Problems
in
Matrix Calculus

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Preface

The purpose of this book is to supply a collection of problems in matrix calculus.

Prescribed books for problems.

1) Matrix Calculus and Kronecker Product:
A Pratcial Approach to Linear and Multilinear Algebra, 2nd edition

by Willi-Hans Steeb and Yorick Hardy
World Scientific Publishing, Singapore 2011
ISBN 978 981 4335 31 7
`http://www.worldscibooks.com/mathematics/8030.html`

2) Problems and Solutions in Introductory and Advanced Matrix Calculus

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2006
ISBN 981 256 916 2
`http://www.worldscibooks.com/mathematics/6202.html`

3) Continous Symmetries, Lie Algebras, Differential Equations and Computer Algebra, second edition

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2007
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6515.html`

4) Problems and Solutions in Quantum Computing and Quantum Information, second edition

by Willi-Hans Steeb and Yorick Hardy
World Scientific, Singapore, 2006
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6077.html`

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Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of natural numbers including 0
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\lvert z\rvert$	modulus of complex number z
	$\lvert x+iy\rvert = (x^2+y^2)^{1/2}, \ x,y \in \mathbf{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbf{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\lVert . \rVert$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbf{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbf{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\mathrm{rank}(A)$	rank of matrix A

A^T	transpose of matrix A
$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices A and B
$[A, B] := AB - BA$	commutator for square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator for square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

The Pauli spin matrices are used extensively in the book. They are given by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In some cases we will also use σ_x, σ_y and σ_z to denote σ_1, σ_2 and σ_3.

Chapter 1

Basic Operations

Problem 1. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the *standard basis* in $\mathbb{R}^3$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(i) Consider the normalized vectors

$$\mathbf{a} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \qquad \mathbf{b} = \frac{1}{\sqrt{3}}(-\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3),$$

$$\mathbf{c} = \frac{1}{\sqrt{3}}(-\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3), \qquad \mathbf{d} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3).$$

These vectors are the unit vectors giving the direction of the four bonds of an atom in the *diamond lattice*. Show that the four vectors are linearly dependent.
(ii) Find the scalar products $\mathbf{a}^T\mathbf{b}$, $\mathbf{b}^T\mathbf{c}$, $\mathbf{c}^T\mathbf{d}$, $\mathbf{d}^T\mathbf{a}$. Discuss.

Problem 2. Consider the 4×4 matrix A and the vector $\mathbf{b}$ in $\mathbb{R}^4$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

(i) Are the vectors in $\mathbb{R}^4$ $\mathbf{b}$, $A\mathbf{b}$, $A^2\mathbf{b}$, $A^3\mathbf{b}$ linearly independent?

(ii) Show that the matrix A is invertible. Look at the column vectors of the matrix A

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Find the inverse of A.

Problem 3. Consider the normalized vector $\mathbf{v}$ in $\mathbb{R}^3$ and the permutation matrix P, respectively

$$\mathbf{v} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Are the vectors $\mathbf{v}$, $P\mathbf{v}$, $P^2\mathbf{v}$ linearly independent?

Problem 4. (i) Consider the Hilbert space $M_2(\mathbb{R})$ of the 2×2 matrices over $\mathbb{R}$. Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

are linearly independent.
(ii) Use the Gram-Schmidt orthonormalization technique to find an orthonormal basis for $M_2(\mathbb{R})$.

Problem 5. Consider the vector space of 2×2 matrices over $\mathbb{R}$ and the matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

(i) Are these matrices linearly independent?
(ii) An $n \times n$ matrix is called normal if $MM^* = M^*M$. Which of the matrices A_j $(j = 1, 2, 3, 4)$ are normal matrices?

Problem 6. Are the four 2×2 matrices

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \qquad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

linearly independent?

Problem 7. Let σ_1, σ_2, σ_3 be the Pauli spin matrices and $\sigma_0 = I_2$. Consider the vector space of 2×2 matrices with underlying field $\mathbb{C}$. Show that the 2×2 matrices

$$\frac{1}{\sqrt{2}} I_2, \quad \frac{1}{\sqrt{2}} \sigma_1, \quad \frac{1}{\sqrt{2}} \sigma_2, \quad \frac{1}{\sqrt{2}} \sigma_3$$

are linearly independent.

Problem 8. Consider the normalized vector $\mathbf{v}_0 = (1 \quad 0 \quad 0)^T$ in $\mathbb{R}^3$. Find three normalized vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ such that

$$\sum_{j=0}^{3} \mathbf{v}_j = \mathbf{0}, \qquad \mathbf{v}_j^T \mathbf{v}_k = -\frac{1}{3} \quad (j \neq k).$$

Problem 9. (i) Find four normalized vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ in $\mathbb{R}^3$ such that

$$\mathbf{a}_j^T \mathbf{a}_k = \frac{4}{3}\delta_{jk} - \frac{1}{3} = \begin{cases} 1 & \text{for } j = k \\ -1/3 & \text{for } j \neq k \end{cases}.$$

(ii) Calculate the vector and the matrix

$$\sum_{j=1}^{4} \mathbf{a}_j, \qquad \frac{3}{4}\sum_{j=1}^{4} \mathbf{a}_j \mathbf{a}_j^T.$$

Discuss.

Problem 10. One can describe a *tetrahedron* in the vector space $\mathbb{R}^3$ by specifying vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ normal to its faces with lengths equal to the faces' area. Give an example.

Problem 11. Find the set of all four (column) vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ in $\mathbb{R}^2$ such that the following conditions are satisfied

$$\mathbf{v}_1^T \mathbf{u}_2 = 0, \quad \mathbf{v}_2^T \mathbf{u}_1 = 0, \quad \mathbf{v}_1^T \mathbf{u}_1 = 1, \quad \mathbf{v}_2^T \mathbf{u}_2 = 1.$$

Problem 12. Let $\mathbf{u}$, $\mathbf{v}$ be (column) vectors in $\mathbb{R}^n$. What does

$$A = \sqrt{|(\mathbf{u}^T \mathbf{u})(\mathbf{v}^T \mathbf{v}) - (\mathbf{u}^T \mathbf{v})^2|}$$

calculate?

Problem 13. Consider the 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $a_{11}, a_{12} \in \mathbb{R}$. Can the expression

$$A^3 + 3AC(A+C) + C^3$$

be simplified for computation?

Problem 14. Let A, B be 2×2 matrices. Let $AB = 0_2$ and $BA = 0_2$. Can we conclude that at least one of the two matrices is the 2×2 zero matrix? Prove or disprove.

Problem 15. Let A, C be $n \times n$ matrices over $\mathbb{R}$. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{b}$, $\mathbf{d}$ be column vectors in $\mathbb{R}^n$. Write the system of equations

$$(A + iC)(\mathbf{x} + i\mathbf{y}) = (\mathbf{b} + i\mathbf{d})$$

as a $2n \times 2n$ set of real equations.

Problem 16. Let A, B be $n \times n$ symmetric matrices over $\mathbb{R}$. What is the condition on A, B such that AB is symmetric?

Problem 17. Let A, B be positive definite matrices. Is AB also positive definite? If not, what is the condition on A, B such that AB is positive definite.

Problem 18. Let $m \geq 1$ and $N \geq 2$. Assume that $N > m$. Let X be an $N \times m$ matrix over $\mathbb{R}$ such that $X^* X = I_m$, where I_m is the $m \times m$ unit matrix.
(i) We define

$$P := XX^*.$$

Calculate P^2, P^* and $\mathrm{tr}(P)$.
(ii) Give an example for such a matrix X, where $m = 1$ and $N = 2$.

Problem 19. (i) Compute the matrix product

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 4 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(ii) Write the quadratic polynomial

$$3x_1^2 - 8x_1x_2 + 2x_2^2 + 6x_1x_3 - 3x_3^2$$

in matrix form.

Problem 20. Given the 2×2 matrix A. Find all 2×2 matrices X such that

$$AX = XA.$$

Problem 21. Let A be an $n \times n$ matrix over $\mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$. Show that

$$\Re(\mathbf{x}^* A\mathbf{x}) \equiv \frac{1}{2}\mathbf{x}^*(A + A^*)\mathbf{x}.$$

Problem 22. Let A, B be normal $n \times n$ matrices. Assume that $AB^* = B^*A$ and $BA^* = A^*B$.
(i) Show that their sum $A + B$ is normal.
(ii) Show that their product AB is normal.

Problem 23. Let B be an $n \times n$ hermitian matrix. Is iB skew-hermitian?

Problem 24. Let A be an $n \times n$ *normal matrix*, i.e. $AA^* = A^*A$. Show that $\ker(A) = \ker(A^*)$, where ker denotes the kernel.

Problem 25. Let A be an $n \times n$ hermitian matrix. Show that A^m is a hermitian matrix for all $m \in \mathbb{N}$.

Problem 26. Let A be a hermitian $n \times n$ matrix and $A \neq 0$. Show that $A^m \neq 0$ for all $m \in \mathbb{N}$.

Problem 27. An $n \times n$ matrix is called *normal* if $AA^* = A^*A$. Obviously, a hermitian matrix is normal. Give a 3×3 matrix which is normal but not hermitian.

Problem 28. Let A be an $n \times n$ matrix with $A^2 = 0_n$. Is the matrix $I_n + A$ invertible?

Problem 29. Let A be an $n \times n$ matrix with $A^3 = 0$. Show that $I_n + A$ has an inverse.

Problem 30. Let A, B be $n \times n$ matrices and c a constant. Assume that the inverses of $(A - cI_n)$ and $(A + B - cI_n)$ exist. Show that

$$(A - cI_n)^{-1}B(A + B - cI_n)^{-1} \equiv (A - cI_n)^{-1} - (A + B - cI_n)^{-1}.$$

Problem 31. Represent the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \qquad \text{(relative to the natural basis)}$$

relative to the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Problem 32. Consider the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Let n be a positive integer. Calculate $R^n(\theta)$.

Problem 33. An $n \times n$ matrix K is called a *Cartan matrix* if it satisfies the following properties
(i) $K_{jj} = 2$ for $j = 1, \ldots, N$.
(ii) K_{jk} is a nonpositive integer if $j \neq k$.
(iii) $K_{jk} = 0$ if and only if $K_{jk} = 0$.
(iv) K is positive definite, i.e. it has rank n.

Find a 2×2 Cartan matrix.

Problem 34. Let B, C be $n \times n$ matrices and 0_n the $n \times n$ zero matrix. Consider the $2n \times 2n$ matrix

$$A = \begin{pmatrix} 0_n & B \\ C & 0_n \end{pmatrix}.$$

Find A^2.

Problem 35. Find a 2×2 matrix which is normal but not hermitian.

Problem 36. Consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}$. Find the condition on α, β such that the inverse matrix exists. Find the inverse in this case.

Problem 37. A 3×3 matrix over $\mathbb{R}$ is orthogonal if and only if the columns of A form an orthogonal basis in $\mathbb{R}^3$. Show that the matrix

$$\begin{pmatrix} \sqrt{3}/3 & 0 & -\sqrt{6}/3 \\ \sqrt{3}/3 & \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{6}/6 \end{pmatrix}$$

is orthogonal.

Problem 38. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Can one find a permutation matrix such that $A = PBP^T$?

Problem 39. Let A be an $n \times n$ matrix. Then A can be written as $A = HU$, where H is an $n \times n$ positive semi-definite matrix and U a unitary matrix. Show that $H^2U = UH^2$ if A is normal, i.e. $A^*A = AA^*$.

Problem 40. Can one find an orthogonal matrix over $\mathbb{R}$ such that

$$R^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ?$$

Problem 41. Let 0_n be the $n \times n$ zero matrix and I_n be the $n \times n$ identity matrix. Find an invertible $2n \times 2n$ matrix T such that

$$T^{-1} \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} T = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

Problem 42. Find all 2×2 matrices g over $\mathbb{C}$ such that

$$\det(g) = 1, \qquad \eta g^* \eta = g^{-1}$$

where η is the diagonal matrix $\eta = \mathrm{diag}(1, -1)$.

Problem 43. The $(n+1) \times (n+1)$ *Hadamard matrix* $H(n)$ of any dimension is generated recursively as follows

$$H(n) = \begin{pmatrix} H(n-1) & H(n-1) \\ H(n-1) & -H(n-1) \end{pmatrix}$$

where $n = 1, 2, \ldots$ and $H(0)$ is the 1×1 matrix $H(0) = (1)$.
(i) Find $H(1)$, $H(2)$, and $H(3)$. Find the eigenvalues of $H(1)$ and $H(2)$.
(ii) Find the inverse of $H(1)$, $H(2)$ and $H(3)$.

Problem 44. Let M be an $2n \times 2n$ matrix with $n \geq 1$. Then M can be written in *block form*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $n\times n$ matrices. Assume that M^{-1} exists and that the $n\times n$ matrix D is also nonsingular. Find M^{-1} using this condition.

Problem 45. Let A be an $m\times n$ matrix with $m \geq n$. Assume that A has rank n. Show that there exists an $m\times n$ matrix B such that the $n\times n$ matrix B^*A is nonsingular. The matrix B can be chosen such that $B^*A = I_n$.

Problem 46. Let A be an $m\times n$ matrix over $\mathbb{R}$. Let

$$\mathbf{x} := (1, x, x^2, \ldots, x^{m-1})^T, \qquad \mathbf{y} := (1, y, y^2, \ldots, y^{n-1})^T.$$

Find the extrema of the function

$$p(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{y}.$$

Problem 47. Let A be an $n\times n$ matrix over $\mathbb{C}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ considered as column vectors. Is

$$\mathbf{v}^* A\mathbf{u} = \mathbf{u}^* A^* \mathbf{v}\,?$$

Problem 48. Let $\mathbf{u}$, $\mathbf{v}$ be normalized (column) vectors in $\mathbb{C}^n$. Let A be an $n\times n$ positive semidefinite matrix over $\mathbb{C}$. Show that

$$(\mathbf{u}^*\mathbf{v})(\mathbf{u}^* A\mathbf{v}) \geq 0.$$

Problem 49. Let A be an arbitrary $n\times n$ matrix over $\mathbb{R}$. Can we conclude that A^2 is positive semi-definite?

Problem 50. Show that if hermitian matrices S and T are positive semi-definite and commute ($ST = TS$), then their product ST is also positive semi-definite. We have to show that

$$(ST\mathbf{u})^*\mathbf{u} \geq 0$$

for all $\mathbf{u} \in \mathbb{C}^n$.

Problem 51. Let $\epsilon \in [0, 1]$. Show that the 2×2 matrix

$$\Pi = \begin{pmatrix} \epsilon & \sqrt{\epsilon - \epsilon^2} \\ \sqrt{\epsilon - \epsilon^2} & 1-\epsilon \end{pmatrix}$$

is a projection matrix. What are the eigenvalues of Π. Be clever.

Problem 52. Let $A \in \mathbb{R}^{m\times n}$ be a nonzero matrix. Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ be vectors such that $c := \mathbf{y}^T A\mathbf{x} \neq 0$. Show that the matrix

$$B := A - c^{-1}A\mathbf{x}\mathbf{y}^T A$$

has rank exactly one less than the rank of A.

Problem 53. Let A, B be $n \times n$ idempotent matrices. Show that $A + B$ are idempotent if and only if $AB = BA = 0$.

Problem 54. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that $A + B$ is invertible. Show that

$$\begin{aligned} (A+B)^{-1}A &= I_n - (A+B)^{-1}B \\ A(A+B)^{-1} &= I_n - B(A+B)^{-1}. \end{aligned}$$

Problem 55. Let $\alpha, \beta \in \mathbb{C}$. What is the condition on α, β such that

$$A(\alpha,\beta) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

is a normal matrix?

Problem 56. Find all invertible 2×2 matrices S such that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} S \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 57. (i) Consider the two-dimensional Euclidean space and let $\mathbf{e}_1$, $\mathbf{e}_2$ be the standard basis

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider the vectors

$$\mathbf{v}_0 = \mathbf{0}, \quad \mathbf{v}_1 = \frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_2 = -\frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2.$$

$$\mathbf{v}_3 = -\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_4 = \frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_5 = -\mathbf{e}_1, \quad \mathbf{v}_6 = \mathbf{e}_1.$$

Find the distance between the vectors and select the vectors pairs with the shortest distance.

Problem 58. Given four points $\mathbf{x}_i$, $\mathbf{x}_j$, $\mathbf{x}_k$, $\mathbf{x}_\ell$ (pairwise different) in $\mathbb{R}^2$. One can define their *cross-ratio*

$$r_{ijk\ell} := \frac{|\mathbf{x}_i - \mathbf{x}_j||\mathbf{x}_k - \mathbf{x}_\ell|}{|\mathbf{x}_i - \mathbf{x}_\ell||\mathbf{x}_k - \mathbf{x}_j|}.$$

Show that the cross-rations are invariant under *conformal transformation*.

Problem 59. Consider the vector space $M_2(\mathbb{R})$ of 2×2 matrices over $\mathbb{R}$. Can one find a basis of $M_2(\mathbb{R})$ such that all four matrices are normal and invertible?

Problem 60. Find all 2×2 matrices A over $\mathbb{C}$ such that $A^2 = -I_2$, $A^* = -A$. Extend to 3×3 matrices.

Problem 61. Let $a, b \in \mathbb{R}$ and $a \neq 0$. Find the inverse of the transformation

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Problem 62. Consider the 4×4 matrix

$$A(\alpha, \beta, \gamma) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ -\sin\beta\sinh\alpha & \cos\beta & 0 & -\sin\beta\cosh\alpha \\ \sin\gamma\cos\beta\sinh\alpha & \sin\gamma\sin\beta & \cos\gamma & \sin\gamma\cos\beta\cosh\alpha \\ \cos\gamma\cos\beta\sinh\alpha & \cos\gamma\sin\beta & -\sin\gamma & \cos\gamma\cos\beta\cosh\alpha \end{pmatrix}.$$

(i) Is each column a normalized vector in $\mathbb{R}^4$?
(ii) Calculate the scalar product between the column vectors. Discuss.

Problem 63. (i) Consider the 3×3 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find all 3×3 matrices A such that $PAP^T = A$.
(ii) Consider the 4×4 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find all 4×4 matrices A such that $PAP^T = A$.

Problem 64. Let $0 \le \theta < \pi/4$. Note that $\sec(x) := 1/\cos(x)$. Consider the matrix

$$A(\theta) = \begin{pmatrix} \sec(2\theta) & -i\tan(2\theta) \\ i\tan(2\theta) & \sec(2\theta) \end{pmatrix}.$$

Is the matrix hermitian? Is the matrix orthogonal? Is the matrix unitary? Is the inverse of $A(\theta)$ given by $A(-\theta)$?

Problem 65. Are the 4×4 matrices

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 1 \end{pmatrix}, \qquad \widetilde{P} = I_4 - P$$

projection matrices? If so describe the subspaces of $\mathbb{R}^4$ they project into.

Problem 66. Let A be a positive definite $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{x} \in \mathbb{R}^n$. Show that $A + \mathbf{x}\mathbf{x}^T$ is positive definite.

Problem 67. Let A be a positive definite $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{x} \in \mathbb{R}^n$. Show that $A + \mathbf{x}\mathbf{x}^T$ is positive definite.

Problem 68. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $A^2 = 0_n$. Find the inverse of $I_n + iA$.

Problem 69. Write the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

as a linear combination of the Pauli spin matrices and the 2×2 identity matrix.

Problem 70. Consider the 2×2 matrices

$$S = \begin{pmatrix} r & t \\ t & r \end{pmatrix}, \qquad R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Calculate RSR^T. Discuss.

Problem 71. The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ point to the vertices of a equilateral *triangle*

$$\mathbf{u} = \begin{pmatrix} 1/\sqrt{3} \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1/(2\sqrt{3}) \\ 1/2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1/(2\sqrt{3}) \\ -1/2 \end{pmatrix}.$$

Find the area of this triangle.

Problem 72. Consider the 4×4 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & * \\ 1 & -1 & -1 & * \\ 1 & -1 & 1 & * \\ 1 & 1 & -1 & * \end{pmatrix}.$$

Find the 4-th column non-zero vector in the matrix A so that this vector is orthogonal to each of three other column vectors of the matrix.

Problem 73. Assume that two planes in $\mathbb{R}^3$ given by

$$kx_1 + \ell x_2 + mx_3 + n = 0, \qquad k'x_1 + \ell' x_2 + m'x_3 + n' = 0$$

be the mirror images with respect to a third plane in $\mathbb{R}^3$ given by

$$ax_1 + bx_2 + cx_3 + d = 0.$$

Show that

$$\begin{pmatrix} k' \\ \ell' \\ m' \end{pmatrix} = \frac{1}{a^2+b^2+c^2} \begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & -a^2+b^2-c^2 & 2bc \\ 2ac & 2bc & -a^2-b^2+c^2 \end{pmatrix} \begin{pmatrix} k \\ \ell \\ m \end{pmatrix}.$$

Problem 74. (i) Find the area of the set

$$S_2 := \{ (x_1, x_2) \, : \, 1 \geq x_1 \geq x_2 \geq 0 \}.$$

(ii) Find the volume of the set

$$S_3 := \{ (x_1, x_2, x_3) \, : \, 1 \geq x_1 \geq x_2 \geq x_3 \geq 0 \}.$$

Extend the n-dimensions.

Problem 75. Let A be a hermitian $n \times n$ matrix over $\mathbb{C}$ with $A^2 = I_n$. Find the matrix

$$(A^{-1} + iI_n)^{-1}.$$

Problem 76. Find the 2×2 matrices F and F' from the two equations

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(F + iF'), \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(F - iF').$$

Find the anticommutator of F and F', i.e. $[F, F']_+ \equiv FF' + F'F$.

Problem 77. Let A be an $n \times n$ matrix over $\mathbb{C}$. Find the condition on A such that

$$(I_n + iA)(I_n - iA) = I_n.$$

This means that $(I_n - iA)$ is the inverse of $(I_n + iA)$.

Problem 78. Show that the 4×4 matrices

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 1 \end{pmatrix}$$

and $I_4 - P$ are projection matrices.

Problem 79. Find the corresponding permutation matrix for the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

Problem 80. Consider the vector space $\mathbb{R}^d$. Suppose that $\{\mathbf{v}_j\}_{j=1}^d$ and $\{\mathbf{w}_k\}_{k=1}^d$ are two bases in $\mathbb{R}^d$. Then there is an invertible $d \times d$ matrix

$$T = (t_{jk})_{j,k=1}^d$$

so that

$$\mathbf{v}_j = \sum_{k=1}^d t_{jk}\mathbf{w}_k, \quad j = 1, 2, \ldots, d.$$

The bases $\{\mathbf{v}_j\}_{j=1}^d$ and $\{\mathbf{w}_k\}_{k=1}^d$ are said to have the same *orientation* if $\det(T) > 0$. If $\det(T) < 0$, then they have the opposite orientation. Consider the two bases in $\mathbb{R}^2$

$$\left\{\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \qquad \left\{\mathbf{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \mathbf{w}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.$$

Find the orientation.

Problem 81. (i) Let I_n be the $n \times n$ matrix. Show that the $2n \times 2n$ matrix

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix}$$

is invertible. Find the inverse.

(ii) Show that the $2n \times 2n$ matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n \\ iI_n & -iI_n \end{pmatrix}$$

invertible. Find the inverse.

Problem 82. Let $\phi_1, \phi_2 \in \mathbb{R}$. Consider the vector in $\mathbb{R}^3$

$$\mathbf{v}(\phi_1, \phi_2) = \begin{pmatrix} \cos(\phi_1)\cos(\phi_2) \\ \sin(\phi_2)\cos(\phi_1) \\ \sin(\phi_1) \end{pmatrix}$$

(i) Find the 3×3 matrix $\mathbf{v}(\phi_1, \phi_2)\mathbf{v}^T(\phi_1, \phi_2)$. What type of matrix do we have?
(ii) Find the eigenvalues of the 3×3 matrix $\mathbf{v}(\phi_1, \phi_2)\mathbf{v}^T(\phi_1, \phi_2)$. Compare with $\mathbf{v}^T(\phi_1, \phi_2)\mathbf{v}(\phi_1, \phi_2)$.

Problem 83. Let A, B be $n \times n$ matrices over $\mathbb{C}$. We define (Jordan product)

$$A \cdot B = \frac{1}{2}(AB + BA).$$

Show that $A \cdot B$ is commutative and satisfies

$$A \cdot (A^2 \cdot B) = A^2 \cdot (A \cdot B)$$

but is not associative in general, i.e. $(A \cdot B) \cdot C \neq A \cdot (B \cdot C)$ in general.

Problem 84. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Is

$$\mathrm{tr}(AA^*BB^*) \leq \mathrm{tr}(AA^*)\mathrm{tr}(BB^*)\,?$$

Problem 85. Consider the six 3×3 permutation matrices denoted by P_0, P_1, P_2, P_3, P_4, P_5, where P_0 is the 3×3 identity matrix. Find all triples $(Q_1, Q_2.Q_3)$ of these permutation matrices such that

$$Q_1Q_2Q_3 = Q_3Q_2Q_1$$

where $Q_1 \neq Q_2$, $Q_2 \neq Q_3$, $Q_3 \neq Q_1$.

Problem 86. (i) Find all 2×2 matrices A such that

$$A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A.$$

(ii) Impose the additional conditions such that $\mathrm{tr}(A) = 0$ and $\det(A) = +1$.

(iii) Find all 4×4 matrices B such that

$$B\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) B.$$

Problem 87. Find all hermitian 2×2 matrices H such that

$$H^2 = 0_2.$$

Problem 88. Can one a find θ such that

$$\begin{pmatrix} \cos(\theta)e^{-i\theta} & i\sin(\theta)e^{-i\theta} \\ i\sin(\theta)e^{-i\theta} & \cos(\theta)e^{-i\theta} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}.$$

Problem 89. Can one find 4×4 matrices A and B such that

$$A\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} ?$$

Of course at least one of the matrices A and B must be singular. The underlying field could be $\mathbb{R}$ or $\mathrm{char}(\mathbb{F}) = 2$.

Problem 90. Find all 2×2 matrices such that $A^2 \neq 0_2$ but $A^3 = 0_2$.

Problem 91. Let $\alpha \in \mathbb{R}$. Is the matrix

$$T(\alpha) = \begin{pmatrix} 1-\alpha & 1+\alpha \\ -(1+\alpha) & 1-\alpha \end{pmatrix}$$

invertible for all α?

Problem 92. Consider the four 8×8 binary matrices

$$S_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1 \end{pmatrix}.$$

Find the 8×8 matrices Q_0, Q_1, Q_2, Q_3 such that

$$Q_0 S_0 Q_0^T = S_1, \quad Q_1 S_1 Q_1^T = S_2, \quad Q_2 S_2 Q_2^T = S_3, \quad Q_3 S_3 Q_3^T = S_0.$$

Problem 93. Can one find a (column) vector in $\mathbb{R}^2$ such that $\mathbf{v}\mathbf{v}^T$ is an invertible 2×2 matrix?

Problem 94. Find the conditions on $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$ such that the 4×4 matrix is invertible

$$A(\epsilon_1, \epsilon_2, \epsilon_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \epsilon_1 & \epsilon_2 & \epsilon_3 \\ 0 & \epsilon_2 & -1 & 0 \\ 0 & \epsilon_3 & 0 & -1 \end{pmatrix}$$

is invertible.

Problem 95. Let A, B be invertible $n \times n$ matrices. We define

$$A \bullet B := AB - A^{-1}B^{-1}.$$

Find matrices A, B such that $A \bullet B$ is invertible.

Problem 96. Find all non-hermitian 2×2 matrices A such that $AA^* = I_2$.

Problem 97. The *standard simplex* Δ_n is defined by the set in $\mathbb{R}^n$

$$\Delta_n := \{ (x_1, \ldots, x_n)^T \, : \, x_j \geq 0, \, \sum_{j=1}^{n} x_j = 1 \}.$$

Consider n affinely independent points $B_1, \ldots, B_n \in \Delta_n$. They span an $(n-1)$-simplex denoted by $\Lambda = \mathrm{Con}(B_1, \ldots, B_n)$, that is

$$\Lambda = \mathrm{Con}(B_1, \ldots, B_n) = \{ \lambda_1 B_1 + \cdots + \lambda_n B_n \, : \, \sum_{j=1}^{n} \lambda_j = 1, \; \lambda_1, \ldots, \lambda_n \geq 0 \}.$$

The set corresponds to an invertible $n \times n$ matrix $[\Lambda]$ whose columns are B_1, $\ldots$, B_n. Conversely, consider the matrix $C = (b_{jk})$, where $C_k = (b_{1k}, \ldots, b_{nk})^T$

$(k = 1, \ldots, n)$. If $\det(C) \neq 0$ and the sum of the entries in each column is 1, then the matrix C corresponds to an $(n-1)$-simplex $\mathrm{Con}(B_1, \ldots, B_n)$ in Δ_n. Let C_1 and C_2 be $n \times n$ matrices with nonnegative entries and all the columns of each matrix add up to 1.
(i) Show that $C_1 C_2$ and $C_2 C_1$ are also such matrices.
(ii) Are the $n^2 \times n^2$ matrices $C_1 \otimes C_2$, $C_2 \otimes C_1$ such matrices?

Problem 98. Let $\mathbf{v}$ be a normalized (column) vector in $\mathbb{C}^n$. Consider the $n \times n$ matrix

$$A = \mathbf{v}\mathbf{v}^* - \frac{1}{2} I_n.$$

(i) Find A^* and AA^*.
(ii) Is the matrix A invertible?

Problem 99. Let $\omega > 0$ (fixed) and $t \geq 0$. Is the 4×4 matrix

$$P(t) = \frac{1}{4} \begin{pmatrix} 1+3e^{-\omega t} & 1-e^{-\omega t} & 1-e^{-\omega t} & 1-e^{-\omega t} \\ 1-e^{-\omega t} & 1+3e^{-\omega t} & 1-e^{-\omega t} & 1-e^{-\omega t} \\ 1-e^{-\omega t} & 1-e^{-\omega t} & 1+3e^{-\omega t} & 1-e^{-\omega t} \\ 1-e^{-\omega t} & 1-e^{-\omega t} & 1-e^{-\omega t} & 1+3e^{-\omega t} \end{pmatrix}$$

a stochastic matrix (probabilistic matrix)? Find the eigenvalues of $P(t)$. Let

$$\boldsymbol{\pi} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find the vectors $\boldsymbol{\pi} P(t)$, $(\boldsymbol{\pi} P(t))P(t)$ etc.. Is $P(t_1 + t_2) = P(t_1)P(t_2)$?

Problem 100. Let $\alpha \in [0, 1]$. Consider the stochastic matrix

$$P(\alpha) = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}.$$

Let $n = 1, 2, \ldots$. Show that

$$P^n(\alpha) = \frac{1}{2} \begin{pmatrix} 1+(2\alpha-1)^n & 1-(2\alpha-1)^n \\ 1-(2\alpha-1)^n & 1+(2\alpha-1)^n \end{pmatrix}.$$

Problem 101. Can any skew-hermitian matrix K be written as $K = iH$, where H is a hermitian matrix?

Problem 102. Are the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

nonnegative? Extend to the $n \times n$ case.

Problem 103. Let $a \geq b \geq 0$ and integers. Find the rank of the 4×4 matrix

$$M(a,b) = \begin{pmatrix} a & a & b & b \\ a & b & a & b \\ b & a & b & a \\ b & b & a & a \end{pmatrix}$$

Is the matrix nonnegative?

Problem 104. Find all 2×2 matrices A and B such that

$$A^2 = B^2 = I_2, \qquad AB = -I_2.$$

Problem 105. Find all 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

with $a_{12} \neq 0$ such that $A^2 = I_2$.

Problem 106. Consider the real symmetric matrix A and the vector $\mathbf{v}$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{v} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Calculate $S_j = \mathbf{v}^* A^j \mathbf{v}$ for $j = 1, 2, 3$. Can A be reconstructed from S_1, S_2, S_3 and the information that A is real symmetric?

Problem 107. Let $a, b \in \mathbb{R}$ and $c^2 := a^2 + b^2$ with $c^2 > 0$. Consider the matrix

$$M(a,b) = \frac{1}{c} \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Find the matrix $M(a,b)M(a,b)$. Discuss.

Problem 108. Let $a, b, c, d \in \mathbb{R}$. Let

$$M = aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3.$$

Find M^2 and express it as a linear combination of M and I_2.

Problem 109. Consider the vectors in $\mathbb{R}^2$

$$\mathbf{v}_k = \begin{pmatrix} \cos((k-1)\pi/4 \\ \sin((k-1)\pi/4) \end{pmatrix}, \quad k = 1, 3, 5, 7$$

and

$$\mathbf{v}_k = \sqrt{2}\begin{pmatrix} \cos((k-1)\pi/4) \\ \sin(((k-1)\pi/4) \end{pmatrix}, \quad k = 2,4,6,8$$

which play a role for the lattice Boltzmann model. Find the angles between the vectors. Find the angles between the vectors.

Problem 110. (i) Consider the normalized vectors in $\mathbb{R}^4$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

(i) Do the vectors form a basis in $\mathbb{R}^4$?
(ii) Find the 4×4 matrix $v_1v_1^* + v_2v_2^* + v_3v_3^* + v_4v_4^*$ and then the eigenvalues.
(iii) Find the 4×4 matrix $v_1v_2^* + v_2v_3^* + v_3v_4^* + v_4v_1^*$ and then the eigenvalues.
(iv) Consider the normalized vectors in $\mathbb{R}^4$

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(v) Do the vectors form a basis in $\mathbb{R}^4$?
(vi) Find the 4×4 matrix $w_1w_1^* + w_2w_2^* + w_3w_3^* + w_4w_4^*$ and then the eigenvalues.
(vii) Find the 4×4 matrix $w_1w_2^* + w_2w_3^* + w_3w_4^* + w_4w_1^*$ and then the eigenvalues.

Problem 111. Let $z = x + iy$ with $x, y \in \mathbb{R}$ and thus $\bar{z} = x - iy$. Consider the maps

$$x + iy \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = A(x,y)$$

and

$$1 \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad i \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(i) Calculate z^2 and $A^2(x,y)$. Discuss.
(ii) Find the eigenvalues and normalized eigenvectors of $A(x,y)$.
(iii) Calculate $A(x,y) \otimes A(x,y)$ and find the eigenvalues and normalized eigenvectors.

Problem 112. Let $t \neq 0$. Find the inverse of the matrix

$$M = \begin{pmatrix} t & s \\ 0 & 1 \end{pmatrix}.$$

Problem 113. Consider the normalized vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

and the vector

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

in the Hilbert space $\mathbb{C}^2$. Find

$$\sum_{j=1}^{3} |\mathbf{v}_j^* \mathbf{w}|^2.$$

Problem 114. Let S, T be $n \times n$ matrices over $\mathbb{C}$ with

$$S^2 = I_n, \qquad (TS)^2 = I_n.$$

Thus S and T are invertible. Show that $STS^{-1} = T^{-1}$, $ST^{-1}S = T$.

Problem 115. Let A be a positive definite $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{x} \in \mathbb{R}^n$. Show that $A + \mathbf{x}\mathbf{x}^T$ is positive definite.

Problem 116. (i) Show that the four 2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

form a basis in the vector space of the 2×2 matrices. Which of these matrices are nonnormal?
(ii) Show that the nine 3×3 matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

form a basis in the vector space of the 3×3 matrices. This basis is called the *spiral basis* (and can be extended to any dimension). Which of these matrices are nonnormal?

Problem 117. (i) Consider the 3×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Can one find 2×3 matrices B such that

$$AB = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) Consider the 3×2 matrix

$$X = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Can one find 2×3 matrices Y such that

$$XY = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 118. Find nonzero 2×2 matrices X and Y such that $X + cY$ is invertible for all $c \in \mathbb{C}$.

Problem 119. Consider the four 2×2 matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A^*, \quad B^*.$$

Do the matrices form a basis in the vector space of 2×2 matrices over $\mathbb{R}$?

Problem 120. Consider a three state Markov model (states S_1, S_2, S_3) with the transition matrix between the states given by the stochastic matrix

$$T = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}.$$

Given the system is in the state S_3 at $t = 0$. Find the probabilty $p(S_3, S_3, S_3, S_1, S_1, S_3, S_2, S_3)$.

Nonnormal Matrices

Problem 121. Let A be an $n \times n$ hermitian matrix and P be an $n \times n$ projection matrix. Then PAP is again a hermitian matrix. Is this still true if A is a normal matrix, i.e. $AA^* = A^*A$?

Problem 122. An $n \times n$ matrix over $\mathbb{C}$ is called normal if $MM^* = M^*M$. Let $a, b \in \mathbb{C}$. What is the condition on a, b such that the 2×2 matrix

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

is normal?

Problem 123. Two $n \times n$ matrices A, B are called *similar* if there exists an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Show that the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are similar. Show that the matrices are A, B are nonnormal. Find the commutator $[A, B] = AB - BA$. Is $[A, B]$ nonnormal?

Problem 124. Let A be an $n \times n$ matrix over $\mathbb{C}$ with $A^2 = I_n$. Can we conclude that A is *normal*?

Nilpotent Matrices

Problem 125. An $n \times n$ matrix is called *nilpotent* if some power of it is the zero matrix, i.e. there is a positive integer p such that $A^p = 0_n$. Show that every nonzero nilpotent matrix is nondiagonalizable.

Problem 126. Consider the 4×4 matrix

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Calculate N^2, N^3, N^4. Is the matrix nilpotent?

Problem 127. Is the product of two $n \times n$ nilpotent matrices nilpotent?

Vector Product

Problem 128. Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

be two normalized vectors in $\mathbb{R}^3$. Assume that $\mathbf{x}^T\mathbf{y} = 0$, i.e. the vectors are orthogonal. Is the vector $\mathbf{x} \times \mathbf{y}$ a unit vector again? Here $\times$ denotes the *vector product*. We have

$$\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \tag{1}$$

and

$$x_1^2 + x_2^2 + x_3^2 = 1, \qquad y_1^2 + y_2^2 + y_3^2 = 1 \tag{2}$$

and

$$\mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 = 0. \tag{3}$$

Problem 129. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \equiv \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

where $\cdot$ denotes the scalar product and $\times$ the vector product.

Problem 130. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ be column vectors in $\mathbb{R}^3$. Show that

$$(\mathbf{v}_1 \times \mathbf{v}_2)^T(\mathbf{v}_3 \times \mathbf{v}_4) \equiv (\mathbf{v}_1^T\mathbf{v}_3)(\mathbf{v}_2^T\mathbf{v}_4) - (\mathbf{v}_2^T\mathbf{v}_3)(\mathbf{v}_1^T\mathbf{v}_4).$$

A special case with $\mathbf{v}_3 = \mathbf{v}_1$, $\mathbf{v}_4 = \mathbf{v}_2$ is

$$(\mathbf{v}_1 \times \mathbf{v}_2)^T(\mathbf{v}_1 \times \mathbf{v}_2) \equiv (\mathbf{v}_1^T\mathbf{v}_1)(\mathbf{v}_2^T\mathbf{v}_2) - (\mathbf{v}_2^T\mathbf{v}_1)(\mathbf{v}_1^T\mathbf{v}_2).$$

Problem 131. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and σ_1, σ_2, σ_3 be the Pauli spin matrices. We define

$$\mathbf{a} \cdot \boldsymbol{\sigma} := a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3.$$

What is the condition on $\mathbf{a}$, $\mathbf{b}$ such that

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) \equiv (\mathbf{a} \cdot \mathbf{b})I_2 + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}?$$

Here $\times$ denotes the vector product and I_2 is the 2×2 identity matrix.

Problem 132. Consider the three linear independent normalized column vectors in $\mathbb{R}^3$

$$\mathbf{a}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

(i) Find the "volume"

$$V_{\mathbf{a}} := \mathbf{a}_1^T (\mathbf{a}_2 \times \mathbf{a}_3).$$

(ii) From the three vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ we form the 3×3 matrix

$$A = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Find the determinant and trace. Discuss.
(iii) Find the vectors

$$\mathbf{b}_1 = \frac{1}{V_{\mathbf{a}}} \mathbf{a}_2 \times \mathbf{a}_3, \quad \mathbf{b}_2 = \frac{1}{V_{\mathbf{a}}} \mathbf{a}_3 \times \mathbf{a}_1, \quad \mathbf{b}_3 = \frac{1}{V_{\mathbf{a}}} \mathbf{a}_1 \times \mathbf{a}_2$$

where $\times$ denotes the vector product. Are the vectors linearly independent?

Problem 133. Let $\mathbf{x} \in \mathbb{R}^3$ and $\times$ be the vector product.
(i) Find all solutions of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(ii) Find all solutions of

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

Problem 134. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Show that

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) \equiv (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2.$$

Problem 135. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be vectors in $\mathbb{R}^3$. Show that (*Lagrange identity*)

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \det \begin{pmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{pmatrix}.$$

Problem 136. (i) Consider a *tetrahedron* defined by the triple of linearly independent vectors $\mathbf{v}_j \in \mathbb{R}^3$, $j = 1, 2, 3$. Show that the normal vectors to the faces defined by two of these vectors, normalized to the area of the face, is given by

$$\mathbf{n}_1 = \frac{1}{2}\mathbf{v}_2 \times \mathbf{v}_3, \quad \mathbf{n}_2 = \frac{1}{2}\mathbf{v}_3 \times \mathbf{v}_1, \quad \mathbf{n}_3 = \frac{1}{2}\mathbf{v}_1 \times \mathbf{v}_2.$$

(ii) Show that

$$\mathbf{v}_1 = \frac{2}{3V}\mathbf{n}_2 \times \mathbf{n}_3, \quad \mathbf{v}_2 = \frac{2}{3V}\mathbf{n}_3 \times \mathbf{n}_1, \quad \mathbf{v}_3 = \frac{2}{3V}\mathbf{n}_1 \times \mathbf{n}_2$$

where V is the volume of the tetrahedron given by

$$V = \frac{1}{3!}(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = \sqrt{\frac{2}{9}(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3}.$$

(iii) Apply it to normalized vectors

$$\mathbf{n}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{n}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

which form an orthonormal basis in $\mathbb{R}^3$.

Problem 137. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be three normalized linearly independent vectors in $\mathbb{R}^3$. Give an interpretation of A defined by

$$\cos(\frac{1}{2}A) = \frac{1 + \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_2 \cdot \mathbf{v}_3 + \mathbf{v}_3 \cdot \mathbf{v}_1}{\sqrt{2(1 + \mathbf{v}_1 \cdot \mathbf{v}_2)(1 + \mathbf{v}_2 \cdot \mathbf{v}_3)(1 + \mathbf{v}_3 \cdot \mathbf{v}_1)}}$$

where $\cdot$ denotes the scalar product. Consider first the case where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ denote the standard basis in $\mathbb{R}^3$.

Problem 138. Let A be an 3×3 matrix over $\mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Find the conditions on A, $\mathbf{u}$, $\mathbf{v}$ such that

$$A(\mathbf{u} \times \mathbf{v}) = (A\mathbf{u}) \times (A\mathbf{v}).$$

Problem 139. Consider the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in $\mathbb{R}^3$. Show that

$$\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3) = 0$$

if $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent.

Problem 140. Consider four nonzero vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ in $\mathbb{R}^3$. Let

$$\mathbf{w} := (\mathbf{v}_1 \times \mathbf{v}_2) \times (\mathbf{v}_3 \times \mathbf{v}_4) \neq \mathbf{0}.$$

Find $\mathbf{w} \times (\mathbf{v}_1 \times \mathbf{v}_2)$ and $\mathbf{w} \times (\mathbf{v}_3 \times \mathbf{v}_4)$. Discuss. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ span a plane in $\mathbb{R}^3$ and $\mathbf{v}_3$ and $\mathbf{v}_4$ span a plane in $\mathbb{R}^3$.

Problem 141. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) = \mathbf{0}.$$

Problem 142. Consider the normalized vectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_1 \times \mathbf{v})2$ form an orthonormal basis in $\mathbb{R}^3$?

Chapter 2

Linear Equations

Problem 1. (i) Find all 2×2 matrices A over $\mathbb{R}$ with

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = 1 \tag{1}$$

and

$$A\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2}$$

(ii) Do these matrices form a group under matrix multiplication?

Problem 2. Find all solutions of the linear system

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + x_4 &= 3 \\ 2x_1 - 3x_2 + x_3 + 5x_4 &= -4 \\ 7x_1 - 10x_3 + 13x_4 &= 0. \end{aligned}$$

Problem 3. Consider the area-preserving map of the two-dimensional torus (modulo 1)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} 4 & 15 \\ 1 & 4 \end{pmatrix}$$

where $\det A = 1$ (area-preserving). Consider a rational point on the torus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} n_1/p \\ n_2/p \end{pmatrix}$$

where p is a prime number (except 2, 3, 5) and n_1, n_2 are integers between 0 and $p-1$. One finds that the orbit has the following property. It is periodic and its period T depends on p alone. Consider $p = 7$, $n_1 = 2$, $n_2 = 3$. Find the orbit and the period T.

Problem 4. Solve the linear equation

$$(\, x_1 \quad x_2 \quad x_3 \,) \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (\, x_1 \quad x_2 \quad x_3 \,)\,.$$

Problem 5. *Gordan's theorem* tells us the following. Let A be an $m \times n$ matrix over $\mathbb{R}$ and $\mathbf{c}$ be an n-vector in $\mathbb{R}^n$. Then exactly one of the following systems has a solution:

System 1: $A\mathbf{x} < \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$.
System 2: $A^T\mathbf{p} = \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$ for some $\mathbf{p} \in \mathbb{R}^m$.

Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find out whether system (1) or system (2) has a solution.

Problem 6. *Gordan's theorem* tells us the following: Let A be an $m \times n$ matrix over $\mathbb{R}$. Exactly one of the following systems has a solution:

System 1: $A\mathbf{x} < \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$
System 2: $A^T\mathbf{p} = \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$ for some nonzero $\mathbf{p} \in \mathbb{R}^n$.

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Find out whether system (1) or system (2) has a solution.

Problem 7. *Farkas' theorem* tells us the following. Let A be an $m \times n$ matrix over $\mathbb{R}$ and $\mathbf{c}$ be an n-vector in $\mathbb{R}^n$. Then exactly one of the following systems has a solution:

System 1: $A\mathbf{x} \leq \mathbf{0}$ and $\mathbf{c}^T\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$.

System 2: $A^T\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$ for some $\mathbf{y} \in \mathbb{R}^m$.

Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find out whether system (1) or system (2) has a solution.

Problem 8. Apply the Gauss-Seidel method to solve the linear system

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 9. Let A be an $n \times n$ matrix. Consider the linear equation $A\mathbf{x} = \mathbf{0}$. If the matrix A has rank r, then there are $n - r$ linearly independent solutions of $A\mathbf{x} = \mathbf{0}$. Let $n = 3$ and

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the rank of A and the linearly independent solutions.

Problem 10. Consider the curve described by the equation

$$2x^2 + 4xy - y^2 + 4x - 2y + 5 = 0 \tag{1}$$

relative to the natural basis (standard basis $\mathbf{e}_1 = (1\ \ 0)^T$, $\mathbf{e}_2 = (0\ \ 1)^T$).
(i) Write the equation in matrix form.
(ii) Find an orthogonal change of basis so that the equation relative to the new basis has no crossterms, i.e. no $x'y'$ term. This change of coordinate system does not change the origin.

Problem 11. Consider the 2×2 matrix

$$\begin{pmatrix} b & -a \\ a & b \end{pmatrix}$$

with $a, b \in \mathbb{R}$ and positive determinant, i.e. $a^2 + b^2 > 0$.
(i) Solve the equation

$$\begin{pmatrix} b & -a \\ a & b \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

for the vector $(x_1 \;\; y_1)^T$ with a given vector $(x_0 \;\; y_0)^T$.
(ii) Let

$$M := \begin{pmatrix} b & -a \\ a & b \end{pmatrix}^{-1} \begin{pmatrix} b & a \\ -a & b \end{pmatrix}$$

and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Calculate $M^T JM$.

Problem 12. Suppose that V is a vector space over a field $\mathbb{F}$ and $U \subset V$ is a subspace. We define an equivalence relation $\sim$ on V by $x \sim y$ iff $x - y \in U$. Let $V/U = V/\sim$. Define addition and scalar multiplication on V/U by $[x] + [y] = [x] + [y]$, $c[x] = [cx]$, where $c \in \mathbb{F}$ and

$$[x] = \{\, y \in V \; : \; y \sim x \,\}.$$

Show that these operations do not depend on which representative x we choose.

Problem 13. Consider the vector space $V = \mathbb{C}^2$ and the subspace $U = \{(x_1, x_2) \; : \; x_1 = 2x_2 \,\}$. Find V/U.

Problem 14. Find all solutions of the system of linear equations

$$\begin{pmatrix} 5 & -2 & -4 \\ -2 & 2 & 2 \\ -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Problem 15. Let $b > a$. Consider the system of linear equations

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & x_2^n & \dots & x_n^n \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} b - a \\ (b^2 - a^2)/2 \\ (b^3 - a^3)/3 \\ \vdots \\ (b^{n+1} - a^{n+1})/(n+1) \end{pmatrix}.$$

Let $n = 2$, $a = 0$, $b = 1$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$. Find w_0, w_1, w_2.

Problem 16. Let Y, X, A, B, C, E $n \times n$ matrices over $\mathbb{R}$. Consider the system of matrix equations

$$Y + CE + DX = 0_n, \qquad AE + BX = 0_n.$$

Assume that A has an inverse. Eliminate the matrix E and solve the system for Y

Problem 17. Let V be a vector space over a field $\mathbb{F}$. Let W be a subspace of V. We define an *equivalence relation* $\sim$ on V by stating that $v_1 \sim v_2$ if $v_1 - v_2 \in W$. The *quotient space* V/W is the set of equivalence classes $[v]$ where $v_1 - v_2 \in W$. Thus we can say that v_1 is equivalent to v_2 modulo W if $v_1 = v_2 + w$ for some $w \in W$. Let

$$V = \mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \; : \; x_1, x_2 \in \mathbb{R} \right\}$$

and

$$W = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \; : \; x_1 \in \mathbb{R} \right\}.$$

(i) Is

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 1 \end{pmatrix}?$$

(ii) Give the quotient space.

Problem 18. For the *three-body problem* the following linear transformation plays a role

$$\begin{aligned} X(x_1, x_2, x_3) &= \frac{1}{3}(x_1 + x_2 + x_3) \\ x(x_1, x_2, x_3) &= \frac{1}{\sqrt{2}}(x_1 - x_2) \\ y(x_1, x_2, x_3) &= \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3). \end{aligned}$$

(i) Find the inverse transformation.
(ii) Introduce polar coordinates

$$x(r, \phi) = r \sin\phi, \quad y(r, \phi) = r \cos\phi, \quad r^2 = \frac{1}{3}((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2).$$

Express $(x_1 - x_2)$, $(x_2 - x_3)$, $(x_3 - x_1)$ using this coordinates.

Problem 19. Let $\alpha \in [0, 2\pi)$. Find all solutions of the linear equation

$$\begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Thus x_1 and x_2 depends on α.

Problem 20. Consider the partial differential equation (*Laplace equation*)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on} \quad [0,1] \times [0,1]$$

with the boundary conditions

$$u(x,0)=1, \quad u(x,1)=2, \quad u(0,y)=1, \quad u(1,y)=2.$$

Apply the *central difference scheme*

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{j,k} \approx \frac{u_{j-1,k}-2u_{j,k}+u_{j+1,k}}{(\Delta x)^2}, \quad \left(\frac{\partial^2 u}{\partial y^2}\right)_{j,k} \approx \frac{u_{j,k-1}-2u_{j,k}+u_{j,k+1}}{(\Delta y)^2}$$

and then solve the linear equation. Consider the cases $\Delta x = \Delta y = 1/3$ and $\Delta x = \Delta y = 1/4$.

Problem 21. Let $\mathbf{n}$ and $\mathbf{p}$ be vectors in $\mathbb{R}^n$ with $\mathbf{n} \neq \mathbf{0}$. The set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ which satisfy the equation

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

is called a hyperplane through the point $\mathbf{p} \in \mathbb{R}$. We call $\mathbf{n}$ a normal vector for the hyperplane and call $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ a normal equation for the hyperplane. Find $\mathbf{n}$ and $\mathbf{p}$ in $\mathbb{R}^4$ such that we obtain the hyperplane given by

$$x_1 + x_2 + x_3 + x_4 = \frac{7}{2}.$$

Note that any hyperplane of the Euclidean space $\mathbb{R}^n$ has exactly two unit normal vectors.

Problem 22. (i) The equation of a line in the Euclidean space $\mathbb{R}^2$ passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1).$$

Apply this equation to the points in $\mathbb{R}^2$ given by $(x_1, y_1) = (1, 1/2)$, $(x_2, y_2) = (1/2, 1)$. Consider the unit square with the corner points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ and the map

$$(0,0) \to 0, \ (0,1) \to 0, \ (1,0 \to 0, \ (1,1) \to 1.$$

We can consider this as a 2 input AND-gate. Show that the line constructed above classifies this map.

(ii) The equation of a plane in $\mathbb{R}^3$ passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) in $\mathbb{R}^3$ is given by

$$\det \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix} = 0.$$

Apply this equations to the points

$$(1,1,1/2), \quad (1,1/2,1), \quad (1/2,1,1).$$

Consider the unit cube in $\mathbb{R}^3$ with the corner points (vertices)

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1)$$

$$(1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1)$$

and the map where all corner points are mapped to 0 except for $(1,1,1)$ which is mapped to 1. We can consider this as a 3 input AND-gate. Show that the plane constructed in (i) separates these solutions.

Problem 23. Find the system of linear equations for a and b given by

$$\frac{x+15}{(x+3)(x-1)} = \frac{a}{x+3} + \frac{b}{x-1}.$$

Solve the system of linear equations.

Problem 24. Consider the linear equation written in matrix form

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

First show that the determinant of the 3×3 matrix is nonzero. Apply two different methods (Gauss elimination and the Leverrier's method) to find the solution. Compare the two methods and discuss.

Problem 25. Let A be a given 3×3 matrix over $\mathbb{R}$ with $\det(A) \neq 0$. Is the transformation

$$x'(x,y) = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}$$

$$y'(x,y) = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}$$

invertible? If so find the inverse.

Problem 26. Let $k = 1,2,3,4$ and $x_0 = 1$, $x_5 = 0$. Solve

$$x_{k-1} - 2x_k + x_{k+1} = 0.$$

Problem 27. Find the solution of the system of linear equations

$$x_1 + x_2 + x_3 = 0, \quad x_1 + 2x_2 + x_3 = 1, \quad 2x_1 + x_2 + x_3 = 2.$$

Problem 28. Consider the polynomial

$$p(x) = a + bx + c^2.$$

Find a, b, c from the conditions

$$p(0) = 0, \quad p(1) = 1, \quad p(2) = 0.$$

Problem 29. Let $A \in \mathbb{C}^{n\times m}$ with $n \geq m$ and $\text{rank}(A) = m$.
(i) Show that the $m \times m$ matrix A^*A is invertible.
(ii) We set $P := A(A^*A)^{-1}A$. Show that P is a projection matrix, i.e. $P^2 = P$ and $P = P^*$.

Chapter 3

Traces, Determinants and Hyperdeterminants

Problem 1. Find all 2×2 matrices A over $\mathbb{C}$ that satisfy the three conditions

$$\mathrm{tr}(A) = 0, \quad A = A^*, \quad A^2 = I_2.$$

Problem 2. Find all 2×2 matrices A such that

$$A^2 = \mathrm{tr}(A)A.$$

Calculate $\det(A)$ and $\det(A^2)$ of such a matrix.

Problem 3. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Calculate the trace of σ_1, σ_2, σ_3, $\sigma_1\sigma_2$, $\sigma_1\sigma_3$, $\sigma_2\sigma_3$, $\sigma_1\sigma_2\sigma_3$.

Problem 4. Let A be an $n \times n$ matrix with $A^2 = I_n$. Let B be a matrix with $AB = -BA$, i.e. $[A, B]_+ = 0_n$.
(i) Show that $\mathrm{tr}(B) = 0$.
(ii) Find $\mathrm{tr}(A \otimes B)$.

Problem 5. (i) Consider the two 2×2 matrices

$$A = \begin{pmatrix} a_{11} & 1 \\ a_{21} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 1 \end{pmatrix}.$$

The first column of the matrices A and B agree, but the second column of the two matrices differ. Is

$$\det(A+B) = 2(\det(A) + \det(B))?$$

Problem 6. Let A, B be 2×2 matrices. Assume that $\det(A) = 0$ and $\det(B) = 0$. Can we conclude that $\det(A+B) = 0$?

Problem 7. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be column vectors in $\mathbb{R}^3$. We form the 3×3 matrix

$$M = (\,\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}\,).$$

Is $\det(M) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$, where $\times$ denotes the vector product and $\cdot$ the scalar product?

Problem 8. The oriented volume of an n-simplex in n-dimensional Euclidean space with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, $\dots$, $\mathbf{v}_n$ is given by

$$\frac{1}{n!}\det(S)$$

where S is the $n \times n$ matrix

$$S := (\mathbf{v}_1 - \mathbf{v}_0 \;\; \mathbf{v}_2 - \mathbf{v}_0 \;\; \dots \;\; \mathbf{v}_{n-1} - \mathbf{v}_0 \;\; \mathbf{v}_n - \mathbf{v}_0\,).$$

Thus each column of the $n \times n$ matrix is the difference between the vectors representing two vertices.
(i) Let

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}.$$

Find the oriented volume.
(ii) Let

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find the oriented volume.

Problem 9. The area A of a *triangle* given by the coordinates of its vertices

$$(x_0, y_0), \qquad (x_1, y_1), \qquad (x_2, y_2)$$

is

$$A = \frac{1}{2}\det\begin{pmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}.$$

(i) Let $(x_0, y_0) = (0,0)$, $(x_1, y_1) = (1,0)$, $(x_2, y_2) = (0,1)$. Find A.
(ii) A *tetrahedron* is a polyhedron composed of four triangular faces, three of which meet at each vertex. A tetrahedron can be defined by the coordinates of the vertices

$$(x_0, y_0, z_0), \quad (x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3).$$

The volume V of the tetrahedron is given by

$$V = \frac{1}{6} \det \begin{pmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix}.$$

Let

$$(x_0, y_0, z_0) = (0,0,0), \quad (x_1, y_1, z_1) = (0,0,1),$$
$$(x_2, y_2, z_2) = (0,1,0), \quad (x_3, y_3, z_3) = (1,0,0).$$

Find the volume V.
(iii) Let

$$(x_0, y_0, z_0) = (+1,+1,+1), \quad (x_1, y_1, z_1) = (-1,-1,+1),$$
$$(x_2, y_2, z_2) = (-1,+1,-1), \quad (x_3, y_3, z_3) = (+1,-1,-1).$$

Find the volume V.

Problem 10. Let A, B be $n \times n$ matrices. Assume that $[A, B] = A$. What can be said about the trace of A?

Problem 11. Find all linearly independent diagonal 3×3 matrices over $\mathbb{R}$ with trace zero.

Problem 12. Let A be a 2×2 matrix over $\mathbb{C}$. Assume that $A^2 = I_2$ and thus $\mathrm{tr}(A^2) = 2$. What can be said about the trace of A?

Problem 13. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that

$$\mathrm{tr}(AB) = 0.$$

(i) Can we conclude that $\mathrm{tr}(AB^*) = 0$?
(ii) Consider the case that B is skew-hermitian.

Problem 14. Let

$$A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{u}), \qquad B = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{v})$$

be $n \times n$ matrices, where the first $n-1$ columns $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}$ are the same and for the last column $\mathbf{u} \neq \mathbf{v}$. Show that

$$\det(A+B) = 2^{n-1}(\det(A) + \det(B)).$$

Problem 15. An $n \times n$ tridiagonal matrix ($n \geq 3$) has nonzero elements only in the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal. The determinant of an $n \times n$ tridiagonal matrix can be calculated by the recursive formula

$$\det(A) = a_{n,n} \det[A]_{\{1,\ldots,n-1\}} - a_{n,n-1}a_{n-1,n} \det[A]_{\{1,\ldots,n-2\}}$$

where $\det[A]_{\{1,\ldots,k\}}$ denotes the k-th principal minor, that is, $[A]_{\{1,\ldots,k\}}$ is the submatrix by the first k rows and columns of A. The cost of computing the determinant of a tridiagonal matrix using this recursion is linear in n, while the cost is cubic for a general matrix. Apply this recursion relation to calculate the determinant of the 4×4 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}.$$

Problem 16. (i) Let A be a 2×2 matrix over $\mathbb{R}$. Assume that

$$\operatorname{tr}(A) = \operatorname{tr}(A^2) = \operatorname{tr}(A^3) = \operatorname{tr}(A^4) = 0.$$

Can we conclude that A is the 2×2 zero matrix?
(ii) Assume that A is a normal matrix and satisfies these conditions. Can we conclude that A is the 2×2 zero matrix?

Problem 17. Consider the symmetric $n \times n$ band matrix ($n \geq 3$)

$$M_n = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 1 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix}$$

with the elements in the field $\mathbb{F}_2$. Show that

$$\det(M_n) = \det(M_{n-1}) - \det(M_{n-2})$$

with the initial conditions $\det M_3 = \det M_4 = 1$. Show that the solution is

$$\det(M_n) = \frac{2\sqrt{3}}{3} \cos\left(\frac{n\pi}{3} - \frac{\pi}{6}\right) \pmod 2.$$

Problem 18. Let H be the 8×8 matrix

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & I_4 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let A, D, X, Y be 4×4 matrices over $\mathbb{C}$ and

$$M = \begin{pmatrix} A & X \\ Y & D \end{pmatrix}.$$

Find the conditions on the matrix M such that

$$HM + M^*H = 0_8$$

and $\mathrm{tr}A - \mathrm{tr}D = 0$.

Problem 19. Consider the symmetric 3×3 matrix

$$A(\alpha) = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

(i) Find the maxima and minima of the function

$$f(\alpha) = \det(A(\alpha)).$$

(ii) For which values of α is the matrix noninvertible?

Problem 20. Let A, B be $n \times n$ hermitian positive definite matrices. Show that

$$\mathrm{tr}(AB) > 0.$$

Problem 21. Let A, B be $n \times n$ matrices over $\mathbb{R}$. Assume that A is invertible. Let t be a nonzero real number. Show that

$$\det(A + tB) = t^n \det(A) \det(A^{-1}B + t^{-1}I_n).$$

Problem 22. Let A be an $n \times n$ invertible matrix over $\mathbb{R}$. Show that A^T is also invertible. Is $(A^T)^{-1} = (A^{-1})^T$?

Problem 23. Consider the nonnormal matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find $(\det(A^*A))^{1/2}$ and $(\det(B^*B))^{1/2}$.

Problem 24. Let A be an 2×2 matrix over $\mathbb{R}$. Let I_2 be the 2×2 unit matrix and $\mu \in \mathbb{R}$. Find the determinant of the 4×4 matrix

$$\begin{pmatrix} -\mu I_2 & A \\ A^T & -\mu I_2 \end{pmatrix}.$$

Problem 25. Let A be an 2×2 matrix over $\mathbb{R}$. Calculate

$$r = \mathrm{tr}(A^2) - (\mathrm{tr}(A))^2.$$

What are the conditions on a_{jk} such that $r = 0$?

Problem 26. Let A be a 2×2 symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

over $\mathbb{R}$. We define

$$\frac{\partial}{\partial a_{12}} A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that

$$\frac{\partial}{\partial a_{12}} \mathrm{tr} A^2 = \mathrm{tr}\left(\frac{\partial}{\partial a_{12}} A^2\right) = \mathrm{tr}\left(2A\frac{\partial A}{\partial a_{12}}\right).$$

Problem 27. Let A, B be $n\times n$ matrices over $\mathbb{C}$. Is

$$\mathrm{tr}(A^*B) = \mathrm{tr}(AB^*)\,?$$

Problem 28. Let A, B be 2×2 matrices. Show that

$$[A,B]_+ \equiv AB + BA = (\mathrm{tr}(AB) - \mathrm{tr}(A)\mathrm{tr}(B))I_2 + \mathrm{tr}(A)B + \mathrm{tr}(B)A.$$

Can this identity be extended to 3×3 matrices?

Problem 29. Find all nonzero 2×2 matrices A and B such that

$$BA^* = \mathrm{tr}(AA^*)A.$$

Problem 30. Consider the Hilbert space $M_4(\mathbb{C})$ of all 4×4 matrices over $\mathbb{C}$ with the scalar product $\langle A, B\rangle := \operatorname{tr}(AB^*)$, where $A, B \in M_4(\mathbb{C})$. The γ-matrices are given by

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We define the 4×4 matrices

$$\sigma_{jk} := \frac{i}{2}[\gamma_j, \gamma_k], \qquad j < k$$

where $j = 1, 2, 3$, $k = 2, 3, 4$ and $[\,,\,]$ denotes the commutator.
(i) Calculate σ_{12}, σ_{13}, σ_{14}, σ_{23}, σ_{24}, σ_{34}.
(ii) Do the 16 matrices

$$I_4,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_5\gamma_1,\ \gamma_5\gamma_2,\ \gamma_5\gamma_3,\ \gamma_5\gamma_4,\ \sigma_{12},\ \sigma_{13},\ \sigma_{14},\ \sigma_{23},\ \sigma_{24},\ \sigma_{34}$$

form a basis in the Hilbert space $M_4(\mathbb{C})$? If so is the basis orthogonal?

Problem 31. Let n be even. Consider a skew-symmetric $n \times n$ matrix over $\mathbb{R}$. Let B be a symmetric $n \times n$ matrix over $\mathbb{R}$ with the entries of $B = (b_{jk})$ given by $b_{jk} = b_j b_k$ ($b_j, b_k \in \mathbb{R}$). Show that

$$\det(A + B) = \det(A).$$

Problem 32. Find the determinant and the inverse of the matrix

$$\begin{pmatrix} e^x \cos(x) & e^x \sin(x) \\ -e^{-x} \sin(x) & e^{-x} \cos(x) \end{pmatrix}.$$

Problem 33. Let A, B be 2×2 matrices. Show that

$$\det(A + B) = \det(A) + \det(B) + \det\begin{pmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{pmatrix} + \det\begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Problem 34. Let $n \geq 2$. An invertible integer matrix, $A \in GL(n, \mathbb{Z})$, generates a toral automorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ via the formula

$$f \circ \pi = \pi \circ A, \qquad \pi : \mathbb{R}^n \to \mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n.$$

The set of fixed points of f is given by

$$\mathrm{Fix}(f) := \{ x^* \in \mathbb{T}^n \, : \, f(x^*) = x^* \}.$$

Let $\sharp\mathrm{Fix}(f)$ be the number of fixed points of f. Now we have: if $\det(I_n - A) \neq 0$, then

$$\sharp\mathrm{Fix}(f) = |\det(I_n - A)|.$$

Let $n = 2$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that $\det(I_2 - A) \neq 0$ and find $\sharp\mathrm{Fix}(f)$.

Problem 35. Calculate the determinant of the 4×4 matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

using the *exterior product*. This means calculate

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Problem 36. (i) Let $\alpha \in \mathbb{R}$. Find the determinant of the matrices

$$A(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}.$$

(ii) Let $\alpha \in \mathbb{R}$. Find the determinant of the matrices

$$A(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} \cosh\alpha & i\sinh\alpha \\ -i\sinh\alpha & \cosh\alpha \end{pmatrix}.$$

Problem 37. The 3×3 diagonal matrices over $\mathbb{R}$ with trace equal to 0 form a vector space. Provide a basis for this vector space. Using the scalar product

$\mathrm{tr}(AB^T)$ for $n \times n$ matrices A, B over $\mathbb{R}$ the elements of the basis should be orthogonal to each other.

Problem 38. Let A be a $n \times n$ matrix with $\det A = -1$. Find $\det(A^{-1})$.

Problem 39. The *Hilbert-Schmidt norm* of an $n \times n$ matrix over $\mathbb{C}$ is given by

$$\|A\|_2 = \sqrt{\mathrm{tr}(A^*A)}.$$

Another norm is the *trace norm* given by

$$\|A\|_1 = \mathrm{tr}\sqrt{(A^*A)}.$$

Calculate the two norms for the matrix

$$A = \begin{pmatrix} 0 & -2i \\ i & 0 \end{pmatrix}.$$

Problem 40. The $n \times n$ permutation matrices form a group under matrix multiplications. Show that

$$\det(I_n - P) = 0$$

for any $n \times n$ permutation matrices.

Problem 41. What is the condition on entries of the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}$$

such that the matrix is invertible?

Problem 42. Let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3.$$

What does

$$\frac{1}{2} \det \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

calculate?

Problem 43. Let A be a 3×3 matrix over $\mathbb{R}$. Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Assume that $AP = A$. Is A invertible?

Problem 44. Let $A = (a_{ij})$ be a $2n \times 2n$ skew-symmetric matrix. The *Pfaffian* is defined as

$$\mathrm{Pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} a_{\sigma(2j-1),\sigma(2j)}$$

where S_{2n} is the symmetric group and $\mathrm{sgn}(\sigma)$ is the signature of permutation σ. Consider the case with $n = 2$, i.e.

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Calculate $\mathrm{Pf}(A)$.

Problem 45. Let A be a skew-symmetric $2n \times 2n$ matrix. For the Pfaffian we have the properties

$$(\mathrm{Pf(A)})^2 = \det(A), \qquad \mathrm{Pf}(BAB^T) = \det(B)\mathrm{Pf}(A)$$
$$\mathrm{Pf}(\lambda A) = \lambda^n \mathrm{Pf}(A), \qquad \mathrm{Pf}(A^T) = (-1)^n \mathrm{Pf}(A).$$

where B is an arbitrary $2n \times 2n$ matrix. Let J be a $2n \times 2n$ skew-symmetric matrix with $\mathrm{Pf}(J) \neq 0$. Let B be a $2n \times 2n$ matrix such that $B^T J B = J$. Show that $\det(B) = 1$.

Problem 46. Consider the *Legendre polynomials* P_j, where

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$p_3(x) = \frac{1}{2}(5x^3 - 3x), \quad p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Show that

$$\det \begin{pmatrix} p_0(x) & p_1(x) & p_2(x) \\ p_1(x) & p_2(x) & p_3(x) \\ p_2(x) & p_3(x) & p_4(x) \end{pmatrix} = (1 - x^2)^3 \begin{pmatrix} p_0(0) & 0 & p_2(0) \\ 0 & p_2(0) & 0 \\ p_2(0) & 0 & p_4(0) \end{pmatrix}.$$

Problem 47. Let $n \geq 2$. Consider the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 3 & 4 & \dots & n & 1 \\ 3 & 4 & 5 & \dots & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & 1 & 2 & \dots & n-2 & n-1 \end{pmatrix}.$$

Show that

$$\det(A) = (-1)^{n(n-1)/2}\frac{1}{2}(n+1)n^{n-1}.$$

Problem 48. Let $n \geq 2$. Consider the $n \times n$ matrix

$$A(x) = \begin{pmatrix} c_1 & x & x & \dots & x & x \\ x & c_2 & x & \dots & x & x \\ x & x & c_3 & \dots & x & x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x & x & \dots & x & c_n \end{pmatrix}.$$

Show that

$$\det(A) = (-1)^n(P(n) - xP'(x))$$

where $P(x) = (x - c_1)(x - c_2)\cdots(x - c_n)$.

Problem 49. Let A_1, A_2, A_3 be 2×2 matrices over $\mathbb{R}$. We define

$$c_j = \mathrm{tr}(A_j) \qquad j = 1, 2, 3$$
$$c_{jk} = \mathrm{tr}(A_jA_k) \qquad j, k = 1, 2, 3$$

(i) Given the coefficients c_j, c_{jk} reconstruct the matrices A_1, A_2, A_3.
(ii) Apply the result to the case

$$c_1 = c_2 = c_3 = 0$$

$$c_{11} = 0, \quad c_{12} = 1, \quad c_{13} = 0,$$
$$c_{21} = 1, \quad c_{22} = 0, \quad c_{23} = 0,$$
$$c_{31} = 0, \quad c_{32} = 0, \quad c_{33} = 2.$$

Problem 50. Let V_1 be a hermitian $n \times n$ matrix. Let V_2 be a positive semidefinite $n \times n$ matrix. Let k be a positive integer. Show that

$$\mathrm{tr}((V_2V_1)^k)$$

can be written as $\mathrm{tr}(V^k)$, where $V := V_2^{1/2}V_1V_2^{1/2}$.

Problem 51. Consider the 2×2 matrix

$$M = \begin{pmatrix} \cosh(r) - \sinh(r)\cos(2\theta) & -\sinh(r)\sin(2\theta) \\ -\sinh(r)\sin(2\theta) & \cosh(r) + \sinh(r)\cos(2\theta) \end{pmatrix}.$$

Find the determinant of M. Thus show that the inverse of M exists. Find the inverse of M.

Problem 52. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Is $\mathrm{tr}(AB^*) = \mathrm{tr}(A^*B)$?

Problem 53. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that

$$\mathrm{tr}(A^k) = \mathrm{tr}(A^{k-1}) + \mathrm{tr}(A^{k-2}), \qquad k = 3, 4, \ldots .$$

Problem 54. Let A be an $n \times n$ matrix. Assume that the inverse of A exists, i.e. $\det(A) \neq 0$. Then the inverse $B = A^{-1}$ can be calculated as

$$\frac{\partial}{\partial a_{jk}} \ln(\det(A)) = b_{kj}.$$

Apply this formula to the 2×2 matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Problem 55. Show that the determinant of the matrix

$$A = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

is nonzero. Find the inverse of the matrix.

Problem 56. Consider the 2×2 matrix over $\mathbb{C}$

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Calculate $\det(CC^*)$ and show that $\det(CC^*) \geq 0$.

Problem 57. Let $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an analytic function, where $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^n \times \mathbb{R}^n$. The *Monge-Ampere determinant* $M(\Phi)$ is defined by

$$M(\Phi) := \det \begin{pmatrix} \Phi & \partial\Phi/\partial x_1 & \ldots & \partial\Phi/\partial x_n \\ \partial\Phi/\partial y_1 & \partial^2\Phi/\partial x_1 \partial y_1 & \ldots & \partial^2\Phi/\partial x_n \partial y_1 \\ \vdots & \vdots & \ddots & \vdots \\ \partial\Phi/\partial y_n & \partial^2\Phi/\partial x_1 \partial y_n & \ldots & \partial^2\Phi/\partial x_n \partial y_n \end{pmatrix}.$$

Let $n = 2$ and

$$\Phi(x_1, x_2, y_1, y_2) = x_1^2 + x_2^2 + (x_1 y_1)^2 + (x_2 y_2)^2 + y_1^2 + y_2^2.$$

Find the Monge-Ampere determinant and the conditions on x_1, x_2, y_1, y_2 such that $M(\Phi) = 0$.

Problem 58. (i) Let $z \in \mathbb{C}$. Find the determinant of

$$A = \begin{pmatrix} 1 & z \\ \bar{z} & z\bar{z} \end{pmatrix}.$$

Is the matrix

$$P_2 = I_2 - \frac{1}{1 + z\bar{z}} A$$

a projection matrix?
(ii) Let $z_1, z_2 \in \mathbb{C}$. Find the determinant of

$$B = \begin{pmatrix} 1 & z_1 & z_2 \\ \bar{z}_1 & z_1\bar{z}_1 & z_2\bar{z}_1 \\ \bar{z}_2 & z_1\bar{z}_2 & z_2\bar{z}_2 \end{pmatrix}.$$

Is the matrix

$$P_3 = I_3 - \frac{1}{1 + z_1\bar{z}_1 + z_2\bar{z}_2} B$$

a projection matrix?

Problem 59. Find the determinant of the 4×4 matrices

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

Problem 60. Let T be the 2×2 matrix

$$T = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Calculate $\ln(\det(I_2 - T))$ using the right-hand side of the identity

$$\ln(\det(I_2 - T)) = -\sum_{k=1}^{\infty} \frac{1}{k} \mathrm{tr}(T^k).$$

Problem 61. Let A be an $n \times n$ matrix. Assume that

$$\mathrm{tr}(A^j) = 0, \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Can we conclude that $\det(A) = 0$?

Problem 62. Consider the golden mean number $\tau = (\sqrt{5} - 1)/2$ and the matrix

$$F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}.$$

Find $\mathrm{tr}(F)$ and $\det(F)$. Since $\det(F) \neq 0$ we have an inverse. Find F^{-1}.

Problem 63. Let A be an $n \times n$ matrix and B be an invertible $n \times n$ matrix. Show that

$$\det(I_n + A) = \det(I_n + BAB^{-1}).$$

Problem 64. Let A be an 2×2 matrix. Show that

$$\det(I_2 + A) = 1 + \mathrm{tr}(A) + \det(A).$$

Can the result extended to $\det(I_3 + A)$?

Problem 65. Let A, B be $n \times n$ matrices over $\mathbb{R}$. Assume that $A = A^T$ (symmetric) and $B = -B^T$ (skew-symmetric). Show that $[A, B]$ is symmetric.

Problem 66. Let A be a 2×2 matrix over $\mathbb{C}$. Let

$$t_1 = \mathrm{tr}(A), \quad t_2 = \mathrm{tr}(A^2), \quad t_3 = \mathrm{tr}(A^3), \quad t_4 = \mathrm{tr}(A^4).$$

Can we reconstruct A from t_1, t_2, t_3, t_4?

Problem 67. The *Levi-Civita symbol* (also called completely antisymmetric constant tensor) is defined by

$$\epsilon_{j_1, j_2, \ldots, j_n} := \begin{cases} +1 & \text{if} \quad j_1, j_2, \ldots, j_n \text{ is an even permutation of } 12\cdots n \\ -1 & \text{if} \quad j_1, j_2, \ldots, j_n \text{ is an odd permutation of } 12\cdots n \\ 0 & \text{otherwise} \end{cases}$$

Let δ_{jk} be the Kronecker delta. Show that

$$\epsilon_{j_1, j_2, \ldots, j_n} \epsilon_{k_1, k_2, \ldots, k_n} = \det \begin{pmatrix} \delta_{j_1 k_1} & \delta_{j_2 k_1} & \ldots & \delta_{j_n k_1} \\ \delta_{j_1 k_2} & \delta_{j_2 k_2} & \ldots & \delta_{j_n k_2} \\ \vdots & & & \vdots \\ \delta_{j_1 k_n} & \delta_{j_2 k_n} & \ldots & \delta_{j_n k_n} \end{pmatrix}.$$

Problem 68. Let $x, \epsilon \in \mathbb{R}$. Find the determinant of the symmetric $n \times n$ matrix

$$A = \begin{pmatrix} x+\epsilon & x & \cdots & x \\ x & x+\epsilon & & x \\ \vdots & & \ddots & \\ x & x & & x+\epsilon \end{pmatrix}.$$

Problem 69. Let $\epsilon \in \mathbb{R}$. Let $A(\epsilon)$ be an invertible $n \times n$ matrix. Assume that the entries a_{jk} are analytic functions of ϵ. Show that

$$\mathrm{tr}\left(A^{-1}(\epsilon)\frac{d}{d\epsilon}A(\epsilon)\right) = \frac{1}{\det(A(\epsilon))}\frac{d}{d\epsilon}\det(A(\epsilon)).$$

Problem 70. Let $\{\, \mathbf{e}_j \,\}$ be the three orthonormal vectors in $\mathbb{Z}^3$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We consider the face-centered cubic lattice as a sublattice of $\mathbb{Z}^3$ generated by the three primitive vectors

$$\mathbf{e}_1 + \mathbf{e}_2, \qquad \mathbf{e}_1 + \mathbf{e}_3, \qquad \mathbf{e}_2 + \mathbf{e}_3.$$

(i) Form the 3×3 matrix

$$(\mathbf{e}_1 + \mathbf{e}_2 \quad \mathbf{e}_1 + \mathbf{e}_3 \quad \mathbf{e}_2 + \mathbf{e}_3).$$

(ii) Show that this matrix has an inverse and find the inverse.

Problem 71. Let A, B, C be $n \times n$ matrices. Show that

$$\mathrm{tr}([A,B]C) = \mathrm{tr}(A[B,C]).$$

Problem 72. (i) Let M be a 2×2 matrix over $\mathbb{R}$. Assume that $\mathrm{tr}(M) = 0$. Show that

$$M^2 = -\det(M)I_2.$$

(ii) Show that

$$e^M = \cos(\sqrt{M})I_2 + \frac{\sin(\sqrt{\det(M)})}{\sqrt{\det(M)}}M.$$

If $\det(M) = 0$ then $\sin(0)/0 = 1$. Both $\cos(\alpha)$ and $\sin(\alpha)/\alpha$ are even functions of α and thus $\exp(M)$ is independent of the choice of the square root of $\det(M)$.

Problem 73. Consider the $m \times m$ matrix $F(\mathbf{x}) = (f_{jk}(\mathbf{x}))$ $(j, k = 1, 2, \ldots, m)$, where $f_{jk} : \mathbb{R}^n \to \mathbb{R}$ are analytic functions. Assume that $F(\mathbf{x})$ is invertible for all $\mathbf{x} \in \mathbb{R}^n$. Then we have the identities $(j = 1, 2, \ldots, m)$

$$\frac{\partial(\det(F(\mathbf{x})))}{\partial x_j} \equiv \det(F(\mathbf{x}))\mathrm{tr}\left(F^{-1}(\mathbf{x})\frac{\partial F(\mathbf{x})}{\partial x_j}\right)$$

and

$$\frac{\partial F^{-1}(\mathbf{x})}{\partial x_j} \equiv -F^{-1}(\mathbf{x})\frac{\partial F(\mathbf{x})}{\partial x_j}F^{-1}(\mathbf{x}).$$

The differentiation is understood entrywise. Apply the identities to the matrix $(m = 2, n = 1)$

$$F(x) = \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}.$$

Problem 74. Let $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable functions. Find the determinant of the 3×3 matrix $A = (a_{jk})$

$$a_{jk} := \frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j}.$$

Problem 75. Consider the 3×3 permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Can we find a 3×3 matrix A such that $[P, A]$ is invertible?

Problem 76. Let $n \geq 2$. Consider the $n \times n$ symmetric tridiagonal matrix over $\mathbb{R}$

$$A_n = \begin{pmatrix} c & 1 & 0 & 0 & \cdots & & & & 0 \\ 1 & c & 1 & 0 & \cdots & & & & 0 \\ 0 & 1 & c & 1 & \cdots & & & & 0 \\ \cdots & & & & \cdots & & & & \cdots \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ \cdots & & & & \cdots & & & & \\ 0 & 0 & 0 & & \cdots & 1 & c & 1 & 0 \\ 0 & 0 & 0 & & \cdots & 0 & 1 & c & 1 \\ 0 & 0 & 0 & & \cdots & 0 & 0 & 1 & c \end{pmatrix}$$

where $c \in \mathbb{R}$. Find the determinant of A_n.

Problem 77. An $n \times n$ matrix A is called *idempotent* if $A^2 = A$. Show that

$$\mathrm{rank}(A) = \mathrm{tr}(A).$$

Problem 78. Let A, B, C be $n \times n$ matrices over $\mathbb{C}$. Let $\mathbf{a}_j$, $\mathbf{b}_j$, $\mathbf{c}_j$ ($j = 1, 2, \ldots, n$) be the j-th column of A, B, C, respectively. Show that if for some $k \in \{1, 2, \ldots, n\}$

$$\mathbf{c}_k = \mathbf{a}_k + \mathbf{b}_k$$

and

$$\mathbf{c}_j = \mathbf{a}_j = \mathbf{b}_j, \quad j = 1, \ldots, k-1, k+1, \ldots, n$$

then

$$\det(C) = \det(A) + \det(B).$$

Problem 79. Let R be an nonsingular $n \times n$ matrix over $\mathbb{C}$. Let A be an $n \times n$ matrix over $\mathbb{C}$ of rank one.
(i) Show that the matrix $R + A$ is nonsingular if and only if

$$\mathrm{tr}(R^{-1}A) \neq -1.$$

(ii) Show that in this case we have

$$(R + A)^{-1} = R^{-1} - (1 + \mathrm{tr}(R^{-1}A))^{-1} R^{-1} A R^{-1}.$$

(iii) Simplify to the case that $R = I_n$.

Problem 80. Let A be an $n \times n$ diagonal matrix over $\mathbb{C}$. Let B be an $n \times n$ matrix over $\mathbb{C}$ with $b_{jj} = 0$ for all $j = 1, \ldots, n$. Can we conclude that all diagonal elements of the commutator $[A, B]$ are 0?

Problem 81. (i) Find a nonzero 2×2 matrix V such that

$$V^2 = \mathrm{tr}(V)V.$$

(ii) Can such a matrix be invertible?

Problem 82. Consider the $(n+1) \times (n+1)$ matrix over $\mathbb{C}$

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & \ldots & 0 & z_1 \\ 0 & 1 & 0 & \ldots & \ldots & 0 & z_2 \\ 0 & 0 & 1 & \ldots & \ldots & 0 & z_3 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \ldots & 1 & z_n \\ z_1 & z_2 & z_3 & \ldots & \ldots & z_n & 1 \end{pmatrix}.$$

Find the determinant. What is the condition on the z_j's such that A is invertible?

Problem 83. Let $\phi \in \mathbb{R}$. Consider the unitary matrix

$$U(\phi) = \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix}.$$

Find the minima and maxima of the function $\mathrm{tr}(U(\phi))$.

Problem 84. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that B is invertible. Find

$$\det(I_n + BAB^{-1}).$$

Problem 85. Let $z_k = x_k + iy_k$, where $x_k, y_k \in \mathbb{R}$ and $k = 1, \dots, n$. Find the $2n \times 2n$ matrix A such that

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \\ \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Find the determinant of the matrix A.

Problem 86. Let A, B be $n \times n$ matrices over $\mathbb{C}$. We define the product

$$A \star B := \frac{1}{2}(AB + BA) - \frac{1}{n}\mathrm{tr}(AB)I_n.$$

(i) Find the trace of $A \star B$.
(ii) Is the product commutative? Is the product associative?

Problem 87. Let A, B be $n \times n$ matrices over $\mathbb{R}$ with $\det(A) = 1$ and $\det(B) = 1$. This means A, B are elements of the Lie group $SL(n, \mathbb{R})$. Can we conclude that

$$\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B).$$

Problem 88. Let A, B be two 2×2 matrices. We define the product

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

(i) Find the determinant and trace of $A \star B$. Express the result using $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\det(A)$, $\det(B)$.
(ii) Assume that the inverse of A and B exists. Is

$$(A \star B)^{-1} = A^{-1} \star B^{-1}\,?$$

Problem 89. Let A be an $n \times n$ invertible matrix over $\mathbb{C}$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Then we have the identity

$$\det(A + \mathbf{x}\mathbf{y}^*) \equiv \det(A)(1 + \mathbf{y}^* A^{-1}\mathbf{x}).$$

Can we conclude that $A + \mathbf{x}\mathbf{y}^*$ is also invertible?

Problem 90. Consider a triangle embedded in $\mathbb{R}^3$. Let $\mathbf{v}_j = (x_j, y_j, z_j)$ $(j = 1, 2, 3)$ be the coordinates of the vertices. Then the area A of the triangle is given by

$$A = \frac{1}{2}\|(\mathbf{v}_2 - \mathbf{v}_1) \times (\mathbf{v}_1 - \mathbf{v}_3)\| = \frac{1}{2}\|(\mathbf{v}_3 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_2)\|$$

where $\times$ denotes the vector product and $\|.\|$ denotes the Euclidean norm. The area of the triangle can also be found via

$$A = \frac{1}{2}\sqrt{\left(\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right)^2 + \left(\det\begin{pmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{pmatrix}\right)^2 + \left(\det\begin{pmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{pmatrix}\right)^2}\,.$$

Consider

$$\mathbf{v}_1 = (1, 0, 0), \quad \mathbf{v}_2 = (0, 1, 0), \quad \mathbf{v}_3 = (0, 0, 1).$$

Find the area of the triangle using both expressions. Discuss. The triangle could be one of the faces of a tetrahedron.

Problem 91. A *tetrahedron* has four triangular faces. Given the coordinates of the four vertices

$$(x_0, y_0, z_0), \quad (x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3)$$

the volume of the tetrahedron is given by

$$V = \frac{1}{3!}\det\begin{pmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix}.$$

(i) Given the four vertices $(1, 0, -\sqrt{2})$, $(2, 0, 0)$, $(0, 0, 0)$, $(1, -\sqrt{2}, 0)$ find the volume.

(ii) Derive an equation for surface area of a tetrahedron given by coordinates. Apply it to the vertices given in (i).

Problem 92. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\dots$, $\mathbf{v}_m$ be vectors in $\mathbb{R}^n$. Show that the parallelepiped determined by those vectors has m-dimensional area

$$\sqrt{\det(V^T U)}$$

where V is the $n \times m$ matrix with $\mathbf{v}_1$, $\mathbf{v}_2$, $\dots$, $\mathbf{v}_m$ as its columns.

Problem 93. Let H be a hermitian $n \times n$ matrix. Show that $\det(H + iI_n) \neq 0$.

Problem 94. Find the determinant of the matrices

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}.$$

Extended to $n \times n$ matrices. Then consider the limit $n \to \infty$.

Problem 95. Let $a_j \in \mathbb{R}$ with $j = 1, 2, 3$. Consider the 4×4 matrices

$$A = \frac{1}{2} \begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_1 & 0 & a_2 & 0 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & a_3 & 0 \end{pmatrix}, \qquad B = \frac{1}{2i} \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & a_2 & 0 \\ 0 & -a_2 & 0 & a_3 \\ 0 & 0 & -a_3 & 0 \end{pmatrix}.$$

Find the spectrum of A and B. Find the spectrum of $[A, B]$.

Problem 96. (i) Let A be a 2×2 matrix over $\mathbb{C}$. Given

$$t_1 = \operatorname{tr}(A), \quad t_2 = \operatorname{tr}(A^2), \quad t_3 = \operatorname{tr}(A^3), \quad t_4 = \operatorname{tr}(A^4).$$

Can we reconstruct A from t_1, t_2, t_3, t_4. Does it depend on whether the matrix A is normal?
(ii) Let A be a 2×2 matrix over $\mathbb{C}$. Given

$$d_1 = \det(A), \quad d_2 = \det(A^2), \quad d_3 = \det(A^3), \quad d_4 = \det(A^4).$$

Can we reconstruct A from d_1, d_2, d_3, d_4. Does it depend on whether the matrix A is normal?

Problem 97. Let $A = (a_{jk})$ be an $n \times n$ skew-symmetric matrix over $\mathbb{R}$, i.e. $j, k = 1, \ldots, n$. Let $B = (b_{jk})$ be an $n \times n$ symmetric matrix over $\mathbb{R}$ defined by $b_{jk} = b_j b_k$, i.e. $j, k = 1, \ldots, n$. Let n be even. Show that

$$\det(A + B) = \det(A).$$

Problem 98. Consider the 3×3 matrix M with entries

$$(M)_{jk} = x_k^{j-1}, \quad j, k = 1, 2, 3$$

Find the determinant of this matrix.

Problem 99. Let

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

be an invertible matrix over $\mathbb{R}$. Thus

$$S^{-1} = \frac{1}{\det(S)} \begin{pmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{pmatrix}$$

with $\det(S) = s_{11}s_{22} - s_{12}s_{21}$. Find the condition on the entries s_{jk} such that

$$S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assume that $\det(S) = 1$.

Problem 100. Let f_{11}, f_{22}, f_{33}, f_{44} be analytic functions $f_{jj} : \mathbb{R} \to \mathbb{R}$. Consider the 4×4 matrix

$$M = \begin{pmatrix} f_{11} & 0 & 0 & 0 \\ 1 & f_{22} & 0 & 1 \\ 0 & 0 & f_{33} & 1 \\ f'_{11} & f'_{22} & f'_{33} & 0 \end{pmatrix}$$

where $'$ denotes differentiation with respect to x. Find the determinant of the matrix and write down the ordinary differential equation which follows from $\det(M) = 0$. Find solutions of the differential equation.

Problem 101. (i) Find all 2×2 matrices A_1, A_2, A_3 over $\mathbb{C}$ such that

$$\mathrm{tr}(A_1) = \mathrm{tr}(A_2) = \mathrm{tr}(A_3) = 0$$

and

$$\mathrm{tr}(A_1A_2) = \mathrm{tr}(A_2A_3) = \mathrm{tr}(A_3A_1) = 0.$$

(ii) Find all 3×3 matrices B_1, B_2, B_3 over $\mathbb{C}$ such that

$$\operatorname{tr}(B_1) = \operatorname{tr}(B_2) = \operatorname{tr}(B_3) = 0$$

and

$$\operatorname{tr}(B_1 B_2) = \operatorname{tr}(B_2 B_3) = \operatorname{tr}(B_3 B_1) = 0.$$

Problem 102. Find all 2×2 matrices A such that

$$\det(A) = 1, \quad \operatorname{tr}(A) = 0.$$

Do these matrices form a group under matrix multiplication?

Problem 103. Find the determinant of the matrix

$$A(\alpha) = \begin{pmatrix} 0 & \cos(\alpha) & \sin(\alpha) \\ \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \end{pmatrix}.$$

Problem 104. Let $n \geq 1$. Consider the $2^n \times 2^n$ matrices A and B. Assume that $A^2 = I_{2^n}$, $B^2 = I_{2^n}$ and $[A, B]_+ \equiv AB + BA = 0_{2^n}$. Show that

$$\det(A + B) = 2^{2^n/2}.$$

Problem 105. Let $f_j : \mathbb{R} \to \mathbb{R}$ ($j = 1, \ldots, n$) be analytic functions. Consider the determinant (Wronskian)

$$W(x) = \det \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

denotes $f_j(n-1)$ denotes the $(n-1)$ derivative of f_j. If the Wronskian of these functions is not identially 0 on $\mathbb{R}$, then these functions form a linearly independent set. Let $n = 3$ and

$$f_1(x) = x, \quad f_2(x) = e^x, \quad f_3(x) = e^{2x}.$$

Find the Wronskian. Discuss.

Problem 106. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that B is invertible. Show that there exists $c \in \mathbb{C}$ such that $A + cB$ is not invertible.

Hint. Start of with the identity $A + cB \equiv (AB^{-1} + cI_n)B$ and apply the determinant.

Problem 107. The *permanent* and the determinant of an $n \times n$ matrix M over $\mathbb{C}$ are respectively defined as

$$\mathrm{perm}(M) := \sum_{\pi \in S_n} \left(\prod_{j=1}^{n} M_{j,\pi(j)} \right), \qquad \det(M) := \sum_{\pi \in S_n} (-1)^{sgn(\pi)} \left(\prod_{j=1}^{n} M_{j,\pi(j)} \right)$$

where S_n denotes the symmetric group on a set of n symbols. For an $n \times n$ matrix A and an $m \times m$ matrix B we know that

$$\det(A \otimes B) \equiv (\det(A))^m (\det(B))^n$$

where $\otimes$ denotes the Kronecker product. Study $\mathrm{perm}(A \otimes B)$.

Problem 108. Let A, B be 2×2 matrices over $\mathbb{C}$. What are the conditions on A, B such that

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)\,?$$

Problem 109. Let A be a 4×4 matrix we write as

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where A_1, A_2, A_3, A_4 are the (block) 2×2 matrices in the matrix A. Consider the map

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \mapsto \begin{pmatrix} A_1 & A_2^T \\ A_3^T & A_4 \end{pmatrix}$$

where ^{T} denotes the transpose. Is the trace preserved under this map? Is the determinant preserved under this map?

Hyperdeterminant

Problem 110. The *hyperdeterminant* $\mathrm{Det}(A)$ of the three-dimensional array $A = (a_{ijk}) \in \mathbb{R}^{2\times 2\times 2}$ can be calculated as follows

$$\begin{aligned}\mathrm{Det}(A) = \frac{1}{4}\Bigg(&\det\left(\begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} + \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}\right) \\ &- \det\left(\begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} - \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}\right)\Bigg)^2 \\ &-4\det\begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix}\det\begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}.\end{aligned}$$

Assume that only one of the coefficients a_{ijk} is nonzero. Calculate the hyperdeterminant.

Problem 111. Let $\epsilon_{00} = \epsilon_{11} = 0$, $\epsilon_{01} = 1$, $\epsilon_{10} = -1$, i.e. we consider the 2×2 matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the determinant of a 2×2 matrix $A_2 = (a_{ij})$ with $i, j = 0, 1$ can be defined as

$$\det(A_2) := \frac{1}{2}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{\ell=0}^{1}\sum_{m=0}^{1}\epsilon_{ij}\epsilon_{\ell m}a_{i\ell}a_{jm}.$$

Thus

$$\det(A_2) = a_{00}a_{11} - a_{01}a_{10}.$$

In analogy the *hyperdeterminant* of the $2\times 2\times 2$ array $A_3 = (a_{ijk})$ with $i, j, k = 0, 1$ is defined as

$$\mathrm{Det}A_3 := -\frac{1}{2}\sum_{ii'=0}^{1}\sum_{jj'=0}^{1}\sum_{kk'=0}^{1}\sum_{mm'=0}^{1}\sum_{nn'=0}^{1}\sum_{pp'=0}^{1}\epsilon_{ii'}\epsilon_{jj'}\epsilon_{kk'}\epsilon_{mm'}\epsilon_{nn'}\epsilon_{pp'}a_{ijk}a_{i'j'm}a_{npk'}a_{n'p'm'}.$$

Calculate $\mathrm{Det}A_3$.

Problem 112. Given a $2\times 2\times 2$ *hypermatrix*

$$A = (a_{jk\ell}), \qquad j, k, \ell = 0, 1$$

and the 2×2 matrix

$$S = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix}.$$

The multiplication AS which is again a 2×2 hypermatrix is defined by

$$(AS)_{jk\ell} := \sum_{r=0}^{1} a_{jkr}s_{r\ell}.$$

Assume that $\det(S) = 1$, i.e. $S \in SL(2, \mathbb{C})$. Show that $\mathrm{Det}(AS) = \mathrm{Det}(A)$. This is a typical problem to apply computer algebra. Write a SymbolicC++ program or Maxima program that solves the problem.

Chapter 4

Eigenvalues and Eigenvectors

Problem 1. (i) Let A, B be 2×2 matrices over $\mathbb{R}$ and vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^2$ such that

$$A\mathbf{x} = \mathbf{y}, \qquad B\mathbf{y} = \mathbf{x}$$

$\mathbf{x}^T\mathbf{y} = 0$ and $\mathbf{x}^T\mathbf{x} = 1$, $\mathbf{y}^T\mathbf{y} = 1$. Show that AB and BA have an eigenvalue $+1$.
(ii) Find all 2×2 matrices A, B which satisfy the conditions given in (i). Use

$$\mathbf{x} = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix}.$$

Problem 2. Find the trace, rank, determinant and eigenvalues of the hermitian 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Problem 3. Let A be an arbitrary 2×2 matrix. Show that

$$A^2 - A\mathrm{tr}(A) + I_2 \det(A) = 0$$

and therefore

$$(\mathrm{tr}A)^2 = \mathrm{tr}(A^2) + 2\det(A).$$

Hint. Apply the *Cayley-Hamilton theorem*.

Problem 4. Let A be an $n \times n$ matrix. The matrix A is called *nilpotent* if there is a positive integer r such that $A^r = 0_n$.
(i) Show that the smallest integer r such that $A^r = 0_n$ is smaller or equal to n.
(ii) Find the characteristic polynomial of A.

Problem 5. Find all 2×2 matrices over $\mathbb{R}$ that admit only one eigenvector.

Problem 6. Let $\mathbf{x}$ be a nonzero column vector in $\mathbb{R}^n$ and $n \geq 2$. Consider the $n \times n$ matrix $\mathbf{x}\mathbf{x}^T$. Find one nonzero eigenvalue and the corresponding eigenvector of this matrix.

Problem 7. Consider the 2×2 matrix

$$A(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad a \in \mathbb{R}.$$

Can one find a condition on the parameter a so that A has only one eigenvector?

Problem 8. If $\{ A_j \}_{j=1}^m$ is a commuting family of matrices that is to say $A_j A_k = A_k A_j$ for every pair from the set, then there exists a unitary matrix V such that for all A_j in the set the matrix

$$\widetilde{A}_j = V^* A_j V$$

is upper triangular. Apply this to the matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Problem 9. Consider the 2×2 matrix

$$A = \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & 1/4 \end{pmatrix}.$$

Let (spectral radius)

$$\rho(A) := \max_{1 \leq j \leq 2} |\lambda_j|$$

where λ_j are the eigenvalues of A.
(i) Check that $\rho(A) < 1$.
(ii) If $\rho(A) < 1$, then

$$(I_2 - A)^{-1} = I_2 + A + A^2 + \cdots$$

Calculate $(I_2 - A)^{-1}$.
(iii) Calculate

$$(I_2 - A)(I_2 + A + A^2 + \cdots + A^k).$$

Problem 10. Consider a symmetric 2×2 matrix A over $\mathbb{R}$ with $a_{11} > 0$, $a_{22} > 0$, $a_{12} < 0$ and $a_{jj} > |a_{12}|$ for $j = 1, 2$. Is the matrix A positive definite?

Problem 11. Let A be a positive definite $n \times n$ matrix. Show that A^{-1} exists and is also positive definite.

Problem 12. Let $c_j \in \mathbb{R}$. Find the eigenvalues of the matrices

$$\begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}.$$

Generalize to the $n \times n$ case.

Problem 13. Let n be a positive integer. Consider the 3×3 matrix with rows of elements summing to unity

$$M = \frac{1}{n} \begin{pmatrix} n-a-b & a & b \\ a & n-2a-c & a+c \\ c & a & n-a-c \end{pmatrix}$$

where the values of a, b, c are such that, $0 \leq a$, $0 \leq b$, $a + b \leq n$, $2a + c \leq n$. Thus the matrix is a stochastic matrix. Find the eigenvalues of M.

Problem 14. (i) Find the eigenvalues and normalized eigenvectors of the 3×3 matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(ii) Use the normalized eigenvectors to construct a 3×3 matrix R such that RMR^{-1} is a diagonal matrix.
(iii) Can M be written as

$$M = \sum_{j=1}^{3} \lambda_j \mathbf{v}_j \mathbf{v}_j^T$$

where λ_1, λ_2, λ_3 are the eigenvalues and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are the (column) normalized eigenvectors of M. Prove or disprove.

Problem 15. (i) Find the eigenvalues of the symmetric matrices

$$A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(ii) Extend the results from (i) to find the largest eigenvalue of the symmetric $n \times n$ matrix

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

Problem 16. Find the eigenvalues of the 4×4 symmetric matrix

$$\begin{pmatrix} -1 & \alpha & 0 & 0 \\ \alpha & -1/2 & \alpha & 0 \\ 0 & \alpha & 1/2 & \alpha \\ 0 & 0 & \alpha & 1 \end{pmatrix}.$$

Discuss the eigenvalues $\lambda_j(\alpha)$ as functions of α. Can the eigenvalues cross as function of α?

Problem 17. Consider the $n \times n$ cyclic matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \ldots & a_{1n-1} & a_{1n} \\ a_{1n} & a_{11} & a_{12} & a_{13} & \ldots & a_{1n-2} & a_{1n-1} \\ a_{1n-1} & a_{1n} & a_{11} & a_{12} & \ldots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n} & a_{11} \end{pmatrix}$$

where $a_{jk} \in \mathbb{R}$. Show that

$$\frac{1}{\sqrt{n}} \begin{pmatrix} \epsilon^{2k} \\ \epsilon^{4k} \\ \vdots \\ \epsilon^{2(n-1)k} \\ 1 \end{pmatrix}, \qquad \epsilon \equiv e^{i\pi/n}, \quad 1 \le k \le n.$$

is a normalized eigenvector of A. Find the eigenvalues.

Problem 18. Let $a, b \in \mathbb{R}$. Find on inspection two eigenvectors and the corresponding eigenvalues of the 4×4 matrix

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & 0 & b \\ 0 & 0 & a & b \\ b & b & b & 0 \end{pmatrix}.$$

Problem 19. Let $a, b \in \mathbb{R}$. Find on inspection two eigenvectors and the corresponding eigenvalues of the 4×4 matrix

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & 0 & b \\ 0 & 0 & -a & b \\ b & b & b & 0 \end{pmatrix}.$$

Problem 20. Let $z \in \mathbb{C}$. Find the eigenvalues and eigenvectors of the 3×3 matrix

$$A = \begin{pmatrix} 0 & z & \bar{z} \\ \bar{z} & 0 & z \\ z & \bar{z} & 0 \end{pmatrix}.$$

Discuss the dependence of the eigenvalues on z.

Problem 21. Find the eigenvalues and normalized eigenvectors of the matrix ($\phi \in [0, 2\pi)$)

$$A(\phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\phi} \\ 1 & e^{-i\phi} \end{pmatrix}.$$

Is the matrix invertible? Make the decision by looking at the eigenvalues. If so find the inverse matrix.

Problem 22. Consider the two permutation matrices

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Show that the two matrices have the same (normalized) eigenvectors. Find the commutator $[S, T]$.

Problem 23. Consider the following 3×3 matrix A and vector $\mathbf{v}$ in $\mathbb{R}^3$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} \sin(\alpha) \\ \sin(2\alpha) \\ \sin(3\alpha) \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq n\pi$ with $n \in \mathbb{Z}$. Show that using this vector we can find the eigenvalues and eigenvectors of A. Start of with $A\mathbf{v} = \lambda\mathbf{v}$.

Problem 24. Consider the symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Find a invertible matrix B such that $B^{-1}AB$ is a diagonal matrix.

Problem 25. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the 4×4 gamma matrices

$$\gamma_1 = \begin{pmatrix} 0_2 & \sigma_1 \\ -\sigma_1 & 0_2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0_2 & \sigma_2 \\ -\sigma_2 & 0_2 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0_2 & \sigma_3 \\ -\sigma_3 & 0_2 \end{pmatrix}$$

and

$$\gamma_0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}.$$

Find $\gamma_1\gamma_2\gamma_3\gamma_0$ and $\mathrm{tr}(\gamma_1\gamma_2\gamma_3\gamma_0)$.

Problem 26. Let $c \in \mathbb{R}$ and consider the symmetric 3×3 matrix

$$A = \begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix}.$$

(i) Show that c is an eigenvalue of A and find the corresponding eigenvector.
(ii) Find the two other eigenvalues and eigenvectors.

Problem 27. Let $c \in \mathbb{R}$. Consider the symmetric 4×4 matrix

$$A = \begin{pmatrix} 1 & c & 0 & 0 \\ c & 2 & 2c & 0 \\ 0 & 2c & 3 & c \\ 0 & 0 & c & 4 \end{pmatrix}.$$

(i) Find the characteristic equation.
(ii) Show that

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 10 \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 &= 35 - 6c^2 \\ \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 &= 50 - 30c^2 \\ \lambda_1\lambda_2\lambda_3\lambda_4 &= 24 - 30c^2 + c^4 \end{aligned}$$

where λ_1, λ_2, λ_3, λ_4 denote the eigenvalues.

Problem 28. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find the rank of the matrix. Explain.
(ii) Find the determinant and trace of the matrix.
(iii) Find all eigenvalues of the matrix.
(iv) Find one eigenvector.
(v) Is the matrix positive semidefinite?

Problem 29. Find the eigenvalues of the 3×3 matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

using the trace and the determinant of the matrix and the information that two eigenvalues are the same.

Problem 30. Let B be a 2×2 matrix with eigenvalues λ_1 and λ_2. Find the eigenvalues of the 4×4 matrix

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{v}$ be an eigenvector of B with eigenvalue λ. What can be said about an eigenvector of the 4×4 matrix X given by eigenvector $\mathbf{v}$ and eigenvalue of B.

Problem 31. Consider the 2×2 identity matrix I_2 and the 2×2 matrix

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the eigenvalues λ_1, λ_2 and the corresponding normalized eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of I_2. Then find the eigenvalues μ_1, μ_2 and the corresponding normalized eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of N. Using the normalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}_1$, $\mathbf{v}_2$ form the 2×2 matrix

$$H = \begin{pmatrix} \mathbf{u}_1^* \mathbf{v}_1 & \mathbf{u}_1^* \mathbf{v}_2 \\ \mathbf{u}_2^* \mathbf{v}_1 & \mathbf{u}_2^* \mathbf{v}_2 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of H. Discuss.

Problem 32. Let A, B be two $n \times n$ matrices over $\mathbb{C}$. The set of all matrices of the form $A - \lambda B$ with $\lambda \in \mathbb{C}$ is said to be a *pencil*. The eigenvalues of the pencil are elements of the set $\lambda(A, B)$ defined by

$$\lambda(A, B) := \{ z \in \mathbb{C} : \det(A - zB) = 0 \}.$$

If $\lambda \in \lambda(A, B)$ and

$$A\mathbf{x} = \lambda B\mathbf{x}, \quad \mathbf{x} \neq \mathbf{0}$$

then $\mathbf{x}$ is referred to as an eigenvector of $A - \lambda B$. Note that λ may be finite, empty or infinite.

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Find the eigenvalue of the pencil.

Problem 33. Let $a, b \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$M = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix}.$$

Problem 34. Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the matrix

$$A(\alpha) = \begin{pmatrix} -1 & \alpha & 0 \\ \alpha & 0 & \alpha \\ 0 & \alpha & 1 \end{pmatrix}.$$

Discuss the dependence of the eigenvalues and eigenvectors of α.

Problem 35. Let $\mathbf{u}$, $\mathbf{v}$ be nonzero column vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{v}$. Consider the $n \times n$ matrix A over $\mathbb{R}$

$$A = \mathbf{u}\mathbf{u}^T + \mathbf{u}\mathbf{v}^T - \mathbf{v}\mathbf{u}^T - \mathbf{v}\mathbf{v}^T.$$

Find the nonzero eigenvalues of A and the corresponding eigenvector.

Problem 36. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $c \in \mathbb{C} \setminus \{0\}$. What are the eigenvalues of cA?

Problem 37. Let A be an $n \times n$ matrix over $\mathbb{R}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues. What can be said about the eigenvalues of the $2n \times 2n$ matrix

$$\begin{pmatrix} 0_n & A \\ A^T & 0_n \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix?

Problem 38. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Let $\alpha, \beta \in \mathbb{C}$. Assume that $A^2 = I_n$ and $B^2 = I_n$ and $AB + BA = 0$. What can be said about the eigenvalues of $\alpha A + \beta B$?

Problem 39. Let A be an $n \times n$ normal matrix, i.e. $AA^* = A^*A$. Let $\mathbf{u}$ be an eigenvector of A, i.e. $A\mathbf{u} = \lambda\mathbf{u}$. Show that $\mathbf{u}$ is also an eigenvector of A^* with eigenvalue $\overline{\lambda}$, i.e.

$$A^*\mathbf{u} = \overline{\lambda}\mathbf{u}.$$

Problem 40. Show that eigenvectors of a normal matrix A corresponding to distinct eigenvalues are orthogonal.

Problem 41. Let A, B be square matrices. Show that AB and BA have the same eigenvalues.

Problem 42. Show that if A is an $n \times m$ matrix and if B is an $m \times n$ matrix, then $\lambda \neq 0$ is an eigenvalue of the $n \times n$ matrix AB if and only if λ is an eigenvalue of the $m \times m$ matrix BA. Show that if $m = n$ then the conclusion is true even for $\lambda = 0$.

Problem 43. Let $A^T = (1/2\,,\, 1/2)^T$. Find the eigenvalues of AA^T and A^TA.

Problem 44. We know that a hermitian matrix has only real eigenvalues. Can we conclude that a matrix with only real eigenvalues is hermitian?

Problem 45. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that the eigenvalues of A^*A are nonnegative.

Problem 46. Let $0 \leq x < 1$. Consider the $N \times N$ matrix C (correlation matrix) with the entries

$$C_{jk} := x^{|j-k|}, \quad j, k = 1, \ldots, N.$$

Find the eigenvalues of C. Show that if $N \to \infty$ the distribution of its eigenvalues becomes a continuous function of $\phi \in [0, 2\pi]$

$$\lambda(\phi) = \frac{1 - x^2}{1 - 2x\cos\phi + x^2}.$$

Problem 47. Let n be a positive integer. Consider the 2×2 matrix

$$T_n = \begin{pmatrix} 2n & 4n^2 - 1 \\ 1 & 2n \end{pmatrix}.$$

Show that the eigenvalues of T_n are real and not of absolute value 1.

Problem 48. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that A is normal if and only if there exists an $n \times n$ unitary matrix U and an $n \times n$ diagonal matrix D such that $D = U^{-1}AU$. Note that $U^{-1} = U^*$.

Problem 49. Let A be a normal $n \times n$ matrix over $\mathbb{C}$.
(i) Show that A has a set of n orthonormal eigenvectors.
(ii) Show that if A has a set of n orthonormal eigenvectors, then A is normal.

Problem 50. The Leverrier's method finds the characteristic polynomial of an $n \times n$ matrix. Find the characteristic polynomial for

$$A \otimes B, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

using this method. How are the coefficients c_i of the polynomial related to the eigenvalues?

Problem 51. Consider the symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

over $\mathbb{R}$. Write down the characteristic polynomial $\det(\lambda I_3 - A)$ and express it using the trace and determinant of A.

Problem 52. Let L_n be the $n \times n$ matrix

$$L_n = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \ddots & -1 \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}.$$

Find the eigenvalues.

Problem 53. The *Pascal matrix* of order n is defined as

$$P_n := \left(\frac{(i+j-2)!}{(i-1)!(j-1)!} \right), \qquad i,j = 1,\ldots,n.$$

Thus

$$P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

(i) Find the determinant of P_2, P_3, P_4. Find the inverse of P_2, P_3, P_4.
(ii) Find the determinant for P_n. Is P_n an element of the group $SL(n,\mathbb{R})$?

Problem 54. Let A be an $m \times n$ matrix ($m < n$) over $\mathbb{R}$.
(i) Show that at least one eigenvalue of the $n \times n$ matrix A^TA is equal to 0.
(ii) Show that the eigenvalues of the $m \times m$ matrix AA^T are also eigenvalues of A^TA.

Problem 55. Find the determinant and eigenvalues of the matrices

$$A_2 = \begin{pmatrix} 0 & a_{12} \\ 1 & a_{22} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & a_{13} \\ 1 & 0 & a_{23} \\ 0 & 1 & a_{33} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}.$$

Extend to the n-dimensional case.

Problem 56. Let $j = 1/2, 1, 3/2, 2, \ldots$ and $\phi \in \mathbb{R}$. Consider the $(2j+1) \times (2j+1)$ matrices

$$H = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \\ & & & \ddots & & \\ & & & & & 1 \\ e^{i\phi} & 0 & & & & 0 \end{pmatrix}$$

$$D = \mathrm{diag}(1,\ \omega,\ \omega^2,\ \ldots, \omega^{2j})$$

where $\omega := \exp(i2\pi/(2j+1))$. Is H unitary? Find $\omega DH - HD$.

Problem 57. (i) Find the eigenvalues of the 3×3 matrix

$$A(\alpha) = \begin{pmatrix} e^{\alpha} & 1 & 1 \\ 1 & e^{\alpha} & 1 \\ 1 & 1 & e^{\alpha} \end{pmatrix}.$$

For which values of α is the matrix $A(\alpha)$ not invertible.
(ii) Extend (i) to the $n \times n$ matrix

$$B(\alpha) = \begin{pmatrix} e^\alpha & 1 & \cdots & 1 \\ 1 & e^\alpha & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & e^\alpha \end{pmatrix}.$$

This matrix plays a role for the *Potts model*.

Problem 58. Let A be an 2×2 matrix over $\mathbb{R}$. Let

$$\operatorname{tr}(A) = c_1, \qquad \operatorname{tr}(A^2) = c_2.$$

Can $\det(A)$ be calculated from c_1, c_2?

Problem 59. Let A be an $n \times n$ matrix with entries $a_{jk} \geq 0$ and with positive spectral radius ρ. Then there is a (column) vector $\mathbf{x}$ with $x_j \geq 0$ and a (column) vector $\mathbf{y}$ such that the following conditions hold:

$$A\mathbf{x} = \rho\mathbf{x}, \quad \mathbf{y}^T A = \rho\mathbf{y}, \quad \mathbf{y}^T\mathbf{x} = 1.$$

Consider the 2×2 matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Show that B has a positive spectral radius. Find the vectors $\mathbf{x}$ and $\mathbf{y}$.

Problem 60. Let A be an $n \times n$ matrix over $\mathbb{C}$. If λ is not an eigenvalue of A, then the matrix $(A - \lambda I_n)$ has an inverse, namely the *resolvent*

$$R_\lambda = (A - \lambda I_n)^{-1}.$$

Let λ_j be the eigenvalues of A. For $|\lambda| \geq a$, where a is any positive constant greater than all the $|\lambda_j|$ the resolvent can be expanded as

$$R_\lambda = -\frac{1}{\lambda}\left(I_n + \frac{1}{\lambda}A + \frac{1}{\lambda^2}A^2 + \cdots\right).$$

Calculate

$$-\frac{1}{2\pi i}\oint_{|\lambda|=a} \lambda^m R_\lambda d\lambda, \qquad m = 0, 1, 2, \ldots.$$

Problem 61. Show that the resolvent satisfies the so-called *resolvent equation*

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu.$$

Problem 62. Let $\tau = (1+\sqrt{5})/2$ be the *golden ratio*. Consider the modular matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the eigenvalues λ_1, λ_2 and the corresponding normalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$. Find the projection matrices Π_1 and Π_2 onto the associated eigendirections.

Problem 63. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that $A^2 = -I_n$. What can be said about the eigenvalues of A?

Problem 64. An $n \times n$ unitary matrix U is defined by

$$UU^* = I_n \quad \text{or} \quad U^* = U^{-1}.$$

What can be concluded about the eigenvalues of U if $U^* = U^T$?

Problem 65. Let $\alpha \in \mathbb{R}$. Consider the symmetric matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+\alpha & 0 & 1-\alpha & 0 \\ 0 & 1+\alpha & 0 & 1-\alpha \\ 1-\alpha & 0 & 1+\alpha & 0 \\ 0 & 1-\alpha & 0 & 1+\alpha \end{pmatrix}.$$

Find an inertible matrix B such that

$$A = B^{-1}DB$$

where D is a diagonal matrix and thus find the eigenvalues of A.

Problem 66. The additive inverse eigenvalue problem is as follows: Let A be an $n \times n$ symmetric matrix over $\mathbb{R}$ with $a_{jj} = 0$ for $j = 1, 2, \ldots, n$. Find a real diagonal $n \times n$ matrix D such that the matrix $A + D$ has the prescribed eigenvalues $\lambda_1, \ldots, \lambda_n$. The number of solutions for the real matrix D varies from 0 to $n!$. Consider the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the prescribed eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$. Can one find a D?

Problem 67. Let A be an $n \times n$ matrix over $\mathbb{A}$. Assume that A is normal. Show that A has a set of n orthonormal eigenvectors.

Problem 68. Let $\phi \in \mathbb{R}$. Consider the $n \times n$ matrix

$$H = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & & \\ & & & & & 1 \\ e^{i\phi} & & & & & 0 \end{pmatrix}.$$

(i) Show that the matrix is unitary.
(ii) Find the eigenvalues of H.
(iii) Consider the $n \times n$ diagonal matrix

$$G = \mathrm{diag}(1, \, \omega, \, \omega^2, \, \cdots \, \omega^{n-1})$$

where $\omega := \exp(i2\pi/n$. Find $\omega GH - HG$.

Problem 69. Consider the symmetric 6×6 matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

This matrix plays a role in the construction of the *icosahedron* which is a regular polyhedron with 20 identical equilateral triangular faces, 30 edges and 12 vertices.
(i) Find the eigenvalues of this matrix.
(ii) Consider the matrix $A + \sqrt{5}I_6$. Find the eigenvalues.
(iii) The matrix $A + \sqrt{5}I_6$ induces an Euclidean structure on the quotient space $\mathbb{R}^6/\ker(A + \sqrt{5}I_6)$. Find the dimension of $\ker(A + \sqrt{5}I_6)$.

Problem 70. Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of

$$A(\alpha) = \frac{1}{\sqrt{1+\alpha^2}} \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix}.$$

For which α is $A(\alpha)$ not invertible?

Problem 71. Let $\epsilon \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \epsilon & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon & 0 & 0 & 0 \end{pmatrix}.$$

Extend to the $n \times n$ case.

Problem 72. Let A, B be two $n \times n$ matrices over $\mathbb{C}$.
(i) Show that every eigenvalue of AB is also an eigenvalue of BA.
(ii) Can we conclude that every eigenvector of AB is also an eigenvector of BA?

Problem 73. (i) Find the eigenvalues and eigenvectors of the orthogonal matrices

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad S = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of RS.

Problem 74. Find the eigenvalues and normalized eigenvectors of the 2×2 matrix

$$\begin{pmatrix} \cos\theta & -e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix}.$$

Problem 75. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Extend to the general case n odd.

Problem 76. (i) Consider the matrices

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad B = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ a_3 & 0 & a_4 \end{pmatrix}.$$

How are the eigenvalues of A and B related?
(ii) Let

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

be a density matrix. Is

$$\rho = \begin{pmatrix} \rho_{11} & 0 & \rho_{12} \\ 0 & 0 & 0 \\ \rho_{21} & 0 & \rho_{22} \end{pmatrix}$$

a density matrix?

Problem 77. Let $a, b, c \in \mathbb{R}$. Find the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -ic & ib \\ b & ic & 0 & -ia \\ c & -ib & ia & 0 \end{pmatrix}.$$

Problem 78. Find all 2×2 matrices over the real numbers with only one 1-dimensional eigenspace, i.e. all eigenvectors are linearly dependent.

Problem 79. Let A be an $n \times n$ matrix over $\mathbb{C}^n$. Let λ be an eigenvalue of A. A generalized eigenvector $\mathbf{x} \in \mathbb{C}^n$ of A corresponding to the eigenvalue λ is a nontrivial solution of

$$(A - \lambda I_n)^j \, \mathbf{x} = \mathbf{0}_n$$

for some $j \in \{1, 2, \ldots\}$, where I_n is the $n \times n$ identity matrix and $\mathbf{0}_n$ is the n-dimensional zero vector. For $j = 1$ we find the eigenvectors. It follows that $\mathbf{x}$ is a generalized eigenvector of A corresponding to λ if and only if

$$(A - \lambda I_n)^n \, \mathbf{x} = \mathbf{0}_n.$$

Find the eigenvectors and generalized eigenvectors of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Problem 80. Find all 2×2 matrices over $\mathbb{R}$ which commute with

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

What is the relation between the eigenvectors of these matrices?

Problem 81. Let n be odd and $n \geq 3$. Consider the matrices

$$A_3 = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}, \qquad A_5 = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} \end{pmatrix}$$

and generally

$$A_n = \begin{pmatrix} 1/\sqrt{2} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & \dots & 0 & 0 & 0 & \dots & 1/\sqrt{2} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1/\sqrt{2} & 0 & 1/\sqrt{2} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 1/\sqrt{2} & 0 & -1/\sqrt{2} & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 1/\sqrt{2} & \dots & 0 & 0 & 0 & \dots & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A_3, A_5. The solve the general case.

Problem 82. Assume we know the eigenvalues λ_1, λ_2 of the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

over $\mathbb{C}$. What can be said about the eigenvalues of the 3×3 matrix

$$B = \begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & c & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix}$$

where $c \in \mathbb{C}$.

Problem 83. Let $\epsilon \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & \epsilon \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \epsilon \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & \epsilon \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Extend to $n \times n$ matrices.

Problem 84. Let A, B, C, D, E, F, G, H be 2×2 matrices over $\mathbb{C}$. We define the product

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \star \begin{pmatrix} E & F \\ G & H \end{pmatrix} := \begin{pmatrix} A & O_2 & O_2 & B \\ 0_2 & E & F & 0_2 \\ 0_2 & G & H & 0_2 \\ C & 0_2 & 0_2 & D \end{pmatrix}.$$

Thus the right-hand side is an 8×8 matrix. Assume we know the eigenvalues and eigenvectors of the two 4×4 matrices on the left-hand side. What can be said about the eigenvalues and eigenvectors of the 8×8 matrix of the right-hand side.

Problem 85. The symmetric 3×3 matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

plays a role for the chemical compounds ZnS and $NaCl$. Find the eigenvalues and eigenvectors of A. Then find the inverse of A. Find all $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{x}$.

Problem 86. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a normal matrix over $\mathbb{C}$ with eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, respectively. What can be said about the eigenvalues and eigenvectors of the 3×3 matrices

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}, \quad B_2 = \begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & 1 & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix}, \quad B_3 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}?$$

Problem 87. Let I_n be the $n\times n$ identity matrix and

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

A matrix $S \in \mathbb{R}^{2n\times 2n}$ is called a *symplectic matrix* if

$$S^T J S = J.$$

(i) Show that symplectic matrices are nonsingular.
(ii) Show that the product of two symplectic matrices S_1 and S_2 is also symplectic.
(iii) Show that if S is symplectic S^{-1} and S^T are also symplectic.
(iv) Let S be a symplectic matrix. Show that if $\lambda \in \sigma(S)$, then $\lambda^{-1} \in \sigma(S)$, where $\sigma(S)$ denotes the spectrum of S.

Problem 88. Let A, B be $n\times n$ matrices over $\mathbb{C}$ and $\mathbf{u}$ a nonzero vector in $\mathbb{C}^n$. Assume that $[A,B] = A$ and $A\mathbf{u} = \lambda\mathbf{u}$. Find $(AB)\mathbf{u}$.

Problem 89. Consider the Hilbert space $\mathbb{C}^n$. Let A, B, C be $n\times n$ matrices acting in $\mathbb{C}^n$. We consider the nonlinear eigenvalue problem

$$A\mathbf{u} = \lambda B\mathbf{u} + \lambda^2 C\mathbf{u}$$

where $\mathbf{u} \in \mathbb{C}^n$ and $\mathbf{u} \neq \mathbf{0}$.
(i) Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Find the solutions of the nonlinear eigenvalue problem

$$\sigma_1 \mathbf{u} = \lambda \sigma_2 \mathbf{u} + \lambda^2 \sigma_3 \mathbf{u}$$

where $\mathbf{u} \in \mathbb{C}^2$ and $\mathbf{u} \neq \mathbf{0}$.
(ii) Consider the basis of the simple Lie algebra $s\ell(2, \mathbb{R})$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Solve the nonlinear eigenvalue problem

$$H\mathbf{u} = \lambda E\mathbf{u} + \lambda^2 F\mathbf{u}$$

where $\mathbf{u} \in \mathbb{C}^2$ and $\mathbf{u} \neq \mathbf{0}$.
(iii) Consider the basis of the simple Lie algebra $so(3, \mathbb{R})$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solve the nonlinear eigenvalue problem.

Problem 90. (i) Let A, B be $n \times n$ matrices over $c \in \mathbb{C}$ with $[A, B] = 0$, where $[A, B]$ denotes the commutator of A and B. Calculate $[A + cI_n, B + cI_n]$, where $c \in \mathbb{C}$ and I_n is the $n \times n$ identity matrix.
(ii) Let $\mathbf{x}$ be an eigenvector of the $n \times n$ matrix A with eigenvalue λ. Show that $\mathbf{x}$ is also an eigenvector of $A + cI_n$, where $c \in \mathbb{C}$.

Problem 91. Consider the $n \times n$ tridiagonal matrix

$$\hat{H} = \begin{pmatrix} \epsilon_1 & 1 & . & . & & \\ 1 & \epsilon_2 & 1 & . & & \\ & 1 & \epsilon_3 & 1 & . & \\ & & & & \ddots & \\ & & & & . & 1 \\ & & & & 1 & \epsilon_n \end{pmatrix}.$$

It is used to describe an electron on a linear chain of length n. Find the eigenvalues. Find the eigenvectors. Make the ansatz

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

for the eigenvectors and find a recursion relation for c_j/c_{j+1}.

Problem 92. Find the eigenvalues and eigenvectors of the Hamilton operator

$$\hat{H} = E_0 I_2 - B_1\sigma_1 - B_2\sigma_2 - B_3\sigma_3.$$

Problem 93. Let $\epsilon \in \mathbb{R}$. Find the eigenvalues of

$$A(\epsilon) = \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix}.$$

Do the eigenvalues cross as a function of ϵ?

Problem 94. Let $\epsilon \in [0,1]$. Consider the 2×2 matrix

$$A(\epsilon) = \frac{1}{\sqrt{1+\epsilon^2}} \begin{pmatrix} 1 & \epsilon \\ \epsilon & -1 \end{pmatrix}.$$

For $\epsilon = 0$ we have the Pauli spin matrix σ_3 and for $\epsilon = 1$ we have the Hadamad matrix. Find the eigenvalues and eigenvectors of $A(\epsilon)$.

Problem 95. (i) Consider the matrix

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Find the function (characteristic polynomial)

$$p(\lambda) = \det(A - \lambda I_2).$$

Find the eigenvalues of A by solving $p(\lambda) = 0$. Find the minima of the function

$$f(\lambda) = |p(\lambda)|.$$

Discuss.
(ii) Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the function (characteristic polynomial)

$$p(\lambda) = \det(A - \lambda I_3).$$

Find the eigenvalues of A by solving $p(\lambda) = 0$. Find the minima of the function

$$f(\lambda) = |p(\lambda)|.$$

Discuss.

Problem 96. Let A be an $n \times n$ normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and pairwise orthogonal eigenvectors $\mathbf{u}_j$ $(j = 1, 2, \ldots, n)$. Then

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \mathbf{u}_j^*.$$

Find $\exp(A)$ and $\sin(A)$.

Problem 97. Consider the normalized vector in $\mathbb{C}^3$

$$\mathbf{n} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}.$$

(i) Calculate the 2×2 matrix

$$U(\theta,\phi) = \mathbf{n} \cdot \boldsymbol{\sigma} \equiv n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$$

where σ_1, σ_2, σ_3 are the Pauli spin matrices.
(ii) Is the matrix $U(\theta,\phi)$ unitary? Find the trace and the determinant. Is the matrix $U(\theta,\phi)$ hermitian?
(iii) Find the eigenvalues and normalized eigenvectors of $U(\theta,\phi)$.

Problem 98. Let A be an $n \times n$ normal matrix. Assume that λ_j $(j = 1, \ldots, n)$ are the eigenvalues of A. Calculate

$$\prod_{k=1}^{n}(1 + \lambda_k)$$

without using the eigenvalues.

Problem 99. Let A be an $n \times n$ matrix over $\mathbb{C}$. Then any eigenvalue of A satisfies the inequality

$$|\lambda| \leq \max_{1 \leq j \leq n} \sum_{k=1}^{n} |a_{jk}|.$$

Write a C++ program that calculates the right-hand side of the inequality for a given matrix. Apply the complex class of STL. Apply it to the matrix

$$A = \begin{pmatrix} i & 0 & 0 & i \\ 0 & 2i & 2i & 0 \\ 0 & 3i & 3i & 0 \\ 4i & 0 & 0 & 4i \end{pmatrix}.$$

Problem 100. (i) Find the eigenvalues of the matrices

$$A_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 4 & 0 & 0 & 4 \end{pmatrix}.$$

Extend to the $n \times n$ case.
(ii) Find the eigenvalues of the matrices

$$B_2 = \begin{pmatrix} 1 & 1 \\ 1/2 & 1/2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}.$$

Extend to the $n \times n$ case.

Problem 101. The $2n \times 2n$ symplectic matrix is defined by

$$S = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. The matrix S is unitary and skew-hermitian. Find the eigenvalues of S from this information.

Problem 102. Find the condition on a_{11}, a_{12}, b_{11}, b_{12} such that

$$\begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{12} & b_{11} & 0 \\ a_{12} & 0 & 0 & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

i.e. we have an eigenvalue equation.

Problem 103. Find the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}.$$

Problem 104. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & a_{23} & 0 & a_{34} \\ a_{14} & a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Problem 105. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Problem 106. Find the eigenvalues and eigenvectors of $\sigma_1\sigma_2\sigma_3$.

Problem 107. Find the eigenvalues of the 7×7 matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} & 0 & 0 & a_{17} \\ 0 & a_{22} & a_{23} & 0 & a_{25} & a_{26} & 0 \\ 0 & a_{32} & a_{33} & 0 & a_{35} & a_{36} & 0 \\ a_{41} & 0 & 0 & a_{44} & 0 & 0 & a_{47} \\ 0 & a_{52} & a_{53} & 0 & a_{55} & a_{56} & 0 \\ 0 & a_{62} & a_{63} & 0 & a_{65} & a_{66} & 0 \\ a_{71} & 0 & 0 & a_{74} & 0 & 0 & a_{77} \end{pmatrix}.$$

Problem 108. (i) Find the eigenvalues and eigenvectors of

$$\sigma_3 + i\sigma_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

(ii) Is this matrix normal?

Problem 109. Let A be an $n \times n$ matrix over $\mathbb{C}$, which satisfies

$$A^2 \equiv AA = cA$$

where $c \in \mathbb{C}$ is a constant. Obviously the equation is satisfied by the zero matrix with $c = 0$. Assume that $A \neq 0_n$. Then we have a "type of eigenvalue equation".
(i) Is c an eigenvalue of A.
(ii) Take the determinant of both sides of the equation. Discuss. Study the cases that A is invertible and non-invertible.
(iii) Study the case

$$A(z) = \begin{pmatrix} e^{-z} & 1 \\ 1 & e^{z} \end{pmatrix}, \quad z \in \mathbb{C}.$$

(iv) Study

$$(A \otimes A)^2 = c(A \otimes A).$$

(v) Let A be a 2×2 matrix and

$$A \star A := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Study the case $(A \star A)^2 = c(A \star A)$.
(vi) Study the case that $A^3 = cA$.

Problem 110. (i) Consider the Pauli spin matrices for describing a spin-$\frac{1}{2}$ system

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the matrix

$$\sigma_3 + i\sigma_1.$$

Is the matrix normal? Find the eigenvalues and eigenvectors of the matrix. Discuss. Find the eigenvalues and eigenvectors of $\sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_1$.
(ii) Consider the Pauli spin matrices for describing a spin-1 system

$$s_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider the matrix

$$s_3 + is_1.$$

Is the matrix normal? Find the eigenvalues and eigenvectors of the matrix. Discuss. Find the eigenvalues and eigenvectors of $s_3 \otimes s_3 + is_1 \otimes s_1$.

Problem 111. Let s_1, s_2, s_3 be the $(2s+1) \times (2s+1)$ spin matrices for spin $s = 1/2, s = 1, s = 3/2, s = 2, \ldots$.
(i) For $s = 1/2$ we have the 2×2 matrices

$$s_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$. Calculate the eigenvalues and eigenvectors of

$$n_1 s_1 + n_2 s_2 + n_3 s_3.$$

(ii) For $s = 1$ we have the 3×3 matrices

$$s_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad s_2 = \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad s_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$. Calculate the eigenvalues and eigenvectors of

$$n_1 s_1 + n_2 s_2 + n_3 s_3.$$

Problem 112. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 0 & 0 & a_{13} \\ 1 & 0 & a_{23} \\ 0 & 1 & a_{33} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}.$$

Problem 113. Let K be an $n \times n$ skew-hermitian matrix with eigenvalues μ_1, $\dots$, μ_n (counted according to multiplicity) and the corresponding normalized eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$, where $\mathbf{u}_j^* \mathbf{u}_k = 0$ for $k \neq j$. Then K can be written as

$$K = \sum_{j=1}^{n} \mu_j \mathbf{u}_j \mathbf{u}_j^*$$

and $\mathbf{u}_j \mathbf{u}_j^* \mathbf{u}_k \mathbf{u}_k^* = 0$ for $k \neq j$ and $j, k = 1, 2, \dots, n$. Note that the matrices $\mathbf{u}_j \mathbf{u}_j^*$ are projection matrices and

$$\sum_{j=1}^{n} \mathbf{u}_j \mathbf{u}_j^* = I_n.$$

(i) Calculate $\exp(K)$.
(ii) Every $n \times n$ unitary matrix can be written as $U = \exp(K)$, where K is a skew-hermitian matrix. Find U from a given K.
(iii) Use the result from (ii) to find for a given U a possible K.
(iv) Apply the result from (ii) and (iii) to the unitary 2×2 matrix

$$U(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

(v) Apply the result from (ii) and (iii) to the 2×2 unitary matrix

$$V(\theta, \phi) = \begin{pmatrix} \cos\theta & -e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix}.$$

(vi) Every hermitian matrix H can be written as $H = iK$, where K is a skew-hermitian matrix. Find H for the examples given above.

Problem 114. Consider a symmetric matrix over $\mathbb{R}$. We impose the following conditions. The diagonal elements are all zero. The non-diagonal elements can only be $+1$ or -1. Show that such a matrix can only have integer values as eigenvalues. An example would be

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

with eigenvalues 3 and -1 (three times).

Problem 115. Let A be an $n \times n$ normal matrix over $\mathbb{C}$. How would one apply genetic algorithms to find the eigenvalues of A. This means we have to construct a fitness function f with the minima as the eigenvalues. The eigenvalue equation is given by $A\mathbf{x} = z\mathbf{x}$ ($z \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq \mathbf{0}$). The characteristic equation is

$$p(z) \equiv \det(A - zI_n) = 0.$$

What would be a fitness function? Apply it to the matrices

$$B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 116. Let A, B be hermitian matrices over $\mathbb{C}$ and eigenvalues λ_1, $\dots$, λ_n and μ_1, $\dots$, μ_n, respectively. Assume that $\mathrm{tr}(AB) = 0$ (scalar product). What can be said about the eigenvalues of $A + B$?

Problem 117. Consider the skew-symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Find the eigenvalues. Let 0_3 be the 3×3 zero matrix. Let A_1, A_2, A_3 be skew-symmetric 3×3 matrices over $\mathbb{R}$. Find the eigenvalues of the 9×9 matrix

$$B = \begin{pmatrix} 0_3 & -A_3 & A_2 \\ A_3 & 0_3 & -A_1 \\ -A_2 & A_1 & 0_3 \end{pmatrix}.$$

Problem 118. Consider the 4×4 Haar matrix

$$K = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

Find all 4×4 hermitian matrices H such that $KHK^T = H$.

Problem 119. Consider the reverse-diagonal $n \times n$ matrix

$$A(\phi_1, \ldots, \phi_n) = \begin{pmatrix} 0 & 0 & \ldots & 0 & e^{i\phi_1} \\ 0 & 0 & \ldots & e^{i\phi_2} & 0 \\ \vdots & \vdots & & & \\ 0 & e^{i\phi_{n-1}} & \ldots & 0 & 0 \\ e^{i\phi_n} & 0 & \ldots & 0 & 0 \end{pmatrix}$$

where $\phi_j \in \mathbb{R}$ $(j = 1, \ldots, n)$. Find the eigenvalues and eigenvectors. Is the matrix unitary?

Problem 120. Let $a_{11}, a_{22} \in \mathbb{R}$ and $a_{12} \in \mathbb{C}$. Consider the hermitian matrix

$$H = \begin{pmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{pmatrix}$$

with the real eigenvalues λ_1 and λ_2. What conditions are impost on the matrix elements of H if $\lambda_1 = \lambda_2$?

Problem 121. (i) Consider the spin matrices for describing a spin-$\frac{1}{2}$ system

$$s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the spin matrices for describing a spin-1 system

$$p_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad p_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find the spectrum (eigenvalues and eigenvector) of the hermitian matrix

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = s_1 \otimes p_1 \otimes s_1 + s_2 \otimes p_2 \otimes s_2 + s_3 \otimes p_3 \otimes s_3.$$

Thus $\hat{K}$ is a 12×12 matrix with $\mathrm{tr}(\hat{K}) = 0$.

Problem 122. $s\ell(3, \mathbb{R})$ is the rank 2 Lie algebra with Cartan matrix

$$C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of C.

Problem 123. Find the eigenvalues and normalized eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Problem 124. Let $\phi_k \in \mathbb{R}$. Consider the matrices

$$A(\phi_1,\phi_2,\phi_3,\phi_4) = \begin{pmatrix} 0 & e^{i\phi_1} & 0 & 0 \\ e^{i\phi_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\phi_3} \\ 0 & 0 & e^{i\phi_4} & 0 \end{pmatrix},$$

$$B(\phi_5,\phi_6,\phi_7,\phi_8) = \begin{pmatrix} 0 & 0 & e^{i\phi_5} & 0 \\ 0 & 0 & 0 & e^{i\phi_6} \\ e^{i\phi_7} & 0 & 0 & 0 \\ 0 & e^{i\phi_8} & 0 & 0 \end{pmatrix}$$

and $A(\phi_1,\phi_2,\phi_3,\phi_4)B(\phi_5,\phi_6,\phi_7,\phi_8)$. Find the eigenvalues of these matrices.

Problem 125. Let U be an $n \times n$ unitary matrix, i.e. $UU^* = I_n$. Assume that $U = U^T$. What can be said about the eigenvalues of such a matrix?

Problem 126. Let I_n be the $n \times n$ identity matrix. Find the eigenvalues of the $2n \times 2n$ matrix

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix}.$$

Problem 127. Consider the unitary matrix

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the skew-hermitian matrix K such that $U = \exp(K)$.

Problem 128. Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Prove or disprove that exactly two eigenvalues are 0.

Problem 129. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that the eigenvectors corresponding to distinct eigenvalues are linearly independent.

Problem 130. We know that any $n \times n$ unitary matrix has only eigenvalues λ with $|\lambda| = 1$. Assume that a given $n \times n$ matrix has only eigenvalues with $|\lambda| = 1$. Can we conclude that the matrix is unitary?

Problem 131. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 \\ a_{14} & 0 & 0 & 0 \end{pmatrix}.$$

Problem 132. (i) Let $t_j \in \mathbb{R}$ for $j = 1, 2, 3, 4$. Find the eigenvalues and eigenvectors of

$$\hat{H} = \begin{pmatrix} 0 & t_1 & 0 & t_4 e^{i\phi} \\ t_1 & 0 & t_2 & 0 \\ 0 & t_2 & 0 & t_3 \\ t_4 e^{-i\phi} & 0 & t_3 & 0 \end{pmatrix}.$$

(ii) Let $t_j \in \mathbb{R}$ for $j = 1, \ldots, 5$. Find the eigenvalues and eigenvectors of

$$\hat{H} = \begin{pmatrix} 0 & t_1 & 0 & 0 & t_5 e^{i\phi} \\ t_1 & 0 & t_2 & 0 & 0 \\ 0 & t_2 & 0 & t_3 & 0 \\ 0 & 0 & t_3 & 0 & t_4 \\ t_5 e^{-i\phi} & 0 & 0 & t_4 & 0 \end{pmatrix}.$$

Problem 133. Let $\mathbf{v}$ be a nonzero column vector in $\mathbb{R}^n$. Matrix multiplication is associative. Then we have

$$(\mathbf{v}\mathbf{v}^T)\mathbf{v} = \mathbf{v}(\mathbf{v}^T\mathbf{v}).$$

Discuss.

Problem 134. (i) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 \\ 0 & b_{32} & b_{33} & 0 & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & 0 & 0 \\ 0 & 0 & 0 & b_{54} & b_{55} & 0 \\ 0 & 0 & 0 & 0 & b_{65} & b_{66} \end{pmatrix}.$$

These matrices are the so-called staircase matrices. Extend the results to the $n \times n$ case.

Problem 135. (i) Let A be an invertible $n\times n$ matrix over $\mathbb{C}$. Assume we know the eigenvalues and eigenvectors of A. What can be said about the eigenvalues and eigenvectors of $A+A^{-1}$?
(ii) Apply the result from (i) to the permutation matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 136. (i) Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the 6×6 matrix

$$B = \begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 & a_{16} \\ 0 & b_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} & 0 \\ b_{61} & 0 & 0 & 0 & 0 & b_{66} \end{pmatrix}.$$

Problem 137. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}.$$

Extend to the $n \times n$ case.

Problem 138. Find the eigenvalues of the 6×6 matrix

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 \\ 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 \end{pmatrix}.$$

Problem 139. (i) Let $\alpha \in \mathbb{R}$. Consider the matrices

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

Find the trace and determinant of these matrices. Show that for the matrix $A(\alpha)$ the eigenvalues depend on α but the eigenvectors do not. Show that for the matrix $B(\alpha)$ the eigenvalues do not depend on α but the eigenvectors do.
(ii) Let $\alpha \in \mathbb{R}$. Consider the matrices

$$C(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \qquad D(\alpha) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ \sinh\alpha & -\cosh\alpha \end{pmatrix}.$$

Find the trace and determinant of these matrices. Show that for the matrix $C(\alpha)$ the eigenvalues depend on α but the eigenvectors do not. Show that for the matrix $D(\alpha)$ the eigenvalues do not depend on α but the eigenvectors do.

Problem 140. Find the lowest eigenvalue of the 4×4 symmetric matrix ($x \in \mathbb{R}$)

$$\begin{pmatrix} 0 & -x\sqrt{5} & 0 & 0 \\ -x\sqrt{5} & 4 & -2x & -2x \\ 0 & -2x & 4-2x & -x \\ 0 & -2x & -x & 8-2x \end{pmatrix}.$$

Problem 141. Let $m > 0$ and $\theta \in \mathbb{R}$. Consider the three 3×3 matrices

$$M_1 = m \begin{pmatrix} 0 & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \\ 0 & \cos\theta & 0 \end{pmatrix}, \quad M_2 = m \begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & \cos\theta \\ \sin\theta & \cos\theta & 0 \end{pmatrix},$$

$$M_3 = m \begin{pmatrix} 0 & \sin\theta & \cos\theta \\ \sin\theta & 0 & 0 \\ \cos\theta & 0 & 0 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of the matrices. These matrices play a role for the Majorana neutrino.

Problem 142. Let A, B be real symmetric and block tridiagonal 4×4 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{12} & b_{22} & b_{23} & 0 \\ 0 & b_{23} & b_{33} & b_{34} \\ 0 & 0 & b_{34} & b_{44} \end{pmatrix}.$$

Assume that B is positive definite. Solve the eigenvalue problem

$$A\mathbf{v} = \lambda B\mathbf{v}.$$

Problem 143. Let A be an $n \times n$ matrix over $\mathbb{C}$. What is the condition on A such that all eigenvalues are 0 and A admits only one eigenvector.

Problem 144. Let A be a 2×2 matrix. Assume that $\det(A) = 0$ and $\operatorname{tr}(A) = 0$. What can be said about the eigenvalues of A. Is such a matrix normal?

Problem 145. Let $a, b, c \in \mathbb{R}$. Consider the symmetric matrix

$$\begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors.

Problem 146. Let I_n be the $n \times n$ unit matrix and I_2 the 2×2 unit matrix. Consider the $n \times$ matrix

$$J_n = \begin{pmatrix} 0 & 1 & 0 & & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & \ddots & \ddots & & \\ & & & & & 1 \\ & & & & & 0 \end{pmatrix}.$$

Hence an arbitrary Jordan block is given by $zI_n + J_n$, where $z \in \mathbb{C}$. Find the eigenvalues of

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \otimes I_n + I_2 \otimes J_n.$$

Problem 147. (i) Let I_2 be the 2×2 identity matrix and σ_3 be the Pauli spin matrix. Find the eigenvalues and normalized eigenvectors of the 4×4 matrix

$$A(t) = \begin{pmatrix} \cosh(2t)I_2 & \sinh(2t)\sigma_3 \\ \sinh(2t)\sigma_3 & \cosh(2t)I_2 \end{pmatrix}.$$

(ii) Let I_2 be the 2×2 identity matrix, 0_2 be the 2×2 zero matrix and σ_3 be the Pauli spin matrix. Find the eigenvalues and normalized eigenvectors of the 6×6 matrix

$$B(t) = \begin{pmatrix} \cosh(2t)I_2 & 0_2 & \sinh(2t)\sigma_3 \\ 0_2 & I_2 & 0_2 \\ \sinh(2t)\sigma_3 & 0_2 & \cosh(2t)I_2 \end{pmatrix}.$$

Can the results from (i) be utilized here?

Problem 148. Find the eigenvalues of the 6×6 matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}}I_2 & 0_2 & \frac{1}{\sqrt{2}}I_2 \\ 0_2 & I_2 & 0_2 \\ \frac{1}{\sqrt{2}}I_2 & 0_2 & -\frac{1}{\sqrt{2}}I_2 \end{pmatrix}.$$

Problem 149. Find all 2×2 matrices A over $\mathbb{C}$ which admit the normalized eigenvectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with the corresponding eigenvalues λ_1 and λ_2.

Problem 150. Consider the *quadratic form*

$$7x_1^2 + 6x_2^2 + 5x_3^2 - 4x_1x_2 - 4x_2x_3 + 14x_1 - 8x_2 + 10x_3 + 6 = 0.$$

Write this equation in matrix form and find the eigenvalues and normalized eigenvectors of the 3×3 matrix.

Problem 151. Let M be an $n \times n$ invertible matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting degeneracy). Find the eigenvalues of

$$M_+ = \frac{1}{2}(M + M^{-1}) \quad \text{and} \quad M_- = \frac{1}{2}(M - M^{-1}).$$

Problem 152. (i) Let $x_1, x_2, x_3 \in \mathbb{R}$. What is the condition such that the 3×3 matrix

$$A(x_1, x_2, x_3) = \begin{pmatrix} 0 & x_1 & 0 \\ 0 & 0 & x_2 \\ x_3 & 0 & 0 \end{pmatrix}.$$

is normal?
(ii) Find the eigenvalues and normalized eigenvectors of A.

Problem 153. Let $a, b, c, d \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the matrices

$$M_1 = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & b \\ 0 & -b & 0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & b & 0 \\ 0 & -b & 0 & c \\ 0 & 0 & -c & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ -a & 0 & b & 0 & 0 \\ 0 & -b & 0 & c & 0 \\ 0 & 0 & -c & 0 & d \\ 0 & 0 & 0 & -d & 0 \end{pmatrix}.$$

Problem 154. (i) Study the eigenvalue problem for the symmetric matrices over $\mathbb{R}$

$$A_3 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

Extend the n dimensions

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

Problem 155. Study the eigenvalue problem for 3×3, 4×4, 5×5 matrices where the entries on the diagonal are 0 and the entries on the counter-diagonal are 0, i.e.

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_{12} & b_{13} & 0 \\ b_{21} & 0 & 0 & b_{24} \\ b_{31} & 0 & 0 & b_{34} \\ 0 & b_{42} & b_{43} & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & c_{12} & c_{13} & c_{14} & 0 \\ c_{21} & 0 & c_{23} & 0 & c_{25} \\ c_{31} & c_{32} & 0 & c_{34} & c_{35} \\ c_{41} & 0 & c_{43} & 0 & c_{45} \\ 0 & c_{52} & c_{53} & c_{54} & 0 \end{pmatrix}.$$

Extend to arbitrary n. Consider the case n odd and n even separately.

Problem 156. Let I_n be the $n \times n$ identity matrix and 0_n be the $n \times n$ zero matrix. Find the eigenvalues and eigenvectors of the $2n \times 2n$ matrices

$$A = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \qquad B = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}.$$

Problem 157. Let A be an normal 2×2 matrix with eigenvalues λ_1 and λ_2. What can be said about the eigenvalues of the 3×3 matrix

$$B = \begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & 0 & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix}.$$

Problem 158. Let $\epsilon \in [0,1]$. Consider the matrix

$$U(\epsilon) = \frac{1}{\sqrt{1+\epsilon^2}} \begin{pmatrix} 1 & \epsilon \\ \epsilon & -1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors. The matrix interpolates between the Pauli matrix σ_3 ($\epsilon = 0$) and the Hadamard matrix ($\epsilon = 1$).

Problem 159. Let $\tau = 2\cos(\pi/5)$. Find the eigenvalues and eigenvectors of the three Cartan matrices

$$A_2 = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}.$$

Problem 160. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 \\ a_{14} & 0 & 0 & 0 \end{pmatrix}.$$

Problem 161. Find the eigenvalues and eigenvectors of 4×4 matrix

$$A(z) = \begin{pmatrix} 1 & 1 & 1 & z \\ 1 & 1 & 1 & z \\ 1 & 1 & 1 & z \\ \bar{z} & \bar{z} & \bar{z} & 1 \end{pmatrix}.$$

Problem 162. Find all 2×2 matrices A over $\mathbb{C}$ which admit the normalized eigenvectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with the corresponding eigenvalues λ_1 and λ_2 and $\lambda_1 \neq \lambda_2$.

Problem 163. Consider the Hadamard matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1).$$

The eigenvalues of the Hadamard matrix are given by $+1$ and -1 with the corresponding normalized eigenvectors

$$\frac{1}{\sqrt{8}}\begin{pmatrix}\sqrt{4+2\sqrt{2}}\\ \sqrt{4-2\sqrt{2}}\end{pmatrix}, \qquad \frac{1}{\sqrt{8}}\begin{pmatrix}\sqrt{4-2\sqrt{2}}\\ -\sqrt{4+2\sqrt{2}}\end{pmatrix}.$$

How can this information be used to find the eigenvalues and eigenvectors of the Bell matrix

$$B=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&-1&0\\1&0&0&-1\end{pmatrix}.$$

Problem 164. Let U be an $n\times n$ unitary matrix. Assume also that $U^*=-U$, i.e. the matrix is also skew-hermitian. Find the eigenvalues of such a matrix.

Problem 165. Let $\alpha\in[0,1]$. Find the inverse, eigenvalues and eigenvectors of the 4×4 matrix

$$A(\alpha)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\i\alpha&0&0&1\end{pmatrix}.$$

Problem 166. Let $A=(a_{jk})$ be an $n\times n$ semi-positive definite matrix with $\sum_{k=1}^n a_{jk}<1$ for $j=1,\dots,n$. Show that $0\le\lambda<1$ for all eigenvalues λ of A.

Problem 167. Let A be an $n\times n$ stochastic matrix. Show that it has an eigenvalue equal to 1.

Problem 168. Given an $n\times n$ permutation matrix P.
(i) Assume that n is even and $\mathrm{tr}(P)=0$. Can we conclude that half of the eigenvalues of such a matrix are $+1$ and the other half are -1?
(ii) Assume that n is odd and $\mathrm{tr}(P)=0$. Can we conclude that the eigenvalues are given by the n solutions of $\lambda^n=1$?

Problem 169. Let A be an $n\times n$ matrix over $\mathbb{C}$. The characteristic polynomial of A is defined by

$$\Delta(\lambda):=\det(\lambda I_n-A)=\lambda^n+\sum_{k=1}^n(-1)^k\sigma(k)\lambda^{n-k}.$$

The eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are the solutions of the characteristic equation $\Delta(\lambda) = 0$. The coefficients $\sigma(j)$ $(j = 1, \ldots, n)$ are given by

$$\sigma(1) = \sum_{j=1}^{n} \lambda_j = \mathrm{tr}(A)$$
$$\sigma(2) = \sum_{j<k} \lambda_j \lambda_k$$
$$\vdots$$
$$\sigma(n) = \prod_{j=1}^{n} \lambda_j = \det(A).$$

Another set of symmetric polynomials is given by the traces of powers of the matrix A, namely

$$s(j) = \sum_{k=1}^{n} (\lambda_k)^j = \mathrm{tr}(A^j) \quad j = 1, \ldots, n.$$

One has (so-called Newton relation)

$$j\sigma(j) - s(1)\sigma(j-1) + \cdots + (-1)^{j-1} s(j-1)\sigma(1) + (-1)^j s(j) = 0, \quad j = 1, \ldots, n.$$

Consider the 3×3 matrix

$$A = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Calculate $s(1)$, $s(2)$, $s(3)$ from the traces of the powers of A. Then apply the Newton relation to find $\sigma(1)$, $\sigma(2)$, $\sigma(3)$.

Problem 170. (i) Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the 6×6 matrix

$$B = \begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 & b_{16} \\ 0 & b_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} & 0 \\ b_{61} & 0 & 0 & 0 & 0 & b_{66} \end{pmatrix}.$$

Problem 171. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find the rank of the matrix. Explain.
(ii) Find the determinant and trace of the matrix.
(iii) Find all eigenvalues of the matrix.
(iv) Find one eigenvector.
(v) Is the matrix positive semidefinite?

Problem 172. Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the symmetric 3×3 and 4×4 matrices, respectively

$$A_3(\alpha) = \begin{pmatrix} \alpha & -1 & 0 \\ -1 & \alpha & -1 \\ 0 & -1 & \alpha \end{pmatrix}, \quad A_4(\alpha) = \begin{pmatrix} \alpha & -1 & 0 & 0 \\ -1 & \alpha & -1 & 0 \\ 0 & -1 & \alpha & -1 \\ 0 & 0 & -1 & \alpha \end{pmatrix}.$$

Extend to n dimensions.

Problem 173. Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the symmetric 4×4 matrix, respectively

$$A_4(\alpha) = \begin{pmatrix} \alpha & -1 & 0 & -1 \\ -1 & \alpha & -1 & 0 \\ 0 & -1 & \alpha & -1 \\ -1 & 0 & -1 & \alpha \end{pmatrix}.$$

Extend to n dimensions.

Problem 174. Let $n \geq 2$ and even. Consider an $n \times n$ hermitian matrix A. Thus the eigenvalues are real. Assume we have the information that if λ is an eigenvalue then $-\lambda$ is also an eigenvalue of A. How can the calculation of the eigenvalues be simplified with this information?

Problem 175. Study the properties of the matrices

$$A_{\pm} = \begin{pmatrix} -1/2 & \mp 3/2 \\ \pm 1/2 & -1/2 \end{pmatrix}$$

i.e. find the determinant, trace, inverse, $A_{\pm}^3$ and the eigenvalues.

Problem 176. Find the eigenvalues and normalized eigenvectors of the 4×4 matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Are the four column vectors

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1cr \quad 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1cr0 \\ -1 \end{pmatrix}$$

in the matrix A entangled?

Problem 177. (i) Find the eigenvalues and eigenvectors of the stochastic matrix

$$S = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the stochastic matrix

$$S = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ p_{DA} & p_{DB} & p_{DC} & 0 \end{pmatrix}$$

where $p_{AA} = p_{DA} = 2 - \sqrt{3}$, $p_{AB} = p_{DB} = \sqrt{3} - \sqrt{2}$, $p_{AC} = p_{DC} = \sqrt{2} - 1$.

Problem 178. Find the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

Problem 179. Let $n \geq 2$. Consider the *tridiagonal matrix*

$$A_n = \begin{pmatrix} a_1 & b_1 & 0 & & \\ c_1 & a_2 & b_2 & & \\ 0 & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix}.$$

We set $D_0 = 1$ and $D_1 = a_1$. For $n \geq 2$ we set $D_n = \det(A_n)$, i.e. the determinant of A_n. Then the determinant D_n $(n \geq 2)$ satisfies the recurrence relation

$$D_n = a_n D_{n-1} - c_{n-1} b_{n-1} D_{n-2}, \quad n = 2, 3, \ldots$$

If we set $a_1 = a_2 = \cdots = a_n = -\lambda$ (λ will be the eigenvalue) then we obtain the characteristic polynomial for the matrix

$$M_n = \begin{pmatrix} 0 & b_1 & 0 & & & \\ c_1 & 0 & b_2 & & & \\ 0 & c_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & 0 \end{pmatrix}.$$

Apply it to the matrix

$$H = \begin{pmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 1 & & \\ 0 & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}$$

and then solve the characteristic equation to find the eigenvalues.

Problem 180. Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. Let $\mathbf{u}_1, \mathbf{u}_2$ be an orthonormal basis in $\mathbb{C}^2$. We define the matrices

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^* + \lambda_2 \mathbf{u}_2 \mathbf{u}_2, \qquad B = \mu_1 \mathbf{u}_1 \mathbf{u}_1^* + \mu_2 \mathbf{u}_2 \mathbf{u}_2^*.$$

Find the commutator $[A, B]$. Find the conditions on λ_1, λ_2, μ_1, μ_2 such that $[A, B] = 0_2$.

Problem 181. Find the eigenvalues and normalized eigenvectors of the 2×2 matrix

$$\begin{pmatrix} \cos\theta & -e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix}.$$

Problem 182. Consider the 4×4 orthogonal matrices

$$A_{12}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_{23}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_{34}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

Find the eigenvalues of $V(\theta) = V_{12}(\theta)V_{23}(\theta)V_{34}(\theta)$.

Problem 183. Prove or disprove: For any permutation matrix their eigenvalues form a group under multiplication. Hint. Each permutation matrix admits the eigenvalue $+1$. Why?

Problem 184. Consider an $n \times n$ permutation matrix P. Obviously $+1$ is always an eigenvalue since the column vector with all n entries equal to $+1$ is an eigenvector. Now want to apply a *brute force method* and give a C++ implementation to figure out whether -1 is an eigenvalue. We run over all column vectors $\mathbf{v}$ of length n, where the entries can only be $+1$ or -1, where of course the cases with all entries $+1$ or all entries -1 can be omitted. Thus the number of column vectors we have to run through are $2^n - 2$. The condition then to be checked is

$$P\mathbf{v} = -\mathbf{v}.$$

If true we have an eigenvalues -1 with the corresponding eigenvector $\mathbf{v}$.

Problem 185. Find all 2×2 matrices over $\mathbb{C}$ which admit the normalized eigenvectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with the corresponding eigenvalues λ_1 and λ_2.

Problem 186. Consider the 3×3 matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A and B. Are the matrices similar?

Problem 187. Consider the 3×3 matrices

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Find the eigenvalues of A and B and the eigenvalues of the commutator $[A, B]$.

Problem 188. (i) What can be said about the eigenvalues and eigenvectors of a nonzero hermitian $n \times n$ matrix A with $\det(A) = 0$ and $\operatorname{tr}(A) = 0$.
(ii) Give one eigenvalue of A. Give one eigenvalue of $A \otimes A$.
(iii) Consider the case $n = 3$ for the matrix A. Find all the eigenvalues and eigenvectors.

Problem 189. Let $n \geq 2$ and $j, k = 0, 1, \ldots, n-1$. Consider

$$M(j,k) = \begin{cases} 1 & \text{for } j < k \\ 0 & \text{for } j = k \\ -1 & \text{for } j > k \end{cases}$$

(i) Write down the matrix M for $n = 2$ and find the eigenvalues and normalized eigenvectors.
(ii) Write down the matrix M for $n = 3$ and find the eigenvalues and normalized eigenvectors.

Problem 190. Find the eigenvalues and eigenvectors of the 3×3 matrix $A = (a_{jk})$ $(j, k = 1, 2, 3)$

$$a_{jk} = \sqrt{j(k-1)}.$$

Problem 191. Let $\phi \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the 3×3 matrices

$$A(\phi) = \begin{pmatrix} 0 & 0 & e^{i\phi} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B(\phi) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e^{i\phi} & 0 & 0 \end{pmatrix}.$$

Problem 192. Let $n \geq 2$ and n even. Find the eigenvalues of the $n \times n$ matrices

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & -1 \\ -1 & 2 & -1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & -1 \\ -1 & 0 & \ddots & 0 & -1 & 2 \end{pmatrix}.$$

Problem 193. Let A be an $n \times n$ diagonalizable matrix over $\mathbb{C}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and diagonalization $A = TDT^{-1}$, where D is a diagonal matrix such that $D_{jj} = \lambda_j$ for all $j = 1, \ldots, n$. Show that

$$\begin{pmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{nn} \end{pmatrix} = (T \bullet (T^{-1})^T) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$$

where $\bullet$ is the Hadamard product.

Problem 194. (i) Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the permutation matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 195. Let $j, k = 0, 1, 2, 3$. Write down the 4×4 matrix A with the entries

$$A_{jk} = \cos\left(\frac{\pi j - \pi k}{2}\right)$$

and find the eigenvalues.

Problem 196. Consider the 4×4 matrices over $\mathbb{R}$

$$A_1 = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{11} & a_{12} & 0 \\ 0 & a_{12} & a_{11} & a_{12} \\ 0 & 0 & a_{12} & a_{11} \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{12} & a_{11} & a_{12} & 0 \\ 0 & a_{12} & a_{11} & a_{12} \\ a_{14} & 0 & a_{12} & a_{11} \end{pmatrix}.$$

Find the eigenvalues of A_1 and A_2.

Problem 197. Find the eigenvalues and eigenvectors of the 3×3 matrix

$$A = \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

Problem 198. (i) Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the 6×6 matrix

$$B = \begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 & b_{16} \\ 0 & b_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} & 0 \\ b_{61} & 0 & 0 & 0 & 0 & b_{66} \end{pmatrix}.$$

Problem 199. Find the eigenvalues of the matrices

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2/3 & 1/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3/4 & 2/4 & 1/4 \\ 3/4 & 3/4 & 2/4 & 1/4 \\ 2/4 & 2/4 & 2/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

Extend to n-dimensions.

Problem 200. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that the matrix A admits the inverse matrix A^{-1} iff $A\mathbf{v} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.

Problem 201. The *permanent* of an $n \times n$ matrix A is defined as

$$\mathrm{Per}(A) = \sum_{\sigma \in S_n} \prod_{j=1}^{n} a_{j,\sigma(j)}$$

where S_n is the set of all permutation of n elements. Find the permanent of a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Problem 202. Consider the 3×3 symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Find the largest eigenvalue and the corresponding eigenvector using the power method. Start from the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 203. Let I_3 be the 3×3 unit matrix and 0_3 be the 3×3 zero matrix. Find the eigenvalues of the 6×6 matrix

$$\begin{pmatrix} 0_3 & I_3 \\ -I_3 & 0_3 \end{pmatrix}.$$

Problem 204. Consider the tridiagonal $n \times n$ matrix

$$A = \begin{pmatrix} a_1 & b_2 & 0 & \dots & 0 & 0 \\ c_2 & a_2 & b_3 & \dots & 0 & 0 \\ 0 & c_3 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_n \\ 0 & 0 & 0 & \dots & c_n & a_n \end{pmatrix}$$

with $a_1, a_j, b_j, c_j \in \mathbb{C}$ $(j = 1, 2, \dots, n)$. It has in general n complex eigenvalues the n roots of the characteristic polynomial $p(\lambda)$. Show that this polynomial can be evaluated by the recursive formula

$$\begin{aligned} p_k(\lambda) &= (\lambda - a_k)p_{k-1} - b_k c_k p_{k-2}(\lambda), \quad k = 2, 3, \dots, n \\ p_1(\lambda) &= \lambda - a_1 \\ p_0(\lambda) &= 1. \end{aligned}$$

Problem 205. Let $\gamma \in \mathbb{R}$ and $\gamma^2 < 1$. Consider the 2×2 matrix

$$K = \sigma_1 - i\gamma\sigma_3.$$

Show that the eigenvalues of K and K^* are given by $\pm\sqrt{1-\gamma^2}$. Find the normalized eigenvecors.

Problem 206. (i) Find the eigenvalues of the double 2×2 stochastic matrix

$$\begin{pmatrix} \sin^2(\theta) & \cos^2(\theta) \\ \cos^2(\theta) & \sin^2(\theta) \end{pmatrix}.$$

(ii) Find the eigenvalues of the double 3×3 stochastic matrix

$$\begin{pmatrix} \sin^2(\theta) & 0 & \cos^2(\theta) \\ \cos^2(\theta) & \sin^2(\theta) & 0 \\ 0 & \cos^2(\theta) & \sin^2(\theta) \end{pmatrix}.$$

(iii) Find the eigenvalues of the double stochastic 4×4 matrix

$$\begin{pmatrix} \sin^2(\theta) & 0 & 0 & \cos^2(\theta) \\ \cos^2(\theta) & \sin^2(\theta) & 0 & 0 \\ 0 & \cos^2(\theta) & \sin^2(\theta) & 0 \\ 0 & 0 & \cos^2(\theta) & \sin^2(\theta) \end{pmatrix}.$$

Problem 207. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}.$$

Problem 208. (i) Find the eigenvalues of the 2×2 matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix}.$$

(ii) Find the eigenvalues of the 3×3 matrices

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}.$$

(iii) Find the eigenvalues of the 4×4 matrices

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{pmatrix}.$$

Problem 209. Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Find the eigenvalues and normalized eigenvectors.
(ii) Find the smallest positive integer k such that $P^k = I_4$.

Problem 210. Let A, B be two nonzero $n \times n$ matrices over $\mathbb{C}$. Let $A\mathbf{v} = \lambda\mathbf{v}$ be the eigenvalue equation for A. Assume that $[A, B] = 0_n$. Then from $[A, B]\mathbf{v} = \mathbf{0}$ it follows that

$$[A, B]\mathbf{v} = (AB - BA)\mathbf{v} = A(B\mathbf{v}) - B(A\mathbf{v}) = \mathbf{0}.$$

Therefore $A(B\mathbf{v}) = \lambda(B\mathbf{v}$. If $B\mathbf{v} \neq \mathbf{0}$ we find that $B\mathbf{v}$ is an eigenvector of A with eigenvalue λ. Apply it to $A = \sigma_1 \otimes \sigma_2$ and $B = \sigma_3 \otimes \sigma_3$.

Problem 211. Let A, B be two nonzero $n \times n$ matrices over $\mathbb{C}$. Let $A\mathbf{v} = \lambda\mathbf{v}$ be the eigenvalue equation for A. Assume that $[A, B]_+ = 0_n$. Then from $[A, B]_+\mathbf{v} = \mathbf{0}$ it follows that

$$[A, B]_+\mathbf{v} = (AB + BA)\mathbf{v} = A(B\mathbf{v}) + B(A\mathbf{v}) = \mathbf{0}.$$

Therefore $A(B\mathbf{v}) = -\lambda(B\mathbf{v}$. If $B\mathbf{v} \neq \mathbf{0}$ we have an eigenvalue equation with eigenvalue $-\lambda$. Apply it to $A = \sigma_1$ and $B = \sigma_2$, where $[\sigma_1, \sigma_2]_+ = 0_2$.

Problem 212. Find the eigenvalues of the 7×7 matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Problem 213. Consider the spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$S_{\alpha,1} = S_\alpha \otimes I_3 \otimes I_3, \quad S_{\alpha,2} = I_3 \otimes S_\alpha \otimes I_3, \quad S_{\alpha,3} = I_3 \otimes I_3 \otimes S_\alpha$$

with $\alpha = 1, 2, 3$. Let

$$\mathbf{S}_1 = \begin{pmatrix} S_1 \otimes I_3 \otimes I_3 \\ S_2 \otimes I_3 \otimes I_3 \\ S_3 \otimes I_3 \otimes I_3 \end{pmatrix}, \quad \mathbf{S}_2 = \begin{pmatrix} I_3 \otimes S_1 \otimes I_3 \\ I_3 \otimes S_2 \otimes I_3 \\ I_3 \otimes S_3 \otimes I_3 \end{pmatrix}, \quad \mathbf{S}_3 = \begin{pmatrix} I_3 \otimes I_3 \otimes S_1 \\ I_3 \otimes I_3 \otimes S_2 \\ I_3 \otimes I_3 \otimes S_3 \end{pmatrix}$$

and

$$\mathbf{S}_2 \times \mathbf{S}_3 = \begin{pmatrix} I_3 \otimes S_2 \otimes S_3 - I_3 \otimes S_3 \otimes S_2 \\ I_3 \otimes S_3 \otimes S_1 - I_3 \otimes S_1 \otimes S_3 \\ I_3 \otimes S_1 \otimes S_2 - I_3 \otimes S_2 \otimes S_1 \end{pmatrix}$$

Thus

$$\mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3) = \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{S}_3 - \mathbf{S}_1 \otimes \mathbf{S}_3 \otimes \mathbf{S}_2 + \mathbf{S}_2 \otimes \mathbf{S}_3 \otimes \mathbf{S}_1 - \mathbf{S}_2 \otimes \mathbf{S}_1 \otimes \mathbf{S}_3 + \mathbf{S}_3 \otimes \mathbf{S}_1 \otimes \mathbf{S}_2 - \mathbf{S}_3 \otimes \mathbf{S}_2 \otimes \mathbf{S}_1.$$

Find the eigenvalues of $\mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3)$.

Problem 214. Find the eigenvalues and normalized eigenvectors of the 4×4 matrix

$$\begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix}.$$

Problem 215. Let $n \geq 2$. Consider the $(n-1) \times n$ matrix A over $\mathbb{R}$. Then AA^T is an $(n-1) \times (n-1)$ matrix over $\mathbb{R}$ and $A^T A$ is an $n \times n$ matrix over $\mathbb{R}$. Show that the $(n-1)$ eigenvalues of AA^T are also eigenvalues of $A^T A$ and $A^T A$ additionally admits the eigenvalue 0.

Problem 216. Study the eigenvalue problem for the symmetric matrices over $\mathbb{R}$

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}.$$

Extend to infinity.

Problem 217. Let H be an hermitian $n \times n$ matrix and U an $n \times n$ unitary matrix such that $UHU^* = H$. We call H invariant under U. From $UHU^* = H$ it follows that $[H, U] = 0_n$. If $\mathbf{v}$ is an eigenvector of H, i.e. $H\mathbf{v} = \lambda\mathbf{v}$, then $U\mathbf{v}$ is also an eigenvector of H since

$$H(U\mathbf{v}) = (HU)\mathbf{v} = U(H\mathbf{v}) = \lambda(U\mathbf{v}).$$

The set of all unitary matrices U_j $(j = 1, \ldots, m)$ that leave a given hermitian matrix invariant, i.e. $U_j H U_j^* = H$ $(j = 1, \ldots, m)$ form a group under matrix multiplication.
(i) Find all 2×2 hermitian matrices H such that $[H, U] = 0_2$, where $U = \sigma_1$.
(ii) Find all 2×2 hermitian matrices H such that $[H, U] = 0_2$, where $U = \sigma_2$.
(iii) Find all 4×4 hermitian matrices H such that $[H, U] = 0_4$, where $U = \sigma_1 \otimes \sigma_1$.
(ii) Find all 4×4 hermitian matrices H such that $[H, U] = 0_4$, where $U = \sigma_2 \otimes \sigma_2$.

Problem 218. Find the eigenvalues and normalized eigenvectors of the 2×2 matrix

$$M(\theta) = \begin{pmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix}.$$

Problem 219. Consider the skew-symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Find the eigenvalues. Let 0_3 be the 3×3 zero matrix. Let A_1, A_2, A_3 be skew-symmetric 3×3 matrices over $\mathbb{R}$. Find the eigenvalues of the 9×9 matrix

$$B = \begin{pmatrix} 0_3 & -A_3 & A_2 \\ A_3 & 0_3 & -A_1 \\ -A_2 & A_1 & 0_3 \end{pmatrix}.$$

Problem 220. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Find $A \otimes B$, $B \otimes A$.
(ii) Find

$$\operatorname{tr}(A \otimes B), \qquad \operatorname{tr}(B \otimes A).$$

Find

$$\det(A \otimes B), \qquad \det(B \otimes A).$$

(iii) Find the eigenvalues of A and B.
(iv) Find the eigenvalues of $A \otimes B$ and $B \otimes A$.
(v) Find $\operatorname{rank}(A)$, $\operatorname{rank}(B)$ and $\operatorname{rank}(A \otimes B)$.

Problem 221. Let λ_1, λ_2 and λ_3 be the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Find $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ without calculating the eigenvalues of A or A^2.

Problem 222. Consider the semi-simple Lie group $SL(2, \mathbb{R})$. Let $A \in SL(2, \mathbb{R})$. Then $A^{-1} \in SL(2, \mathbb{R})$. Explain why. Show that A and A^{-1} have the same eigenvalues. Is this still true for $A \in SL(3, \mathbb{R})$?

Problem 223. Let A_n be the $n \times n$ matrices of the form

$$A_1 = 1, \quad A_2 = \begin{pmatrix} 0 & t \\ t & r \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & t \\ 0 & 1 & 0 \\ t & 0 & r \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & t & 0 \\ 0 & t & r & 0 \\ t & 0 & 0 & r \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & t & 0 & r & 0 \\ t & 0 & 0 & 0 & r \end{pmatrix}.$$

Thus the even dimensional matrix A_{2n} has t along the skew-diagonal and r along the lower main diagonal. Otherwise the entries are 0. The odd dimensional matrix A_{2n+1} has t along the skew-diagonal except 1 at the centre and r along the lower main diagonal. Otherwise the entries are 0. Find the eigenvalues of these matrices.

Problem 224. Let $\mu_1, \mu_2 \in \mathbb{R}$. Consider the 2×2 matrix

$$A(\mu_1^2, \mu_2^2) = \begin{pmatrix} \mu_1^2 \cos^2(\theta) + \mu_2^2 \sin^2(\theta) & (\mu_2^2 - \mu_1^2)\cos(\theta)\sin(\theta) \\ (\mu_2^2 - \mu_1^2)\sin(\theta)\cos(\theta) & \mu_1^2 \sin^2(\theta) + \mu_2^2 \cos^2(\theta) \end{pmatrix}.$$

(i) Find the trace and determinant of $A(\mu_1^2, \mu_2^2)$.
(ii) Find the eigenvalues of $A(\mu_1^2, \mu_2^2)$.
Remark. Any positive semidefinite 2×2 matrix over $\mathbb{R}$ can be written in this form. Try the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 225. Consider the 4×4 matrix

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \end{pmatrix}.$$

Find the eigenvalues.

Problem 226. Let A be an $n \times n$ hermitian matrix with n odd. Thus the eigenvalues are real. Assume that exactly one eigenvalue is 0 and $\mathrm{tr}(A) = 0$. What are the conditions on A such that the remaining (real) eigenvalues are symmetric around 0, i.e. for each eigenvalue $\lambda \neq 0$ one has the eigenvalue $-\lambda$. Consider first the 3×3 case. Can the spectral theorem be applied?

Chapter 5

Commutators and Anticommutators

Problem 1. Let A, B be 2×2 symmetric matrices over $\mathbb{R}$. Assume that $AA^T = I_2$ and $BB^T = I_2$. Is

$$[A, B] = 0_2\,?$$

Prove or disprove.

Problem 2. Let A, B, X, Y be $n \times n$ matrices over $\mathbb{C}$. Assume that

$$AX - XB = Y.$$

(i) Let $z \in \mathbb{C}$. Show that

$$(A - zI_n)X - X(B - zI_n) = Y.$$

(ii) Assume that $A - zI_n$ and $B - zI_n$ are invertible. Show that

$$X(B - zI_n)^{-1} - (A - zI_n)^{-1}X = (A - zI_n)^{-1}Y(B - zI_n)^{-1}.$$

Problem 3. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that $[A, B] = 0_n$. Let U be a unitary matrix. Calculate $[U^*AU, U^*BU]$.

Problem 4. Can we find nonzero symmetric 2×2 matrices H and A over $\mathbb{R}$ such that

$$[H, A] = \mu A$$

where $\mu \in \mathbb{R}$ and $\mu \neq 0$?

Problem 5. Let A, B be $n \times n$ hermitian matrices. Is $i[A, B]$ hermitian?

Problem 6. A *truncated Bose annihilation operator* is defined as the $n \times n$ $(n \geq 2)$ matrix

$$B_n = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \sqrt{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

(i) Calculate $B_n^* B_n$.
(ii) Calculate the commutator $[B_n, B_n^*]$.

Problem 7. Find nonzero 2×2 matrices A, B such that $[A, B] \neq 0_2$, but

$$[A, [A, B]] = 0_2, \qquad [B, [A, B]] = 0_2.$$

Problem 8. Let A, B be symmetric $n \times n$ matrices over $\mathbb{R}$. Show that $[A, B]$ is skew-symmetric over $\mathbb{R}$.

Problem 9. Let A, B be $n \times n$ matrices. Show that

$$AB \equiv \frac{1}{2}([A, B] + [A, B]_+).$$

Problem 10. Let A, B be $n \times n$ matrices. Suppose that

$$[A, B] = 0_n, \qquad [A, B]_+ = 0_n$$

and that A is invertible. Show that B must be the zero matrix.

Problem 11. Let

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find all 2×2 matrices A such that $[A, C] = 0_2$, where 0_2 is the 2×2 zero matrix.

Problem 12. Let A, B, C be $n \times n$ matrices. Show that

$$\operatorname{tr}([A, B]C) \equiv \operatorname{tr}(A[B, C]).$$

Problem 13. Find all nonzero 2×2 matrices A, B such that

$$[A, B] = A + B.$$

Problem 14. Find all nonzero 2×2 matrices J_+, J_-, J_z such that

$$[J_z, J_+] = J_+, \quad [J_z, J_-] = -J_-, \quad [J_+, J_-] = 2J_z$$

where $(J_+)^* = J_-$.

Problem 15. Find all nonzero 2×2 matrices K_+, K_-, K_z such that

$$[K_z, K_+] = K_+, \quad [K_z, K_-] = -K_-, \quad [K_+, K_-] = -2K_z$$

where $(K_+)^* = K_-$.

Problem 16. Find all nonzero 2×2 matrices A_1, A_2, A_3 such that

$$[A_1, A_2] = 0, \quad [A_1, A_3] = A_1, \quad [A_2, A_3] = A_2.$$

Problem 17. Let H be a nonzero $n \times n$ hermitian matrix. Let E be a nonzero $n \times n$ matrix. Assume that

$$[H, E] = aE$$

where $a \in \mathbb{R}$ and $a \neq 0$. Show that E cannot be hermitian.

Problem 18. Let A and B be positive semi-definite matrices. Can we conclude that $[A, B]_+ \equiv AB + BA$ is positive semi-definite.

Problem 19. Let A, B be $n \times n$ matrices. Given the expression

$$A^2B + AB^2 + B^2A + BA^2 - 2ABA - 2BAB.$$

Write the expression in a more compact form using commutators.

Problem 20. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that $A^2 = I_n$ and $B^2 = I_n$.
(i) Find the commutators $[AB + BA, A]$, $[AB + BA, B]$.
(ii) Give an example of such matrices for $n = 2$ and $A \neq B$.

Problem 21. Consider the 2×2 matrices

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad B(\beta) = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}$. Calculate the commutator $[A(\alpha), B(\beta)]$. What is the condition on α, β such that $[A(\alpha), B(\beta)] = 0_2$?

Problem 22. Let A_1, A_2, A_3 be $n \times n$ matrices over $\mathbb{C}$. The *ternary commutator* $[A_1, A_2, A_3]$ (also called the *ternutator*) is defined as

$$\begin{aligned}[A_1, A_2, A_3] &:= \sum_{\pi \in S_3} \mathrm{sgn} A_{\pi(1)} A_{\pi(2)} A_{\pi(3)} \\ &\equiv A_1A_2A_3 + A_2A_3A_1 + A_3A_1A_2 - A_1A_3A_2 - A_2A_1A_3 - A_3A_2A_1.\end{aligned}$$

(i) Let $n = 2$ and consider the Pauli spin matrices σ_1, σ_2, σ_3. Calculate the ternutator

$$[\sigma_1, \sigma_2, \sigma_3].$$

(ii) Calculate

$$A_1 \otimes A_2 \otimes A_3 + A_2 \otimes A_3 \otimes A_1 + A_3 \otimes A_1 \otimes A_2 - A_1 \otimes A_3 \otimes A_2 - A_2 \otimes A_1 \otimes A_3 - A_3 \otimes A_2 \otimes A_1.$$

Problem 23. Let A, B, C be 2×2 matrices. Find the conditions such that $[A, B, C] = 0$.

Problem 24. Let A, B, H be $n \times n$ matrices such that

$$[H, A] = 0, \qquad [H, B] = 0.$$

Show that

$$[H \oplus I_n + I_n \oplus H, A \oplus B] = 0$$

where $\oplus$ denotes the direct sum.

Problem 25. Show that any two 2×2 matrices which commute with the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

commute with each other.

Problem 26. Let A_1, A_2 be $m \times m$ matrices over $\mathbb{C}$. Let B_1, B_2 be $n \times n$ matrices over $\mathbb{C}$. Show that

$$[A_1 \oplus B_1, A_2 \oplus B_2] = ([A_1, A_2]) \oplus ([B_1, B_2])$$

where $\oplus$ denotes the direct sum.

Problem 27. Let A, B be an $n \times n$ matrix over $\mathbb{C}$. Let $\mathbf{u} \in \mathbb{C}^n$ considered as column vector. Is

$$[\mathbf{u}^* A \boldsymbol{\partial}, \mathbf{u}^* B \boldsymbol{\partial}] = \mathbf{u}^* [A, B] \boldsymbol{\partial}\,?$$

Here [,] denotes the commutator and

$$\boldsymbol{\partial} = \begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix}.$$

Problem 28. Let $c \in \mathbb{R}$ and A be an 2×2 matrix over $\mathbb{R}$. Find the commutator of the 3×3 matrices

$$c \oplus A, \qquad A \oplus c$$

where $\oplus$ denotes the direct sum.

Problem 29. Consider the 3×3 matrices

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & b_1 \\ 0 & b_2 & 0 \\ b_3 & 0 & 0 \end{pmatrix}.$$

Can we find a_j, b_j ($j = 1, 2, 3$) such that the commutator $[A, B]$ is invertible?

Problem 30. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Can we conclude that

$$\|[A, B]\| \le \|A\|\,\|B\|\,?$$

Problem 31. Consider $(m+n) \times (m+n)$ matrices of the form

$$\begin{pmatrix} m \times m & m \times n \\ n \times m & n \times n \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} \widetilde{B}_1 & 0 \\ 0 & \widetilde{B}_2 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & F_1 \\ F_2 & 0 \end{pmatrix}, \quad \widetilde{F} = \begin{pmatrix} 0 & \widetilde{F}_1 \\ \widetilde{F}_2 & 0 \end{pmatrix}.$$

Find the commutators $[B.\widetilde{B}]$, $[B, F]$ and the anticommutator $[F, \widetilde{F}]_+$.

Problem 32. Can one find non-invertible 2×2 matrices A and B such the commutator $[A, B]$ is invertible?

Problem 33. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that B is invertible. Show that

$$[A, B^{-1}] \equiv -B^{-1}[A, B]B^{-1}.$$

Problem 34. Let A, B be $n \times n$ matrices. Show that

$$AB \equiv \frac{1}{2}[A,B] + \frac{1}{2}[A,B]_+.$$

Problem 35. (i) Let A, B be 2×2 skew-symmetric matrices over $\mathbb{R}$. Find the commutator $[A,B]$.
(ii) Let A, B be 3×3 skew-symmetric matrices over $\mathbb{R}$. Find the commutator $[A,B]$.

Problem 36. Find two linearly independent 2×2 matrices A, B such that

$$-A = [B,[B,A]], \qquad -B = [A,[A,B]].$$

Problem 37. Let A be an $n \times n$ matrix and 0_n be the $n \times n$ zero matrix. Find the commutator

$$\left[\begin{pmatrix} 0_n & A \\ A & 0_n \end{pmatrix}, \begin{pmatrix} 0_n & A \\ -A & 0_n \end{pmatrix} \right]$$

and the anticommutator

Problem 38. Find all 2×2 matrices over $\mathbb{C}$ such that the commutator is an invertible diagonal matrix D, i.e. $d_{11} \neq 0$ and $d_{22} \neq 0$.

Problem 39. Let A, B be invertible $n \times n$ matrices over $\mathbb{C}$. Assume that $[A,B] = 0_n$. Can we conclude that $[A^{-1},B^{-1}] = 0_n$?

Problem 40. Consider the 3×3 matrices over $\mathbb{C}$

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}.$$

(i) Calculate the commutator $[A,B]$ and $\det([A,B])$.
(ii) Set $a_{11} = e^{i\phi_1}$, $a_{22} = e^{i\phi_2}$, $a_{33} = e^{i\phi_3}$. Find the condition on ϕ_1, ϕ_2, ϕ_3, b_{12}, b_{13}, b_{21}, b_{23}, b_{31}, b_{32} such that $[A,B]$ is unitary.

Problem 41. Let A, B be $n \times n$ matrices and T a (fixed) invertible $n \times n$ matrix. We define the bracket

$$[A,B]_T := ATB - BTA.$$

Let

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find $[X,Y]_T$, $[X,H]_T$, $[Y,H]_T$.

Problem 42. Can one find a 2×2 matrix A over $\mathbb{R}$ such that

$$[A^T, A] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Problem 43. Consider the set of 3×3 matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Calculate the anticommutator and thus show that we have a basis of a *Jordan algebra*.

Problem 44. Consider the invertible matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2i\pi/3} & 0 \\ 0 & 0 & e^{-2i\pi/3} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Is the matrix $[A,B]$ invertible?

Problem 45. A classical 3×3 matrix representation of the algebra $iso(1,1)$ is given by

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the commutators and anticommutators.

Problem 46. Find the conditions on the two 2×2 hermitian matrices A, B such that

$$[A\otimes B, P] = 0_4$$

where P is the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 47. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$ with $\text{tr}(A) = 0$. Show that A can be written as commutator, i.e. there are $n \times n$ matrices X and Y such that

$$A = XY - YX.$$

Problem 48. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$ with $\text{tr}(A) = 0$. Show that A can be written as commutator, i.e., there are $n \times n$ matrices X and Y such that $A = [X, Y]$.

Problem 49. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$ with $\text{tr}A = 0$. Show that A can be written as commutator, i.e., there are $n \times n$ matrices X and Y such that $A = [X, Y]$.

Problem 50. Let A, B be hermitian matrices, i.e. $A^* = A$ and $B^* = B$. Then in general $A + iB$ is non-normal. What are the conditions on A and B such that $A + iB$ is normal?

Problem 51. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that if A and B commute and if A is normal, then A^* and B commute.

Problem 52. Let σ_j $(j = 0, 1, 2, 3)$ be the Pauli spin matrices, where σ_0 is the 2×2 identity matrix. Form the four 4×4 matrices

$$\gamma_k = \begin{pmatrix} 0_2 & \sigma_k \\ -\sigma_k & 0_2 \end{pmatrix}, \qquad k = 0, 1, 2, 3$$

where 0_2 is the 2×2 identity matrix.
(i) Are the matrices γ_k linearly independent?
(ii) Find the eigenvalues and eigenvectors of the γ_k's.
(iii) Are the matrices γ_k invertible. Use the result from (ii). If so, find the inverse.
(iv) Find the commutators $[\gamma_k, \gamma_\ell]$ for $k, \ell = 0, 1, 2, 3$. Find the anticommutators $[\gamma_k, \gamma_\ell]_+$ for $k, \ell = 0, 1, 2, 3$.
(v) Can the matrices γ_k be written as the Kronecker product of two 2×2 matrices?

Problem 53. Let A be an $n \times n$ matrix over $\mathbb{C}$. It can be written as $A = HU$, where H is a non-negative definite hermitian matrix and U is unitary. Show that the matrices H and A commutate if and only if A is normal.

Problem 54. Let A, B be $n \times n$ matrices over $\mathbb{C}$ such that $A^2 = I_n$, $B^2 = I_n$, where I_n is the $n \times n$ unit matrix. Now assume that

$$[A, B]_+ \equiv AB + BA = 0_n$$

i.e. the anticommutator of A and B vanishes. Show that there is no solution for A and B if n is odd.

Problem 55. Let T be an invertible matrix. Show that

$$T^{-1}AT \equiv A + T^{-1}[A,T].$$

Problem 56. (i) Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the three non-normal matrices

$$A = \sigma_1 + i\sigma_2, \quad B = \sigma_2 + i\sigma_3, \quad C = \sigma_3 + i\sigma_1.$$

Find the commutators and anti-commutators. Discuss.
(ii) Consider the three non-normal matrices

$$X = \sigma_1 \otimes \sigma_1 + i\sigma_2 \otimes \sigma_2, \quad Y = \sigma_2 \otimes \sigma_2 + i\sigma_3 \otimes \sigma_3, \quad Z = \sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_1.$$

Find the commutators and anti-commutators. Discuss.

Problem 57. (i) Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the nonnormal matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \frac{1}{2}\begin{pmatrix} 3 & i \\ i & 1 \end{pmatrix}$$

where $\det(A) = \det(B) = \det(C) = 1$, i.e. A, B, C are elements of the Lie group $SL(2,\mathbb{C})$. Show that

$$[A, A^*] = \sigma_3, \quad [B, B^*] = \sigma_1, \quad [C, C^*] = \sigma_2.$$

(ii) Consider the unitary matrices

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Show that

$$B = UAU^*, \qquad C = VBV^*.$$

(iii) Consider the nonnormal and noninvertible matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad Z = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

All have trace zero and thus are elements of the Lie algebra $s\ell(2,\mathbb{C})$. Show that

$$[X, X^*] = \sigma_3, \quad [Y, Y^*] = \sigma_1, \quad [Z, Z^*] = \sigma_2.$$

Hint. Obviously we have $X = A - I_2$, $Y = B - I_2$, $Z = C - I_2$.
(iv) Show that

$$Y = UXU^*, \qquad Z = VYV^*.$$

(v) Study the commutators

$$[X \otimes X, X^* \otimes X^*], \qquad [X \otimes X^*, X^* \otimes X],$$

$$[Y \otimes Y, Y^* \otimes Y^*], \qquad [Y \otimes Y^*, Y^* \otimes Y],$$

$$[Z \otimes Z, Z^* \otimes Z^*], \qquad [Z \otimes Z^*, Z^* \otimes Z].$$

Problem 58. (i) Can one find $n \times n$ matrices A and B over $\mathbb{C}$ such that the following conditions are satisfied

$$[A, B] = 0_n, \qquad [A, B]_+ = 0_n$$

and both A and B are invertible?
(ii) Can one find $n \times n$ matrices A and B over $\mathbb{C}$ such that the following conditions are satisfied

$$[A, B] = 0_n, \qquad [A, B]_+ = I_n$$

and both A and B are invertible?

Problem 59. Consider the hermitian 3×3 matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the commutators $[L_1, L_2]$, $[L_2, L_3]$, $[L_3, L_1]$. Find the 3×3 matrix

$$A = c_{23}[L_2, L_3]_+ + c_{13}[L_3, L_1]_+ + c_{12}[L_1, L_2]_+$$

where $[,]_+$ denotes the anti-commutator.

Problem 60. Let A, B, C be $n \times n$ matrices. Show that

$$[A, [B, C]_+] = [[A, B], C]_+ + [B, [A, C]]_+.$$

Problem 61. Find all 2×2 matrices A, B, C such that $[A, B] \neq 0_2$, $[A, C] \neq 0_2$, $[B, C] \neq 0_2$ and

$$[A, [B, C]] = 0_2.$$

Problem 62. Show that one can find a 3×3 matrix over $\mathbb{R}$ such that

$$A^2 A^T + A^T A^2 = 2A, \quad A A^T A = 2A, \quad A^3 = 0_3$$

and

$$\frac{1}{2}[A^T, A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Problem 63. Find all nonzero 2×2 matrices A, B such that

$$[A, B]_+ = 0_2, \qquad \mathrm{tr}(AB^*) = 0.$$

Problem 64. Consider the matrices

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{pmatrix}.$$

Find the conditions on d_1, d_2, d_3, a, b such that

$$[D, M] = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & b \\ 0 & -b & 0 \end{pmatrix}.$$

Problem 65. (i) Find all 2×2 matrices A and B such that

$$[A, B] = A - B.$$

(ii) Find all 2×2 matrices A and B such that

$$[A \otimes A, B \otimes B] = A \otimes A - B \otimes B.$$

Problem 66. Given the 4×4 matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$. Find A^2, B^2 and the anticommutator $[A, B]_+$.

Problem 67. (i) Let $\alpha, \beta \in \mathbb{C}$. Show that the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \beta & -\beta^2 \\ 1 & -\beta \end{pmatrix}$$

satisfy the conditions

$$[A,B]_+ = I_2, \qquad [A,A]_+ = 0_2, \qquad [B,B]_+ = 0_2.$$

(ii) Let $\alpha, \beta \in \mathbb{C}$. Show that the 2×2 matrices

$$A = \begin{pmatrix} \alpha & 1 \\ -\alpha^2 & -\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfy the conditions

$$[A,B]_+ = I_2, \qquad [A,A]_+ = 0_2, \qquad [B,B]_+ = 0_2.$$

Problem 68. (i) Find all 2×2 matrices A over $\mathbb{C}$ such that

$$[A, A^*]_+ = I_2$$

where $[,]_+$ denotes the anti-commutator.
(ii) Find all 2×2 matrices B over $\mathbb{C}$ such that

$$[B, B^*]_+ = I_2, \qquad [B, B^*] = \sigma_3$$

where $[,]$ denotes the commutator and σ_3 is the third Pauli spin matrix.

Problem 69. Find 4×4 matrices C and $2\times$ matrices A such that

$$[C, A \otimes I_2 + I_2 \otimes A] = 0_4.$$

Problem 70. (i) Consider the two 2×2 matrices (counter diagonal matrices)

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}.$$

Find the condition on A and B such that the commutator $[A,B]$ vanishes, i.e. $[A,B] = 0_2$.
(ii) Consider the two 3×3 matrices (counter diagonal matrices)

$$A = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & 0 \\ b_{31} & 0 & 0 \end{pmatrix}.$$

Find the condition on A and B such that the commutator $[A,B]$ vanishes, i.e. $[A,B] = 0_3$.
(iii) Extend to n dimensions.

Problem 71. Find all nonzero 2×2 matrices A and B such that

$$[A, [A, B]] = 0_2.$$

Problem 72. Can we find 3×3 matrices A and B such that $[A, B]_+ = 0_3$ and $A^2 = B^2 = I_3$?

Problem 73. Find all 2×2 matrices A, B over $\mathbb{C}$ such that

$$[A, B] = A + B.$$

Problem 74. Find all 2×2 matrices A over $\mathbb{C}$ such that

$$[A, A^*]_+ = I_2, \quad [A, A^*] = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 75. Let A be an arbitrary 2×2 matrix. Calculate the commutator

$$[A \otimes \sigma_3, I_2 \otimes A].$$

Chapter 6

Decomposition of Matrices

Problem 1. We consider 3×3 matrices over $\mathbb{R}$. An orthogonal matrix Q such that $\det Q = 1$ is called a *rotation matrix*. Let $1 \leq p < r \leq 3$ and ϕ be a real number. An orthogonal 3×3 matrix $Q_{pr}(\phi) = (q_{ij})_{1 \leq i,j \leq 3}$ given by

$$\begin{aligned} q_{pp} &= q_{rr} = \cos\phi \\ q_{ii} &= 1 \quad \text{if } \ i \neq p, r \\ q_{pr} &= -q_{rp} = -\sin\phi \\ q_{ip} &= q_{pi} = q_{ir} = q_{ri} = 0 \quad i \neq p, r \\ q_{ij} &= 0 \quad \text{if } \ i \neq p, r \ \text{ and } \ j \neq p, r \end{aligned}$$

will be called a plane rotation through ϕ in the plane span (e_p, e_r). Let $Q = (q_{ij})_{1 \leq i,j \leq 3}$ be a rotation matrix. Show that there exist angles $\phi \in [0, \pi)$, $\theta, \psi \in (-\pi, \pi]$ called the *Euler angles* of Q such that

$$Q = Q_{12}(\phi) Q_{23}(\theta) Q_{12}(\psi). \tag{1}$$

Problem 2. For any $n \times n$ matrix A over $\mathbb{C}$, there exists a positive semi-definite matrix H and a unitary matrix such that $A = HU$ (*polar decomposition*). If A is nonsingular, then H is positive definite and U and H are unique. Find the polar decomposition for

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}.$$

Problem 3. If $A \in \mathbb{R}^{n\times n}$, then there exists an orthogonal $Q \in \mathbb{R}^{n\times n}$ such that

$$Q^TAQ = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{mn} \end{pmatrix}$$

where each R_{ii} is either a 1×1 matrix or a 2×2 matrix having complex conjugate eigenvalues. Find Q for the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 4 & 0 \end{pmatrix}.$$

Then calculate Q^TAQ.

Problem 4. Let $n \geq 2$ and $n = 2k$. Let A be an $n \times k$ matrix and

$$A^*A = I_k$$

where I_k is the $k \times k$ unit matrix. Find the $n \times n$ matrix AA^* using the singular value decomposition. Calculate $\mathrm{tr}(AA^*)$.

Problem 5. Let $n \geq 2$ and $n = 2k$. Let A be an $n \times k$ matrix and

$$A^*A = I_k$$

where I_k is the $k \times k$ unit matrix. Let S be a positive definite $n \times n$ matrix. Show that

$$1 \leq \frac{\mathrm{tr}(A^*S^2A)}{\mathrm{tr}((A^*SA)^2)}.$$

Problem 6. Consider the symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and the orthogonal matrix

$$O = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate $\widetilde{A} = O^{-1}AO$. Can we find an angle ϕ such that $\widetilde{a}_{12} = \widetilde{a}_{21} = 0$?

Problem 7. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that A^{-1} exits. Given the singular value decomposition of A, i.e. $A = UWV^T$. Find the singular value decomposition for A^{-1}.

Problem 8. Find the cosine-sine decomposition of the 4×4 unitary matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Problem 9. Find a cosine-sine decomposition of the Hadamard matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problem 10. (i) Consider the 4×4 matrices

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Can one find 4×4 permutation matrices P, Q such that

$$\Omega = P\widetilde{\Omega}Q\,?$$

(ii) Consider the $2n \times 2n$ matrices

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 & \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & -1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}$$

and

$$\widetilde{\Omega} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. Can one find $2n \times 2n$ permutation matrices P, Q such that $\Omega = P\widetilde{\Omega}Q$?

Problem 11. Let A be an $m \times m$ matrix. Let B be an $n \times n$ matrix. Let X be an $m \times n$ matrix such that

$$AX = XB. \tag{1}$$

We can find non-singular matrices V and W such that

$$V^{-1}AV = J_A, \qquad W^{-1}BW = J_B$$

where J_A, J_B are the Jordan canonical form of A and B, respectively. Show that from (1) it follows that

$$J_A Y = Y J_B$$

where $Y := V^{-1}XW$.

Problem 12. Find the Cosine-Sine decomposition of

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Problem 13. Given the 3×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the singular value decomposition of A.

Chapter 7

Functions of Matrices

Let A be an $n \times n$ matrix over $\mathbb{C}$. The function $\exp(A)$ can be calculated from

$$\exp(A) = \sum_{j=0}^{\infty} fracA^j j!$$

and

$$\exp(A) = \lim_{m \to \infty} \left(I_n + \frac{A}{m} \right)^m .$$

For $\sinh(A)$ and $\cosh(A)$ we have

$$\sinh(A) = \sum_{j=0}^{\infty} \frac{A^{2j+1}}{(2j+1)!}$$

$$\cosh(A) = \sum_{j=0}^{\infty} \frac{A^{2j}}{(2j)!}.$$

Problem 1. Consider the matrix ($\zeta \in \mathbb{R}$)

$$S(\zeta) = \begin{pmatrix} \cosh(\zeta) & 0 & 0 & \sinh(\zeta) \\ 0 & \cosh(\zeta) & \sinh(\zeta) & 0 \\ 0 & \sinh(\zeta) & \cosh(\zeta) & 0 \\ \sinh(\zeta) & 0 & 0 & \cosh(\zeta) \end{pmatrix}.$$

(i) Show that the matrix is invertible, i.e. find the determinant.
(ii) Calculate the inverse of $S(\zeta)$.

(iii) Calculate

$$A := \frac{d}{d\zeta}S(\zeta)\bigg|_{\zeta=0}$$

and then calculate $\exp(\zeta A)$.
(iv) Do the matrices form a group under matrix multiplication?

Problem 2. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$. Let $\mathbf{q}$ and $\mathbf{J}$ be column vectors in $\mathbb{R}^n$. Calculate

$$Z(\mathbf{J}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dq_1 \cdots dq_n \exp\left(-\frac{1}{2}\mathbf{q}^T A\mathbf{q} + \mathbf{J}^T\mathbf{q}\right).$$

Note that

$$\int_{-\infty}^{\infty} dq e^{-(aq^2+bq+c)} = \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/(4a)}. \tag{1}$$

Problem 3. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$, i.e. $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Calculate

$$\int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A\mathbf{x}) d\mathbf{x}.$$

Problem 4. Consider the 2×2 matrix

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.$$

Calculate A^n, where $n \in \mathbb{N}$.

Problem 5. Let A be an $n \times n$ matrix over $\mathbb{C}$. The $n \times n$ matrix B over $\mathbb{C}$ is a square root of A iff $B^2 = A$. The number of square roots of a given matrix A may be zero, finite or infinite. Does the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

admit a square root?

Problem 6. Show that for any Pauli spin matrix σ_1, σ_2, σ_3 we have

$$\sin(\theta\sigma_j) = \sin(\theta)\sigma_j.$$

Problem 7. Let M be an $n \times n$ matrix with $m_{jk} = 1$ for all $j, k = 1, 2, \ldots, n$. Let $s \in \mathbb{C}$. Find $\exp(sM)$. Then consider the special case $sn = i\pi$

Problem 8. Let X, Y be $n \times n$ matrices. Show that

$$[e^X, Y] = \sum_{k=1}^{\infty} \frac{[X^k, Y]}{k!}.$$

Problem 9. Let A, B be $n \times n$ matrices. Show that

$$e^{A+B} - e^A \equiv \int_0^1 e^{(1-t)A} B e^{t(A+B)} dt.$$

Problem 10. Let A be an $n \times n$ matrix. Then $\exp(A)$ can also be calculated as

$$e^A = \lim_{m \to \infty} \left(I_n + \frac{A}{m} \right)^m .$$

Use this definition to show that

$$\det(e^A) \equiv e^{\mathrm{tr(A)}}.$$

Problem 11. Let $A_1, A_2, \ldots, A_p$ be $n \times n$ matrices over $\mathbb{C}$. The generalized *Trotter formula* is given by

$$\exp\left(\sum_{j=1}^{n} A_j \right) = \lim_{n \to \infty} f_n(\{ A_j \}) \tag{1}$$

where the n-th approximant $f_n(\{ A_j \})$ is defined by

$$f_n(\{ A_j \}) := \left(\exp\left(\frac{1}{n} A_1 \right) \exp\left(\frac{1}{n} A_2 \right) \cdots \exp\left(\frac{1}{n} A_p \right) \right)^n .$$

Let $p = 2$ and

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate the left and right-hand side of (1).

Problem 12. Let $\alpha, \beta \in \mathbb{R}$. Calculate

$$\exp \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}.$$

Problem 13. Consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $t \in \mathbb{R}$. Find $\exp(tA)$.

Problem 14. Let A, B be $n \times n$ matrices over $\mathbb{C}$ and $\alpha \in \mathbb{C}$. The *Baker-Campbell-Hausdorff formula* states that

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha[A, B] + \frac{\alpha^2}{2!}[A, [A, B]] + \cdots = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!}\{A^j, B\} = \widetilde{B}(\alpha)$$

where $[A, B] := AB - BA$ and

$$\{A^j, B\} := [A, \{A^{j-1}, B\}]$$

is the repeated commutator.
(i) Extend the formula to

$$e^{\alpha A} B^k e^{-\alpha A}$$

where $k \geq 1$.
(ii) Extend the formula to

$$e^{\alpha A} e^B e^{-\alpha A}.$$

Problem 15. Consider the $n \times n$ matrix ($n \geq 2$)

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Calculate

$$f(0)I_n + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 + \cdots + \frac{f^{n-1}}{(n-1)!}A^{n-1}$$

where $'$ denotes differentiation. Discuss.

Problem 16. Let A, B be $n \times n$ matrices. Show that

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

if $AB = BA$.

Problem 17. Consider the 3×3 matrix

$$A(\alpha) = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

Find $\exp(A)$.

Problem 18. Let A, B be $n \times n$ matrices with $A^2 = I_n$ and $B^2 = I_n$. Assume that the anticommutator of A and B vanishes, i.e.

$$[A, B]_+ = AB + BA = 0_n.$$

Let $a, b \in \mathbb{C}$. Calculate e^{aA+bB}.

Problem 19. Let A, B be $n \times n$ matrices with $A^2 = I_n$ and $B^2 = I_n$. Assume that the commutator of A and B vanishes, i.e.

$$[A, B] = AB - BA = 0_n.$$

Let $a, b \in \mathbb{C}$. Calculate e^{aA+bB}.

Problem 20. Consider the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}.$$

Find $\exp(tA)$.

Problem 21. Can one find $n \times n$ matrices A such that ($\epsilon \in \mathbb{R}$)

$$\exp(i\epsilon A) = I_n + (\cos(\epsilon) - 1)A^2 + i\sin(\epsilon)A\,?$$

Problem 22. Let $\alpha, \beta \in \mathbb{C}$. Let

$$M(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}.$$

(i) Calculate $\exp(M(\alpha, \beta))$.
(ii) For which values of $\alpha, \beta \in \mathbb{C}$ is the matrix nonnormal? Simplify the result for $\alpha = i\pi$ and β arbitrary. Is the matrix $M(\alpha = i\pi, \beta)$ nonnormal? Is the matrix $\exp(M(\alpha = i\pi, \beta)$ nonnormal?

Problem 23. Consider the two-dimensional rotation matrix

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

with $0 \le \theta \le \pi$. Find a square root $R^{1/2}$ of R, i.e a matrix S such that $S^2 = R$.

Problem 24. Let $f:\mathbb{R}\to\mathbb{R}$ be an analytic function. Let $\theta\in\mathbb{R}$, $\mathbf{n}$ a normalized vector in $\mathbb{R}^3$ and σ_1, σ_2, σ_3 the Pauli spin matrices. We define

$$\mathbf{n}\cdot\boldsymbol{\sigma} := n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3.$$

Then

$$f(\theta\mathbf{n}\cdot\boldsymbol{\sigma}) \equiv \frac{1}{2}(f(\theta)+f(-\theta))I_2 + \frac{1}{2}(f(\theta)-f(-\theta))(\mathbf{n}\cdot\boldsymbol{\sigma}).$$

Apply this identity to $f(x)=\sin(x)$.

Problem 25. Let $a,b\in\mathbb{R}$ and

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Calculate $\exp(M)$.

Problem 26. Let A be an $n\times n$ matrix over $\mathbb{C}$. Assume that all eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ are pairwise distinct. Then e^{tA} can be calculated as follows (*Lagrange interpolation*)

$$e^{tA} = \sum_{j=1}^{n} e^{\lambda_j t}\prod_{\substack{k=1\\k\neq j}}^{n}\frac{(A-\lambda_k I_n)}{(\lambda_j-\lambda_k)}.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate e^{tA} using this method.

Problem 27. Let A be an $n\times n$ matrix over $\mathbb{C}$. Assume that all eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n$ are pairwise distinct. Then e^{tA} can be calculated as follows (*Newton interpolation*)

$$e^{tA} = e^{\lambda_1 t}I_n + \sum_{j=2}^{n}[\lambda_1,\ldots,\lambda_j]\prod_{k=1}^{j-1}(A-\lambda_k I_n).$$

The divided differences $[\lambda_1,\ldots,\lambda_j]$ depend on t and are defined recursively by

$$[\lambda_1,\lambda_2] := \frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}$$

$$[\lambda_1,\ldots,\lambda_{k+1}] := \frac{[\lambda_1,\ldots,\lambda_k]-[\lambda_2,\ldots,\lambda_{k+1}]}{\lambda_1-\lambda_{k+1}}, \quad k\geq 2.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate e^{tA} using this method.

Problem 28. Let A, B, C be $n \times n$ matrices over $\mathbb{C}$ such that $A^2 = I_n$, $B^2 = I_n$ and $C^2 = I_n$. Furthermore assume that

$$[A, B]_+ \equiv AB + BA = 0_n, \quad [B, C]_+ \equiv BC + CB = 0_n, \quad [C, A]_+ \equiv CA + AC = 0_n$$

i.e. the anticommutators vanish. Let $\alpha, \beta, \gamma \in \mathbb{C}$. Calculate $e^{\alpha A + \beta B + \gamma C}$ using

$$e^{\alpha A + \beta B + \gamma C} = \sum_{j=0}^{\infty} \frac{(\alpha A + \beta B + \gamma C)^j}{j!}.$$

Problem 29. Let A, B be $n \times n$ matrices. Then we have the identity

$$\det(e^A e^B e^{-A} e^{-B}) \equiv \exp(\operatorname{tr}([A, B]))$$

where $[A, B] := AB - BA$ defines the commutator. Show that

$$\det(e^A e^B e^{-A} e^{-B}) = 1.$$

Problem 30. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Show that

$$\exp(e) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \exp(-f) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

(ii) Show that

$$\exp(e) \exp(-f) \exp(e) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Problem 31. Let A be an $n \times n$ matrix. The *characteristic polynomial*

$$\det(\lambda I_n - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = p(\lambda)$$

is closely related the the *resolvent* $(\lambda I_n - A)^{-1}$ trough the formula

$$(\lambda I_n - A)^{-1} = \frac{N_1 \lambda^{n-1} + N_2 \lambda^{n-2} + \cdots + N_n}{\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n} = \frac{N(\lambda)}{p(\lambda)}$$

where the adjugate matrix $N(\lambda)$ is a polynomial in λ of degree $n-1$ with constant $n \times n$ coefficient matrices $N_1, \ldots, N_n$. The *Laplace transform* of the matrix exponential is the resolvent

$$\mathcal{L}(e^{tA}) = (\lambda I_n - A)^{-1}.$$

The N_k matrices and a_k coefficients may be computed recursively as follows

$$\begin{aligned}
N_1 &= I_n, \qquad a_1 = -\frac{1}{1}\operatorname{tr}(AN_1) \\
N_2 &= AN_1 + a_1 I_n, \qquad a_2 = -\frac{1}{2}\operatorname{tr}(AN_2) \\
&\vdots \\
N_n &= AN_{n-1} + a_{n-1} I_n, \qquad a_n = -\frac{1}{n}\operatorname{tr}(AN_n) \\
0 &= AN_n + a_n I_n.
\end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find the N_k matrices and the coefficients a_k and thus calculate the resolvent.

Problem 32. Let A be an $n \times n$ matrix. The *characteristic polynomial*

$$\det(\lambda I_n - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = p(\lambda)$$

is closely related the the *resolvent* $(\lambda I_n - A)^{-1}$ trough the formula

$$(\lambda I_n - A)^{-1} = \frac{N_1\lambda^{n-1} + N_2\lambda^{n-2} + \cdots + N_n}{\lambda^n + a_1\lambda^{n-1} + \cdots + a_n} = \frac{N(\lambda)}{p(\lambda)}$$

where the adjugate matrix $N(\lambda)$ is a polynomial in λ of degree $n-1$ with constant $n \times n$ coefficient matrices $N_1, \ldots, N_n$. The Laplace transform of the matrix exponential is the resolvent

$$\mathcal{L}(e^{tA}) = (\lambda I_n - A)^{-1}.$$

The N_k matrices and a_k coefficients may be computed recursively as follows

$$\begin{aligned}
N_1 &= I_n, \qquad a_1 = -\frac{1}{1}\operatorname{tr}(AN_1) \\
N_2 &= AN_1 + a_1 I_n, \qquad a_2 = -\frac{1}{2}\operatorname{tr}(AN_2) \\
&\vdots \\
N_n &= AN_{n-1} + a_{n-1} I_n, \qquad a_n = -\frac{1}{n}\operatorname{tr}(AN_n) \\
0 &= AN_n + a_n I_n.
\end{aligned}$$

Show that

$$\mathrm{tr}(\mathcal{L}(e^{tA})) = \frac{1}{p(\lambda)}\frac{dp(\lambda)}{d\lambda}.$$

Problem 33. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$, i.e. all the eigenvalues, which are real, are positive. We also have $A^T = A$. Consider the analytic function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \exp\left(-\frac{1}{2}\mathbf{x}^T A^{-1}\mathbf{x}\right).$$

Calculate the *Fourier transform* of f. The Fourier transform is defined by

$$\hat{f}(\mathbf{k}) := \int_{\mathbb{R}^n} f(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}$$

where $\mathbf{k}\cdot\mathbf{x} \equiv \mathbf{k}^T\mathbf{x} \equiv k_1x_1 + \cdots + k_nx_n$ and $d\mathbf{x} = dx_1 \dots dx_n$. The inverse Fourier transform is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}$$

where $d\mathbf{k} = dk_1 \dots dk_n$. Note that we have with $a > 0$

$$\int_{\mathbb{R}} e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{a}}e^{(b^2-4ac)/(4a)}.$$

Problem 34. Let A be an $n \times n$ matrix. Suppose f is an analytic function inside on a closed contour Γ which encircles $\lambda(A)$, where $\lambda(A)$ denotes the eigenvalues of A. We define $f(A)$ to be the $n \times n$ matrix

$$f(A) = \frac{1}{2\pi i}\oint_{\Gamma} f(z)(zI_n - A)^{-1}dz.$$

This is a matrix version of the *Cauchy integral theorem*. The integral is defined on an element-by-element basis $f(A) = (f_{jk})$, where

$$f_{jk} = \frac{1}{2\pi i}\oint_{\Gamma} f(z)\mathbf{e}_j^T(zI_n - A)^{-1}\mathbf{e}_k dz$$

where $\mathbf{e}_j$ $(j = 1, 2, \dots, n)$ is the standard basis in $\mathbb{C}^n$. Let $f(z) = z^2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate $f(A)$.

Problem 35. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Calculate

$$e^A B e^A.$$

Set $f(\epsilon) = e^{\epsilon A} B e^{\epsilon A}$, where ϵ is a real parameter. Then differentiate with respect to ϵ. For $\epsilon = 1$ we have $e^A B e^A$.

Problem 36. Let A, B be positive definite matrices. Then we have the integral representation ($x \geq 0$)

$$\ln(A + xB) - \ln(A) \equiv \int_0^\infty (A + uI_n)^{-1} xB(A + xB + uI_n)^{-1} du.$$

Let

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Calculate the left and right-hand side of the integral representation.

Problem 37. Let $\epsilon \in \mathbb{R}$. Calculate

$$f(\epsilon) = e^{-\epsilon\sigma_2} \sigma_3 e^{\epsilon\sigma_2}.$$

Hint. Differentiate the matrix-valued function f with respect to ϵ and solve the initial value problem of the resulting ordinary differential equation.

Problem 38. Find the square root of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. find the matrices X such that $X^2 = A$.

Problem 39. Let A be an arbitrary $n \times n$ matrix. Can we conclude that

$$\exp(A^*) = (\exp(A))^* \,?$$

Problem 40. Let A be an invertible $n \times n$ matrix over $\mathbb{R}$. Consider the functions

$$E_j = \frac{1}{2}(A\mathbf{c}_j - \mathbf{e}_j)^T (A\mathbf{c}_j - \mathbf{e}_j)$$

where $j = 1, \ldots, n$, $\mathbf{c}_j$ is the j-th column of the inverse matrix of A, $\mathbf{e}_j$ is the j-th column of the $n \times n$ identity matrix. This means $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis (as column vectors) in $\mathbb{R}^n$. The $\mathbf{c}_j$ are determined by minimizing the E_j with respect to the $\mathbf{c}_j$. Apply this method to find the inverse of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Problem 41. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that A is hermitian, i.e. $A^* = A$. Thus A has only real eigenvalues. Assume that

$$A^5 + A^3 + A = 3I_n.$$

Show that $A = I_n$.

Problem 42. Let $\mathbf{f}$ be a function from U, an open subset of $\mathbb{R}^m$, to $\mathbb{R}^n$. Assume that the component function f_j $(j = 1, \ldots, n)$ possess first order partial derivatives. Then we can associate the $n \times m$ matrix

$$\left(\left. \frac{\partial f_j}{\partial x_k} \right|_{\mathbf{p}} \right), \qquad j = 1, \ldots, n \quad k = 1, \ldots, m$$

where $\mathbf{p} \in U$. The matrix is is called the *Jacobian matrix* of $\mathbf{f}$ at the point $\mathbf{p}$. When $m = n$ the determinant of the square matrix $\mathbf{f}$ is called the *Jacobian* of $\mathbf{f}$. Let

$$A = \{ r \in \mathbb{R} : r > 0 \}, \qquad B = \{ \theta \in \mathbb{R} : 0 \le \theta < 2\pi \}$$

and $\mathbf{f} : A \times B \to \mathbb{R}^2$ with $f_1(r, \theta) = r \cos\theta$, $f_2(r, \theta) = r \sin\theta$. Find the Jacobian matrix and the Jacobian.

Problem 43. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Find all nonzero 2×2 matrices A such that

$$AJ = JA.$$

Problem 44. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let T be a nilpotent matrix over $\mathbb{C}$ satisfying

$$T^* A + AT = 0.$$

Show that

$$(e^T)^* A e^T = A.$$

Problem 45. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Let $\beta, \epsilon \in \mathbb{R}$. Show that

$$\exp(\beta(A+B)) \equiv \exp(\beta A)\left(I_n + \int_0^\beta d\epsilon e^{-\epsilon A} B e^{\epsilon(A+B)}\right).$$

Problem 46. Consider the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $\alpha \in \mathbb{R}$. Find $\exp(\alpha A)$.

Problem 47. Let $\epsilon \in \mathbb{R}$. Let

$$I_n - \epsilon A$$

be a positive definite matrix. Calculate

$$\exp(\operatorname{tr}(\ln(I_n - \epsilon A))$$

using the identity $\det e^M \equiv \exp(\operatorname{tr}(M))$.

Problem 48. Consider the Pauli spin matrices σ_1, σ_2 and σ_3. Can one find an $\alpha \in \mathbb{R}$ such that

$$\exp(i\alpha\sigma_3)\sigma_1 \exp(-i\alpha\sigma_3) = \sigma_2\,?$$

Problem 49. (i) Let $a, b \in \mathbb{R}$. Let

$$K = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Find $\exp(iK)$.
(ii) Use the result to find a, b such that

$$\exp(iK) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 50. Let P be an $n \times n$ projection matrix. Let $\epsilon \in \mathbb{R}$. Calculate

$$\exp(\epsilon P).$$

Problem 51. (i) Let $P_1, P_2, \ldots, P_n$ be an $n \times n$ projection matrices. Assume that $P_j P_k = 0$ $(j \neq k)$ for all $j, k = 1, 2, \ldots, n$. Let $\epsilon_j \in \mathbb{R}$ with $j = 1, 2, \ldots, n$. Calculate

$$\exp(\epsilon_1 P_1 + \epsilon_2 P_2 + \cdots + \epsilon_n P_n).$$

(ii) Assume additionally that

$$P_1 + P_2 + \cdots + P_n = I_n.$$

Simplify the result from (i) using this condition.

Problem 52. Let A, B be $n \times n$ hermitian matrices. There exists $n \times n$ unitary matrices U and V (depending on A and B) such that

$$\exp(iA)\exp(iB) = \exp(iUAU^{-1} + iVBV^{-1}).$$

Consider $n = 2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Find U and V. Note that A and B are also unitary and represend the NOT-gate and Hadamard gate, respectively. Furthermore

$$[A, B] = \sqrt{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Problem 53. Let $a, b \in \mathbb{C}$ and

$$M(a, b) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Calculate $\exp(M(a, b))$.

Problem 54. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that A and B commute with the commutator $[A, B]$. Then

$$\exp(A + B) = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A, B]\right).$$

Can this formula be applied to the matrices

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 55. Let $\epsilon \in \mathbb{R}$. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Expand

$$e^{\epsilon A} e^{\epsilon B} e^{-\epsilon A} e^{-\epsilon B}$$

up to second order in ϵ.

Problem 56. Let $\alpha, \beta \in \mathbb{R}$. Consider the 2×2 matrix

$$B = \begin{pmatrix} -i\alpha & -\beta \\ -\beta & i\alpha \end{pmatrix}.$$

Find $\exp(tB)$, where $t \in \mathbb{R}$ and thus solve the initial value problem of the matrix differential equation

$$\frac{dA}{dt} = BA(t).$$

Problem 57. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that for all eigenvalues λ we have $\Re(\lambda) < 0$. Let B be an arbitrary $n \times n$ matrix over $\mathbb{C}$. Let

$$R := \int_0^\infty e^{tA^*} B e^{tA}.$$

Show that the matrix R satisfies the matrix equation

$$RA + A^* R = -B.$$

Problem 58. Let A, B be $n \times n$ matrices over $\mathbb{C}$ such that $[A, B] = A$. What can be said about the commutator

$$[e^A, e^B] \, ?$$

Problem 59. Consider the positive semidefinite matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Use the right-hand side of the identity

$$\det(A) \equiv \exp(\operatorname{tr}(\ln(A)))$$

to calculate $\det(A)$.

Problem 60. Let A, B be $n \times n$ matrices over $\mathbb{C}$ and $A^2 = I_n$, $B^2 = I_n$. Calculate

$$\exp(z_1 A + z_2 B)$$

where $z_1, z_2 \in \mathbb{C}$.

Problem 61. Let $z \in \mathbb{C}$. Let A, B be $n \times n$ matrices over $\mathbb{C}$. We say that B is invariant with respect to A if

$$e^{zA} B e^{-zA} = B.$$

Obviously e^{-zA} is the inverse of e^{zA}. Show that, if this condition is satisfied, one has $[A,B]=0_n$, where 0_n is the $n\times n$ zero matrix. If e^{zA} would be unitary we have $UBU^*=B$.

Problem 62. Let $z\in\mathbb{C}$ and

$$A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{11} \end{pmatrix}.$$

(i) Calculate $\exp(zA)$, $\exp(-zA)$ and $\exp(zA)B\exp(-zA)$.
(ii) Calculate the commutator $[A,B]$.

Problem 63. Let $z\in\mathbb{C}$ and

$$A=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} \\ -b_{12} & b_{11} \end{pmatrix}.$$

(i) Calculate $\exp(zA)$, $\exp(-zA)$ and $\exp(zA)B\exp(-zA)$.
(ii) Calculate the commutator $[A,B]$.

Problem 64. Let $z\in\mathbb{C}$ and

$$A=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{11} \end{pmatrix}.$$

(i) Calculate $\exp(zA)$, $\exp(-zA)$ and $\exp(zA)B\exp(-zA)$.
(ii) Calculate the commutator $[A,B]$.

Problem 65. Consider the Pauli spin matrices σ_1, σ_2, σ_3. Find the skew-hermitian matrices Σ_1, Σ_2, Σ_3 such that

$$\sigma_1=\exp(\Sigma_1), \quad \sigma_2=\exp(\Sigma_2), \quad \sigma_3=\exp(\Sigma_3).$$

Find the commutators $[\Sigma_1,\Sigma_2]$, $[\Sigma_2,\Sigma_3]$, $[\Sigma_3,\Sigma_1]$ and compare with the commutators $[\sigma_1,\sigma_2]$, $[\sigma_2,\sigma_3]$, $[\sigma_3,\sigma_1]$.

Problem 66. Let $\alpha\in\mathbb{R}$. Consider the matrix

$$A(\alpha)=\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

(i) Show that the matrix is orthogonal.
(ii) Find the determinant of $A(\alpha)$. Is the matrix an element of $SO(2,\mathbb{R})$?
(iii) Do these matrices form a group under matrix multiplication?
(iv) Calculate

$$X=\frac{d}{d\alpha}A(\alpha)\bigg|_{\alpha=0}.$$

Calculate $\exp(\alpha X)$ and compare this matrix with $A(\alpha)$. Discuss.
(v) Let $\beta \in \mathbb{R}$ and

$$B(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}.$$

Is the matrix $A(\alpha) \otimes B(\beta)$ orthogonal? Find the determinant of $A(\alpha) \otimes B(\alpha)$. Is this matrix an element of $SO(4,\mathbb{R})$?

Problem 67. We know that for any $n \times n$ matrix A over $\mathbb{C}$ the matrix $\exp(A)$ is invertible with the inverse $\exp(-A)$. What about $\cos(A)$ and $\cosh(A)$?

Problem 68. (i) Let $\epsilon \in \mathbb{R}$. Let A be an $n \times n$ matrix over $\mathbb{C}$. Find

$$\lim_{\epsilon \to 0} \frac{\sinh(2\epsilon A)}{\sinh(\epsilon)}.$$

(ii) Assume that $A^2 = I_n$. Calculate

$$\frac{\sinh(2\epsilon A)}{\sinh(\epsilon)}.$$

(iii) Assume that $A^2 = 0_n$. Calculate

$$\frac{\sinh(2\epsilon A)}{\sinh(\epsilon)}.$$

Problem 69. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Is $\sin(A)$ invertible? Is $\cos(A)$ invertible? Is $\sin(B)$ invertible? Is $\cos(B)$ invertible?

Problem 70. Is $\cos(A)$ invertible for all $n \times n$ matrices A over $\mathbb{C}$?

Problem 71. Let A be a nilpotent matrix. Is the matrix $\cos(A)$ invertible?

Problem 72. Consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find $\cosh(A)$, $\sinh(A)$, $\cosh(B)$, $\sinh(B)$. Which of these matrices are invertible?

Problem 73. Let A be an $n \times n$ matrix over $\mathbb{C}$. Find

$$\lim_{\epsilon \to 0} \frac{\sinh(\epsilon A)}{\sinh(\epsilon)}.$$

Problem 74. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Write the matrix A in the form

$$A = I_3 + B, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and calculate e^A using $e^A = e^{I_3} e^B$.

Problem 75. Consider the 3×3 matrix

$$A = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

Calculate $\exp(i\phi A)$, where $\phi \in \mathbb{R}$.

Problem 76. Let A be an $n \times n$ matrix. The *characteristic polynomial*

$$\det(\lambda I_n - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = p(\lambda)$$

is closely related the the *resolvent* $(\lambda I_n - A)^{-1}$ trough the formula

$$(\lambda I_n - A)^{-1} = \frac{N_1 \lambda^{n-1} + N_2 \lambda^{n-2} + \cdots + N_n}{\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n} = \frac{N(\lambda)}{p(\lambda)}$$

where the adjugate matrix $N(\lambda)$ is a polynomial in λ of degree $n-1$ with constant $n \times n$ matrices $N_1, \ldots, N_n$. The *Laplace transform* of the matrix exponential is the resolvent

$$\mathcal{L}(e^{tA}) = (\lambda I_n - A)^{-1}.$$

The N_k matrices and a_k coefficients may be computed recursively as follows

$$\begin{aligned}
N_1 &= I_n, \qquad a_1 = -\frac{1}{1}\mathrm{tr}(AN_1) \\
N_2 &= AN_1 + a_1 I_n, \qquad a_2 = -\frac{1}{2}\mathrm{tr}(AN_2) \\
&\vdots \\
N_n &= AN_{n-1} + a_{n-1} I_n, \qquad a_n = -\frac{1}{n}\mathrm{tr}(AN_n) \\
0 &= AN_n + a_n I_n.
\end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find the N_k matrices and the coefficients a_k and thus calculate the resolvent.

Problem 77. Let A be a normal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and corresponding normalized pairwise orthogonal eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$. Let $\mathbf{w}, \mathbf{v} \in \mathbb{C}^n$ (column vectors). Find

$$\mathbf{w}^* e^A \mathbf{v}$$

by expanding $\mathbf{w}$ and $\mathbf{v}$ with respect to the basis $\mathbf{u}_j$ $(j = 1, \dots, n)$.

Problem 78. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find $\cos(A)$ and the inverse of this matrix. Find $\cos(B)$ and the inverse of this matrix. Find the commutators $[A, B]$ and $[\cos(A), \cos(B)]$. Discuss.

Problem 79. Let V be the 2×2 matrix

$$V = v_0 I_2 + v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3$$

where $v_0, v_1, v_2, v_3 \in \mathbb{R}$. Consider the equation

$$\exp(i\epsilon V) = (I_2 - iW)(I_2 + iW)^{-1}$$

where ϵ is real. Find W as a function of V.

Problem 80. Consider the rotation matrix

$$R(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

where ω is the fixed frequency. Find the matrix

$$H(t) = i\hbar \frac{dR(t)}{dt} R^T(t)$$

and show it is hermitian.

Problem 81. (i) Let σ_1 be the Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate

$$\exp\left(-\frac{1}{2}i\pi(\sigma_1 - I_2)\right).$$

(ii) Find all 2×2 matrices A and $c \in \mathbb{C}$ such that

$$\exp(c(A - I_2)) = A.$$

Problem 82. Let B be an $n \times n$ matrix with $B^2 = I_n$. Show that

$$\exp\left(-\frac{1}{2}i\pi(B - I_n)\right) \equiv B.$$

Problem 83. Let A, B be $n \times n$ matrices over $\mathbb{C}$ and

$$\exp(A)\exp(B) = \exp(C).$$

Then the matrix C can be given as an infinite series of commutators of A and B. Let $z \in \mathbb{C}$. We write

$$\exp(zA)\exp(zB) = \exp(C(zA, zB))$$

where

$$C(zA, zB) = \sum_{j=1}^{\infty} c_j(A, B)z^j.$$

Show that the expansion up to fourth order is given by

$$\begin{aligned}
c_1(A, B) &= A + B \\
c_2(A, B) &= \frac{1}{2}[A, B] \\
c_3(A, B) &= \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B] \\
c_4(A, B) &= -\frac{1}{24}[A, [B, [A, B].
\end{aligned}$$

Problem 84. The 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are nonnormal, i.e. $AA^* \neq A^*A$ and $BB^* \neq B^*B$. Note that $A^* = B$. Are the matrices

$$\exp(A), \qquad \exp(B)$$

normal? Are the matrices $\sin(A)$, $\sin(B)$, $\cos(A)$, $\cos(B)$ normal?

Problem 85. Let $a, b \in \mathbb{C}$. Find

$$\exp\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}.$$

Problem 86. Find the unitary matrix

$$U(t) = e^{i\phi\sin(\omega t)\sigma_1}$$

with the Pauli spin matrix σ_1.

Problem 87. Let N be a nilpotent $n \times n$ matrix with $N^j = 0_n$ with $j \geq 1$. Is

$$\ln(I_n - N) = -\left(N + \frac{1}{2}N + \cdots + \frac{1}{j-1}N^{j-1}\right)$$

and $\exp(\ln(I_n - N) = I_n - N$.

Problem 88. Let $H = H_1 + H_2 + H_3$ and H_1, H_2, H_3 be $n \times n$ hermitian matrices. Show that

$$e^{-\beta H} = e^{-\beta H_1/2}e^{-\beta H_2/2}e^{-\beta H_3}e^{-\beta H_2/2}e^{-\beta H_1/2} + \left(-\frac{1}{2}\beta\right)^3 S$$

where

$$S = \frac{1}{6}([[H_2 + H_3, H_1], H_1 + 2H_2 + 2H_3] + [[H_3, H_2], H_2 + 2H_3]).$$

Problem 89. Consider the matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Calculate the matrix

$$U = \exp\left(i\frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)\right).$$

Is the matrix U unitary? Prove or disprove. If so find the group generated by U.

Problem 90. Consider the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

over the complex numbers. Find $\exp(A)$.

Problem 91. Let $z \in \mathbb{C}$. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that $A^3 = zA$. Find $\exp(A)$.

Problem 92. Let $a_{12}, a_{13}, a_{23} \in \mathbb{R}$. Consider the skew-symmetric matrix

$$A(a_{12}, a_{13}, a_{23}) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

Find $\exp(A)$.

Problem 93. Consider the matrix

$$A(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Is

$$A(\alpha)\frac{dA(\alpha)}{d\alpha} = \frac{dA(\alpha)}{d\alpha}A(\alpha)\,?$$

Problem 94. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that the inverse of A and $(A+B)$ exist. Show that

$$(A+B)^{-1} = A^{-1} - A^{-1}B(A+B)^{-1}.$$

Apply the identity to $A = \sigma_3$ and $B = \sigma_1$.

Problem 95. (i) Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $z \in \mathbb{C}$. Calculate

$$\cosh(z\sigma_j), \qquad \sinh(z\sigma_j) \qquad j = 1, 2, 3.$$

(ii) Let $\alpha \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Calculate

$$\cosh\left(\alpha\frac{d}{dx}\right)f(x), \qquad \sinh\left(\alpha\frac{d}{dx}\right)f(x).$$

Problem 96. Let A, B be $n \times n$ matrices. We know that

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$$

Find

$$e^A e^B e^{-A}.$$

Problem 97. Consider the matrix

$$A(z_1, z_2) = \begin{pmatrix} 0 & 1 \\ z_1 & z_2 \end{pmatrix}.$$

Find

$$\exp(zA(z_1, z_2)).$$

Problem 98. Consider the 2×2 matrix

$$A(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ -\sinh(\alpha) & -\cosh(\alpha) \end{pmatrix}.$$

Find the maxima and minima of the function

$$f(\alpha) = \operatorname{tr}(A^2(\alpha)) - (\operatorname{tr}(A(\alpha)))^2.$$

Problem 99. Consider the 3×3 matrix

$$S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Is the matrix hermitian? Find the eigenvalues and eigenvectors of S_2.
(ii) Is $S^3 = S$? Prove or disprove.
(iii) Let $\phi \in \mathbb{R}$. Find

$$\exp(i\phi S_2).$$

Problem 100. Let $x, y \in \mathbb{R}$. We know that

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Is

$$\sin(A+B) = \sin(A)\cos(B) + \cos(B)\sin(B).$$

Prove or disprove.

Problem 101. Let $A_1, \dots, A_m$ be $n \times n$ matrices. Show that

$$e^{A_1} \cdots e^{A_m} = \exp(\frac{1}{2}\sum_{j<k}^{m}[A_j, A_k]) \exp(A_1 + \cdots + A_m)$$

if the matrices A_j $(j = 1, \ldots, m)$ satisfy

$$[[A_j, A_k], A_\ell] = 0 \quad \text{for all } j, k, \ell \in \{1, \ldots, m\}.$$

Problem 102. (i) Consider the three (hermitian) spin-1 matrices

$$S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

all with the eigenvalues $+1$, 0 and -1. Show that $S_j^3 = S_j$.
(ii) Let $\phi \in \mathbb{R}$. Show that

$$\exp(i\phi S_j) = I_3 + i\sin(\phi)S_j - (1 - \cos(\phi))S_j^2$$

which is a unitary matrix.

Problem 103. Let A, B be $n \times n$ matrices over $\mathbb{C}$ with $A^2 = B^2 = I_n$ and $[A, B]_+ = 0_n$, i.e. the anticommutator vanishes. Let $z \in \mathbb{C}$. The *Lie-Trotter formula* is given by

$$\exp(z(A + B)) = \lim_{p \to \infty} \left(e^{zA/p} e^{zB/p} \right)^p.$$

Calculate $e^{z(A+B)}$ using the right-hand side.

Problem 104. The exponential of an $n \times n$ matrix A is defined as

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Calculate

$$\exp\left(\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

in two different ways. Compare and discuss.

Problem 105. Let $\mathbf{v}_1$, $\mathbf{v}_2$ be an orthonormal set in $\mathbb{C}^2$. Consider the 2×2 matrix

$$A = -i\mathbf{v}_1\mathbf{v}_1^* + i\mathbf{v}_2\mathbf{v}_2^*.$$

Thus $I_2 = \mathbf{v}_1\mathbf{v}_1^* + \mathbf{v}_2\mathbf{v}_2^*$. Find K such that $\exp(K) = A$.

Problem 106. Let $z_1, z_2, z_3 \in \mathbb{C}$. Consider the 2×2 matrix

$$A(z_1, z_2, z_3) = \begin{pmatrix} z_1 & z_2 \\ z_3 & 0 \end{pmatrix}.$$

Calculate

$$\exp(A(z_1, z_2, z_3))$$

using

$$\exp(A(z_1, z_2, z_3)) = \sum_{j=0}^{\infty} \frac{A^j(z_1, z_2, z_3)}{j!}.$$

Problem 107. Any 2×2 matrix can be written as a linear combination of the Pauli spin matrices and the 2×2 identity matrix

$$A = aI_2 + b\sigma_1 + c\sigma_2 + d\sigma_3$$

where $a, b, c, d \in \mathbb{C}$.
(i) Find A^2 and A^3.
(ii) Use the result from (i) to find all matrices A such that $A^3 = \sigma_1$.

Problem 108. Let A, B be $n \times n$ hermitian matrices. Then there exit $n \times n$ unitary matrices U and V such that

$$\exp(iA)\exp(iB) = \exp(iUAU^{-1} + iVBV^{-1}).$$

Let $n = 2$, $A = \sigma_1$, $B = \sigma_3$. Find U and V.

Problem 109. Let A, B be $n \times n$ matrices with $\|B\| \ll \|A\|$. Then we have the expansion

$$e^{A+B} = e^A \left(I_n + \int_0^1 e^{-tA} B e^{tA} dt \right) + \cdots$$

Apply the equation to

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

Problem 110. Consider the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

which is a unitary matrix.
(i) Apply the spectral theorem to find the skew-hermitian matrix K such that $B = e^K$.
(ii) Apply the Cayley-Hamilton theorem to find the skew-hermitian matrix K such that $B = e^K$.

Problem 111. Find the square root of

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

applying the spectral theorem.

Problem 112. (i) Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the 2×2 matrix

$$A(\alpha) = \frac{1}{\sqrt{1+\alpha^2}} \begin{pmatrix} \alpha & 1 \\ 1 & -\alpha \end{pmatrix}.$$

For $\alpha = 0$ we obtain the Pauli spin matrix σ_1, for $\alpha = 1$ we have the Hadamard matrix and for $\alpha \to \infty$ we obtain the Pauli spin matrix σ_3.
(ii) Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the 2×2 matrix

$$B(\alpha) = \begin{pmatrix} \tanh(\alpha) & 1/\cosh(\alpha) \\ 1/\cosh(\alpha) & -\tanh(\alpha) \end{pmatrix}$$

The matrices $A(\alpha)$ and $B(\alpha)$ are connected via the invertible transformation $\alpha \to \sinh(\alpha)$.

Problem 113. (i) Find the square roots of

$$-i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

(ii) Find the square roots of

$$(-i\sigma_1) \otimes (-i\sigma_1) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 114. Let $\alpha \in \mathbb{R}$ and P be an $n \times n$ projection matrix. Let $Q = P^{1/2}$ be a square root of P, i.e. $Q^2 = P$. Find

$$U(\alpha) = \exp(i\alpha P^{1/2}).$$

Problem 115. Calculating $\exp(A)$ we can also use the Cayley-Hamilton theorem and the Putzer method. Using the Cayley-Hamilton theorem we can write

$$f(A) = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_2A^2 + a_1A + a_0I_n \tag{1}$$

where the complex numbers a_0, a_1, $\ldots$, a_{n-1} are determined as follows: Let

$$r(\lambda) := a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

which is the right-hand side of (1) with A^j replaced by λ^j ($j = 0, 1, \ldots, n-1$). For each distinct eigenvalue λ_j of the matrix A, we consider the equation

$$f(\lambda_j) = r(\lambda_j). \tag{2}$$

If λ_j is an eigenvalue of multiplicity k, for $k > 1$, then we consider also the following equations

$$f'(\lambda)|_{\lambda=\lambda_j} = r'(\lambda)|_{\lambda=\lambda_j}, \quad \cdots \quad , f^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j} = r^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j}.$$

Any unitary matrix U can be written as $U = \exp(iK)$, where K is hermitian. Apply this method to find K for the Hadamard gate

$$U_H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problem 116. Let $z \in \mathbb{C}$. Construct all 2×2 matrices A and B over $\mathbb{C}$ such that

$$\exp(zA)B\exp(-zA) = e^{-z}B.$$

Problem 117. Let A be an $n \times n$ matrix over $\mathbb{C}$. Consider the Taylor series

$$(I_n + A)^{1/2} = I_n + \frac{1}{2}A - \frac{1}{2\cdot 4}A^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}A^3 - \cdots$$

and

$$(I_n + A)^{-1/2} = I_n - \frac{1}{2}A + \frac{1\cdot 3}{2\cdot 4}A^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}A^3 + \cdots$$

What is the condition (the norm) on A such that the Taylor series exist? Can it be applied to the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

Note that for $n = 1$ we have the condition $-1 < A \leq +1$.

Problem 118. Let U be an $n \times n$ unitary matrix. Let $H = U + U^*$. Calculate $\exp(zH)$.

Problem 119. Let A be an $n \times n$ matrix. The *characteristic polynomial*

$$\det(\lambda I_n - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = p(\lambda)$$

is closely related the the *resolvent* $(\lambda I_n - A)^{-1}$ trough the formula

$$(\lambda I_n - A)^{-1} = \frac{N_1\lambda^{n-1} + N_2\lambda^{n-2} + \cdots + N_n}{\lambda^n + a_1\lambda^{n-1} + \cdots + a_n} = \frac{N(\lambda)}{p(\lambda)}$$

where the adjuagte matrix $N(\lambda)$ is a polynomial in λ of degree $n-1$ with constant $n \times n$ matrices $N_1, cdots, N_n$. The Laplace transform of the matrix exponential is the resolvent

$$\mathcal{L}(e^{tA}) = (\lambda I_n - A)^{-1}.$$

The N_k matrices and a_k coefficients may be computed recursively as follows

$$\begin{aligned}
N_1 &= I_n, \qquad a_1 = -\frac{1}{1}\mathrm{tr}(AN_1) \\
N_2 &= AN_1 + a_1 I_n, \qquad a_2 = -\frac{1}{2}\mathrm{tr}(AN_2) \\
&\vdots \\
N_n &= AN_{n-1} + a_{n-1}I_n, \qquad a_n = -\frac{1}{n}\mathrm{tr}(AN_n) \\
0 &= AN_n + a_n I_n.
\end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find the N_k matrices and the coefficients a_k and thus calculate the resolvent.

Problem 120. Let A, B, C be $n \times n$ positive semidefinite matrices. We define

$$A \circ B := A^{1/2} B A^{1/2}$$

where $A^{1/2}$ denotes the unique positive square root of A. Is

$$A \circ (B \circ C) = (A \circ B) \circ C\,?$$

Prove or disprove.

Problem 121. Let $\alpha > 0$ and

$$A(\alpha) = \begin{pmatrix} 0 & 0 & 0 & i/\alpha \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ \alpha i & 0 & 0 & 0 \end{pmatrix}.$$

Find $\exp(\pi A(\alpha))$.

Chapter 8

Cayley-Hamilton Theorem

Consider an $n \times n$ matrix A over $\mathbb{C}$ and the polynomial

$$p(\lambda) = \det(A - \lambda I_n)$$

with the characteristic equation $p(\lambda) = 0$. The Cayley-Hamilton theorem states that substituting the matrix A in the characteristic polynomial results in the $n \times n$ zero matrix.

Problem 1. Apply the *Cayley-Hamilton theorem* to the 3×3 matrix A and express the result using the trace and determinant of A.

Problem 2. Let A be an $n \times n$ matrix. Let

$$c(z) := \det(zI_n - A) = z^n - \sum_{k=0}^{n-1} c_k z^k$$

be the characteristic polynomial of A. Apply the *Cayley-Hamilton theorem* $c(A) = 0$ to calculate $\exp(A)$.

Problem 3. (i) Let A be an $n \times n$ matrix with $A^3 = I_n$. Calculate $\exp(A)$ using

$$\exp(A) = \sum_{j=0}^{\infty} A^j/(j!).$$

(ii) Let

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $B^3 = I_3$. Calculate $\exp(B)$ using the result from (i).
(iii) Calculate $\exp(B)$ applying the *Cayley-Hamilton theorem*.

Problem 4. The Cayley-Hamilton theorem can also be used to calculate $\exp(A)$ and other entire functions for an $n \times n$ matrix. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let f be an *entire function*, i.e., an analytic function on the whole complex plane, for example $\exp(z)$, $\sin(z)$, $\cos(z)$. An infinite series expansion for $f(A)$ is not generally useful for computing $f(A)$. Using the *Cayley-Hamilton theorem* we can write

$$f(A) = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_2A^2 + a_1A + a_0I_n \tag{1}$$

where the complex numbers a_0, a_1, $\dots$, a_{n-1} are determined as follows: Let

$$r(\lambda) := a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

which is the right-hand side of (1) with A^j replaced by λ^j, where $j = 0, 1, \dots, n-1$ of each distinct eigenvalue λ_j of the matrix A, we consider the equation

$$f(\lambda_j) = r(\lambda_j). \tag{2}$$

If λ_j is an eigenvalue of multiplicity k, for $k > 1$, then we consider also the following equations

$$\begin{aligned} f'(\lambda)|_{\lambda=\lambda_j} &= r'(\lambda)|_{\lambda=\lambda_j} \\ f''(\lambda)|_{\lambda=\lambda_j} &= r''(\lambda)|_{\lambda=\lambda_j} \\ \cdots &= \cdots \\ f^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j} &= r^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j}. \end{aligned}$$

(i) Apply this technique to find $\exp(A)$ with

$$A = \begin{pmatrix} c & c \\ c & c \end{pmatrix}, \qquad c \in \mathbb{R}, \quad c \neq 0.$$

(ii) Use the method given above to calculate $\exp(iK)$, where the hermitian 2×2 matrix K is given by

$$K = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}, \qquad a, c \in \mathbb{R}, \quad b \in \mathbb{C}.$$

Problem 5. Let $z_1, z_2, z_3 \in \mathbb{C}$. Assume that at least two of the three complex numbers are non-zero. Consider the matrix

$$A = \begin{pmatrix} z_1 & z_2 \\ z_3 & 0 \end{pmatrix}.$$

Calculate $\exp(A)$ applying the Cayley-Hamilton theorem. The characteristic equation for A is given by

$$\lambda^2 - z_1\lambda - z_2 z_3 = 0$$

with the eigenvalues

$$\lambda_+ = \frac{1}{2}z_1 + \sqrt{z_1^2 + 4z_2 z_3}, \quad \lambda_- = \frac{1}{2}z_1 - \sqrt{z_1^2 + 4z_2 z_3}.$$

Cayley-Hamilton theorem then states that

$$A^2 - z_1 A - z_1 z_3 I_2 = 0_2.$$

Problem 6. Calculate

$$\sec \begin{pmatrix} \frac{\pi}{\sqrt{2}} & 0 & 0 & \frac{\pi}{\sqrt{2}} \\ 0 & \frac{\pi}{\sqrt{2}} & \frac{\pi}{\sqrt{2}} & 0 \\ 0 & \frac{\pi}{\sqrt{2}} & -\frac{\pi}{\sqrt{2}} & 0 \\ \frac{\pi}{\sqrt{2}} & 0 & 0 & -\frac{\pi}{\sqrt{2}} \end{pmatrix}.$$

Chapter 9

Linear Differential Equations

Problem 1. Let A be an $n \times n$ matrix over $\mathbb{R}$. Consider the initial value problem of the system of linear differential equations

$$\frac{d\mathbf{u}(t)}{dt} + A\mathbf{u}(t) = \mathbf{g}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0 \tag{1}$$

where $\mathbf{g}(t) = (g_1(t), g_2(t), \ldots, g_n(t))^T$. The solution of the initial value problem is

$$\mathbf{u}(t) = e^{-tA}\mathbf{u}_0 + \int_0^t e^{-(t-\tau)A}\mathbf{g}(\tau)d\tau. \tag{2}$$

(i) Discretize the system with the implicit Euler method with step size h.
(ii) Compare the two solutions of the two systems for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

initial values $\mathbf{u}_0 = (1\,,\,1\,,\,1)^T$ with $\mathbf{g}(t) = (1,\,0,\,1)^T$ and the step size $h = 0.1$.

Problem 2. Let L and K be two $n \times n$ matrices. Assume that the entries depend on a parameter t and are differentiable with repect to t. Assume that $K^{-1}(t)$ exists for all t. Assume that the time-evolution of L is given by

$$L(t) = K(t)L(0)K^{-1}(t).$$

(i) Show that $L(t)$ satisfies the matrix differential equation

$$\frac{dL}{dt} = [L, B](t)$$

where $[\,,\,]$ denotes the commutator and

$$B = -\frac{dK}{dt}K^{-1}(t).$$

(ii) Show that if $L(t)$ is hermitian and $K(t)$ is unitary, then the matrix $B(t)$ is skew-hermitian.

Problem 3. Consider a system of linear ordinary differential equations with periodic coefficients

$$\frac{d\mathbf{u}}{dt} = A(t)\mathbf{u}, \qquad \mathbf{u} = \begin{pmatrix} u_2 \\ u_2 \end{pmatrix}$$

where $A(t)$ is a 2×2 matrix of periodic functions with period T. By the classical *Floquet theory*, any fundamental matrix $\Phi(t)$, which is defined as a nonsingular matrix satisfying the matrix differential equation

$$\frac{d\Phi}{dt} = A(t)\Phi(t)$$

can be expressed as

$$\Phi(t) = P(t)\exp(TR).$$

Here $P(t)$ is nonsingular matrix of periodic functions with the same period T, and R, a constant matrix, whose eigenvalues λ_1 and λ_2 are called the characteristic exponents of the periodic sytem (1). For a choice of fundamental matrix $\Phi(t)$, we have

$$\exp(TR) = \Phi(t_0)\Phi(t_0 + T)$$

which does not depend on the initial time t_0. The matrix $\exp(TR)$ is called the *monodromy matrix* of the periodic system (1). Calculate

$$\operatorname{tr}\exp(TR).$$

Problem 4. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(i) Calculate $\exp(tA)$, where $t \in \mathbb{R}$.
(ii) Find the solution of the initial value problem of the differential equation

$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with the initial conditions $u_1(t=0) = u_{10}$, $u_2(t=0) = u_{20}$. Use the result from (i).

Problem 5. Solve the initial value problem for the matrix differential equation

$$[B, A(\epsilon)] = \frac{dA}{d\epsilon}$$

where $A(\epsilon)$ and B are 2×2 matrices with

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 6. Consider the initial problem of the matrix differential equation

$$\frac{dX}{dt} = A(t)X, \qquad X(0) = I_n$$

where $A(t)$ is an $n \times n$ matrix which depends smoothly on t and I_n is the $n \times n$ identity matrix. It is known that the solution of this matrix differential equation can locally be written as

$$X(t) = \exp(\Omega(t))$$

where $\Omega(t)$ is obtained as an infinite series

$$\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t).$$

This is the so-called Magnus expansion.
Implement this recursion in SymbolicC++ and apply it to

$$A(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Problem 7. Let $a, b \in \mathbb{R}$. Consider the linear matrix differential equation

$$\frac{d^2X}{dt^2} + a\frac{dX}{dt} + bX = 0.$$

Find the solution of the initial value problem.

Problem 8. Let A be an $n \times n$ matrix over $\mathbb{R}$. The autonomous system of first order differential equations $d\mathbf{u}/dt = A\mathbf{u}$ admits the solution of the initial value problem $\mathbf{u}(t) = \exp(A)\mathbf{u}(0)$. Differentiation of the differential equations yields the second order system

$$\frac{d^2\mathbf{u}}{dt^2} = A\frac{d\mathbf{u}}{dt} = A^2\mathbf{u}.$$

Thus we can write

$$\frac{d\mathbf{u}}{dt} = \mathbf{v} = A\mathbf{u}, \quad \frac{d\mathbf{v}}{dt} = A^2\mathbf{u} = A\mathbf{v}$$

or in matrix form

$$\begin{pmatrix} d\mathbf{u}/dt \\ d\mathbf{v}/dt \end{pmatrix} = \begin{pmatrix} 0_n & I_n \\ A^2 & 0_n \end{pmatrix} \begin{pmatrix} \mathbf{u}(0) \\ \mathbf{v}(0) \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. Find the solution of the initial value problem. Assume that A is invertible.

Problem 9. (i) Let A be an $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{u} = (u_1, \ldots, u_n)^T$. Solve the initial value problem of the system of linear differential equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}$$

using the Laplace transform, where $\mathbf{u}(t = 0) = \mathbf{u}(0)$. Note that (applying integration by parts)

$$\begin{aligned} \mathcal{L}(d\mathbf{u}/dt)(s) &= \int_0^\infty e^{-st}\frac{d\mathbf{u}}{dt}dt \\ &= e^{-st}\mathbf{u}(t)\big|_{t=0}^{t=\infty} + s\int_0^\infty e^{-st}\mathbf{u}(t)dt \\ &= s\mathbf{U}(s) - \mathbf{u}(0). \end{aligned}$$

(ii) Apply it to the 2×2 Hadamard matrix

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Problem 10. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(i) Calculate $\exp(tA)$, where $t \in \mathbb{R}$.
(ii) Find the solution of the initial value problem of the differential equation

$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with the initial conditions $u_1(t = 0) = u_{10}$, $u_2(t = 0) = u_{20}$. Use the result from (i).

Chapter 10

Norms and Scalar Products

Problem 1. Let U_1, U_2 be unitary $n \times n$ matrices. Let $\mathbf{v}$ be a normalized vector in $\mathbb{C}^n$. Consider the norm of a $k \times k$ matrix M

$$\|M\| = \max_{\|\mathbf{x}\|=1} \|M\mathbf{x}\|$$

where $\|\mathbf{x}\|$ denotes the Euclidean norm. Show that if $\|U_1 - U_2\| \leq \epsilon$ then

$$\|U_1\mathbf{v} - U_2\mathbf{v}\| \leq \epsilon.$$

Problem 2. Given the 2×2 matrix

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Calculate

$$\|A(\alpha)\| = \sup_{\|\mathbf{x}\|=1} \|A(\alpha)\mathbf{x}\|.$$

Problem 3. Let A be an $n \times n$ matrix. Let $\rho(A)$ be the spectral radius of A. Then we have

$$\rho(A) \leq \min\{\max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|, \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|\}.$$

Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}.$$

Calculate $\rho(A)$ and the right-hand side of the inequality.

Problem 4. Consider the Hilbert space $\mathbb{C}^n$. We define a norm of an $n \times n$ matrix A over $\mathbb{C}$

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

where the right-hand side denotes the Euclidean norm. Let U be an $n \times n$ unitary matrix. Show that $\|U\| = 1$.

Problem 5. Let A be an $n \times n$ positive semidefinite (and thus hermitian) matrix. Is

$$\|A^{1/2}\| = \|A\|^{1/2}\,?$$

Problem 6. Let A be an $n \times n$ positive semidefinite matrix. Show that

$$|\mathbf{x}^* A\mathbf{y}| \le \sqrt{\mathbf{x}^* A\mathbf{x}}\sqrt{\mathbf{y}^* A\mathbf{y}}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}$.

Problem 7. Let $t \in \mathbb{R}$. Consider the symmetric matrix over $\mathbb{R}$

$$A(t) = \begin{pmatrix} t & 1 & 0 \\ 1 & t & 1 \\ 0 & 1 & t \end{pmatrix}.$$

Find the condition on t such that $\rho(A(t)) < 1$, where $\rho(A(t))$ denotes the spectral radius of $A(t)$.

Problem 8. (i) Let A be an $n \times n$ positive semidefinite matrix. Show that $(I_n + A)^{-1}$ exists.
(ii) Let B be an arbitrary $n \times n$ matrix. Show that the inverse of $I_n + B^* B$ exists.

Problem 9. Let A be an $n \times n$ matrix. One approach to calculate $\exp(A)$ is to compute an eigenvalue decomposition $A = XBX^{-1}$ and then apply the formula $e^A = Xe^B X^{-1}$. We have using the *Schur decomposition*

$$U^* AU = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) + N$$

where U is unitary, the matrix $N = (n_{jk})$ is a strictly upper triangular ($n_{jk} = 0, j \ge k$) and $\lambda(A) = \{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of A. Using the *Padé approximation* to calculate e^A we have

$$R_{pq} = (D_{pq}(A))^{-1} N_{pq}(A)$$

where

$$N_{pq}(A) := \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!} A^j$$

$$D_{pq}(A) := \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} (-A)^j.$$

Let

$$A = \begin{pmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Calculate $\|R_{11} - e^A\|$, where $\| \cdot \|$ denotes the 2-norm.

Problem 10. Let A be an $n \times n$ matrix with $\|A\| < 1$. Then $\ln(I_n + A)$ exists. Show that

$$\| \ln(I_n + A)\| \leq \frac{\|A\|}{1 - \|A\|}.$$

Problem 11. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$\|[A, B]\| \leq 2\|A\| \, \|B\|$$

where $[\,,\,]$ denotes the commutator.

Problem 12. Let A be an $n \times n$ matrix. Let

$$B = \begin{pmatrix} A & I_n & 0_n \\ I_n & A & I_n \\ 0_n & I_n & A \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix. Calculate B^2 and B^3.

Problem 13. Denote by $\| \cdot \|_{HS}$ the Hilbert-Schmidt norm and by $\| \cdot \|_O$ the operator norm, i.e.

$$\|A\|_{HS} := \sqrt{\operatorname{tr}(AA^*)}, \qquad \|A\|_O := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \sup_{\substack{\lambda \in \mathbb{C},\, \|\mathbf{x}\|=1 \\ (A^*A)\mathbf{x} = \lambda \mathbf{x}}} \sqrt{\lambda}$$

where A is an $m \times n$ matrix over $\mathbb{C}$, $\mathbf{x} \in \mathbb{C}^n$ and $\|\mathbf{x}\|$ is the Euclidean norm.
(i) Calculate

$$\left\| \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \right\|_{HS} \quad \text{and} \quad \left\| \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \right\|_{O}.$$

(ii) Let A be an $m \times n$ matrix over $\mathbb{C}$. Find the conditions on A such that

$$\|A\|_{HS} = \|A\|_O.$$

Problem 14. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Then

$$\|[A,B]\| \leq \|AB\| + \|BA\| \leq 2\|A\|\,\|B\|$$

where $\|\ \|$ denotes the norm. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Calculate $\|[A,B]\|$, $\|AB\|$, $\|BA\|$, $\|A\|$, $\|B\|$ and thus verify the inequality for these matrices. The norm is given by $\|C\| = \sqrt{\mathrm{tr}(CC^*)}$.

Problem 15. Let A be an $n \times n$ matrix. The *logarithmic norm* is defined by

$$\mu[A] := \lim_{h \to 0^+} \frac{\|I_n + hA\| - 1}{h}.$$

Let

$$\|A\| := \sup_{\mathbf{x}=1} |A\mathbf{x}\|.$$

Let A be the $n \times n$ identity matrix I_n. Find $\mu[I_n]$.

Problem 16. Find a 2×2 unitary matrix U such that

$$U^* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Problem 17. Consider the Hilbert space $\mathbb{R}^n$. The scalar product $\langle \mathbf{x}, \mathbf{y} \rangle$ $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \mathbf{y} = \sum_{j=1}^{n} x_j y_j.$$

Thus the norm is given by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. Show that

$$|\mathbf{x}^T \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

Problem 18. Let $\{\, \mathbf{v}_1, \ldots, \mathbf{v}_m \,\}$ be a linearly independent set of vectors in the normed space $\mathbb{R}^n$ with $m \leq n$.

(i) Show that there is a number $c > 0$ such that for every choice of real numbers $c_1, \ldots, c_m$ we have

$$\|c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m\| \geq c\left(|c_1| + \cdots + |c_m|\right). \tag{1}$$

(ii) Consider $\mathbb{R}^2$ and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find a c for this case.

Problem 19. Let A be an $n \times n$ hermitian matrix. Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Consider the equation

$$A\mathbf{u} - \lambda\mathbf{u} = \mathbf{v}.$$

(i) Show that for λ nonreal (i.e. it has an imaginary part) the vector $\mathbf{v}$ cannot vanish unless $\mathbf{u}$ vanishes.
(ii) Show that for λ nonreal we have

$$\|(A - \lambda I_n)^{-1}\mathbf{v}\| \leq \frac{1}{|\Im\lambda|}\|\mathbf{v}\|.$$

Problem 20. (i) Let A and C be invertible $n \times n$ matrices over $\mathbb{R}$. Let B be an $n \times n$ matrix over $\mathbb{R}$. Assume that

$$\|A\| \leq \|B\| \leq \|C\|.$$

Is B invertible?
(ii) Let A, B, C be invertible $n \times n$ matrices over $\mathbb{R}$ with

$$\|A\| \leq \|B\| \leq \|C\|.$$

Is

$$\|A^{-1}\| \geq \|B^{-1}\| \geq \|C^{-1}\|\,?$$

Problem 21. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $\|A\| < 1$, where

$$\|A\| := \sup_{\|\mathbf{x}\|=\mathbf{1}} \|A\mathbf{x}\|.$$

Show that the matrix $B = I_n + A$ is invertible, i.e. $B \in GL(n.\mathbb{R})$. To show that the expansion

$$I_n - A + A^2 - A^3 + \cdots$$

converges apply

$$\begin{aligned}\|A^m - A^{m+1} + A^{m+2} - \cdots \pm A^{m+k-1}\| &\leq \|A^m\| \cdot \|1 + \|A\| + \cdots + \|A\|^{k-1}\| \\ &= \|A\|^m \frac{1 - \|A\|^k}{1 - \|A\|}.\end{aligned}$$

Then calculate $(I_n + A)(I_n - A + A^2 - A^3 + \cdots)$.

Problem 22. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let $\|.\|$ be a subordinate matrix norm for which $\|I_n\| = 1$. Assume that $\|A\| < 1$.
(i) Show that $(I_n - A)$ is nonsingular.
(ii) Show that $\|(I_n - A)^{-1}\| \leq (1 - \|A\|)^{-1}$.

Problem 23. Let A be an $n \times n$ matrix. Assume that $\|A\| < 1$. Show that

$$\|(I_n - A)^{-1} - I_n\| \leq \frac{\|A\|}{1 - \|A\|}.$$

Problem 24. Let A be an $n \times n$ nonsingular matrix and B an $n \times n$ matrix. Assume that $\|A^{-1}B\| < 1$.
(i) Show that $A - B$ is nonsingular.
(ii) Show that

$$\frac{\|A^{-1} - (A - B)^{-1}\|}{\|A^{-1}\|} \leq \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|}.$$

Problem 25. Let M be an $m \times n$ matrix over $\mathbb{C}$. The Frobenius norm of M is given by

$$\|M\|_F := \sqrt{\mathrm{tr}(M^*M)} = \sqrt{\mathrm{tr}(MM^*)}.$$

Let U_m be $m \times m$ unitary matrix and U_n be an $n \times n$ unitary matrix. Show that

$$\|U_m M\|_F = \|MU_n\|_F = \|M\|.$$

Show that $\|M\|_F$ is the square root of the sum of the squares of the singular values of M.

Problem 26. Let M be an $m \times n$ matrix over $\mathbb{C}$. Find the rank-1 $m \times n$ matrix A over $\mathbb{C}$ which minimizes

$$\|M - A\|_F.$$

Hint: Find the singular value decomposition of $M = U\Sigma V^*$ and find A' with rank 1 which minimizes

$$\|\Sigma - A'\|_F.$$

Apply the method to

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 27. Let A be an $n \times n$ nonsingular matrix and B an $n \times n$ matrix. Assume that $\|A^{-1}B\| < 1$.
(i) Show that $A - B$ is nonsingular.
(ii) Show that

$$\frac{\|A^{-1} - (A-B)^{-1}\|}{\|A^{-1}\|} \leq \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|}.$$

Problem 28. Let A be an $n \times n$ matrix over $\mathbb{C}$. The spectral radius of the matrix A is the non-negative number $\rho(A)$ defined by

$$\rho(A) := \max\{\, |\lambda_j(A)| \,:\, 1 \leq j \leq n\,\}$$

where $\lambda_j(A)$ $(j = 1, 2, \ldots, n)$ are the eigenvalues of A. We define the norm of A as

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

where $\|A\mathbf{x}\|$ denotes the Euclidean norm. Is $\rho(A) \leq \|A\|$? Prove or disprove.

Problem 29. Let A be an $n \times n$ matrix over $\mathbb{R}$. The *spectral norm* is

$$\|A\|_2 := \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}.$$

It can be shown that $\|A\|_2$ can also be calculated as

$$\|A\|_2 = \sqrt{\text{largest eigenvalue of } A^T A}.$$

Note that the eigenvalues of $A^T A$ are real and nonnegative. Let

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}.$$

Calculate $\|A\|_2$ using this method.

Chapter 11

Graphs and Matrices

Problem 1. A graph $G(V, E)$ is a set of nodes V (points, vertices) connected by a set of links E (edges, lines). We assume that there are n nodes. The adjancy $(n \times n)$ matrix $A = A(G)$ takes the form with 1 in row i, column j if i is connected to j, and 0 otherwise. Thus A is a symmetric matrix. Associated with A is the degree distribution, a diagonal matrix with row-sums of A along the diagonal, and 0's elsewhere. We assume that $d_{ii} > 0$ for all $i = 1, 2, \ldots, n$. We define the Laplacian as $L := D - A$. Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Give an interpretation of A, A^2, A^3.
(ii) Find D and L.
(iii) Show that L admits the eigenvalue $\lambda_0 = 0$ (lowest eigenvalue) with eigenvector $\mathbf{x} = (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)^T$.

Problem 2. A graph $G(V, E)$ is a set of nodes V (points, vertices) connected by a set of links E (edges, lines). We assume that there are n nodes. The adjacency $(n \times n)$ matrix $A = A(G)$ takes the form with 1 in row i, column j if i is connected to j, and 0 otherwise. Thus A is a symmetric matrix. Associated with A is the degree distribution D, a diagonal matrix with row-sums of A along the diagonal, and 0's elsewhere. D describes how many connections each node has. We define the *Laplacian* as $L := D - A$. Let $A = (a_{ij})$, i.e. a_{ij} are the entries of adjacency matrix. Find the mimimum of the weighted sum

$$S = \frac{1}{2}\sum_{i,j=1}^{n}(x_i - x_j)^2 a_{ij}$$

with the constraint $\mathbf{x}^T\mathbf{x} = 1$, where $\mathbf{x}^T = (x_1, x_2, \ldots, x_n)$. Use the Lagrange multiplier method. The sum is over all pairs of squared distances between nodes which are connected, and so the solution should result in nodes with large numbers of inter-connections being clustered together.

Problem 3. Find the eigenvalues of the three adjacent matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

provided by three simple graphs. Find the "energy" $E(G)$ of each graph defined by

$$E(G) = \sum_{j=1}^{3} |\lambda_j|.$$

Problem 4. Let G be a graph with vertices $V_G = \{1, 2, \ldots, n_G\}$ and edges $E_G \subseteq V_G \times V_G$. The $n_G \times n_G$ adjacency matrix A_G is given by $(A_G)_{ij} = \chi_{E_G}(i,j) \in \{0,1\}$ where $i, j \in V$. In the following products of graphs G_1 and G_2 with $n_{G_1}n_{G_2} \times n_{G_2}n_{G_2}$ adjacency matrices. The rows and columns are indexed by $V_{G_1} \times V_{G_2}$, where we use the ordering $(i,j) \le (k,l)$ if $i < k$ or, $i = k$ and $j \le l$.
The cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph with vertices

$$V_{G_1 \times G_2} = V_{G_1} \times V_{G_2}$$

and edges

$$\begin{aligned} E_{G_1 \times G_2} = & \{\, ((a,b),(a,b')) \,:\, a \in V_{G_1} \text{ and } (b,b') \in E_{G_2} \,\} \\ & \cup \{\, ((a,b),(a',b)) \,:\, (a,a') \in E_{G_1} \text{ and } b \in V_{G_2} \,\}\,. \end{aligned}$$

It follows that

$$(A_{G_1 \times G_2})_{(i,j),(k,l)} = \delta_{ij}(A_{G_2})_{(k,l)} \boxplus \delta_{kl}(A_{G_1})_{(i,j)}$$

where $\boxplus$ is the usual addition with the convention that $1 \boxplus 1 = 1$. Thus $A_{G_1 \times G_2} = (A_{G_1} \otimes I_{n_{G_2}}) \boxplus (I_{n_{G_2}} \otimes A_{G_2})$.

The lexicographic product $G_1 \bullet G_2$ of two graphs G_1 and G_2 is the graph with vertices

$$V_{G_1 \bullet G_2} = V_{G_1} \times V_{G_2}$$

and edges

$$\begin{aligned} E_{G_1 \bullet G_2} = \{\, ((a,b),(a,b')) \,:\, a \in V_{G_1} \text{ and } (b,b') \in E_{G_2} \,\} \\ \cup \{\, ((a,b),(a',b')) \,:\, (a,a') \in E_{G_1} \text{ and } b,b' \in V_{G_2} \,\}\,. \end{aligned}$$

Thus $E_{G_1 \times G_2} \subseteq E_{G_1 \bullet G_2}$. It follows that

$$(A_{G_1 \times G_2})_{(i,j),(k,l)} = \delta_{ij}(A_{G_2})_{(k,l)} \boxplus (A_{G_1})_{(i,j)}.$$

Thus $A_{G_1 \times G_2} = (A_{G_1} \otimes \mathbf{1}_{n_{G_2}}) \boxplus (I_{n_{G_2}} \otimes A_{G_2})$. Here $\mathbf{1}_{n_{G_1}}$ is the $n_{G_2} \times n_{G_2}$ with every entry equal to 1.
The tensor product $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertices

$$V_{G_1 \otimes G_2} = V_{G_1} \times V_{G_2}$$

and edges

$$E_{G_1 \otimes G_2} = \{\, ((a,b),(a',b')) \,:\, (a,a') \in E_{G_1} \text{ and } (b,b') \in E_{G_2} \,\}\,.$$

It follows that

$$(A_{G_1 \otimes G_2})_{(i,j),(k,l)} = (A_{G_1})_{(i,j)}(A_{G_2})_{(k,l)} = (A_{G_1} \otimes A_{G_2})_{(i-1)n_{G_2}+k,(j-1)n_{G_2}+l}.$$

Thus $A_{G_1 \otimes G_2} = A_{G_1} \otimes A_{G_2}$ where $\otimes$ is the Kronecker product of matrices.
The normal product $G_1 \star G_2$ of two graphs G_1 and G_2 is the graph with vertices

$$V_{G_1 \star G_2} = V_{G_1} \times V_{G_2}$$

and edges

$$E_{G_1 \star G_2} = E_{G_1 \times G_2} \cup E_{G_1 \otimes G_2}.$$

Thus

$$A_{G_1 \star G_2} = A_{G_1 \times G_2} \boxplus A_{G_1 \otimes G_2} = (A_{G_1} \otimes I_{n_{G_2}}) \boxplus (I_{n_{G_2}} \otimes A_{G_2}) \boxplus A_{G_1} \otimes A_{G_2}.$$

(i) Show that the Euler path is not preserved (in general) under these operations.
(ii) Show that the Hamilton path is not preserved (in general) under these operations.

Chapter 12

Hadamard Product

Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times m$ matrices with entries in some fields. Then the *Hadamard product* is the entrywise product of A and B, that is, the $m \times n$ matrix $A \bullet B$ whose (i, j) entry is $a_{ij}b_{ij}$. We have the properties. Suppose A, B, C are matrices of the same size and λ is a scalar. Then

$$\begin{aligned} A \bullet B &= B \bullet A \\ A \bullet (B + C) &= A \bullet B + A \bullet C \\ A \bullet (\lambda B) &= \lambda(A \bullet B). \end{aligned}$$

If A, B be $n \times n$ diagonal matrices, then $A \bullet B = AB$. If A, B are $n \times n$ positive definite matrices and (a_{jj}) are the diagonal entries of A, then

$$\det(A \bullet B) \geq \det B \prod_{j=1}^{n} a_{jj} \tag{1}$$

with equality if and only if A is a diagonal matrix.

Problem 1. Let

$$A = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 13 & 4 \\ 4 & 4 \end{pmatrix}.$$

First show that A and B are positive definite and then calculate the left and right-hand side of (1).

Problem 2. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate the Hadamard product $A \bullet B$. Show that

$$\|A \bullet B\| \leq \|A^* A\| \, \|B^* B\|$$

where the norm is given by the Hilbert-Schmidt norm.

Problem 3. Let A, B, C and D^T be $n \times n$ matrices over $\mathbb{R}$. The Hadamard product is defined by $(A \bullet B)_{ij} := a_{ij} b_{ij}$. Show that

$$\operatorname{tr}((A \bullet B)(C^T \bullet D) = \operatorname{tr}((A \bullet B \bullet C)D).$$

Problem 4. If V and W are matrices of the same order, then their Schur product $V \bullet W$ is defined by (entrywise multiplication)

$$(V \bullet W)_{j,k} := V_{j,k} W_{j,k}.$$

If all entries of V are nonzero, then we say that X is Schur invertible and define its Schur inverse, $V^{(-)}$, by $V^{(-)} \bullet V = J$, where J is the matrix with all 1's.

The vector space $M_n(\mathbb{F})$ of $n \times n$ matrices acts on itself in three distinct ways: if $C \in M_n(\mathbb{F})$ we can define endomorphisms X_C, Δ_C and Y_C by

$$X_C M := CM, \qquad \Delta_C M := C \bullet M, \qquad Y_C := MC^T.$$

Let A, B be $n \times n$ matrices. Assume that X_A is invertible and Δ_B is invertible in the sense of Schur. Note that X_A is invertible if and only if A is, and Δ_B is invertible if and only if the Schur inverse $B^{(-)}$ is defined. We say that (A, B) is a *one-sided Jones pair* if

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B.$$

We call this the *braid relation.* Give an example for a one-sided Jones pair.

Problem 5. Let A, B be $n \times n$ matrices. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors in $\mathbb{C}^n$. We form the n^2 column vectors

$$(A\mathbf{e}_j) \bullet (B\mathbf{e}_k), \qquad j, k = 1, \ldots, n.$$

If A is invertible and B is Schur invertible, then for any j

$$\{\, (A\mathbf{e}_1) \bullet (B\mathbf{e}_j), \;\; (A\mathbf{e}_2) \bullet (B\mathbf{e}_j), \;\ldots\; (A\mathbf{e}_n) \bullet (B\mathbf{e}_j) \,\}$$

is a basis of the vector space $\mathbb{C}^n$. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Find these bases for these matrices.

Problem 6. Let U be an $n \times n$ unitary matrix. Can we conclude that $U \bullet U^*$ is a unitary matrix?

Problem 7. Let $B = (b_{jk})$ be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Thus, there is nonsingular $n \times n$ matrix A such that

$$B = A(\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n))A^{-1}.$$

Show that

$$\begin{pmatrix} b_{11} \\ b_{22} \\ \vdots \\ b_{nn} \end{pmatrix} = (A \bullet (A^{-1})^T) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$$

where $\bullet$ is the Hadamard product (Schur product, entrywise product). Thus the vector of eigenvalues of B is transformed to the vector of its diagonal entries by the ceofficient matrix $A \bullet (A^{-1})^T$.

Problem 8. Let A, B, C, D be $n \times n$ matrices over $\mathbb{R}$. Let

$$\mathbf{s}^T = (1 \quad 1 \quad \ldots \quad 1)$$

be a row vector in $\mathbb{R}^n$. Show that

$$\mathbf{s}^T(A \bullet B)(C^T \bullet D)\mathbf{s} = \mathrm{tr}(C\Gamma D)$$

where $\Gamma = (\gamma_{ij})$ is a diagonal matrix with $\gamma_{jj} = \sum_{i=1}^n a_{ij}b_{ij}$ with $j = 1, 2, \ldots, n$.

Problem 9. Given two matrices A and B of the same size. We use $A \bullet B$ to denote the Schur product. If all entries of A are nonzero, then we say that A is *Schur invertible* and define its Schur inverse, $A^{(-)}$ by

$$A_{ij}^{(-)} := \frac{1}{A_{ij}}.$$

Equivalently, we have $A^{(-)} \bullet A = J$, where J is the matrix with all ones. An $n \times n$ matrix W is a *type-II matrix* if

$$WW^{(-)^T} = nI_n$$

where I_n is the $n \times n$ identity matrix. Find such a matrix for $n = 2$.

Problem 10. Let A be an invertible $n \times n$ matrix. Can we conclude that

$$A \bullet A^{-1}$$

is invertible?

Problem 11. The $(n+1) \times (n+1)$ *Hadamard matrix* $H(n)$ of any dimension is generated recursively as follows

$$H(n) = \begin{pmatrix} H(n-1) & H(n-1) \\ H(n-1) & -H(n-1) \end{pmatrix}$$

where $n = 1, 2, \ldots$ and $H(0) = (1)$. Find $H(1)$, $H(2)$ and $H(3)$.

Problem 12. Show that

$$\mathrm{tr}(A(B \bullet C)) \equiv (\mathrm{vec}(A^T \bullet B))^T \mathrm{vec}(C).$$

Problem 13. Let $\circ$ be the Hadamard product. Let A be a positive semidefinite $n \times n$ matrix. Let B be an $n \times n$ matrix with $\|B\| \leq 1$, where $\| \, . \, \|$ denotes the spectral norm. Show that

$$\max\{ \, \|A \circ B\| \, : \, \|B\| \leq 1 \, \} = \max a_{jj}$$

where $\| \, \|$ denotes the spectral norm.

Problem 14. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$\mathrm{rank}(A \bullet B) \leq (\mathrm{rank}(A))(\mathrm{rank}(B)).$$

Problem 15. Show that the Hadamard product is linear.

Problem 16. Let A, B be $m \times n$ matrices and D and E be $m \times m$ and $n \times n$ diagonal matrices, respectively. Show that

$$D(A \bullet B)E = A \bullet (DBE).$$

Problem 17. Let A, B, C, D be 2×2 matrices. Let $\bullet$ be the *Hadamard product.* Is

$$(A \otimes B) \bullet (C \otimes D) = (A \bullet C) \otimes (B \bullet D)\,?$$

Problem 18. Let A, B be $n \times n$ positive semidefinite matrices. Show that $A \bullet B$ is also positive semidefinite.

Chapter 13

Unitary Matrices

An $n \times n$ matrix U over $\mathbb{C}$ is called a unitary matrix if $U^* = U^{-1}$. Thus $UU^* = I_n$ for a unitary matrix. If U and V are unitary $n \times n$ matrices, then UV is an $n \times n$ unitary matrix. If U is an $n \times n$ unitary matrix and V is an $m \times m$ unitary matrix, then $U \otimes V$ is a unitary matrix, where $\otimes$ denotes the Kronecker product. The columns in a unitary matrix are pairwise orthonormal. The $n \times n$ unitary matrices form a group under matrix multiplication. Let K be a skew-hermitian matrix, then $\exp(K)$ is a unitary matrix. For any unitary matrix U we can find a skew-hermitian matrix such that $U = \exp(K)$. If U is an $n \times n$ unitary matrix and V is an $m \times m$ unitary matrix, then $U \oplus V$ is a unitary matrix, where $\oplus$ denotes the direct sum product. If U is a 2×2 unitary matrix and V is an 2×2 unitary matrix, then $U \star V$ is a unitary matrix, where $\star$ denotes the star operation.

Problem 1. (i) Let A, B be $n \times n$ matrices over $\mathbb{R}$. Show that one can find a $2n \times 2n$ unitary matrix U such that

$$U \begin{pmatrix} A & B \\ -B & A \end{pmatrix} U^* = \begin{pmatrix} A + iB & 0_n \\ 0_n & A - iB \end{pmatrix}.$$

Here 0_n denotes the $n \times n$ zero matrix.
(ii) Use the result from (i) to show that

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det(A + iB)\overline{\det(A + iB)} \geq 0.$$

Problem 2. Let $\mathbf{u}$ be a column vector in $\mathbb{C}^n$ with $\mathbf{u}^*\mathbf{u} = 1$, i.e. the vector is normalized. Consider the matrix

$$U = I_n - 2\mathbf{u}\mathbf{u}^*.$$

(i) Show that U is hermitian.
(ii) Show that U is unitary.

Problem 3. Can one find a 2×2 unitary matrix such that

$$U \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} U^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Problem 4. Let σ_1, σ_2, σ_3 be the Pauli spin matrices and

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

be the Hadamard matrix. Find

$$U_H \sigma_1 U_H^*, \qquad U_H \sigma_2 U_H^*, \qquad U_H \sigma_3 U_H^*.$$

Problem 5. Find all 2×2 hermitian and unitary matrices A, B such that

$$AB = e^{i\pi} BA.$$

Problem 6. Find all $(n+1) \times (n+1)$ matrices A such that

$$A^* U A = U$$

where U is the unitary matrix

$$U = \begin{pmatrix} 0 & 0 & i \\ 0 & I_{n-1} & 0 \\ -i & 0 & 0 \end{pmatrix}$$

and $\det(A) = 1$. Consider first the case $n = 2$.

Problem 7. Consider the 2×2 hermitian matrices A and B with $A \neq B$ with the eigenvalues λ_1, λ_2; μ_1, μ_2; and the corresponding normalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$; $\mathbf{v}_1$, $\mathbf{v}_2$, respectively. Form from the normalized eigenvectors the 2×2 matrix

$$\begin{pmatrix} \mathbf{u}_1^*\mathbf{v}_1 & \mathbf{u}_1^*\mathbf{v}_2 \\ \mathbf{u}_2^*\mathbf{v}_1 & \mathbf{u}_2^*\mathbf{v}_2 \end{pmatrix}.$$

Is this matrix unitary? Find the eigenvalues of this matrix and the corresponding normalized eigenvectors of the 2×2 matrix. How are the eigenvalues and eigenvectors are linked to the eigenvalues and eigenvectors of A and B?

Problem 8. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $a_j \in \mathbb{R}$ with $j = 0, 1, 2, 3$ and

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1.$$

Show that

$$U = e^{i\phi}(a_0 I_2 + a_1 i\sigma_1 + a_2 i\sigma_2 + a_3 i\sigma_3)$$

is a unitary matrix, where $\phi \in \mathbb{R}$.

Problem 9. Let I_n be the $n \times n$ unit matrix. Is the $2n \times 2n$ matrix

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ I_n & -iI_n \end{pmatrix}$$

unitary?

Problem 10. Consider the two 2×2 unitary matrices

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Can one find a unitary 2×2 matrix V such that

$$U_1 = VU_2V^* \,?$$

Problem 11. Let U be an $n \times n$ unitary matrix.
(i) Is $U + U^*$ invertible?

(ii) Is $U + U^*$ hermitian?
(iii) Calculate $\exp(\epsilon(U + U^*))$, where $\epsilon \in \mathbb{R}$

Problem 12. Let U be an $n \times n$ unitary matrix. Then $U + U^*$ is a hermitian matrix. Can any hermitian matrix represented in this form?

Problem 13. (i) Find the condition on the $n \times n$ matrix A over $\mathbb{C}$ such that $I_n + A$ is a unitary matrix.
(ii) Let B be an 2×2 matrix over $\mathbb{C}$. Find all solutions of the equation

$$B + B^* + BB^* = 0_2.$$

Problem 14. Find all 2×2 invertible matrices A such that

$$A + A^{-1} = I_2.$$

Problem 15. Let $z_1, z_2, w_1, w_2 \in \mathbb{C}$. Consider the 2×2 matrices

$$U = \begin{pmatrix} 0 & z_1 \\ z_2 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & w_1 \\ w_2 & 0 \end{pmatrix}$$

where $z_1\bar{z}_1 = 1$, $z_2\bar{z}_2 = 1$, $w_1\bar{w}_1 = 1$, $w_2\bar{w}_2 = 1$. This means the matrices U, V are unitary. Find the condition on z_1, z_2, w_1, w_2 such that the commutator $[U, V]$ is again a unitary matrix.

Problem 16. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Find the conditions on $\alpha_1, \alpha_2, \alpha_3$ such that

$$U = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$$

is a unitary matrix.

Problem 17. Consider $n \times n$ unitary matrices. A scalar product of two $n \times n$ matrices U, V can be defined as

$$\langle U, V \rangle := \frac{1}{n}\mathrm{tr}(UV^*).$$

Find two 2×2 unitary matrices U, V such that

$$\langle U, V \rangle = \frac{1}{2}.$$

Problem 18. Let

$$\{\, |a_0\rangle,\, |a_1\rangle,\, \ldots\, ,|a_{n-1}\rangle \,\}$$

be an orthonormal basis in the Hilbert space $\mathbb{C}^n$. The discrete Fourier transform

$$|b_j\rangle = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \omega^{jk} |a_k\rangle, \qquad j = 0, 1, \ldots, n$$

where $\omega := \exp(2\pi i/n)$ is the primitive n-th root of unity.
(i) Apply the discrete Fourier transform to the standard basis in $\mathbb{C}^4$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(ii) Apply the discrete Fourier transform to the Bell basis in $\mathbb{C}^4$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Problem 19. (i) Consider the Pauli spin matrices $\sigma_0 = I_2$, σ_1, σ_2, σ_3. The matrices are unitary and hermitian. Is the 4×4 matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_1 \\ \sigma_2 & \sigma_3 \end{pmatrix}$$

unitary?
(ii) Is the 4×4 matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{pmatrix}$$

unitary?

Problem 20. Let U be an 2×2 unitary matrix. Is the 4×4 matrix

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U + \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2$$

unitary?

Problem 21. The Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are hermitian and unitary. Together with the 2×2 identity matrix $\sigma_0 = I_2$ they form an orthogonal basis in Hilbert space of the 2×2 matrices over $\mathbb{C}$ with the scalar product $\mathrm{tr}(AB^*)$. Let X be an $n \times n$ hermitian matrix. Then $(X + iI_n)^{-1}$ exists and

$$U = (X - iI_n)(X + iI_n)^{-1}$$

is unitary. This is the so-called Cayley transform of X. Find the Cayley transform of the Pauli spin matrices and the 2×2 identiy matrix. Show that these matrices also form an orthogonal basis in the Hilbert space.

Problem 22. Consider the unitary matrix with determinant +1

$$U(r, \phi) = \begin{pmatrix} \cosh(r) & e^{i\phi}\sinh(r) \\ e^{-i\phi}\sinh(r) & \cosh(r) \end{pmatrix}$$

where $r, \phi \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors. Construct another unitary matrix using these normalized eigenvectors as columns of this matrix.

Problem 23. Show that the two matrices

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

are conjugate in the Lie group $SU(2)$.

Problem 24. A wave-scattering problem can be described by its scattering matrix U. In a stationary problem, U relates the outgoing-wave to the ingoing-wave amplitudes. The condition of flux conservation implies unitary of U, i.e.

$$UU^\dagger = I$$

where I is the identity operator. If, additionally, the scattering problem is invariant under the operatiom of time reversal, we also have $U = U^T$, i.e. U is symmetric. Find all 2×2 unitary matrices that also satisfy $U = U^T$. Do these matrices form a subgroup of the Lie group $U(2)$?

Problem 25. Let U be an $n \times n$ unitary matrix. Let V be an $n \times n$ unitary matrix such that $V^{-1}UV = D$ is a diagonal matrix D. Is $V^{-1}U^*V$ a diagonal matrix?

Problem 26. Let U be an $n \times n$ unitary matrix. Is $U + U^*$ invertible?

Problem 27. Let U, V be two $n \times n$ unitary matrices. Then we can define a scalar product via

$$\langle U, V \rangle := \frac{1}{n}\mathrm{tr}(UV^*).$$

Find 2×2 unitary matrices U, V such that $\langle U, V \rangle = 1/2$.

Problem 28. Let $\omega := \exp(2\pi i/4)$. Consider the 3×3 unitary matrices

$$\Sigma = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} -i/2 & (1+i)/2 & 1/2 \\ (1+i)/2 & 0 & (1-i)/2 \\ 1/2 & (1-i)/2 & i/2 \end{pmatrix}.$$

Do the matrices of the set

$$\Lambda := \{\, \Sigma^j C^k \Omega^\ell \,:\, 0 \le j \le 3,\, 0 \le k \le 1,\, 0 \le \ell \le 2 \,\}$$

form a group under matrix multiplication?

Problem 29. Let $\omega := \exp(2\pi i/4)$. Consider the 4×4 unitary matrices

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $c > 0$. The four-state *Potts quantum chain* is defined by the Hamilton operator

$$\hat{H} = -\frac{1}{\pi\sqrt{c}} \sum_{j=1}^{N} \big((\sigma_j + \sigma_j^2 + \sigma_j^3) + c(\Gamma_j \Gamma_{j+1}^3 + \Gamma_j^2 \Gamma_{j+1}^2 + \Gamma_j^3 \Gamma_{j+1})\big)$$

where N is the number of sites and one imposes cyclic boundary conditions $N + 1 \equiv 1$. Let $N = 2$. Find the eigenvalues and eigenvectors of $\hat{H}$.

Problem 30. Let U be an $n \times n$ unitary matrix and A an arbitrary $n \times n$ matrix. Then we know that

$$Ue^A U^{-1} = e^{UAU^{-1}}.$$

Calculate $Ue^A U$ with $U \neq U^{-1}$.

Problem 31. Consider the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

which is a unitary matrix. Each column vector of the matrix is a fully entangled state. Are the normalized eigenvectors of B are also fully entangled states?

Problem 32. Consider the unitary matrix

$$U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_{11}} & e^{i\phi_{12}} \\ e^{i\phi_{21}} & e^{i\phi_{22}} \end{pmatrix}.$$

Calculate the proudct $U(\phi_{11},\phi_{12},\phi_{21},\phi_{22})U(\psi_{11},\psi_{12},\psi_{21},\psi_{22})$ and find the conditions on ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} and ψ_{11}, ψ_{12}, ψ_{21}, ψ_{22} such that we have again a matrix of this form.

Problem 33. Consider the Hamilton operator $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_0 = \hbar\omega\sigma_3, \qquad \hat{H}_1 = \hbar\omega\sigma_1.$$

Let U and U_0 be the unitary matrices

$$U = \exp(-i\hat{H}t/\hbar), \qquad U_0 = \exp(-i\hat{H}_0 t/\hbar).$$

Let n be a positive integer. The *Moller wave operators*

$$\Omega_\pm := \lim_{n\to\pm\infty} U^n U_0^{-n}.$$

Owing to their intertwining property the Moller wave operators transform the eigenvectors of the free dynamics $U_0 = \exp(-i\hat{H}_0 t/\hbar)$ into eigenvectors of the interacting dynamics $U = \exp(-i\hat{H}t/\hbar)$. Find $\Omega_\pm$.

Problem 34. Consider the unitary matrices

$$U_1(\phi_1,\phi_2,\phi_3) = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}, \qquad U_2(\phi_4,\phi_5,\phi_6) = \begin{pmatrix} 0 & 0 & e^{i\phi_4} \\ 0 & e^{i\phi_5} & 0 \\ e^{i\phi_6} & 0 & 0 \end{pmatrix}.$$

What is the condition on $\phi_1,\ldots,\phi_6$ such that $[U_1,U_2] = 0_3$?

Problem 35. Consider the matrices

$$U = \begin{pmatrix} e^{i\alpha}\cos\theta & e^{i\beta}\sin\theta & 0 \\ -e^{-i\beta}\sin\theta & e^{-i\alpha} & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \\ 1 & 1 & 0 \end{pmatrix}.$$

Find the commutator $[U,N]$.

Problem 36. Let $n \geq 2$ and even. Let U be a unitary antisymmetric $n\times n$ matrix. Show that there exists a unitary matrix V such that

$$V^T U V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\oplus$ denotes the direct sum.

Problem 37. Let U be a unitary and symmetric matrix. Show that there exists a unitary and symmetric matrix V such that $U = V^2$.

Problem 38. Is the matrix

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \exp(i2\pi/3) & \exp(i4\pi/3) \\ 1 & \exp(i4\pi/3) & \exp(i2\pi/3) \end{pmatrix}$$

unitary? Find the eigenvalues and eigenvectors of U.

Problem 39. (i) Let $\tau = (\sqrt{5}-1)/2$ be the golden mean number. Consider the 2×2 matrices

$$B_1 = \begin{pmatrix} e^{-i7\pi/10} & 0 \\ 0 & -e^{-i3\pi/10} \end{pmatrix}, \quad B_2 = \begin{pmatrix} -\tau e^{-i\pi/10} & -i\sqrt{\tau} \\ -i\sqrt{\tau} & -\tau e^{i\pi/10} \end{pmatrix}.$$

The matrices are invertible. Are the matrices unitary? Is $B_1 B_2 B_1 = B_2 B_1 B_2$?
(ii) Show that using computer algebra

$$B_2^{-2} B_1^4 B_2^{-1} B_1 B_2^{-1} B_1 B_2 B_1^{-2} B_2 B_1^{-1} B_2^{-5} B_1 B_2^{-1} \approx \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Problem 40. Let U be an $n \times n$ unitary matrix. Show that $|\det(U)| = 1$.

Problem 41. Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that if λ is an eigenvalue of A, then $\bar{\lambda}$ is also an eigenvalue of A. Give an example for a 2×2 matrix, where the eigenvalues are complex.

Problem 42. Let V be an $n \times n$ normal matrix over $\mathbb{C}$. Assume that all its eigenvalues have absolute value of 1, i.e. they are of the form $e^{i\phi}$. Show that V is unitary.

Problem 43. (i) What are the conditions on $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \mathbb{R}$ such that

$$U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_{11}} & e^{i\phi_{12}} \\ e^{i\phi_{21}} & e^{i\phi_{22}} \end{pmatrix}$$

is a unitary matrix?
(ii) What are the condition on $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \mathbb{R}$ such that $U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22})$ is an element of $SU(2)$?

Problem 44. Let $\alpha, \beta \in \mathbb{R}$. Are the 4×4 matrices

$$U = \begin{pmatrix} e^{i\alpha}\cosh\beta & 0 & 0 & \sinh\beta \\ 0 & e^{-i\alpha}\cosh\beta & \sinh\beta & 0 \\ 0 & \sinh\beta & e^{i\alpha}\cosh\beta & 0 \\ \sinh\beta & 0 & 0 & e^{-i\alpha}\cosh\beta \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & e^{i\alpha}\cosh\beta & -e^{i\alpha}\sinh\beta & 0 \\ -e^{-i\alpha}\cosh\beta & 0 & 0 & e^{-i\alpha}\sinh\beta \\ e^{i\alpha}\sinh\beta & 0 & 0 & -e^{i\alpha}\cosh\beta \\ 0 & -e^{-i\alpha}\sinh\beta & e^{-i\alpha}\cosh\beta & 0 \end{pmatrix},$$

unitary?

Problem 45. Is the matrix

$$U = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

unitary?

Problem 46. (i) The electronic scattering matrix has the form

$$S(\phi_1,\phi_2,\phi_3,\gamma) = e^{i\phi_1\sigma_0}e^{i\phi_2\sigma_3}e^{i\gamma\sigma_2}e^{i\phi_3\sigma_3}$$

where $\phi_1,\phi_2,\phi_3 \in [0,2\pi)$, $\gamma \in [0,\pi/2)$. Find $S(\phi_1,\phi_2,\phi_3,\gamma)$.
(ii) Find

$$T(\phi_1,\phi_2,\phi_3,\gamma) = e^{i\phi_1\sigma_0} \otimes e^{i\phi_2\sigma_3} \otimes e^{i\gamma\sigma_2} \otimes e^{i\phi_3\sigma_3}.$$

Problem 47. Let I_n be the $n \times n$ idenity matrix. Is the $2n \times 2n$ matrix

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} I_n & I_n \\ iI_n & -iI_n \end{pmatrix}$$

unitary? Find the eigenvalues of V.

Problem 48. Consider the unitary matrix

$$U = \frac{1}{\sqrt{2}}(I_2 \otimes I_2 + i\sigma_1 \otimes \sigma_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate

$$U(\sigma_1 \otimes I_2)U^{-1}, \quad U(\sigma_2 \otimes I_2)U^{-1}, \quad U(\sigma_3 \otimes I_2)U^{-1},$$
$$U(\sigma_1 \otimes \sigma_1)U^{-1}, \quad U(\sigma_2 \otimes \sigma_2)U^{-1}, \quad U(\sigma_3 \otimes \sigma_3)U^{-1}.$$

Discuss.

Problem 49. Let $\alpha,\beta,\gamma \in \mathbb{R}$. Show that any $U \in SU(2)$ can be written as

$$U = \exp(i\gamma\sigma_3)\exp(i\beta\sigma_1)\exp(i\alpha\sigma_3).$$

Problem 50. Let
$$Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv i\sigma_2.$$
Find all 2×2 matrices S such that
$$YSY^{-1} = S^T, \qquad S = S^*.$$

Problem 51. Can one find a 2×2 unitary matrix U such that
$$\begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & -\cos\theta \end{pmatrix} = U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^* \,?$$

Problem 52. (i) Consider an $n \times n$ unitary matrix $U = (u_{jk})$ with $j, k = 1, 2, \dots, n$. Show that
$$S = (s_{jk}) = (u_{jk}\bar{u}_{jk})$$
is a double stochastic matrix.
(ii) Given a double stochastic $n \times n$ matrix S. Can we construct the unitary matrix U which generates the double stochastic matrix as described in (i).

Problem 53. Consider the unitary matrices
$$U_1(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \qquad U_2(\phi) = \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}.$$
What is the condition on ϕ such that the commutator of $U_1(\phi)$ and $U_2(\phi)$ vanishes, i.e. $[U_1, U_2] = 0_2$.

Problem 54. We define the 2×2 matrix
$$\sigma_\phi := \cos(\phi)\sigma_1 + \sin(\phi)\sigma_2 = \begin{pmatrix} 0 & \cos(\phi) - i\sin(\phi) \\ \cos(\phi) + i\sin(\phi) & 0 \end{pmatrix}.$$
Calculate
$$\exp(-i\theta\sigma_\phi/2).$$

Problem 55. Let U be a unitary matrix with $U = U^T$. Show that U can be written as $U = V^T V$, where V is unitary.

Problem 56. Let
$$\sigma := \frac{1}{2}(1 - \sqrt{5}), \qquad \tau := \frac{1}{2}(1 + \sqrt{5})$$

with τ the golden mean number. Are the 4×4 matrices

$$U_1 = -\frac{1}{2}\begin{pmatrix} 1 & -\tau & -\sigma & 0 \\ \tau & 1 & 0 & \sigma \\ \sigma & 0 & 1 & -\tau \\ 0 & -\sigma & \tau & 1 \end{pmatrix}, \qquad U_2 = -\frac{1}{2}\begin{pmatrix} 1 & -\tau & -\sigma & 0 \\ \tau & 1 & 0 & -\sigma \\ \sigma & 0 & 1 & \tau \\ 0 & \sigma & -\tau & 1 \end{pmatrix}$$

unitary? Prove or disprove.

Problem 57. Is the matrix

$$U(\theta,\phi) = \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{i\phi} \\ \sin(\theta)e^{-i\phi} & -\cos(\theta) \end{pmatrix}$$

unitary? If so find the inverse.

Problem 58. Find the square roots of the Pauli spin matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Problem 59. Consider the unit vectors in $\mathbb{C}^3$

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

i.e.

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \qquad |w_1|^2 + |w_2|^2 + |w_3|^2 = 1.$$

Assume that (complex unit cone)

$$z_1 w_1 + z_2 w_2 + z_3 w_3 = 0.$$

Show that $U \in SU(3)$ can be written as

$$U = \begin{pmatrix} \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

where

$$u_j := \sum_{k,\ell=1}^{3} \epsilon_{jk\ell} z_k \bar{w}_\ell.$$

Problem 60. Let U be a unitary matrix. Then the determinant must be of the form $e^{i\phi}$. Find the determinant of $U + U^*$.

Problem 61. (i) Let $0 \leq r_{jk} \leq 1$ and $\phi_{jk} \in \mathbb{R}$ $(j,k = 1,2)$. Find the conditions on r_{jk}, ϕ_{jk} such that the 2×2 matrix

$$U(r_{jk},\phi_{jk}) = \begin{pmatrix} r_{11}e^{i\phi_{11}} & r_{12}e^{i\phi_{12}} \\ r_{21}e^{i\phi_{21}} & r_{22}e^{i\phi_{22}} \end{pmatrix}$$

is unitary. Then simplify to the special case $\phi_{jk} = 0$ for $j,k = 1,2$.

Problem 62. (i) Let $\phi_1, \phi_2 \in \mathbb{R}$. Show that

$$U(\phi_1,\phi_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & -e^{-i\phi_2} \\ e^{i\phi_2} & e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $U(\phi_1,\phi_2)$ an element of $SU(2)$? Find the eigenvalues and eigenvectors of $U(\phi_1,\phi_2)$. Do the eigenvalues and eigenvectors depend on the ϕ's? Show that

$$V(\phi_1,\phi_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & e^{-i\phi_2} \\ e^{i\phi_2} & -e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $V(\phi_1,\phi_2)$ an element of $SU(2)$? Find the eigenvalues and eigenvectors of $V(\phi_1,\phi_2)$. Do the eigenvalues and eigenvectors depend on the ϕ's?
(ii) Let $\phi_1,\phi_2,\phi_3,\phi_4 \in \mathbb{R}$. Show that

$$U(\phi_1,\phi_2,\phi_3,\phi_4) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & 0 & 0 & -e^{-i\phi_2} \\ 0 & e^{i\phi_3} & -e^{-i\phi_4} & 0 \\ 0 & e^{i\phi_4} & e^{-i\phi_3} & 0 \\ e^{i\phi_2} & 0 & 0 & e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $U(\phi_1,\phi_2,\phi_3,\phi_4)$ an element of $SU(4)$? Find the eigenvalues and eigenvectors of $U(\phi_1,\phi_2,\phi_3,\phi_4)$. Do the eigenvalues and eigenvectors depend on the ϕ's? Let $\phi_1,\phi_2,\phi_3,\phi_4 \in \mathbb{R}$. Show that

$$V(\phi_1,\phi_2,\phi_3,\phi_4) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & 0 & 0 & e^{-i\phi_2} \\ 0 & e^{i\phi_3} & e^{-i\phi_4} & 0 \\ 0 & e^{i\phi_4} & -e^{-i\phi_3} & 0 \\ e^{i\phi_2} & 0 & 0 & -e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $V(\phi_1,\phi_2,\phi_3,\phi_4)$ an element of $SU(4)$? Find the eigenvalues and eigenvectors of $V(\phi_1,\phi_2,\phi_3,\phi_4)$. Do the eigenvalues and eigenvectors depend on the ϕ's?
(iii) Let $\phi_1,\phi_2 \in \mathbb{R}$. Show that

$$U(\phi_1,\phi_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & 0 & -e^{-i\phi_2} \\ 0 & \sqrt{2} & 0 \\ e^{i\phi_2} & 0 & e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $U(\phi_1,\phi_2)$ an element of $SU(2)$? Find the eigenvalues and eigenvectors of $U(\phi_1,\phi_2)$. Do the eigenvalues and eigenvectors depend on the ϕ's? Show that

$$V(\phi_1,\phi_2)=\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_1} & 0 & e^{-i\phi_2} \\ 0 & \sqrt{2} & 0 \\ e^{i\phi_2} & 0 & -e^{-i\phi_1} \end{pmatrix}$$

is unitary. Is $V(\phi_1,\phi_2)$ an element of $SU(2)$? Find the eigenvalues and eigenvectors of $V(\phi_1,\phi_2)$. Do the eigenvalues and eigenvectors depend on the ϕ's?

Problem 63. Show that the matrix

$$U=\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{pmatrix}$$

is unitary. Find the eigenvalues and normalized eigenvectors of U. Write down the spectral decomposition of U. Let

$$|\psi\rangle=\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find $U|\psi\rangle$, $U^2|\psi\rangle$, $U^3|\psi\rangle$. Study $U^n|\psi\rangle$ with $n\to\infty$.

Problem 64. Find a unitary matrix U with $\det(U)=-1$ such that

$$U\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 65. (i) Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $\alpha_j\in\mathbb{R}$ for $j=1,2,3$. Find the 2×2 matrices

$$U_k := (I_2 - i\alpha_k\sigma_k)(I_2+i\alpha_k\sigma_k)^{-1}$$

where $k=1,2,3$. Are the matrices unitary?
(ii) Find the 4×4 matrices

$$V_k := (I_4 - i\alpha_k\sigma_k\otimes\sigma_k)(I_4+i\alpha_k\sigma_k\otimes\sigma_k)^{-1}$$

where $k=1,2,3$. Are the matrices unitary?

Problem 66. Find all 3×3 unitary matrices of the form

$$U=\begin{pmatrix} u_{11} & u_{12} & 0 \\ u_{21} & u_{22} & u_{23} \\ 0 & u_{32} & u_{33} \end{pmatrix}.$$

Problem 67. Can one find a unitary matrix U with $\det(U) = 1$, i.e. U is an element of $SU(2)$, such that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = U \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem 68. What is the condition on the phases ϕ_1, ϕ_2, ϕ_3 such that

$$U(\phi_1, \phi_2, \phi_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\phi_1} \\ e^{i\phi_2} & e^{i\phi_3} \end{pmatrix}$$

is a unitary matrix?

Problem 69. Can one find a 2×2 unitary matrix such that

$$U \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} U^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} ?$$

Motivate. If so find the matrix U. Discuss.

Problem 70. Can one 2×2 find unitary matrices such that the rows and columns add up to one. Of course the 2×2 identity matrix is one of them.

Problem 71. Let $n \geq 2$. Consider the $n \times n$ matrix (counting from $(0,0)$)

$$(A_{jk}) = \frac{1}{\sqrt{n}}(e^{i\pi(j-k)})$$

where $j, k = 0, 1, \ldots, n-1$. Is the matrix unitary? Study first the cases $n = 2$ and $n = 3$.

Problem 72. (i) Is the matrix

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & -i & -1 & 0 \\ i & 0 & 0 & -1 \end{pmatrix}$$

unitary?
(ii) Is the matrix

$$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

unitary?

Problem 73. Let n be a prime number and $j,k=1,\ldots,n$. Show that the matrices

$$
\begin{aligned}
(U_1)_{jk} &= \frac{1}{\sqrt{n}}\exp((2\pi i/n)(j+k-1)^2)\\
&\vdots\\
(U_r)_{jk} &= \frac{1}{\sqrt{n}}\exp((2\pi i/n)r(j+k-1)^2)\\
&\vdots\\
(U_{n-1})_{jk} &= \frac{1}{\sqrt{n}}\exp(((2\pi i/n)(n-1)(j+k-1)^2)\\
(U_n)_{jk} &= \frac{1}{\sqrt{n}}\exp((2\pi i/n)jk)\\
(U_{n+1})_{jk} &= \delta_{jk}
\end{aligned}
$$

are unitary.

Problem 74. Is the 2×2 matrix

$$U=\begin{pmatrix} u_1+iu_2 & u_3+iu_4 \\ -u_3+iu_4 & u_1-iu_2 \end{pmatrix}$$

with

$$u_1=\cos(\alpha),\quad u_2=\sin(\alpha)\cos(\beta),$$

$$u_3=\sin(\alpha)\sin(\beta)\cos(\gamma),\quad u_4=\sin(\alpha)\sin(\beta)\sin(\gamma)$$

unitary?

Problem 75. Find the eigenvalues of the unitary operator

$$U=\exp\left(-i\frac{\pi}{4}b^\dagger b\otimes\sigma_3\right).$$

Note that $e^{i\pi/4}=(1+i)/\sqrt{2}$, $e^{-i\pi/4}=(1-i)/\sqrt{2}$.

Problem 76. Is the 3×3 matrix

$$U=\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

a unitary matrix?

Problem 77. Are the 4×4 matrices

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

untary?

Problem 78. Show that any unitary $n \times n$ matrix is conjugate to a diagonal matrix of the form

$$\mathrm{diag}(\, e^{i\phi_1} \;\; e^{i\phi_2} \;\; \dots \;\; e^{i\phi_n}\,).$$

Problem 79. (i) What can be said about the eigenvalues of a matrix which is unitary and hermitian?
(ii) What can be said about th eigenvalues of a matrix which is unitary and skew-hermitian?
(iii) What can be said about the eigenvalues of a matrix which is unitary and $U^T = U$?
(iv) What can be said about the eigenvalues of a matrix which is unitary and $U^3 = U$?

Problem 80. Is the matrix

$$U(\theta, \phi) = \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{i\phi} \\ -\sin(\theta)e^{-i\phi} & \cos(\theta) \end{pmatrix}$$

unitary?

Problem 81. Consider the rotation matrix

$$R(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Find all 2×2 matrices

$$M(\beta) = \begin{pmatrix} f_{11}(\beta) & f_{12}(\beta) \\ f_{21}(\beta) & f_{22}(\beta) \end{pmatrix}$$

such that $R(\alpha)M(\beta)R^{-1}(\alpha) = M(\beta + \alpha)$.

Problem 82. Are the matrices

$$V_1 = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

unitary?

Problem 83. Find all $n \times n$ matrices T over $\mathbb{R}$ such that

$$U = I_n + iT$$

is a unitary matrix.

Problem 84. Let U be an $N \times N$ unitary matrix. Then

$$\det(U - \lambda I_N) = \prod_{j=1}^{N} (e^{i\phi_j} - \lambda)$$

where $\phi_j \in [0, 2\pi)$. Let $N = 2$. Find λ for the matrix

$$U = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Then calculate $|e^{i\phi_1} - e^{i\phi_2}|$.

Problem 85. What are the conditions on $\phi_1, \phi_2, \phi_3, \phi_4 \in \mathbb{R}$ such that the matrix

$$U(\phi_1, \phi_2, \phi_3, \phi_4) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_1} & e^{i\phi_2} \\ e^{i\phi_3} & e^{i\phi_4} \end{pmatrix}$$

is unitary?

Problem 86. Let U be a unitary $d \times d$ matrix. Then

$$\det(U - \lambda I_d) = \prod)j = 1^d (e^{i\phi_j} - \lambda).$$

Use this expression to calculate the eigenvalues of the 2×2 matrix

$$U = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Problem 87. Find a unitary 2×2 matrix such that

$$\sigma_2 U - U \sigma_3 = 0_2.$$

Problem 88. Find a unitary matrix U which can be written as a direct sum of two 2×2 matrices and

$$U \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \equiv U \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 89. In the Hilbert space $\mathbb{C}^4$ the Bell states

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

(i) Let $\omega = e^{2\pi i/4}$. Apply the Fourier transformation

$$U_F = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\1 & \omega & \omega^2 & \omega^3\\1 & \omega^2 & 1 & \omega^2\\1 & \omega^3 & \omega^2 & \omega\end{pmatrix}$$

to the Bell states and study the entanglement of these states.
(ii) Apply the Haar wavelet transformation

$$U_H = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & -1 & -1\\\sqrt{2} & -\sqrt{2} & 0 & 0\\0 & 0 & \sqrt{2} & -\sqrt{2}\end{pmatrix}$$

to the Bell states and study the entanglement of these states.
(iii) Apply the Walsh-Hadamard transformation

$$U_W = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\-1 & -1 & 1 & 1\\-1 & 1 & 1 & -1\\1 & -1 & 1 & -1\end{pmatrix}$$

to the Bell states and study the entanglement of these states.

Extend to the Hilbert space $\mathbb{C}^{2^n}$ with the first Bell state given by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & \cdots & 0 & 1\end{pmatrix}^T$$

Problem 90. Is the matrix

$$U = \frac{1}{\sqrt{3}}(I_2 \otimes I_2 \otimes I_2 + i\sigma_1 \otimes \sigma_1 \otimes \sigma_1 + i\sigma_3 \otimes \sigma_3 \otimes \sigma_3)$$

unitary?

Problem 91. Let A be an $n \times n$ hermitian matrix with $A^2 = I_n$.
(i) Is $U = \frac{1}{\sqrt{2}}(I_n + iA)$ unitary?

(ii) Is $U = \frac{1}{\sqrt{2}}(I_n - iA)$ unitary?
(iii) Let $\phi \in [0, 2\pi)$. What is the condition on ϕ such that

$$V(\phi) = \frac{1}{\sqrt{2}}(e^{i\phi}I_n + e^{-i\phi}A)$$

is unitary?

Problem 92. Let U be a unitary 2×2 matrix. Can the matrix be reconstructed from

$$\mathrm{tr}(U), \quad \mathrm{tr}(U^2), \quad \mathrm{tr}(U^3)?$$

Problem 93. Is the 3×3 matrix

$$U = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ i & i & -i \\ i & -i & -i \end{pmatrix}$$

unitary?

Chapter 14

Numerical Methods

Problem 1. Let A be an invertible $n \times n$ matrix over $\mathbb{R}$. Consider the system of linear equation $A\mathbf{x} = \mathbf{b}$ or

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, n.$$

Let $A = C - R$. This is called a *splitting* of the matrix A and R is the defect matrix of the splitting. Consider the iteration

$$C\mathbf{x}^{(k+1)} = R\mathbf{x}^{(k)} + \mathbf{b}, \qquad k = 0, 1, 2, \ldots .$$

Let

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \qquad \mathbf{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The iteration converges if $\rho(C^{-1}R) < 1$, where $\rho(C^{-1}R)$ denotes the spectral radius of $C^{-1}R$. Show that $\rho(C^{-1}R) < 1$. Perform the iteration.

Problem 2. Let A be an $n \times n$ matrix over $\mathbb{R}$ and let $\mathbf{b} \in \mathbb{R}^n$. Consider the linear equation $A\mathbf{x} = \mathbf{b}$. Assume that $a_{jj} \neq 0$ for $j = 1, 2, \ldots, n$. We define the diagonal matrix $D = \mathrm{diag}(a_{jj})$. Then the linear equation $A\mathbf{x} = \mathbf{b}$ can be written as

$$\mathbf{x} = B\mathbf{x} + \mathbf{c}$$

with $B := -D^{-1}(A-D)$, $\mathbf{c} := D^{-1}\mathbf{b}$. The *Jacobi method* for the solution of the linear equation $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{c}, \qquad k = 0, 1, \ldots$$

where $\mathbf{x}^{(0)}$ is any initial vector in $\mathbb{R}^n$. The sequence converges if

$$\rho(B) := \max_{j=1,\ldots,n} |\lambda_j(B)| < 1$$

where $\rho(B)$ is the spectral radius of B. Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(i) Show that the Jacobi method can applied for this matrix.
(ii) Find the solution of the linear equation with $\mathbf{b} = (1\ 1\ 1)^T$.

Problem 3. Let A be an $n \times n$ matrix over $\mathbb{R}$. The (p,q) *Padé approximation* to $\exp(A)$ is defined by

$$R_{pq}(A) := (D_{pq}(A))^{-1} N_{pq}(A)$$

where

$$N_{pq}(A) = \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!} A^j$$

$$D_{pq}(A) = \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} (-A)^j.$$

Nonsingularity of $D_{pq}(A)$ is assured if p and q are large enough or if the eigenvalues of A are negative. Find the Padé approximation for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $p = q = 2$. Compare with the exact solution.

Problem 4. Let A be an $n \times n$ matrix over $\mathbb{R}$. Then we have the Taylor expansion

$$\sin(A) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^{2k+1}, \qquad \cos(A) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} A^{2k}.$$

To calculate $\sin(A)$ and $\cos(A)$ from a truncated Taylor series approximation is only worthwhile near the origin. We can use the repeated application of the *double angle formula*

$$\cos(2A) \equiv 2\cos^2(A) - I_n, \qquad \sin(2A) \equiv 2\sin(A)\cos(A).$$

We can find $\sin(A)$ and $\cos(A)$ of a matrix A from a suitably truncated Taylor series approximates as follows

$$S_0 = \text{Taylor approximate to } \sin(A/2^k)$$

$$C_0 = \text{Taylor approximate to } \cos(A/2^k)$$

and the recursion

$$S_j = 2S_{j-1}C_{j-1}, \qquad C_j = 2C_{j-1}^2 - I_n$$

where $j = 1, 2, \ldots$. Here k is a positive integer chosen so that, say $\|A\|_\infty \approx 2^k$. Apply this recursion to calculate sine and cosine of the 2×2 matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Use $k = 2$.

Problem 5. Let A be an $n \times n$ matrix. We define the $j - k$ approximant of $\exp(A)$ by

$$f_{j,k}(A) := \left(\sum_{\ell=0}^{k} \frac{1}{\ell!} \left(\frac{A}{j} \right)^\ell \right)^j. \tag{1}$$

We have the inequality

$$\|e^A - f_{j,k}(A)\| \le \frac{1}{j^k (k+1)!} \|A\|^{k+1} e^{\|A\|} \tag{2}$$

and $f_{j,k}(A)$ converges to e^A, i.e.

$$\lim_{j \to \infty} f_{j,k}(A) = \lim_{k \to \infty} f_{j,k}(A) = e^A.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Find $f_{2,2}(A)$ and e^A. Calculate the right-hand side of the inequality (2).

Problem 6. The *power method* is the simplest algorithm for computing eigenvectors and eigenvalues Consider the vector space $\mathbb{R}^n$ with the Euclidean norm

$\|\mathbf{x}\|$ of a vector $\mathbf{x} \in \mathbb{R}$. The iteration is as follows: Given a nonsingular $n \times n$ matrix M and a vector $\mathbf{x}_0$ with $\|\mathbf{x}_0\| = 1$. One defines

$$\mathbf{x}_{t+1} = \frac{M\mathbf{x}_t}{\|M\mathbf{x}_t\|}, \quad t = 0, 1, \ldots$$

This defines a dynamical system on the sphere S^{n-1}. Since M is invertible we have

$$\mathbf{x}_t = \frac{M^{-1}\mathbf{x}_{t+1}}{\|M^{-1}\mathbf{x}_{t+1}\|}, \quad t = 0, 1, \ldots$$

(i) Apply the power method to the nonnormal matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) Apply the power method to the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 7. Let A be an $n \times n$ matrix over $\mathbb{R}$ and let $\mathbf{u}$ be an n-vector in $\mathbb{R}^n$ (column vector) with $\mathbf{u} \neq \mathbf{u}$. In numerical linear algebra we often have to compute

$$\left(I_n - \frac{2\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}} \right) \tag{1}$$

where I_n is the $n \times n$ identity matrix. Naively we would form the matrix $(I_n - 2\mathbf{u}\mathbf{u}^T/\mathbf{u}^T\mathbf{u})$ from the vector $\mathbf{u}$ and then form the matrix product explicitly with A. This would require $O(m^3)$ flops. Provide a faster computation for expression (1).

Problem 8. Consider an $n \times n$ symmetric tridiagonal matrix over $\mathbb{R}$. Let $f_n(\lambda) := \det(A - \lambda I_n)$ and

$$f_k(\lambda) = \det \begin{pmatrix} \alpha_1 - \lambda & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 - \lambda & \beta_2 & \cdots & 0 \\ 0 & \beta_2 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \alpha_{k-1} - \lambda & \beta_{k-1} \\ 0 & \cdots & 0 & \beta_{k-1} & \alpha_k - \lambda \end{pmatrix}$$

for $k = 1, 2, \ldots, n$ and $f_0(\lambda) = 1$, $f_{-1}(\lambda) = 0$. Then

$$f_k(\lambda) = (\alpha - \lambda) f_{k-1}(\lambda) - \beta_{k-1}^2 f_{k-2}(\lambda)$$

for $k = 2, 3, \ldots, n$. Find $f_4(\lambda)$ for the 4×4 matrix

$$\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

Chapter 15

Binary Matrices

Problem 1. For a 2×2 binary matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{jk} \in \{0, 1\}$$

we define the determinant as

$$\det A = (a_{11} \cdot a_{22}) \oplus (a_{12} \cdot a_{21})$$

where $\cdot$ is the AND-operation and $\oplus$ is the XOR-operation.
(i) Find the determinant for the following 2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(ii) Find the determinant for the following 2×2 matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 2. The determinant of a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is given by

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

For a binary matrix B we replace this expression by

$$\det B = (b_{11} \cdot b_{22} \cdot b_{33}) \oplus (b_{12} \cdot b_{23} \cdot b_{31}) \oplus (b_{13} \cdot b_{21} \cdot b_{32}) \\ \oplus (b_{13} \cdot b_{22} \cdot b_{31}) \oplus (b_{11} \cdot b_{23} \cdot b_{32}) \oplus (b_{12} \cdot b_{21} \cdot b_{33}).$$

(i) Calculate the determinant for the binary matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Calculate the determinant for the binary matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Problem 3. The finite field $GF(2)$ consists of the elements 0 and 1 (bits) which satisfies the following addition (XOR-operation) and multiplication (AND-operation) tables

$\oplus$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Find the determinant of the *binary matrices*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 4. A boolean function $f : \{0,1\}^n \to \{0,1\}$ can be transformed from the domain $\{0,1\}$ into the spectral domain by a linear transformation

$$T\mathbf{y} = \mathbf{s}$$

where T is a $2^n \times 2^n$ orthogonal matrix, $\mathbf{y} = (y_0, y_1, \dots, y_{2^n-1})^T$, is the two valued ($\{+1,-1\}$ with $0 \leftrightarrow 1$, $1 \leftrightarrow -1$) truth table vector of the boolean function and s_j $(j = 0, 1, \ldots, 7)$ are the spectral coefficients $(\mathbf{s} = (s_0, s_1, \ldots, s_{2^n-1})^T)$. Since T is invertible we have

$$T^{-1}\mathbf{s} = \mathbf{y}.$$

For T we select the Hadamard matrix. The $2^n \times 2^n$ Hadamard matrix $H(n)$ is recursively defined as

$$H(n) = \begin{pmatrix} H(n-1) & H(n-1) \\ H(n-1) & -H(n-1) \end{pmatrix}, \qquad n = 1, 2, \ldots$$

with $H(0) = (1)$ $(1 \times 1$ matrix). The inverse of $H(n)$ is given by

$$H^{-1}(n) = \frac{1}{2^n} H(n).$$

Now any boolean function can be expanded as the arithmetical polynomial

$$f(x_1, \ldots, x_n) = \frac{1}{2^{n+1}} \left(2^n - s_0 - s_1(-1)^{x_n} - s_2(-1)^{x_{n-1}} - \cdots - s_{2^n-1}(-1)^{x_1 \oplus x_2 \oplus \cdots \oplus x_n}\right)$$

where $\oplus$ denotes the modulo-2 addition.

Consider the boolean function $f : \{0, 1\}^3 \to \{0, 1\}$ given by

$$f(x_1, x_2, x_3) = \bar{x}_1 \cdot \bar{x}_2 \cdot \bar{x}_3 + \bar{x}_1 \cdot x_2 \cdot \bar{x}_3 + x_1 \cdot x_2 \cdot \bar{x}_3.$$

Find the truth table, the vector $\mathbf{y}$ and then calculate, using $H(3)$, the spectral coefficients s_j, $(j = 0, 1, \ldots, 7)$.

Problem 5. Consider the binary matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Calculate the Hadamard product $A \bullet B$.

Problem 6. Consider the two permutation matrices (NOT-gate and XOR-gate)

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Can we generate all other permutation matrices from these two permutation matrices?

Problem 7. How many 3×3 binary matrices can one form which contain three 1's? Write down these matrices. Which of them are invertible?

Chapter 16

Groups

Problem 1. (i) Find the group generated by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

under matrix multiplication.
(ii) Find the group generated by the 6×6 matrix

$$B = \begin{pmatrix} 0_2 & 0_2 & I_2 \\ I_2 & 0_2 & 0_2 \\ 0_2 & I_2 & 0_2 \end{pmatrix}$$

under matrix multiplication.

Problem 2. The Pauli matrix σ_1 is not only hermitian, unitary and his own inverse, but also a permutation matrix. Find all 2×2 matrices A such that

$$\sigma_1^{-1} A \sigma_1 = A.$$

Problem 3. Let $z \in \mathbb{C}$ and $z \neq 0$.
(i) Do the 2×2 matrices

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$$

form a group under matrix multiplication?
(ii) Do the 3×3 matrices

$$\begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & z \\ 0 & 1 & 0 \\ z^{-1} & 0 & 0 \end{pmatrix}$$

form a group under matrix multiplication?

Problem 4. Find all 3×3 permutation matrices P such that

$$P^{-1} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Show that these matrices form a group, i.e. a subgroup of the 3×3 permutation matrices.

Problem 5. (i) Let $c \in \mathbb{R}$ and $c \neq 0$. Do the matrices

$$A(c) = \begin{pmatrix} c & c & c & c \\ c & c & c & c \\ c & c & c & c \\ c & c & c & c \end{pmatrix}$$

form a group under matrix multiplication?
(ii) Find the eigenvalues of $A(c)$.

Problem 6. Do the matrices

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U(\phi) = \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}.$$

form a group under matrix multiplication?

Problem 7. Consider the Pauli spin matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Do the sixteen 4×4 matrices ($j = 0, 1, 2, 3$)

$$\begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_j \end{pmatrix}, \quad \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & -\sigma_j \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ -\sigma_j & 0_2 \end{pmatrix}$$

form a group under matrix multiplication?

(ii) Do the sixteen 4×4 matrices ($j = 0, 1, 2, 3$)

$$\begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_j \end{pmatrix}, \quad \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & -\sigma_j \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ -\sigma_j & 0_2 \end{pmatrix}$$

form a Lie algebra under the commutator?

Problem 8. For the vector space of the $n \times n$ matrices over $\mathbb{R}$ we can introduce a scalar product via

$$\langle A, B \rangle := \operatorname{tr}(AB^T).$$

Consider the Lie group $SL(2, \mathbb{R})$ of the 2×2 matrices with determinant 1. Find $X, Y \in SL(2, \mathbb{R})$ such that

$$\langle X, Y \rangle = 0$$

where neither X nor Y can be 2×2 identity matrix.

Problem 9. Consider the diagonal matrix D and the permutation matrix P

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(2i\pi/3) & 0 \\ 0 & 0 & \exp(-2i\pi/3) \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(i) What group is generated by D and P?
(ii) Calculate the commutator $[D, P]$. Discuss.

Problem 10. Do the numbers 1, i, -1, $-i$ form a group under multiplication? Do the vectors

$$\frac{1}{2}\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -i \\ 1 \\ i \\ -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 \\ -i \\ 1 \\ i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}$$

form an orthonormal basis in $\mathbb{C}^4$?

Problem 11. Let I_2, 0_2 be the 2×2 identity and zero matrix, respectively. Find the group generated by the 4×4 matrix

$$A = \begin{pmatrix} 0_2 & -I_2 \\ I_2 & 0_2 \end{pmatrix}.$$

Problem 12. Do the matrices

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}$$

with $\det(A) \neq 0$ form a group under matrix multiplication?

Problem 13. Consider the permutation group S_3. The matrix representation of the permutations (12) and (13) are

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (13) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Show that the other permutations of S_3 can be constructed from products of these two matrices.

Problem 14. Do the set of 2×2 matrices

$$\begin{pmatrix} e^{i(\alpha+\beta)} \cosh(\tau) & e^{i(\alpha-\beta)} \sinh(\tau) \\ e^{-i(\alpha-\beta)} \sinh(\tau) & e^{-i(\alpha+\beta)} \cosh(\tau) \end{pmatrix}$$

form a group under matrix multiplication, where $\tau, \alpha, \beta \in \mathbb{R}$?

Problem 15. Let $0 \leq \alpha < \pi/4$. Consider the transformation

$$X(x, y, \alpha) = \frac{1}{\sqrt{\cos(2\alpha)}}(x \cos(\alpha) + iy \sin(\alpha))$$
$$Y(x, y, \alpha) = \frac{1}{\sqrt{\cos(2\alpha)}}(-ix \sin(\alpha) + y \cos(\alpha)).$$

(i) Show that $X^2 + Y^2 = x^2 + y^2$.
(ii) Do the matrices

$$\frac{1}{\sqrt{\cos(2\alpha)}} \begin{pmatrix} \cos(\alpha) & i \sin(\alpha) \\ -i \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

form a group under matrix multiplication?

Problem 16. Let $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. Do the matrices

$$M(\alpha, \beta) = \begin{pmatrix} \cos(\alpha) & \beta^{-1} \cos(\alpha) \\ -\beta \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

form a group under matrix multiplication?

Problem 17. In the Lie group $U(N)$ of the $N \times N$ unitary matrices one can find two $N \times N$ matrices U and V such that

$$UV = e^{2\pi i/N} VU.$$

Any $N \times N$ hermitian matrix H can be written in the form

$$H = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} h_{jk} U^j V^k.$$

Using the expansion coefficients h_{jk} one can associate to the hermitian matrix H the function

$$h(q,q) = \sum_{j=0}^{N-1} \sum_{k=1}^{N-1} h_{jk} e^{2\pi i(jp+kq)}$$

where $p = 0, 1, \ldots, N-1$ and $q = 0, 1, \ldots, N-1$. Consider the case $N = 2$ and

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Consider the hermitian and unitary matrix

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Find $h(p,q)$.
(ii) Consider the hermitian and projection matrix

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find $h(p,q)$.

Problem 18. Consider the 3×3 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(i) Find P^2, P^3. Do the matrices P, P^2, P^3 form a group under matrix multiplication?
(ii) Find the eigenvalues and eigenvectors of P. Do the eigenvalues of P form a group under multiplication?

Problem 19. Consider the two 3×3 permutation matrices

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Can the remaining four 3×3 matrices be generated from C_1 and A using matrix multiplication?

Problem 20. Consider the two 4×4 permutation matrices

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Can the remaining twenty-two 4×4 matrices be generated from C_1 and A using matrix multiplication?

Problem 21. Can one find a 4×4 permutation matrix P such that

$$P \begin{pmatrix} \alpha & 0 & \alpha \\ 0 & \alpha & 0 \\ \alpha & 0 & \alpha \end{pmatrix} P^T = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & \alpha \\ 0 & \alpha & \alpha \end{pmatrix}.$$

Problem 22. (i) Consider the matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the condition on A such that $\sigma_1 A \sigma_1 = A$. Assume that A is also invertible. Do these matrices from a group?
(ii) Find the condition on A such that

$$(\sigma_1 \otimes \sigma_1)(A \otimes A)(\sigma_1 \otimes \sigma_1).$$

Problem 23. Can one find an α such that

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}?$$

Problem 24. Consider the permutation matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the condition on a 3×3 matrix A such that

$$CAC^T = A.$$

Note that $C^T = C^{-1}$.

Problem 25. Consider the permutation matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find the condition on a 4×4 matrix A such that

$$CAC^T = A.$$

Note that $C^T = C^{-1}$.

Problem 26. The Lie group $0(2)$ is generated by a rotation R_1 and a reflection R_2

$$R_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Find the trace, determinant and eigenvalues of R_1 and R_2.

Problem 27. (i) Consider the group G of all 4×4 permutation matrices. Show that

$$\frac{1}{|G|} \sum_{g \in G} g$$

is a projection matrix. Here $|G|$ denotes the number of elements in the group.
(ii) Consider the subgroup given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Show that

$$\frac{1}{|G|} \sum_{g \in G} g$$

is a projection matrix.

Problem 28. Let A be the 2×2 matrix

$$A = \begin{pmatrix} a_{11}e^{i\phi_{11}} & a_{12}e^{i\phi_{12}} \\ a_{21}e^{i\phi_{21}} & a_{22}e^{i\phi_{22}} \end{pmatrix}$$

where $a_{jk} \in \mathbb{R}$, $a_{jk} > 0$ for $j,k = 1,2$ and $a_{12} = a_{21}$. We also have $\phi_{jk} \in \mathbb{R}$ for $j,k = 1,2$ and impose $\phi_{12} = \phi_{21}$. What are the conditions on a_{jk} and ϕ_{jk} such that $I_2 + iA$ is a unitary matrix?

Problem 29. Let $\alpha, \beta, \phi \in \mathbb{R}$ and $\alpha, \beta \neq 0$. Consider the matrices

$$A(\alpha,\beta,\phi) = \begin{pmatrix} \alpha\cos\phi & -\beta\sin\phi \\ \beta^{-1}\sin\phi & \alpha^{-1}\cos\phi \end{pmatrix}.$$

Do the matrices form a group under matrix multiplication?

Problem 30. Show that the four 2×2 matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

from a group under matrix multiplication. Is the group abelian?

Problem 31. Let $x \in \mathbb{R}$. Is the matrix

$$A(x) = \begin{pmatrix} \cos x & 0 & \sin x & 0 \\ 0 & \cos x & 0 & \sin x \\ -\sin x & 0 & \cos x & 0 \\ 0 & -\sin x & 0 & \cos x \end{pmatrix}$$

an orthogonal matrix?

Problem 32. (i) Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix}.$$

where $a_{11}, a_{12} \in \mathbb{R}$. Find all invertible 2×2 matrices S over $\mathbb{R}$ such that

$$SAS^{-1} = A.$$

Obviously the identity matrix I_2 would be such as matrix.
(ii) Do the matrices S form a group under matrix multiplication? Prove or disprove.
(iii) Use the result form (i) to calculate

$$(S \otimes S)(A \otimes A)(S \otimes S)^{-1}.$$

Discuss.

Problem 33. Let $\alpha, \beta, \gamma \in \mathbb{R}$. Do the 3×3 matrices

$$M(\alpha,\beta,\gamma) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & \beta \\ -\sin(\alpha) & \cos(\alpha) & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

form a group under matrix multiplication?

Problem 34. Is the invertible matrix

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

an element of the Lie group $SO(4)$?

Problem 35. Do the eight 2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

form a group under matrix multiplication? If not add the matrices so that one has a group.

Problem 36. The *Heisenberg group* is the set of upper 3×3 matrices of the form

$$H = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c can be taken from some (arbitrary) commutative ring.
(i) Find the inverse of H.
(ii) Given two elements x, y of a group G, we define the *commutator* of x and y, denoted by $[x, y]$ to be the element $x^{-1}y^{-1}xy$. If a, b, c are integers (in the ring $\mathbb{Z}$ of the integers) we obtain the discrete Heisenberg group H_3. It has two generators

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find

$$z = xyx^{-1}y^{-1}.$$

Show that $xz = zx$ and $yz = zy$.
(iii) The derived subgroup (or commutator subgroup) of a group G is the subgroup $[G, G]$ generated by the set of commutators of every pair of elements of G. Find $[G, G]$ for the Heisenberg group.
(iv) Let

$$A = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

and $a, b, c \in \mathbb{R}$. Find $\exp(A)$.
(v) The Heisenberg group is a simple connected Lie group whose Lie algebra consists of matrices

$$L = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the commutators $[L, L']$ and $[L, L'], L']]$, where $[L, L'] := LL' - L'L$.

Problem 37. Find all 2×2 matrices S over $\mathbb{C}$ with determinant 1 (i.e. they are elements of $SL(2, \mathbb{C})$) such that

$$S^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Obviously, the 2×2 identity matrix is such an element.

Problem 38. There are six 3×3 permutation matrices which form a group under matrix multiplication.
(i) Can the six elements be generated from the two permutation matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

using matrix multiplication?
(ii) Does A, A^2, A^3 provide a subgroup?

Problem 39. There are twenty-four 4×4 permutation matrices which form a group under matrix multiplication.
(i) Can the 24 elements be generated from the two permutation matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

using matrix multiplication?
(ii) Does A, A^2, A^3, A^4 provide a subgroup?

Problem 40. Let $\alpha, \beta, \gamma \in \mathbb{R}$. The *Heisenberg group* $H_3(\mathbb{R})$ consists of all 3×3 upper trian matrices of the form

$$M(\alpha, \beta, \gamma) = \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}$$

with matrix multiplication as composition. Let $t \in \mathbb{R}$. Consider the matrices

$$A(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad C(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Show that $\{A(t) : t \in \mathbb{R}\}$, $\{B(t) : t \in \mathbb{R}\}$, $\{C(t) : t \in \mathbb{R}\}$ are one-parameter subgroups in $H_3(\mathbb{R})$.
(ii) Show that $\{C(t) : t \in \mathbb{R}\}$ is the center of $H_3(\mathbb{R})$.

Problem 41. Consider the Pauli spin matrices σ_1, σ_2, σ_3. Let $x_1, x_2, x_3 \in \mathbb{R}$. Show that

$$e^{i(x_1\sigma_1+x_2\sigma_2+x_3\sigma_3)} = I_2\cos(r) + \frac{\sin(r)}{r} i(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$.

Problem 42. (i) Consider the 2×2 matrix

$$J := -i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Show that any 2×2 $A \in SL(2, \mathbb{C})$ satisfies

$$A^T J A = J$$

where J denotes the transposed matrix of A.
(ii) Let A satisfying $A^T J A = J$. Is

$$(A \otimes A)^T (J \otimes J)(A \otimes A) = J \otimes J\,?$$

Is

$$(A \oplus A)^T (J \oplus J)(A \oplus A) = J \oplus J\,?$$

Is

$$(A \star A)^T (J \star J)(A \star A) = J \star J\,?$$

Problem 43. The group $SL(2, \mathbb{F}_3)$ consists of unimodular 2×2 matrices with integer entries taken modulo three. Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

be an element of $SL(2, \mathbb{F}_3)$. Find the the inverse of A.

Problem 44. (i) Do the 4×4 matrices

$$g(\phi, \theta) = \begin{pmatrix} e^{i\phi}\cosh(\theta) & 0 & 0 & \sinh(\theta) \\ 0 & e^{-i\phi}\cosh(\theta) & \sinh(\theta) & 0 \\ 0 & \sinh(\theta) & e^{i\phi}\cosh(\theta) & 0 \\ \sinh(\theta) & 0 & 0 & e^{-i\phi}\cosh(\theta) \end{pmatrix}$$

form a group?
(ii) Do the 4×4 matrices

$$g(\phi, \theta) = \begin{pmatrix} e^{i\phi}\cos(\theta) & 0 & 0 & -\sin(\theta) \\ 0 & e^{-i\phi}\cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & e^{i\phi}\cos(\theta) & 0 \\ \sin(\theta) & 0 & 0 & e^{-i\phi}\cos(\theta) \end{pmatrix}$$

form a group?

Problem 45. Consider the cyclic 3×3 matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(i) Show that the matrices C, C^2, C^3 form a group under matrix multiplication. Is the group commutative?
(ii) Find the eigenvalues of C and show that the form a group under multiplication.
(iii) Find the normalized eigenvalues of C. Show that the form a orthonormal basis in $\mathbb{C}^3$.
(iv) Use the eigenvalues and normalized eigenvectors to write down the spectral decomposition of C.
(v) Use the result from (iv) to find K such $C = \exp(K)$.
(vi) Use these results to find L such that $C^2 = \exp(L)$.

Problem 46. (i) Let $x \in \mathbb{R}$. Show that the matrix

$$A(x) = \begin{pmatrix} \cos(x) & 0 & -\sin(x) & 0 \\ 0 & \cos(x) & 0 & -\sin(x) \\ \sin(x) & 0 & \cos(x) & 0 \\ 0 & \sin(x) & 0 & \cos(x) \end{pmatrix}$$

is invertible. Find the inverse. Do these matrices form a group under matrix multiplication?
(ii) Let $x \in \mathbb{R}$. Show that the matrix

$$B(x) = \begin{pmatrix} \cosh(x) & 0 & \sinh(x) & 0 \\ 0 & \cosh(x) & 0 & \sinh(x) \\ \sinh(x) & 0 & \cosh(x) & 0 \\ 0 & \sinh(x) & 0 & \cosh(x) \end{pmatrix}$$

is invertible. Find the inverse. Do these matrices form a group under matrix multiplication.

Problem 47. For the vector space of the $n \times n$ matrices over $\mathbb{R}$ we can introduce a scalar product via

$$\langle A, B \rangle := \operatorname{tr}(AB^T).$$

Consider the Lie group $SL(2,\mathbb{R})$ of the 2×2 matrices with determinant 1. Find $X, Y \in SL(2,\mathbb{R})$ such that

$$\langle X, Y \rangle = 0$$

where neither X nor Y can be 2×2 identity matrix.

Problem 48. Which group is generated by

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ?$$

First find the inverse of A.

Problem 49. The *free group* Γ_2 with two generators g_1 and g_2 has the matrix representation

$$g_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Obviously g_1 and g_2 are elements of $SL(2,\mathbb{R})$. Find the inverse of g_1 and g_2. Calculate

$$g_1 g_2^{-1} g_1 g_2 g_1^{-1} g_2.$$

Problem 50. Consider the 3×3 matrix

$$A(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}.$$

(i) Show that $A(\theta)$ is an element of $SO(3,\mathbb{R})$.
(ii) Find the eigenvalues and normalized eigenvectors of $A(\theta)$.
(iii) Find the eigenvalues and normalized eigenvectors of $A(\theta) \otimes A(\theta)$.

Problem 51. (i) Find the group (matrix multiplication) generated by the 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Is the group commutative?
(ii) Find the determinant of all these matrices from (i). Do these numbers form a group under multiplication?
(iii) Find all the eigenvalues of these matrices. Do these numbers form a group under multiplication?
(iv) Let $\alpha \in \mathbb{R}$. Find $\exp(\alpha A)$. Is $\exp(\alpha A)$ an element of $SL(4,\mathbb{R})$?

Problem 52. Consider the two spin-1 matrices

$$L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\theta, \phi \in \mathbb{R}$. Calculate

$$T(\theta,\phi) = \exp(-i\phi L_3)\exp(-i\theta L_2).$$

Is $T(\theta,\phi)$ an elements of $SO(3,\mathbb{R}$?

Problem 53. Let $\alpha \in \mathbb{R}$. Consider the spin matrix

$$S_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

(i) Find

$$A(\alpha) = \exp(-i\alpha S_2).$$

(ii) Calculate

$$B = \left.\frac{dA(\alpha)}{d\alpha}\right|_{\alpha=0}.$$

Then find

$$C(\alpha) = \exp(\alpha B).$$

Discuss.
(iii) Find

$$D(\alpha) = \exp(-i\alpha S_2 \otimes S_2).$$

(iv) Calculate

$$E = \left.\frac{dD(\alpha)}{d\alpha}\right|_{\alpha=0}.$$

Then find

$$G(\alpha) = \exp(\alpha E).$$

Discuss.

Problem 54. Let $\alpha, \theta \in \mathbb{R}$. Do the matrices

$$\begin{pmatrix} \cosh(\alpha) & e^{i\theta}\sinh(\alpha) \\ e^{-i\theta}\sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$$

form a group under matrix multiplication? Are the matrices unitary?

Problem 55. Let $x, y, z \in \mathbb{Z}$. Do the matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

form a group under matrix multiplication?

Problem 56. Find the group generated by the 3×3 matrices

$$G_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Set $G_0 = G_1^2 = I_3$.

Problem 57. (i) Show that the matrix

$$U = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$$

is unitary.
(ii) Find the eigenvalues and eigenvectors of U.
(iii) Find the group generated by U. Find the group generated by $U \otimes U$.

Problem 58. Consider the six 3×3 permutation matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Find two of these permutation matrices which generate all six permutation matrices.

Problem 59. Consider the six 3×3 permutation matrices. Which two of the matrices generate all the other ones.

Problem 60. (i) Find all invertible 2×2 matrices over $\mathbb{R}$ such that

$$T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} T^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(ii) Do these matrices form a group?

Problem 61. (i) Show that the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

are similar. This means find an invertible 2×2 matrix S, i.e. $S \in GL(2,\mathbb{R})$, such that $SAS^{-1} = B$.
(ii) Is there an invertible 2×2 matrix S such that

$$(S \otimes S)(A \otimes A)(S^{-1} \otimes S^{-1}) = B \otimes B.$$

Problem 62. Find the group generated by the two 2×2 matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

under matrix multiplication. Is the group commutative?

Problem 63. Find all $M \in GL(2,\mathbb{C})$ such that

$$M^{-1} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Thus we consider the invariance of the Pauli matrix σ_2. Show that these matrices form a group under matrix multiplication.

Problem 64. Find the group generated by the matrices

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

under matrix multiplication.

Problem 65. Let $\omega^3 = 1$. What group is generated by the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}$$

under matrix multiplication? Is the matrix

$$P = \frac{1}{3}(A + B + C)$$

a projection matrix?

Chapter 17

Lie Groups

Problem 1. (i) Find all 2×2 matrices A over $\mathbb{R}$ such that

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = 1$$

(i.e. A is an element of the Lie group $SL(2,\mathbb{R})$) and

$$A\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(ii) Do these matrices form a group under matrix multiplication?

Problem 2. The generators of the braid group B_3 are given by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus σ_1 and σ_2 are elements of the Lie group $SL(2,\mathbb{R})$.
(i) Find σ_1^{-1} and σ_2^{-1}. Find $\sigma_1\sigma_2$ and $\sigma_1^{-1}\sigma_2$.
(ii) Is $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$?

Problem 3. Let $\alpha \in \mathbb{R}$. Consider the hermitian matrix which is an element of the noncompact Lie group $SO(1,1)$

$$A(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}.$$

Find the *Cayley transform*

$$B = (A - iI_2)(A + iI_2)^{-1}.$$

Note that B is a unitary matrix and therefore an element of the compact Lie group $U(n)$. Find $B(\alpha \to \infty)$.

Problem 4. If $A \in SL(2,\mathbb{R})$, then it can be uniquely be written in the form

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \exp \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Find this decomposition for the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Problem 5. The unit sphere

$$S^3 := \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \, : \, \sum_{j=1}^{4} x_j^2 = 1 \}$$

we identitify with the Lie group $SU(2)$

$$(x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

(i) Map the standard basis of $\mathbb{R}^4$ into $SU(2)$ and express these matrices using the Pauli spin matrices and the 2×2 identity matrix.
(ii) Map the Bell basis

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

into $SU(2)$ and express these matrices using the Pauli spin matrices and the 2×2 identity matrix.

Problem 6. Is the 3×3 matrix

$$O(\theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi & -\sin\phi & -\cos\theta\cos\phi \\ \sin\theta\sin\phi & \cos\phi & \cos\theta\sin\phi \\ \cos\theta & 0 & \sin\theta \end{pmatrix}$$

an element of the compact Lie group $SO(3)$?

Problem 7. Are the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

elements of $SL(3,\mathbb{R})$ and $SL(5,\mathbb{R})$, respectively? We have to test that $\det(A) = 1$ and $\det(B) = 1$.

Problem 8. (i) Let $\alpha \in \mathbb{R}$. Do the matrices

$$A(\alpha) = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}$$

form a group under matrix multiplication?
(i) Let $\alpha \in \mathbb{R}$. Do the matrices

$$A(\alpha) = \begin{pmatrix} \cosh\alpha & i\sinh\alpha \\ -i\sinh\alpha & \cosh\alpha \end{pmatrix}$$

form a group under matrix multiplication?

Problem 9. Consider the Lie group $SL(n,\mathbb{C})$, i.e. the $n \times n$ matrices over $\mathbb{C}$ with determinant 1. Can we find $A, B \in SL(n,\mathbb{C})$ such that $[A,B]$ is an element of $SL(n,\mathbb{C})$?

Problem 10. Consider the 2×2 matrices

$$A(\alpha) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Both are elements of the non-compact Lie group $SL(2,\mathbb{C})$. Can one finds $\alpha \in \mathbb{C}$ such that the commutator $[A(\alpha), B]$ is again an element of $SL(2,\mathbb{C})$?

Problem 11. (i) Let A, B be elements of $SL(n,\mathbb{R})$. Is $A \otimes B$ an element of $SL(n^2,\mathbb{R})$.
(ii) Let A, B be elements of $SL(n,\mathbb{R})$. Is $A \oplus B$ an element of $SL(2n,\mathbb{R})$.
(iii) Let A, B be elements of $SL(2,\mathbb{R})$. Is

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}$$

an element of $SL(4,\mathbb{R})$?

Problem 12. (i) The matrix

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

is an element of the Lie group $SO(2,\mathbb{R})$. Find the spectral decomposition of $A(\alpha)$.

(ii) The matrix

$$B(\beta) = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}$$

is an element of the Lie group $SO(1,1,\mathbb{R})$. Find the spectral decomposition of $B(\beta)$.
(iii) Find the spectral decomposition of $A(\alpha) \otimes B(\beta)$.

Problem 13. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that if $A^T = -A$, then $e^A \in O(n,\mathbb{C})$.

Problem 14. The Lie group $SU(2)$ is defined by

$$SU(2) := \{ U \;\, 2 \times 2\,\text{matrix} \,:\, UU^* = I_2,\ \det U = 1 \}.$$

Let (3-sphere)

$$S^3 := \{ (x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \;:\; x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}.$$

Show that $SU(2)$ can be identified as a real manifold with the 3-sphere S^3.

Problem 15. Both

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad B(\beta) = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}$$

are elements of the Lie group $SL(2,\mathbb{R})$. Are

$$A(\alpha) \otimes B(\beta), \quad A(\alpha) \oplus B(\beta), \quad A(\alpha) \star B(\beta)$$

elements of $SL(4,\mathbb{R})$?

Problem 16. The Lie group $SU(1,1)$ consists of all 2×2 matrices T over the complex numbers with

$$TMT^* = M, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \det(T) = 1.$$

Find the conditions on $\xi_0, \xi_1, \xi_2, \xi_3 \in \mathbb{R}$ such that

$$T = \begin{pmatrix} \xi_0 + i\xi_3 & \xi_1 + i\xi_2 \\ \xi_1 - i\xi_2 & \xi_0 - i\xi_3 \end{pmatrix}$$

is an element of $SU(1,1)$.

Problem 17. Let n be an integer with $n \geq 2$. Let p, q be integers with $p, q \geq 1$ and $n = p + q$. Let I_p be the $p \times p$ identity matrix and I_q be the $q \times q$ identity matrix. Let

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \equiv I_p \oplus (-I_q).$$

The Lie group $O(p, q)$ is the set of all $n \times n$ matrices defined by

$$O(p,q) := \{ A \in GL(n, \mathbb{R}) \, : \, A^T I_{p,q} A = I_{p,q} \}.$$

Show that this is the group of all invertible linear maps of $\mathbb{R}^n$ that preserves the quadratic form

$$\sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{n} x_j y_j.$$

Problem 18. Consider the Pauli spin matrices σ_1, σ_2, σ_3. Then $i\sigma_1$, $i\sigma_2$, $i\sigma_3$ are elements of the Lie group $SU(2)$. Consider the unitary matrix (constructed from the four Bell states)

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}.$$

Show that

$$U(i\sigma_1 \otimes i\sigma_1)U, \quad U(i\sigma_2 \otimes i\sigma_2)U, \quad U(i\sigma_3 \otimes i\sigma_3)U$$

are elements of the Lie group $SO(4)$.

Problem 19. Consider the differentiable manifold

$$S^3 = \{ (x_1, x_2, x_3, x_4) \, : \, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}.$$

(i) Show that the matrix

$$U(x_1, x_2, x_3, x_4) = -i \begin{pmatrix} x_3 + ix_4 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 + ix_3 \end{pmatrix}$$

is unitary. Show that the matrix is an element of $SU(2)$.
(ii) Consider the parameters (θ, ψ, ϕ) with $0 \leq \theta < \pi$, $0 \leq \psi < 4\pi$, $0 \leq \phi < 2\pi$. Show that

$$x_1(\theta, \psi, \phi) + ix_2(\theta, \psi, \phi) = \cos(\theta/2) e^{i(\psi+\phi)/2}$$
$$x_3(\theta, \psi, \phi) + ix_4(\theta, \psi, \phi) = \sin(\theta/2) e^{i(\psi-\phi)/2}$$

is a parametrization. Thus the matrix given in (i) takes the form

$$-i\begin{pmatrix} \sin(\theta/2)e^{i(\psi-\phi)/2} & \cos(\theta/2)e^{-i(\psi+\phi)/2} \\ \cos(\theta/2)e^{i(\psi+\phi)/2} & -\sin(\theta/2)e^{-i(\psi-\phi)/2} \end{pmatrix}.$$

(iii) Let $(\xi_1, \xi_2, \xi_3) = (\theta, \psi, \phi)$ with $0 \leq \theta < \pi$, $0 \leq \psi < 4\pi$, $0 \leq \phi < 2\pi$. Show that

$$\frac{1}{24\pi^2}\int_0^\pi d\theta \int_0^{4\pi} d\psi \int_0^{2\pi} d\phi \sum_{j,k,\ell=1}^{3} \epsilon_{jk\ell}\mathrm{tr}(U^{-1}\frac{\partial U}{\partial \xi_j}U^{-1}\frac{\partial U}{\partial \xi_k}U^{-1}\frac{\partial U}{\partial \xi_\ell}) = 1$$

where $\epsilon_{123} = \epsilon_{321} = \epsilon_{132} = +1$, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$ and 0 otherwise.
(iv) Consider the metric tensor field

$$g = dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3 + dx_4 \otimes dx_4.$$

Using the parametrization show that

$$g_{S^3} = \frac{1}{4}(d\theta \otimes d\theta + d\psi \otimes d\psi + d\phi \otimes d\phi + \cos(\theta)d\psi \otimes d\phi + \cos(\theta)d\phi \otimes d\psi).$$

(v) Consider the differential one forms e_1, e_2, e_3 defined by

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -x_4 & -x_3 & x_2 & x_1 \\ x_3 & -x_4 & -x_1 & x_2 \\ -x_2 & x_1 & -x_4 & x_3 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{pmatrix}.$$

Show that

$$g_{S^3} = de_1 \otimes de_1 + de_2 \otimes de_2 + de_3 \otimes de_3.$$

(vi) Show that

$$de_j = \sum_{k,\ell=1}^{3} \epsilon_{jk\ell} e_k \wedge e_\ell$$

i.e. $de_1 = 2e_2 \wedge e_3$, $de_2 = 2e_3 \wedge e_1$, $de_3 = 2e_1 \wedge e_2$.

Problem 20. The Lie group $SU(2,2)$ is defined as the group of transformation on the four dimensional complex space $\mathbb{C}^4$ leaving invariant the indefinite quadratic form

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2.$$

The Lie algebra $su(2,2)$ is defined as the 4×4 matrices X with trace 0 and $X^*L + LX = 0_4$, where L is the 4×4 matrix

$$L = \begin{pmatrix} -I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix}.$$

Is

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

an element of the Lie algebra $su(2,2)$. Find $\exp(zX)$. Discuss.

Problem 21. (i) Can any element of the Lie group $SU(1,1)$ be written as

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} \cosh(\zeta/2) & \sinh(\zeta/2) \\ \sinh(\zeta/2) & \cosh(\zeta/2) \end{pmatrix} \begin{pmatrix} e^{i\beta/2} & 0 \\ 0 & e^{-i\beta/2} \end{pmatrix} ?$$

Each element in the this product is an element of the Lie group $SU(1,1)$.
(ii) Can any element of the compact Lie group $SU(2)$ be written as

$$\begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \begin{pmatrix} \cos(\beta/2) & \sin(\beta/2) \\ -\sin(\beta/2) & \cos(\beta/2) \end{pmatrix} ?$$

Problem 22. (i) Consider the non-compact Lie group $SO(1,1,\mathbb{R})$ with the element

$$A(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}.$$

Find the inverse of $A(\alpha)$. Find the eigenvalues and eigenvectors of $A(\alpha)$.
(ii) Let $\oplus$ be the direct sum. Find the determinant, eigenvalues and normalized eigenvectors of the 3×3 matrix

$$(1) \oplus A(\alpha).$$

(iii) Find the determinant, eigenvalues and normalized eigenvectors of the matrix

$$\begin{pmatrix} \cosh(\alpha) & 0 & \sinh(\alpha) \\ 0 & 1 & 0 \\ \sinh(\alpha) & 0 & \cosh(\alpha) \end{pmatrix}.$$

Find the inverse of this matrix.

Problem 23. Let $\theta \in \mathbb{R}$.
(i) Is the 2×2 matrix

$$M_1(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

unitary? Prove or disprove. If so is the matrix an element of the Lie group $SU(2)$?

(ii) Is the 8×8 matrix

$$M_2(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

unitary? Prove or disprove. If so is the matrix an element of the Lie group $SU(4)$?
(iii) Let $\oplus$ be the direct sum. Is the 6×6 matrix

$$M_3(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

unitary? Prove or disprove. If so is the matrix an element of the Lie group $SU(4)$?

Problem 24. Write down two 2×2 matrices A and B which are elements of the Lie group $O(2, \mathbb{R})$ but **not** elements of $SO(2, \mathbb{R})$.
(i) Is AB an element of the Lie group $SO(2, \mathbb{R})$?
(ii) Is $A \otimes B$ an element of the Lie group $SO(4, \mathbb{R})$?
(iii) Is $A \oplus B$ an element of $SO(4, \mathbb{R})$?

Problem 25. Let $\alpha \in \mathbb{R}$. Consider the 2×2 matrix

$$A(\alpha) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\alpha} & -e^{\alpha} \\ e^{-\alpha} & e^{-\alpha} \end{pmatrix}.$$

(i) Find the trace, determinant, eigenvalues and normalized eigenvectors of the matrix $A(\alpha)$.
(ii) Calculate

$$X := \left. \frac{dA(\alpha)}{d\alpha} \right|_{\alpha=0}.$$

Find $\exp(\alpha X)$ and compare with $A(\alpha)$, i.e. is $\exp(\alpha X) = A(\alpha)$? Discuss.

Problem 26. Consider the matrix ($\alpha \in \mathbb{R}$)

$$A(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}.$$

(i) Is $A(\alpha)$ an element of $SO(2)$?
(ii) Consider the transformation

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A(\alpha) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Is $(x_1')^2 + (x_2')^2 = x_1^2 + x_2^2$? Prove or disprove.

(iii) We define

$$A(\alpha) \star A(\alpha) = \begin{pmatrix} \cos(\alpha) & 0 & 0 & \sin(\alpha) \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & \sin(\alpha) & -\cos(\alpha) & 0 \\ \sin(\alpha) & 0 & 0 & -\cos(\alpha) \end{pmatrix}.$$

Is $A(\alpha) \star A(\alpha)$ an element of $SO(4)$.
(iv) Is $A(\alpha) \otimes A(\alpha)$ an element of $SO(4)$?
(v) Find

$$X = \frac{dA(\alpha)}{d\alpha}\bigg|_{\alpha=0}$$

and then $B(\alpha) = \exp(\alpha X)$. Compare $B(\alpha)$ and $A(\alpha)$ and discuss.

Problem 27. Let $z = x + iy$ with $x, y \in \mathbb{R}$. Consider the 2×2 matrix

$$\begin{pmatrix} \cos(z) & -\sin(z) \\ \sin(z) & \cos(z) \end{pmatrix}$$

One has

$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$
$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

Let $x = 0$. Then we arrive at the matrix

$$M(y) = \begin{pmatrix} \cosh(y) & -i\sinh(y) \\ i\sinh(y) & \cosh(y) \end{pmatrix}.$$

(i) Do these matrices ($y \in \mathbb{R}$ form a group under matrix multiplication?
(ii) Calculate

$$X = \frac{d}{dy} M(y)\bigg|_{y=0}$$

and then $\exp(yX)$ with $y \in \mathbb{R}$. Discuss.

Problem 28. Let $\alpha, \beta \in \mathbb{R}$. Do the 3×3 matrices

$$A(\alpha, \beta) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\beta) & \sinh(\beta) \\ 0 & \sinh(\beta) & \cosh(\beta) \end{pmatrix}$$

form a group under matrix multiplication? For $\alpha = \beta = 0$ we have the indenity matrix.

Problem 29. Let $A, B \in SL(2, \mathbb{R})$.

(i) Is $\mathrm{tr}A = \mathrm{tr}A^{-1}$? Prove or disprove.
(ii) Is

$$\mathrm{tr}(AB) = \mathrm{tr}A\,\mathrm{tr}B - \mathrm{tr}(AB^{-1})\,?$$

Prove or disprove.

Problem 30. The group of complex rotations $O(n,\mathbb{C})$ is defined as the group of all $n \times n$ complex matrices O, such that $OO^T = O^TO = I_n$, where ^{T} means transpose. These transformations preserve the real scalar product

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_j y_j$$

so that $(O\mathbf{x}) \cdot O\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are complex vectors in general, i.e. $x_j, y_j \in \mathbb{C}$.
(i) Show that the matrix ($\alpha \in \mathbb{R}$)

$$O = \begin{pmatrix} \cosh\alpha & i\sinh\alpha \\ -i\sinh\alpha & \cosh\alpha \end{pmatrix}$$

is an element of $O(n,\mathbb{C})$.
(ii) Find the partial derivatives under complex orthogonal transformations $O \in O(n,\mathbb{C})$

$$w_j(\mathbf{x}) := (O\mathbf{x})_j = \sum_{k=1}^{n} O_{jk} x_k, \qquad j = 1, 2, \ldots, n$$

i.e. $\partial/\partial w_j$ with $j = 1, 2, \ldots, n$.

Problem 31. The Lie group $SL(2,\mathbb{C})$ consists of all 2×2 matrices over $\mathbb{C}$ with determinant equal to 1. The group is not compact. Explain why. The maximal compact Lie subgroup of $SL(2,\mathbb{C})$ is $SU(2)$. Give a 2×2 matrix A which is an element of $SL(2,\mathbb{C})$, but not an element of $SU(2)$. .

Problem 32. Consider the Lie group $SL(2,\mathbb{R})$, i.e. the set of all real 2×2 matrices with determinant equal to 1. A dynamical system in $SL(2,\mathbb{R})$ can be defined by

$$M_{k+2} = M_k M_{k+1} \qquad k = 0, 1, 2, \ldots$$

with the initial matrices $M_0, M_1 \in SL(2,\mathbb{R})$. Let $F_k := \mathrm{tr}M_k$. Is

$$F_{k+3} = F_{k+2}F_{k+1} - F_k \qquad k = 0, 1, 2, \ldots ?$$

Prove or disprove.
Hint. Use that property that for any 2×2 matrix A we have

$$A^2 - A\mathrm{tr}(A) + I_2 \det(A) = 0.$$

Chapter 18

Lie Algebras

Problem 1. Let L be a finite dimensional Lie algebra and $Z(L)$ the center of L. Show that $\mathrm{ad} : L \to g\ell(L)$ is a homomorphisim of the Lie algebra L with kernel $Z(L)$.

Problem 2. Consider the Lie algebra $s\ell(2,\mathbb{R})$ with the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that the Lie algebra $s\ell(2,\mathbb{R})$ has no proper nontrivial ideals.

Problem 3. Consider the Lie algebra $s\ell(2,\mathbb{R})$ with the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find a basis of $s\ell(2,\mathbb{R})$ with all basis elements are normal matrices.

Problem 4. Consider the 2×2 matrices over $\mathbb{R}$

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Calculate the commutator $C = [A, B]$ and check whether C can be written as a linear combination of A and B. If so we have a basis of a Lie algebra.

Problem 5. Show that $s\ell(2,\mathbb{F})$ has no non-trivial ideals if $\mathrm{char}(\mathbb{F}) \neq 2$.

Problem 6. Consider the simple Lie algebra $s\ell(2,\mathbb{R})$ and the basis E, F, H with the commutators

$$[E,F]=H, \quad [H,E]=2E, \quad [H,F]=-2F.$$

Let $U(s\ell(2,\mathbb{R}))$ be the universal enveloping algebra. Then any element of $U(s\ell(2,\mathbb{R}))$ can be expressed as a sum of product of the form $F^jH^kE^\ell$ where $j,k,\ell\geq 0$. Show that

$$EF^2=-2F+2FH+F^2E.$$

Hint: Utilize that $EF^2\equiv[E,F^2]+F^2E$.

Problem 7. A basis of the Lie algebra $s\ell(2,\mathbb{R})$ is given by

$$E=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Which of these matrices are nonnormal?
(ii) Use linear combinations to find a basis where all elements are normal matrices.

Problem 8. Study the Lie algebra $s\ell(2,\mathcal{F})$, where $\mathrm{char}\mathcal{F}=2$.

Problem 9. A Lie algebra is simple if it contains no nontrivial ideals and semi-simple if its contains no nontrivial abelian (commutative) ideals. Is the Lie algebra $s\ell(2,\mathbb{R})$ with the basis

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

simple?

Problem 10. Do the matrices

$$E_+=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_-=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

form a basis for the Lie algebra $s\ell(2,\mathbb{R})$.

Problem 11. Consider the Pauli spin matrices

$$\sigma_0=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Do the sixteen 4×4 matrices $(j=0,1,2,3)$

$$\begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_j \end{pmatrix}, \quad \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & -\sigma_j \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ -\sigma_j & 0_2 \end{pmatrix}$$

form a group under matrix multiplication?
(ii) Do the sixteen 4×4 matrices ($j = 0, 1, 2, 3$)

$$\begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_j \end{pmatrix}, \quad \begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & -\sigma_j \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ -\sigma_j & 0_2 \end{pmatrix}$$

form a Lie algebra under the commutator?

Problem 12. The isomorphism of the Lie algebras $s\ell(2,\mathbb{C})$ and $so(3,\mathbb{C})$ has the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & b-c & -i(b+c) \\ c-b & 0 & 2ia \\ i(b+c) & -2ia & 0 \end{pmatrix}.$$

Let $z \in \mathbb{C}$. Find

$$\exp\left(z\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right), \qquad \exp\left(z\begin{pmatrix} 0 & b-c & -i(b+c) \\ c-b & 0 & 2ia \\ i(b+c) & -2ia & 0 \end{pmatrix}\right).$$

Problem 13. In the decomposition of the simple Lie algebra $s\ell(3,\mathbb{R})$ one finds the 3×3 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & -a_{11}-a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$$

where $a_{jk}, b_{jk} \in \mathbb{R}$. Find the commutators $[A, A']$, $[B, B']$ and $[A, B]$. Discuss.

Problem 14. We know that

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is an ordered basis of the simple Lie algebra $s\ell(2,\mathbb{R})$ with

$$[X, H] = -2X, \quad [X, Y] = H, \quad [Y, H] = 2Y.$$

Consider

$$\widetilde{X} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find the commutators

$$[\widetilde{X}, \widetilde{H}], \quad [\widetilde{X}, \widetilde{Y}], \quad [\widetilde{Y}, \widetilde{H}].$$

Problem 15. The simple Lie algebra $s\ell(2,\mathbb{R})$ has a basis given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the commutation relations

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

The *universal enveloping algebra* $U(s\ell(2,\mathbb{R}))$ of the Lie algebra $s\ell(2,\mathbb{R})$ is the associative algebra with generators H, E, F and the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H.$$

Find a basis of the Lie algebra $s\ell(2,\mathbb{R})$ so that all matrices are invertible. Find the inverse matrices of these matrices. Give the commutation relations.

Problem 16. A *Chevalley basis* for the semisimple Lie algebra $s\ell(3,\mathbb{R})$ is given by

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $Y_j = X_j^T$ for $j = 1,2,3$. The Lie algebra has rank 2 owing to H_1, H_2 and $[H_1,H_2] = 0$. Another basis could be formed by looking at the linear combinations

$$U_j = X_j + Y_j, \qquad V_j = X_j - V_j.$$

(i) Find the table of the commutator.
(ii) Calculate the vectors of commutators

$$\begin{pmatrix} [H_1,X_1] \\ [H_2,X_1] \end{pmatrix}, \qquad \begin{pmatrix} [H_1,X_2] \\ [H_2,X_2] \end{pmatrix}, \qquad \begin{pmatrix} [H_1,X_3] \\ [H_2,X_3] \end{pmatrix}$$

and thus find the roots.

Problem 17. Given the spin matrices

$$s_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad s_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider the 4×4 matrices

$$\begin{pmatrix} 0_2 & s_- \\ 0_2 & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & 0_2 \\ s_+ & 0_2 \end{pmatrix}, \quad \begin{pmatrix} s_+ & 0_2 \\ 0_2 & s_+ \end{pmatrix}, \quad \begin{pmatrix} s_- & 0_2 \\ 0_2 & s_- \end{pmatrix}.$$

Calculate the commutators of these matrices and extend the set so that one finds a basis of a Lie algebra.

Problem 18. Let L_1 and L_2 be two Lie algebras. Let $\varphi : L_1 \to L_2$ be a Lie algebra homomorphism. Show that $\mathrm{im}(\varphi)$ is a Lie subalgebra of L_2 and $\ker(\varphi)$ is an ideal in L_1.

Problem 19. Let $m, n \in \{-1, 0, 1\}$. Classify the Lie algebra with the generators L_{-1}, L_0, L_1 given

$$i[L_n, L_m] = (n-m)L_{n+m}.$$

Problem 20. (i) The standard basis for the vector space of the 2×2 matrices is given by

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We define the star composition of two 2×2 matrices as the 4×4 matrix

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Show that the sixteen 4×4 matrices $E_{jk} \star E_{\ell m}$ $(j, k, \ell, m = 1, 2)$ form a basis in the vector space of the 4×4 matrices.
(ii) The matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis of the Lie algebra $s\ell(2, \mathbb{R})$. Do the nine 4×4 matrices

$$X \star X, \;\; X \star Y, \;\; X \star H, \;\; Y \star X, \;\; Y \star Y, \;\; Y \star H, \;\; H \star X, \;\; H \star Y, \;\; H \star H$$

form a basis of a Lie algebra?

Problem 21. Consider the Lie algebra of real-skew symmetric 3×3 matrices

$$A = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}.$$

Let R be a real orthogonal 3×3 matrix, i.e. $RR^T = I_3$. Show that RAR^T is a real-skew symmetric matrix.

Problem 22. The matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis of the simple Lie algebra $s\ell(2, \mathbb{R})$. Define the matrices

$$\Delta(H) = H \otimes I_2 + I_2 \otimes H, \quad \Delta(E) = E \otimes H^{-1} + H \otimes E, \quad \Delta(F) = F \otimes H^{-1} + H \otimes F.$$

Find the commutators

$$[\Delta(H), \Delta(E)], \quad [\Delta(H), \Delta(F)], \quad [\Delta(E), \Delta(F)].$$

Discuss.

Problem 23. The *Heisenberg group* is the set of upper 3×3 matrices of the form

$$H = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c can be taken from some (arbitrary) commutative ring.
(i) Find the inverse of H.
(ii) Given two elements x, y of a group G, we define the *commutator* of x and y, denoted by $[x, y]$ to be the element $x^{-1}y^{-1}xy$. If a, b, c are integers (in the ring $\mathbb{Z}$ of the integers) we obtain the discrete Heisenberg group H_3. It has two generators

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find

$$z = xyx^{-1}y^{-1}.$$

Show that $xz = zx$ and $yz = zy$.
(iii) The derived subgroup (or commutator subgroup) of a group G is the subgroup $[G, G]$ generated by the set of commutators of every pair of elements of G. Find $[G, G]$ for the Heisenberg group.
(iv) Let

$$A = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

and $a, b, c \in \mathbb{R}$. Find $\exp(A)$.

(v) The Heisenberg group is a simple connected Lie group whose Lie algebra consists of matrices

$$L = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the commutators $[L, L']$ and $[L, L'], L']]$, where $[L, L'] := LL' - L'L$.

Problem 24. Consider the 3×3 matrices

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show that the matrices form a basis of a Lie algebra.

Problem 25. Consider the two 2×2 matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where A is the Pauli spin matrix σ_3 and B the Pauli spin matrix σ_1. The two matrices A, B are linearly independent. Let A, B be the generators of a Lie algebra. Classify the Lie algebra generated.

Problem 26. The Lie group $SU(1,1)$ consists of all 2×2 matrices T over the complex numbers with

$$TMT^* = M, \quad M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \det(T) = 1.$$

Find the conditions on $\xi_0, \xi_1, \xi_2, \xi_3 \in \mathbb{R}$ such that

$$T = \begin{pmatrix} \xi_0 + i\xi_3 & \xi_1 + i\xi_2 \\ \xi_1 - i\xi_2 & \xi_0 - i\xi_3 \end{pmatrix}$$

is an element of $SU(1,1)$.

Problem 27. (i) Find the Lie algebra generated by the 2×2 matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

(ii) Find the Lie algebra generated by the 3×3 matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 28. Can one find nonzero 2×2 matrices L_1, L_2, L_3 over $\mathbb{C}$ such that

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = 0_2, \quad [L_3, L_1] = L_2.$$

Problem 29. The Lie group $SU(2,2)$ is defined as the group of transformation on the four dimensional complex space $\mathbb{C}^4$ leaving invariant the indefinite quadratic form

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2.$$

The Lie algebra $su(2,2)$ is defined as the 4×4 matrices X with trace 0 and $X^* L + LX = 0_4$, where L is the 4×4 matrix

$$L = \begin{pmatrix} -I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix}.$$

Is

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

an element of the Lie algebra $su(2,2)$. Find $\exp(zX)$. Discuss.

Problem 30. Show that a basis of the Lie algebra $s\ell(2, \mathbb{C})$ is given by

$$\frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 31. Consider the non-commutative two-dimensional Lie algebra with $[A, B] = A$ where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that the 4×4 matrices

$$\{A \otimes I_2 + I_2 \otimes A, \;\; B \otimes I_2 + I_2 \otimes B\}$$

also form a noncommutative two-dimensional Lie algebra under the commutator.

Problem 32. The $g\ell(1|1)$ superalgebra involves two even (denoted by h and z) and two odd (denoted by e, f) generators. The following commutation and anti-commutation relations hold

$$[z,e]=[z,f]=[z,h]=0, \quad [h,e]=e, \quad [h,f]=-f, \quad [e,f]_+=z$$

and $e^2=f^2=0$. Find a 2×2 matrix representation.

Problem 33. Consider the two 2×2 matrices

$$A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where A is the Pauli spin matrix σ_3 and B the Pauli spin matrix σ_1. The two matrices A, B are linearly independent. Let A, B be the generators of a Lie algebra. Classify the Lie algebra generated.

Problem 34. A basis of the simple Lie algebra $s\ell(2,\mathbb{R})$ is given by the traceless 2×2 matrices

$$X_1=\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2=\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_3=\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(i) Find the commutators $[X_1,X_2]$, $[X_2,X_3]$, $[X_3,X_1]$.
(ii) Let $z\in\mathbb{C}$. Find

$$e^{zX_1}, \quad e^{zX_2}, \quad e^{zX_3}.$$

(iii) Let $u,v,r\in\mathbb{R}$. Show that

$$\begin{aligned} g(u,v,r) &= e^{uX_3}e^{rX_2}e^{vX_3} \\ &= e^{r/2}\begin{pmatrix} -\sin(u/2)\sin(v/2) & \cos(v/2)\sin(u/2) \\ -\cos(u/2)\sin(v/2) & \cos(u/2)\cos(v/2) \end{pmatrix} \\ &\quad + e^{-r/2}\begin{pmatrix} \cos(u/2)\cos(v/2) & \cos(u/2)\sin(v/2) \\ -\cos(v/2)\sin(u/2) & -\sin(u/2)\sin(v/2) \end{pmatrix}. \end{aligned}$$

(iv) Find $g(u,v,r)^{-1}$.

Problem 35. Let E_{jk} $(j,k=1,2,3,4)$ be the standard basis in the vector space of 4×4 matrices. This means that E_{jk} is the matrix with $+1$ at entry (jk) (jth column and kth row) and 0 otherwise. Show that the 15 matrices

$$X_1=E_{12}, \quad X_2=E_{23}, \quad X_3=E_{13}, \quad X_4=E_{34}, \quad X_5=E_{24}, \quad X_6=E_{14}$$

$$Y_1=E_{21}, \quad Y_2=E_{32}, \quad Y_3=E_{31}, \quad Y_4=E_{43}, \quad Y_5=E_{42}, \quad Y_6=E_{41}$$

$$H_1 = \frac{1}{4}\mathrm{diag}(3,-1,-1,-1,-1),$$
$$H_2 = \frac{1}{2}\mathrm{diag}(1,1,-1,-1),$$
$$H_3 = \frac{1}{4}\mathrm{diag}(1,1,1,-3).$$

Show that these matrices form a basis (Cartan-Weyl basis) of the Lie algebra $s\ell(4,\mathbb{C})$.

Problem 36. Do the eight 3×3 skew-hermitian matrices

$$\Gamma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\Gamma_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Gamma_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},$$

$$\Gamma_7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \Gamma_8 = \frac{1}{\sqrt{6}}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}$$

together with the 3×3 unit matrix form an orthonormal basis in the vector space of 3×3 matrices over the complex number. Find all pairwise scalar products $\langle A, B\rangle := \mathrm{tr}(AB^*)$. Discuss.

Problem 37. The semisimple Lie algebra $s\ell(3,\mathbb{R}$ has dimension 8. The standard basis is given by

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_{13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the commutator table.

Problem 38. Consider a four dimensional vector space with basis e_1, e_2, e_3, e_4. Assume that the non-zero commutators are given

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3.$$

Do these relations define a Lie algebra? If so find the adjoint representation.

Problem 39. Let A, B, C be nonzero $n \times n$ matrices. Assume that $[A, B] = 0_n$ and $[C, A] = 0_n$. Can we conclude that $[B, C] = 0_n$? Is the Jacobi identity

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0_n$$

of any use?

Problem 40. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Applying the star operation $\star$ we obtain

$$\sigma_1 \star \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 \star \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_3 \star \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(i) Find

$$[\sigma_1, \sigma_2], \quad [\sigma_2, \sigma_3], \quad [\sigma_3, \sigma_1], \quad [\sigma_1 \star \sigma_1, \sigma_2 \star \sigma_2], \quad [\sigma_2 \star \sigma_2, \sigma_3 \star \sigma_3], \quad [\sigma_3 \star \sigma_3, \sigma_1 \star \sigma_1].$$

Discuss.

(ii) Find

$$[\sigma_1, \sigma_2]_+, \quad [\sigma_2, \sigma_3]_+, \quad [\sigma_3, \sigma_1]_+, \quad [\sigma_1 \star \sigma_1, \sigma_2 \star \sigma_2]_+, \quad [\sigma_2 \star \sigma_2, \sigma_3 \star \sigma_3]_+, \quad [\sigma_3 \star \sigma_3, \sigma_1 \star \sigma_1]_+.$$

Discuss.

Problem 41. Consider the semisimple Lie algebra $s\ell(n+1, \mathbb{F})$. Let $E_{i,j}$ ($i, j \in \{1, 2, \ldots, n+1\}$) denote the standard basis, i.e. $(n+1) \times (n+1)$ matrices with all entries zero except for the entry in the i-th row and j-th column which is one. We can form a Cartan-Weyl basis with

$$H_j := E_{j,j} - E_{j+1,j+1}, \qquad j \in \{1, 2, \ldots, n\}.$$

Show that $E_{i,j}$ are *root vectors* for $i \neq j$, i.e. there exists $\lambda_{H,i,j} \in \mathbb{F}$ such that

$$[H, E_{i,j}] = \lambda_{H,i,j} E_{i,j}$$

for all $H \in \text{span}\{ H_1, \ldots, H_n \}$.

Problem 42. Consider the vector space $\mathbb{R}^3$ and the *vector product* $\times$. The vector product is not associative. The *associator* of three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is defined by

$$\text{ass}(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) := (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} - \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

The associator measures the failure of associativity.
(i) Consider the unit vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find the associator.
(ii) Consider the normalized vectors

$$\mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Find the associator.

Problem 43. Consider vectors in the vector space $\mathbb{R}^3$ and the vector product. Consider the mapping of the vectors in $\mathbb{R}^3$ into 3×3 skew-symmetric matrices

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}.$$

Calculate the vector product

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \times \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$

and the commutator $[M_1, M_2]$, where

$$M_1 = \begin{pmatrix} 0 & c_1 & -b_1 \\ -c_1 & 0 & a_1 \\ b_1 & -a_1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & c_2 & -b_2 \\ -c_2 & 0 & a_2 \\ b_2 & -a_2 & 0 \end{pmatrix}.$$

Discuss.

Chapter 19

Inequalities

Problem 1. Let A be an $n \times n$ positive semidefinite matrix. Let B be an $n \times n$ positive definite matrix. Then we have *Klein's inequality*

$$\mathrm{tr}(A(\ln(A) - \ln(B))) \geq \mathrm{tr}(A - B).$$

(i) Let

$$A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Calculate the left-hand side and the right-hand side of the inequality.
(ii) When is the inequality an equality?

Problem 2. Let A, B be $n \times n$ hermitian matrices. Then (*Golden-Thompson-Symanzik inequality*)

$$\mathrm{tr} e^{A+B} \leq \mathrm{tr}(e^A e^B).$$

Let $A = \sigma_3$ and $B = \sigma_1$. Calculate the left and right-hand side of the inequality.

Problem 3. Let A, B, C be positive definite $n \times n$ matrices. Then (*Lieb inequality*)

$$\mathrm{tr}(e^{\ln(A) - \ln(B) + \ln(C)}) \leq \mathrm{tr} \int_0^\infty A(B + uI_n)^{-1} C (B + uI_n)^{-1} du.$$

(i) Let

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}.$$

Calculate the left-hand side and right-hand side of the inequality.
(ii) Give a sufficient condition such that one has an equality.

Problem 4. Let A be an $n \times n$ skew-symmetric matrix over $\mathbb{R}$. Show that

$$\det(I_n + A) \geq 1$$

with equality holding if and only if $A = 0$.

Problem 5. Let $\mathbf{v}$ be a normalized (column) vector in $\mathbb{C}^n$ and let A be an $n \times n$ hermitian matrix. Is

$$\mathbf{v}^* e^A \mathbf{v} \geq e^{\mathbf{v}^* A \mathbf{v}}$$

for all normalized $\mathbf{v}$? Prove or disprove.

Problem 6. Let A be an $n \times n$ matrix over $\mathbb{C}$ and $\mathbf{v}, \mathbf{u} \in \mathbb{C}^n$. Is

$$\|A\mathbf{v} - A\mathbf{u}\| \leq \|A\| \cdot \|\mathbf{v} - \mathbf{u}\|\,?$$

Problem 7. Let H be an $n \times n$ hermitian matrix and $\mathbf{v}$ be a normalized (column) vector in $\mathbb{C}^n$. Is

$$\mathbf{v}^* e^H \mathbf{v} \geq e^{\mathbf{v}^* H \mathbf{v}}\,?$$

Problem 8. Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that there exists nonnull vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbb{R}^n$ such that

$$\frac{\mathbf{x}_1^T A \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \leq \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \leq \frac{\mathbf{x}_2^T A \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2}$$

for every nonnull vector $\mathbf{x}$ in $\mathbb{R}^n$.

Problem 9. Let A be an $n \times n$ matrix over $\mathbb{C}$. Is

$$\|I_n + A\| \leq 1 + \|A\|\,?$$

Chapter 20

Braid Group

Let $n \geq 2$. The *braid group* $\mathcal{B}_n$ of n strings has $n-1$ generators $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ (pairwise distinct) satisfying the relations

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \quad \text{for} \quad j = 1, 2, \ldots, N-2 (Yang-Baxterrelation)$$
$$\sigma_j \sigma_k = \sigma_k \sigma_j \quad \text{for} \quad |j-k| \geq 2$$
$$\sigma_j \sigma_j^{-1} = \sigma_j^{-1} \sigma_j = e$$

where e is the identity element. Thus it is generated by elements σ_j (σ_j interchanges elements j and $j+1$). Thus actually one should write σ_{12}, σ_{23}, $\ldots$, σ_{n-1n} instead of $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. The braid group $\mathcal{B}_n$ is a generalization of the permutation group.

The word written in terms of letters, generators from the set

$$\{\, \sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1} \,\}$$

gives a particular braid. The length of the braid is the total number of used letters, while the minimal irreducible length (referred sometimes as the primitive word) is the shortest non-contractible length of a particular braid which remains after applying all the group relations given above.

Let $n \geq 2$. The *braid group* on n strings, denoted by B_n, is an abstract group defined via the presentation

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$$

with the relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } 1 \leq i \leq n-2. \end{aligned}$$

The generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ are called the standard generators of B_n.

The pure braid group denoted by P_n is defined as the kernel of the homomorphism $B_n \to S_n$ defined by $\sigma_i \to (i, i+1)$, $(1 \leq i \leq n-1$. It is finitely generated by the elements

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \quad 1 \leq i < j \leq n.$$

Problem 1. Consider the braid group $\mathcal{B}_5$ with the generators σ_1, σ_2, σ_3, σ_4. Simplify

$$\sigma_1^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2\sigma_3\sigma_2\sigma_3^{-1}\sigma_4\sigma_1.$$

Problem 2. Consider the braid group $\mathcal{B}_3$. A faithful representation for the generators σ_1 and σ_2 is

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Both are elements of $SL(2,\mathbb{Z})$. Find the inverse of σ_1 and σ_2. Do the elements σ_1, σ_2, σ_1^{-1}, σ_2^{-1} and the 2×2 identity matrix form a group under matrix multiplication?

Problem 3. Consider the unitary 2×2 matrices

$$A(\theta) = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \qquad B(\theta) = \begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{pmatrix}$$

where $\theta \in \mathbb{R}$.
(i) Find the conditions on $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ such that (braid like relation)

$$A(\theta_1)B(\theta_2)A(\theta_3) = B(\theta_3)A(\theta_2)B(\theta_1).$$

(ii) Do the matrices $A(\theta)$ form a group under matrix multiplication?
(iii) Do the matrices $B(\theta)$ form a group under matrix multiplication?

Problem 4. Find all invertible 2×2 matrices A, B such that (*braid-like relation*)

$$ABA = BAB.$$

Problem 5. Can one find 2×2 matrices A and B with $[A, B] \neq 0$ and satisfying the braid-like relation

$$ABBA = BAAB.$$

Problem 6. (i) Do the 2×2 unitary matrices

$$A = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & ie^{-i\pi/4} \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

satisfy the *braid-like relation*

$$ABA = BAB.$$

(ii) Find the smallest $n \in \mathbb{N}$ such that $A^n = I_2$.
(iii) Find the smallest $m \in \mathbb{N}$ such that $B^m = I_2$.

Problem 7. If V and W are matrices of the same order, then their Schur product $V \bullet W$ is defined by (entrywise multiplication)

$$(V \bullet W)_{j,k} := V_{j,k} W_{j,k}.$$

If all entries of V are nonzero, then we say that X is Schur invertible and define its Schur inverse, $V^{(-)}$, by $V^{(-)} \bullet V = J$, where J is the matrix with all 1's.

The vector space $M_n(\mathbb{F})$ of $n \times n$ matrices acts on itself in three distinct ways: if $C \in M_n(\mathbb{F})$ we can define endomorphisms X_C, Δ_C and Y_C by

$$X_C M := CM, \qquad \Delta_C M := C \bullet M, \qquad Y_C := MC^T.$$

Let A, B be $n \times n$ matrices. Assume that X_A is invertible and Δ_B is invertible in the sense of Schur. Note that X_A is invertible if and only if A is, and Δ_B is invertible if and only if the Schur inverse $B^{(-)}$ is defined. We say that (A, B) is a *one-sided Jones pair* if

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B.$$

We call this the *braid relation*. Give an example for a one-sided Jones pair.

Problem 8. The braid linking matrix B is a square symmetric $k \times k$ matrix defined by $B = (b_{ij})$ with b_{ii} the sum of half-twists in the i-th branch, b_{ij} the sum of the crossings between the i-th and the j-th branches of the ribbon graph with standard insertion. Thus the i-th diagonal element of B is the local torsion of the i-th branch. The off-diagonal elements of B are twice the linking numbers of the ribbon graph for the i-th and j-th branches. Consider the braid linking matrix

$$B = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Discuss. Draw a graph.

Problem 9. Consider the five 4×4 matrices

$$B_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad B_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Is

$$B_1B_2B_1 = B_2B_1B_2, \quad B_2B_3B_2 = B_3B_2B_3, \quad B_3B_4B_3 = B_4B_3B_4, \quad B_4B_5B_4 = B_5B_4B_5\,?$$

Problem 10. Let $n \geq 3$ and let $\sigma_1, \ldots, \sigma_{n-1}$ be the generators. The *braid group* B_n on n-strings where $n \geq 3$ has a finite presentation of B_n given by

$$\langle \sigma_1, \ldots, \sigma_{n-1} \,:\, \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$$

where $1 \leq i, j < n-1$, $|i-j| > 1$ or $j = n-1$. Here $\sigma_i\sigma_j = \sigma_j\sigma_i$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ are called the braid relations. The second one is also called the Yang-Baxter equation.
(i) Consider B_3, $a = \sigma_1\sigma_2\sigma_1$ and $b = \sigma_1\sigma_2$. Show that $a^2 = b^3$.
(ii) Consider B_3. The cosets $[\sigma_1]$ of σ_1 and $[\sigma_2]$ of σ_2 map to the 2×2 matrices

$$[\sigma_1] \mapsto R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad [\sigma_2] \mapsto L^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

where $L, R \in SL(2,\mathbb{Z})$. Thus $L^{-1}, R^{-1} \in SL(2,\mathbb{Z})$. Show that

$$RL^{-1}R = L^{-1}RL^{-1}.$$

Problem 11. (i) Do the matrices

$$S_1 = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}$$

satisfy the braid-like relation $S_1S_2S_1 = S_2S_1S_2$.
(ii) Do the matrices $S_1 \otimes S_1$ and $S_2 \otimes S_2$ satisfy the braid-like relation

$$(S_1 \otimes S_1)(S_2 \otimes S_2)(S_1 \otimes S_1) = (S_2 \otimes S_2)(S_1 \otimes S_1)(S_2 \otimes S_2)\,?$$

Problem 12. Consider the five 4×4 matrices

$$B_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$B_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{pmatrix}, \quad B_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Are the matrices unitary? Is (braid-like relation)

$$B_j B_{j+1} B_j = B_{j+1} B_j B_{j+1}, \quad j = 1, 2, 3, 4$$

Problem 13. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the 4×4 matrix

$$R = a(\lambda, \mu)\sigma_1 \otimes \sigma_1 + b(\lambda, \mu)(\sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)$$

where

$$a(\lambda, \mu) = \frac{1}{4}\frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2}, \qquad b(\lambda, \mu) = \frac{1}{2}\frac{\lambda\mu}{\lambda^2 - \mu^2}.$$

Does R satisfy the braid like relation

$$(R \otimes I_2)(I_2 \otimes R)(R \otimes I_2) = (I_2 \otimes R)(R \otimes I_2)(I_2 \otimes R)\,?$$

Problem 14. Consider the 4×4 matrix

$$R(u) = \begin{pmatrix} a(u) & 0 & 0 & d(u) \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ d(u) & 0 & 0 & a(u) \end{pmatrix}.$$

What is the condition on $a(u)$, $b(u)$, $c(u)$, $d(u)$ such that $R(u)$ satisfies the braid like relation

$$(R(u) \otimes I_2)(I_2 \otimes R(u))(R(u) \otimes I_2) = (I_2 \otimes R(u))(R(u) \otimes I_2)(I_2 \otimes R(u))\,?$$

Problem 15. Consider the five 4×4 matrices

$$B_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

$$B_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{pmatrix}, \quad B_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Show that

$$B_1B_2B_1 = B_2B_1B_2, \quad B_2B_3B_2 = B_3B_2B_3, \quad B_3B_4B_3 = B_4B_3B_4, \quad B_4B_5B_4 = B_5B_4B_5$$

and

$$B_1B_3 = B_3B_1, \quad B_2B_4 = B_4B_2, \quad B_3B_5 = B_5B_3.$$

Problem 16. (i) Let R be an $m \times m$ matrix and I_n be the $n \times n$ identity matrix. Consider the braid-like relation

$$(R \otimes I_n)(I_n \otimes R)(R \otimes I_n) = (I_n \otimes R)(R \otimes I_n)(I_n \otimes R).$$

Let $m = 4$ and $n = 2$. Does

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

satisfy the braid like relation?
(ii) Does

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

satisfy the braid like relation? Or what is the condition on a, b, c, d so that the condition is satisfied?
(iii) Does

$$R = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$$

satisfy the braid like relation? Or what is the condition on a, b, c, d so that the condition is satisfied?

Problem 17. Let T be an $n \times n$ matrix and R be an $n^2 \times n^2$ matrix. Consider the equation

$$R(T \otimes T) = (T \otimes T)R.$$

(i) Let $n = 2$ and

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find all 4×4 matrices R which satisfy the equation.

(ii) Let $n = 2$ and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find all T which satisfy the equation.

Problem 18. (i) Let A, B be invertible $n \times n$ matrices with $AB \neq BA$. Assume that

$$ABA = BAB \quad \text{and} \quad ABBA = I_n.$$

Show that $A^4 = B^4 = I_n$.
(ii) Find all 2×2 matrices A and B which satisfy the conditions given in (i).
(iii) Find all 3×3 matrices A and B which satisfy the conditions given in (i).
(iv) Find all 4×4 matrices A and B which satisfy the conditions given in (i).

Problem 19. Find all 4×4 matrices A, B with $[A, B] \neq 0_4$ satisfying the conditions

$$ABA = BAB, \qquad ABBA = I_4.$$

The first condition is the braid relation and the second condition $ABBA = I_4$ runs under "Dirac game".

Problem 20. (i) Do the matrices

$$S_1 = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}$$

satisfy the braid-like relation $S_1 S_2 S_1 = S_2 S_1 S_2$.
(ii) Do the matrices $S_1 \otimes S_1$ and $S_2 \otimes S_2$ satisfy the braid-like relation

$$(S_1 \otimes S_1)(S_2 \otimes S_2)(S_1 \otimes S_1) = (S_2 \otimes S_2)(S_1 \otimes S_1)(S_2 \otimes S_2)\,?$$

Problem 21. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Is

$$AB^{-1}A = B^{-1}AB^{-1}\,?$$

Problem 22. Consider the braid group $\mathcal{B}_n$. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\dots$, $\mathbf{e}_n$ be the standard basis in $\mathbb{R}^n$. Then $\mathbf{u} \in \mathbb{R}^n$ can be written as

$$\mathbf{u} = \sum_{k=1}^{n} c_k \mathbf{e}_k.$$

Consider the linear operators B_j $(j = 1, 2, \dots, n-1)$ in $\mathbb{R}^n$ $(\alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha, \gamma \neq 0)$ defined by

$$B_j \mathbf{u} := c_1 \mathbf{e}_1 + \cdots + (\alpha c_{j+1} + \beta)\mathbf{e}_j + (\gamma c_j + \delta)\mathbf{e}_{j+1} + \cdots + c_n \mathbf{e}_n$$

and the corresponding inverse operator B_j^{-1}

$$B_j^{-1}\mathbf{u} = c_1 \mathbf{e}_1 + \cdots + \frac{1}{\gamma}(c_{j+1} - \delta)\mathbf{e}_j + \frac{1}{\alpha}(c_j - \beta)\mathbf{e}_{j+1} + \cdots + c_n \mathbf{e}_n.$$

Show that the linear operators B_j satisfy the braid condition

$$B_j B_{j+1} B_j = B_{j+1} B_j B_{j+1}$$

if $\gamma\beta + \delta = \alpha\delta + \beta$.

Problem 23. Let $\mathcal{B}_n$ denote the braid group on n strands. $\mathcal{B}_n$ is generated by the elementary braids $\{ b_1, b_2, \dots, b_{n-1} \}$ with the *braid relations*

$$b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1}, \qquad 1 \leq j < n-1,$$

$$b_j b_k = b_k b_j, \qquad |j-k| \geq 2.$$

Let $\{ \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \}$ denote the standard basis in $\mathbb{R}^n$. Then $\mathbf{u} \in \mathbb{R}^n$ can be written as

$$\mathbf{u} = \sum_{k=1}^{n} c_k \mathbf{e}_k, \qquad c_1, c_2, \dots, c_n \in \mathbb{R}.$$

Consider the operators B_j $(\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha, \gamma \neq 0)$ defined by

$$B_j \mathbf{u} := c_1 \mathbf{e_1} + \ldots + (\alpha c_{j+1} + \beta)\mathbf{e}_j + (\gamma c_{j+1} + \delta)\mathbf{e}_{j+1} + \ldots + c_n \mathbf{e}_n$$

and the corresponding inverse operation

$$B_j^{-1}\mathbf{u} := c_1 \mathbf{e_1} + \ldots + \frac{1}{\gamma}(c_{j+1} - \delta)\mathbf{e}_j + \frac{1}{\alpha}(\gamma c_j - \beta)\mathbf{e}_{j+1} + \ldots + c_n \mathbf{e}_n.$$

Use computer algebra to show that B_1, B_2, $\dots$, B_{n-1} saitisfy the braid condition

$$B_j B_{j+1} B_j \mathbf{u} = B_{j+1} B_j B_{j+1} \mathbf{u}$$

if

$$\gamma\beta + \delta = \alpha\delta + \beta.$$

Problem 24. Consider the braid group B_3 with the generators $\{\sigma_1, \sigma_2\}$ and the relation

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

Let $t \neq 0$. Show that a matrix representation is given by

$$\sigma_1 = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}$$

with

$$\sigma_1^{-1} = \begin{pmatrix} -1/t & 1/t \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -1/t \end{pmatrix}.$$

Let

$$\sigma = \sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}.$$

Find

$$f(t) = \det(\sigma - I_2).$$

Find minima and maxima of f.

Chapter 21

vec Operator

Problem 1. Consider the 2×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Let $B = A^T$. Thus B is a 3×2 matrix. Find the 6×6 permutation matrix P such that

$$\mathrm{vec}(B) = P\mathrm{vec}(A).$$

Problem 2. Let A be an $m \times n$ matrix over $\mathbb{C}$ and B be a $s \times t$ matrix over $\mathbb{C}$. Find the permutation matrix P such that

$$\mathrm{vec}(A \otimes B) = P(\mathrm{vec}(A) \otimes \mathrm{vec}(B)).$$

Problem 3. Let A be an $m \times n$ matrix over $\mathbb{C}$. Using

$$\mathrm{vec}_{m\times n} A := \sum_{j=1}^{n} \mathbf{e}_{j,n} \otimes (A\mathbf{e}_{j,n}) = (I_n \otimes A) \sum_{j=1}^{n} \mathbf{e}_{j,n} \otimes \mathbf{e}_{j,n}$$

and

$$\mathrm{vec}^{-1}_{m\times n}\mathbf{x} = \sum_{i=1}^{m} \sum_{j=1}^{n} \left((\mathbf{e}_{j,n} \otimes \mathbf{e}_{i,m})^* \, \mathbf{x} \right) \mathbf{e}_{i,m} \otimes \mathbf{e}^*_{j,n}.$$

Show that

$$\mathrm{vec}^{-1}_{m\times n}(\mathrm{vec}_{m\times n} A) = A.$$

Problem 4. Let A be an $m \times n$ matrix over $\mathbb{C}$ and B be a $s \times t$ matrix over $\mathbb{C}$. Show that

$$A \otimes B = \mathrm{vec}^{-1}_{ms\times nt}\left(L_{A,s\times t}\left(\mathrm{vec}_{s\times t} B\right)\right)$$

where

$$L_{A,s\times t} := (I_n \otimes I_t \otimes A \otimes I_s) \sum_{j=1}^{n} \mathbf{e}_{j,n} \otimes I_t \otimes \mathbf{e}_{j,n} \otimes I_s.$$

Problem 5. (i) Let $AX + XB = C$, where C is an $m \times n$ matrix over $\mathbb{R}$. What are the dimensions of A, B, and X?
(ii) Solve the equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + X \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

for the real valued matrix X.

Problem 6. Let A be an $m \times n$ matrix over $\mathbb{C}$ and B be an $s \times t$ matrix over $\mathbb{C}$. Define

$$R(A \otimes B) := \mathrm{vec}A\,(\mathrm{vec}B)^T.$$

Find an algebraic expression for R. Find

$$R\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right).$$

R is the reshaping operator.

Problem 7. Show that

$$\mathrm{tr}(ABCD) \equiv (\mathrm{vec}(D^T))(A \otimes C^T)\mathrm{vec}(B^T).$$

Problem 8. (i) Let K be a given $n \times n$ nonnormal matrix over $\mathbb{C}$. We want to find all unitary $n \times n$ matrices such that

$$UKU^* = K^*.$$

Now we can write $KU^* = U^*K^*$ and therefore

$$KU^* - U^*K^* = 0_n$$

where 0_n is the $n \times n$ zero matrix. This is a matrix equation (a special case of Sylvester) and using the vec-operator and Kronecker product we can cast the matrix equation into a vector equation. Write down this linear equation.

(ii) Apply it to the case $n = 2$ with

$$K = \begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}.$$

Problem 9. Let A, B be $n \times n$ matrices. Show that

$$\mathrm{vec}(A \bullet B) = \mathrm{vec}(A) \bullet \mathrm{vec}(B)$$

where $\bullet$ denotes the Hadamard product.

Problem 10. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$\mathrm{tr}(\mathrm{vec}(B)(\mathrm{vec}(A))^T) = \mathrm{tr}(A^T B).$$

Problem 11. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Consider the maps

$$\begin{aligned} f_1(A,B) &= A \times B \\ f_2(A,B) &= \mathrm{vec}(A)(\mathrm{vec}(B))^T \\ f_3(A,B) &= \mathrm{vec}(A)(\mathrm{vec}(B^T)^T \\ f_4(A,B) &= \mathrm{vec}(B)(\mathrm{vec}(A^T))^T. \end{aligned}$$

Is

$$\mathrm{tr}(f_1(A,B)) = \mathrm{tr}(f_2(A,B)), \quad \mathrm{tr}(f_2(A,B)) = \mathrm{tr}(f_3(A,B)), \quad \mathrm{tr}(f_3(A,B)) = \mathrm{tr}(f_4(A,B)).$$

Chapter 22

Star Product

Consider the 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

We define the composition (star product)

$$A \star B := \begin{pmatrix} b_{11} & 0 & 0 & b_{12} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ b_{21} & 0 & 0 & b_{22} \end{pmatrix}.$$

The extension to 4×4 matrix is: Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & a_{23} & b_{24} \\ b_{31} & a_{32} & a_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$

Then $A \star B$ is the 8×8 matrix

$$\begin{pmatrix} b_{11} & b_{12} & 0 & 0 & 0 & 0 & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 & b_{23} & b_{24} \\ 0 & 0 & a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ 0 & 0 & a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\ b_{31} & b_{32} & 0 & 0 & 0 & 0 & b_{33} & b_{34} \\ b_{41} & b_{42} & 0 & 0 & 0 & 0 & b_{43} & b_{44} \end{pmatrix}.$$

Problem 1. (i) What can be said about the trace of $A \star B$? What can be said about the determinant of $A \star B$?
(ii) Let A_1, A_2, A_3, A_4 be a basis in the vector space of 2×2 matrices over $\mathbb{C}$. Let B_1, B_2, B_3, B_4 be a basis in the vector space of 2×2 matrices over $\mathbb{C}$. Do the 16 matrices $A_j \star B_k$ $(j, k = 1, 2, 3, 4)$ form a basis in the vector space of 4×4 matrices?
(iii) Given the eigenvalues of A and B. What can be said about the eigenvalues of $A \star B$?
(iv) Can one find 4×4 permutation matrices P and Q such that

$$P(A \star B)Q = A \oplus B\,?$$

Here $\oplus$ denotes the direct sum

Problem 2. Consider the 2×2 matrices A, B over $\mathbb{C}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Let $A \star B$ be the star product.
(i) Answer the following questions: Let A and B be normal matrices. Is $A \star B$ normal. Let A and B be invertible matrices. Is $A \star B$ an invertible matrix? Let A and B be unitary matrices. Is $A \star B$ a unitary matrix? Let A and B be nilpotent matrices. Is $A \star B$ a nilpotent matrix? Answer these questions also for $A \star A$.
(ii) What is the conditions on A and B such that

$$A \star B = A \otimes B\,?$$

Problem 3. Let A, B be 2×2 matrices. We define

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Can one find a permutation matrix such that

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix} = P(A \star B)P^T.$$

Problem 4. (i) Let A, B be 2×2 matrices. Let $A \star B$ be the star product. The 2×2 matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a group under matrix multiplication. Do the four 4×4 matrices $A \star A$, $A \star B$, $B \star A$, $B \star B$ form a group under matrix multiplication?
(ii) Let G be a finite group represented by 2×2 matrices. Let the order be n with the group elements $g_1 = e$, g_2, $\ldots$, g_n. Do the 4×4 matrices $g_j \star g_k$ $(j, k = 1, \ldots, n)$ form a group under matrix multiplication.

Problem 5. Let A, B be 2×2 matrices. Let $A \star B$ be the star product. Show that one can find a 4×4 permutation matrix P such that

$$P(A \star B)P^T = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}.$$

Problem 6. Let A, B be invertible 2×2 matrices. Let $A \star B$ be the star product. Is $A \star B$ invertible?

Problem 7. (i) The 2×2 matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a group under matrix multiplication. Do the four 4×4 matrices $A \star A$, $A \star B$, $B \star A$, $B \star B$ form a group under matrix multiplication?
(ii) Let G be a finite group represented by 2×2 matrices. Let the order be n with the group elements $g_1 = e$, g_2, $\ldots$, g_n. Do the 4×4 matrices $g_j \star g_k$ $(j, k = 1, \ldots, n)$ form a group under matrix multiplication.

Problem 8. Among others one can form a 4×4 matrix from two 2×2 matrices A and B using the direct sum $A \oplus B$, the Kronecker product $A \otimes B$ and the star product

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Given the eigenvalues and eigenvectors of A and B. What can be said about the eigenvalues and eigenvectors of $A \oplus B$, $A \otimes B$, $A \star B$?

Problem 9. (i) Let A, B be invertible 2×2 matrices. Is $A \star B$ invertible?
(ii) Let U and V be elements of $SU(2)$. Is $U \star V$ an element of $SU(4)$?
(iii) Let X and Y be elements of $SL(2, \mathbb{R})$. Is $X \star Y$ an element of $SL(4, \mathbb{R})$?

Problem 10. Let A, B be normal 2×2 matrices with eigenvalues λ_1, λ_2 and μ_1, μ_2, respectively. What can be said about the eigenvalues of $A \star B - B \star A$?

Problem 11. (i) Given the eigenvalues of A and B. What can be said about the eigenvalues of $A \star B$?
(ii) Can one find 4×4 permutation matrices P and Q such that

$$P(A \star B)Q = A \oplus B\,?$$

Here $\oplus$ denotes the direct sum

Problem 12. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where A_j, B_j $(j = 1, 2, 3, 4)$ are 2×2 matrices. We define the product

$$A \star B := \begin{pmatrix} A_1 & 0_2 & 0_2 & A_2 \\ 0_2 & B_1 & B_2 & 0_2 \\ 0_2 & B_3 & B_4 & 0_2 \\ A_3 & 0_2 & 0_2 & A_4 \end{pmatrix}$$

where 0_2 is the 2×2 zero matrix. Thus $A \star B$ is an 8×8 matrix.
(i) Assume that A and B are invertible. Is $A \star B$ invertible?
(ii) Assume that A, B are unitary. Is $A \star B$ unitary?

Problem 13. Let A, B be the 4×4 matrices

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where A_j, B_j $(j = 1, 2, 3, 4)$ are 2×2 matrices. We define the product

$$A \star B := \begin{pmatrix} A_1 & 0_2 & 0_2 & A_2 \\ 0_2 & B_1 & B_2 & 0_2 \\ 0_2 & B_3 & B_4 & 0_2 \\ A_3 & 0_2 & 0_2 & A_4 \end{pmatrix}$$

where 0_2 is the 2×2 zero matrix. Thus $A \star B$ is an 8×8 matrix.
(i) Assume that A and B are invertible. Is $A \star B$ invertible?
(ii) Assume that A, B are unitary. Is $A \star B$ unitary?

Problem 14. Let A, B be 3×3 matrices. We define the composition

$$A \Diamond B := \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & a_{13} \\ 0 & b_{11} & b_{12} & b_{13} & 0 \\ a_{21} & b_{21} & a_{22}b_{22} & b_{23} & a_{23} \\ 0 & b_{31} & b_{32} & b_{33} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{33} \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Find the eigenvalues of M and $M \Diamond M$.

Problem 15. Let P and Q be 2×2 projection matrices. Is the 4×4 matrix $P \star Q$ a projection matrix? Apply it to $P \star P$ where

$$P = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 16. Let A and B be 2×2 matrices and $A \star B$ the star product. Let λ_1, λ_2 be the eigenvalues of A and μ_1, μ_2 be the eigenvalues of B, respectively. What can be said about the eigenvalues of $A \star B$? Apply the result to the matrix

$$\begin{pmatrix} \cosh(\beta) & 0 & 0 & \sinh(\beta) \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ \sinh(\beta) & 0 & 0 & \cosh(\beta) \end{pmatrix}.$$

Problem 17. Among others one can form a 4×4 matrix from two 2×2 matrices A and B using the direct sum $A \oplus B$, the Kronecker product $A \otimes B$ and the star product $A \star B$. Given the eigenvalues and eigenvectors of A and B. What can be said about the eigenvalues and eigenvectors of $A \oplus B$, $A \otimes B$, $A \star B$?

Problem 18. The matrix

$$A(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

admits the eigenvalues $\lambda_+(\alpha) = e^{i\alpha}$ and $\lambda_-(\alpha) = e^{-i\alpha}$ with the corresponding normalized eigenvectors

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Let

$$A(\alpha) \star A(\alpha) = \begin{pmatrix} \cos(\alpha) & 0 & 0 & -\sin(\alpha) \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ \sin(\alpha) & 0 & 0 & \cos(\alpha) \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of $A(\alpha) \star A(\alpha)$.

Problem 19. Consider the star product $A \star B$ of two 2×2 matrices A and B and the product

$$A \diamond B = \begin{pmatrix} b_{11} & 0 & 0 & b_{12} \\ b_{21} & 0 & 0 & b_{22} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \end{pmatrix}.$$

Is there a 4×4 permutation matrix P such that $P(A \star B)P^T = A \diamond B$?

Problem 20. Let A be a 3×3 matrix and B be a 2×2 matrix. We define

$$A \star B := \begin{pmatrix} b_{11} & 0 & 0 & 0 & b_{12} \\ 0 & a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{31} & a_{32} & a_{33} & 0 \\ b_{21} & 0 & 0 & 0 & b_{22} \end{pmatrix}.$$

Let

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad B = I_3.$$

Find $A \star B$ and calculate the eigenvalues and normalized eigenvectors of $A \star I_3$.

Chapter 23

Nonnormal Matrices

A square matrix M over $\mathbb{C}$ is called normal if

$$MM^* = M^*M.$$

A square matrix M over $\mathbb{C}$ is called nonnormal if

$$MM^* \neq M^*M.$$

Examples for nonnormal matrices are

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If M is a nonnormal matrix, then M^* and M^T are nonnormal matrices. If M is nonnormal and invertible, then M^{-1} is nonnormal. If M is nonnormal, then $M + M^*$, MM^*, the commutator $[M, M^*]$ and the anti-commutator $[M, M^*]$ are normal matrices.

Problem 1. Let A be a nonnormal $n \times n$ matrix over $\mathbb{C}$.
(i) Show that $A + A^*$ is a normal matrix.
(ii) Show that AA^* is a normal matrix.

Problem 2. (i) Let $\theta \in \mathbb{R}$. Consider the matrix

$$A(\theta) = \begin{pmatrix} 0 & \sin(\theta) \\ \cos(\theta) & 0 \end{pmatrix}.$$

(i) Is the matrix $A(\theta)$ nonnormal for all θ?
(ii) What is the condition on θ that the matrix $A(\theta)$ is invertible?
(iii) Is the matrix $A(\theta) \otimes A(\theta)$ nonnormal for all θ?

Problem 3. Let A be a non-zero 2×2 matrix with $A^2 = 0_2$. Are these matrices nonnormal?

Problem 4. (i) Let $\theta \in \mathbb{R}$. Consider the matrix

$$A(\theta) = \begin{pmatrix} 0 & \sinh(\theta) \\ \cosh(\theta) & 0 \end{pmatrix}.$$

(i) Is the matrix $A(\theta)$ non-normal for all θ?
(ii) What is the condition on θ that the matrix $A(\theta)$ is invertible?

Problem 5. Let $z_1, z_2, z_3 \in \mathbb{C}$. Consider the 2×2 matrix

$$A(z_1, z_2, z_3) = \begin{pmatrix} z_1 & z_2 \\ z_3 & 0 \end{pmatrix}.$$

What is the condition on z_1, z_2, z_3 such that $A(z_1, z_2, z_3)$ is a normal matrix?

Problem 6. (i) Is $A \bullet A^*$ a normal matrix, where $\bullet$ denotes the Hadamard product?
(ii) Let U be a unitary $n \times n$ matrix. Show that UAU^* is also nonnormal.
(iii) Is the commutator $[A, A^*]$ a normal matrix? Is the anti-commutator $[A, A^*]_+$ a normal matrix?
(iv) Is $A \otimes A^*$ a normal matrix, where $\otimes$ denotes the Kronecker product?
(v) Is $A \oplus A^*$ a normal matrix, where $\oplus$ denotes the direct sum?
(vi) Let A also be invertible. Is A^{-1} also nonnormal?
(vii) Is $\exp(A)$ nonnormal?
(viii) Is $\sinh(A)$ nonnormal?
(ix) Is A^2 nonnormal?

Problem 7. Let $s_j := 2\sin(2\pi j/5)$ with $j = 1, 2, \ldots, 5$. Consider the 5×5 matrix

$$M = \begin{pmatrix} s_1 & 1 & 0 & 0 & -1 \\ -1 & s_2 & 1 & 0 & 0 \\ 0 & -1 & s_3 & 1 & 0 \\ 0 & 0 & -1 & s_4 & 1 \\ 1 & 0 & 0 & -1 & s_5 \end{pmatrix}.$$

Show that the matrix is nonnormal. Find the eigenvalues and eigenvectors. Is the matrix diagonalizable?

Problem 8. Let $\epsilon \neq 0$. Show that the matrix

$$A = \begin{pmatrix} 1 & \epsilon \\ 0 & -1 \end{pmatrix}$$

is nonnormal. Give the eigenvalues and eigenvectors.

Problem 9. Let $a > 0$, $b \geq 0$ and $\phi \in [0, \pi]$. What are the conditions a, b, ϕ such that

$$A(a, b, \phi) = \begin{pmatrix} 0 & a \\ e^{i\phi}b & 0 \end{pmatrix}$$

is a normal matrix?

Problem 10. Let A, B be nonnormal matrices. Is $A \otimes B$ nonnormal? Is $A \oplus B$ nonnormal?

Problem 11. Can we conclude that an invertible matrix is normal?

Problem 12. Show that all non-diagonalizable matrices are nonnormal.

Problem 13. Show that not all nonnormal matrices are non-diagonalizable.

Problem 14. Find all 2×2 matrices over $\mathbb{C}$ which are nonnormal but diagonalizable.

Problem 15. Let A be a nonnormal invertible matrix. How do we construct a matrix B such that $A = \exp(B)$. Study first the two examples

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 1/2 \end{pmatrix}.$$

Both are elements of the Lie group $SL(2, \mathbb{R})$.

Problem 16. Let $a_{11}, a_{22}, \epsilon \in \mathbb{R}$. What is the condition on a_{11}, a_{22}, ϵ such that the matrix
$$A = \begin{pmatrix} a_{11} & e^{\epsilon} \\ e^{-\epsilon} & a_{22} \end{pmatrix}$$
is normal?

Problem 17. Let A be a nonnormal matrix. Is $[A, A^*]$ a normal matrix? Is $[A, A^*]_+$ a nonnormal matrix?

Problem 18. Let A be a normal $n \times n$ matrix and let S be an invertible nonnormal matrix. Is SAS^{-1} a normal matrix? Study the case that
$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$
Of course the eigenvalues of A and SAS^{-1} are the same.

Problem 19. Consider the invertible nonnormal matrix
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$
Is the matrix $A \otimes A^{-1}$ normal?

Problem 20. Consider the nonnormal matrix
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
Find $\sinh(A)$ and $\sinh^{-1}(A) = \text{arcsinh}(A)$.

Problem 21. Let A, B be nonzero $n \times n$ hermitian matrices. We form the matrix K as
$$K := A + iB.$$
This matrix is nonnormal if $[A, B] \neq 0$ and normal if $AB = BA$. In the following we assume that K is nonnormal. Assume that we can find a unitary $n \times n$ matrix U such that
$$UKU^* = K^*.$$
What can be said about the eigenvalues of K? In physics such a unitary matrix is called a "quasi-symmetry operator".

Problem 22. Consider the nonnormal matrix
$$A = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}.$$

Find a unitary matrix U such that $UAU^* = A^*$.

Problem 23. Consider the nonnormal matrix

$$K = \begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}.$$

Find a unitary matrix U such $UKU^* = K^*$. Find the eigenvalues and normalized eigenvectors of K and K^*. Discuss.

Problem 24. Can one find a 2×2 matrix A such that

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}?$$

Hint. The condition provides four equations. Since the determinant of the matrix of the right-hand side is equal to 0, the determinant of A, i.e. $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ must also be 0.

Problem 25. Can one find a 2×2 matrix A such that

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}?$$

Hint. The condition provides four equations. Since the determinant of the matrix of the right-hand side is equal to 1, the determinant of A, i.e. $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ must be $+1$ or -1.

Problem 26. Let A, B be normal matrices. Can we conclude that AB is a normal matrix?

Problem 27. (i) Consider the nonnormal 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Can we find a 2×2 matrix X such that $X^2 = A$, i.e. X would be the square root of A.
(ii) Consider the nonnormal 3×3 matrix

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Can we find a 3×3 matrix Y such that $Y^2 = B$, i.e. Y would be the square root of B.

Problem 28. Let A be an normal $n \times n$ matrix. Is the matrix $A - iI_n$ normal?

Problem 29. Find all nonnormal 2×2 matrices A such that

$$AA^* + A^*A = I_2.$$

An example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 30. (i) Can one find a nonormal matrix A such that

$$e^A e^{A^*} = e^{A^*} e^A \,?$$

(ii) Can one find a nonormal matrix A such that

$$e^A e^{A^*} = e^{A+A^*} \,?$$

Problem 31. Let $s \in \{1/2, 1, 3/2, 2, \ldots\}$ be the spin quantum number. The spin matrices s_1 and s_3 are defined as the $(2s+1) \times (2s+1)$ matrices $s_1 = (s_+ + s_-)/2$ and $s_3 = \mathrm{diag}(s, s-1, \ldots, -s)$ where the entries of s_+ and s_- are all zero except for the entries given by (here rows and columns are numbered $s, s-1, s-2, \ldots, -s$)

$$(s_+)_{m+1,m} = \sqrt{(s-m)(s+m+1)} \qquad m = s-1, s-2, \ldots, -s$$

and $s_- = s_+^T$.

Calculate $[s_1, s_3]$ in terms of s. Find the $\|[s_1, s_3]\|^2$ in terms of s, where $\|A\| := \sqrt{\mathrm{tr}(AA^*)}$ is the Frobenius norm.

Problem 32. A measure of nonnormality is given by

$$m(A) := \|A^*A - AA^*\|$$

where $\|\cdot\|$ denotes some matrix norm. Let s_3 and s_1 be real valued and symmetric $n \times n$ matrices (for example, spin matrices). Calculate $m(s_3 + \exp(i\phi)s_1)$.

Problem 33. Let A be a nonnormal matrix, i.e. $AA^* \neq A^*A$
(i) Is $\mathrm{tr}(AA^*) = \mathrm{tr}(A^*A)$?
(ii) Is $\mathrm{tr}e^{AA^*} = \mathrm{tr}e^{A^*A}$?

Problem 34. Consider the non-normal matrix

$$A = \begin{pmatrix} i\pi & 1 \\ 0 & -i\pi \end{pmatrix}.$$

Is

$$e^{A+A^*} = e^A e^{A^*} ?$$

Problem 35. Let Q be a non-normal invertible $n \times n$ matrix. Is

$$Q \otimes Q^{-1}$$

non-normal?

Problem 36. Let $\alpha, \beta \in \mathbb{C}$. What are the conditions on α, β such that the matrix

$$M(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$$

is normal?

Problem 37. Let M be an $n \times n$ nonnormal matrix.
(i) Is

$$\begin{pmatrix} 0_n & M^* \\ M & 0_n \end{pmatrix}$$

nonnormal?
(ii) Is

$$\begin{pmatrix} 0_n & -M^* \\ M & 0_n \end{pmatrix}$$

nonnormal?

Problem 38. Let $z \in \mathbb{C}$. Is the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 - e^z & e^{-z} \end{pmatrix}$$

nonnormal?

Problem 39. Prove or disprove the following statements.
(i) If the $n \times n$ matrix A is nonnormal, then there exists no matrix B such that $B^2 = A$.
(ii) Let A, B be nonzero $n \times n$ matrices with $B^2 = A$. Then the matrix A is normal.
(iii) An $n \times n$ nonnormal matrix is not diagonalizable.

Problem 40. We know that all real symmetric matrices are diagonalizable. Are all complex symmetric matrices are diagonalizable?

Problem 41. An $n \times n$ matrix A is called nonnormal if $AA^* \neq A^*A$.

(i) Let U be a unitary matrix. Is $\widetilde{A} = UAU^{-1}$ nonnormal for all U.
(ii) Let $Q \in SL(n, \mathbb{C})$. Can one find Q such that $\widetilde{A} = QAQ^{-1}$ is normal?

Problem 42. Is every invertible matrix normal? Prove or disprove.

Problem 43. Let $\alpha \in \mathbb{R}$. Find the condition on α such that

$$\begin{pmatrix} 0 & \cos(\alpha) \\ \sin(\alpha) & 0 \end{pmatrix}$$

is a normal matrix.

Chapter 24

Spectral Theorem

Problem 1. Let A be a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and pairwise orthonormal eigenvectors $\mathbf{a}_j$ (column vectors), i.e. $\mathbf{a}_j^* \mathbf{a}_k = \delta_{jk}$. Then we can write A as (spectral decomposition)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{a}_j \mathbf{a}_j^*.$$

Analogously for a normal matrix B we have

$$B = \sum_{k=1}^{n} \mu_k \mathbf{b}_k \mathbf{b}_k^*.$$

(i) Find the condition on λ_j, $\mathbf{a}_j$ and μ_k, $\mathbf{b}_k$ such that $\mathrm{tr}(AB^*) = 0$, i.e. the two matrices are orthogonal to each other.
(ii) Find the condition on λ_j, $\mathbf{a}_j$ and μ_k, $\mathbf{b}_k$ such that $[A, B] = 0_n$, i.e. the the commutator of the matrices vanishes.

Problem 2. Let A be a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and pairwise orthonormal eigenvectors $\mathbf{a}_j$ (column vectors), i.e. $\mathbf{a}_j^* \mathbf{a}_k = \delta_{jk}$. Then we can write A as (spectral decomposition)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{a}_j \mathbf{a}_j^*.$$

Analogously for a normal matrix B we have

$$B = \sum_{k=1}^{n} \mu_k \mathbf{b}_k \mathbf{b}_k^*.$$

Let $z \in \mathbb{C}$. Use the spectral decomposition to calculate

$$e^{zA}Be^{-zA}.$$

Problem 3. Let A be an $n \times n$ normal matrix over $\mathbb{C}$ with eigenvalues λ_1, $\dots$, λ_n and corresponding pairwise orthonormal eigenvectors $\mathbf{u}_j$ $(j = 1, \dots, n)$. Then the matrix A can be written as (spectral decomposition)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \mathbf{u}_j^* \equiv \sum_{j=1}^{n} \lambda_j |u_j\rangle\langle u_j|.$$

(i) Let $z \in \mathbb{C}$. Use this spectral decomposition to calculate $\exp(zA)$.
(ii) Apply it to $A = \sigma_1$.

Problem 4. (i) Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors and thus the spectral decomposition of P.
(ii) Find the matrix X such that $\exp(X) = P$.

Problem 5. (i) Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of P and thus the spectral decomposition of P.
(ii) Find the matrix X such that $\exp(X) = P$.

Problem 6. Consider the two 3×3 permutation matrices (which are of course then also unitary matrices)

$$U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We want to find efficiently K_1 and K_2 such that $U_1 = e^{K_1}$ and $U_2 = e^{K_2}$. We would apply the spectral decomposition theorem to find K_1, i.e.

$$K_1 = \sum_{j=1}^{3} \ln(\lambda_j) \mathbf{v}_j \mathbf{v}_j^*$$

where λ_j are the eigenvalues of U_1 and $\mathbf{v}_j$ are the corresponding normalized eigenvectors. But then to find K_2 we would apply the property that $U_1^2 = U_2$. Or could we actually apply that $U_2 = U_1^T$? Note that U_1, U_2, I_3 form a commutative subgroup of the group of 3×3 permutation under matrix multiplication.

Problem 7. Let A be a hermitian $n \times n$ matrix. Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are pairwise different. Then the normalized eigenvectors $\mathbf{u}_j$ ($j = 1, \ldots, n$) satisfy $\mathbf{u}_j^* \mathbf{u}_k = 0$ for $j \neq k$ and $\mathbf{u}_j^* \mathbf{u}_j = 1$. We have (spectral theorem)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \mathbf{u}_j^*.$$

Let $\mathbf{e}_k$ ($k = 1, \ldots, n$) be the standard basis in $\mathbb{C}^n$. Calculate $U^* A U$, where

$$U = \sum_{k=1}^{n} \mathbf{u}_k \mathbf{e}_k^*.$$

Problem 8. Let A be a positive definite $n \times n$ matrix. Thus all the eigenvalues are real and positive. Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are pairwise different. Then the normalized eigenvectors $\mathbf{u}_j$ ($j = 1, \ldots, n$) satisfy $\mathbf{u}_j^* \mathbf{u}_k = 0$ for $j \neq k$ and $\mathbf{u}_j^* \mathbf{u}_j = 1$. We have (spectral theorem)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \mathbf{u}_j^*.$$

Let $\mathbf{e}_k$ ($k = 1, \ldots, n$) be the standard basis in $\mathbb{C}^n$. Calculate

$$\ln(A).$$

Note that the unitary matrix

$$U = \sum_{k=1}^{n} \mathbf{u}_k \mathbf{e}_k^*$$

transforms A into a diagonal matrix, i.e. $\widetilde{A} = U^* A U$ is a diagonal matrix.

Problem 9. The *spectral theorem* for $n \times n$ normal matrices over $\mathbb{C}$ is as follows: A matrix A is normal if and only if there exists an $n \times n$ unitary matrix U and a diagonal matrix D such that $D = U^* A U$. Use this theorem to prove that the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is not normal.

Problem 10. Consider the normal matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of A and thus the spectral decomposition. Thus this result to calculate $\exp(zA)$, where $z \in \mathbb{C}$.

Problem 11. (i) Find the spectral decomposition of the normal matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(ii) Find the spectral decomposition of $\exp(A)$.

Problem 12. (i) Find the eigenvalues, normalized eigenvectors and spectral decomposition of the permutation matrices

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(ii) Use the spectral decomposition to find the matrices A_1 and A_2 such that $P_1 = \exp(A_1)$, $P_2 = \exp(A_2)$.

Problem 13. Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(i) Find the eigenvalues and normalized eigenvectors of P.
(ii) Show that the eigenvalues form a group under multiplication.
(iii) Use the result from (i) and the spectral representation to find K such that $P = \exp(K)$.

Problem 14. Let A be an $n \times n$ matrix over $\mathbb{C}$. An $n \times n$ matrix B is called a square root of A if $B^2 = A$. Find the square roots of the 2×2 identity matrix applying the spectral theorem. The eigenvalues of I_2 are $\lambda_1 = 1$ and $\lambda_2 = 1$. As normalized eigenvectors choose

$$\begin{pmatrix} e^{i\phi_1}\cos(\theta) \\ e^{i\phi_2}\sin(\theta) \end{pmatrix}, \quad \begin{pmatrix} e^{i\phi_1}\sin(\theta) \\ -e^{i\phi_2}\cos(\theta) \end{pmatrix}$$

which form an orthonormal basis in $\mathbb{C}^2$. Four cases $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (1,1)$, $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (1,-1)$, $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (-1,1)$, $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (-1,-1)$ have to be studied. The first and last case are trivial. So study the second case $(\sqrt{\lambda_1}, \sqrt{\lambda_2}) = (1,-1)$. The second case and the third case are "equivalent".

Problem 15. Consider the vectors in $\mathbb{C}^2$ (sometimes called *spinors*)

$$\mathbf{v}_1 = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} \sin(\theta/2)e^{-i\phi/2} \\ -\cos(\theta/2)e^{i\phi/2} \end{pmatrix}.$$

(i) First show that they are normalized and orthonormal.
(ii) Assume that for the vector $\mathbf{v}_1$ is an eigenvector with the corresponding eigenvalue $+1$ and that the vector $\mathbf{v}_2$ is an eigenvector with the corresponding eigenvalue -1. Apply the spectral theorem to find the corresponding 2×2 matrix.
(iii) Since the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis in $\mathbb{C}^2$ we can form an orthonormal basis in $\mathbf{C}^4$ via the Kronecker product

$$\mathbf{w}_{11} = \mathbf{v}_1 \otimes \mathbf{v}_1, \quad \mathbf{w}_{12} = \mathbf{v}_1 \otimes \mathbf{v}_2, \quad \mathbf{w}_{21} = \mathbf{v}_2 \otimes \mathbf{v}_1, \quad \mathbf{w}_{22} = \mathbf{v}_2 \otimes \mathbf{v}_2.$$

Assume that the eigenvalue for $\mathbf{w}_{11}$ is $+1$, for $\mathbf{w}_{12}$ -1, for $\mathbf{w}_{21}$ -1 and for $\mathbf{w}_{22}$ $+1$. Apply the spectral theorem to find the corresponding 4×4 matrix.

Problem 16. Given a normal 5×5 matrix which provides the characteristic equation

$$-\lambda^5 + 4\lambda^3 - 3\lambda = 0$$

with the eigenvalues

$$\lambda_1 = -\sqrt{3}, \quad \lambda_2 = -1, \quad \lambda_3 = 0, \quad \lambda_4 = 1, \quad \lambda_5 = \sqrt{3}$$

and the corresponding normalized eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} \sqrt{3}/6 \\ -1/2 \\ 1/\sqrt{3} \\ -1/2 \\ \sqrt{3}/6 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/2 \\ -1/2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ -1/2 \\ -1/2 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} \sqrt{3}/6 \\ 1/2 \\ 1/\sqrt{3} \\ 1/2 \\ \sqrt{3}/6 \end{pmatrix},$$

Reconstruct the matrix applying the spectral theorem.

Chapter 25

Mutually Unbiased Bases

Two orthonormal basis in the Hilbert space $\mathbb{C}^d$

$$\mathcal{A} = \{\, \mathbf{e}_1, \dots, \mathbf{e}_d \,\}, \quad \mathcal{B} = \{\, \mathbf{f}_1, \dots, \mathbf{f}_d \,\}$$

are called mutually unbiased if for every $1 \le j, k \le d$

$$|\langle \mathbf{e}_j, \mathbf{f}_k \rangle| = \frac{1}{\sqrt{d}}.$$

Problem 1. For the Pauli spin matrices σ_3, σ_1, σ_2 with eigenvalues $+1$ and -1 the normalized eigenvectors are given by

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{B}_2 = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\},$$

$$\mathcal{B}_3 = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}.$$

This is a set of three mutually unbiased basis.
(i) Consider now $\sigma_3 \otimes \sigma_3$, $\sigma_1 \otimes \sigma_1$, $\sigma_2 \otimes \sigma_2$. Show that

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$\mathcal{B}_3 = \left\{ \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} \otimes \begin{pmatrix} 1 \\ i \end{pmatrix},\ \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix},\ \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} \otimes \begin{pmatrix} 1 \\ i \end{pmatrix},\ \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$$

provides mutually unbiased basis in the Hilbert space $\mathbb{C}^4$.
(ii) Consider the case in the Hilbert space $\mathbb{C}^8$ with

$$\sigma_3 \otimes \sigma_3 \otimes \sigma_3, \quad \sigma_1 \otimes \sigma_1 \otimes \sigma_1, \quad \sigma_2 \otimes \sigma_2 \otimes \sigma_2.$$

Extend the n Kronecker products.

Problem 2. (i) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two finite dimensional Hilbert spaces with $\dim(\mathcal{H}_1) = d_1$ and $\dim(\mathcal{H}_2) = d_2$. Let $\mathcal{B}_{1,1} = \{|j_1\rangle\}$ and $\mathcal{B}_{1,2} = \{|j_2\rangle\}$ ($j_1, j_2 = 1, \ldots, d_1$) be two orthonormal bases in the Hilbert space $\mathcal{H}_1$. They are called *mutually unbiased* iff

$$|\langle j_1 | j_2 \rangle|^2 = \frac{1}{d_1} \quad \text{for all} \quad j_1, j_2 = 1, \ldots, d_1.$$

Let $\mathcal{B}_{2,1} = \{|k_1\rangle\}$ and $\mathcal{B}_{2,2} = \{|k_2\rangle\}$ ($k_1, k_2 = 1, \ldots, d_2$) be orthonormal bases in the Hilbert space $\mathcal{H}_2$. Assume in the following that $\mathcal{B}_{1,1}$ and $\mathcal{B}_{1,2}$ are mutually unbiased bases in the Hilbert space $\mathcal{H}_1$ and $\mathcal{B}_{2,1}$ and $\mathcal{B}_{2,2}$ are mutually unbiased bases in the Hilbert space $\mathcal{H}_2$, respectively. Show that

$$\{\, |j_1\rangle \otimes |k_1\rangle \,\}, \quad \{\, |j_2\rangle \otimes |k_2\rangle \,\}, \quad j_1, j_2 = 1, \ldots, d_1, \ \ k_1, k_2 = 1, \ldots, d_2$$

are mutually unbiased bases in the finite dimensional product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = d_1 \cdot d_2$ and the scalar product in the product Hilbert space

$$(\langle j_1| \otimes \langle k_1|)(|j_2\rangle \otimes |k_2\rangle) = \langle j_1|j_2\rangle\langle k_1|k_2\rangle.$$

(ii) Apply the result from (i) to $\mathcal{H}_1 = \mathbb{C}^2$ with

$$\mathcal{B}_{1,1} = \{\, |1\rangle_{1,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ |2\rangle_{1,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$$

$$\mathcal{B}_{1,2} = \{\, |1\rangle_{1,2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ |2\rangle_{1,2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \}$$

and $\mathcal{H}_2 = \mathbb{C}^3$ with

$$\mathcal{B}_{2,1} = \{\, |1\rangle_{2,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ |2\rangle_{2,1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ |3\rangle_{2,1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$$

$$\mathcal{B}_{2,2} = \{\, |1\rangle_{2,2} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ |2\rangle_{2,2} = \begin{pmatrix} 1/\sqrt{3} \\ -i/2 - 1/(2\sqrt{3}) \\ i/2 - 1/(2\sqrt{3}) \end{pmatrix},\ |3\rangle_{2,2} = \begin{pmatrix} 1/\sqrt{3} \\ i/2 - 1/(2\sqrt{3}) \\ i/2 - 1/(2\sqrt{3}) \end{pmatrix} \}.$$

(iii) Consider the Hilbert space $M(2,\mathbb{C})$ of the 2×2 matrices over the complex numbers and the mutually unbiased bases

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Construct mutually unbiased bases applying the Kronecker product for the Hilbert space $M(4,\mathbb{C})$.

Problem 3. Let $d \geq 2$. Consider the Hilbert space $\mathbb{C}^d$ and $\mathbf{e}_0$, $\mathbf{e}_1$, $\dots$, $\mathbf{e}_{d-1}$ be the standard basis

$$\mathbf{e}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \mathbf{e}_{d-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Let $\omega_d := e^{i2\pi/d}$. Then we can form new orthonormal bases via

$$\mathbf{v}_{m;b} \equiv |m;b\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} \omega^{bn(n-1)/2-nm} \mathbf{e}_n, \quad b, m = 0, 1, \dots, d-1$$

where the d labelled by the b are the bases and m labels the state within a basis. Thus we find mutually unbiased bases.

(i) Consider $d = 2$ and $b = 0$. Find the basis $|0;0\rangle$, $|1;0\rangle$.

(ii) Consider $d = 3$ and $b = 0$. Find the basis $|1;0\rangle$, $|2;0\rangle$, $|3;0\rangle$.

(iii) Consider $d = 4$ and $b = 0$. Find the basis $|0;0\rangle$, $|1;0\rangle$, $|2;0\rangle$, $|3;0\rangle$. Find out whether the states can be written as Kronecker product of two vectors in $\mathbb{C}^2$, i.e. whether the states are entangled or not?

Chapter 26

Integration

Let A be an $n \times n$ matrix over $\mathbb{C}$. If $f : \mathbb{C} \to \mathbb{C}$ is analytic in a region Ω containing the spectrum of A, then $f(A)$ can be defined through the *Cauchy integral formula*

$$f(A) = \frac{1}{2\pi i} \int_{\Omega} (zI_n - A)^{-1} f(z) dz.$$

Problem 1. Let P_j $(j = 0, 1, 2, \ldots)$ be the Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots .$$

Calculate the infinite dimensional matrix $A = (A_{jk})$

$$A_{jk} = \int_{-1}^{+1} P_j(x) \frac{dP_k(x)}{dx} dx$$

where $j, k = 0, 1, \ldots$. Consider the matrix A as a linear operator in the Hilbert space $\ell_2(\mathbb{N}_0)$. Is $\|A\| < \infty$?

Problem 2. Let $\mathbb{C}^{n \times N}$ be the vector space of all $n \times N$ complex matrices. Let $Z \in \mathbb{C}^{n \times N}$. Then $Z^* \equiv \bar{Z}^T$, where ^{T} denotes transpose. One defines a *Gaussian measure* μ on $\mathbb{C}^{n \times N}$ by

$$d\mu(Z) := \frac{1}{\pi^{nN}} \exp(-\mathrm{tr}(ZZ^*)) dZ$$

where dZ denotes the Lebesgue measure on $\mathbb{C}^{n \times N}$. The *Fock space* $\mathcal{F}(\mathbb{C}^{n \times N})$ consists of all entire functions on $\mathbb{C}^{n \times N}$ which are square integrable with respect to the Gaussian measure $d\mu(Z)$. With the scalar product

$$\langle f|g\rangle := \int_{\mathbb{C}^{n\times N}} f(Z)\overline{g(Z)}d\mu(Z), \qquad f,g \in \mathcal{F}(\mathbb{C}^{n\times N})$$

one has a Hilbert space. Show that this Hilbert space has a *reproducing kernel* K. This means a continuous function $K(Z,Z') : \mathbb{C}^{n\times N} \times \mathbb{C}^{n\times N} \to \mathbb{C}$ such that

$$f(Z) = \int_{\mathbb{C}^{n\times N}} K(Z,Z')f(Z')d\mu(Z')$$

for all $Z \in \mathbb{C}^{n\times N}$ and $f \in \mathcal{F}(\mathbb{C}^{n\times N})$.

Problem 3. Let A be an $n\times n$ positive definite matrix over $\mathbb{R}$, i.e. $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Calculate

$$\int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A\mathbf{x})d\mathbf{x}.$$

Problem 4. Consider the matrices

$$A(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \quad B(t) = \int_0^t A(s)ds.$$

Find the commutator $[A(t),B(t)]$. Discuss. What is the condition such that $[A(t),B(t)] = 0_2$.

Problem 5. Let $\alpha \in \mathbb{R}$. Consider an $n\times n$ matrix $A(\alpha)$, where the entries of A depends smoothly on α. Then one has the identity

$$\frac{d}{d\alpha}e^{A(\alpha)} \equiv \int_0^1 e^{(1-s)A(\alpha)}\frac{dA(\alpha)}{d\alpha}e^{sA(\alpha)}ds \equiv \int_0^1 e^{sA(\alpha)}\frac{dA(\alpha)}{d\alpha}e^{(1-s)A(\alpha)}ds.$$

Let $n = 2$ and

$$A(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Calculate the right-hand sides of the identities.

Chapter 27

Differentiation

Problem 1. Let j be a positive integer. Let A, B be $n \times n$ matrices over $\mathbb{R}$. Calculate

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}((A + \epsilon B)^j - A^j).$$

Calculate

$$\frac{d}{d\epsilon} \operatorname{tr}(A + \epsilon B)^j \Big|_{\epsilon=0}.$$

Problem 2. Find the partial differential equation given by the condition

$$\det \begin{pmatrix} 0 & \partial u/\partial x_1 & \partial u/\partial x_2 \\ \partial u/\partial x_1 & \partial^2 u/\partial x_1^2 & \partial^2 u/\partial x_1 \partial x_2 \\ \partial u/\partial x_2 & \partial^2 u/\partial x_2 \partial x_1 & \partial^2 u/\partial x_2^2 \end{pmatrix}.$$

Find a nontrivial solution of the partial differential equation.

Problem 3. Consider the 2×2 matrix

$$A(\epsilon) = \begin{pmatrix} f_1(\epsilon) & f_2(\epsilon) \\ f_3(\epsilon) & f_4(\epsilon) \end{pmatrix}$$

where f_j ($j = 1, 2, 3, 4$) are smooth functions and $\det(A(\epsilon)) > 0$ for all ϵ. Show that

$$\operatorname{tr}((dA(\epsilon))A(\epsilon)^{-1}) = d(\ln(\det(A(\epsilon))))$$

where d is the exterior derivative.

Problem 4. Let $\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t) \in \mathbb{R}^3$. Solve the initial value problem of the nonlinear autonomous system of first order differential equations

$$\frac{d\mathbf{u}_1}{dt} = \mathbf{u}_2 \times \mathbf{u}_3, \quad \frac{d\mathbf{u}_2}{dt} = \mathbf{u}_3 \times \mathbf{u}_1, \quad \frac{d\mathbf{u}_3}{dt} = \mathbf{u}_1 \times \mathbf{u}_2$$

where $\times$ denotes the vector product.

Problem 5. Let $\mathbf{u}(t) \in \mathbb{R}^3$. Solve the initial value problem for the differential equation

$$\frac{d^2\mathbf{u}}{dt^2} = \mathbf{u} \times \frac{d\mathbf{u}}{dt}$$

where $\times$ denotes the vector product.

Problem 6. Let $f_{jk} : \mathbb{R} \to \mathbb{R}$ be analytic functions, where $j, k = 1, 2$. Find the differential equations for f_{jk} such that

$$\begin{pmatrix} f_{11}(\epsilon) & f_{12}(\epsilon) \\ f_{21}(\epsilon) & f_{22}(\epsilon) \end{pmatrix} \frac{d}{d\epsilon} \begin{pmatrix} f_{11}(\epsilon) & f_{12}(\epsilon) \\ f_{21}(\epsilon) & f_{22}(\epsilon) \end{pmatrix} = \left(\frac{d}{d\epsilon} \begin{pmatrix} f_{11}(\epsilon) & f_{12}(\epsilon) \\ f_{21}(\epsilon) & f_{22}(\epsilon) \end{pmatrix} \right) \begin{pmatrix} f_{11}(\epsilon) & f_{12}(\epsilon) \\ f_{21}(\epsilon) & f_{22}(\epsilon) \end{pmatrix}.$$

Problem 7. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an analytic functions. Consider the matrices

$$A(t) = \begin{pmatrix} e^{i\phi(t)} & 1 \\ 1 & e^{id\phi(t)/dt} \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & e^{i\phi(t)} \\ e^{id\phi(t)/dt} & 1 \end{pmatrix}.$$

(i) Find the differential equation for ϕ from the condition

$$\operatorname{tr}(AB) = 0.$$

(ii) Find the differential equation for ϕ from the condition

$$\det(AB) = 0.$$

Problem 8. Consider the invertible 2×2 matrix

$$A(\theta) = \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}.$$

Show that

$$d(\ln(\det(A))) = \operatorname{tr}(A^{-1}dA)$$

where d denotes the exterior derivative.

Problem 9. (i) Consider the analytic function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = \sinh(x_2), \quad f_2(x_1, x_2) = \sinh(x_1).$$

Show that this function admits the (only) fixed point $(0,0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

(ii) Consider the analytic function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$

$$g_1(x_1, x_2) = \sinh(x_1), \quad g_2(x_1, x_2) = -\sinh(x_2).$$

Show that this function admits the (only) fixed point $(0,0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

(iii) Multiply the two matrices found in (i) and (ii).
(iv) Find the composite function $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{h}(\mathbf{x}) = (\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x})).$$

Show that this function also admits the fixed point $(0,0)$. Find the functional matrix at this fixed point

$$\left.\begin{pmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}.$$

Compare this matrix with the matrix found in (iii).

Problem 10. (i) Consider the 2×2 matrix

$$V(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

Calculate $dV(t)/dt$ and then find the commutator $[dV(t)/dt, V(t)]$.
(ii) Let $V(t)$ be a 2×2 matrix where all the entries are smooth functions of t. Calculate $dV(t)/dt$ and then find the conditions on the entries such that $[dV(t)/dt, V(t)] = 0_2$.

Problem 11. Let $f_j(x_1, x_2)$ $(j = 1, 2, 3)$ be real valued smooth functions. Consider the matrix

$$N(x_1, x_2) = f_1\sigma_1 + f_2\sigma_2 + \sigma_3 \equiv \begin{pmatrix} f_3 & -if_2 + f_1 \\ if_2 + f_1 & -f_3 \end{pmatrix}.$$

Find dN, N^*. Then calculate $d(N^* dN)$. Find the conditions of f_1, f_2, f_3 such that

$$d(N^* dN) = 0_2$$

where 0_2 is the 2×2 zero matrix.

Problem 12. (i) Consider the 2×2 matrix

$$F(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

with $\det(F(\alpha)) = 1$. Is the determinant preserved under repeated differentiation of $F(\alpha)$ with respect to α?
(ii) Consider the 2×2 matrix

$$G(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix}$$

with $\det(G(\alpha)) = -1$. Is the determinant preserved under repeated differentiation of $G(\alpha)$ with respect to α?

Problem 13. Let A be an $n \times n$ matrix. Assume that the inverse of A exists, i.e. $\det(A) \neq 0$. Then the inverse $B = A^{-1}$ can be calculated as

$$\frac{\partial}{\partial a_{jk}} \ln(\det(A)) = b_{kj}.$$

Apply this formula to the 2×2 matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Problem 14. Consider the four 6×6 matrices

$$\beta_0 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and

$$\psi = (\, E_1 \quad E_2 \quad E_3 \quad cB_1 \quad cB_2 \quad cB_3 \,)^T .$$

Show that

$$\beta_0 \frac{\partial}{c\partial t}\psi + \beta_1 \frac{\partial}{\partial x_1}\psi + \beta_2 \frac{\partial}{\partial x_2}\psi + \beta_3 \frac{\partial}{\partial x_3}\psi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

are Maxwell's equations

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B}, \qquad -\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}.$$

Chapter 28

Hilbert Spaces

Problem 1. The vector space of all $n \times n$ matrices over $\mathbb{C}$ form a Hilbert space with the scalar product defined by

$$\langle A, B \rangle := \operatorname{tr}(AB^*).$$

This implies a norm $\|A\|^2 = \operatorname{tr}(AA^*)$.
(i) Consider the Lie group $U(n)$. Find two unitary 2×2 matrices U_1, U_2 such that $\|U_1 - U_2\|$ takes a maximum.
(ii) Are the matrices

$$U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

such a pair?

Problem 2. Consider the four 2×2 matrices

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

(i) Show that they form an orthonormal basis in the Hilbert space of the 2×2 matrices with the scalar product $\langle X, Y \rangle = \operatorname{tr}(XY^*)$.
(ii) Find the multiplication table.

Problem 3. Consider the Hilbert space $M_d(\mathbb{C})$ of $d \times d$ matrices with scalar product $\langle A, B \rangle := \operatorname{tr}(AB^*)$, $A, B \in M_d(\mathbb{C})$. Consider an orthogonal basis of d^2 $d \times d$ hermitian matrices $B_1, B_2, \ldots, B_{d^2}$, i.e.

$$\langle B_j, B_k \rangle = \operatorname{tr}(B_j B_k) = d\delta_{jk}$$

since $B_k^* = B_k$ for a hermitian matrix. Let M be a $d \times d$ hermitian matrix. Let

$$m_j = \mathrm{tr}(B_j M) \qquad j = 1, \ldots, d^2.$$

Given m_j and B_j ($j = 1, \ldots, d^2$). Find M.

Problem 4. Consider the Hilbert space $\mathcal{H}$ of the 2×2 matrices over the complex numbers with the scalar product

$$\langle A, B \rangle := \mathrm{tr}(AB^*), \qquad A, B \in \mathcal{H}.$$

Show that the rescaled Pauli matrices $\mu_j = \frac{1}{\sqrt{2}}\sigma_j$, $j = 1, 2, 3$

$$\mu_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus the rescaled 2×2 identity matrix

$$\mu_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $\mathcal{H}$.

Problem 5. Can we find an invertible 2×2 matrix S over the real numbers such that

$$S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ?$$

Chapter 29

Miscellaneous

Problem 1. (i) For $n = 4$ the transform matrix for the *Daubechies wavelet* is given by

$$D_4 = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & -c_2 & c_1 & -c_0 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & -c_0 & c_3 & -c_2 \end{pmatrix}, \qquad \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1+\sqrt{3} \\ 3+\sqrt{3} \\ 3-\sqrt{3} \\ 1-\sqrt{3} \end{pmatrix}.$$

Is D_4 orthogonal? Prove or disprove.
(ii) For $n = 8$ the transform matrix for the Daubechies wavelet is given by

$$D_8 = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & 0 & 0 & 0 & 0 \\ c_3 & -c_2 & c_1 & -c_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & c_1 & c_2 & c_3 & 0 & 0 \\ 0 & 0 & c_3 & -c_2 & c_1 & -c_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & c_3 & -c_2 & c_1 & -c_0 \\ c_2 & c_3 & 0 & 0 & 0 & 0 & c_0 & c_1 \\ c_1 & -c_0 & 0 & 0 & 0 & 0 & c_3 & -c_2 \end{pmatrix}.$$

Is D_8 orthogonal? Prove or disprove.

Problem 2. Consider the $2n \times 2n$ matrix

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

We define that the $2n \times 2n$ matrix H over $\mathbb{R}$ is *Hamiltonian* if $(JH)^T = JH$. We define that the $2n \times 2n$ matrix S over $\mathbb{R}$ is *symplectic* if $S^T JS = J$. Show that if H is Hamiltonian and S is symplectic, then the matrix $S^{-1}HS$ is Hamiltonian.

Problem 3. Let A be an $n \times n$ matrix over $\mathbb{R}$. Consider the $2n \times 2n$ matrices

$$S = \begin{pmatrix} I_n & I_n \\ A & I_n + A \end{pmatrix}, \quad \widetilde{S} = \begin{pmatrix} A + 2I_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

Can we find an invertible $2n \times 2n$ matrix T such that $\widetilde{S} = T^{-1}ST$?

Problem 4. Let

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}.$$

Find P^*P. Show that P^*JP is a diagonal matrix.

Problem 5. Let J be the $2n \times 2n$ matrix

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

We define symplectic **G**-reflectors to be those $2n \times 2n$ symplectic matrices that have a $(2n-1)$-dimensional fixed-point subspace. It can be shown that any symplectic **G**-reflector can be expressed in the form

$$G = I_{2n} + \beta \mathbf{u}\mathbf{u}^T J \tag{1}$$

for some $0 \neq \beta \in \mathbb{F}$, $\mathbf{0} \neq \mathbf{u} \in \mathbb{F}^{2n}$ and $\mathbf{u}$ is considered as a column vector. The underlying field is $\mathbb{F}$. Conversely, any G given by (1) is always a symplectic **G**-reflector. Show that $\det(G) = +1$.

Problem 6. Consider the two polynomials

$$p_1(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad p_2(x) = b_0 + b_1 x + \cdots + b_m x^m$$

where $n = \deg(p_1)$ and $m = \deg(p_2)$. Assume that $n > m$. Let

$$r(x) = \frac{p_2(x)}{p_1(x)}.$$

We expand $r(x)$ in powers of $1/x$, i.e.

$$r(x) = \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots$$

From the coefficients $c_1, c_2, \ldots, c_{2n-1}$ we can form an $n \times n$ *Hankel matrix*

$$H_n = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix}.$$

The determinant of this matrix is proportional to the *resultant* of the two polynomials. If the resultant vanishes, then the two polynomials have a non-trivial greatest common divisor. Apply this theorem to the polynomials

$$p_1(x) = x^3 + 6x^2 + 11x + 6, \qquad p_2(x) = x^2 + 4x + 3.$$

Problem 7. Consider the 3×3 matrix

$$A = \begin{pmatrix} -1/2 & -\sqrt{3}/6 & \sqrt{6}/3 \\ -\sqrt{3}/6 & -5/6 & -\sqrt{2}/3 \\ \sqrt{6}/3 & -\sqrt{2}/3 & 1/3 \end{pmatrix}.$$

Show that $A^T = A^{-1}$ by showing that the column of the matrix are normalized and pairwise orthonormal.

Problem 8. Find the Moore-Penrose pseudo inverses of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Problem 9. Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that there exists nonnull vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbb{R}^n$ such that

$$\frac{\mathbf{x}_1^T A \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \leq \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \leq \frac{\mathbf{x}_2^T A \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2}$$

for every nonnull vector $\mathbf{x}$ in $\mathbb{R}^n$.

Problem 10. A generalized Kronecker delta can be be defined as follows

$$\delta_{I,J} := \begin{cases} 1 & \text{if } J = (j_1, \dots, j_r) \text{ is an even permutation of } I = (i_1, \dots, i_r) \\ -1 & \text{if } J \text{ is an odd permutation of } I \\ 0 & \text{if } J \text{ is not a permutation of } I \end{cases}$$

Find $\delta_{126,621}$, $\delta_{126,651}$, $\delta_{125,512}$.

Problem 11. Let $c_j > 0$ for $j = 1, \dots, n$. Show that the $n \times n$ matrices

$$\left(\frac{\sqrt{c_j c_k}}{c_j + c_k} \right), \qquad \left(\frac{1/c_j + 1/c_k}{\sqrt{c_j c_k}} \right)$$

$(k = 1, \dots, n)$ are positive definite.

Problem 12. Let $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ be nontrivial involutions. This means that $R = R^{-1} \neq \pm I_m$ and $S = S^{-1} \neq I_n$. A matrix $A \in \mathbb{C}^{m\times n}$ is called (R,S)-symmetric if $RAS = A$. Consider the case $m = n = 2$ and the Pauli spin matrices

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Find all 2×2 matrices A over $\mathbb{C}$ such that $RAS = A$.

Problem 13. Let $X \in \mathbb{R}^{n\times n}$. Show that X can be written as

$$X = A + S + cI_n$$

where A is antisymmetric ($A^T = -A$), S is symmetric ($S^T = S$) with $\mathrm{tr}(S) = 0$ and $c \in \mathbb{R}$.

Problem 14. Consider vectors in the vector space $\mathbb{R}^3$ and the vector product. Consider the mapping of the vectors in $\mathbb{R}^3$ into 3×3 skew-symmetric matrices

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}.$$

Calculate the vector product

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \times \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

and the commutator $[M_1, M_2]$, where

$$M_1 = \begin{pmatrix} 0 & c_1 & -b_1 \\ -c_1 & 0 & a_1 \\ b_1 & -a_1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & c_2 & -b_2 \\ -c_2 & 0 & a_2 \\ b_2 & -a_2 & 0 \end{pmatrix}.$$

Discuss.

Problem 15. Let

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

be elements of $\mathbb{C}^2$. Solve the equation $\mathbf{z}^*\mathbf{w} = \mathbf{w}^*\mathbf{z}$.

Problem 16. Let A, B be $n\times n$ matrices over $\mathbb{C}$. We define the quasi-multiplication

$$A \bullet B := \frac{1}{2}(AB + BA).$$

Obviously $A \bullet B = B \bullet A$. Show that

$$(A^2 \bullet B) \bullet A = A^2 \bullet (B \bullet A).$$

This is called the *Jordan identity.*

Problem 17. Let σ_j ($j = 0, 1, 2, 3$) be the Pauli spin matrices, where $\sigma_0 = I_2$. Does the set of 4×4 matrices

$$\begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_k \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_k & 0_2 \end{pmatrix}, \quad j, k = 0, 1, 2, 3$$

form a group under matrix multiplication. If not add the elements to find a group. Here 0_2 is the 2×2 zero matrix.

Problem 18. Let A be a symmetric 2×2 matrix over $\mathbb{R}$

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}.$$

Thus $a_{01} = a_{10}$. Assume that

$$a_{00}a_{01} = a_{01}^2, \qquad a_{00}a_{11} = a_{01}a_{11}.$$

Find all matrices A that satisfy these conditions.

Problem 19. Let A, B be 3×3 matrices. We define the composition

$$A \Diamond B := \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & a_{13} \\ 0 & b_{11} & b_{12} & b_{13} & 0 \\ a_{21} & b_{21} & a_{22}b_{22} & b_{23} & a_{23} \\ 0 & b_{31} & b_{32} & b_{33} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{33} \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Find the eigenvalues of M and $M \Diamond M$.

Problem 20. Find a 3×3 matrix A over $\mathbb{R}$ which satisfies

$$A^2A^T + A^TA^2 = 2A, \qquad AA^TA = 2A, \qquad A^3 = 0.$$

Thus the matrix is nilpotent.

Problem 21. Find the Cayley transform of the Hermitian matrix

$$H = \begin{pmatrix} h_{11} & h_{12} \\ \bar{h}_{12} & h_{22} \end{pmatrix}, \qquad h_{11}, h_{22} \in \mathbb{R}, \quad h_{12} \in \mathbb{C}.$$

Problem 22. Let S be an invertible $n \times n$ matrix. Find the inverse of the $2n \times 2n$ matrix

$$\begin{pmatrix} 0_n & S^{-1} \\ S & 0_n \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix.

Problem 23. Consider the $n \times n$ matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad B = \mathrm{diag}(1\ \omega\ \omega^2\ \dots\ \omega^{n-1})$$

where ω is the n-th primitive root of unity. We have $A^n = B^n = I_n$ and $\omega^n = 1$. We have

$$AB = \omega BA.$$

Let $R = A \otimes I_n$ and $S = B \otimes I_n$. Find RS. Let $X = A \otimes A$ and $Y = B \otimes B$. Find XY. Find the commutator $[X, Y]$.

Problem 24. Find all 2×2 matrices C over $\mathbb{R}$ such that

$$C^T C + C C^T = I_2, \qquad C^2 = 0_2.$$

Problem 25. Let $\delta_j, \eta_j \in \mathbb{R}$ with $j = 1, 2, 3$. Any 3×3 unitary symmetric matrix U can be written in the product form

$$U = \begin{pmatrix} e^{i\delta_1} & 0 & 0 \\ 0 & e^{i\delta_2} & 0 \\ 0 & 0 & e^{i\delta_3} \end{pmatrix} \begin{pmatrix} \eta_1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & \eta_2 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & \eta_3 \end{pmatrix} \begin{pmatrix} e^{i\delta_1} & 0 & 0 \\ 0 & e^{i\delta_2} & 0 \\ 0 & 0 & e^{i\delta_3} \end{pmatrix}$$

where $\gamma_{jk} = N_{jk} \exp(i\beta_{jk})$ with $N_{jk}, \beta_{jk} \in \mathbb{R}$. It follows that

$$U_{jj} = \eta_j \exp(2i\delta_j), \qquad U_{jk} = N_{jk} \exp(i(\delta_j + \delta_k + \beta_{jk})).$$

The unitary condition $UU^* = I_3$ provides

$$\sum_{k \neq j}^{3} N_{jk}^2 + \eta_j = 1, \quad j = 1, 2, 3$$

and

$$N_{12}(\eta_1 \exp(i\beta_{12}) + \eta_2 \exp(-i\beta_{12})) = N_{13}N_{23}\exp(i(\pi + \beta_{23} - \beta_{13}))$$

and cyclic $(1 \to 2 \to 3 \to 1)$. Write the unitary symmetric matrix

$$W = \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{pmatrix}$$

in this form.

Problem 26. Consider the map $\mathbf{f} : \mathbb{C}^2 \mapsto \mathbb{R}^3$

$$\begin{pmatrix} \cos(\theta) \\ e^{i\phi}\sin(\theta) \end{pmatrix} \mapsto \begin{pmatrix} \sin(2\theta)\cos(\phi) \\ \sin(2\theta)\sin(\phi) \\ \cos(2\theta) \end{pmatrix}.$$

(i) Consider the map for the special case $\theta = 0$, $\phi = 0$.
(ii) Consider the map for the special case $\theta = \pi/4$, $\phi = \pi/4$.

Problem 27. (i) Consider the hermitian 3×3 matrices to describe a particle with *spin-1*

$$S_1 := \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 := \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 := \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

With $S_+ := S_1 + iS_2$, $S_- := S_1 - iS_2$ we find

$$S_+ = \sqrt{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_- = \sqrt{2}\hbar\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

An example of a spin-1 particle is the photon. Let $\mathbf{m}$, $\mathbf{n}$ be normalized vectors in $\mathbb{R}^3$ which are orthogonal, i.e. $\mathbf{m}^T\mathbf{n} = 0$. Find the eigenvalues of the 3×3 matrix

$$K = (\mathbf{m} \cdot \mathbf{S})^2 - (\mathbf{n} \cdot \mathbf{S})^2$$

where $\mathbf{m} \cdot \mathbf{S} = m_1S_1 + m_2S_2 + m_3S_3$.
(ii) Show that

$$P_{\mathbf{m}} = I_3 - (\mathbf{m} \cdot \mathbf{S})^2$$

is a projection operator.

Problem 28. Let A, B be 2×2 matrices over $\mathbb{R}$. Find A, B such that

$$\min \| [A, B] - I_2 \|$$

where $[\,,\,]$ denotes the commutator and for the norm $\|\;\|$ consider the Frobenius norm and max-norm.

Problem 29. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that A^{-1} exists. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $\mathbf{u}, \mathbf{v}$ are considered as column vectors. (i) Show that if

$$\mathbf{v}^T A^{-1} \mathbf{u} = -1$$

then $A + \mathbf{u}\mathbf{v}^T$ is not invertible.
(ii) Assume that $\mathbf{v}^T A^{-1} \mathbf{u} \neq -1$. Show that

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}}.$$

Problem 30. Can we find an invertible 2×2 matrix S over the real numbers such that

$$S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ?$$

Problem 31. Let A and B be $n \times n$ matrices over $\mathbb{R}$. Assume that $A \neq B$, $A^3 = B^3$ and $A^2 B = B^2 A$. Is $A^2 + B^2$ invertible?

Problem 32. Let

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

and I_2 be the 2×2 identity matrix. For $j \geq 1$, let d_j be the greatest common divisor of the entries of $A^j - I_2$. Show that

$$\lim_{j \to \infty} d_j = \infty.$$

Problem 33. Let $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$. Consider the hermitian matrix

$$K = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}.$$

Show that the matrix can be written as linear combination of the 2×2 identity matrix and the Pauli spin matrices.

Problem 34. (i) Consider the polynomial

$$p(x) = x^2 - sx + d, \quad s, d \in \mathbb{C}.$$

Find a 2×2 matrix A such that its characteristic polynomial is p.
(ii) Consider the polynomial

$$q(x) = -x^3 + sx^2 - qx + d, \quad s, q, d \in \mathbb{C}.$$

Find a 3×3 matrix B such that its characteristic polynomial is q.

Problem 35. Let A, B be $n \times n$ matrices over $\mathbb{C}$. The matrix A is called *similar* to the matrix B if there is a $n \times n$ invertible matrix S such that

$$A = S^{-1}BS.$$

If A is similar to B, then B is also similar to A, since $B = SAS^{-1}$.
(i) Consider the two matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Are the matrices similar?
(ii) Consider the two matrices

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Are the matrices similar?

Problem 36. (i) Consider the matrix

$$R = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}.$$

Show that $R^{-1} = R^* = R$. Use these properties and $\mathrm{tr}(R)$ to find all the eigenvalues of R. Find the eigenvectors.
(ii) Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Calculate RA_1R^{-1} and RA_2R^{-1}. Discuss.

Problem 37. (i) Find the conditions on the 2×2 matrices over $\mathbb{C}$ such that

$$ABA = BAB.$$

Find solutions where $AB \neq BA$, i.e. $[A, B] \neq 0_2$.

(ii) Find the conditions on the 2×2 matrices A and B such that

$$A \otimes B \otimes A = B \otimes A \otimes B.$$

Find solutions where $AB \neq BA$, i.e. $[A, B] \neq 0_2$.

Problem 38. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that A^{-1} exists. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $\mathbf{u}, \mathbf{v}$ are considered as column vectors. (i) Show that if

$$\mathbf{v}^T A^{-1} \mathbf{u} = -1$$

then $A + \mathbf{u}\mathbf{v}^T$ is not invertible.
(ii) Assume that $\mathbf{v}^T A^{-1} \mathbf{u} \neq -1$. Show that

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}}.$$

Problem 39. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let

$$U(\alpha, \beta, \gamma) = e^{-i\alpha\sigma_3/2} e^{-i\beta\sigma_2/2} e^{-i\gamma\sigma_3/2}$$

where α, β, γ are the three *Euler angles* with the range $0 \leq \alpha < 2\pi$, $0 \leq \beta \leq \pi$ and $0 \leq \gamma < 2\pi$. Show that

$$U(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i\alpha/2} \cos(\beta/2) e^{-i\gamma/2} & -e^{-i\alpha/2} \sin(\beta/2) e^{i\gamma/2} \\ e^{-i\alpha/2} \sin(\beta/2) e^{-i\gamma/2} & e^{i\alpha/2} \cos(\beta/2) e^{i\gamma/2} \end{pmatrix}. \tag{1}$$

Problem 40. Let A and B be $n \times n$ matrices over $\mathbb{R}$. Assume that $A \neq B$, $A^3 = B^3$ and $A^2 B = B^2 A$. Is $A^2 + B^2$ invertible?

Problem 41. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that A and $A + B$ are invertible. Show that

$$(A + B)^{-1} \equiv A^{-1} - A^{-1} B (A + B)^{-1}.$$

Apply the identity to $A = \sigma_1$, $B = \sigma_3$.

Problem 42. Let A be an $n \times n$ matrix over $\mathbb{R}$ and let $\mathbf{u}$ be an n-vector in $\mathbb{R}^n$ (column vector) with $\mathbf{u} \neq \mathbf{u}$. In numerical linear algebra we often have to compute

$$\left(I_n - \frac{2\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}} \right) \tag{1}$$

where I_n is the $n \times n$ identity matrix. Naively we would form the matrix $(I_n - 2\mathbf{u}\mathbf{u}^T/\mathbf{u}^T\mathbf{u})$ from the vector $\mathbf{u}$ and then form the matrix product explicitly with

A. This would require $O(m^3)$ flops. Provide a faster computation for expression (1).

Problem 43. Find all 2×2 matrices A_1, A_2, A_3 such that

$$A_1 A_2 = A_2 A_3, \qquad A_3 A_1 = A_2 A_3.$$

Problem 44. Let A be an $n \times n$ normal matrix with pairwise different eigenvalues. Are the matrices

$$P_j = \prod_{k=1, j \neq k}^{n} \frac{A - \lambda_k I_n}{\lambda_j - \lambda_k}$$

projection matrices?

Problem 45. Let $n \geq 2$ and $\omega = \exp(2\pi i/n)$. Consider the diagonal and permutation matrices, respectively

$$D = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \omega^{n-1} \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

(i) Show that $D^n = P^n = I_n$.
(ii) Show that the set of matrices

$$\{ D^j P^k \, : \, j, k = 0, 1, 2, \dots, n-1 \}$$

form a basis of the vector space of $n \times n$ matrices.
(iii) Show that

$$PD = \omega DP, \qquad P^j D^k = \omega^{jk} D^k P^j.$$

(iv) Find the matrix

$$X = \zeta P + \zeta^{-1} P^{-1} + \eta D + \eta^{-1} D^{-1}$$

and calculate the eigenvalues.

Problem 46. Let $z \in \mathbb{C}$. Consider the 2×2 matrices

$$A = \begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & z \\ z & -1 \end{pmatrix}, \quad C = \begin{pmatrix} z & 1 \\ 1 & z \end{pmatrix}, \quad D = \begin{pmatrix} z & 1 \\ -1 & z \end{pmatrix}.$$

Find the condition on z such that A, B, C, D are invertible.

Problem 47. Let $s = 1/2, 1, 3/2, 2, \ldots$ be the spin. Let $n = 2s+1$, i.e. for $s = 1/2$ we have $n = 2$, for $s = 1$ we have $n = 3$ etc. Consider the $n \times n$ matrix $V_s = (V_{jk})$ with

$$V_{jk} = \exp(c(s-j+1)(s-k+1))$$

where $j, k = 1, 2, \ldots, n$ and c is a positive constant.
(i) Let $s = 1/2$, i.e. $n = 2$. Let

$$R_{1/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Find $R_{1/2}^{-1} V_{1/2} R_{1/2}$.
(ii) Let s be positive integer with $n = 2s+1$ and the $n \times n$ matrix

$$R_s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \iddots & \\ & & 1 & 0 & 1 & & \\ & & 0 & \sqrt{2} & 0 & & \\ & & 1 & 0 & -1 & & \\ & \iddots & & & & \ddots & \\ 1 & & & & & & -1 \end{pmatrix}$$

Find $R_s V_s R_s$.
(iii) Let s be $1/2, 3/2, \ldots$ with $n = 2s+1$ and the $n \times n$ matrix

$$R_s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & 1 \\ & \ddots & & & \iddots & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \\ & \iddots & & & \ddots & \\ 1 & & & & & -1 \end{pmatrix}.$$

Find $R_s V_s R_s$.

Problem 48. Consider the *Bell matrix*

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

(i) Find all matrices A such that $BAB^* = A$.
(ii) Find all matrices A such that BAB^* is a diagonal matrix.

Problem 49. Let $n \geq 2$. An invertible integer matrix, $A \in GL_n(\mathbb{Z})$, generates a toral automorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ via the formula

$$f \circ \pi = \pi \circ A, \qquad \pi : \mathbb{R}^n \to \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n.$$

The set of fixed points of f is given by

$$\#\mathrm{Fix}(f) := \{\, x^* \in \mathbb{T}^n \;:\; f(x^*) = x^* \,\}$$

Now we have: if $\det(I_n - A) \neq 0$, then

$$\#\mathrm{Fix}(f) = |\det(I_n - A)|.$$

Let $n = 2$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that $\det(I_2 - A) \neq 0$ and find $\#\mathrm{Fix}(f)$.

Problem 50. Consider the symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Find a invertible matrix B such that $B^{-1}AB$ is a diagonal matrix.

Problem 51. Let V_1 be a hermitian $n \times n$ matrix. Let V_2 be a positive semidefinite $n \times n$ matrix. Let k be a positive integer. Show that

$$\mathrm{tr}((V_2 V_1)^k)$$

can be written as $\mathrm{tr}(V^k)$, where $V := V_2^{1/2} V_1 V_2^{1/2}$.

Problem 52. Can one find a 2×2 unitary matrix such that

$$U \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} U^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Problem 53. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let A, B be two arbitrary 2×2 matrices. Is

$$\frac{1}{2}\mathrm{tr}(AB) \equiv \sum_{j=1}^{3} \left(\frac{1}{2}\mathrm{tr}(\sigma_j A)\right) \left(\frac{1}{2}\mathrm{tr}(\sigma_j B)\right)?$$

Problem 54. In the following we count from $(0,0)$ to $(n-1, n-1)$ for $n \times n$ matrices. Let $\omega := \exp(2\pi i/n)$. Consider the $n \times n$ matrices

$$H = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & & \\ 0 & & & & 1 \\ 1 & 0 & & & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix}.$$

Then $H^n = G^n = I_n$, $HH^* = GG^* = I_n$, $HG = \omega GH$. Let U be the unitary matrix

$$U = \frac{1}{\sqrt{n}}(\omega^{jk})$$

where $j, k = 0, 1, \ldots, n-1$. Show that $UHU^{-1} = G$.

Problem 55. The *standard simplex* Δ_n is defined by the set in $\mathbb{R}^n$

$$\Delta_n := \{\, (x_1, \ldots, x_n)^T \,:\, x_j \geq 0,\ \sum_{j=1}^{n} x_j = 1 \,\}.$$

Consider n affinely independent points $B_1, \ldots, B_n \in \Delta_n$. They span an $(n-1)$-simplex denoted by $\Lambda = \mathrm{Con}(B_1, \ldots, B_n)$, that is

$$\Lambda = \mathrm{Con}(B_1, \ldots, B_n) = \{\, \lambda_1 B_1 + \cdots + \lambda_n B_n \,:\, \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_1, \ldots, \lambda_n \geq 0 \,\}.$$

The set corresponds to an invertible $n \times n$ matrix whose columns are $B_1, \ldots, B_n$. Conversely, consider the matrix $C = (b_{jk})$, where $C_k = (b_{1k}, \ldots, b_{nk})^T$ ($k = 1, \ldots, n$). If $\det(C) \neq 0$ and the sum of the entries in each column is 1, then the matrix C corresponds to an $(n-1)$-simplex $\mathrm{Con}(B_1, \ldots, B_n)$ in Δ_n. Let C_1 and C_2 be $n \times n$ matrices with nonegative entries and all the columns of each matrix add up to 1.
(i) Show that $C_1 C_2$ and $C_2 C_1$ are also such matrices.
(ii) Are the $n^2 \times n^2$ matrices $C_1 \otimes C_2$, $C_2 \otimes C_1$ such matrices?

Problem 56. Consider the matrices

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $n = 0, 1, 2, \ldots$. Study the sequence of matrices

$$A_{n+1} = A_n B_n, \qquad B_{n+1} = A_n.$$

Discuss. Is the sequence of matrices periodic?

Problem 57. Consider the alphabet $\Sigma = \{U, V, W\}$, axiom: $\omega = U$ and the set of production rules

$$U \mapsto UVW, \quad V \mapsto UV, \quad W \mapsto U.$$

(i) Apply it to $U = \sigma_1$, $V = \sigma_2$, $W = \sigma_3$ and matrix multiplication. Is the sequence periodic?

(ii) Apply it to $U = \sigma_1$, $V = \sigma_2$, $W = \sigma_3$ and the Kronecker product.

Problem 58. (i) Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that the inverse of A and $A + B$ exists. Show that

$$(A + B)^{-1} \equiv A^{-1} - A^{-1}B(A + B)^{-1}.$$

Apply the identity to

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 59. Let M be an inertible $n \times n$ matrix. Then we can form

$$M_+ = \frac{1}{2}(M + M^{-1}), \qquad M_- = \frac{1}{2}(M - M^{-1}).$$

Let

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Find M_+ and M_-. M is a nonnormal matrix. Are the matrices M_+ and M_- normal?

Problem 60. Let A, B be $n \times n$ matrices over $\mathbb{C}$. The two $n \times n$ matrices A and B have a common eigenvector if and only if the $n \times n$ matrix

$$\sum_{j,k=1}^{n-1} [A^j, B^k]^* [A^j, B^k]$$

is singular, i.e. the determinant is equal to 0.
(i) Apply the theorem to the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(ii) Apply the theorem to the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad B = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) Give a computer algebra implementation of this condition for two given matrices.

Problem 61. Let M be the Minkowski space endowed with the standard coordinates x_0, x_1, x_2, x_3 with the metric tensor field

$$g = -dx_0 \otimes dx_0 + dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dx_3 \otimes dx_3$$

and the quadratic form

$$q(x) = -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2.$$

Let $H(2)$ be the vector space of 2×2 hermitian matrices. Consider the map $\varphi : M \to H(2)$

$$\varphi(x) = X = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$$

which is an isomorphism. Find the determinant of X. Find the Cayley transform of X, i.e.

$$U = (X + iI_2)^{-1}(X - iI_2)$$

with the inverse

$$X = i(I_2 + U)(I_2 - U)^{-1}.$$

Problem 62. Consider the four-dimensional real vector space with a basis e_0, e_1, e_2, e_3. The *Gödel quaternion algebra* $\mathbb{G}$ is defined by the non-commutative multiplication

$$\begin{aligned} &e_0 e_k = e_k = e_k e_0, \qquad (e_0)^2 = e_0 \\ &e_j e_k = (-1)^\ell \epsilon_{jk\ell} e_\ell - (-1)^k \delta_{jk} e_0, \quad j,k,\ell = 1,2,3. \end{aligned}$$

Let q_0, q_1, q_2, q_3 be arbitrary real numbers. We call the vector

$$q = q_o e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$$

a Gödel quaternion. We define the basis

$$e_{11} := \frac{1}{2}(e_0 + e_3), \qquad e_{22} := \frac{1}{2}(e_0 - e_3)$$

$$e_{12} := \frac{1}{2}(e_1 + e_2), \qquad e_{21} := \frac{1}{2}(e_1 - e_2).$$

Show that the e_{jk} satisfy the multiplication law

$$e_{jk} e_{rs} = \delta_{kr} e_{js}, \qquad j,k,r,s = 1,2.$$

Show that every Gödel quaternion q can be written as

$$q = \sum_{j,k=1}^{2} q_{jk} e_{jk}.$$

Show that

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a matrix representation.

Problem 63. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that $[A, B] \neq 0_n$. Can we conclude that

$$[e^A, e^B] \neq 0_n\,?$$

Problem 64. Consider the hermitian 3×3 matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Find A^2 and A^3. We know that

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3, \quad \mathrm{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad \mathrm{tr}(A^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3.$$

Use Newton's method to solve this system of three equations to find the eigenvalues of A.

Problem 65. The matrices g_1 and g_2 play a role for the matrix representation of the braid group B_4

$$g_1 = \begin{pmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t^{-1} \end{pmatrix}, \quad g = \begin{pmatrix} 1-t & -t^{-1} & t^{-1} \\ 1-t^2 & -t^{-1} & 0 \\ 1 & -t^{-1} & 0 \end{pmatrix}$$

as generators. Let $g_2 = g_1 g^{-1}$. Find the eigenvalues and eigenvectors of g_1, g_2.

Problem 66. (i) Given an $m \times n$ matrix over $\mathbb{R}$. Write a C++ program that finds the maximum value in each row and then the minimum value of these values.
(ii) Given an $m \times n$ matrix over $\mathbb{R}$. Write a C++ program that finds the minimum value in each row and then the maximum value of these values.

Problem 67. Given an $m \times n$ matrix over $\mathbb{C}$. Find the elements with the largest absolute values and store the entries (j, k) $(j = 0, 1, \ldots, m-1)$ $k = 0, 1, \ldots, n-1)$ which contain the elements with the largest absolute value.

Problem 68. (i) Let $j_0, j_1, k_0, k_1 \in \{0, 1\}$. Consider the tensor

$$T^{j_0, j_1}_{k_0, k_1}.$$

Give a $1-1$ map that maps $T^{j_0,j_1}_{k_0,k_1}$ to a $2^2 \times 2^2$ matrix $S = (s_{\ell_0,\ell_1})$ with $\ell_0, \ell_1 = 0, 1, 2, 3$, i.e

$$\begin{pmatrix} t^{00}_{00} & t^{00}_{01} & t^{00}_{10} & t^{00}_{11} \\ t^{01}_{00} & t^{01}_{01} & t^{01}_{10} & t^{01}_{11} \\ t^{10}_{00} & t^{10}_{01} & t^{10}_{10} & t^{10}_{11} \\ t^{11}_{00} & t^{11}_{01} & t^{11}_{10} & t^{11}_{11} \end{pmatrix} \mapsto \begin{pmatrix} s_{00} & s_{01} & s_{02} & s_{03} \\ s_{10} & s_{11} & s_{12} & s_{13} \\ s_{20} & s_{21} & s_{22} & s_{23} \\ s_{30} & s_{31} & s_{32} & s_{33} \end{pmatrix}.$$

(ii) Let $j_0, j_1, j_2, k_0, k_1, k_2 \in \{0, 1\}$. Consider the tensor

$$T^{j_0,j_1,j_2}_{k_0,k_1,k_2}.$$

Give a $1-1$ map that maps $T^{j_0,j_1,j_2}_{k_0,k_1,k_2}$ to a $2^3 \times 2^3$ matrix $S = (s_{\ell_0,\ell_1})$ with $\ell_0, \ell_1 = 0, 1, \ldots, 2^3 - 1$.
(iii) Let $n \geq 2$ and $j_0, j_1, \ldots, j_n, k_0, k_1, \ldots, k_n \in \{0, 1\}$. Consider the tensor

$$T^{j_0,j_1,\ldots,j_{n-1}}_{k_0,k_1,\ldots,k_{n-1}}.$$

(i) Give a $1-1$ map that maps $T^{j_0,j_1,\ldots,j_{n-1}}_{k_0,k_1,\ldots,k_{n-1}}$ to a $2^n \times 2^n$ matrix $S = (s_{\ell_0,\ell_1})$ with $\ell_0, \ell_1 = 0, 1, \ldots, 2^n - 1$.
(ii) Give a SymbolicC++ implementation. The user provides n.

Problem 69. Show that any rank one positive semidefinite $n \times n$ matrix A can be written as $A = \mathbf{v}\mathbf{v}^T$, where $\mathbf{v}$ is some (column) vector in $\mathbb{C}^n$.

Problem 70. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Let $\mathbf{v}$ be a normalized (column) vector in $\mathbb{C}^n$. Let $\langle A \rangle := \mathbf{v}^* A \mathbf{v}$ and $\langle B \rangle := \mathbf{v}^* B \mathbf{v}$. We have the identity

$$AB \equiv (A - \langle A \rangle I_n)(B - \langle B \rangle I_n) + A\langle B \rangle + B\langle A \rangle - \langle A \rangle \langle B \rangle I_n.$$

We approximate AB as

$$AB \approx A\langle B \rangle + B\langle A \rangle - \langle A \rangle \langle B \rangle I_n.$$

Let

$$A = \sigma_1, \quad B = \sigma_2, \quad \mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(i) Find AB and $A\langle B \rangle + B\langle A \rangle - \langle A \rangle \langle B \rangle I_n$ and the distance (Frobenius norm) between the two matrices.
(ii) Apply the result to the case $n = 2$ and

$$A = \sigma_1, \quad B = \sigma_2, \quad \mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Problem 71. The $(n+1)\times(n+1)$ *Hadamard matrix* $H(n)$ of any dimension is generated recursively as follows

$$H(n) = \begin{pmatrix} H(n-1) & H(n-1) \\ H(n-1) & -H(n-1) \end{pmatrix}$$

where $n = 1, 2, \ldots$ and $H(0) = (1)$. Find $H(1)$, $H(2)$. and $H(3)$.

Problem 72. Consider the symmetric 6×6 matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

This matrix plays a role in the construction of the *icosahedron* which is a regular polyhedron with 20 identical equilateral triangular faces, 30 edges and 12 vertices.
(i) Find the eigenvalues of this matrix.
(ii) Consider the matrix $A + \sqrt{5}I_6$. Find the eigenvalues.
(iii) the matrix $A + \sqrt{5}I_6$ induces an Euclidean structure on the quotient space $\mathbb{R}^6/\ker(A + \sqrt{5}I_6)$. Find $\ker(A + \sqrt{5}I_6)$.

Problem 73. Let A be an arbitrary $n \times n$ matrix over $\mathbb{R}$. Can we conclude that A^2 is positive semi-definite?

Problem 74. Show that the matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

as the product of two 4×4 matrices A and B such that each of these matrices has precisely two non-zero entries in each row.

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