Ideal Quantum Gases of Massive Particles

Free, massive particles $\hat{H} = \sum_{j=1}^{N} \frac{\hat{p}_j^2}{2m}$

One particle Hamiltonian: $\hat{H} = \frac{\hat{p}^2}{2m}$

$$\Rightarrow \langle \underline{r} | \Psi_{\underline{k}} \rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\underline{k}\cdot\underline{r}}.$$

$$V = L \times L \times L \qquad \underline{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$\sum_{\underline{k},\sigma} \rightarrow \frac{V}{(2\pi)^3} \int d^3k \; ; \quad \epsilon_{\underline{k}} = \frac{\hbar^2 k^2}{2m}$$

$$\langle n_{\underline{k}} \rangle = \frac{1}{e^{\beta(\epsilon_{\underline{k}} - \mu)} \pm 1}$$

$+$ Fermions

$-$ Bosons.

$$N = \sum_{\underline{k}} \langle n_{\underline{k}} \rangle$$

$$= \frac{V}{(2\pi)^3} \int \frac{1}{e^{\beta(\hbar^2 k^2/2m - \mu)} \pm 1} d^3k \qquad \beta = \frac{1}{k_B T}.$$

$$\left(x^2 = \beta \frac{\hbar^2 k^2}{2m} \right)$$

$$\Rightarrow \frac{N}{V} \lambda^3 = \frac{1}{\pi^{3/2}} \int_0^\infty \frac{1}{e^{(x^2 - \beta\mu)} \pm 1} d^3x$$

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

High energy $\rightarrow x^2 \gg \beta\mu \rightarrow$ classical limit

Then $\frac{N}{V}\lambda^3 = n\lambda^3 = \frac{1}{\pi^{3/2}}\int_0^\infty \left(e^{\beta\mu - x^2} \mp e^{2\beta\mu - 2x^2}\right) d^3x$

$$= e^{\beta\mu} \mp \underbrace{\frac{1}{2^{3/2}} e^{2\beta\mu}}_{\text{Qu. correction.}} + \dots$$

Equation of state $\qquad PV = k_B T \ln Z_{gr}$

$$Z_{gr} \begin{cases} = \prod_{\underline{k}} \left(1 + e^{\beta(\mu - \epsilon_{\underline{k}})}\right) & \text{Fermions} \\ = \prod_{\underline{k}} \frac{1}{1 - e^{\beta(\mu - \epsilon_{\underline{k}})}} & \text{Bosons} \quad \mu - \epsilon_{\underline{k}} < 0. \end{cases}$$

$$P = \frac{k_B T}{V} \sum_{\underline{k}} \ln\left(1 + e^{\beta(\mu - \epsilon_{\underline{k}})}\right) \qquad \text{Fermions}$$

$$= -\frac{k_B T}{V} \sum_{\underline{k}} \ln\left(1 - e^{\beta(\mu - \epsilon_{\underline{k}})}\right) \qquad \text{Bosons.}$$

$$\sum_{\underline{k}} \to \frac{V}{(2\pi)^3} \int d^3k$$

$$\frac{P}{k_B T} = \frac{1}{V} \ln Z_{gr} = \pm \frac{1}{\lambda^3 \pi^{3/2}} \int \underbrace{\ln\left(1 \pm e^{-x^2 + \beta\mu}\right)} d^3x$$

$x^2 = \frac{\hbar^2 k^2}{2m}$

$$\ln(1+\delta) \simeq \delta - \frac{\delta^2}{2}$$

$$\mp e^{-x^2 + \beta\mu} - \frac{1}{2} e^{-2x^2 + 2\beta\mu}$$

$$\frac{P}{k_B T} = \underbrace{\frac{1}{\lambda^3}\Big(e^{\beta\mu}}_{\text{Classical}} \mp \underbrace{\frac{1}{2^{5/2}} e^{2\beta\mu}\Big)}_{\text{Quantum.}}$$

$$n = \frac{1}{\lambda^3}\left(e^{\beta\mu} \mp \frac{1}{2^{3/2}} e^{2\beta\mu}\right).$$

$$P = n + A n^2 \quad \Rightarrow \quad \mp \frac{1}{\lambda^3}\frac{1}{2^{3/2}} + \frac{A}{\lambda^6 \lambda^3} = \mp \frac{1}{\lambda^3}\frac{1}{2^{5/2}}$$

$$A = \lambda^3\left(\mp \frac{1}{2^{5/2}} \pm \frac{1}{2^{3/2}}\right) = \pm \lambda^3 \frac{1}{2^{5/2}}$$

Hence $$\underline{\frac{P}{k_B T} = n\left(1 \pm \lambda^3 2^{-5/2} n\right).} \qquad \begin{matrix} + & \text{Fermions} \\ - & \text{Bosons.} \end{matrix}$$

Summary

Massive quantum particles, dilute limit

$$N = \sum_{\underline{k}} \langle n_{\underline{k}} \rangle = \frac{V}{(2\pi)^3} \int \frac{1}{e^{\beta\left(\frac{\hbar^2 \underline{k}^2}{2m} - \mu\right)} \pm 1} \, d^3 k \qquad \begin{matrix} + \; F \\ - \; B. \end{matrix}$$

$$\beta \frac{\hbar^2 k^2}{2m} - \beta\mu \gg 1$$

$$n = \frac{1}{\lambda^3}\left(e^{\beta\mu} \mp \frac{1}{2^{3/2}} e^{2\beta\mu}\right).$$

$$PV = k_B T \ln \mathcal{Z}_{gr} = \frac{1}{\lambda^3}\left(\underbrace{e^{\beta\mu}}_{\text{Classical}} \mp \underbrace{\frac{1}{2^{5/2}} e^{2\beta\mu}}_{\text{Quantum}}\right).$$

$$\frac{P}{k_B T} = n\left(1 \pm \lambda^3 2^{-5/2} n\right). \qquad \begin{matrix} + \; F \\ - \; B. \end{matrix}$$