Problems and Solutions
for
Partial Differential Equations

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Chapter 1

Linear Partial Differential Equations

Problem 1. Show that the fundamental solution of the *drift diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}$$

is given by

$$u(x,t) = \exp\left(-\frac{1}{\sqrt{4\pi t}}\frac{(x-x_0+2t)^2}{4t}\right).$$

Problem 2. (i) Show that

$$D_x^m(f \cdot 1) = \frac{\partial^m f}{\partial x^m}. \tag{1}$$

(ii) Show that

$$D_x^m(f \cdot g) = (-1)^m D_x^m(g \cdot f). \tag{2}$$

(iii) Show that

$$D_x^m(f \cdot f) = 0, \quad \text{for} \quad m \quad \text{odd} \tag{3}$$

Problem 3. (i) Show that

$$D_x^m D_t^n(\exp(k_1 x - \omega_1 t) \cdot \exp(k_2 x - \omega_2 t)) =$$

$$(k_1 - k_2)^m(-\omega_1 + \omega_2)^n \exp((k_1 + k_2)x - (\omega_1 + \omega_2)t) \tag{1}$$

This property is very useful in the calculation of soliton solutions.

(ii) Let $P(D_t, D_x)$ be a polynomial in D_t and D_x. Show that

$$P(D_x, D_t)(\exp(k_1x - \omega_1t) \cdot \exp(k_2x - \omega_2t)) =$$

$$\frac{P(k_1 - k_2, -\omega_1 + \omega_2)}{P(k_1 + k_2, -\omega_1 - \omega_2)} P(D_x, D_t)(\exp((k_1 + k_2)x - (\omega_1 + \omega_2)t) \cdot 1) \tag{2}$$

Problem 4. Consider a free particle in two dimensions confined by the boundary

$$G := \{\, (x, y) \,:\, |xy| = 1 \,\}.$$

Solve the eigenvalue problem

$$\Delta\psi + k^2\psi = 0$$

where

$$k^2 = \frac{2mE}{\hbar^2}$$

with

$$\psi_G = 0.$$

Problem 5. Consider an electron of mass m confined to the $x - y$ plane and a constant magnetic field $\mathbf{B}$ parallel to the z-axis, i.e.

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}.$$

The Hamilton operator for this two-dimensional electron is given by

$$\hat{H} = \frac{(\hat{\mathbf{p}} + e\mathbf{A})^2}{2m} = \frac{1}{2m}((\hat{p}_x + eA_x)^2 + (\hat{p}_y + eA_y)^2)$$

where $\mathbf{A}$ is the vector potential with

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}, \qquad \hat{p}_y = -i\hbar\frac{\partial}{\partial y}.$$

(i) Show that $\mathbf{B}$ can be obtained from

$$\mathbf{A} = \begin{pmatrix} 0 \\ xB \\ 0 \end{pmatrix}$$

or

$$\mathbf{A} = \begin{pmatrix} -yB \\ 0 \\ 0 \end{pmatrix}.$$

(ii) Use the second choice for $\mathbf{A}$ to find the Hamilton operator $\hat{H}$.
(iii) Show that

$$[\hat{H}, \hat{p}_x] = 0.$$

(iv) Let $k = p_x/\hbar$. Make the ansatz for the wave function

$$\psi(x, y) = e^{ikx}\phi(y)$$

and show that the eigenvalue equation $\hat{H}\psi = E\psi$ reduces to

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\omega_c^2}{2}(y - y_0)^2\right)\phi(y) = E\phi(y)$$

where

$$\omega_c := \frac{eB}{m}, \qquad y_0 := \frac{\hbar k}{eB}.$$

(v) Show that the eigenvalues are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c, \qquad n = 0, 1, 2, \ldots.$$

Problem 6. Consider the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2m}\Delta + V(x)\right)\psi.$$

Find the coupled system of partial differential equations for

$$\rho := \psi^*\psi, \qquad v := \Im\left(\frac{\nabla\psi}{\psi}\right).$$

Problem 7. Consider the conservation law

$$\frac{\partial c(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0$$

where

$$j(x,t) = -D(x)\frac{\partial c(x,t)}{\partial x}.$$

The diffusion coefficient $D(x)$ depends as follows on x

$$D(x) = D_0(1 + gx) \qquad g \geq 0.$$

Thus $dD(x)/dx = D_0 g$. Inserting the current j into the convervation law we obtain a *drift diffusion equation*

$$\frac{\partial c(x,t)}{\partial t} = D_0 g \frac{\partial c(x,t)}{\partial x} + D_0(1+gx)\frac{\partial^2 c(x,t)}{\partial x^2}.$$

The initial condition for this partial differential equation is

$$c(x, t=0) = M_0 \delta(x)$$

where δ denotes the delta function. The boundary conditions are

$$j(x = x_0, t) = j(x = +\infty, t) = 0.$$

Solve the one-dimensional drift-diffusion partial differential equation for these initial and boundary conditions using a *product ansatz* $c(x,t) = T(t)X(x)$.

Problem 8. Consider the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = \left(-\frac{1}{2}\Delta + V(\mathbf{x})\right)\psi \tag{1}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Let

$$\rho := |\psi|^2 = \psi\psi^*, \qquad \mathbf{v} := \Im\left(\frac{\nabla\psi}{\psi}\right) = \Im\begin{pmatrix} \frac{1}{\psi}\frac{\partial\psi}{\partial x_1} \\ \frac{1}{\psi}\frac{\partial\psi}{\partial x_2} \\ \frac{1}{\psi}\frac{\partial\psi}{\partial x_3} \end{pmatrix}.$$

Show that the time-dependent Schrödinger equation can be written as the system of partial differential equations (Madelung equations)

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\mathbf{v}\rho) = -\left(\frac{\partial(v_1\rho)}{\partial x_1} + \frac{\partial(v_2\rho)}{\partial x_2} + \frac{\partial(v_3\rho)}{\partial x_3}\right) \tag{2}$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(V(\mathbf{x}) - \frac{\Delta(\rho^{1/2})}{2\rho^{1/2}}\right). \tag{3}$$

Problem 9. Consider the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t) + V(\mathbf{x},t)\psi(\mathbf{x},t).$$

Consider the ansatz

$$\psi(\mathbf{x},t) = \phi(\mathbf{x},t)\exp(imS(\mathbf{x},t)/\hbar)$$

where the functions ϕ and S are real. Find the partial differential equations are ϕ and S.

Problem 10. Consider the Schrödinger equation $\hat{H}\Psi = E\Psi$ of a particle on the torus. A *torus* surface can be parametrized by the azimuthal angle ϕ and its polar angle θ

$$\begin{aligned} x(\phi,\theta) &= (R + a\cos(\theta))\cos(\phi) \\ y(\phi,\theta) &= (R + a\cos(\theta))\sin(\phi) \\ z(\phi,\theta) &= a\sin(\theta) \end{aligned}$$

where R and a are the outer and inner radius of the torus, respectively such that the ratio a/R lies between zero and one.
(i) Find $\hat{H}$.
(ii) Apply the separation ansatz

$$\Psi(\theta,\phi) = \exp(im\phi)\psi(\theta)$$

where m is an integer.

Problem 11. The linear one-dimensional *diffusion equation* is given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0, \quad -\infty < x < \infty$$

where $u(x,t)$ denotes the concentration at time t and position $x \in \mathbb{R}$. D is the diffusion constant which is assumed to be independent of x and t. Given the initial condition $c(x,0) = f(x)$, $x \in \mathbb{R}$ the solution of the one-dimensional diffusion equation is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t|x',0)f(x')dx'$$

where

$$G(x,t|x',t') = \frac{1}{\sqrt{4\pi D(t-t')}}\exp\left(-\frac{(x-x')^2}{4D(t-t')}\right).$$

Here $G(x,t|x',t')$ is called the fundamental solution of the diffusion equation obtained for the initial data $\delta(x-x')$ at $t = t'$, where δ denotes the Dirac delta function.
(i) Let $u(x,0) = f(x) = \exp(-x^2/(2\sigma))$. Find $u(x,t)$.
(ii) Let $u(x,0) = f(x) = \exp(-|x|/\sigma)$. Find $u(x,t)$.

Problem 12. Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable function. Consider the first order vector partial differential equation

$$\nabla \times \mathbf{f} = k\mathbf{f}$$

where k is a positive constant.
(i) Find $\nabla \times (\nabla \times \mathbf{f})$.
(ii) Show that $\nabla \mathbf{f} = 0$.

Problem 13. Consider *Maxwell's equation*

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{div}\mathbf{E} = 0, \quad \mathrm{div}\mathbf{B} = 0$$

with $\mathbf{B} = \mu_0 \mathbf{H}$.
(i) Assume that $E_2 = E_3 = 0$ and $B_1 = B_3 = 0$. Simplify Maxwell's equations.
(ii) Now assume that E_1 and B_2 only depends on x_3 and t with

$$E_1(x_3, t) = f(t)\sin(k_3 x_3), \qquad B_2(x_3, t) = g(t)\cos(k_3 x_3)$$

where k_3 is the third component of the wave vector $\mathbf{k}$. Find the system of ordinary differential equations for $f(t)$ and $g(t)$ and solve it and thus find the *dispersion relation*. Note that

$$\nabla \times \mathbf{E} = \begin{pmatrix} \partial E_3/\partial x_2 - \partial E_2/\partial x_3 \\ \partial E_1/\partial x_3 - \partial E_3/\partial x_1 \\ \partial E_2/\partial x_1 - \partial E_1/\partial x_2 \end{pmatrix}.$$

Problem 14. Let k be a constant. Show that the vector partial differential equation

$$\nabla \times \mathbf{u} = k\mathbf{u}$$

has the general solution

$$\mathbf{u}(x_1, x_2, x_3) = \nabla \times (\mathbf{c}v) + \frac{1}{k}\nabla \times \nabla(\mathbf{c}v)$$

where $\mathbf{c}$ is a constant vector and v satisfies the partial differential equation

$$\nabla^2 v + k^2 v = 0.$$

Problem 15. Consider the partial differential equation of first order

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right) f = 0.$$

Show that f is of the form

$$f(x_1 - x_2, x_2 - x_3, x_3 - x_1).$$

Problem 16. Consider the operators

$$D_x + \frac{\partial}{\partial x}, \qquad D_t = \frac{\partial}{\partial t} + u(x,t)\frac{\partial}{\partial x}.$$

Find the commutator

$$[D_x, D_t]f(x,t)$$

Problem 17. Let $\mathbf{C}$ be a constant column vector in $\mathbb{R}^n$ and $\mathbf{x}$ be a column vector in $\mathbb{R}^n$. Show that

$$\mathbf{C}e^{\mathbf{x}^T\mathbf{C}} \equiv \nabla e^{\mathbf{x}^T\mathbf{C}}$$

where ∇ denotes the gradient.

Problem 18. Consider the partial differential equation (*Laplace equation*)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on} \quad [0,1]\times[0,1]$$

with the boundary conditions

$$u(x,0) = 1, \quad u(x,1) = 2, \quad u(0,y) = 1, \quad u(1,y) = 2.$$

Apply the *central difference scheme*

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{j,k} \approx \frac{u_{j-1,k} - 2u_{j,k} + u_{j+1,k}}{(\Delta x)^2}, \qquad \left(\frac{\partial^2 u}{\partial y^2}\right)_{j,k} \approx \frac{u_{j,k-1} - 2u_{j,k} + u_{j,k+1}}{(\Delta y)^2}$$

and then solve the linear equation. Consider the cases $\Delta x = \Delta y = 1/3$ and $\Delta x = \Delta y = 1/4$.

Problem 19. Consider the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + k^2\psi = 0.$$

(i) Apply the transformation

$$u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy, \qquad \psi(x,y) = \widetilde{\psi}(u(x,y), v(x,y)).$$

(ii) Then introduce polar coordinates $u = r\cos\phi$, $v = r\sin\phi$.

Problem 20. Starting from Maxwell's equations in vacuum show that

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}.$$

Problem 21. Show that the linear partial differential equation

$$\frac{\partial^2 u}{\partial \theta^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

admits the solution $u(t,\theta) = \sin(\theta)\sin(t)$. Does this solution satisfies the boundary condition $u(t,\theta=\pi) = 0$ and the initial condition $u(t=0,\theta) = 0$?

Problem 22. Consider the Hamilton operator

$$\hat{H} = -\frac{\hbar^2}{2}\sum_{j=1}^{3}\frac{1}{m_j}\frac{\partial^2}{\partial q_j^2} + \frac{k}{2}((q_2-q_1-d)^2 + (q_3-q_2-d)^2)$$

where d is the distance between two adjacent atoms. Apply the linear transformation

$$\begin{aligned}
\xi(q_1,q_2,q_3) &= (q_2-q_1)-d \\
\eta(q_1,q_2,q_3) &= (q_3-q_2)-d \\
X(q_1,q_2,q_3) &= \frac{1}{M}(m_1q_1+m_2q_2+m_3q_3)
\end{aligned}$$

where $M := m_1+m_2+m_3$ and show that the Hamilton operator decouples.

Problem 23. Consider the Hamilton operator for three particles

$$\hat{H} = -\frac{1}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) - 6g\delta(x_1-x_2)\delta(x_2-x_3)$$

and the eigenvalue problem $\hat{H}u(x_1,x_2,x_3) = Eu(x_1,x_2,x_3)$. Apply the transformation

$$\begin{aligned}
y_1(x_1,x_2,x_3) &= \sqrt{\frac{2}{3}}\left(\frac{1}{2}(x_1+x_2)-x_3\right) \\
y_2(x_1,x_2,x_3) &= \frac{1}{\sqrt{2}}(x_1-x_2) \\
y_3(x_1,x_2,x_3) &= \frac{1}{3}(x_1+x_2+x_3) \\
\widetilde{u}(y_1(x_1,x_2,x_3),y_2(x_1,x_2,x_3),y_3(x_1,x_2,x_3)) &= u(x_1,x_2,x_3)
\end{aligned}$$

where y_3 is the centre-of mass position of the three particles and y_1, y_2 give their relative positions up to constant factors. Find the Hamilton operator for the new coordinates.

Problem 24. Consider the four dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} = 0.$$

Transform the equation into polar coordinates (r, θ, ϕ, χ)

$$\begin{aligned} x_1(r,\theta,\phi,\chi) &= r\sin(\theta/2)\sin((\phi-\chi)/2) \\ x_2(r,\theta,\phi,\chi) &= r\sin(\theta/2)\cos((\phi-\chi)/2) \\ x_3(r,\theta,\phi,\chi) &= r\cos(\theta/2)\sin((\phi+\chi)/2) \\ x_4(r,\theta,\phi,\chi) &= r\cos(\theta/2)\cos((\phi+\chi)/2) \\ u(\mathbf{x}) &= \widetilde{u}(r(\mathbf{x}),\theta(\mathbf{x}),\phi(\mathbf{x}),\chi(\mathbf{x})) \end{aligned}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$.

Problem 25. A nonrelativistic particle is described by the Schrödinger equation

$$\left(\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{x},t)\right)\psi(\mathbf{x},t) = i\hbar\frac{\partial\psi}{\partial t}$$

where $\hat{\mathbf{p}} := -i\hbar\nabla$. Write the wave function ψ in *polar form*

$$\psi(\mathbf{x},t) = R(\mathbf{x},t)\exp(iS(\mathbf{x},t)/\hbar)$$

where R, S are real functions and $R(\mathbf{x},t) \geq 0$. Give an interpretation of $\rho = R^2$.

Problem 26. The time-dependent Schrödinger equation for the one-dimensional free particle case can be written in either position or momentum space as

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

$$\frac{p^2}{2m}\phi(p,t) = i\hbar\frac{\partial\phi(x,t)}{\partial t}.$$

Consider the momentum space approach with the solution

$$\phi(p,t) = \phi_0(p)\exp(-ip^2t/2m\hbar)$$

where $\phi(p,0) = \phi_0(p)$ is the initial momentum distribution. We define $\hat{x} := i\hbar(\partial/\partial p)$. Find $\langle\hat{x}\rangle(t)$.

Problem 27. Consider the partial differential equation

$$\frac{\partial^2}{\partial\mathbf{x}\partial t}(\ln(\det(I_n + tD\mathbf{f}(\mathbf{x}))) = \mathbf{0}$$

where $D\mathbf{f}(\mathbf{x})$ ($\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$) is the Jacobian matrix. Find the solution of the initial value problem.

Problem 28. The partial differential equation

$$\left(\frac{\partial u}{\partial x_1}\right)^2\left(\frac{\partial^2 u}{\partial x_2\partial x_3}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\left(\frac{\partial^2 u}{\partial x_1\partial x_3}\right)^2+\left(\frac{\partial u}{\partial x_3}\right)^2\left(\frac{\partial^2 u}{\partial x_1\partial x_2}\right)^2$$
$$-2\left(\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1\partial x_3}\frac{\partial^2 u}{\partial x_2\partial x_3}+\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3}\frac{\partial^2 u}{\partial x_1\partial x_2}\frac{\partial^2 u}{\partial x_2\partial x_3}+\frac{\partial u}{\partial x_2}\frac{\partial u}{\partial x_3}\frac{\partial^2 u}{\partial x_1\partial x_2}\frac{\partial^2 u}{\partial x_1\partial x_3}\right)$$
$$+4\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\frac{\partial u}{\partial x_3}\frac{\partial^3 u}{\partial x_1\partial x_2\partial x_3}=0$$

has some link to the Bateman equation. Find the Lie symmetries. There are Lie-Bäcklund symmetries? Is there a Legendre transformation to linearize this partial differential equation?

Problem 29. Consider the two-dimensional heat equation

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}$$

Show that

$$T(x,y,t)=\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{x^2+(y-1)^2}{4t}\right)-\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{x^2+(y+1)^2}{4t}\right)$$

satisfies this partial differential equation.

Problem 30. Let $\epsilon>0$. Consider the *Fokker-Planck equation*

$$\frac{\partial u}{\partial t}=\epsilon\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}(xu)$$

with $u\geq 0$ for all x.
(i) Find steady state solutions, i.e. find solutions of the ordinary differential equation

$$\epsilon\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}(xu)=0.$$

(ii) Find time dependent solution of the inital value problem with

$$u(t=0,x)=N_{\sigma_0}(x-\mu_0)=\frac{1}{\sqrt{2\pi\sigma_0}}e^{-(x-\mu_0)^2/(2\sigma_0)}$$

with $\sigma_0\geq 0$ the variance and μ_0 the mean. With $\sigma=0$ we have the delta function at 0.

Problem 31. Consider the Schrödinger equation

$$\left(-\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}+2c\sum_{j<k}\delta(x_j-x_k)\right)u(\mathbf{x})=Eu(\mathbf{x})$$

describing a one-dimensional Bose gas with the δ-fnction repulsive interaction. Show that for $N = 2$ the eigenvalue problem can be solved with a exponential ansatz.

Problem 32. (i) Show that the power series

$$P(t,\lambda) = \sum_{n=0}^{\infty} \rho_n \lambda^n \cos(\sqrt{n}t)$$

satisfies the linear partial differential equation (diffusion type equation)

$$\frac{\partial^2 P(t,\lambda)}{\partial t^2} = -\lambda \frac{\partial}{\partial \lambda} P(t,\lambda)$$

with boundary conditions

$$P(0,\lambda) = \sum_{n=0}^{\infty} \rho_n \lambda^n = \rho(\lambda), \quad \frac{\partial P(0,\lambda)}{\partial t} = 0, \quad P(t,0) = \rho(0).$$

(ii) Perform a *Laplace transformation*

$$\widetilde{P}(z,\lambda) := \int_0^{\infty} e^{-zt} P(t,\lambda) dt$$

and show that $\widetilde{P}(z,\lambda)$ obeys the differential equation

$$z^2 \widetilde{P}(z,\lambda) - z\rho(\lambda) = -\lambda \frac{\partial}{\partial \lambda} \widetilde{P}(z,\lambda)$$

with boundary condition

$$\widetilde{P}(z,0) = \frac{\rho(0)}{z}$$

and

$$\widetilde{P}(z,\lambda) = \int_0^{\lambda} \exp(-z^2 \ln(\lambda/x)) \frac{\rho(x)}{x} dx.$$

Chapter 2

Nonlinear Partial Differential Equations

Problem 1. Consider the system of quasi-linear partial differential equations

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial y} + u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

Show that this system arise as compatibility conditions $[L, M] = 0$ of an overdetermined system of linear equations $L\Psi = 0$, $M\Psi = 0$, where $\Psi(x, y, t, \lambda$ is a function, λ is a spectral parameter, and the *Lax pair* is given by

$$L = \frac{\partial}{\partial t} - v\frac{\partial}{\partial x} - \lambda\frac{\partial}{\partial y}, \qquad M = \frac{\partial}{\partial y} + u\frac{\partial}{\partial x} - \lambda\frac{\partial}{\partial x}.$$

Problem 2. Find the *traveling wave solution*

$$u(x, t) = f(x - ct) \qquad c = \text{constant} \tag{1}$$

of the one-dimensional *sine-Gordon equation*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin u\,. \tag{2}$$

Problem 3. Show that the nonlinear nondispersive part of the *Korteweg-de Vries equation*

$$\frac{\partial u}{\partial t} + (\alpha + \beta u)\frac{\partial u}{\partial x} = 0 \qquad (1)$$

possesses *shock wave solutions* that are intrinsically implicit

$$u(x,t) = f(x - (\alpha + \beta u(x,t))t) \qquad (2)$$

with the initial value problem $u(x, t = 0) = f(x)$, where α, $\beta \in \mathbb{R}$.

Problem 4. Consider the *Korteweg-de Vries-Burgers equation*

$$\frac{\partial u}{\partial t} + a_1 u \frac{\partial u}{\partial x} + a_2 \frac{\partial^2 u}{\partial x^2} + a_3 \frac{\partial^3 u}{\partial x^3} = 0 \qquad (1)$$

where a_1, a_2 and a_3 are non-zero constants. It contains dispersive, dissipative and nonlinear terms.
(i) Find a solution of the form

$$u(x,t) = \frac{b_1}{(1 + \exp(b_2(x + b_3 t + b_4)))^2} \qquad (2)$$

where b_1, b_2, b_3 and b_4 are constants determined by a_1, a_2, a_3 and a_4.
(ii) Study the case $t \to \infty$ and $t \to -\infty$.

Problem 5. The *Fisher equation* is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au(1-u) \qquad (1)$$

where the positive constant a is a measure of intensity of selection. (i) Consider the substitution

$$v(x, \tau(t)) = \frac{1}{6} u(x,t), \qquad \tau(t) = 5t. \qquad (2)$$

(i) Show that (1) takes the form

$$5\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + av(1 - 6v). \qquad (3)$$

(ii) Consider the following ansatz

$$v(x,t) := \sum_{j=0}^{\infty} v_j(x,t)\phi^{j-p}(x,t) \qquad (4)$$

is single-valued about the solution movable singular manifold $\phi = 0$. This means p is a positive integer, recursion relationships for v_j are self-consistent,

and the ansatz (4) has enough free functions in the sense of the Cauchy-Kowalevskia theorem.
(iii) Try to truncate the expansion (4) with $v_j = 0$ for $j \geq 2$.

Problem 6. Consider Fisher's equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

Show that

$$u(x,y,t) = \left(1 + \exp\left(\frac{(x - y/\sqrt{2}) - (5/\sqrt{6})t}{\sqrt{6}}\right)\right)^{-2}$$

is a traveling wave solution of this equation. Is the the solution an element of $L_2(\mathbb{R}^2)$ for a fixed t?

Problem 7. Consider the nonlinear partial differential equation

$$\left(\frac{\partial \phi}{\partial y}\right)^2 \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial y^2} - 2\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} = 0. \qquad (1)$$

(i) Show that this equation has an implicit solution for ϕ of the form

$$x = X(\phi, y, t) = f(\phi, t)y + h(\phi, t) \qquad (2)$$

where f and h are arbitrary differentiable functions of ϕ and t.
(ii) Show that (1) may also be solved by means of a Legendre transformation.

Problem 8. The *sine-Gordon equation* is the equation of motion for a theory of a single, dimensionless scalar field u, in one space and one time dimension, whose dynamics is determined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(u_t^2 - c^2 u_x^2) + \frac{m^4}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{m}u\right) - \mu. \qquad (1)$$

Here c is a limiting velocity while m, λ, and μ are real parameters. u_t and u_x are the partial derivatives of ϕ with respect to t and x, respectively. In the terminology of quantum field theory, m is the mass associated with the normal modes of the linearized theory, while λ/m^2 is a dimensionless, coupling constant that measures the strength of the interaction between these normal modes. In classical theory m is proportional to the characteristic frequency of these normal modes.

(i) Let

$$x \to \frac{x}{m}, \qquad t \to \frac{t}{m}, \qquad u \to mu\sqrt{\lambda} \tag{2}$$

and set $c = 1$. Show that then the Lagrangian density becomes

$$\mathcal{L} = \frac{m^4}{2\lambda}((u_t^2 - u_x^2) + 2\cos u) - \mu \tag{3}$$

with the corresponding Hamiltonian density being given by

$$\mathcal{H} = \frac{m^2}{2\lambda}(u_t^2 + u_x^2 - 2\cos u) + \mu. \tag{4}$$

(ii) By choosing

$$\mu = \frac{m^4}{\lambda} \tag{5}$$

show that the minimum energy of the theory is made zero and (4) can be written as

$$\mathcal{H} = \frac{m^4}{2\lambda}(u_t^2 + u_x^2 + 2(1 - \cos u)). \tag{6}$$

Problem 9. Find the solution to the nonlinear partial differential equation

$$\left(1 - \left(\frac{\partial u}{\partial t}\right)^2\right)\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x \partial t} - \left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

which satisfies the initial conditions (Cauchy problem)

$$u(t=0) = a(x), \qquad \frac{\partial u}{\partial t}(t=0) = b(x). \tag{2}$$

Equation (1) can describe processes which develop in time, since it is of the hyperbolic type if $1 + (\partial u/\partial x)^2 - (\partial u/\partial t)^2 > 0$. Show that the hyperbolic condition for (1) implies for the initial conditions that

$$1 + a'^2(x) - b^2(x) > 0. \tag{3}$$

Problem 10. Consider the *Korteweg-de Vries equation*

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

From (1) we can derive the iteration scheme

$$u^{(j+1)} = \frac{1}{6}\frac{\left(u_t^{(j)} + u_{xxx}^{(j)}\right)}{u_x^{(j)}}, \qquad j = 0, 1, 2, \ldots \tag{2}$$

where

$$u_x^{(j)} := \frac{\partial u^{(j)}}{\partial x}. \tag{3}$$

(i) Let

$$u^{(0)}(x,t) = \ln(x - ct). \tag{4}$$

Show that (4) converges within two steps to an exact solution

$$u(x,t) = -\frac{c}{6} + 2(x-ct)^{-2} \tag{5}$$

of the Korteweg-de Vries equation (1).
(ii) Show that

$$u^{(0)}(x,t) = (x-ct)^3 \tag{6}$$

also converges to the solution (5).
(iii) Let

$$u^{(0)}(x,t) = \cos(a(x-ct))^{-k}. \tag{7}$$

Show that within two steps of the iteration we arrive at

$$u(x,t) = \frac{4}{3}a^2 - \frac{c}{6} + 2a^2\tan^2(a(x-ct)). \tag{8}$$

(iv) Show that by demanding that $u \to 0$ for $|x| \to \infty$, provides $a = i\sqrt{c}/2$ and (5) becomes the well known soliton solution

$$u(x,t) = -\frac{c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right). \tag{9}$$

Problem 11. The *Zakharov-Kuznetsov equation* for ion acoustic waves and solitons propagating along a very strong external and uniform magnetic field is, for a two-component plasma

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(\Delta u) = 0 \tag{1}$$

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{2}$$

Here u is the normalized deviation of the ion density from the average. Exact solitonlike solutions exist in one, two and three space dimensions. They depend on the independent variables through the combination

$$x - ct, \qquad \rho := ((x-ct)^2 + y^2)^{1/2}, \quad r := ((x-ct)^2 + y^2 + z^2)^{1/2}, \tag{3}$$

respectively, where

$$\Delta u - \left(c - \frac{u}{2}\right)u = 0 \tag{4}$$

and

$$\Delta := \frac{\partial^2}{\partial x^2}, \qquad \Delta := \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right), \qquad \Delta := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) \tag{5}$$

for the three cases. (i) Show that for the one-dimensional case we obtain the soliton solution

$$u(x,t) = 3c\,\mathrm{sech}^2(c^{1/2}(x - ct - x_0)/2). \tag{6}$$

(ii) Show that the flat soliton (6) is unstable with respect to nonaligned perturbations.

Problem 12. Show that the *nonlinear Schrödinger equation*,

$$i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + 2\sigma |w|^2 w = 0, \qquad \sigma = \pm 1 \tag{1}$$

has one-zone solutions

$$w(x,t) = \sqrt{f(\theta(x,t))}\exp(i\varphi(x,t)), \qquad \varphi = \psi + h(\theta), \tag{2}$$

$$\theta(x,t) := kx - \omega t, \qquad \psi := \kappa x - \Omega t \tag{3}$$

where $f(\theta)$ and $h(\theta)$ are elliptic functions.

Problem 13. We consider the diffusion equation with nonlinear quadratic recombination

$$\frac{\partial u}{\partial t} = -\alpha u^2 + D_x \frac{\partial^2 u}{\partial x^2} + D_y \frac{\partial^2 u}{\partial y^2} + D_z \frac{\partial^2 u}{\partial z^2} \tag{1}$$

where D_x, D_y and D_z are constants. This equation is relevant for the evolution of plasmas or of charge carries in solids, of generating functions. Find the condition on c_1, c_2 such that

$$u(x,y,z,t) = \frac{6}{5\alpha}\frac{c_1}{x^2/D_x + y^2/D_y + z^2/D_z + c_2(t + t_0)} +$$

$$\frac{24}{\alpha}\frac{x^2/D_x + y^2/D_y + z^2/D_z}{(x^2/D_x + y^2/D_y + z^2/D_z + c_2(t - t_0))^2} \tag{2}$$

satisfies (1).

Problem 14. The Burgers equation determines the motion of a pressureless fluid subjected to dissipation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

(i) Show that any solution v of the diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \tag{2}$$

yields a solution of the Burgers equation via the *Hopf-Cole transformation*

$$u = -\frac{2}{v}\frac{\partial v}{\partial x}. \tag{3}$$

(ii) Solving (1) with respect to u we can introduce the iteration formula

$$u^{(j+1)} = \frac{1}{u_x^{(j)}}(u_{xx}^{(j)} - u_t^{(j)}), \qquad j = 0, 1, 2, \ldots \tag{4}$$

where $u_t \equiv \partial u/\partial t$ etc.. Show that for $u^{(0)} = v^n$ we find the sequence

$$u^{(1)} = (n-1)\frac{1}{v}\frac{\partial v}{\partial x} \tag{5}$$

$$u^{(2)} = -\frac{2}{v}\frac{\partial v}{\partial x} \qquad \text{fixed point} \tag{6}$$

Remark. Thus the Hopf-Cole transformation is an attractor (fixed point) of the iteration (4).

Problem 15. Consider the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right). \tag{1}$$

(i) Show that this equation admits a solution of the form

$$u(x,t) = \begin{cases} \dfrac{1}{x_1}\left(1 - \left(\dfrac{x}{x_1}\right)^2\right) & \text{for } t > 0, \left|\dfrac{x}{x_1}\right| < 1 \\ 0 & \text{for } t > 0, \left|\dfrac{x}{x_1}\right| > 1 \end{cases} \tag{2}$$

where

$$x_1 := (6t)^{1/3} \tag{3}$$

Remark. The solution exhibits a wave-like behaviour although it is not a wave of constant shape. The leading edge of this wave, that is where $u = 0$, is at $x = x_1$ and the speed of propagation is proportional to $t^{-2/3}$.

Problem 16. Show that the nonlinear partial differential equation (Fisher's equation)

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)u + u = u^2 \tag{1}$$

admits the travelling wave solution

$$u(x,t)=\frac{1}{4}\left(1-\tanh\left(\frac{1}{2\sqrt{6}}\left(x-\frac{5}{\sqrt{6}}vt\right)\right)\right)^2. \qquad (2)$$

Problem 17. Show that the coupled system of partial differential equations

$$i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}=nE, \qquad \frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial x^2}=\frac{\partial^2}{\partial x^2}(|E|^2) \qquad (1)$$

admits the solution

$$E(x,t)=E_0(x-st)\exp\left(\frac{is}{2}x+i\left(\lambda^2-\frac{s^2}{4}\right)t\right) \qquad (2a)$$

$$n(x,t)=-\frac{2\lambda^2}{\cosh^2\lambda(x-st-x_0)} \qquad (2b)$$

where

$$E_0(x-st)=\frac{\lambda\sqrt{2(1-s)^2}}{\cosh(\lambda(x-st-x_0))} \qquad (2c)$$

The soliton solution represents a moving one-dimensional plasma density well which *Langmuir oscillations* are locked.

Problem 18. The *Navier-Stokes equation* is given by

$$\frac{\partial \mathbf{u}}{\partial t}=-\mathbf{u}\cdot\nabla\mathbf{u}+\nu\nabla^2\mathbf{u}-\frac{1}{\rho}\nabla p$$

where $\mathbf{u}$ denotes the solenoidal

$$\nabla\cdot\mathbf{u}\equiv\mathrm{div}\mathbf{u}=\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}=0$$

flow velocity field, ν and ρ are the constant kinematic viscosity and fluid density. Here p is the fluid pressure. We have

$$\mathbf{u}\cdot\nabla\mathbf{u}:=\begin{pmatrix} u_1\frac{\partial u_1}{\partial x}+u_2\frac{\partial u_1}{\partial y}+u_3\frac{\partial u_1}{\partial z}\\ u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_2}{\partial y}+u_3\frac{\partial u_2}{\partial z}\\ u_1\frac{\partial u_3}{\partial x}+u_2\frac{\partial u_3}{\partial y}+u_3\frac{\partial u_3}{\partial z}\end{pmatrix}$$

Find the time evolution of

$$\mathbf{v}:=\mathrm{curl}\mathbf{u}\equiv\nabla\times\mathbf{u}=\begin{pmatrix}\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\\ \frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\\ \frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\end{pmatrix}.$$

Problem 19. Show that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{1}$$

can be linearized into

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \bar{x} \tag{2}$$

with

$$\bar{t}(x,t) = t, \qquad \bar{x}(x,t) = u(x,t), \qquad \bar{u}(\bar{x}(x,t),\bar{t}(x,t)) = x. \tag{3}$$

Problem 20. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = u^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Show that (1) linearizes under the transformation

$$\bar{x}(x,t) = \int_{-\infty}^{x} \frac{1}{u(s,t)} ds, \qquad \bar{t}(x,t) = t, \qquad \bar{u}(\bar{x}(x,t),\bar{t}(x,t)) = u(x,t). \tag{2}$$

It is assumed that u and all its spatial derivatives vanish at $-\infty$.

Problem 21. Show that the equation

$$\frac{\partial u}{\partial t} = u^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

is transformed under the *Cole-Hopf transformation*

$$\bar{u}(\bar{x}(x,t),\bar{t}(x,t)) = \frac{1}{u}\frac{\partial u}{\partial x}, \qquad \bar{t}(x,t) = t, \qquad \bar{x}(x,t) = x \tag{2}$$

into *Burgers equation*

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left(\bar{u}^2 + \frac{\partial \bar{u}}{\partial \bar{x}}\right). \tag{3}$$

Problem 22. Show that the the equation

$$\frac{\partial u}{\partial t} = u^2 \frac{\partial^2 u}{\partial x^2} + cu^2 \frac{\partial u}{\partial x} \tag{1}$$

is linearized by the transformation

$$\bar{x}(x,t) = \int_{-\infty}^{x} u(s,t)^{-1} ds, \qquad \bar{t}(x,t) = t, \qquad \bar{u}(\bar{x}(x,t),\bar{t}(x,t)) = u(x,t). \tag{2}$$

Problem 23. Show that the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \frac{3}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{3}{2}u\frac{\partial^2 u}{\partial x^2} + \frac{3}{4}u^2\frac{\partial u}{\partial x} \tag{1}$$

can be derived from the linear equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial t^3} \tag{2}$$

and the transformation

$$\phi(x,t) = \exp\left(\frac{1}{2}\int^x u(s,t)ds\right). \tag{3}$$

Problem 24. Using the transformation

$$\bar{x}(x,t) = \int_{-\infty}^{x} \frac{ds}{u(s,t)}, \qquad \bar{t}(x,t) = t, \qquad \bar{u}(\bar{x}(x,t),\bar{t}(x,t)) = u(x,t) \tag{1}$$

the equation

$$\frac{\partial u}{\partial t} = u^3\frac{\partial^3 u}{\partial x^3} \tag{2}$$

can be recast into the differential equation

$$\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial^3 \bar{u}}{\partial \bar{x}^3} - 3\frac{1}{\bar{u}^2}\frac{\partial \bar{u}}{\partial \bar{x}}\left(\frac{1}{2}\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 - \bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right) = 0. \tag{3}$$

(ii) Show that using the Cole-Hopf transformation

$$\bar{u}(\bar{x},\bar{t}) = \frac{1}{v(\bar{x},\bar{t})}\frac{\partial v(\bar{x},\bar{t})}{\partial \bar{x}} \tag{4}$$

(3) can further be reduced to the modified Korteweg-de Vries equation in v. Hint. We have

$$\frac{\partial \bar{x}}{\partial t} = -\int_{-\infty}^{x} \frac{1}{u^2(s,t)}\frac{\partial u(s,t)}{\partial t}ds = -\int_{-\infty}^{x} u(s,t)\frac{\partial^3 u(s,t)}{\partial s^3}ds = u\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right). \tag{5}$$

Problem 25. Show that the transformation

$$\bar{u}(\bar{x},\bar{t}) = \frac{\partial u}{\partial x}, \qquad \bar{x}(x,t) = u(x,t), \qquad \bar{t}(x,t) = t \tag{1}$$

transforms the nonlinear heat equation

$$\frac{\partial u}{\partial t} = A(u)\frac{\partial^2 u}{\partial x^2} \tag{2}$$

into

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \bar{u}^2 \frac{\partial}{\partial \bar{x}} \left(A(\bar{x}) \frac{\partial \bar{u}}{\partial \bar{x}} \right). \tag{3}$$

Problem 26. Consider the nonlinear partial differential equation

$$\Delta u - f(u)(\nabla u)^2 + \mathbf{a}(\mathbf{x},t)\nabla u + b(\mathbf{x},t)\frac{\partial u}{\partial t} = 0 \tag{1}$$

where ∇ is the gradient operator in the variables $x_1, \ldots, x_n$, $\Delta := \nabla\nabla$, $f(u)$ and $b(\mathbf{x},t)$ are given functions, and $\mathbf{a}(\mathbf{x},t)$ is a given n-dimensional vector. Show that the transformation

$$\int_{u_0}^{u(\mathbf{x},t)} \left(\exp\left(-\int_{u_0}^{s} f(z)dz \right) \right) ds - v(\mathbf{x},t) = 0. \tag{2}$$

reduces (1) to the linear partial differential equation

$$\Delta v + \mathbf{a}(\mathbf{x},t)\nabla v + b(\mathbf{x},t)\frac{\partial v}{\partial t} = 0. \tag{3}$$

Hint. From (2) we find

$$\frac{\partial v}{\partial t} = \exp\left(-\int_{u_0}^{u(\mathbf{x},t)} f(z)dz \right) \frac{\partial u}{\partial t} \tag{4a}$$

$$\frac{\partial v}{\partial x_1} = \exp\left(-\int_{u_0}^{u(\mathbf{x},t)} f(z)dz \right) \frac{\partial u}{\partial x_1} \tag{4b}$$

$$\frac{\partial^2 v}{\partial x_1^2} = \exp\left(-\int_{u_0}^{u(\mathbf{x},t)} f(z)dz \right) \frac{\partial^2 u}{\partial x_1^2} - \exp\left(-\int_{u_0}^{u(\mathbf{x},t)} f(z)dz \right) f(u) \left(\frac{\partial u}{\partial x_1} \right)^2 . \tag{4c}$$

Problem 27. Consider the two-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = \sin u$$

It may be regarded as describing solitary waves in a two-dimensional Josephson junction. The *Lamb substitution* is given by

$$u(x,y,t) = 4\tan^{-1}[M(x,y,t)]. \tag{1}$$

Find the equation for M.

Problem 28. We consider the nonlinear *d'Alembert equation*

$$\Box u = F(u) \tag{1}$$

where $u = u(\mathbf{x})$, $\mathbf{x} = (x_0, x_1, \ldots, x_n)$,

$$\Box := \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \tag{2}$$

and $F(u)$ is an arbitrary differentiable function.
(i) Consider the transformation

$$u(\mathbf{x}) = \Phi(w(\mathbf{x})) \tag{3}$$

where $w(\mathbf{x})$ and $\Phi(w)$ are new unknown functions. Show that (1) takes the form

$$\frac{d\Phi}{dw}\Box w + \frac{d^2\Phi}{dw^2} w_\mu w^\mu = F(\Phi) \tag{4}$$

where

$$w_\mu w^\mu := \left(\frac{\partial w}{\partial x_0}\right)^2 - \left(\frac{\partial w}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial w}{\partial x_n}\right)^2. \tag{5}$$

(ii) Show that (4) is equivalent to the following equation

$$\frac{d\Phi}{dw}\left(\Box w - \lambda \frac{\dot{P}_n}{P_n}\right) + \frac{d^2\Phi}{dw^2}(w_\mu w^\mu - \lambda) + \lambda\left(\frac{d^2\Phi}{dw^2} + \frac{d\Phi}{dw}\frac{\dot{P}_n}{P_n}\right) - F(\Phi) = 0 \tag{6}$$

where $P_n(w)$ is an arbitrary polynomial of degree n in w, and $\lambda = -1, 0, 1$. Moreover $\dot{P}_n \equiv dP_n/dw$.
(iii) Assume that Φ satisfies

$$\lambda\left(\frac{d^2\Phi}{dw^2} + \frac{d\Phi}{dw}\frac{\dot{P}_n}{P_n}\right) = F(\Phi). \tag{7}$$

Show that (6) takes the form

$$\frac{d\Phi}{dw}\left(\Box w - \lambda \frac{\dot{P}_n}{P_n}\right) + \frac{d^2\Phi}{dw^2}(w_\mu w^\mu - \lambda) = 0. \tag{8}$$

(iv) Show that a solution of the system

$$\Box w = \lambda \frac{\dot{P}_n}{P_n}, \qquad w_\mu w^\mu = \lambda \tag{9}$$

is also a solution of (8), and in this way we obtain a solution of (1) provided Φ satisfies (7).

Problem 29. Consider the nonlinear wave equation in one space dimension

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u + u^3 = 0. \tag{1}$$

(i) Show that (1) can be derived from the Lagrangian density

$$\mathcal{L}(u_t, u_x, u) = \frac{1}{2}(u_t^2 - u_x^2) - g(u) \tag{2}$$

with $g(u)$ the potential function

$$g(u) = \frac{1}{4}(u^2-1)^2 \equiv \frac{1}{4}(u^4 - 2u^2 + 1). \tag{3}$$

(ii) **Definition.** A *conservation law* associated to (1) is an expression of the form

$$\frac{\partial T(u(x,t))}{\partial t} + \frac{\partial X(u(x,t))}{\partial x} = 0 \tag{4}$$

where T is the *conserved density* and X the *conserved flux*. T and X are functionals of u and its derivatives.
(ii) Show that the quantities

$$T_1 := \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) + \frac{1}{4}(u^2-1)^2 \geq 0, \qquad T_2 := -\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} \tag{5}$$

are conserved densities of (1), provided that $u^2 - 1$, $\partial u/\partial t$, and $\partial u/\partial x$ tend to zero sufficiently fast as $|x| \to +\infty$.

Hint. The corresponding fluxes are

$$X_1 = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}, \qquad X_2 = \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) - \frac{1}{4}(u^2-1)^2. \tag{6}$$

T_1 and T_2 correspond to the energy and momentum densities, respectively.

The *Euler-Lagrange equation* is given by

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0 \tag{7}$$

where $\mathcal{L}$ is the Lagrange density (2).

Problem 30. Consider the nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u + u^3 = 0. \tag{1}$$

(i) Show that (1) admits the solution

$$u_K(x-vt) = \tanh((\gamma/\sqrt{2})(x-vt-x_0)) = -u_{\bar{K}}(x-vt) \tag{2}$$

where

$$\gamma^2 \equiv (1-v^2)^{-1} \tag{3}$$

and x_0 is a constant. These solutions are of *kink* (K) and *antikink* ($\bar{K}$) type traveling at constant velocity v. These solutions do not tend to zero at infinity, but they do connect two minimum states of the potential $\frac{1}{4}(u^2-1)^2$.
(ii) The energy and momentum densities T_1 and T_2 for the kink and antikink are obtained by substituting (2) into (5) of problem 1. Show that

$$T_1(x,t) = \frac{\gamma^2}{2}\cosh^{-4}((\gamma/\sqrt{2})(x-vt-x_0)) \tag{4}$$

$$T_2(x,t) = \frac{\gamma^2 v}{2}\cosh^{-4}((\gamma/\sqrt{2})(x-vt-x_0)). \tag{5}$$

(iii) Show that by integrating over x we find the energy and the momentum

$$E := \int_{-\infty}^{+\infty} T_1 dx = \frac{4}{3}\frac{\gamma}{\sqrt{2}}, \qquad P := \int_{-\infty}^{+\infty} T_2 dx = \frac{4}{3}\frac{\gamma v}{\sqrt{2}}. \tag{6}$$

(iv) Show that the densities T_1 and T_2 are localized in space, in contrast with u_K and $u_{\bar{K}}$.
(v) We associate the mass

$$M^2 := E^2 - P^2 \tag{7}$$

to (1) and the *energy center*

$$X_c := \frac{\displaystyle\int_{-\infty}^{+\infty} xT_1 dx}{\displaystyle\int_{-\infty}^{+\infty} T_1 dx}. \tag{8}$$

Show that the kink and antikink solutions of (6) take the values

$$M^2 = \frac{8}{9}, \qquad X_c = vt + x_0. \tag{9}$$

Problem 31. We say that a partial differential equation described by the field $u(x,y)$ is *hodograph invariant* if it does not change its form by the *hodograph transformations*

$$\bar{x}(x,y) = u(x,y), \qquad \bar{y}(x,y) = y, \qquad \bar{u}(\bar{x}(x,y), \bar{y}(x,y)) = x. \tag{1}$$

Show that the *Monge-Ampere equation* for the surface $u(x,y)$ with a constant total curvature K

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = K\left(1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) \tag{2}$$

is hodograph invariant.

Problem 32. Show that the nonlinear equation of the *Born-Infeld type* for a scalar field u is obtained by varying the Lagrangian

$$\mathcal{L}(u_x, u_y, u_z, u_t) = 1 - (1 + u_x^2 + u_y^2 + u_z^2 - u_t^2)^{1/2} \tag{1}$$

and has the following form

$$\left(1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 - \left(\frac{\partial u}{\partial t}\right)^2\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2}\right)$$

$$-\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial t}\right)^2\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 u}{\partial x \partial y}$$

$$-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial z} - 2\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t \partial x} + 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial t \partial y} + 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial t \partial z} = 0. \tag{2}$$

Problem 33. The *sine-Gordon equation* is the equation of motion for a theory of a single, dimensionless scalar field u, in one space and one time dimension, whose dynamics is determined by the Lagrangian density

$$\mathcal{L}(u_t, u_x, u) = \frac{1}{2}(u_t^2 - c^2 u_x^2) + \frac{m^4}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{m}u\right) - \mu. \tag{1}$$

Here c is a limiting velocity while m, λ, and μ are real parameters. u_t and u_x are the partial derivatives of u with respect to t and x, respectively. In the terminology of quantum field theory, m is the mass associated with the normal modes of the linearized theory, while λ/m^2 is a dimensionless, coupling constant that measures the strength of the interaction between these normal modes. In classical theory m is proportional to the characteristic frequency of these normal modes. Let

$$x \to \frac{x}{m}, \qquad t \to \frac{t}{m}, \qquad u \to \frac{mu}{\sqrt{\lambda}} \tag{2}$$

and set $c = 1$. (i) Show that then the Lagrangian density becomes

$$\mathcal{L}(u_t, u_x, u) = \frac{m^4}{2\lambda}((u_t^2 - u_x^2) + 2\cos u) - \mu \tag{3}$$

with the corresponding Hamiltonian density being given by

$$\mathcal{H} = \frac{m^4}{2\lambda}(u_t^2 + u_x^2 - 2\cos u) + \mu. \tag{4}$$

(ii) Show that by choosing

$$\mu = m^4/\lambda \tag{5}$$

the minimum energy of the theory is made zero and (4) can be written as

$$\mathcal{H} = \frac{m^4}{2\lambda}(u_t^2 + u_x^2 + 2(1 - \cos u)). \tag{6}$$

Problem 34. Show that the following theorem holds. The *conservation law*

$$\frac{\partial}{\partial t}(T(\partial u/\partial x, \partial u/\partial t, u)) + \frac{\partial}{\partial x}(F(\partial u/\partial x, \partial u/\partial t, u)) = 0 \tag{1}$$

is transformed to the reciprocally associated conservation law

$$\frac{\partial}{\partial t'}(T'(\partial u/\partial x', \partial u/\partial t', u)\} + \frac{\partial}{\partial x'}(F'(\partial u/\partial x', \partial u/\partial t', u)\} = 0 \tag{2}$$

by the reciprocal transformation

$$dx'(x,t) = Tdx - Fdt, \qquad t'(x,t) = t, \tag{3}$$

$$T'(\partial u/\partial x', \partial u/\partial t', u) = \frac{1}{T(\partial u/\partial x, \partial u/\partial t, u)} \tag{4a}$$

$$F'(\partial u/\partial x', \partial/\partial t', u) = \frac{-F(\partial u/\partial x, \partial u/\partial t, u)}{T(\partial/\partial x; \partial/\partial t; u)}. \tag{4b}$$

(ii) Show that

$$\frac{\partial}{\partial x} = \frac{1}{T'}\frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial t} = \frac{F'}{T'}\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}. \tag{5}$$

Problem 35. Consider the coupled *two-dimensional nonlinear Schrödinger equation*

$$i\frac{\partial u}{\partial t} - \beta\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^2 u}{\partial y^2} - \delta u^* u u - 2wu = 0 \tag{1a}$$

$$\beta\frac{\partial^2 w}{\partial x^2} + \gamma\frac{\partial^2 w}{\partial y^2} + \beta\delta\frac{\partial^2}{\partial x^2}(u^* u) = 0 \tag{1b}$$

where β, γ, δ are arbitrary constants. Consider the following transformation

$$\bar{x}(x,y,t) = \frac{x}{t}, \qquad \bar{y}(x,y,t) = \frac{y}{t}, \qquad \bar{t}(x,y,t) = -\frac{1}{t}, \tag{2a}$$

$$u(x,y,t) = \frac{1}{t}\exp\left(\frac{-ix^2}{4\beta t} + \frac{iy^2}{4\gamma t}\right)\bar{u}(\bar{x}(x,y,t),\bar{y}(x,y,t),\bar{t}(x,y,t)) \qquad (2b)$$

$$w(x,y,t) = \frac{1}{t^2}w(\bar{x}(x,y,t),\bar{y}(x,y,t),\bar{t}(x,y,t)). \qquad (2c)$$

Show that $\bar{u}$ and $\bar{w}$ satisfy the same equation with the subscripts x, y, t replaced by $\bar{x}$, $\bar{y}$, $\bar{t}$.

Problem 36. An simplified analog of the *Boltzmann equation* is constructed as follows. It is one-dimensional and the velocities of the molecules are allowed to take two discrete values, $\pm c$, only. Thus the distribution function in the Boltzmann equation, $f(x,v,t)$, is replaced by two functions $u_+(x,t)$ and $u_-(x,t)$ denoting the density of particles with velocity $+c$ or $-c$, repsectively, at point x and time t. The gas is not confined, but x varies over all points of the real line $(-\infty,\infty)$. It is further assumed that there are only two types of interaction, viz., two + particles go over into two − particles, and vice versa, the probalitity for both processes to occur within one unit of time being the same number σ. Then the Boltzmann equation, which in absence of external forces reads

$$\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\nabla f(\mathbf{x},\mathbf{v},t) = \int d^3\mathbf{v}_1 d\Omega |\mathbf{v}-\mathbf{v}_1|\sigma(|\mathbf{v}-\mathbf{v}_1|,\theta)[f'f_1' - ff_1] \quad (1)$$

where f' is the final distribution, i.e., the distribution after a collision. Under the assumption described above (1) translates into a system of two equations

$$\frac{\partial u_+}{\partial t} + c\frac{\partial u_+}{\partial x} = \sigma(u_-^2 - u_+^2), \qquad \frac{\partial u_-}{\partial t} - c\frac{\partial u_-}{\partial x} = \sigma(u_+^2 - u_-^2) \qquad (2)$$

where c and σ are positive constants. This model is called the *Carleman model*. The Carleman model is rather unphysical. However with its aid one can prove almost all those results which one would like to obtain for the Boltzmann equation itself – as, for instance, the existence of solutions for a wide class of initial conditions or a rigorous treatment of the hydrodynamic limit. (i) Show that as for the Boltzmann equation, the *H theorem* holds for the Caleman model: The quantity

$$-\int (u_+(x,t)\ln u_+(x,t) + u_-(x,t)\ln u_-(x,t))dx \qquad (3)$$

never decreases in time. (ii) Show that there exists the following generalizations of the H theorem. Let f be concave function, defined on the half-line $(0,\infty)$ which is once continuously differentiable. Let

$$S_f := \int (f(u_+) + f(u_-))dx \qquad (4)$$

Show that

$$\frac{d}{dt} S_f(u) \geq 0. \tag{5}$$

Thus, not only does entropy never decrease, but the same is true for all quasientropies.
(iii) Show that as a consequence, all *Renyi entropies*

$$S_\alpha := (1-\alpha)^{-1} \ln \int (u_+^\alpha(x,t) + u_-^\alpha(x,t)) dx \tag{6}$$

never decrease. In information-theretical language, this means that all sensible measures of the lack of information are nondecreasing. In other words, information is lost, or chaos is approached, in the strongest possible way.

Problem 37. For the functions $u, v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ we consider the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = v^2 - u^2, \qquad \frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = u^2 - v^2 \tag{1a}$$

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x). \tag{1b}$$

This is the *Carleman model* introduced above.
(i) Define

$$S := u + v, \qquad D := u - v \tag{2}$$

Show that S and D satisfy the system of partial differential equations

$$\frac{\partial S}{\partial t} + \frac{\partial D}{\partial x} = 0, \qquad \frac{\partial D}{\partial t} + \frac{\partial S}{\partial x} = -2DS \tag{3}$$

and the conditions $u \geq 0$, $v \geq 0$ take the form

$$S \geq 0 \quad \text{and} \quad S^2 - D^2 \geq 0. \tag{4}$$

(ii) Find explict solutions assuming that S and D are conjugate harmonic functions.

Problem 38. The Carleman model as an approximation of the Boltzmann equation is given by the non-linear equations

$$\frac{\partial u_+}{\partial t} + c\frac{\partial u_+}{\partial x} = \sigma(u_-^2 - u_+^2), \tag{1a}$$

$$\frac{\partial u_-}{\partial t} - c\frac{\partial u_-}{\partial x} = \sigma(u_+^2 - u_-^2). \tag{1b}$$

(i) Show that if $u_\pm(x,0) \geq 0$, then $u_\pm(x,t) \geq 0$ for all $t > 0$.

(ii) Show that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}}(u_+ + u_-)dx = 0. \tag{2}$$

The two properties have to hold, of course, in any model that is considered to be of some physical relevance.
(iii) Let

$$H \equiv \int_{\mathbb{R}}(u_+ \ln u_+ + u_- \ln u_-)dx \tag{3}$$

Show that the analogue of Boltzmann's H-function is given by

$$\frac{dH}{dt} \leq 0 \tag{4}$$

(iv) Show that

$$\frac{\partial}{\partial t}(u_+ \ln u_+ + u_- \ln u_-) + c\frac{\partial}{\partial x}(u_+ \ln u_+ - u_- \ln u_-) \leq 0. \tag{5}$$

$u_+ \ln u_+ + u_- \ln u_-$ has to be interpreted as the negative *entropy density* and $c(u_+ \ln u_+ - u_- \ln u_-)$ as the negative *entropy flux*.
(v) We define

$$S_f := \int (f(u_+) + f(u_-))dx \tag{6}$$

where f is a concave (or convex, respectively) function. In the special case

$$f(s) = -s \ln s$$

S_f is the expression for entropy i.e., $-H$. Show that

$$\frac{dS_f}{dt} \leq 0$$

if f is convex. Show that

$$\frac{dS_f}{dt} \geq 0$$

if f is concave.

Problem 39. The *Broadwell model* can then be written as

$$\frac{\partial f_+}{\partial t} + \frac{\partial f_+}{\partial x} = \frac{1}{\epsilon}(f_0^2 - f_+ f_-) \tag{1a}$$

$$\frac{\partial f_0}{\partial t} = \frac{1}{\epsilon}(f_+ f_- - f_0^2) \tag{1b}$$

$$\frac{\partial f_-}{\partial t} - \frac{\partial f_-}{\partial x} = \frac{1}{\epsilon}(f_0^2 - f_+ f_-). \tag{1c}$$

The parameter ϵ can qualitatively be understood as the mean free path. This system of equations serves as a model for the *Boltzmann equation*. The limit $\epsilon \to 0$ corresponds to a vanishing mean free parth and the fluid regime, while $\epsilon \to \infty$ approaches free molecular flow. The locally conserved spatial densities

$$\rho := f_+ + 2f_0 + f_-, \qquad \rho u := f_+ - f_- \tag{2}$$

corresponding to mass and x momentum. Show that these are governed by the local conservation laws

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \qquad \frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}(f_+ + f_-) = 0. \tag{3}$$

Problem 40. Consider the hyperbolic system of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = 0 \tag{1}$$

where $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable. A convex function $\eta(\mathbf{u})$ is called an entropy for (1) with entropy flux $q(\mathbf{u})$ if

$$\frac{\partial}{\partial t}\eta(\mathbf{u}) + \frac{\partial}{\partial x}q(\mathbf{u}) = 0 \tag{2}$$

holds identically for any smooth vector field $\mathbf{u}(x,t)$ which staisfies (1).
(i) Show that (2) follows from (1) if

$$\sum_{j=1}^{m} \frac{\partial \eta}{\partial u_j}\frac{\partial f_j}{\partial u_k} = \frac{\partial q}{\partial u_k}, \qquad k = 1, \ldots, m. \tag{3}$$

(ii) Show that for $m = 1$, every convex function $\eta(n)$ is an entropy for (1) with entropy flux

$$q(u) = \int_0^u \eta'(\omega) df(\omega). \tag{4}$$

Problem 41. Consider the so-called *sine-Hilbert equation*

$$H\left(\frac{\partial u}{\partial t}\right) = -\sin u \tag{1}$$

where the integral operator H is defined by

$$Hu(x,t) := \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x} dy. \tag{2}$$

This is the so-called *Hilbert transform.* P denotes the *Cauchy principal value.* Let f be a continuous function, except at the singularity c. Then the Cauchy principal value is defined by

$$P\int_{-\infty}^{\infty} f(x) := \lim_{\epsilon\to 0}\left(\int_{-\infty}^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^{\infty} f(x)dx\right). \tag{3}$$

The Cauchy principal value can be found by applying the *residue theorem.* Let

$$u(x,t) := i\ln\left(\frac{f^*(x,t)}{f(x,t)}\right) \tag{4}$$

where

$$f(x,t) := \prod_{j=1}^{N}(x - x_j(t)) \tag{5a}$$

$$\Im x_j(t) > 0 \qquad j = 1,2,\ldots,N, \qquad x_n \neq x_m \quad \text{for } n \neq m. \tag{5b}$$

Here $x_j(t)$ are complex functions of t and * denotes complex conjugation.
(i) Show that

$$H\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\ln(f^* f)\right) \tag{6}$$

follows from (2), (4) and (5).
(ii) Show that (1) is transformed into the form

$$\sum_{j=1}^{N}\frac{1}{x - x_j(t)}\frac{dx_j}{dt} + \sum_{j=1}^{N}\frac{1}{x - x_j^*(t)}\frac{dx_j^*}{dt} = \frac{1}{2i}\left(\frac{\prod_{j=1}^{N}(x - x_j^*(t))}{\prod_{j=1}^{N}(x - x_j(t))} - \frac{\prod_{j=1}^{N}(x - x_j(t))}{\prod_{j=1}^{N}(x - x_j^*(t))}\right). \tag{7}$$

(iii) Show that (7) is equivalent to the equation

$$\frac{\partial}{\partial t}(f^* f) = \frac{1}{2i}(f^2 - f^{*2}) = \Im(f^2). \tag{8}$$

(iv) Show that by multiplying both sides of (7) by $x - x_n(t)$ and then putting $x = x_n$, we obtain

$$\frac{dx_n}{dt} = \frac{1}{2i}\frac{\prod_{j=1}^{N}(x_n(t) - x_j^*(t))}{\prod_{j=1(j\neq n)}^{N}(x_n(t) - x_j(t))} \qquad n = 1,2,\ldots,N. \tag{9}$$

Problem 42. Consider the *Fitzhugh-Nagumo equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a) \tag{1}$$

where a is a constant. Without loss of generality we can set $-1 \leq a < 1$. Insert the ansatz

$$u(x,t) = f(x,t)w(z(x,t)) + g(x,t) \tag{2}$$

into (1) and require that $w(z)$ satisfies an ordinary differential equation. This is the so-called *direct method* and z is the so-called *reduced variable.*

Problem 43. Consider the *nonlinear Dirac equations* of the form

$$i\sum_{\mu=0}^{3}\gamma^{\mu}\frac{\partial}{\partial x_{\mu}}\psi - M\psi + F(\bar{\psi}\psi)\psi = 0. \tag{1}$$

The notation is the following

$$\psi : \mathbb{R}^4 \to \mathbb{C}^4, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{2}$$

M is a positive constant,

$$\bar{\psi}\psi := (\gamma^0\psi, \psi) \equiv \psi_1^*\psi_1 + \psi_2^*\psi_2 - \psi_3^*\psi_3 - \psi_4^*\psi_4 \tag{3}$$

where $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{C}^4$ and the γ^{μ}'s are the 4×4 matrices of the Pauli-Dirac representation, given by

$$\gamma^0 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^k := \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \quad \text{for} \quad k = 1,2,3 \tag{4}$$

where the *Pauli matrices* are given by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

and $F : \mathbb{R} \to \mathbb{R}$ models the nonlinear interaction. (i) Consider the ansatz (*standing waves*, *stationary states*)

$$\psi(t, \mathbf{x}) = e^{i\omega t}\mathbf{u}(\mathbf{x}) \tag{6}$$

where $x_0 = t$ and $\mathbf{x} = (x_1, x_2, x_3)$. Show that $\mathbf{u} : \mathbb{R}^3 \to \mathbb{C}^4$ satisfies the equation

$$i\sum_{k=1}^{3}\gamma^k\frac{\partial}{\partial x_k}\mathbf{u} - M\mathbf{u} + \omega\gamma^0\mathbf{u} + F(\bar{\mathbf{u}}\mathbf{u})\mathbf{u} = \mathbf{0}. \tag{7}$$

(ii) Let

$$\mathbf{u}(\mathbf{x}) := \begin{pmatrix} v(r)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ iw(r)\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} \end{pmatrix}. \tag{8}$$

Here $r = |\mathbf{x}|$ and (θ, ϕ) are the angular parameters. Show that w and v satisfy the nonautonomous planar dynamical system

$$\frac{dw}{dr} + \frac{2w}{r} = v(F(v^2 - w^2) - (M - \omega)) \tag{9a}$$

$$\frac{dv}{dr} = w(F(v^2 - w^2) - (M + \omega)). \tag{9b}$$

(ii) Hint. Notice that

$$\bar{\mathbf{u}}\mathbf{u} = u_1^* u_1 + u_2^* u_2 - u_3^* u_3 - u_4^* u_4 \tag{10}$$

Inserting (8) yields

$$\bar{\mathbf{u}}\mathbf{u} = v^2 - w^2. \tag{11}$$

Problem 44. Consider a one-dimensional system to describe the *electron-beam plasma system*

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e(u_e + V)) = 0 \tag{1a}$$

$$\frac{\partial u_e}{\partial t} + (u_e + V)\frac{\partial u_e}{\partial x} = -\frac{e}{m_e}\mathcal{E} - \frac{1}{m_e n_e}\frac{\partial p_e}{\partial x} \tag{1b}$$

$$\frac{\partial \mathcal{E}}{\partial t} + (u_e + V)\frac{\partial \mathcal{E}}{\partial x} = 4\pi e n_i (u_e + V). \tag{1c}$$

where n_e is the density of the beam-electron fluids, m_e the beam-electron mass, u_e the bulk fluid velocity, $p = K_b T_e n_e$ the particle fluid pressure, e the charge on an electron and $\mathcal{E} = -\nabla\phi$. We study the simplest case of uniform plasma in the absence of an external electromagnetic field, and assume that the positive ions are taken to form a fixed, neutralizing background of uniform density $n_i = N_0 = \text{const}$ throughout the present analysis. The electrons move with a beam drift velocity V corresponding to the ions. Assume that electrostatic perturbation is sinusoidal

$$n_e(x,t) = N_0\left(1 + \frac{n(t)}{N_0}(\sin(kx) + \cos(kx))\right), \qquad n(t) \le \frac{1}{2}N_0 \tag{2a}$$

$$u_e(x,t) = u(t)(\cos(kx) - \sin(2kx)), \qquad \mathcal{E}(x,t) = E(t)(\cos(kx) - \sin(kx)). \tag{2b}$$

Define

$$X(t) := n(t), \quad Y(t) := \frac{1}{2}ku(t), \quad Z(t) := \frac{e}{m_e p}\frac{k}{2}E(t), \quad q := \frac{K_b T_e}{m_e N_0}\frac{k^2}{2}, \quad p := \frac{4\pi e^2}{m_e}. \tag{3}$$

Show that when (2) and (3) are substituted into (1a)-(1c), we obtain

$$\frac{dX}{dt} = kVX - XY + 2N_0Y$$
$$\frac{dY}{dt} = -qX + Y^2 - pZ$$
$$\frac{dZ}{dt} = N_0Y - YZ + kVZ.$$

Problem 45. Consider the partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta\frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0 \qquad (1)$$

subject to periodic boundary conditions in the interval $[0, L]$, with initial conditions $u(x,0) = u_0(x)$. We only consider solutions with zero spatial average. We recall that for $L \leq 2\pi$ all initial conditions evolve into $u(x,t) = 0$. We expand the solution for u in the Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t)\exp(ik_nx) \qquad (2)$$

where $k_n := 2n\pi/L$ and the expansion coefficients satisfy

$$a_{-n}(t) = \bar{a}_n(t). \qquad (3)$$

Here $\bar{a}$ denotes the complex conjugate of a. Since we choose solutions with zero average we have $a_0 = 0$. (i) Show that inserting the series expansion (2) into (1) we obtain the following system for the time evolution of the Fourier amplitudes

$$\frac{da_n}{dt} + (k_n^4 - k_n^2 - i\delta k_n^3)a_n + \frac{1}{2}ik_n\sum_{m=0}^{\infty}(a_ma_{n-m} + \bar{a}_ma_{n+m}) = 0. \qquad (4)$$

(ii) Show that keeping only the first five modes we obtain the system

$$\frac{da_1}{dt} + (\mu_1 - i\delta k^3)a_1 + ik(\bar{a}_1a_2 + \bar{a}_2a_3 + \bar{a}_3a_4 + \bar{a}_4a_5) = 0 \qquad (5a)$$

$$\frac{da_2}{dt} + (\mu_2 - 8i\delta k^3)a_2 + ik(a_1^2 + 2\bar{a}_1a_3 + 2\bar{a}_2a_4 + 2\bar{a}_3a_5) = 0 \qquad (5b)$$

$$\frac{da_3}{dt} + (\mu_3 - 27i\delta k^3)a_3 + 3ik(a_1a_2 + \bar{a}_1a_4 + \bar{a}_2a_5) = 0 \qquad (5c)$$

$$\frac{da_4}{dt} + (\mu_4 - 64i\delta k^3)a_4 + 2ik(a_2^2 + 2a_1a_3 + 2\bar{a}_1a_5) = 0 \qquad (5d)$$

$$\frac{da_5}{dt} + (\mu_5 - 125i\delta k^3)a_5 + 5ik(a_1a_4 + a_2a_3) = 0 \tag{5e}$$

where

$$k := 2\frac{\pi}{L}, \qquad \mu_n := k_n^4 - k_n^2. \tag{6}$$

Problem 46. The modified *Boussinesq-Oberbeck equations* are given by

$$\frac{\partial \triangle \psi}{\partial t} = \sigma \frac{\partial \theta}{\partial x} + \sigma \triangle (\triangle \psi) - \left(\frac{\partial \psi}{\partial x} \frac{\partial \triangle \psi}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \triangle \psi}{\partial x} \right)$$

$$\frac{\partial \theta}{\partial t} = R\frac{\partial \psi}{\partial x} + \triangle\theta - \left(\frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} \right) + -\frac{\bar{\epsilon}}{T_i}\left[-2R\frac{\partial \theta}{\partial z} + \left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 \right] \tag{1}$$

where σ is the Prandtl number, R the Rayleigh number, ψ the stream function, and θ a function measuring the difference between the profile of temperature and a profile linearly decreasing with height and time, namely

$$\theta = T - T_0 + \frac{\bar{\epsilon}}{T_i} F^2 t + Rz. \tag{2}$$

Let L be the horizontal extenion of the convection cells, $l = H/L$ and $R_c = \pi^4(1+l^2)^3/l^2$ the critical Rayleigh number for the onset of convection. One sets $r = R/R_c$. Suppose that the temperature T_1 is fixed and that T_0 increases. We define the dimensionless parameter

$$\alpha := \frac{T_0 - T_1}{T_1} \frac{1}{r}. \tag{3}$$

This quantity is a constant once the fluid and the experimental setting are chosen. The dimensionless temperatures appearing in the last term of the second of equations (1) are now expressed by

$$T_0 = R_c \frac{(\alpha r + 1)}{\alpha}, \qquad T_1 = \frac{R_c}{\alpha}. \tag{4}$$

The boundary conditions are

$$\psi = \triangle\psi = 0 \quad \text{at } z = 0,\ H$$

$$\theta = 0, \quad \text{at } z = 0,\ H$$

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{at} \quad x = kL, \quad k \in \mathbb{Z} \tag{5}$$

Show that the equation

$$\frac{dX}{dt} = -\sigma X + \sigma Y, \qquad \frac{dY}{dt} = rX - Y - XZ + \epsilon W \left(1 + \frac{Z}{r}\right)$$

$$\frac{dW}{dt} = -W - \epsilon Y\left(1 + \frac{Z}{r}\right), \qquad \frac{dZ}{dt} = -bZ + XY \tag{6}$$

can be derived from (1) and the boundary condition (5) via a Fourier expansion of the function $\psi(x, z, t)$ and $\theta(x, z, t)$.

Problem 47. Consider the complex *Ginzburg-Landau equation*

$$\frac{\partial w}{\partial t} = (1 + ic_1)\frac{\partial^2 w}{\partial x^2} + w - (a + ic_2)|w|^2 w. \tag{1}$$

(i) Show that setting $a = -1$ in (1), and writing

$$w(x, t) = R(x, t)\exp(i\Theta(x, t)) \tag{2}$$

one obtains two real equations which, after suitable linear combination and division by c_1^2, can be written as

$$\epsilon^2\frac{\partial R}{\partial t} + \epsilon R\frac{\partial \Theta}{\partial t} = \left((1 + \epsilon^2)\left(\frac{\partial^2}{\partial x^2} - \left(\frac{\partial \Theta}{\partial x}\right)^2\right) + \epsilon^2 + (\beta + \epsilon^2)R^2\right)R \quad (3a)$$

$$-\frac{1}{2}\epsilon\frac{\partial}{\partial t}R^2 + \epsilon^2 R^2\frac{\partial \Theta}{\partial t} = (1 + \epsilon^2)\frac{\partial}{\partial x}\left(r^2\frac{\partial \Theta}{\partial x}\right) - \epsilon(1 + (1 - \beta)R^2)R^2. \quad (3b)$$

Here we have introduced $c_2 := -\beta c_1$ and $\epsilon := 1/c_1$. (ii) Make an expansion in ϵ of the form

$$R := R_0 + \epsilon^2 R_2 + \cdots, \qquad \Theta := \epsilon^{-1}(\Theta_{-1} + \epsilon^2\Theta_1 + \cdots) \tag{4}$$

Problem 48. Consider the cubic *nonlinear one-dimensional Schrödinger equation*,

$$i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + Qw|w|^2 = 0 \tag{1}$$

where Q is a constant. (i) Show that a discretization with the *periodic boundary conditions* $w_{j+N} \equiv w_j$ is given by

$$i\frac{dw_j}{dt} + \frac{w_{j+1} + w_{j-1} - 2w_j}{h^2} + Q|w_j|^2 w_j^{(k)} = 0, \qquad k = 1, 2, \ldots \tag{2}$$

where

$$\text{(a)} \qquad w_j^{(1)} := w_j \qquad \text{and} \qquad \text{(b)} \quad w_j^{(2)} := \frac{1}{2}(w_{j+1} + w_{j-1}). \tag{3}$$

(ii) Show that both schemes are of second-order accuracy. (iii) Show that in case (2a) there are first integrals, the L^2 norm,

$$I := \sum_{j=0}^{N-1} |w_j|^2 \tag{4}$$

and the Hamilton function

$$H = -i \sum_{j=0}^{N-1} \left(\frac{|w_{j+1} - w_j|^2}{h^2} - \frac{1}{2} Q |w_j|^4 \right). \tag{5}$$

The Poisson brackets are the standard ones. Thus when $N = 2$ the system is integrable. This system has been used as a model for a nonlinear dimer.
(iv) Show that the Hamilton function of scheme (2b) is given by ($h = 1$)

$$H = -i \sum_{j=0}^{N-1} (w_j^*(w_{j-1} + w_{j+1}) - \frac{4}{Q} \ln(1 + \frac{1}{2} Q w_j w_j^*) \tag{6}$$

together with the nonstandard Poisson brackets

$$\{q_m, p_n\} := (1 + \frac{1}{2} Q q_n p_n) \delta_{m,n} \tag{7}$$

and

$$\{q_m, q_n\} = \{p_m, p_n\} = 0. \tag{8}$$

Problem 49. Consider the *sine-Gordon equation* in $3 + 1$ dimensions

$$\Box \chi = \sin \chi \tag{1}$$

where

$$\Box := \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} - \frac{\partial}{\partial t^2} \tag{2}$$

and $\chi(x, y, z, t)$ is a real valued scalar field. The sinh-Gordon equation is given by

$$\Box \chi = \sinh \chi \tag{3}$$

Let

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

be the *Pauli spin matrices* and

$$a := \sigma_0 \frac{\partial}{\partial x} + i\sigma_1 \frac{\partial}{\partial y} + i\sigma_3 \frac{\partial}{\partial z} + \sigma_2 \frac{\partial}{\partial t}. \tag{5}$$

Let $\bar{a}$ denote complex conjugates. Let $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$, $-\infty < \lambda < \infty$ and $-\infty < \tau < \infty$ be arbitrary parameters. We set

$$U := \exp(i\theta\sigma_1 \exp(-i\phi\sigma_2 e^{-\tau\sigma_1})). \tag{6}$$

Assume that α and β are solutions of (1) and (2), respectively. Let

$$u := \alpha - i\beta. \tag{7}$$

The Bäcklund transformation $\hat{B}$ is then given by

$$\alpha \longrightarrow i\beta = \hat{B}(\phi, \theta, \tau)\alpha \tag{8}$$

where $\hat{B}(\phi, \theta, \tau)$ is the Bäcklund transformation operator. The functions α and β are related by

$$\frac{1}{2}au = \sin\left(\frac{1}{2}\bar{u}\right)U. \tag{9}$$

The Bäcklund transformation works as follows. Let α (respectively β) be a solution of (1) (respectively 3)) then solve (9) for β (respectively α). The solution then solves (3) (respectively (1)).
(i) Show for any U such that

$$\bar{U} = U^{-1} \tag{10}$$

all solutions of (9) must be of the form (i.e. plane travelling wave with speed v less than one)

$$u(x,y,z,t) = f(\eta), \qquad \eta := kx + ly + mz - \omega t \tag{11}$$

where k, l, m, ω are real constants and

$$k^2 + l^2 + m^2 - \omega^2 = 1, \qquad v = \frac{\omega^2}{k^2 + l^2 + m^2}.$$

(ii) Thus show that we have a Bäcklund transformation between the ordinary differential equations

$$\frac{d^2\alpha}{d\eta^2} = \sin\alpha, \qquad \frac{d^2\beta}{d\eta^2} = \sinh\beta$$

defined by

$$\frac{du}{d\eta} = 2e^{i\psi}\sin\left(\frac{1}{2}\bar{u}\right).$$

Problem 50. The equation which describes small amplitude waves in a dispersive medium with a slight deviation from one-dimensionality is

$$\frac{\partial}{\partial x}\left(4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) \pm 3\frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

The + refers to the two-dimensional Korteweg-de Vries equation. The $-$ refers to the two-dimensional Kadomtsev-Petviashvili equation. Let (formulation of the *inverse scattering transform*)

$$u(x,y,t) := 2\frac{\partial}{\partial x}K(x,x;y,t) \tag{2}$$

where

$$K(x,z;y,t) + F(x,z;y,t) + \int_x^\infty K(x,s;y,t)F(s,z;y,t)ds = 0 \tag{3}$$

and F satsifies the system of linear partial differential equations

$$\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3} + \frac{\partial F}{\partial t} = 0, \qquad \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} + \sigma\frac{\partial F}{\partial y} = 0 \tag{4}$$

where $\sigma = 1$ for the two-dimensional Korteweg de Vries equation and $\sigma = i$ for the Kadomtsev Petviashvili equation. Find solutions of the form

$$F(x,z;y,t) = \alpha(x;y,t)\beta(z;y,t) \tag{5}$$

and

$$K(x,z;y,t) = L(x;y,t)\beta(z;y,t). \tag{6}$$

Problem 51. Show that Hirota's operators $D_x^n(f\cdot g)$ and $D_x^m(f\cdot g)$ given in (1) and (2) can be written as

$$D_x^n(f\cdot g) = \sum_{j=0}^{n}\frac{(-1)^{(n-j)}n!}{j!(n-j)!}\frac{\partial^j f}{\partial x^j}\frac{\partial^{n-j}g}{\partial x^{n-j}}, \tag{3}$$

$$D_x^m D_t^n(f\cdot g) = \sum_{j=0}^{m}\sum_{i=0}^{n}\frac{(-1)^{(m+n-j-i)}m!}{j!(m-j)!}\frac{n!}{i!(n-i)!}\frac{\partial^{i+j}f}{\partial t^i\partial x^j}\frac{\partial^{n+m-i-j}g}{\partial t^{n-i}\partial x^{m-j}}. \tag{4}$$

Problem 52. Consider the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

(i) Consider the dependent variable transformation

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2}. \tag{2}$$

Show that f satisfies the differential equation

$$f\frac{\partial^2 f}{\partial x \partial t} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial t} + f\frac{\partial^4 f}{\partial x^4} - 4\frac{\partial f}{\partial x}\frac{\partial^3 f}{\partial x^3} + 3\left(\frac{\partial^2 f}{\partial x^2}\right)^2 = 0. \tag{3}$$

(ii) Show that this equation in f can be written in bilinear form

$$(D_x D_t + D_x^4)(f \cdot f) = 0. \tag{4}$$

Problem 53. The *Sawada-Kotera equation* is given by

$$\frac{\partial u}{\partial t} + 45u^2\frac{\partial u}{\partial x} + 15\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 15u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0. \tag{1}$$

Show that using the dependent variable transformation

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} \tag{2}$$

and integrating once with respect to x we arrive at

$$(D_x D_t + D_x^6)(f \cdot f) = 0. \tag{3}$$

Problem 54. The *Kadomtsev-Petviashvili equation* is given by

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

Show that using the transformation

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} \tag{2}$$

the Kadomtsev-Petviashvili equation takes the form

$$(D_x D_t + D_x^4 + 3D_y^2)(f \cdot f) = 0. \tag{3}$$

Problem 55. Consider the system of partial differential equations

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\\
\frac{\partial v}{\partial t} &= \beta v\frac{\partial u}{\partial x} + cu\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\\
\frac{\partial w}{\partial t} &= \gamma w\frac{\partial u}{\partial x} + \delta u\frac{\partial w}{\partial x}
\end{aligned}$$

where $\alpha, \beta, \gamma, c, \delta$ are constants. Show that the system admits the converved densities

$$\begin{aligned} H_0 &= u \\ H_1 &= v + \frac{1}{2}(\beta - c)u^2 \\ H_2 &= uv + \frac{1}{\gamma - \delta} w + \left(\frac{\alpha + \beta}{2} - c \right) \frac{u^3}{3}, \qquad \gamma \neq \delta \\ H_3 &= w, \qquad \gamma = \delta. \end{aligned}$$

Problem 56. The *polytropic gas dynamics* in $1+1$ dimensions is of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= u \frac{\partial u}{\partial x} + C \rho^{\Gamma} \frac{\partial \rho}{\partial x} \\ \frac{\partial \rho}{\partial t} &= \frac{\partial}{\partial x} (\rho u) \end{aligned}$$

where x is the space coordinate, t is the (minus physical) time coordinate, u is the velocity, ρ the density and $\Gamma = \gamma - 2$. Here γ is the polytropic exponent. The constant C can be removed by a rescaling of ρ. Express this system of partial differential equations applying the *Riemann invariants*

$$r_{1,2}(x,t) = u(x,t) \pm \frac{2}{\Gamma + 1} \rho^{(\Gamma+1)/2}(x,t). \qquad \Gamma \neq -1.$$

Problem 57. The Korteweg-de Vries equation is given by

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

For a steady-state pulse solution we make the ansatz

$$u(x,t) = -\frac{b}{2} \text{sech}^2 \left(\frac{\sqrt{c}}{2}(x - ct) \right)$$

where

$$\text{sech}(y) := \frac{1}{\cosh(y)}.$$

Find the condition on b and c such that this ansatz is a solution of the Korteweg-de Vries equation.

Problem 58. Consider the Korteweg-de Vries equation and its solution given in the previous problem. Show that

$$\int_{-\infty}^{\infty} \sqrt{|u(x,t)|} dx = \pi.$$

Hint. We have

$$\int \operatorname{sech}(s)ds \equiv \int \frac{1}{\cosh(s)}ds = 2\arctan(e^s).$$

Problem 59. Consider the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Show that

$$u(x,t) = -12\frac{4\cosh(2x-8t)+\cosh(4x-64t)+3}{(3\cosh(x-28t)+\cosh(3x-36t))^2}$$

is a solution of the Korteweg-de Vries equation. This is a so-called two soliton solution.

Problem 60. Consider the one-dimensional Euler equations

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0\\ \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0\\ \frac{\partial p}{\partial t} + \gamma p\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} &= 0\end{aligned}$$

where $u(x,t)$ is the velocity field, $\rho(x,t)$ is the density field and $p(x,t)$ is the pressure field. Here t is the time, x is the space coordinate and γ is the ratio of specific heats. Find the linearized equation around $\widetilde{u}$, $\widetilde{p}$, $\widetilde{\rho}$.

Problem 61. Consider the classical it Heisenberg ferromagnetic equation

$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2}$$

where $\mathbf{S} = (S_1, S_2, S_3)^T$, $S_1^2+S_2^2+S_3^2 = 1$ and $\times$ denotes the vector product The natural boundary conditions are $\mathbf{S}(x,t) \to (0,0,1)$ as $|x| \to \infty$.
(i) Find partial differential equation under the stereographic projection

$$S_1 = \frac{2u}{Q}, \quad S_2 = \frac{2v}{Q}, \quad S_3 = \frac{-1+u^2+v^2}{Q}$$

where $Q = 1 + u^2 + v^2$.
(ii) Perform a Painlevé test.

(iii) The Heisenberg ferromagnetic equation in the form given for u, v is gauge equivalent to the one-dimensional Schrödinger equation

$$\frac{\partial u}{\partial t}+\frac{\partial^2 v}{\partial x^2}+2(u^2+v^2)v=0$$
$$\frac{\partial v}{\partial t}-\frac{\partial^2 u}{\partial x^2}-2(u^2+v^2)u=0.$$

Both systems of differential equations arise as consistency conditions of a system of linear partial differential equations

$$\frac{\partial \psi}{\partial x}=U\psi, \qquad \frac{\partial \psi}{\partial t}=V\psi$$

where $\psi=(\psi_1,\psi_2)^T$ and U and V are 2×2 matrices. The consistency condition is given by

$$\frac{\partial U}{\partial t}-\frac{\partial V}{\partial x}+[U,V]=0.$$

Two systems of nonlinear partial differential equations that are integrable if there is an invertible 2×2 matrix g which depends on x and t such that

$$U_1=gU_2g^{-1}+\frac{\partial g}{\partial x}g^{-1}, \qquad V_1=gV_2g^{-1}+\frac{\partial g}{\partial t}g^{-1}.$$

Are the resonances of two gauge equivalent systems the same?

Problem 62. The system of partial differential equations of the system of *chiral field equations* can be written as

$$\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{u}}{\partial x}-\mathbf{u}\times(J\mathbf{v})=\mathbf{0}$$
$$\frac{\partial \mathbf{v}}{\partial t}-\frac{\partial \mathbf{v}}{\partial x}-\mathbf{v}\times(J\mathbf{u})=\mathbf{0}$$

where $\mathbf{u}=(u_1,u_2,u_3)^T$, $\mathbf{v}=(v_1,v_2,v_3)$, $\mathbf{u}^2=\mathbf{v}^2=1$, $J=\mathrm{diag}(j_1,j_2,j_3)$ is a 3×3 diagonal matrix and $\times$ denotes the vector product. Consider the linear mapping $M:\mathbb{R}^3\to so(3)$

$$M(\mathbf{u})=\begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}$$

where $so(3)$ is the simple Lie algebra of the 3×3 skew-symmetric matrices. Rewrite the system of partial differential equations using $M(\mathbf{u})$.

Problem 63. The *Landau-Lifshitz equation*

$$\frac{\partial \mathbf{u}}{\partial t}=\mathbf{u}\times\frac{\partial^2 \mathbf{u}}{\partial x^2}+\mathbf{u}\times(K\mathbf{v})$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{u}^2 = 1$, $K = \mathrm{diag}(k_1, k_2, k_3)$ is a 3×3 diagonal matrix and $\times$ denotes the vector product. Consider the linear mapping $M : \mathbb{R}^3 \to so(3)$

$$M(\mathbf{u}) = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}$$

where $so(3)$ is the simple Lie algebra of the 3×3 skew-symmetric matrices. Rewrite the system using $M(\mathbf{u})$.

Problem 64. Consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x}.$$

Let

$$u(x,t) = -\frac{\phi_t}{\phi_x}.$$

Find the partial differential equation for ϕ.

Problem 65. (i) Show that the *Burgers equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x}$$

admits the Lax reprsentation

$$\left(\frac{\partial}{\partial x} + \frac{u}{2} \right) \psi = \lambda \psi, \qquad \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + u \frac{\partial \psi}{\partial x}$$

where u is a smooth function of x and t.
(ii) Show that

$$[T, K]\psi = 0$$

also provides the Burgers equation, where

$$T := \frac{\partial^2}{\partial x^2} + u \frac{\partial}{\partial x} - \frac{\partial}{\partial t}, \qquad K := \frac{\partial}{\partial x} - \lambda + \frac{u}{2}.$$

Problem 66. Consider the Burgers equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x}.$$

Consider the operator (so-called *recursion operator*)

$$R := D + \frac{1}{2} \frac{\partial u}{\partial x} D^{-1} + \frac{u}{2}$$

where

$$D := \frac{\partial}{\partial x}, \qquad D^{-1} f(x) := \int^x f(s) ds.$$

(i) Show that applying the recursion operator R to the right-hand side of the Burgers equation results in the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \frac{3}{2} u \frac{\partial^2 u}{\partial x^2} + \frac{3}{4} u^2 \frac{\partial^2 u}{\partial x}.$$

(ii) Show that this partial differential equation can also be derived from the linear partial differential equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial x^3}$$

and the transformation

$$\phi(x,t) = \exp\left(\frac{1}{2} \int^x u(s,t) ds\right).$$

Note that

$$D^{-1}\left(\frac{\partial u}{\partial x}\right) = u.$$

Problem 67. Find the solution of the system of partial differential equations

$$\frac{\partial f}{\partial x} + f^2 = 0, \qquad \frac{\partial f}{\partial t} + fg = 0$$

$$\frac{\partial g}{\partial x} + fg = 0, \qquad \frac{\partial g}{\partial t} + g^2 = 0.$$

Problem 68. Consider the Kortweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Show that

$$u(x,t) = -2 \frac{\partial^2}{\partial x^2} \log \Delta(x,t)$$

is a solution of the Kortweg-de Vries equation, where

$$\Delta(x,t) = \det\left(\delta_{jk} + \frac{c_j c_k}{\eta_j + \eta_k} \exp(-(\eta_j^3 + \eta_k^3)t - (\eta_j + \eta_k)x)\right)_{j,k=1,\ldots,N}.$$

This is the so-called N-soliton solution.

Problem 69. Consider the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 0.$$

Let $v(x,t) = \exp(u(x,t))$. Find the partial differential equation for v.

Problem 70. Consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + |\psi|^2\psi = 0.$$

Show that

$$\psi(x,t) = 2\nu \mathrm{sech}(s)\exp(i\phi(s,t))$$

with

$$s := 2\nu(x - \zeta(t)), \quad \phi(s,t) := \frac{\mu}{\nu}s + \delta(t)$$

$$\zeta(t) = 2\mu t + \zeta_0, \quad \delta(t) = 4(\mu^2 + \nu^2)t + \delta_0.$$

is a solution (so-called one-soliton solution) of the nonlinear Schrördinger equation.

Problem 71. Consider the system of nonlinear partial differential equations

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u + u(u^2+v^2) = 0, \quad \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - \left(1 - \frac{\kappa}{2}\right)v + v(u^2+v^2) = 0.$$

(i) Show that

$$u(x,t) = \pm\tanh(s/\sqrt{2}), \quad v(x,t) = 0$$

is a solution, where $s := \gamma(x - ct)$.
(ii) Show that

$$u(x,t) = \pm\tanh(\sqrt{\kappa/2}s), \qquad v(x,t) = (1-\kappa)^{1/2}\mathrm{sech}(\sqrt{\kappa/2}s)$$

is a solution, where $s := \gamma(x - ct)$.
(iii) Show that

$$u(x,t) = \pm\tanh(\sqrt{\kappa/2}s), \qquad v(x,t) = -(1-\kappa)^{1/2}\mathrm{sech}(\sqrt{\kappa/2}s)$$

is a solution, where $s := \gamma(x - ct)$.

Problem 72. Consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u = 0, \qquad -\infty < x < \infty$$

Show that
$$u(x,t) = 2\eta \exp(i\phi(x,t))\mathrm{sech}(\psi(x,t))$$
where
$$\phi(x,t) = -2(\xi x + 2(\xi^2 - \eta^2)t) + \phi_0, \quad \psi(x,t) = 2\eta(x + 4\xi t) + \psi_0$$
is a solution of the one-dimensional nonlinear Schrödinger equation.

Problem 73. The Landau-Lifshitz equation describing nonlinear spin waves in a ferromagnet is given by
$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2} + \mathbf{S} \times J\mathbf{S}$$
where
$$\mathbf{S} = (S_1, S_2, S_3)^T, \qquad S_1^2 + S_2^2 + S_3^2 = 1$$
and $J = \mathrm{diag}(J_1, J_2, J_3)$ is a constant 3×3 diagonal matrix. Show that
$$\frac{\partial \mathbf{w}}{\partial x_1} = L\mathbf{w}, \qquad \frac{\partial \mathbf{w}}{\partial x_2} = M\mathbf{w}$$
with $x_1 = x$, $x_2 = -it$ and
$$L = \sum_{\alpha=1}^{3} z_\alpha S_\alpha \sigma_\alpha$$
$$M = i \sum_{\alpha,\beta,\gamma=1}^{3} z_\alpha \sigma_\alpha S_\beta \frac{\partial S_\gamma}{\partial x} \epsilon^{\alpha\beta\gamma} + 2z_1 z_2 z_3 \sum_{\alpha=1}^{3} z_\alpha^{-1} S_\alpha \sigma_\alpha$$
provide a Lax pair. Here σ_1, σ_2, σ_3 are the Pauli spin matrices and the spectral parameters (z_1, z_2, z_3) constitue an algebraic coordinate of an elliptic curve defined by
$$z_\alpha^2 - z_\beta^2 = \frac{1}{4}(J_\alpha - J_\beta), \qquad \alpha, \beta = 1, 2, 3.$$

Problem 74. Consider the nonlinear partial differential equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x}.$$
Consider the generalized Hopf-Cole transformation
$$v(t,x) = w(t)u(t,x)\exp\left(\int_0^s ds u(s,t)\right)$$

with

$$u(t,x) = \frac{v(t,x)}{w(t) + \int_0^s ds v(s,t)}.$$

Find the differential equations for v and w.

Problem 75. Consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2c|u|^2 = 0$$

where $x \in \mathbb{R}$ and $c = \pm 1$. Show that it admits the solution

$$u(x,t) = \frac{A}{\sqrt{t}} \exp\left(it \left(\frac{1}{4}\left(\frac{x}{t}\right)^2 + 2cA^2 \frac{\ln(t)}{t} + \frac{\phi}{t} \right) \right).$$

Problem 76. Let α be a positive constant. The *Kadomtsev Petviashvilli equation* is given by

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) + \alpha \frac{\partial^2 u}{\partial y^2} = 0.$$

Consider the one-soliton solution

$$u(x,y,t) = 2k_1^2 \mathrm{sech}^2(k_1 x + k_2 y - \omega t)$$

where $\mathrm{sech}(x) \equiv 1/\cosh(x) \equiv 2/(e^x + e^{-x})$. Find the *dispersion relation* $\omega(k_1, k_2)$.

Problem 77. Find the partial differential equation given by the condition

$$\det \begin{pmatrix} u & \partial u/\partial z_1 \\ \partial u/\partial z_2 & \partial^2 u/\partial z_1 \partial z_2 \end{pmatrix} = 0.$$

Find a solution of the partial differential equation.

Problem 78. A one-dimensional Schrödinger equation with cubic nonlinearity is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} - g\rho\psi, \qquad \rho := \psi^* \psi$$

where $g > 0$.
(i) Show that the partial differential equation admits the (soliton) solution

$$\psi_s(x,t) = \pm \exp\left(i\frac{mv}{\hbar}(x - ut) \right) \frac{\hbar}{\sqrt{gm}} \frac{\gamma}{\cosh(\gamma(x - vt))}, \quad \gamma^2 = \frac{m^2 v^2}{\hbar^2}\left(1 - \frac{2u}{v}\right).$$

The soliton moves with group velocity v. The phase velocity u must be $u < v/2$.
(ii) Show that the partial differential equation is Galileo-invariant. This means that any solution of partial differential equation can be mapped into another solution via the Galileo boost

$$x \to x - Vt, \qquad \psi(x,t) \to \exp\left(\frac{i}{\hbar}mV\left(x - \frac{1}{2}Vt\right)\right)\psi(t, x - Vt).$$

Show that the soliton can be brought to rest.

Problem 79. The KP-equations are given by

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = -3\alpha^2\frac{\partial^2 u}{\partial y^2}$$

with $\alpha = i$ and $\alpha = -1$. Show that this equation is an integrability condition on

$$L\psi \equiv \alpha\frac{\partial\psi}{\partial y} + \frac{\partial^2\psi}{\partial x^2} + u\psi = 0$$

$$M\psi \equiv \frac{\partial\psi}{\partial t} + 4\frac{\partial^3\psi}{\partial x^3} + 6u\frac{\partial\psi}{\partial x} + 3\left(\frac{\partial u}{\partial x} - \alpha\int_{-\infty}^{x}\frac{\partial u(x',y)}{\partial y}dx'\right) + \beta\psi = 0.$$

Problem 80. Show that the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = (1+u^2)\frac{\partial^2 u}{\partial x^2} - u\left(\frac{\partial u}{\partial x}\right)^2$$

is transformated under the transformation $u \mapsto u/\sqrt{1+u^2}$ into the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\tanh^{-1} u).$$

Problem 81. Consider the metric tensor field

$$g = g_{11}(x_1,x_2)dx_1\otimes dx_1 + g_{12}(x_1,x_2)dx_1\otimes dx_2 + g_{21}(x_1,x_2)dx_2\otimes dx_1 + g_{22}(x_1,x_2)dx_2\otimes dx_2$$

with $g_{12} = g_{21}$ and the g_{jk} are smooth functions of x_1, x_2. Let $\det(g) \equiv g_{11}g_{22} - g_{12}g_{21} \neq 0$ and

$$R = \frac{2}{(\det(g))^2}\det\begin{pmatrix} g_{11} & g_{22} & g_{12} \\ \partial g_{11}/\partial x_1 & \partial g_{22}/\partial x_1 & \partial g_{12}/\partial x_1 \\ \partial g_{11}/\partial x_2 & \partial g_{22}/\partial x_2 & \partial g_{12}/\partial x_2 \end{pmatrix}.$$

Find solutions of the partial differential equation $R = 0$. Find solutions of the partial differential equation $R = 1$.

Problem 82. Consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + 2|w|^2 w = 0.$$

Consider the ansatz

$$w(x,t) = \exp(i\omega t)u(x).$$

Find the ordinary differential equation for u.

Problem 83. Consider the system of partial differential differential equations

$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2} + \mathbf{S} \times (J\mathbf{S}), \quad J = \begin{pmatrix} j_1 & 0 & 0 \\ 0 & j_2 & 0 \\ 0 & 0 & j_3 \end{pmatrix}$$

where $\mathbf{S} = (S_1, S_2, S_3)^T$ and $S_1^2 + S_2^2 + S_3^2 = 1$. Express the partial differental equation using $p(x,t)$ and $p(x,t)$ given by

$$S_1 = \sqrt{1-p^2}\cos(q), \quad S_2 = \sqrt{1-p^2}\sin(q), \quad S_3 = p.$$

Problem 84. Consider the two-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \sin(u).$$

Let

$$u(x_1, x_2, t) = 4\arctan(v(x_1, x_2, t)).$$

Find the partial differential equation for v. Separate this partial differential equation into a linear part and nonlinear part. Solve these partial differential equations to find solutions for the two-dimensional sine-Gordon equation. Note that

$$\sin(4\alpha) \equiv 4\sin(\alpha)\cos(\alpha) - 8\sin^3(\alpha)\cos(\alpha)$$

$$\sin(\arctan(\alpha)) = \frac{\alpha}{\sqrt{1+\alpha^2}}, \quad \cos(\arctan(\alpha)) = \frac{1}{\sqrt{1+\alpha^2}}$$

and therefore

$$\sin(4\arctan(v)) = \frac{4v(1-v^2)}{(1+v^2)^2}.$$

Furthermore

$$\frac{\partial^2}{\partial x_1^2}\arctan(v) = \frac{-2v}{(1+v^2)^2}\left(\frac{\partial v}{\partial x_1}\right)^2 + \frac{1}{1+v^2}\frac{\partial^2 v}{\partial x_1^2}.$$

Problem 85. (i) Find a non-zero vector field in $\mathbb{R}^3$ such that

$$V \cdot \mathrm{curl} V = 0.$$

(ii) Find a non-zero vector field in $\mathbb{R}^3$ such that

$$V \times \mathrm{curl}(V) = \mathbf{0}.$$

Problem 86. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Consider the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u(x))}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Find solutions of the form $u(x,t) = \phi(x - ct)$ (traveling wave solutions) where ϕ is a smooth function. Integrate the obtained ordinary differential equation.

Problem 87. Consider the partial differential equation (Thomas equation)

$$\frac{\partial^2 u}{\partial x \partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial t} = 0$$

where a, b are constants. Show that the equation can be linearized with the transformation

$$u(x,t) = -bx - at + \ln(v(x,t).$$

Problem 88. The Korteweg-de Vries equation is given by

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Setting $u = \partial v/\partial x$ we obtain the equation

$$\frac{\partial^2 v}{\partial x \partial t} - 6\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^4 v}{\partial x^4} = 0.$$

(i) Show that this equation can be dervied from the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}v_x v_t + (v_x)^3 + \frac{1}{2}(v_{xx})^2$$

where the Lagrange equation is given by

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial v_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial v_x}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial v_{xx}}\right) - \frac{\partial \mathcal{L}}{\partial v} = 0.$$

(ii) Show the Hamiltonian density is given by

$$\mathcal{H} = v_t \frac{\partial \mathcal{L}}{\partial v_t} - \mathcal{L} = -(v_x)^3 - \frac{1}{2}(v_{xx})^2.$$

Problem 89. Show that the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

admits the solution

$$u(x,t) = -\frac{c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right).$$

Show that

$$\int_{-\infty}^{\infty} dx \sqrt{|u(x,t)|} = \pi.$$

Problem 90. Consider the one-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \omega^2 \sin(u) = 0.$$

Show that

$$u_+(x,t) = 4\arctan(\exp(d^{-1}(x-vt-X)\cosh(\alpha))$$
$$u_-(x,t) = 4\arctan(\exp(-d^{-1}(x-vt-X)\cosh(\alpha))$$

are solutions of the one-dimensional sine-Gordon equation, where $v = c\tanh(\alpha)$, $d = c/\omega$. Discuss.

Problem 91. Let $\kappa(s,t)$ be the curvature and $\tau(s,t)$ be the torsion with s and t being the arclength and time, respectively. Consider the complex valued function

$$w(s,t) = \kappa(s,t)\exp(i\int_0^s ds'\tau(s',t)).$$

Show that if the motion is described by

$$\frac{\partial}{\partial t}\mathbf{r} - \frac{\partial}{\partial s}\mathbf{r} \times \frac{\partial^2}{\partial s^2}\mathbf{r} = \kappa\mathbf{b}$$

where $\mathbf{b}$ is the binormal unit vector, then $w(s,t)$ satisfies the nonlinear Schrödinger equation

$$i\frac{\partial}{\partial t}w + \frac{\partial^2}{\partial s^2}w + \frac{1}{2}|w|^2 w = 0.$$

Problem 92. Consider the first order system of partial differential equation

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ and the 3×3 matrix is given by

$$\begin{pmatrix} \Gamma u_1 & u_2 & u_3 \\ u_2 & u_1 - \Delta & 0 \\ u_3 & 0 & u_1 - \Delta \end{pmatrix}$$

where Γ and Δ are constants. Along a characteristic curve $C : x = x(s), t = t(s)$ for the system of partial differential equation one has

$$\det(A - \lambda I_3) = 0, \qquad \lambda = \frac{x'(s)}{t'(s)}.$$

Find the Riemann invariants.

Problem 93. Consider the system of partial differential equations

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} = Z^2 - VW$$
$$\frac{\partial W}{\partial t} - \frac{\partial W}{\partial x} = Z^2 - VW$$
$$\frac{\partial Z}{\partial t} = -\frac{1}{2}Z^2 + \frac{1}{2}VW$$

Let $N := V + W + 4Z$ and $J := V - W$. Find $\partial N/\partial t + \partial J/\partial x$ and $\partial J/\partial t + \partial V/\partial x + \partial W/\partial x$.

Problem 94. Consider the Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v}.$$

(i) Show that in the limit $\nu \to 0$ the Navier-Stokes equation are invariant under the scaling transformation ($\lambda > 0$)

$$r \to \lambda r, \quad \mathbf{v} \to \lambda^h \mathbf{v}, \quad t \to \lambda^{1-h} t.$$

(ii) Show that for finite ν one finds invariance of the Navier-Stokes equation if $\nu \to \lambda^{1+h}\nu$.

Problem 95. (i) Show that

$$u(x,t) = \frac{k^2/2}{(\cosh((kx - \omega t)/2))^2}, \quad k^3 = \omega$$

is a solution (solitary wave solution) of the Korteweg de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

(ii) Let

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln(F(x,t)).$$

Show that $F(x,t) = 1 + e^{kx-\omega t}$.

Chapter 3

Lie Symmetry Methods

Problem 1. Show that the partial differential equation

$$\frac{\partial u}{\partial x_2} = u\frac{\partial u}{\partial x_1} \tag{1}$$

is invariant under the transformation

$$\bar{x}_1(x_1, x_2, \epsilon) = x_1, \qquad \bar{x}_2(x_1, x_2, \epsilon) = \epsilon x_1 + x_2 \tag{2a}$$

$$\bar{u}(\bar{x}_1(x_1, x_2), \bar{x}_2(x_1, x_2), \epsilon) = \frac{u(x_1, x_2)}{1 - \epsilon u(x_1, x_2)} \tag{2b}$$

where ϵ is a real parameter.

Problem 2. Consider the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + 12u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0\,.$$

Show that the scaling

$$x' = cx, \quad t' = c^3 t, \quad u(x,t) = c^2 u'(x', t')$$

leaves the Korteweg-de Vries equation invariant.

Problem 3. Consider the partial differential equation

$$\Phi(\Box u, (\nabla u)^2, u) = 0$$

where Φ is an analytic function, u depends on $x_0, x_1, \ldots, x_n$ and

$$\Box u := \frac{\partial u}{\partial x_0} - \frac{\partial u}{\partial x_1} - \cdots - \frac{\partial u}{\partial x_n}, \quad (\nabla u)^2 := \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial u}{\partial x_n}\right)^2 .$$

Show that the equation is invariant under the Poincaré algebra

$$\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

$$x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_0}, x_0 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_0}, \ldots, x_0 \frac{\partial}{\partial x_n} + x_n \frac{\partial}{\partial x_0}$$

$$x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_k}, \quad j, k = 1, 2, \ldots, n \quad j \neq k .$$

Problem 4. Consider the nonlinear partial differential equation (*Born-Infeld equation*)

$$\left(1 - \left(\frac{\partial u}{\partial t}\right)^2\right) \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x \partial t} - \left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right) \frac{\partial^2 u}{\partial t^2} = 0. \qquad (1)$$

Show that this equation admits the following seven Lie symmetry generators

$$Z_1 = \frac{\partial}{\partial x}, \qquad Z_2 = \frac{\partial}{\partial t}, \qquad Z_3 = \frac{\partial}{\partial u}$$

$$Z_4 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}, \qquad Z_5 = u \frac{\partial}{\partial x} - x \frac{\partial}{\partial u}$$

$$Z_6 = u \frac{\partial}{\partial t} + t \frac{\partial}{\partial u}, \qquad Z_7 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} . \qquad (2)$$

Problem 5. Consider the *Harry-Dym equation*

$$\frac{\partial u}{\partial t} - u^3 \frac{\partial^3 u}{\partial x^3} = 0. \qquad (1)$$

(i) Find the Lie symmetry vector fields.
(ii) Compute the flow for one of the Lie symmetry vector fields.

Problem 6. Consider the *Magneto-Hydro-Dynamics equations* and carry out the Lie symmetry analysis. We neglect dissipative effects, and thus restrict the analysis to the ideal case. The equations are given by

$$\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho + \rho \nabla \cdot \mathbf{v} = 0 \qquad (1a)$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)+\nabla(p+\frac{1}{2}\mathbf{H}^2)-(\mathbf{H}\cdot\nabla)\mathbf{H}=\mathbf{0} \tag{1b}$$

$$\frac{\partial \mathbf{H}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{H}+\mathbf{H}\nabla\cdot\mathbf{v}-(\mathbf{H}\cdot\nabla)\mathbf{v}=\mathbf{0} \tag{1c}$$

$$\nabla\cdot\mathbf{H}=0 \tag{1d}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\kappa}\right)+(\mathbf{v}\cdot\nabla)\left(\frac{p}{\rho^\kappa}\right)=0 \tag{1e}$$

with pressure p, mass density ρ, coefficient of viscosity κ, fluid velocity $\mathbf{v}$ and magnetic field $\mathbf{H}$.

Problem 7. The stimulated *Raman scattering equations* in a symmetric form are given by

$$\frac{\partial v_1}{\partial x}=ia_1v_2^*v_3^*, \qquad \frac{\partial v_2}{\partial x}=ia_2v_3^*v_1^*, \qquad \frac{\partial v_3}{\partial t}=ia_3v_1^*v_2^*. \tag{1}$$

The a_i are real coupling constants that can be normalized to ± 1 and we have set $v_1=iA_1^*$, $v_2=A_2$, $v_3=X$, where A_1, A_2 and X are the complex pump, Stokes and material excitation wave envelopes, respectively. The stars denote complex conjugation. Equations (1) are actually a special degenerate case of the full three wave resonant interaction equations. Find the similarity solutions using the similarity ansatz

$$s(x,t):=xt^{-\epsilon} \qquad \text{similarity variable} \tag{2}$$

and

$$v_1(x,t)=t^{-\frac{1+\epsilon}{2}}\rho_1(s)\exp(i\phi_1(s)), \qquad v_2(x,t)=t^{-\frac{1+\epsilon}{2}}\exp(-i\epsilon alnt)\rho_2(s)\exp(i\phi_2(s))$$

$$v_3(x,t)=\frac{1}{x}e^{ialnx}\rho_3(s)\exp(i\phi_3(s)) \tag{3}$$

where $\rho_i\geq 0$, $0\leq\phi_i<2\pi$ and $\epsilon=\pm 1$.

Problem 8. The motion of an inviscid, incompressible-ideal fluid is goverened by the system of equations

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\Delta\mathbf{u}=-\Delta p \tag{1}$$

$$\Delta\cdot\mathbf{u}=0 \tag{2}$$

first obtained by Euler. Here $\mathbf{u}=(u,v,w)$ are the components of the three-dimensional velocity field and p the pressure of the fluid at a position $\mathbf{x}=(x,y,z)$.

(i) Show that the Lie symmetry group of the Euler equations in three dimensions is generated by the vector fields

$$
{}_a = a\frac{\partial}{\partial x} + a'\frac{\partial}{\partial u} - a''x\frac{\partial}{\partial p}, \qquad \mathbf{v}_b = b\frac{\partial}{\partial y} + b'\frac{\partial}{\partial v} - b''y\frac{\partial}{\partial p}
$$

$$
\mathbf{v}_c = c\frac{\partial}{\partial z} + c'\frac{\partial}{\partial w} - c''z\frac{\partial}{\partial p}, \qquad \mathbf{v}_0 = \frac{\partial}{\partial t}
$$

$$
\mathbf{s}_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}, \qquad \mathbf{s}_2 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p}
$$

$$
\mathbf{r}_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \qquad \mathbf{r}_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w}
$$

$$
\mathbf{r}_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}, \qquad \mathbf{v}_q = q\frac{\partial}{\partial p} \tag{4}
$$

where a, b, c, q are arbitrary functions of t.

(ii) Show that these vector fields exponentiate to familiar one-parameter symmetry groups of the Euler equations. For instance, a linear combination of the first three fields $\mathbf{v}_a + \mathbf{v}_b + \mathbf{v}_c$ generates the group transformations

$$
(\mathbf{x}, t, \mathbf{u}p) \to (\mathbf{x} + \epsilon\mathbf{a}, t, \mathbf{u} + \epsilon\mathbf{a}', p - \epsilon\mathbf{x}\cdot\mathbf{a}'' + \frac{1}{2}\epsilon^2\mathbf{a}\cdot\mathbf{a}'')
$$

where ϵ is the group parameter, and $\mathbf{a} := (a, b, c)$. These represent changes to arbitrarily moving coordinate systems, and have the interesting consequence that for a fluid with no free surfaces, the only essential effect of changing to a moving coordinate frame is to add an extra component, namely,

$$
-\epsilon\mathbf{x}\cdot\mathbf{a} + \frac{1}{2}\epsilon^2\mathbf{a}\cdot\mathbf{a}''
$$

to the resulting pressure.

(iii) Show that the group generated by $\mathbf{v}_0$ is that of time translations, reflecting the time-independence of the system. (iv) Show that the next two groups are scaling transformations

$$
\mathbf{s}_1 : (\mathbf{x}, t\mathbf{u}, p) \to \epsilon\mathbf{x}, \epsilon t, \mathbf{u}, p)
$$

$$
\mathbf{s}_2 : (\mathbf{x}, t, \mathbf{u}, p) \to (\mathbf{x}, \epsilon t, \epsilon^{-1}\mathbf{u}, \epsilon^{-2}p).
$$

The vector fields $\mathbf{r}_x$, $\mathbf{r}_y$, $\mathbf{r}_2$ generate the orthogonal group $SO(3)$ of simultaneous rotations of space and associated velocity field; e.g., $\mathbf{r}_x$ is just an infinitesimal rotation around the x axis. The final group indicates that arbitrary functions of t can be added to the pressure.

Problem 9. Consider the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u^2}\frac{\partial u}{\partial x}\right). \tag{1}$$

Within the jet bundle formalism we consider instead of (1) the submanifold

$$F(x,t,u,u_x,u_{xx}) \equiv u_t - \frac{u_{xx}}{u^2} + 2\frac{u_x^2}{u^3} = 0 \tag{2}$$

and its differential consequences

$$F_x(x,t,u,u_x,\cdots) \equiv u_{tx} - \frac{u_{xxx}}{u^2} + \frac{6u_x u_{xx}}{u^3} - \frac{6u_x^3}{u^4} = 0, \ldots \tag{3}$$

with the contact forms

$$\theta = du - u_x dx - u_t dt, \qquad \theta_x = du_x - u_{xx}dx - u_{tx}dt, \ldots \tag{4}$$

The non-linear partial differential equation (1) admits the Lie point symmetries (infinitesimal generator)

$$X_v = -u_x\frac{\partial}{\partial u}, \qquad T_v = u_t\frac{\partial}{\partial u}$$

$$S_v = (-xu_x - 2tu_t)\frac{\partial}{\partial u}, \qquad V_v = (xu_x + u)\frac{\partial}{\partial u}. \tag{5}$$

The subscript v denotes the vertical vector fields. The non-linear partial differential equation (1) also admits *Lie-Bäcklund vector fields*. The first in the hierarchy is

$$U = \left(\frac{u_{xxx}}{u^3} - \frac{9u_x u_{xx}}{u^4} + \frac{12u_x^3}{u^5}\right)\frac{\partial}{\partial u}. \tag{6}$$

Find a similarity solution using a linear combination of the Lie symmetry vector field T_v and U.

Problem 10. Consider the nonlinear partial differential equation

$$\frac{\partial^3 u}{\partial x^3} + u\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

where c is a constant. Show that this partial differential equation admits the Lie symmetry vector field

$$V = t\frac{\partial}{\partial t} + ct\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\,.$$

First solve the first order autonomous system of differential equations (initial value problem) which belongs to the vector field V, i.e.

$$\frac{dt'}{d\epsilon} = t', \qquad \frac{dx'}{d\epsilon} = ct', \qquad \frac{du'}{d\epsilon} = u'\,.$$

From this solution of the initial value problem find the transformation between the prime and unprime system. Then using differentiation and the chain rule show that the prime and unprime system have the same form.

Problem 11. Consider the nonlinear Schrödinger equation

$$i\frac{\partial w}{\partial t} + \Delta w = F(w)$$

with a nonlinearity $F : \mathbb{C} \to \mathbb{C}$ and a complex valued $w(t, x_1, \ldots, x_n)$. We assume that for the given nonlinearity the energy remains bounded. We also assume that there exists a C^1-function $G : \mathbb{R}^+ \to \mathbb{R}$ with $G(0) = 0$ such that $F(z) = G'(|z|^2)z$ for all $z \in \mathbb{C}$.
(i) Show that the nonlinear Schrödinger is translation invariant.
(ii) Show that it is phase invariant, i.e. $w \to e^{i\alpha}w$.
(iii) Show that it is Galilean invariant

$$w(t, \mathbf{x}) \to e^{i\mathbf{v}\cdot\mathbf{x}/2}e^{-i|\mathbf{v}|^2t/4}w(t, \mathbf{x} - \mathbf{v}t)$$

for any velocity $\mathbf{v}$.
(iv) Show that the mass defined by

$$M(w) = \int_{\mathbb{R}^n} |w(t, \mathbf{x})|^2 d\mathbf{x}$$

is a conserved quantity.
(v) Show that the Hamilton density

$$H(w) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla w(t, \mathbf{x})|^2 + \frac{1}{2}G(|w(t, \mathbf{x})|^2)d\mathbf{x}$$

is a conserved quantity.

Problem 12. For a barotropic fluid of index γ the *Navier-Stokes equation* read, in one space dimension

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} + \frac{(\gamma-1)}{2}c\frac{\partial v}{\partial x} = 0 \qquad \text{continuity equation} \qquad (1a)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{2}{(\gamma-1)}c\frac{\partial c}{\partial x} - \frac{\partial^2 v}{\partial x^2} = 0 \qquad \text{Euler's equation} \qquad (1b)$$

where v represents the fluid's velocity and c the sound speed. For the class of solutions characterized by a vanishing presure (i.e., $c = 0$), the above system reduces to the Burgers equation. We assume for simplicity the value $\gamma = 3$ in what follows.
(i) Show that the *velocity potential* Φ exists, and it is given by the following pair of equations

$$\frac{\partial \Phi}{\partial x} = v, \qquad \frac{\partial \Phi}{\partial t} = \frac{\partial v}{\partial x} - \frac{1}{2}(v^2 + c^2). \tag{2}$$

Hint. From (2) with

$$\frac{\partial^2 \Phi}{\partial t \partial x} = \frac{\partial^2 \Phi}{\partial x \partial t}$$

the Euler equation (1b) follows. Thus the condition of integrability of Φ precisely coincides with the Euler equation (1b).
(ii) Consider the similarity ansatz

$$v(x,t) = f(s)\frac{x}{t}, \qquad c(x,t) = g(s)\frac{x}{t} \tag{3}$$

where the similarity variable s is given by

$$s(x,t) := \frac{x}{\sqrt{t}}\,. \tag{4}$$

Show that the Navier-Stokes equation yields the following system of ordinary differential equations

$$s\frac{df}{ds}g + s\left(f - \frac{1}{2}\right)\frac{dg}{ds} + g(2f-1) = 0 \tag{5a}$$

$$2\frac{d^2 f}{ds^2} + \frac{df}{ds}\left(\frac{4}{s} + s(1-2f)\right) + 2f(1-f) = 2g\left(s\frac{dg}{ds} + g\right). \tag{5b}$$

(iii) Show that the continuity equation (1a) admits of a first integral, expressing the law of mass conservation, namely

$$s^2 g(2f-1) = C \tag{6}$$

where C is a constant. (iv) Show that g can be eliminated, and we obtain a second-order ordinary differential equation for the function f.

Problem 13. The nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial u}{\partial x_1} + u^2 = 0 \tag{1}$$

describes the relaxation to a Maxwell distribution. The symmetry vector fields are given by

$$Z_1 = \frac{\partial}{\partial x_1}, \quad Z_2 = \frac{\partial}{\partial x_2}, \quad Z_3 = -x_1 \frac{\partial}{\partial x_1} + u \frac{\partial}{\partial u}, \quad Z_4 = e^{x_2} \frac{\partial}{\partial x_2} - e^{x_2} u \frac{\partial}{\partial u}. \tag{2}$$

Construct a similarity ansatz from the symmetry vector field

$$Z = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2} + c_3 \left(-x_1 \frac{\partial}{\partial x_1} + u \frac{\partial}{\partial u} \right) \tag{3}$$

and find the corresponding ordinary differential equation. Here $c_1, c_2, c_3 \in \mathbb{R}$.

Problem 14. Consider the Kortweg de Vries equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Insert the similarity solution

$$u(x,t) = \frac{w(s)}{(3t)^{2/3}}, \qquad s = \frac{x}{(3t)^{1/3}}$$

where s the similarity variable and find the ordinary differential equation.

Problem 15. Let $\mathbf{u}$ be the velocity field and p the pressure. Show that the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{0}, \quad \mathrm{div}\mathbf{u} = 0$$

admits the Lie symmetry vector fields

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}$$

$$t\frac{\partial}{\partial x_1} + \frac{\partial}{\partial u_1}, \quad t\frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_2}, \quad t\frac{\partial}{\partial x_3} + \frac{\partial}{\partial u_3}$$

$$x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1},$$

$$x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} + u_2 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial u_2},$$

$$x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3} + u_3 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_3}$$

$$2t\frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - u_1 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2} - u_3 \frac{\partial}{\partial u_3} - 2p\frac{\partial}{\partial p}.$$

Find the commutators of the vector fields and thus show that we have a basis of a Lie algebra. Show that $\sqrt{(x_1^2+x_2^2)}/x_3$ is a similarity variable. Do which vector fields does it belong.

Problem 16. The AB system in its canonical form is given by the coupled system of partial differential equations

$$\frac{\partial^2 Q}{\partial\xi\partial\eta} = QS, \qquad \frac{\partial S}{\partial\xi} = -\frac{1}{2}\frac{\partial|Q|^2}{\partial\eta}$$

where ξ and η are semi-characteristic coordinates, Q (complex valued) and S are the wave amplitudes satisfying the normalization condition

$$\left|\frac{\partial Q}{\partial\eta}\right|^2 + S^2 = 1\,.$$

Show that these system of partial differential equations can be written as a compatibility condition

$$\frac{\partial^2\psi_{1,2}}{\partial\xi\partial\eta} = \frac{\partial^2\psi_{1,2}}{\partial\eta\partial\xi}$$

of two linear systems of partial differential equations

$$\frac{\partial\psi_1}{\partial\xi} = F\psi_1 + G\psi_2, \qquad \frac{\partial\psi_1}{\partial\eta} = A\psi_1 + B\psi_2$$

$$\frac{\partial\psi_2}{\partial\xi} = H\psi_1 - F\psi_2, \qquad \frac{\partial\psi_2}{\partial\eta} = C\psi_1 - A\psi_2\,.$$

Problem 17. Show that the system of partial differential equations

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial y}$$

$$\frac{\partial u_2}{\partial t} = \frac{1}{3}\frac{\partial^3 u_1}{\partial y^3} + \frac{8}{3}u_1\frac{\partial u_1}{\partial y}$$

admits the (scaling) Lie symmetry vector field

$$S = -2t\frac{\partial}{\partial t} - y\frac{\partial}{\partial y} + 2u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2}\,.$$

Problem 18. Show that the partial differential equation

$$\frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{\partial^2 u}{\partial x^2}$$

admits the Lie symmetry vector field

$$V = -\frac{\partial}{\partial t} - \frac{1}{2}xu\frac{\partial}{\partial x} + t\frac{\partial}{\partial u}\,.$$

Problem 19. Consider the Kuramoto-Sivashinsky equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^4 u}{\partial x^4} = 0\,.$$

Show that the equation is invariant under the *Galilean transformation*

$$(u, x, t) \mapsto (u + c, x - ct, t)\,.$$

Problem 20. Find the Lie symmetry vector fields for the Monge-Ampére equations

$$\frac{\partial^2 u}{\partial t^2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial^2 u}{\partial x \partial t}\right)^2 = -K$$

where $K = +1$, $K = 0$, $K = -1$.

Problem 21. Let $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{v} = (v_1, v_2, v_3)^T$ and $u_1^2 + u_2^2 + u_3^2 = 1$, $v_1^2 + v_2^2 + v_3^2 = 1$. Let $\times$ be the vector product. The $O(3)$ chiral field equations are the system of partial differential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x} + \mathbf{u} \times R\mathbf{v} = \mathbf{0}, \qquad \frac{\partial \mathbf{v}}{\partial t} - \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v} \times R\mathbf{u} = \mathbf{0}$$

where R is a 3×3 diagonal matrix with non-negative entries. Show that this system of partial differential equations admits a Lax pair (as 4×4 matrices) L, M, i.e. $[L, M] = 0$.

Problem 22. Consider the Schrödinger-Newton equation (MKSA-system)

$$-\frac{\hbar^2}{2m}\Delta\psi + U\psi = \mu\psi$$
$$\Delta U = 4\pi G m^2 |\psi|^2$$

where m is the mass, G the gravitational constant and μ the energy eigenvalues. The normalization consition is

$$\int_{\mathbb{R}^3} |\psi(x_1, x_2, x_3)|^2 dx_1 dx_2 dx_3 = 1\,.$$

Write the Schrödinger-Newton equation in dimension less form. Then find the Lie symmetries of this system of partial differential equation.

Problem 23. Show that the one-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

is invariant under

$$\begin{pmatrix} x'(x,t) \\ ct'(x,t) \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$u'(x'(x,t), t'(x,t)) = u(x,t).$$

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