j = 1,2,3,4$$

and show that we obtain 2 two-dimensional Hilbert spaces under these projections.

Problem 20. Consider the 3×3 matrix

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) The matrix A can be considered as an element of the Hilbert space of the 3×3 matrices with the scalar product $\langle A, B \rangle := \mathrm{tr}(AB^T)$. Find the norm of A with respect to this Hilbert space.
(ii) On the other hand A can be considered as a linear operator in the Hilbert space $\mathbb{R}^3$. Find die norm

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|, \qquad \mathbf{x} \in \mathbb{R}^3.$$

(iii) Find the eigenvalues of A and AA^T. Compare the result with (i) and (ii).

Problem 21. Consider the Hilbert space $\mathbb{R}^3$. Find the spectrum (eigenvalues and normalized eigenvectors) of matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Find $\|A\| := \sup_{\mathbf{x}=1} \|A\mathbf{x}\|$, where $\|.\|$ denotes the norm and $\mathbf{x} \in \mathbb{R}^3$.

Problem 22. Find the spectrum (eigenvalues and normalized eigenvectors) of the 3×3 matrix

$$A = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

Find $\|A\|$, where $\|.\|$ denotes the norms

$$\|A\|_1 := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

$$\|A\|_2 := \sqrt{\mathrm{tr}(AA^*)}.$$

Compare the norms with the eigenvalues. Find $\exp(A)$.

Problem 23. Consider the Hilbert space $M_4(\mathbb{C})$ of all 4×4 matrices over $\mathbb{C}$ with the scalar product $\langle A, B\rangle := \mathrm{tr}(AB^*)$, where $A, B \in M_4(\mathbb{C})$. The γ-matrices are given by

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We define the 4×4 matrices

$$\sigma_{jk} := \frac{i}{2}[\gamma_j, \gamma_k], \qquad j < k$$

where $j = 1, 2, 3$, $k = 2, 3, 4$ and $[\,,\,]$ denotes the commutator.
(i) Calculate σ_{12}, σ_{13}, σ_{14}, σ_{23}, σ_{24}, σ_{34}.
(ii) Do the 16 matrices

$$I_4,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_5\gamma_1,\ \gamma_5\gamma_2,\ \gamma_5\gamma_3,\ \gamma_5\gamma_4,\ \sigma_{12},\ \sigma_{13},\ \sigma_{14},\ \sigma_{23},\ \sigma_{24},\ \sigma_{34}$$

form a basis in the Hilbert space $M_4(\mathbb{C})$? If so is the basis orthogonal?

Problem 24. Find the spectrum (eigenvalues and normalized eigenvectors) of matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find $\|A\|$, where $\|.\|$ denotes the norm.

Problem 25. Let A and B be two arbitrary matrices. Give the definition of the Kronecker product. Let $\mathbf{u}_j$ $(j = 1, 2, \ldots, m)$ be an orthonormal basis in the Hilbert space $\mathbb{R}^m$. Let $\mathbf{v}_k$ $(k = 1, 2, \ldots, n)$ be an orthonormal basis in the Hilbert space $\mathbb{R}^n$. Show that $\mathbf{u}_j \otimes \mathbf{v}_k$ $(j = 1, 2, \ldots, m)$, $(k = 1, 2, \ldots, n)$ is an orthonormal basis in $\mathbb{R}^{m+n}$.

Problem 26. Show that the 2×2 matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $M^2(\mathbb{C})$.

Problem 27. Show that the 2×2 matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

form a basis in the Hilbert space $M^2(\mathbb{R})$. Apply the *Gram-Schmidt technique* to obtain an orthonormal basis.

Problem 28. Consider the 3×3 matrices over the real numbers

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) The matrix A can be considered as an element of the Hilbert space of the 3×3 matrices over the real nunbers with the scalar product

$$\langle B, C \rangle := \mathrm{tr}(BC^T).$$

Find the norm of A with respect to this Hilbert space.
(ii) On the other hand the matrix A can be considered as a linear operator in the Hilbert space $\mathbb{R}^3$. Find the norm

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|, \qquad \mathbf{x} \in \mathbb{R}^3.$$

(iii) Find the eigenvalues of A and $A^T A$. Compare the result with (i) and (ii).

Problem 29. Consider the Hilbert space $\mathbb{C}^2$. The Pauli spin matrices σ_x, σ_y, σ_z act as linear operators in this Hilbert space. Let

$$\hat{H} = \hbar \omega \sigma_3$$

be a Hamilton operator, where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and ω is the frequency. Calculate the time evolution (intial value problem) of

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

i.e.

$$i\hbar \frac{d\sigma_x}{dt} = [\sigma_1, \hat{H}](t).$$

The matrices σ_1, σ_2, σ_3 are the Pauli matrices, where

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Problem 30. Consider the Hilbert space $\mathbb{C}^4$. Consider the Hamilton operator

$$\hat{H} := \hbar\omega \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Find the time-evolution of the operator

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

using the Heisenberg equation of motion

$$i\hbar \frac{d\gamma_3}{dt} = [\gamma_3, \hat{H}](t).$$

Problem 31. Let M be any $n \times n$ matrix. Let $\mathbf{x} = (x_1, x_2, \ldots)^T$. The linear operator A is defined by

$$A\mathbf{x} = (w_1, w_2, \ldots)^T$$

where

$$w_j := \sum_{k=1}^{n} M_{jk} x_k, \qquad j = 1, 2, \ldots, n$$
$$w_j := x_j, \qquad j > n$$

and $\mathcal{D}(A) = \ell_2(\mathbf{N})$. Show that A is self-adjoint if the $n \times n$ matrix M is hermitian. Show that A is unitary if M is unitary.

Problem 32. Consider the Hilbert space $\mathbb{C}^n$. Let $\mathbf{u}_j$, $j = 1, 2, \ldots, n$, and $\mathbf{v}_j$, $j = 1, 2, \ldots, n$ be orthonormal bases in $\mathbb{C}^n$, where $\mathbf{u}_j$, $\mathbf{v}_j$ are considered as column vectors. Show that

$$U = \sum_{j=1}^{n} \mathbf{u}_j \mathbf{v}_j^*$$

is a unitary $n \times n$ matrix.

Problem 33. Consider the Hilbert space $\mathbb{R}^2$. Given the vectors

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix}.$$

The three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are at 120 degrees of each other and are normalized, i.e. $\|\mathbf{u}_j\| = 1$ for $j = 1, 2, 3$. Every given two-dimensional vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3, \qquad c_1, c_2, c_3 \in \mathbb{R}$$

in many different ways. Given the vector $\mathbf{v}$ minimize

$$\frac{1}{2}(c_1^2 + c_2^2 + c_3^2)$$

subject to the two constraints

$$\mathbf{v} - c_1\mathbf{u}_1 - c_2\mathbf{u}_2 - c_3\mathbf{u}_3 = \mathbf{0}.$$

Problem 34. Let A, H be $n \times n$ hermitian matrices, where H plays the role of the Hamilton operator. The Heisenberg equations of motion is given by

$$\frac{dA(t)}{dt} = \frac{i}{\hbar}[H, A(t)].$$

with $A = A(t = 0) = A(0)$. Let E_j ($j = 1, 2, \ldots, n^2$) be an orthonormal basis in the Hilbert space $\mathcal{H}$ of the $n \times n$ matrices with scalar product

$$\langle X, Y \rangle := \operatorname{tr}(XY^*), \qquad X, Y \in \mathcal{H}.$$

Now $A(t)$ can be expanded using this orthonormal basis as

$$A(t) = \sum_{j=1}^{n^2} c_j(t) E_j$$

and H can be expanded as

$$H = \sum_{j=1}^{n^2} h_j E_j.$$

Find the time evolution for the coefficients $c_j(t)$, i.e. dc_j/dt, where $j = 1, 2, \ldots, n^2$.

Problem 35. The sequence space consists of the set of all (bounded or unbounded) sequences of complex numbers

$$x = (x_1, x_2, \ldots)$$

Thus we have a vector space. Can we define a metric in this vector space which is not implied by a norm?

Chapter 3

Hilbert Space $L_2(\Omega)$

Problem 1. A basis in the Hilbert space $L_2[0,1]$ is given by

$$B := \left\{ e^{2\pi i x n} : n \in \mathbb{Z} \right\}.$$

Let

$$f(x) = \begin{cases} 2x & 0 \le x < 1/2 \\ 2(1-x) & 1/2 \le x < 1 \end{cases}$$

Is $f \in L_2[0,1]$? Find the first two expansion coefficients of the Fourier expansion of f with respect to the basis given above.

Problem 2. (i) Consider the Hilbert space $L_2[-1,1]$. Consider the sequence

$$f_n(x) = \begin{cases} -1 \text{ if } & -1 \le x \le -1/n \\ nx \text{ if } & -1/n \le x \le 1/n \\ +1 \text{ if } & 1/n \le x \le 1 \end{cases}$$

where $n = 1, 2, \ldots$. Show that $\{ f_n(x) \}$ is a sequence in $L_2[-1,1]$ that is a Cauchy sequence in the norm of $L_2[-1,1]$.
(ii) Show that $f_n(x)$ converges in the norm of $L_2[-1,1]$ to

$$\operatorname{sgn}(x) = \begin{cases} -1 \text{ if } -1 \le x < 0 \\ +1 \text{ if } \ \ 0 < x \le 1 \end{cases}.$$

(iii) Use this sequence to show that the space $C[-1,1]$ is a subspace of $L_2[-1,1]$ that is not closed.

Problem 3. Let $f \in L_2(\mathbb{R})$. Give the definition of the Fourier transform. Let us call the transformed function $\hat{f}$. Is $\hat{f} \in L_2(\mathbb{R})$? What is preserved under the Fourier transform?

Problem 4. Consider the Hilbert space $L_2[a,b]$, where $a,b \in \mathbb{R}$ and $b > a$. Find the condition on a and b such that

$$\langle \cos(x), \sin(x) \rangle = 0$$

where $\langle\, , \,\rangle$ denotes the scalar product in $L_2[a,b]$.
Hint. Since $b > a$, we can write $b = x + \epsilon$, where $\epsilon > 0$.

Problem 5. Consider the Hilbert space $L_2[0,1]$. The *Legendre polynomials* are defined as

$$P_0(x) = 1, \qquad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1).$$

Show that the first first four elements are given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3-3x).$$

Normalize the four elements. Show that the four elements are pairwise orthonormal.

Problem 6. Let R be a bounded region in n-dimensional space. Consider the eigenvalue problem

$$-\Delta u = \lambda u, \qquad u(q \in \partial R) = 0$$

where ∂R denotes the boundary of R.
(i) Show that all eigenvalues are real and positive
(ii) Show that the eigenfunctions which belong to different eigenvalues are orthogonal.

Problem 7. Consider the inner product space

$$C[a,b] = \{\, f(x) \,:\, f \text{ is continuous on } x \in [a,b] \,\}$$

with the inner product

$$\langle f, g \rangle := \int_a^b f(x) g^*(x) dx.$$

This implies a norm

$$\langle f, f \rangle = \int_a^b f(x) f^*(x) dx = \|f\|^2.$$

Show that $C[a,b]$ is incomplete. This means find a Cauchy sequence in the space $C[a,b]$ which converges to an element which is not in the space $C[a,b]$.

Problem 8. Consider the Hilbert space $L_2[-\pi,\pi]$. Given the function

$$f(x) = \begin{cases} 1 & 0 < x \le \pi \\ 0 & x = 0 \\ -1 & -\pi \le < 0 \end{cases}$$

Obviously $f \in L_2[-\pi,\pi]$. Find the Fourier expansion of f. The orthonormal basis $\mathcal{B}$ is given by

$$\mathcal{B} := \left\{ \phi_k(x) = \frac{1}{\sqrt{2\pi}} \exp(ikx) \;\; k \in \mathbb{Z} \right\}.$$

Find the approximation $a_0\phi_0(x) + a_1\phi_1(x) + a_{-1}\phi_{-1}(x)$, where a_0, a_1, a_{-1} are the Fourier coefficients.

Problem 9. Consider the linear operator A in the Hilbert space $L_2[0,1]$ defined by $Af(x) := xf(x)$. Find the matrix elements

$$\langle P_i, AP_j \rangle$$

for $i,j = 0,1,2,3$, where P_i are the (normalized) Legrende polynomials. Is the matrix A_{ij} symmetric?

Problem 10. Consider the Hilbert space $L_2[0,2\pi)$. Let

$$g(x) = \cos(x), \qquad f(x) = x.$$

Find the conditions on the coefficients of the polynomial

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

such that

$$\langle g(x), p(x) \rangle = 0, \qquad \langle f(x), p(x) \rangle = 0.$$

Solve the equations for a_3, a_2, a_1, a_0.

Problem 11. Consider the Hilbert space $L_2(\mathbb{R})$. Give the definition and an example of an even function in $L_2(\mathbb{R})$. Give the definition and an example of an odd function in $L_2(\mathbb{R})$. Show that any function $f \in L_2(\mathbb{R})$ can be written as a combination of an even and an odd function.

Problem 12. The Chebyshev polynomials $T_n(x)$ of the 1-st kind are defined for $x \in [-1,1]$ and given by

$$T_n(x) = \cos(n \arccos x), \qquad n = 0,1,2,\ldots$$

The Chebyshev polynomials $U_n(x)$ of the 2-nd kind are defined for $x \in [-1,1]$ and given by

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sqrt{1-x^2}}, \qquad n = 0, 1, 2, \ldots .$$

Consider the Hilbert spaces

$$\mathcal{H}_1 = L_2\left([-1,1], \frac{dx}{\pi\sqrt{1-x^2}}\right), \qquad \mathcal{H}_2 = L_2\left([-1,1], \frac{2\sqrt{1-x^2}dx}{\pi}\right)$$

which bases are formed by the Chebyshev polynomials of the 1-st and 2-nd type

$$\Phi_n^{(1)}(x) = \sqrt{2}T_n(x), \quad n \geq 1, \quad \Phi_0^{(1)} = T_0(x) = 1$$
$$\Phi_n^{(2)}(x) = U_n(x), \qquad n \geq 0$$

Find a recursion relation for $\Phi_n^{(1)}$ and $\Phi_n^{(2)}$.

Problem 13. Consider the Hilbert space $L_2[-\pi,\pi]$. Obviously $\cos(x) \in L_2[-\pi,\pi]$. Find the norm $\|\cos(x)\|$. Find nontrivial functions $f, g \in L_2[-\pi,\pi]$ such that

$$\langle f(x), \cos(x)\rangle = 0, \qquad \langle g(x), \cos(x)\rangle = 0$$

and

$$\langle f(x), g(x)\rangle = 0.$$

Problem 14. Consider the Hilbert space $L_2[0,1]$. Find a non-trivial polynomial p

$$p(x) = ax^3 + bx^2 + cx + d$$

such that

$$\langle p, 1\rangle = 0, \qquad \langle p, x\rangle = 0, \qquad \langle p, x^2\rangle = 0.$$

Problem 15. Consider the set of polynomials

$$\{1, x, x^2, \ldots, x^n, \ldots\}.$$

Use the Gram-Schmidt procedure and the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\omega(x)dx, \qquad \omega(x) > 0$$

to obtain the first four orthogonal polynomials when

(i) $a = -1$, $b = 1$, $\omega(x) = 1$ (Legrendre polynomials)
(ii) $a = -1$, $b = 1$, $\omega(x) = (1-x^2)^{-1/2}$ (Chebyshev polynomials)
(iii) $a = 0$, $b = +\infty$, $\omega(x) = e^{-x}$ (Laguerre polynomials)
(iv) $a = -\infty$, $b = +\infty$, $\omega(x) = e^{-x^2}$ (Hermite polynomials)

Problem 16. Consider the function

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \cos(jx).$$

Is f an element of $L_2[-\pi, \pi]$?

Problem 17. Consider the Hilbert space $L_2([0,1])$. The *shifted Legrendre polynomials* , defined on the interval $[0,1]$, are obtained from the Legrendre polynomial by the transformation $y = 2x - 1$. The shifted Legrendre polynomials are given by the recurrence formula

$$P_j(x) = \frac{(2j+1)(2x-1)}{j+1} P_j(x) - \frac{j}{j+1} P_{j-1}(x) \qquad j = 1, 2, \ldots$$

and $P_0(x) = 1$, $P_1(x) = 2x - 1$. They are elements of the Hilbert space $L_2([0,1])$. A function u in the Hilbert space $L_2([0,1])$ can be approximated in the form of a series with $n+1$ terms

$$u(x) = \sum_{j=0}^{n} c_j P_j(x)$$

where the coeffients $c_j \in \mathbb{R}$, $j = 0, 1, \ldots, n$. Consider the Volterra integral equation of first kind

$$\lambda \int_0^x \frac{y(t)}{(x-t)^\alpha} dt = f(x), \qquad 0 \le t \le x \le 1$$

with $0 < \alpha < 1$ and $f \in L_2([0,1])$. Consider the ansatz

$$y_n(x) = a_0 x^\alpha + \sum_{j=0}^{n} c_j P_j(x).$$

to find an approximate solution to the *Volterra integral equation* of first kind ($\alpha = 1/2$)

$$\lambda \int_0^x \frac{y(t)}{\sqrt{x-t}} dt = f(x)$$

where

$$f(x) = \frac{2}{105} \sqrt{x}(105 - 56x^2 + 48x^3).$$

Problem 18. The *Fock space* $\mathcal{F}$ is the Hilbert space of entire functions with inner product given by

$$\langle f|g\rangle := \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}dxdy, \qquad z = x+iy$$

where $\mathbb{C}$ denotes the complex numbers. Therefore the growth of functions in the Hilbert space $\mathcal{F}$ is dominated by $\exp(|z|^2/2)$. Let $f, g \in \mathcal{F}$ with Taylor expansions

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \qquad g(z) = \sum_{j=0}^{\infty} b_j z^j.$$

(i) Find $\langle f|g\rangle$ and $\|f\|^2$.
(ii) Consider the special that $f(z) = \sin(z)$ and $g(z) = \cos(z)$. Calculate $\langle f|g\rangle$.
(iii) Let

$$\mathcal{K}(z,w) := e^{z\overline{w}}, \qquad z, w \in \mathbb{C}$$

Calculate $\langle f(z)|\mathcal{K}(z,w)\rangle$.

Problem 19. Consider the Hilbert space $L_2[0,\pi]$. Let $\|\,.\,\|$ be the norm induced by the scalar product of $L_2[0,\pi]$. Find the constants a, b such that

$$\|\sin(x) - (ax^2+bx)\|$$

is a minimum.

Problem 20. Consider the Hilbert space $L_2(\mathbb{R})$. Let

$$f_n(x) = \frac{x}{1+nx^2}, \qquad n = 1, 2, \ldots .$$

(i) Find $\|f_n(x)\|$ and

$$\lim_{n\to\infty} \|f_n(x)\|.$$

(ii) Does the sequence $f_n(x)$ converge uniformely on the real line?

Problem 21. Let $n = 1, 2, \ldots$. We define the functions $f_n \in L_2[0,\infty)$ by

$$f_n(x) = \begin{cases} \sqrt{n} \text{ for } n \le x \le n+1/n \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

(i) Calculate the norm $\|f_n - f_m\|$ implied by the scalar product. Does the sequence $\{\, f_n \,\}$ converge in the $L_2[0,\infty)$ norm?
(ii) Show that $f_n(x)$ converges pointwise in the domain $[0,\infty)$ and find the limit. Does the sequence converge pointwise uniformly?

(iii) Show that $\{ f_n \}$ $(n = 1, 2, \ldots)$ is an orthonormal system. Is it a basis in the Hilbert space $L_2[0, \infty)$?

Problem 22. Consider the function $f \in L_2[0,1]$

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1/2 \\ 1 - x & \text{for } 1/2 \leq x \leq 1 \end{cases}$$

A basis in the Hilbert space is given by

$$\mathcal{B} := \left\{ 1,\ \sqrt{2}\cos(\pi n x) \quad : \quad n = 1, 2, \ldots \right\}.$$

Find the Fourier expansion of f with respect to this basis. From this expansion show that

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

Problem 23. A particle is enclosed in a rectangular box with impenetrable walls, inside which it can move freely. The Hilbert space is

$$L_2([0,a] \times [0,b] \times [0,c])$$

where $a, b, c > 0$. Find the eigenfunctions and the eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions?

Problem 24. Consider the Hilbert space $L_2[0,1]$. Find a non-trivial function

$$f(x) = ax^3 + bx^2 + cx + d.$$

such that

$$\langle f(x), x \rangle = 0, \qquad \langle f(x), x^2 \rangle = 0, \qquad \langle f(x), x^3 \rangle = 0$$

where $\langle\, , \rangle$ denotes the scalar product

Problem 25. Consider the Hilbert space $L_2[0,1]$. Find a non-trivial function f such that

$$\langle f(x), x \rangle = 0, \qquad \langle f(x), x^2 \rangle = 0, \qquad \langle f(x), x^3 \rangle = 0$$

where $\langle\, , \rangle$ denotes the scalar product

Problem 26. Consider the Hilbert space $L_2[0,1]$ and the polynomials

$$1,\ x,\ x^2,\ x^3,\ x^4\,.$$

Apply the Gram-Schmidt orthogonalization process to these polynomials.

Problem 27. Consider the Hilbert space $L_2(\mathbb{T})$. Let $f \in L_2(\mathbb{T})$. Give an example of a bounded linear functional.

Problem 28. Consider the Hilbert space $L_2(\mathbb{R})$. Show that the Hilbert space is the direct sum of the Hilbert space $\mathcal{M}$ of even functions and the Hilbert space $\mathcal{N}$ of odd functions. Give an example of such functions in this Hilbert space.

Problem 29. Let $a > 0$. Consider the Hilbert space $L_2[0, a]$. Let

$$Af(x) := xf(x)$$

for $f \in L_2[0, a]$. Find the norm of the operator A. We define

$$\|A\| := \sup_{\|f\|=1} \|Af\|.$$

Problem 30. Consider the Hilbert space $L_2[0, 2\pi]$. Let

$$g(x) = \cos(x), \qquad f(x) = x.$$

Find the conditions on the coefficients of the polynomial

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

such that

$$\langle g(x), p(x)\rangle = 0, \qquad \langle f(x), p(x)\rangle = 0.$$

Solve the equations for a_3, a_2, a_1, a_0.

Problem 31. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1 - \cos(2\pi x)}{x}.$$

Using L'Hospital rules we have $f(0) = 0$. Is $f \in L_2(\mathbb{R})$?

Problem 32. Consider the Hilbert space $L_2[-1, 1]$. The *Legendre polynomials* are given by

$$P_j(x) := \frac{1}{2^j j!}\frac{d^j}{dx^j}(x^2 - 1)^j.$$

Find the scalar product

$$\langle P_j(x), P_k(x) \rangle.$$

Problem 33. Consider the Hilbert space $\mathcal{H} = L_2(\mathbb{T})$. This is the vector space of 2π-periodic functions. Then

$$u(x) = \frac{1}{\sqrt{2}}$$

is a constant function which is normalized, i.e. $\|u\| = 1$. Show that the projection operator P_u defined by

$$P_u f := \langle u, f \rangle u$$

maps a function f to its mean. This means

$$P_u f = \langle f \rangle, \qquad \langle f \rangle = \int_0^{2\pi} f(x) dx.$$

Problem 34. Consider the Hilbert space $L_2[-\pi, \pi]$ and the vector space of continiuous real-valued functions $C[-\pi, \pi]$ on the interval $[-\pi, \pi]$. Let $k > 0$ and

$$f_k(x) = \begin{cases} 0 \text{ if } -\pi \le x \le 0 \\ kx \text{ if } 0 \le x \le 1/k \\ 1 \text{ if } 1/k \le x \le \pi \end{cases}$$

The sequence of functions f_k belongs to the vector space $C[-\pi, \pi]$.
(i) Show that $f_n \to \chi$ in the norm of the Hilbert space $L_2[-\pi, \pi]$, where

$$\chi(x) := \begin{cases} 0 \text{ if } -\pi \le x \le 0 \\ 1 \text{ if } \ 0 < x \le \pi \end{cases}$$

so that the sequence $\{ f_k \}$ is a Cauchy sequence in the Hilbert space $L_2[-\pi, \pi]$.
(ii) Show that $\|\chi - g\| > 0$ for every $g \in C[-\pi, \pi]$. Conclude that $C[-\pi, \pi]$ is not a Hilbert space.

Problem 35. Let Ω be the unit disk. A Hilbert space of analytic functions can be defined by

$$\mathcal{H} := \left\{ f(z) \text{ analytic}, \ |z| < 1 \ : \sup_{a<1} \int_{|z|=a} |f(z)|^2 ds < \infty \right\}$$

and the scalar product

$$\langle f, g \rangle := \lim_{a \to 1} \int_{|z|=a} \overline{f(z)} g(z) ds.$$

Let c_n $(n = 0, 1, 2, \ldots)$ be the coefficients of the power-series expansion of the analytic function f. Find the norm of f.

Problem 36. Let $\mathbb{C}^n$ denote the complex Euclidean space. Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{C}^n$ then the scalar product (inner product) is given by

$$\mathbf{z} \cdot \mathbf{w} := \mathbf{z}\mathbf{w}^* = \mathbf{z}\overline{\mathbf{w}}^T$$

where $\overline{\mathbf{z}} = (\overline{z}_1, \ldots, \overline{z}_n)$. Let E_n denote the set of entire functions in $\mathbb{C}^n$. Let F_n denote the set of $f \in E_n$ such that

$$\|f\|^2 := \frac{1}{\pi^n} \int_{\mathbb{C}^n} |f(\mathbf{z})|^2 \exp(-|\mathbf{z}|^2) dV$$

is finite. Here dV is the volume element (Lebesgue mesure)

$$dV = \prod_{j=1}^{n} dx_j dy_j = \prod_{j=1}^{n} r_j dr_j d\theta_j$$

with $z_j = r_j e^{i\theta_j}$. The norm follows from the scalar product of two functions $f, g \in F_n$

$$\langle f, g \rangle := \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(\mathbf{z}) \overline{g(\mathbf{z})} \exp(-|\mathbf{z}|^2) dV.$$

Let

$$\mathbf{z}^m := z_1^{m_1} \cdots z_n^{m_n}$$

where the multiindex m is defined by $m! = m_1! \cdots m_n!$ and $|m| = \sum_{j=1}^{n} m_j$. Find the scalar product

$$\langle \mathbf{z}^m, \mathbf{z}^p \rangle.$$

Problem 37. Let Ψ be a complex-valued differentiable function of ϕ in the interval $[0, 2\pi]$ and $\Psi(0) = \Psi(2\pi)$, i.e. Ψ is an element of the Hilbert space $L_2([0, 2\pi])$. Assume that (normalization condition)

$$\int_0^{2\pi} \Psi^*(\phi) \Psi(\phi) d\phi = 1.$$

Calculate

$$\Im \frac{\hbar}{i} \int_0^{2\pi} \Psi^*(\phi) \phi \frac{d}{d\phi} \Psi(\phi) d\phi$$

where $\Im$ denotes the imaginary part.

Problem 38. The *Legrendre polynomials* are defined on the interval $[-1, 1]$ and defined by the recurrence formula

$$L_j(x) = \frac{2j+1}{j+1} x L_j(x) - \frac{j}{j+1} L_{j-1}(x) \qquad j = 1, 2, \ldots$$

and $L_0(y) = 1$, $L_1(x) = x$. They are elements of the Hilbert space $L_2([-1, 1])$. Calculate the scalar product

$$\langle L_j(x), L_k(x) \rangle$$

for $j, k = 0, 1, \ldots$. Discuss.

Problem 39. Let $f_n : [-1, 1] \to [-1, 1]$ be defined by

$$f_n(x) = \begin{cases} 1 & \text{for } -1 \le x \le 0 \\ \sqrt{1 - nx} & \text{for } 0 \le x \le 1/n \\ 0 & \text{for } 1/n \le x \le 1 \end{cases}$$

Show that $f_n \in L_2[-1, 1]$. Show that f_n is a Cauchy sequence.

Problem 40. Consider the Hilbert space $L_2([-1, 1])$. The *Chebyshev polynomials* are defined by

$$T_n(x) := \cos(n \cos^{-1} x), \quad n = 0, 1, 2, \ldots.$$

They are elements of the Hilbert space $L_2([-1, 1])$. We have

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x.$$

Calculate the scalar products

$$\langle T_0, T_1 \rangle, \quad \langle T_1, T_2 \rangle, \quad \langle T_2, T_3 \rangle.$$

Calculate the integrals

$$\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\sqrt{1 - x^2}} dx$$

for $(m, n) = (0, 1), (m, n) = (1, 2), (m, n) = (2, 3)$.

Problem 41. (i) Consider the Hilbert space $L_2[0, 1]$ with the scalar product $\langle \cdot, \cdot \rangle$. Let $f : [0, 1] \to [0, 1]$

$$f(x) := \begin{cases} 2x & \text{if } x \in [0, 1/2) \\ 2(1 - x) & \text{if } x \in [1/2, 1] \end{cases}$$

Thus $f \in L_2[0,1]$. Calculate the moments μ_k, $k = 0, 1, 2, \ldots$ defined by

$$\mu_k := \langle f(x), x^k \rangle \equiv \int_0^1 f(x) x^k dx.$$

(ii) Show that

$$\sum_{k=0}^{\infty} |\mu_k|^2 < \pi \int_0^1 |f(x)|^2 dx.$$

Problem 42. Let $a, b \in \mathbb{R}$ and $-\infty < a < b < +\infty$. Let f be a function in the class C^1 (i.e., the derivative df/dt exists and is continuous) on the interval $[a, b]$. Thus f is also an element of the Hilbert space $L_2([a, b])$. Show that

$$\lim_{\omega \to \infty} \int_a^b f(t) \sin(\omega t) dt = 0. \tag{1}$$

Problem 43. Consider the *Lie group*

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \; |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

The elements of this Lie group act as analytic automorphism of the disk

$$\Omega := \{ \, |z| < 1 \, \}$$

under

$$z \to zg = \frac{\bar{\alpha} z + \beta}{\bar{\beta} z + \alpha}$$

where $(zg)h = z(gh)$. Let $n \geq 2$. We define

$$\mathcal{H}_n := \{ \, f(z) \text{ analytic in } \Omega, \; \|f\|^2 = \int_\Omega |f(z)|^2 (1 - |z|^2)^{n-2} dx dy < \infty \, \}$$

and

$$U_n(g) f(z) := \frac{1}{(\bar{\beta} z + \alpha)^n} f((\bar{\alpha} z + \beta)/(\bar{\beta} z + \alpha)).$$

Then $\mathcal{H}_n$ is a Hilbert space, i.e., the analytic functions in

$$L_2(\Omega, (1 - |z|^2)^{n-2} dx dy)$$

form a closed subspace. U_n is a representation, i.e.,

$$U_n(gh) = U_n(g) U_n(h)$$

and $U_n(e) = I$, where e is the identity element in $SU(1,1)$ (2×2 unit matrix).
Show that

$$\frac{1}{(1-|z|^2)^2} dx \wedge dy$$

is invariant $z \to zg$.

Problem 44. Consider the problem of a particle in a one-dimensional box. The underlying Hilbert space is $L_2(-a,a)$. Solve the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

as follows: The formal solution is given by

$$\psi(t) = \exp(-i\hat{H}t/\hbar)\psi(0).$$

Expand $\psi(0)$ with respect to the eigenfunctions of the operator $\hat{H}$. The eigenfunctions form a basis of the Hilbert space. Then apply $\exp(-i\hat{H}t/\hbar)$. Calculate the probability

$$P = |\langle \phi, \psi(t) \rangle|^2$$

where

$$\phi(q) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right)$$

and

$$\psi(q,0) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right).$$

Problem 45. Let $f \in L_2(\mathbb{R}^n)$. Consider the following operators

$$\begin{aligned} T_{\mathbf{y}} f(\mathbf{x}) &= f(\mathbf{x}-\mathbf{y}), \qquad \text{translation operator} \\ M_{\mathbf{k}} f(\mathbf{x}) &= e^{i\mathbf{x}\cdot\mathbf{k}} f(\mathbf{x}), \qquad \text{modulation operator} \\ D_s f(\mathbf{x}) &= |s|^{-n/2} f(s^{-1}\mathbf{x}), \quad s \in \mathbb{R}\setminus\{0\} \quad \text{dilation operator} \end{aligned}$$

where $\mathbf{x}\cdot\mathbf{k} = k_1 x_1 + \cdots + x_n k_n$.
(i) Find $\|T_{\mathbf{y}} f\|$, $\|M_{\mathbf{k}}\|$, $\|D_s f\|$, where $\|\ \|$ denotes the norm in $L_2(\mathbb{R}^n)$.
(ii) Find the adjoint operators of these three operators.

Problem 46. Consider the vector space

$$H_1(a,b) := \{\, f(x) \in L_2(a,b) \,:\, f'(x) \in L_2(a,b) \,\}$$

with the norm $g \in H_1(a,b)$)

$$\|g\|_1 := \sqrt{\|g\|_0^2 + \|\partial g/\partial x\|_0^2}.$$

Consider the Hilbert space $L_2(-\pi,\pi)$ and $f(x) = \sin(x)$. Find the norm $\|f\|_1$.

Problem 47. Let $f \in H_1(a,b)$. Then for $a \le x < y \le b$ we have

$$f(y) = f(x) + \int_x^y f'(s)ds.$$

(i) Show that $f \in C[a,b]$.
(ii) Show that

$$|f(y) - f(x)| \le \|f\|_1 \sqrt{|y-x|}.$$

Problem 48. Consider the Hilbert space $L_2[0,\infty)$. The Laguerre polynomials are defined by

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}), \qquad n = 0,1,2,\ldots.$$

The first five Laguerre polynomials are given by

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= 1 - x \\ L_2(x) &= 2 - 4x + x^2 \\ L_3(x) &= 6 - 18x + 9x^2 - x^3 \\ L_4(x) &= 24 - 96x + 72x^2 - 16x^3 + x^4. \end{aligned}$$

Show that the function

$$\phi_n(x) = \frac{1}{n!} e^{-x/2} L_n(x)$$

form an orthonormal system in the Hilbert space $L_2[0,\infty)$.

Problem 49. Consider the Hilbert space $L_2[-\pi,\pi]$. A basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \, : \, k \in \mathbb{Z} \right\}.$$

Find the Fourier expansion of

$$f(x) = 1.$$

Problem 50. (i) Consider the functions

$$f(x) = \frac{1}{1+x^2}, \qquad g(x) = \frac{x}{1+x^2}.$$

Obviously $f, g \in L_2(\mathbb{R})$. Calculate the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx.$$

(ii) Let $\omega > 0$. Consider the functions

$$f(t) = \frac{\sin(\omega t)}{\omega t}, \qquad g(t) = \frac{1-\cos(\omega t)}{\omega t}.$$

Obvioulsy $f(0) = 1$, $g(0) = 0$ and $f, g \in L_2(\mathbb{R})$. Calculate the scalar product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)dt.$$

Problem 51. Consider the Hilbert space $L_2[0,1]$. Let $\mathcal{P}^n$ be the $n+1$-dimensional real linear space of all polynomial of maximal degree n in the variable x, i.e.

$$\mathcal{P}^n = \text{span}\{\, 1, x, x^2, \ldots, x^n \,\}.$$

The linear space $\mathcal{P}^n$ can be spanned by various systems of basis functions. An important basis is formed by the *Bernstein polynomials*

$$\{B_0^n(x), B_1^n(x), \ldots, B_n^n(x)\}$$

of degree n with

$$B_i^n(x) := x^i(1-x)^{n-i}, \qquad i = 0, 1, \ldots, n.$$

The Bernstein polynomials have a unique dual basis

$$\{\, D_0^n(x), D_1^n(x), \ldots, D_n^n(x) \,\}$$

which consists of the $n+1$ dual basis functions

$$D_i^n(x) = \sum_{j=0}^{n} c_{ij} B_j^n(x).$$

The dual basis functions satisfy

$$\langle D_i^n(x), B_j^n(x) \rangle = \delta_{ij}.$$

(i) Calculate the scalar product

$$\langle B_i^m(x), B_j^n(x) \rangle.$$

(ii) Find the coefficients c_{ij}.

Problem 52. Consider Fourier series and analytic (harmonic) functions on the disc

$$\mathbb{D} := \{ z \in \mathbb{C} : |z| \leq 1 \}.$$

A Fourier series can be viewed as the boundary values of a Laurent series

$$\sum_{n=-\infty}^{\infty} c_n z^n.$$

Suppose we are given a function f on $\mathbb{T}$. Find the harmonic extension u of f into $\mathbb{D}$. This means

$$\Delta u = 0 \quad \text{and} \quad u = f \quad \text{on} \quad \partial\mathbb{D} = \mathbb{T}$$

where $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Problem 53. Consider the compact abelian Lie group $U(1)$

$$U(1) = \{ e^{2\pi i\theta} : 0 \leq \theta < 1 \}.$$

The Hilbert space $L_2(U(1))$ is the space $L_2([0,1])$ consisting of all measureable funcrions $f(\theta)$ with period 1 such that

$$\int_0^1 |f(\theta)|^2 d\theta < \infty.$$

Now the set of functions

$$\{ e^{2\pi im\theta} : m \in \mathbb{Z} \}$$

form an orthonormal basis for the Hilbert space $L_2([0,1])$. Thus every $f \in L_2([0,1])$ can be expressed uniquely as

$$f(\theta) = \sum_{m=-\infty}^{+\infty} c_m e^{2\pi im\theta}, \qquad c_m = \int_0^1 f(\theta) e^{-2\pi im\theta} d\theta.$$

Calculate

$$\int_0^1 |f(\theta)|^2.$$

Problem 54. The Hilbert space $L_2(\mathbb{R})$ is the vector space of measurable functions defined almost everywhere on $\mathbb{R}$ such that $|f|^2$ is integrable. $H_1(\mathbb{R})$ is the vector space of functions with first derivatives in $L_2(\mathbb{R})$. Give two examples of such a function.

Problem 55. Consider the Hilbert space $L_2[-\pi, \pi]$. The set of functions

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{-inx} \right\}_{n \in \mathbb{Z}}$$

is an orthonormal basis for $L_2[-\pi, \pi]$. Let

$$K(x, t) = \frac{1}{\sqrt{2\pi}} e^{itx}.$$

For t fixed find the Fourier expansion of this function.

Problem 56. Consider the vector space $C([0,1])$ of continouos functions. We define the *triangle function*

$$\Lambda(x) := \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2 - 2x & 1/2 < x \leq 1 \end{cases}.$$

Let $\Lambda_0(x) := x$ and

$$\Lambda_n(x) := \Lambda(2^j x - k)$$

where $j = 0, 1, 2, \ldots$, $n = 2^j + k$ and $0 \leq k < 2^j$. The functions

$$\{ 1, \Lambda_0, \Lambda_1, \ldots \}$$

are the *Schauder basis* for the vector space $C([0,1])$. Let $f \in C([0,1])$. Then

$$f(x) = a + bx + \sum_{n=1}^{\infty} c_n \Lambda_n(x).$$

(i) Find the Schauder coefficients a, b, c_n.
(ii) Consider $g : [0,1] \to [0,1]$

$$g(x) = 4x(1-x).$$

Find the Schauder coefficients for this function.

Problem 57. Let s be a nonnegative integer. Let $x \in \mathbb{R}$ and h_n ($n = 0, 1, 2, \ldots$ be

$$h_n(x) = \frac{(-1)^n}{2^{n/2}\sqrt{n!}\sqrt[4]{\pi}} \exp(x^2/2) \frac{d^n e^{-x^2}}{dx^n}.$$

Thus h_n for an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$. Consider the sequence

$$f_s(x) = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta} h_n(x)$$

where $s = 0, 1, 2, \ldots$. Show that the sequence converges weakly but not strongly to 0.

Problem 58. Let $\mathbb{C}^{n\times N}$ be the vector space of all $n \times N$ complex matrices. Let $Z \in \mathbb{C}^{n\times N}$. Then $Z^* \equiv \bar{Z}^T$, where ^{T} denotes transpose. One defines a *Gaussian measure* μ on $\mathbb{C}^{n\times N}$ by

$$d\mu(Z) := \frac{1}{\pi^{nN}} \exp(-\mathrm{tr}(ZZ^*))dZ$$

where dZ denotes the Lebesgue measure on $\mathbb{C}^{n\times N}$. The *Fock space* $\mathcal{F}(\mathbb{C}^{n\times N})$ consists of all entire functions on $\mathbb{C}^{n\times N}$ which are square integrable with respect to the Gaussian measure $d\mu(Z)$. With the scalar product

$$\langle f|g\rangle := \int_{\mathbb{C}^{n\times N}} f(Z)\overline{g(Z)}d\mu(Z), \qquad f, g \in \mathbb{F}(\mathbb{C}^{n\times N})$$

one has a Hilbert space. Show that this Hilbert space has a *reproducing kernel* K. This means a continuous function $K(Z, Z') : \mathbb{C}^{n\times N} \times \mathbb{C}^{n\times N} \to \mathbb{C}$ such that

$$f(Z) = \int_{\mathbb{C}^{n\times N}} K(Z, Z')f(Z')d\mu(Z')$$

for all $Z \in \mathbb{C}^{n\times N}$ and $f \in \mathcal{F}(\mathbb{C}^{n\times N})$.

Problem 59. Consider the Hilbert space $L_2[0,\infty)$ and the function $f \in L_2[0,\infty)$

$$f(x) = \exp(-u^{1/4})\sin(u^{1/4}).$$

Find

$$\int_0^\infty f(x)x^n dx, \qquad n = 0, 1, 2, \ldots.$$

Problem 60. Consider the Hilbert space $L_2[0, 2\pi)$ with the scalar product

$$\langle f_1, f_2\rangle = \frac{1}{2\pi}\int_0^{2\pi} f_1(e^{i\theta})\overline{f_2(e^{i\theta})}d\theta.$$

(i) Let $f_1(z) = z$ and $f_2(z) = z^2$. Find $\langle f_1, f_2\rangle$.
(ii) Let $f_1(z) = z^2$ and $f_2(z) = \sin(z)$. Find $\langle f_1, f_2\rangle$.

Problem 61. Consider the Hilbert space $L_2(\mathbb{R}^2)$ with the basis

$$\psi_{mn}(x_1, x_2) = NH_m(x_1)H_n(x_2)e^{-(x_1^2+x_2^2)/2}$$

where $m, n = 0, 1, \ldots$ and N is the normalization factor. Consider the two-dimensional potential

$$V(x_1, x_2) = \frac{a}{4}(x_1^4 + x_2^4) + cx_1x_2.$$

(i) Find all linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$V(T\mathbf{x}) = V(\mathbf{x}).$$

(ii) Show that these 2×2 matrices form a group. Is the group abelian.
(iii) Find the conjugacy classes and the irreducible representations.
(iv) Consider the Hilbert space $L_2(\mathbb{R}^2)$ with the orthogonal basis

$$\psi_{mn}(x_1, x_2) = H_m(x_1)e^{-x_1^2/2}H_n(x_2)e^{-x_2^2/2}$$

where $m, n = 0, 1, 2, \ldots$. Find the invariant subspaces from the projection operators of the irreducible representations.

Problem 62. Consider the Hilbert space $L_2[-\pi, \pi]$. A basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \ : \ k \in \mathbb{Z} \right\}.$$

Find the Fourier expansion of

$$f(x) = 1.$$

Problem 63. Consider the Hilbert space $L_2[0, 1]$. Let $\mathcal{P}^n$ be the $n+1$-dimensional real linear space of all polynomial of maximal degree n in the variable x, i.e.

$$\mathcal{P}^n = \text{span}\{\, 1, x, x^2, \ldots, x^n \,\}.$$

The linear space $\mathcal{P}^n$ can be spanned by various systems of basis functions. An important basis is formed by the *Bernstein polynomials* $\{B_0^n(x), B_1^n(x), \ldots, B_n^n(x)\}$ of degree n with

$$B_i^n(x) := x^i(1-x)^{n-i}, \qquad i = 0, 1, \ldots, n.$$

The Bernstein polynomials have a unique dual basis $\{D_0^(x), D_1^n(x), \ldots, D_n^n(x)\,\}$ which consists of the $n+1$ dual basis functions

$$D_i^n(x) = \sum_{j=0}^{n} c_{ij}B_j^n(x).$$

The dual basis functions satisfy

$$\langle D_i^n(x), B_j^n(x)\rangle = \delta_{ij}.$$

(i) Find the scalar product

$$\langle B_i^m(x), B_j^n(x)\rangle.$$

(ii) Find the coefficients c_{ij}.

Problem 64. Let V be a metric vector space. A *reproducing kernel Hilbert space* on V is a Hilbert space $\mathcal{H}$ of functions on V such that for each $x \in V$, the point evaluation functional

$$\delta_x(f) := f(x), \qquad f \in \mathcal{H}$$

is continouos. A reproducing kernel Hilbert space $\mathcal{H}$ possesses a unique reproducing kernel K which is a function on $V \times V$ characterized by the properties that for all $f \in \mathcal{H}$ and $x \in V$, $K(x, \cdot) \in \mathcal{H}$ and

$$f(x) = \langle f, K(x, \cdot)\rangle_{\mathcal{H}}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product on $\mathcal{H}$. The reproducing kernel K uniquely determines the reproducing kernel Hilbert space $\mathcal{H}$. The reproducing kernel Hilbert space of a reproducing kernel K is denoted by $\mathcal{H}_K$. The *Paley-Wiener space* is defined by

$$S := \{\, f \in C(\mathbb{R}^d) \cap L_2(\mathbb{R}^d) \,:\, \mathrm{supp}\hat{f} \subseteq [-\pi, \pi]^d \,\}$$

is a reproducing kernel Hilbert space. The Fourier transform of $f \in L_1(\mathbb{R}^d)$ is given by

$$\hat{f}(\mathbf{k}) := \frac{1}{(\sqrt{2\pi})^{2d}} \int_{\mathbb{R}^{2d}} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{k}} d\mathbf{x}, \qquad \mathbf{k} \in \mathbb{R}^d$$

where $\mathbf{x}\cdot\mathbf{k} = x_1k_1 + \cdots + x_dk_d$ is the inner product in $\mathbb{R}^d$. The norm on the vector space S inherits from that in $L_2(\mathbb{R}^d)$. Show that the reproducing kernel for the Paley-Wiener space S is the *sinc function*

$$\mathrm{sinc}(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^{d} \frac{\sin(\pi(x_j - y_j))}{\pi(x_j - y_j)}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Problem 65. Can one construct an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$ starting from $(\sigma > 0)$

$$e^{-|x|/\sigma}, \quad xe^{-|x|/\sigma}, \quad x^2e^{-|x|/\sigma}, \quad x^3e^{-|x|/\sigma}, \ldots .$$

Problem 66. Consider the Hilbert space $L_2[-1,1]$. Normalize the function $f(x)=x$ in this Hilbert space.

Problem 67. Show that (Mehler's formula)

$$\exp(-(u^2+v^2-2uvz)/(1-z^2)) = (1-z^2)^{1/2}\exp(-(u^2+v^2))\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(u)H_n(v)$$

where H_n are the Hermite polynomials.

Problem 68. Consider the Hilbert space $L_2(\mathbb{R})$. Let $j,k=1,2,\ldots$. Consider the functions

$$f_j(x)=x^j e^{-|x|}, \qquad f_k(x)=x^k e^{-|x|}.$$

Find the scalar product

$$(f_j(x),f_k(x)) := \int_{-\infty}^{\infty} f_j(x)\bar{f}_k(x)dx = \int_{-\infty}^{\infty} f_j(x)f_k(x)dx.$$

Discuss.

Problem 69. Consider the Hilbert space $L_2(\mathbb{R})$ and the one-dimensional Schrödinger equation (eigenvalue equation)

$$\left(-\frac{d^2}{dx^2}+V(x)\right)u(x)=Eu(x)$$

where the potential V is given by

$$V(x)=x^2+\frac{ax^2}{1+bx^2}$$

where $b>0$. Insert the ansatz

$$u(x)=e^{-x^2/2}v(x)$$

and find the differential equation for v. Discuss. Make a polynomial ansatz for v.

Problem 70. Consider the Hilbert space $L_2(\mathbb{R})$. Let $g>0$. Consider the one-dimensional Schrödinger equation (eigenvalue equation)

$$\left(-\frac{d^2}{dx^2}+x^2+\frac{\lambda x^2}{1+gx^2}\right)u(x)=Eu(x)\,.$$

Find a solution of the second order differential equation by making the ansatz

$$u(x) = A(1 + gx^2) \exp(-x^2/2).$$

Problem 71. (i) Consider the Hilbert space $L_2[-1/2, 1/2]$. Show that the following sets

$$\mathcal{B}_1 := \{ \phi_k(x) = \exp(2\pi i k x),\ k \in \mathbb{Z} \}$$
$$\mathcal{B}_2 := \{ \psi_k(x) = \sqrt{2} \sin(2\pi k x),\ k \in \mathbb{N} \}$$

each form an orthonormal basis in this Hilbert space.
(ii) Expand the step function

$$f(x) = \begin{cases} -1 \text{ for } x \in [-1/2, 0] \\ \ \ 1 \text{ for } x \in [0, 1/2] \end{cases}$$

with respect to the basis $\mathcal{B}_1$ and with respect to the basis $\mathcal{B}_2$. Show that the two expansions are equivalent. Recall that

$$2 \sin(x) \sin(y) \equiv \cos(x - y) - \cos(x + y).$$

Problem 72. Consider the problem of a free particle in a one-dimensional box $[-a, a]$. The underlying Hilbert space is $L_2[-a, a]$. An orthonormal basis in $L_2[-a, a]$ is given by

$$\mathcal{B} = \{ u_k^{(+)}(q),\ u_k^{(-)}(q) \ :\ k \in \mathbb{N} \}$$

where

$$u_k^{(+)} = \frac{1}{\sqrt{a}} \cos\left(\frac{(k - 1/2)\pi q}{a} \right), \qquad u_k^{(-)} = \frac{1}{\sqrt{a}} \sin\left(\frac{k\pi q}{a} \right).$$

The formal solution of the initial value problem of the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

is given by

$$\psi(t) = \exp(-i\hat{H}t/\hbar)\psi(0).$$

Let

$$\psi(q, 0) = \frac{1}{\sqrt{a}} \sin(\pi q/a), \qquad \phi(q) = \frac{1}{\sqrt{a}} \sin(\pi q/a).$$

Find $\exp(-i\hat{H}t/\hbar)$ and $P = |\langle \phi, \psi(t) \rangle|^2$.

Problem 73. Let n be a positive integer. Consider the Hilbert space $L_2[0,n]$ and the function
$$f(x) = e^{-x}.$$
Find $a, b \in \mathbb{R}$ such that
$$\|f(x) - (ax^2 + bx)\|$$
is a minimum. The norm in the Hilbert space $L_2[0,n]$ is induced by the scalar product.

Problem 74. Give a function $f \in L_2([0,\infty))$ such that
$$\int_0^\infty f(x)dx = 1, \qquad \int_0^\infty xf(x)dx = 1.$$

Problem 75. Consider the Hilbert space $L_2[0,2\pi]$. The linear operator $Lf(x) := df(x)/dx$ acts on a dense subset of $L_2[0,2\pi]$. Show that this linear operator is not bounded.

Problem 76. Consider the Hilbert space $L_2(\mathbb{R}^3, d\mathbf{x})$ and let
$$S^2 = \{\, (x_1, x_2, x_3) \,:\, x_1^2 + x_2^2 + x_3^2 = 1 \,\}.$$
In spherical coordinates this Hilbert space has the decomposition
$$L_2(\mathbb{R}^3, d\mathbf{x}) = L_2(\mathbb{R}^+, r^2 dr) \otimes L_2(S^2, \sin(\theta)d\theta d\phi).$$
Let $\hat{I}$ be the identity operator in the Hilbert space $L_2(S^2, \sin(\theta)d\theta d\phi)$. Then the radial momentum operator
$$\hat{P}_r := -i\hbar \frac{1}{r}\left(\frac{\partial}{\partial r}\right) r$$
is identified with the closure of the operator $\hat{P}_r \otimes \hat{I}$ defined on $D(\hat{P}_r) \otimes L_2(S^2, \sin(\theta)d\theta d\phi)$ where
$$D(\hat{P}_r) = \left\{ f \in L_2(\mathbb{R}^+, r^2 dr) \,:\, f \in AC(\mathbb{R}^+),\ \frac{1}{r}\frac{d}{dr} rf(r) \in L_2(\mathbb{R}^+, r^2 dr)\ \lim_{r\to 0} r|f(r)| = 0 \right\}$$
and for each $f \in D(\hat{P}_r)$
$$\hat{P}_r f(r) = -i\hbar \frac{1}{r}\frac{d}{dr}(rf(r))$$
where $\hat{P}_r$ is maximal symmetric in $L_2(\mathbb{R}^+, r^2 dr)$. Show that $\hat{P}_r$ is not self-adjoint.

Problem 77. Consider the Hilbert space $L_2[-\pi, \pi]$ and the functions

$$f(x) = |\sin(x)| \qquad g(x) = |\cos(x)|.$$

Find the distance

$$\|f(x) - g(x)\|$$

in this Hilbert space.

Problem 78. Consider the Hilbert space $L_2(\mathbb{R})$. Show that the spectrum of the *position operator* $\hat{x}$ is the real line denoted by $\mathbb{R}$.

Problem 79. Consider the Hilbert space $L_2(\mathbb{R})$. Is

$$\phi_n(x) = \frac{1}{\sqrt{\pi(1+x^2)}} e^{2in\arctan(x)}, \quad n \in \mathbb{Z}$$

an orthonormal basis in $L_2(\mathbb{R})$?

Problem 80. Consider the Hilbert space $L_2[0, \infty)$. Show that the functions

$$\phi_n(x) = e^{-x/2} L_n(x), \quad n = 0, 1, 2, \ldots$$

form an orthonormal basis in $L_2[0, \infty)$, where L_n are the Laguerre polynomials defined by

$$L_n(x) = \frac{x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k.$$

Problem 81. Consider the Hilbert space $L_2(\mathbb{R})$. Show that the functions

$$\phi_n(x) = \frac{1}{2^{n/2}\sqrt{n!}(\pi)^{1/4}} H_n(x) e^{-x^2/2}, \quad n = 0, 1, 2, \ldots$$

form an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$, where H_n are the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}, \quad n = 0, 1, 2, \ldots$$

Problem 82. Let $b > a$ and $n = 1, 2, \ldots$. Consider the Hilbert space $L_2[a, b]$. Find

$$\int_a^b \sin^2\left(\frac{n\pi(x-a)}{b-a}\right).$$

The functions

$$\phi_n(x) = \sqrt{\frac{2}{b-a}} \sin\left(\frac{n\pi(x-a)}{b-a}\right)$$

form an orthonormal basis in the Hilbert space $L_2[a,b]$.

Problem 83. Consider the Hilbert space $L_2[0,\infty)$ and

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y)\cos(yx)dy, \qquad g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y)\cos(yx)dy.$$

Let $g : \mathbb{R}^+ \to \mathbb{R}$

$$g(y) = e^{-y}$$

Find $f(x)$.

Problem 84. Consider the Hilbert space $L_2[-1,1]$. An orthonormal basis in this Hilbert space is given by

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \ : \ |x| \leq \pi, \ k \in \mathbb{Z} \right\}.$$

Consider the function $f(x) = e^{iax}$ in this Hilbert space, where the constant a is real but not an integer. Apply *Parseval's relation*

$$\|f\|^2 = \sum_{k \in \mathbb{Z}} |\langle f, \phi_k \rangle|^2, \quad \phi_k(x) = \frac{1}{\sqrt{2}} e^{ikx}$$

to show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(a-k)^2} = \frac{\pi^2}{\sin^2(ax)}.$$

Problem 85. Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{x}{\sinh(x)}$$

with $f(0) = 1$. Show that

$$\frac{x}{\sinh(x)} = 1 + 2\sum_{j=1}^{\infty} (-1)^j \frac{x^2}{x^2 + (j\pi)^2}.$$

Chapter 4

Hilbert Space $\ell_2(\mathbb{N})$

Problem 1. Consider the Hilbert space $\ell_2(\mathbb{N})$. Let $\mathbf{x} = (x_1, x_2, \ldots)^T$ be an element of $\ell_2(\mathbb{N})$. We define the linear operator A in $\ell_2(\mathbb{N})$ as

$$A\mathbf{x} = (x_2, x_3, \ldots)^T$$

i.e. x_1 is omitted and the $n+1$st coordinate replaces the nth for $n = 1, 2, \ldots$. Then for the domain we have $\mathcal{D}(A) = \ell_2(\mathbb{N})$. Find $A^*\mathbf{y}$ and the domain of A^*, where $\mathbf{y} = (y_1, y_, \ldots)$. Is A unitary?

Problem 2. Consider the Hilbert space $\ell_2(\mathbb{N})$ and $\mathbf{x} = (x_1, x_2, \ldots)^T$. The linear bounded operator A is defined by

$$A(x_1, x_2, x_3, \ldots, x_{2n}, x_{2n+1}, \ldots)^T = (x_2, x_4, x_1, x_6, x_3, x_8, x_5, \ldots, x_{2n+2}, x_{2n-1}, \ldots)^T.$$

Show that the operator A is unitary. Show that the point spectrum of A is empty and the continuous spectrum is the entire unit circle in the λ-plane.

Problem 3. Consider the Hilbert space $\ell_2(\mathbb{N})$. Suppose that S and T are the right and left shift linear operators on this sequence space, defined by

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots), \qquad T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Show that $T = S^*$.

Problem 4. Find the spectrum of the infinite dimensional matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \dots \\ & & \ddots & & \ddots & \end{pmatrix}.$$

In other words

$$a_{ij} = \begin{cases} 1 \text{ if } i = j+1 \\ 1 \text{ if } i = j-1 \\ 0 \ \text{ otherwise} \end{cases}$$

Problem 5. Let P_j $(j = 0, 1, 2, \ldots)$ be the *Legendre polynomials*

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots .$$

Calculate the infinite dimensional matrix $A = (A_{jk})$

$$A_{jk} = \int_{-1}^{+1} P_j(x) \frac{dP_k(x)}{dx} dx$$

where $j, k = 0, 1, \ldots$. Consider the matrix A as a linear operator in the Hilbert space $\ell_2(\mathbb{N}_0)$. Is $\|A\| < \infty$?

Problem 6. Let $\mathbb{Z}$ be the set of integers. Consider the Hilbert space $\ell_2(\mathbb{Z}^2)$. Let $(m_1, m_2) \in \mathbb{Z}^2$. Let $f(m_1, m_2)$ be an element of $\ell_2(\mathbb{Z}^2)$. Consider the unitary operators

$$Uf(m_1, m_2) := e^{-2\pi i \alpha m_2} f(m_1{+}1, m_2), \quad Vf(m_1, m_2) := e^{-2\pi i \beta m_1} f(m_1, m_2{+}1).$$

They are the so-called magnetic translation operators with phase α and β, respectively. Find the spectrum of U and V. Find the commutator $[U, V]$. The so-called *Harper operator* which is self-adjoint is defined by

$$\hat{H} := U + U^* + V + V^*.$$

Find the spectrum of $\hat{H}$. Consider the case α, β irrational and α, β rational.

Problem 7. The spectrum $\sigma(\hat{H})$ of a linear operator $\hat{H}$ is defined as the set of all λ for which the *resolvent*

$$R(\lambda) = (\lambda I - \hat{H})^{-1}$$

does not exist. If the linear operator $\hat{H}$ is self-adjoint, the spectrum is a subset of the real axis. The Lebesgue decomposition theorem states that

$$\sigma = \sigma_{pp} \cup \sigma_{ac} \cup \sigma_{sing}$$

where σ_{pp} is the countable union of points (the pure point spectrum), σ_{ac} is absolutely continuous with respect to Lebesgue measure and σ_{sing} is singular with respect to Lebesgue measure, i.e. it is supported on a set of measure zero. Consider the Hilbert space $\ell_2(\mathbb{Z})$ and the linear operator

$$\hat{H} = \cdots \otimes I_2 \otimes I_2 \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_3 \otimes I_2 \otimes I_2 \otimes \cdots$$

where σ_1 is at position 0. Find the spectrum of this linear operator.

Problem 8. Let M be any $n \times n$ matrix. Let $\mathbf{x} = (x_1, x_2, \ldots)^T$. The linear operator A is defined by

$$A\mathbf{x} = (w_1, w_2, \ldots)^T$$

where

$$w_j = \sum_{k=1}^{n} M_{jk} x_k, \qquad j = 1, 2, \ldots, n$$
$$w_j = x_j, \qquad j > n$$

and $\mathcal{D}(A) = \ell_2(\mathbb{N})$. Show that A is self-adjoint if the $n \times n$ matrix M is hermitian. Show that A is unitary if M is unitary.

Problem 9. Let Ω be the unit disk. A Hilbert space of analytic functions can be defined by

$$\mathcal{H} := \left\{ f(z) \;\text{ analytic }\; |z| < 1 \; : \; \sup_{a<1} \int_{|z|=a} |f(z)|^2 ds < \infty \right\}$$

and the scalar product

$$\langle f, g \rangle := \lim_{a \to 1} \int_{|z|=a} \overline{f(z)} g(z) ds.$$

Let c_n $(n = 0, 1, 2, \ldots)$ be the coefficients of the power-series expansion of the analytic function f. Find the norm of f.

Problem 10. Let $|n\rangle$ be the number states $(n = 0, 1, \ldots)$. Let $k = 0, 1, \ldots$. Define the operators

$$T_k := \sum_{n=0}^{\infty} |n\rangle\langle 2n + k|.$$

(i) Show that $T_k T_{k'}^\dagger = \delta_{kk'} I$.
(ii) Show that $T_k^\dagger T_k = P_k$ is a projection operator.
(iii) Show that $\sum_{k=0}^{\infty} P_k = I$.
(iv) Is the operator

$$\sum_{k=0}^{\infty} T_k \otimes T_k^\dagger$$

unitary?

Chapter 5

Fourier Transform

Problem 1. Consider the Hilbert space $L_2(\mathbb{R})$. Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{if} \quad -1 \leq x \leq 0 \\ e^{-x} & \text{if} \quad x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2. (i) Find the Fourier transform for

$$f_\alpha(x) = \frac{\alpha}{2}\exp(-\alpha|x|), \qquad \alpha > 0.$$

Discuss α large and α small.
(ii) Calculate

$$\int_{-\infty}^{\infty} f_\alpha(x)dx.$$

Problem 3. Find the Fourier transform of the *hat function*

$$f(t) = \begin{cases} 1-|t| & \text{for} \quad -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 4. Let $f \in L_2(\mathbb{R})$ and $f \in L_1(\mathbb{R}$. Assume that $f(x) = f(-x)$. Can we conclude that $\hat{f}(k) = \hat{f}(-k)$?

Problem 5. Consider the Hilbert space $L_2(\mathbb{R})$. Find the Fourier transform of

$$f(x) = e^{-a|x|}, \qquad a > 0.$$

Problem 6. Consider the Hilbert space $L_2(\mathbb{R})$. Let $a > 0$. Define

$$f_a(x) = \begin{cases} \frac{1}{2a} & |x| < a \\ 0 & |x| > a \end{cases}$$

Calculate

$$\int_{\mathbb{R}} f_a(x) dx$$

and the Fourier transform of f_a. Discuss the result in dependence of a.

Problem 7. Consider the Hilbert space $L_2(\mathbb{R})$. Let

$$\hat{\psi}(\omega) = \begin{cases} 1 & \text{if} \quad 1/2 \le |\omega| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{\phi}(\omega) = e^{-\alpha|\omega|}, \qquad \alpha > 0.$$

(i) Calculate the inverse Fourier transform of $\hat{\psi}(\omega)$ and $\hat{\phi}(\omega)$, i.e.

$$\psi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \hat{\psi}(\omega)$$
$$\phi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \hat{\phi}(\omega).$$

(ii) Calculate the scalar product $\langle \psi(t) | \phi(t) \rangle$ by utilizing the identity

$$2\pi \langle \psi(t) | \phi(t) \rangle = \langle \hat{\psi}(\omega) | \hat{\phi}(\omega) \rangle.$$

Problem 8. Consider the Hilbert space $L_2(\mathbb{R})$ and the function $f \in L_2(\mathbb{R})$

$$f(x) = \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| \ge 1 \end{cases}$$

Calculate $f * f$ and verify the *convolution theorem*

$$\widehat{f * f} = \hat{f} \hat{f}.$$

Problem 9. Let

$$\hat{f}(\omega) = \begin{cases} (1 - \omega^2) & \text{for } |\omega| \le 1 \\ 0 & \text{for } |\omega| > 1 \end{cases}$$

Find $f(t)$.

Problem 10. Let $a > 0$. Find the Fourier transform of the function $f_a : \mathbb{R} \to \mathbb{R}$

$$f_a(x) = \begin{cases} x/a^2 + 1/a & \text{for} \quad -a \leq x \leq 0 \\ -x/a^2 + 1/a & \text{for} \quad 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

Problem 11. Let $a > 0$. Find the Fourier transform of

$$f_a(t) = \frac{1}{\sqrt{a}} e^{-a|t|}.$$

Discuss the cases a large and a small. Is $f_a \in L_2(\mathbb{R})$.

Problem 12. Show that the Fourier transform of the rectangular window of size N

$$w_n = \begin{cases} 1 & \text{for} \quad 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

is

$$W(e^{i\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-i\omega(N-1)/2}.$$

Problem 13. Consider the Hilbert space $L_2(\mathbb{R})$. Let $T > 0$. Consider the function in $L_2(\mathbb{R})$

$$f(t) = \begin{cases} A\cos(\Omega t) & \text{for} \quad -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

where A is a positive constant. Calculate the Fourier transform.

Problem 14. Let $\sigma > 0$. Show that the Fourier transform of the *Gaussian function*

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

is again a Gaussian function

$$\hat{g}_\sigma(k) = e^{-\sigma^2 k^2/2}.$$

We have $\int_{-\infty}^{\infty} g_\sigma(x)dx = 1$. Is

$$\int_{-\infty}^{\infty} \hat{g}_{\sigma_k}(k)dk = 1\,?$$

Problem 15. Show that the analytic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \mathrm{sech}(\pi x)$$

is an element of $L_2(\mathbb{R})$ and $L_1(\mathbb{R})$. Find the Fourier transform of the function.

Problem 16. Let $a > 0$. Find the Fourier transform of

$$\sqrt{2\pi} f_a(x) + \frac{\sin(ax)}{ax}$$

where f_a is the function with 1 for $|x| \le a$ and 0 otherwise.

Problem 17. Consider the Hermite-Gauss functions

$$f_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi}x) \exp(-\pi x^2), \qquad n = 0, 1, 2, \dots$$

where H_n is the nth Hermite polynomial. They for an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$. Do the Fourier transform of the functions form an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$.

Chapter 6

Wavelets

Problem 1. Consider the Hilbert space $L_2[0,1]$ and the function $f(x) = x^2$ in this Hilbert space. Project the function f onto the subspace of $L_2[0,1]$ spanned by the functions $\phi(x)$, $\psi(x)$, $\psi(2x)$, $\psi(2x-1)$, where

$$\phi(x) := \begin{cases} 1 & \text{for} \quad 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(x) := \begin{cases} 1 & \text{for} \quad 0 \le x < 1/2 \\ -1 & \text{for} \quad 1/2 \le x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

This is related to the Haar wavelet expansion of f. The function ϕ is called the father wavelet and ψ is called the mother wavelet.

Problem 2. Consider the function $H \in L_2(\mathbb{R})$

$$H(x) = \begin{cases} 1 & 0 \le x \le 1/2 \\ -1 & 1/2 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$H_{mn}(x) := 2^{-m/2} H(2^{-m}x - n)$$

where $m, n \in \mathbb{Z}$. Draw a picture of H_{11}, H_{21}, H_{12}, H_{22}. Show that

$$\langle H_{mn}(x), H_{kl}(x) \rangle = \delta_{mk}\delta_{nl}, \qquad k, l \in \mathbb{Z}$$

where $\langle\, . \,\rangle$ denotes the scalar product in $L_2(\mathbb{R})$ Expand the function

$$f(x) = \exp(-|x|)$$

with respect to H_{mn}. The functions H_{mn} form an orthonormal basis in $L_2(\mathbb{R})$.

Problem 3. Consider the Hilbert space $L_2[0,1]$ and the *Haar scaling function* (father wavelet)

$$\phi(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let n be a positive integer. We define

$$g_k(x) := \sqrt{n}\phi(nx-k), \qquad k = 0, 1, \dots, n-1.$$

(i) Show that the set of functions $\{ g_0, g_1, \dots, g_{n-1} \}$ is an orthonormal set in the Hilbert space $L_2[0,1]$.
(ii) Let f be a continuous function on the unit interval $[0,1]$. Thus $f \in L_2[0,1]$. Form the projection f_n on the subspace S_n of the Hilbert space $L_2[0,1]$ spanned by $\{ g_0, g_1, \dots g_{n-1} \}$, i.e.

$$f_n = \sum_{k=0}^{n-1} \langle f, g_k \rangle g_k \, .$$

Show that $f_n(x) \to f(x)$ pointwise in x as $n \to \infty$.

Problem 4. The *continuous wavelet transform*

$$Wf(a,b) = \frac{1}{a} \int_{-\infty}^{+\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt, \qquad (a, b \in \mathbb{R},\, a > 0)$$

decomposes the function $f \in L_2(\mathbb{R})$ hierarchically in terms of elementary components $\psi((t-b)/a)$. They are obtained from a single *analyzing wavelet* ψ applying *dilations* and *translations*. Here $\bar{\psi}$ denotes the complex conjugate of ψ and a is the scale and b the shift parameter. The function ψ has to be chosen so that it is well localized both in physical and Fourier space. The signal $f(t)$ can be uniquely recovered by the *inverse wavelet transform*

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} Wf(a,b) \psi\left(\frac{t-b}{a}\right) \frac{da}{a} db$$

if $\psi(t)$ (respectively its Fourier transform $\hat{\psi}(\omega)$ satisfies the *admissibility condition*

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty.$$

Consider the analytic function

$$\psi(t) = te^{-t^2/2}.$$

Does ψ satisfies the admissibility condition?

Problem 5. Consider the function $H \in L_2(\mathbb{R})$

$$H(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ -1 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$H_{mn}(x) := 2^{-m/2} H(2^{-m}x - n)$$

where $m, n \in \mathbb{Z}$. Draw a picture of H_{11}, H_{21}, H_{12}, H_{22}. Show that

$$\langle H_{mn}(x), H_{kl}(x) \rangle = \delta_{mk}\delta_{nl}, \qquad k, l \in \mathbb{Z}$$

where $\langle \, . \, \rangle$ denotes the scalar product in $L_2(\mathbb{R})$. Expand the function

$$f(x) = \exp(-|x|)$$

with respect to H_{mn}. The functions H_{mn} form an orthonormal basis in $L_2(\mathbb{R})$.

Problem 6. Consider the Hilbert space $L_2(\mathbb{R})$. Let $\phi \in L_2(\mathbb{R})$ and assume that ϕ satisfies

$$\int_{\mathbb{R}} \phi(t)\overline{\phi(t-k)} dt = \delta_{0,k}$$

i.e. the integral equals 1 for $k = 0$ and vanishes for $k = 1, 2, \ldots$. Show that for any fixed integer j the functions

$$\phi_{jk}(t) := 2^{j/2}\phi(2^j t - k), \qquad k = 0, \pm 1, \pm 2, \ldots$$

form an orthonormal set.

Problem 7. Consider the function $\phi : \mathbb{R} \to \mathbb{R}$

$$\psi(x) := \begin{cases} 1 & \text{for} \quad x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find $\psi(x) := \phi(2x) - \phi(2x-1)$. Calculate

$$\int_{-\infty}^{\infty} \psi(x) dx.$$

Problem 8. Consider the Littlewood-Paley orthonormal basis of wavelets. The mother wavelet of this set is

$$L(x) := \frac{1}{\pi x}(\sin(2\pi x) - \sin(\pi x)).$$

Then

$$L_{mn}(x) = \frac{1}{2^{m/2}} L(2^{-m} wx - n), \qquad m, n \in \mathbb{Z}$$

generates an orthonormal basis in the Hilbert space $L_2(\mathbb{R})$. Apply the rule of L'Hospital to find $L(0)$.

Problem 9. (i) Consider the Hilbert space $L_2(\mathbb{R})$ and $\phi \in L_2(\mathbb{R})$. The basic scaling function (father wavelet) satisfies a scaling relation of the form

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x - k).$$

Show that the *Hilbert transform* of ϕ

$$H(\phi)(y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\phi(x)}{x - y} dx$$

is a solution of the same scaling relation. Note that the scaling function ϕ may have compact support, the Hilbert transform has support on the real line and decays as y^{-1}.
(ii) Show that the Hilbert transform of the related mother wavelet ψ is also noncompact and decays like y^{-p-1} where

$$\int_{\mathbb{R}} x^m \psi(x) dx = 0$$

for $m = 0, 1, \ldots, p - 1$.

Chapter 7

Linear Operators

Problem 1. Show that an isometric operator need not be a unitary operator.

Problem 2. Consider the Hilbert space $L_2[0,1]$. Show that the linear operator $T : L_2[0,1] \to L_2[0,1]$ defined by

$$Tf(x) = xf(x)$$

is a bounded self-adjoint linear operator without eigenvalues.

Problem 3. Show that if two bounded self-adjoint linear operators S and T on a Hilbert space $\mathcal{H}$ are positive semi-definite and commute ($ST = TS$), then their product ST is positive semi-definite. We have to show that $\langle STf, f \rangle \geq 0$ for all $f \in \mathcal{H}$.

Problem 4. Let $a > 0$. Consider the Hilbert space $L_2[-a,a]$. Consider the Hamilton operator

$$\hat{H} = \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

where

$$V(x) = \begin{cases} 0 & \text{for} \quad |x| \leq a \\ \infty & \text{otherwise} \end{cases}$$

Solve the Schrödinger equation, where the initial function $\psi(t=0)=\phi(x)$ is given by

$$\phi(x)=\begin{cases} x/a^2+1/a & \text{for } -a\le x\le 0 \\ -x/a^2+1/a & \text{for } 0\le x\le a \end{cases}$$

Normalize ϕ. Calculate the probability to find the particle in the state

$$\chi(x)=\frac{1}{\sqrt{a}}\sin\left(\frac{\pi x}{a}\right)$$

after time t. A basis in the Hilbert space $L_2[-a,a]$ is given by

$$\left\{\frac{1}{\sqrt{a}}\sin\left(\frac{n\pi x}{a}\right),\ \frac{1}{\sqrt{a}}\cos\left(\frac{(n-1/2)\pi x}{a}\right)\quad n=1,2,\ldots\right\}.$$

Problem 5. Show that in one dimensional problems the energy spectrum of the bound state is always non-degenerate. Hint. Suppose that the oppsite is true. Let u_1 and u_2 be two linearly independent eigenfunctions with the same energy eigenvalues E, i.e.

$$\frac{d^2u_1}{dx^2}+\frac{2m}{\hbar^2}(E-V)u_1=0,\qquad \frac{d^2u_2}{dx^2}+\frac{2m}{\hbar^2}(E-V)u_2=0.$$

Problem 6. A particle is enclosed in a rectangular box with imprenetrable walls, inside which it can move freely. The Hilbert space is $L_2([0,a]\times[0,b]\times[0,c])$. Find the eigenfunctions and eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions.

Problem 7. Conside the Hilbert space $L_2[0,1]$ and the linear operator $T: L_2[0,1]\to L_2[0,1]$ defined by

$$(Tf)(x):=xf(x).$$

Show that T is self-adjoint and positive definite. Find its positive square root.

Problem 8. Consider the Hilbert space $\ell_2(\mathbb{N})$ and the linear operator T defined by

$$T:(x_1,x_2,x_3,\ldots)\mapsto(0,0,x_3,x_4,\ldots).$$

Is T bounded? Is T self-adjoint? If so is T positive?

Problem 9. In classical mechanics we have

$$\mathbf{L}=\mathbf{r}\times\mathbf{p},\qquad \mathbf{T}=\mathbf{r}\times\mathbf{F}$$

where $\mathbf{T}$ is the torque, $\mathbf{F} = -\nabla V$ (V potential depending only on $\mathbf{r}$) and

$$\frac{d\mathbf{L}}{dt} = \mathbf{T}\,.$$

In quantum mechanics with $\mathbf{p} \to -i\hbar\nabla$, $\mathbf{r} \to \mathbf{r}$ and wave function ψ we have

$$\mathbf{L} = -i\hbar \int_{\mathbb{R}^3} d^3\mathbf{x}\, \psi^*(\mathbf{r} \times \nabla)\psi$$

and

$$\mathbf{T} = -\int_{\mathbb{R}^3} d^3\mathbf{x}\, \psi^*(\mathbf{r} \times \nabla V)\psi$$

since $\mathbf{F} = -\nabla V$. ψ and ψ^* obey the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*\,.$$

Show that

$$\frac{d\mathbf{L}}{dt} = \mathbf{T}.$$

Problem 10. Let $\hat{H}$ be a bounded self-adjoint Hamilton operator with normalized eigenfunctions ϕ_j ($j \in I$) which form an orthonormal basis in the underlying Hilbert space. We can write

$$\psi(t) = \sum_{j\in I} c_j e^{-iE_j t/\hbar}\phi_j$$

where E_j are the eigenvalues of $\hat{H}$. Find $P(t) = \langle\psi(t=0)|\psi(t)\rangle$.

Problem 11. Consider the Hamilton operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + D(1 - e^{-\alpha x})^2 + eEx\cos(\omega t)$$

where $\alpha > 0$. Find the quantum Liouville equation for this Hamilton operator.

Problem 12. Consider the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2m}\Delta + V(x)\right)\psi$$

Find the coupled system of partial differential equations for

$$\rho := \psi^* \psi, \qquad v := \Im \left(\frac{\nabla \psi}{\psi} \right).$$

Problem 13. Consider the Hilbert space $L_2(\mathbb{R})$. Let $f \in L_2(\mathbb{R})$ and $\theta \in \mathbb{R}$. We define the operator $U(\theta)$ as

$$U(\theta) f(x) := e^{i\theta/2} f(x e^{i\theta}).$$

Is the operator $U(\theta)$ unitary?

Problem 14. Consider the Hilbert space $L_2(\mathbb{R})$. Let $k \in \mathbb{Z}$. For $k = 0$ we define $s_0 = 0$, for $k \geq 1$ we define

$$s_k := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

and for $k < 0$ we define $s_k = -s_{-k}$. Let $\epsilon > 0$. Define the indicator functions W_k as

$$W_k(x) := \begin{cases} 1 & \text{for} \quad s_k < x/\epsilon \leq s_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

Let $u \in L_2(\mathbb{R})$. Define the linear operator O as

$$(Ou)(x) := g(x) u(x)$$

where

$$g(x) = -\frac{x}{\epsilon} + \sum_{k \in \mathbb{Z}} \left(\frac{s_k + s_{k+1}}{2} \right) W_k(x).$$

(i) Show that O is a bounded self-adjoint operator for any $\epsilon > 0$.
(ii) Show that the norm of O

$$\|O\| = \sup_{\|u\|=1} \|Ou\|$$

is given by $1/2$.

Chapter 8

Generalized Functions

Problem 1. Consider the function $H : \mathbb{R} \to \mathbb{R}$

$$H(x) := \begin{cases} 1 & 0 \le x \le 1/2 \\ -1 & 1/2 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the derivative of H in the sense of generalized functions. Obviously H can be considered as a regular functional

$$\int_{\mathbb{R}} H(x)\phi(x)dx.$$

Find the Fourier transform of H. Draw a picture of the Fourier transform.

Problem 2. Let $C^m[a,b]$ be the vector space of m-times differentiable functions and the m-th derivative is continuous over the interval $[a,b]$ ($b > a$). We define an inner product (scalar product) of such two functions f and g as

$$\langle f, g \rangle_m := \int_a^b \left(fg + \frac{df}{dx}\frac{dg}{dx} + \cdots + \frac{d^m f}{dx^m}\frac{d^m g}{dx^m} \right) dx.$$

Given (Legendre polynomials)

$$f(x) = \frac{1}{2}(3x^2 - 1), \qquad g(x) = \frac{1}{2}(5x^3 - 3x)$$

and the interval $[-1,1]$, i.e. $a = -1$ and $b = 1$. Show that f and g are orthogonal with respect to the inner product $\langle f, g \rangle_0$. Are they orthogonal with respect to $\langle f, g \rangle_1$?

Problem 3. Let P be the parity operator, i.e.

$$P\mathbf{r} := -\mathbf{r}.$$

Obviously, $P = P^{-1}$. We define

$$O_P u(\mathbf{r}) := u(P^{-1}\mathbf{r}) \equiv u(-\mathbf{r}).$$

The vector $\mathbf{r}$ can be expressed in spherical coordinates as

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

where

$$0 \le \phi < 2\pi \quad 0 \le \theta < \pi.$$

(i) Calculate $P(r,\theta,\phi)$.
(ii) Let

$$Y_{lm}(\theta,\phi) = \frac{(-1)^{l+m}}{2^l l!}\left(\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2}(\sin\theta)^m \frac{d^{l+m}}{d(\cos\theta)^{l+m}}(\sin\theta)^{2l}e^{im\phi}$$

be the *spherical harmonics*. Find

$$O_P Y_{lm}.$$

Problem 4. In the Hilbert space $\mathcal{H} = \ell_2(\mathbb{N}_0)$ Bose annihilation and creation operators denoted by b and $b^\dagger$ are defined as follows: They have a common domain

$$\mathcal{D}(b) = \mathcal{D}(b^\dagger) = \left\{ \xi = (x_0, x_1, x_2, \ldots)^T \,:\, \sum_{j=0}^{\infty} j|x_j|^2 < \infty \right\}.$$

Then $b\eta$ is given by

$$b(x_0, x_1, x_2, \ldots)^T = (x_1, \sqrt{2}x_2, \sqrt{3}x_3, \ldots)^T$$

and $b^\dagger\eta$ is given by

$$b^\dagger(x_0, x_1, x_2, \ldots) = (0, x_0, \sqrt{2}x_1, \sqrt{3}x_2, \ldots).$$

The infinite dimensional vectors

$$u_n = (0, 0, \ldots, 0, 1, 0, \ldots)^T$$

where the 1 is at the n position ($n = 0, 1, 2, \ldots$) form the standard basis in $\mathcal{H} = \ell_2(\mathbb{N}_0)$. Is

$$\xi = (1, 1/2, 1/3, \ldots, 1/n, \ldots)$$

an element of $\mathcal{D}(a)$?

Problem 5. Given a function (signal) $f(\mathbf{t}) = f(t_1, t_2, \ldots, t_n) \in L_2(\mathbb{R}^n)$ of n real variables $\mathbf{t} = (t_1, t_2, \ldots, t_n)$. We define the *symplectic tomogram* associated with the square integrable function f

$$w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \prod_{k=1}^{n} \frac{1}{2\pi|\nu_k|} \left| \int_{\mathbb{R}^n} dt_1 dt_2 \cdots dt_n f(\mathbf{t}) \exp\left(\sum_{j=1}^{n} \left(\frac{i\mu_j}{2\nu_j} t_j^2 - \frac{iX_j}{\nu_j} t_j \right) \right) \right|^2$$

where ($\nu_j \neq 0$ for $j = 1, 2, \ldots, n$)

$$\mathbf{X} = (X_1, X_2, \ldots, X_n), \quad \boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n), \quad \boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n).$$

(i) Prove the equality

$$\int_{\mathbb{R}^n} w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) d\mathbf{X} = \int_{\mathbb{R}^n} |f(\mathbf{t})|^2 d\mathbf{t} \tag{1}$$

for the special case $n = 1$. The tomogram is the probabilty distribution function of the random variable $\mathbf{X}$. This probability distribution function depends on $2n$ extra real parameters $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$.
(ii) The map of the function $f(\mathbf{t})$ onto the tomogram $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is invertible. The square integrable function $f(\mathbf{t})$ can be associated to the density matrix

$$\rho_f(\mathbf{t}, \mathbf{t}') = f(\mathbf{t}) f(\mathbf{t}').$$

This density matrix can be mapped onto the *Ville-Wigner function*

$$W(\mathbf{q}, \mathbf{p}) = \int_{\mathbb{R}^n} \rho_f \left(\mathbf{q} + \frac{\mathbf{u}}{2}, \mathbf{q} - \frac{\mathbf{u}}{2} \right) e^{-i\mathbf{p}\cdot\mathbf{u}} d\mathbf{u}.$$

Show that this map is invertible.
(iii) How is the tomogram $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$ related to the Ville-Wigner function?
(iv) Show that the Ville-Wigner function can be reconstructed from the function $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$.
(v) Show that the density matrix $f(\mathbf{t}) f^*(\mathbf{t}')$ can be found from $w(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu})$.

Problem 6. Starting with the set of polynomials $\{1, x, x^2, \ldots, x^n, \ldots\}$ use the Gram-Schmidt procedure the scalar product (inner product)

$$\langle f, g \rangle = \int_{-1}^{1} (f(x)g(x) + f'(x)g'(x)) dx$$

to find the first five orthogonal polynomials, where f' denotes derivative.

Problem 7. Describe the one-dimensional scattering of a particle incident on a Dirac delta function, i.e.

$$U(q) = U_0 \delta(q)$$

where $u_0 > 0$. Find the transmission and reflection coefficient.

Problem 8. (i) Give the definition of the current density, transmission coefficient, and reflection coefficient.
(ii) Calculate the transmission and the reflection coefficients of a particle having total energy E, at the potential barrier given by

$$V(x) = a\delta(x), \qquad a > 0$$

Problem 9. Show that

$$\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} = \sum_{k=-\infty}^{\infty} \delta(x - 2k\pi)$$

in the sense of generalized functions
Hint. Expand the 2π periodic function

$$f(x) = \frac{1}{2} - \frac{x}{2\pi}$$

into a Fourier series.

Problem 10. (i) Give the definition of a generalized function.
(ii) Calculate the first and second derivative in the sense of generalized function of

$$f(x) = \begin{cases} 0 & x < 0 \\ 4x(1-x) & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

(iii) Calculate the Fourier transform of $f(x) = 1$ in the sense of generalized functions.

Problem 11. Consider the generalized function

$$f(x) = |\cos(x)|.$$

Find the derivative in the sense of generalized functions.

Problem 12. Consider the generalized function

$$f(x) := \begin{cases} \cos(x) & \text{for} \quad x \in [0, 2\pi) \\ 0 & \text{otherwise} \end{cases}$$

Find the first and second derivative of f in the sense of generalized functions.

Problem 13. Find the derivative of $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |x|$$

in the sense of generalized functions.

Problem 14. Find the first three derivatives of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-|x|}$$

in the sense of generalized functions.

Problem 15. The *Sobolev space* of order m, denoted by $H^m(\Omega)$, is defined to be the space consisting of those functions in the Hilbert space $L_2(\Omega)$ that, together with all their weak partial derivatives up to and including those of order m, belong to the Hilbert space $L_2(\Omega)$, i.e.

$$H^m(\Omega) := \{ u : D^\alpha u \in L_2(\Omega) \text{ for all } \alpha \text{ such that } |\alpha| \leq m \}.$$

We consider real-valued functions only, and make $H^m(\Omega)$ an inner product space by introducing the Sobolev inner product $\langle \cdot, \cdot \rangle_{H^m}$ defined by

$$\langle u, v \rangle_{H^m} := \int_\Omega \sum_{|\alpha| \leq m} (D^\alpha u)(D^\alpha v) dx \quad \text{for} \quad u, v \in H^m(\Omega).$$

This inner product generates the Sobolev norm $\| \cdot \|_{H^m}$ defined by

$$\|u\|^2_{H^m} = \langle u|u \rangle_{H^m} = \int_\Omega \sum_{|\alpha| \leq m} (D^\alpha u)^2 dx.$$

Thus $H^0(\Omega) = L_2(\Omega)$. We can write

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \langle D^\alpha, D^\alpha v \rangle_{L_2(\Omega)}.$$

In other words the Sobolev inner product $\langle u, v \rangle_{H^m(\Omega)}$ is equal to the sum of the $L_2(\Omega)$ inner products of $D^\alpha u$ and $D^\alpha v$ over all α such that $|\alpha| \leq m$.
(i) Consider the domain $\Omega = (0, 2)$ and the function

$$u(x) = \begin{cases} x^2 & 0 < x \leq 1 \\ 2x^2 - 2x + 1 & 1 < x < 2. \end{cases}$$

Obviously $u \in L_2(\Omega)$. Find the Sobolev space to which u belongs.

(ii) Find the norm of u.

Problem 16. Let $c > 0$. Consider the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + c\delta^{(n)}(x)\psi = E\psi$$

where $\delta^{(n)}$ ($n = 0, 1, 2, \ldots$) denotes the n-th derivative of the delta function. Derive the joining conditions on the wave function ψ.

Problem 17. The *Morlet wavelet* consists of a plane wave modulated by a Gaussian, i.e.

$$\psi(\eta) = \frac{1}{\pi^{1/4}} e^{i\omega\eta} e^{-\eta^2/2}$$

where ω is the dimensionless frequency. Show that if $\omega = 6$ the admissibility condition is satisfied.

Problem 18. Let

$$f_0(x) = \exp(-x^2/2).$$

We define the mother wavelets f_n as

$$f_n(x) = -\frac{d}{dx} f_{n-1}(x), \qquad n = 1, 2, \ldots$$

Show that the family of f_n's obey the Hermite recursion relation

$$f_n(x) = x f_{n-1}(x) - (n-1) f_{n-2}(x), \qquad n = 2, 3, \ldots.$$

Problem 19. Show that the 2-dimensional complex δ-function can be written as ($\alpha \in \mathbb{C}$)

$$\delta^{(2)}(z) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha \exp(\alpha^* z - z^* \alpha) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha \exp(i(\alpha^* z + z^* \alpha)).$$

Problem 20. Show that

$$\delta(x - x') = \frac{1}{\pi}\left(1 + 2\sum_{k=1}^{\infty}\cos(kx)\cos(kx')\right).$$

Problem 21. Let $a > 0$. Show that

$$\sum_{m=-\infty}^{\infty} \exp(i2\pi m(x+q)/a) \equiv a \sum_{k=-\infty}^{\infty} \delta(x + q - ka).$$

Problem 22. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and x_0 a root of f, i.e. $f(x_0) = 0$. Show that

$$\delta'(f(x)) = \frac{1}{(f'(x_0))^2} \left(\delta'(x - x_0) + \frac{f''(x_0)}{f'(x_0)} \delta(x - x_0) \right)$$

Problem 23. Show that the sum

$$\frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(x) P_\ell(y)$$

of Legendre polynomials P_ℓ is given by the Dirac delta function $\delta(y - x)$ for $-1 \le x \le +1$ and $-1 \le y \le +1$.

Problem 24. Show that

$$\delta(x - x') = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} (\cos(nx)\cos(nx') + \sin(nx)\sin(nx')).$$

Miscellaneous

Problem 25. Let

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

be the Schrödinger equation, where

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + U(r), \qquad \Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

and $\mathbf{r} = (x_1, x_2, x_3)$. Let

$$\rho(\mathbf{r}, t) := \bar{\psi}(\mathbf{r}, t)\psi(\mathbf{r}, t)$$

Find $\mathbf{j}$ such that

$$\text{div}\mathbf{j} + \frac{\partial\rho}{\partial t} = 0.$$

Problem 26. A particle is enclosed in a rectangular box with impenetrable walls, inside which in can move freely. The Hilbert space is

$$L_2([0, a] \times [0, b] \times [0, c])$$

where $a, b, c > 0$. Find the eigenfunctions and eigenvalues. What can be said about the degeneracy, if any, of the eigenfunctions?

Problem 27. Show that in one-dimensional problems the energy spectrum of the bound states is always non-degenerate. Hint. Suppose that the opposite is true. Let u_1, u_2 be two linearly independent eigenfunctions with the same energy eigenvalue E, i.e.

$$\frac{d^2u_1}{dx^2} + \frac{2m}{\hbar^2}(E - V)u_1 = 0$$

$$\frac{d^2u_2}{dx^2} + \frac{2m}{\hbar^2}(E - V)u_2 = 0.$$

Problem 28. Derive the Heisenberg uncertainty relation.

Problem 29. Give the standard postulates in quantum mechanics and discuss the problematic.

Problem 30. Show that in one-dimensional problems the energy spectrum of the bound states is always non-degenerate.

Hint. Suppose that the opposite is true.

Let u_1 and u_2 be two linearly independent eigenfunctions with the same energy eigenvalues E.

$$\frac{d^2u_1}{dx^2} + \frac{2m}{\hbar^2}(E-V)u_1 = 0$$

$$\frac{d^2u_2}{dx^2} + \frac{2m}{\hbar^2}(E-V)u_2 = 0$$

Problem 31. Let $a > 0$ and let $f_a : \mathbb{R} \to \mathbb{R}$ be given by

$$f_a(x) = \begin{cases} x/a^2 + 1/a & \text{for } -a \le x \le 0 \\ -x/a^2 + 1/a & \text{for } \ 0 \le x \le a \end{cases}$$

The function f_a generates regular functional. Find the derivative of f_a in the sense of generalized functions.

Problem 32. Consider a one-dimensional lattice (chain) with lattice constant a. Let k be the sum over the first Brioullin zone we have

$$\frac{1}{N}\sum_{k\in 1.BZ} F(\epsilon(k)) \to \frac{a}{2\pi}\int_{-\pi/a}^{\pi/a} F(\epsilon(k))dk = G$$

where

$$\epsilon(k) = \epsilon_0 - 2\epsilon_1\cos(ka).$$

Using the identity

$$\int_{-\infty}^{\infty} \delta(E-\epsilon(k))F(E)dE \equiv F(\epsilon(k))$$

we can write

$$G = \frac{a}{2\pi}\int_{-\infty}^{\infty} F(E)\left(\int_{-\pi/a}^{\pi/a}\delta(E-\epsilon(k))dk\right)dE.$$

Calculate

$$g(E) = \int_{-\pi/a}^{\pi/a}\delta(E-\epsilon(k))dk$$

where $g(E)$ is called the density of states.

Problem 33. Let $\epsilon > 0$. Consider the Schrödinger eigenvalue equation

$$\left(-\frac{d^2}{dx^2} + 2\epsilon\delta(x)\right)u(x,\epsilon) = E(\epsilon)u(x,\epsilon)$$

with the boundary conditions $u(\pm 1, \epsilon) = 0$. Here ϵ is the coupling constant and determines the penetrability of the potential barrier. Find the eigenfunctions and the eigenvalues.

Problem 34. Show that in the sense of generalized functions

$$\delta(x) = \frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} e^{-|x|/\epsilon}$$

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to \infty} \epsilon \frac{\sin^2(\epsilon x)}{(\epsilon x)^2}$$

$$\delta(x) = \frac{1}{4} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(1 + \frac{|x|}{\epsilon} \right) e^{-|x|/\epsilon}.$$

Problem 35. Give two interpretations of the series of derivatives of δ functions

$$f(k) = 2\pi \sum_{n=0}^{\infty} c_n (-1)^n \delta^{(n)}(k). \tag{1}$$

Problem 36. Show that

$$H(x-a) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \exp(iu(\tau - x)).$$

Problem 37. Show that

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{z} \right) = \pi \delta(z)$$

where

$$\frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Problem 38. Let $a > 0$. Show that

$$\delta(x^2 - a^2) = \frac{1}{2a} (\delta(x-a) + \delta(x+a)).$$

Problem 39. (i) Show that the Fourier transform in the sense of generalized function of the *Dirac comb*

$$\sum_{n \in \mathbb{Z}} \delta(x - n)$$

is again a Dirac comb.
(ii) Find the Fourier transform in the sense of generalized functions of

$$1+\sqrt{2\pi}\delta(x).$$

Problem 40. (i) Consider the nonlinear differential equation

$$3u\frac{du}{dx}=2\frac{du}{dx}\frac{d^2u}{dx^2}+u\frac{d^3u}{dx^3}.$$

Show that $u(x)=e^{-|x|}$ is a solution in the sense of generalized function.
(ii) Consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t}-\frac{\partial^3 u}{\partial x^2\partial t}+3u\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3}.$$

Show that $u(x,t)=c\exp(-|x-ct|)$ (*peakon*) is a solution in the sense of generalized functions.

Problem 41. Let f be a differentiable function with a simple zero at $x=a$ such that $f(x=a)=0$ and $df(x=a)/dx\neq 0$. Let g be a differentiable function with a simple zero at $x=b\neq a$ such that $g(x=b)=0$ and $dg(x=b)/dx\neq 0$. Show that

$$\delta(f(x)g(x))=\frac{1}{|f'(a)g(a)|}\delta(x-a)+\frac{1}{|f(b)g'(b)|}\delta(x-b)$$

where $'$ denotes differentiation.

Problem 42. Consider the non-relativistic hydrogen atom, where a_0 is the Bohr radius and $a=a_0/Z$. The Schrödinger-Coulomb Green function $G(\mathbf{r}_1,\mathbf{r}_2;E)$ corresponding to the energy variable E is the solution of the partial differential equation

$$\left(-\frac{\hbar^2}{2m}\nabla_1^2-\frac{\hbar^2}{amr_1}-E\right)G(\mathbf{r}_1,\mathbf{r}_2;E)=\delta(\mathbf{r}_1-\mathbf{r}_2)$$

with the appropiate boundary conditions. Show that expanding G in terms of spherical harmonics $Y_{\ell m}$

$$G(\mathbf{r}_1,\mathbf{r}_2;E)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}g_\ell(r_1,r_2;E)Y_{\ell m}(\theta_1,\phi_1)Y^*_{\ell m}(\theta_2,\phi_2)$$

we find for the radial part g_ℓ of the Schrödinger-Coulomb Green function

$$\left(\frac{1}{r_1^2}\frac{d}{dr_1}\left(r_1^2\frac{d}{dr_1}\right)-\frac{\ell(\ell+1)}{r_1^2}+\frac{2}{ar_1}-\frac{1}{\nu^2a^2}\right)g_\ell(r_1,r_2;\nu)=-\frac{2m}{\hbar^2}\frac{\delta(r_1-r_2)}{r_1r_2}$$

where $\nu^2 a^2 := -\hbar^2/(2mE)$.
Hint. Utilize the identity

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \frac{\delta(r_1 - r_2)}{r_1 r_2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta_1, \phi_1) Y_{\ell m}^*(\theta_2, \phi_2)$$

Problem 43. Show that (distributional identity on $L_1(\mathbb{R})$)

$$\frac{1}{\pi^2} \int_{\mathbb{R}} \frac{1}{(t-x)(s-x)} dx = \delta(t-s)$$

where the integral is evaluated in the principal value sense.

Problem 44. Let $c > 0$. Show that an integral representation of the delta function is given by

$$\delta(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{tx} dt$$

where the path of the t-integration can be closed to the right or left.

Problem 45. Show that

$$\delta\left(t - \frac{x}{c}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t(t-x/c)} d\omega.$$

Problem 46. Show that in the sense of generalized functions

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(ikx) dx, \qquad \theta(x) = \int_0^{\infty} \delta(\lambda - x) d\lambda.$$

Problem 47. Let $\epsilon > 0$. Show that

$$f_\epsilon(x-a) = \frac{1}{\sqrt{\pi\epsilon}} \exp\left(-\frac{(x-a)^2}{\epsilon}\right)$$

tends to $\delta(x-a)$ in the sense of generalized function if $\epsilon \to 0_+$.

Problem 48. Let J_0 be the Bessel functions. Show that

$$\delta(x)\delta(x) = \frac{1}{2\pi} \int_0^{\infty} k J_0(k\sqrt{(x^2+y^2)}) dk$$

in the sense of generalized functions.

Problem 49. Let $\alpha \in [0,1)$. Show that

$$\int_0^\infty x^{\alpha-1} P\left(\frac{1}{1-x^2}\right) = \frac{\pi}{2}\cot(\pi\alpha/2).$$

Problem 50. Show that

$$\delta(\mathbf{x}-\mathbf{x}') = \lim_{\alpha\to\beta}\left(\frac{2\pi}{\beta-\alpha}\right)^{3/2} \exp\left(-\frac{\alpha\beta}{2(\beta-\alpha)}(\mathbf{x}-\mathbf{x}')^2\right).$$

Problem 51. Let $p \in [0,1]$ and

$$\rho(x) = \frac{1}{2}pe^{-|x|} + (1-p)\delta(x).$$

Then $\rho(x) \geq 0$. Show that in the sense of generalized function

$$\int_{\mathbb{R}} \rho(x)dx = 1.$$

Problem 52. The two-dimensional Dirac comb function is defined by

$$C(x_1,x_1) := \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \delta(x_1-m)\delta(x_2-n).$$

Find the Fourier transform of C in the sense of generalized functions.

Problem 53. Let $T>0$. Consider the sequence of functions

$$f_n(t) = \frac{1}{n!}\frac{n}{T}\left(\frac{nt}{T}\right)^n \exp(-nt/T)$$

where $n = 1,2,\ldots$. Find $f_n(t)$ for $n\to\infty$ in the sense of generalized functions. Find the Laplace transform of $f_n(t)$.

Problem 54. What charge distribution $\rho(r)$ does the spherical symmetric potential

$$V(r) = \frac{e^{-\mu r}}{r}$$

give? For $r \neq 0$ *Poisson's equation* in spherical coordinates is given by

$$\Delta V(\mathbf{r}) = \frac{1}{r}\frac{d^2}{dr^2}(rV(\mathbf{r})) + R(\theta,\phi)V(\mathbf{r}) = -4\pi\rho(\mathbf{r})$$

where $R(\theta, \phi)$ is the differential operator depending on the angles θ, ϕ.

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