Wilson renormalization

Landau - Ginzburg.

$$H = \int \left(\frac{1}{2} (\nabla x(\underline{r}))^2 + \frac{1}{2} r_0 x^2(\underline{r}) + \underbrace{u x^4(\underline{r})}_{} \right) d^d r$$

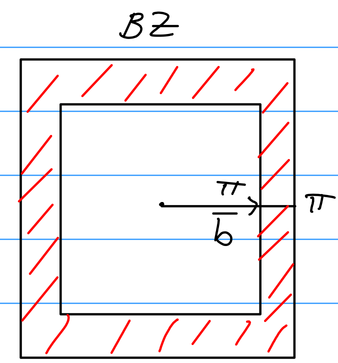
Fourier transform of the last term :

$$\sum_{\underline{r}} x^4(\underline{r})\, d^d r = \frac{1}{N^2} \sum_{\underline{r}} \sum_{\underline{q}_1 \underline{q}_2 \underline{q}_3 \underline{q}_4} x_{\underline{q}_1} x_{\underline{q}_2} x_{\underline{q}_3} x_{\underline{q}_4} e^{i(\underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \underline{q}_4)\underline{r}}$$

$N\delta(\underline{q}_1 + \dots + \underline{q}_4)$

$$= \frac{1}{N} \sum_{\underline{q}_1 \dots \underline{q}_4} x_{\underline{q}_1} x_{\underline{q}_2} x_{\underline{q}_3} x_{\underline{q}_4} \delta(\underline{q}_1 + \dots + \underline{q}_4).$$

$$\rightarrow \quad H = \underbrace{\sum_{|q_i| \leq \pi} \frac{1}{2} (q^2 + r_0) x_{\underline{q}} x_{-\underline{q}}}_{H_g} + \underbrace{\sum_{\underline{q}_1 \dots \underline{q}_4} \frac{u}{N} x_{\underline{q}_1} x_{\underline{q}_2} x_{\underline{q}_3} x_{\underline{q}_4} \delta(\underline{q}_1 + \dots + \underline{q}_4)}_{V}.$$



① $$e^{-H'} = \int \prod_{\frac{\pi}{b} < |q_i| < \pi} dx_{\underline{q}}\, dx_{-\underline{q}}\; e^{-H}$$

b rescaling length

$$e^{-H'} = \int \prod_{B_b} dx_{\underline{q}}\, dx_{-\underline{q}}\; e^{-H_g - V} =$$

B_b: $\frac{\pi}{b} < |q_i| < \pi$

$$e^{-H'} = \int \prod_{B_b} dx_{\underline{q}}\, dx_{-\underline{q}}\; e^{-H_g} \left(1 - V + \frac{1}{2!} V^2 + \dots \right).$$

$$= e^{-H_g'} \left(1 - \langle V \rangle_g + \frac{1}{2!} \langle V^2 \rangle_g + \dots \right).$$

$$\text{where } \langle V \rangle_g = \frac{\int \prod_{B_b} dx_{\underline{q}}\, dx_{-\underline{q}}\, e^{-H_g}\, V}{\underbrace{\int \prod_{B_b} dx_{\underline{q}}\, dx_{-\underline{q}}\, e^{-H_g}}_{e^{-H_g'}}} \quad \text{etc.}$$

$$e^{-H'} = e^{-H_g' - V'} \rightarrow e^{-V'} = 1 - \langle V \rangle_g + \frac{1}{2} \langle V^2 \rangle_g + \cdots$$

$$-V' = \ln\Big(1 \underbrace{- \langle V \rangle_g + \frac{1}{2} \langle V^2 \rangle_g + \cdots}_{x}\Big)$$

Using $\ln(1+x) = x - \frac{x^2}{2} + \ldots$

we obtain $V' = \langle V \rangle_g - \frac{1}{2}\left(\langle V^2 \rangle_g - \langle V \rangle_g^2\right) + \ldots$ $O(V^3)$.

Cumulant expansion.

We need to calculate $\langle V \rangle_g$

$$\langle x_{\underline{q}_1} x_{\underline{q}_2} x_{\underline{q}_3} x_{\underline{q}_4} \rangle_g = \frac{\int \prod_{Bb} dx_{\underline{q}}\, dx_{-\underline{q}}\, e^{-\frac{1}{2}\sum_{\underline{q}\in B_b} x_{\underline{q}} x_{-\underline{q}} (\underline{q}^2 + r_0)}\, x_{\underline{q}_1} x_{\underline{q}_2} x_{\underline{q}_3} x_{\underline{q}_4}}{\int \prod_{Bb} dx_{\underline{q}}\, dx_{-\underline{q}}\, e^{-\frac{1}{2}\sum_{\underline{q}\in Bb} x_{\underline{q}} x_{-\underline{q}} (\underline{q}^2 + r_0)}}$$

From our previous analysis, we may construct two pairs:

$x_{\underline{q}} x_{-\underline{q}}$, $x_{\underline{q}'} x_{-\underline{q}'}$; $\underline{q} \neq \pm \underline{q}'$.

$$\langle x_{\underline{q}} x_{-\underline{q}} x_{\underline{q}'} x_{-\underline{q}'} \rangle = \frac{1}{q^2 + r_0} \frac{1}{q'^2 + r_0}$$

$\underline{q}=\underline{q}'$:
$$\frac{\int dx_{\underline{q}}\, dx_{-\underline{q}}\; e^{-\frac{1}{2}(r_0+q^2)x_{+\underline{q}}x_{-\underline{q}}}\,(x_{\underline{q}}x_{-\underline{q}})^2}{\int dx_{\underline{q}}\, dx_{-\underline{q}}\; e^{-\frac{1}{2}(r_0+q^2)x_{\underline{q}}x_{-\underline{q}}}} = \frac{2}{(r_0+q^2)^2}$$

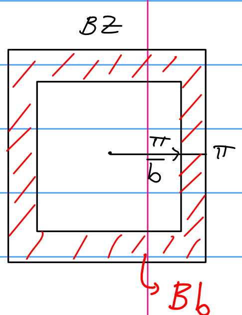
$$\langle x_{\underline{q}_1}x_{\underline{q}_2}x_{\underline{q}_3}x_{\underline{q}_4}\rangle_{\tilde{q}} = \langle x_{\underline{q}_1}x_{\underline{q}_2}\rangle\langle x_{\underline{q}_3}x_{\underline{q}_4}\rangle + \langle x_{\underline{q}_1}x_{\underline{q}_3}\rangle\langle x_{\underline{q}_2}x_{\underline{q}_4}\rangle$$

$\underline{q}\quad -\underline{q}\quad \underline{q}'\quad -\underline{q}'$

$$+\ \langle x_{\underline{q}_1}x_{\underline{q}_4}\rangle\langle x_{\underline{q}_2}x_{\underline{q}_3}\rangle$$

$\underline{q}'\neq\underline{q}$

Wick's theorem

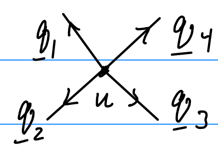


We see that $\delta(\underline{q}_1+\underline{q}_2+\underline{q}_3+\underline{q}_4)$ is always guaranteed.

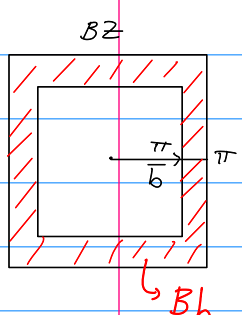
Now consider $\langle V\rangle_{\tilde{q}} = -\frac{u}{N}\langle x_{\underline{q}_1}x_{\underline{q}_2}x_{\underline{q}_3}x_{\underline{q}_4}\rangle_{\tilde{q}}$. We must consider 3 cases

Case 1 all $\underline{q}_i$'s outside B_b

Leads to $-\frac{u}{N}\, x_{\underline{q}_1}x_{\underline{q}_2}x_{\underline{q}_3}x_{\underline{q}_4}\,\delta(\underline{q}_1+\dots+\underline{q}_4)$

Diagram: 

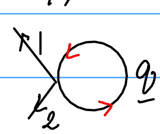
Note $\underline{q}_1+\underline{q}_2+\underline{q}_3+\underline{q}_4=0$
'momentum conservation'



Case 2 — 2 $\underline{q}$'s form a pair with $\underline{q}\in B_b$.
2 other $\underline{q}$'s outside B_b

The choice for the pair can be made in $\binom{4}{2}=6$ ways

This yields: $-\frac{6u}{N}x_{\underline{q}_1}x_{\underline{q}_2}\sum_{\underline{q}\in B_b}\underbrace{\frac{1}{r_0+q^2}}_{\langle x_{\underline{q}}x_{-\underline{q}}\rangle}$; $\underline{q}_1,\underline{q}_2\notin B_b$.



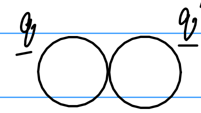
Note that $\underline{q}_1+\underline{q}_2=\underline{0}$ due to momentum conservation

Case ③ We have two pairs, both in B_b. $\underline{q} \neq \underline{q}'$

These can be chosen in 3 ways $\begin{cases} 12, 34 \\ 23, 14 \\ 13, 24 \end{cases}$

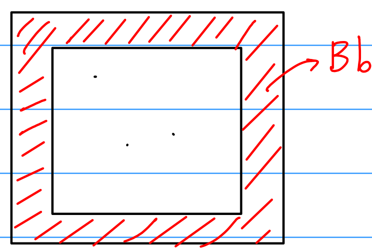
This yields the constant

$$-3\,\frac{u}{N}\left[\sum_{\underline{q}\in B_b}\left(\frac{1}{q^2+r_0}\right)\right]^2 .$$

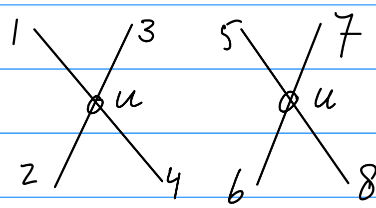
Diagram: 

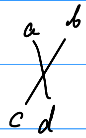
Now consider $\langle V^2\rangle = \frac{u^2}{N^2}\sum_{\underline{q}_1\ldots\underline{q}_8}\langle x_{\underline{q}_1}\ldots x_{\underline{q}_8}\rangle\,\delta(\underline{q}_1+\ldots+\underline{q}_4)\,\delta(\underline{q}_5+\ldots+\underline{q}_8)$.

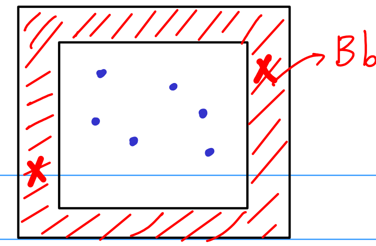
We must consider 4 cases.



① all $\underline{q}_i$ outside B_b: $\langle V^2\rangle$

Diagram: 

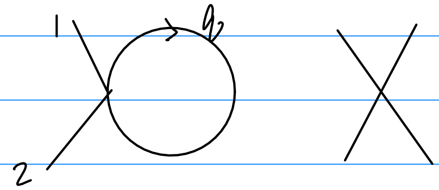
Vertex : a factor $\frac{u}{N}$ and $\delta(\underline{q}_a+\underline{q}_b+\underline{q}_c+\underline{q}_d)$.



② Two $\underline{q}_i$ inside Bb

$\underline{q}_1 \cdots \underline{q}_4 \quad \underline{q}_5 \cdots \underline{q}_8$

2a



$$-6 \times 2 \left(\frac{u}{N}\right)^2 x_{\underline{q}_1} \cdots x_{\underline{q}_6} \sum_{\underline{q} \in Bb} \frac{1}{r_0 + \underline{q}^2} \delta(\underline{q}_1 + \underline{q}_2)\delta(\underline{q}_3 + \cdots + \underline{q}_6)$$

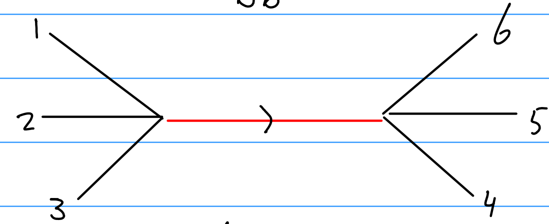
Cumulant expansion:

$$V' = \langle V \rangle - \tfrac{1}{2}\left(\langle V^2 \rangle - \langle V \rangle^2\right) + \cdots$$

All disconnected diagrams disappear.

2b

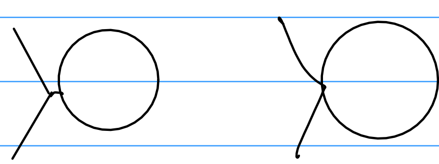
$$\underbrace{\underline{q}_1 \cdots \underline{q}_4 \quad \underline{q}_5 \cdots \underline{q}_8}_{\text{in } Bb}$$



discard.

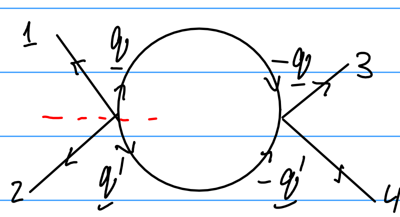
③ 2 pairs $\underline{q}, -\underline{q} \quad \underline{q}', -\underline{q}'$ in Bb

3a



Disconnected.

3b



$\binom{4}{2} = 6$

$\binom{4}{2} = 6$

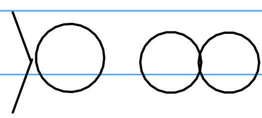
72 possibilities.

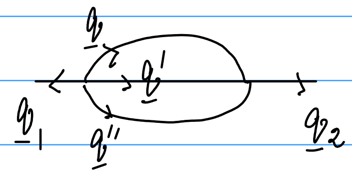
$$\underline{q}_1+\underline{q}_2+\underline{q}+\underline{q}'=0. \qquad \left.\begin{array}{l} \underline{q}_1+\underline{q}_2=-\underline{q}-\underline{q}' \\ q_3+q_4=\underline{q}+\underline{q}' \end{array}\right\}$$

$$\Rightarrow\ -\frac{1}{2}\left(\frac{U}{N}\right)^2\cdot 72\cdot x_{\underline{q}_1}x_{\underline{q}_2}x_{\underline{q}_3}x_{\underline{q}_4}\sum_{\underline{q}\in B_b}\frac{1}{(r_0+\underline{q}^2)}\,\frac{1}{r_0+(\underline{q}_1+\underline{q}_2+\underline{q})^2}\,\delta(\underline{q}_1+\dots+\underline{q}_4)$$

This term 'renormalizes the 4-pt interaction'.

(4) 3 pairs $\underline{q},-\underline{q}$ in B_b.

4a  disconnected $\Rightarrow$ vanishes.

4b 

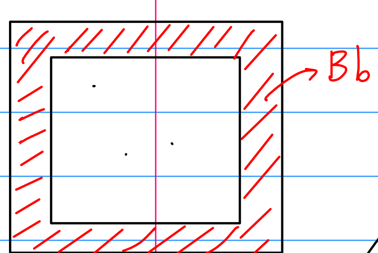
two-pt interaction.

$$-\frac{1}{2}\left(\frac{U}{N}\right)^2 4\times4\times6\ x_{\underline{q}_1}x_{\underline{q}_2}\sum_{\underline{q}\in B_b}\frac{1}{\underline{q}^2+r_0}\sum_{\underline{q}'\in B_b}\frac{1}{\underline{q}'^2+r_0}\,\frac{1}{(\underline{q}_1-\underline{q}-\underline{q}')+r_0}\,\delta(\underline{q}_1+\underline{q}_2)$$

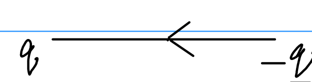
Scales with $db^2=(b-1)^2$.
$\neq$

In summary, after step 1 of the RG procedure:

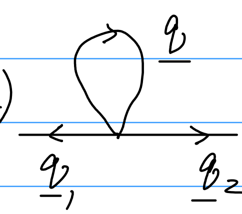
Diagrams:



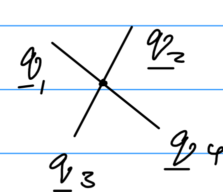
$$H'=-\sum_{\underline{q}\notin B_b}\frac{1}{2}(r_0+\underline{q}^2)x_{\underline{q}}x_{-\underline{q}}$$



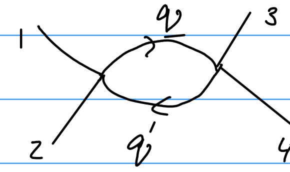
$$-\frac{6U}{N}\sum_{\underline{q}_1,\underline{q}_2\notin B}x_{\underline{q}_1}x_{\underline{q}_2}\underbrace{\sum_{\underline{q}\in B_b}\frac{1}{r_0+\underline{q}^2}}_{NA\sim b-1}\delta(\underline{q}_1+\underline{q}_2)$$



$$+\sum_{\underline{q}_i\notin B_b}\frac{U}{N}x_{\underline{q}_1}x_{\underline{q}_2}x_{\underline{q}_3}x_{\underline{q}_4}\delta(\underline{q}_1+\dots+\underline{q}_4)$$



$$-36\sum_{q_i\notin B_b}\left(\frac{u}{N}\right)^2 x_{q_1}x_{q_2}x_{q_3}x_{q_4}\sum_{q\in B_b}\frac{1}{(r_0+q^2)}\,\frac{1}{r_0+(q_1+q_2+q)^2}\,\delta(q_1+\dots+q_4)$$



$$\frac{1}{r_0+(q+\underbrace{q_1+q_2}_{})^2}=\frac{1}{r_0+q^2}-\cancelto{0}{\frac{2q\cdot(q_1+q_2)}{(r_0+q^2)^2}}+\frac{\big(q\cdot(q_1+q_2)\big)^2+(r_0+q^2)(q_1+q_2)^2}{(r_0+q^2)^3}$$

$(\nabla x)^2x^2.$ irrelevant.

$$=-36\left(\frac{u}{N}\right)^2\sum_{q_i\notin B_b}x_{q_1}x_{q_2}x_{q_3}x_{q_4}\underbrace{\sum_{q\in B_b}\frac{1}{(r_0+q^2)^2}}_{NB\,\sim\, b-1}\delta(q_1+\dots+q_4).$$

$r_0' = r_0 + 12\,u\,A$; $A = A(r_0)$

$u' = u - 36\,u^2 B$

$$H'=-\sum_{|q_i|<\frac{\pi}{b}}\frac{1}{2}(r_0'+q^2)x_q x_{-q}+\frac{u'}{N}\sum_{|q_i|\le\frac{\pi}{b}}x_{q_1}x_{q_2}x_{q_3}x_{q_4}\,\delta(q_1+\dots+q_4)$$

$q_i \to b\,q_i$, i.e. $q_i' = b\,q_i$

Lattice const $1 \to b$

$$N'=\frac{N}{b^d}$$

$$H''=-\sum_{|q_i'|\le\pi}\frac{1}{2}\left(r_0'+q'^2/b^2\right)\lambda^2\,\overset{x(q'/b)}{\tilde{x}_{q'}}\,\tilde{x}_{q'}+\frac{u'}{N'}b^{-d}\sum_{|q'|\le\pi}\lambda^4\,\tilde{x}_{q_1'}\dots\tilde{x}_{q_4'}\,\delta(q_1+\dots+q_4)$$

$\lambda = b$, $q' \to q$, $\tilde{x} \to x$

$$\to H'''=-\sum_{q\in BZ}\frac{1}{2}\left(b^2 r_0'+q^2\right)x_q x_{-q}+\frac{u'}{N'}b^{4-d}\sum_{q_i\in BZ}x_{q_1}\dots x_{q_4}\,\delta(q_1+\dots+q_4).$$

$$r_0' = r_0 + 12\,u\,A$$
$$u' = u - 36\,u^2 B$$

Now: $$\boxed{\begin{aligned} r_0'' &= r_0 b^2 + 12\,u\,A\,b^2 \\ u'' &= u\,b^{4-d} - 36\,u^2 b^{4-d} B \end{aligned}}$$

Gaussian fixed point: $r_0 = u = 0$

Then: $H = \frac{1}{2} \sum_{\underline{q} \in B} q^2\, x_{\underline{q}}\, x_{-\underline{q}}$

Linearize: $$\underline{\underline{A}} = \begin{pmatrix} \partial r_0'/\partial r_0 & \partial r_0'/\partial u \\ \partial u'/\partial r_0 & \partial u'/\partial u \end{pmatrix} = \begin{pmatrix} b^2 & 12 A b^2 \\ 0 & b^{4-d} \cancel{- 72\, \underset{\overset{\shortparallel}{0}}{u}\, b^{4-d}} B \end{pmatrix}$$

Eigen vector: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_1 = b^2 \rightarrow z = 2$ $\quad \nu = 1/z = 1/2$.

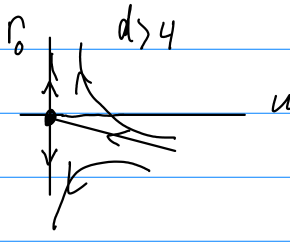
relevant

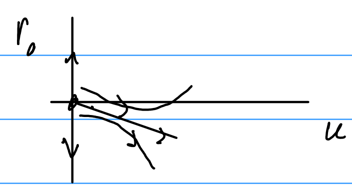
and $\begin{pmatrix} a \\ 1 \end{pmatrix}$: $b^2 a + 12 A b^2 = b^{4-d} a \rightarrow a = \dfrac{-12A}{1 - b^{2-d}}$

$b^{4-d} = \lambda_2$ relevant $d < 4$

irrelev. $d > 4$.

$d < 4$



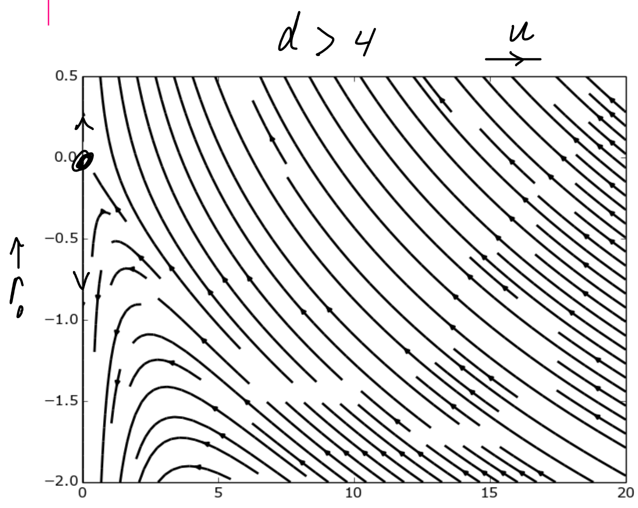


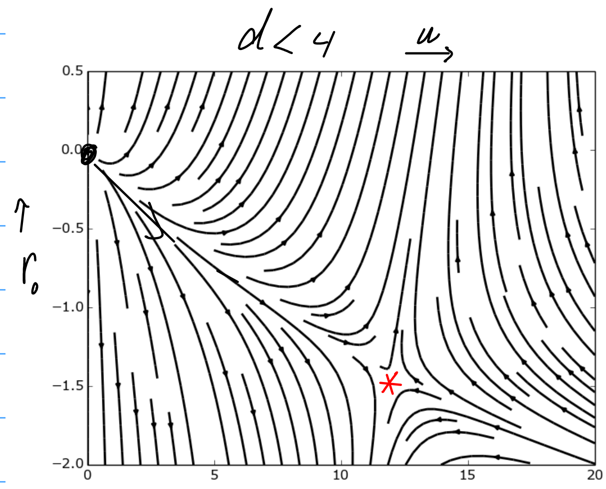
There is also a non-Gaussian fixed point for u and r_0 finite.

$$\boxed{\begin{aligned} r_0^* &= r_0^* b^2 + 12 u^* A b^2 \\ u^* &= u^* b^{4-d} - 36 u^{*2} b^{4-d} B \end{aligned}}$$

From the 2nd eq, we see that $u^* = \dfrac{b^{4-d} - 1}{36 b^{4-d} B}$ $\to$ $u > 0$ $\to b^{4-d} > 1$ $\to d < 4$

$$r_0 = r_0 b^2 + \frac{1}{3}\frac{A}{B} b^2 (1 - b^{d-4}) \to r_0^* = \frac{A}{3B} \frac{1 - b^{d-4}}{1 - b^2} b^2$$





$$A = \sum_{q \in B_b} \frac{1}{r_0 + q^2} \simeq (b-1) \cdot \frac{\pi^{d-1}}{r_0 + \pi^2}$$

$\epsilon = 4 - d$ (Wilson and Fisher, 1974).

$$\begin{aligned} r_0' &= r_0 b^2 + 12 u A b^2 \\ u' &= u b^{\epsilon} - 36 u^2 b^{\epsilon} B \end{aligned}$$

$$u^* = \frac{b^{4-d} - 1}{36 b^{4-d} B} = \frac{b^{\epsilon} - 1}{36 b^{\epsilon} B}$$

$$r_0^* = \frac{A}{3B} \frac{1 - b^{d-4}}{1 - b^2} b^2 = \frac{A}{3B} \frac{1 - b^{-\epsilon}}{b^{-2} - 1}$$

$b^{\epsilon} \approx 1 + \epsilon \ln b$

$$u^* = \frac{1-b^{-\epsilon}}{36B} = \frac{\epsilon \overbrace{\ln b}^{\sim b-1}}{3\underbrace{bB}_{b-1}}$$

$$r_0^* = -\frac{A\epsilon\ln b}{6B\ln b} = -\frac{\epsilon}{6}\frac{A}{B}$$

$$\begin{aligned} u &= u^*+\delta u \\ r_0 &= r_0^*+\delta r_0 \end{aligned} \quad \xrightarrow{RT} \quad \begin{aligned} &u^*+\delta u' \\ &r_0^*+\delta r_0' \end{aligned} \qquad \boxed{\begin{aligned} r_0' &= r_0 b^2 + 12\, u A b^2 \\ u' &= u b^{\epsilon} - 36\, u^2 b^{\epsilon} B(r_0) \end{aligned}}$$

$$\delta u' = \delta u\, b^{\epsilon} - 72 u^*\delta u\, b^{\epsilon} B - \underbrace{\cancel{36 u^{*2} b^{\epsilon}\frac{dB}{dr_0}}}_{\mathcal{O}(\epsilon^2)}\delta r_0$$

$$\left(u^* = \frac{\epsilon\ln b}{36B}\right) \qquad = \delta u\left(b^{\epsilon} - 2\epsilon\ln b\right)$$

$$\delta r_0' = \delta r_0 b^2 + 12 A b^2\delta u + 12\overbrace{\frac{dA}{dr_0}}^{-B}\delta r_0 b^2 u^*$$

$$NA = \sum_{\underline{q}\in B_b}\frac{1}{q^2+r_0} \qquad \frac{dA}{dr_0} = -B$$

$$NB = \sum_{\underline{q}\in B_b}\frac{1}{(q^2+r_0)^2}$$

$$\delta r_0' = \delta r_0 b^2 - \frac{\delta r_0 b^2}{3} + 12 A b^2\delta u$$

Matrix for RT close to r_0^*, u^*:

$$\begin{array}{c|cc} & r_0 & u \\ \hline r_0 & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} b^2 - \frac{\epsilon}{3}\ln b & 12 A b^2 \\ 0 & b^{\epsilon}(1-2\epsilon\ln b) \end{pmatrix}$}} \\ u & & \end{array}$$

$$\Rightarrow \quad \lambda_T = b^2 - \frac{\epsilon}{3}\ln b \simeq b^{2-\epsilon/3} + \mathcal{O}(\epsilon^2) \qquad z = 2-\epsilon/3$$

$$1 > \lambda_u = b^{\epsilon}(1-2\epsilon\ln b) \simeq b^{-\epsilon} + \mathcal{O}(\epsilon^2). \qquad \text{irrelevant}$$

$$\nu = \frac{1}{2} \overset{\epsilon=1}{=} 0.6 \quad \text{or:} \quad \frac{1}{2-\epsilon/3} = \frac{1}{2} + \frac{\epsilon}{12} = 0.583$$

Experiment: $\nu = 0.6 - 0.7$

ϵ- exp to 5^{th} order: 0.631

$$x_{\underline{q}} \to \underline{x}_{\underline{q}} \quad \to \nu = \frac{1}{2} + \frac{1}{4}\frac{n+2}{n+8}\epsilon$$

n dimensions