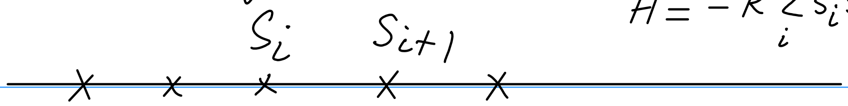
1D Ising model.

$$H = -K\sum_i s_i s_{i+1} - h\sum_i s_i$$



$J = \beta K = K/k_BT$

$B = \beta h = h/k_BT$

$$Z = \sum_{\{s_i=\pm\}} e^{J\sum_{i=1}^{N} s_i s_{i+1} + B\sum_{i=1}^{N} s_i}$$

$s_{N+1} \equiv s_1$ PBC.

$$= \sum_{\{s_i=\pm\}} \prod_i e^{J s_i s_{i+1} + B(s_i+s_{i+1})/2}.$$

$\begin{pmatrix}1\\0\end{pmatrix}$: $s=+$

$\begin{pmatrix}0\\1\end{pmatrix}$: $s=-$

Transfer matrix

$$\langle s_i|\hat{T}|s_{i+1}\rangle = e^{J s_i s_{i+1} + \frac{B}{2}(s_i+s_{i+1})}$$

$$\begin{array}{c|cc} & + & - \\ \hline + & e^{J+B} & e^{-J} \\ - & e^{-J} & e^{J-B} \end{array}$$

$$= \sum_{\{s_i=\pm\}} \langle s_1|\hat{T}\underbrace{|s_2\rangle\langle s_2|}_{\mathbb{1}}\hat{T}\underbrace{|s_3\rangle\langle s_3|}_{\mathbb{1}} \cdots \underbrace{|s_N\rangle\langle s_N|}_{\mathbb{1}}\hat{T}|s_1\rangle$$

$$= \sum_{\{s_1=\pm1\}} \langle s_1|\hat{T}^N|s_1\rangle = \mathrm{Tr}(\hat{T}^N)$$ trace

$$\hat{T}|\varphi_n\rangle = \lambda_n|\varphi_n\rangle. \qquad \langle\varphi_n|\varphi_n\rangle = 1.$$

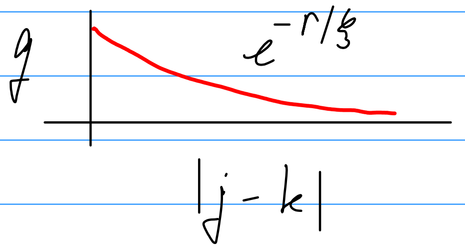
$$\mathrm{Tr}(\hat{T}^N) = \sum_n \langle\varphi_n|\hat{T}^N|\varphi_n\rangle =$$

$$\sum_n \lambda_n^N \overset{\text{large}}{=} \lambda_{max}^N$$

N: length of chain

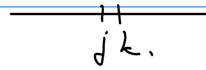
$+ \quad -$

$$g_{jk} = \langle s_j s_k\rangle - \overset{m}{\langle s_j\rangle}\overset{m}{\langle s_k\rangle}$$



$r = |j-k|$

$1 \le |j-k| \ll N$



From now $\lambda_{max} \equiv \lambda_0$, λ_1 is next largest eigenvalue.

$$\hat{s}|s_j\rangle = s_j|s_j\rangle \qquad \begin{array}{c|cc} & + & - \\ \hline + & 1 & 0 \\ - & 0 & -1 \end{array}$$

$$\langle s_j\rangle = \frac{\sum_{s_1=\pm1}\sum_{s_2=\pm1}\cdots\sum_{s_N=\pm}\langle s_1|\hat{T}|s_2\rangle\langle s_2|\ldots\hat{T}|s_j\rangle \boxed{s_j} \langle s_j|\hat{T}|s_{j+1}\rangle\cdots|\hat{T}|s_1}{Tr(\hat{T}^N)}$$

$$= \frac{\sum_{s_1=\pm1}\sum_{s_j=\pm1}\langle s_1|\hat{T}^j\hat{s}|s_j\rangle\langle s_j|\hat{T}^{N-j}|s_1\rangle}{Tr(\hat{T}^N)}$$

Trace is cyclic: $Tr(\hat{A}\hat{B}\hat{C}) = Tr(\hat{B}\hat{C}\hat{A}) = Tr(\hat{C}\hat{A}\hat{B})$

and: $\hat{T}^N = \sum_j |\varphi_j\rangle\lambda_j^N\langle\varphi_j| \approx |\varphi_0\rangle \underset{\uparrow \text{ largest}}{\lambda_0^N}\langle\varphi_0|$ if there are many $\hat{T}$'s

$$= \frac{Tr(\hat{T}^N\hat{s})}{Tr(\hat{T}^N)} = \frac{Tr(|\varphi_0\rangle\lambda_0^N\langle\varphi_0|\hat{s})}{\lambda_0^N} = \langle\varphi_0|\underset{\underset{\sigma_z}{\uparrow}}{\hat{s}}|\varphi_0\rangle \qquad \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\langle s_j s_k\rangle = \frac{Tr(\hat{T}^j\hat{s}\hat{T}^{k-j}\hat{s}\hat{T}^{N-k})}{Tr(\hat{T}^N)} =$$

$$\frac{Tr(\overbrace{|\varphi_0\rangle\lambda_0^j\langle\varphi_0|}^{\hat{T}^j}\overset{s_j}{\hat{s}}\overbrace{|\varphi_0\rangle\lambda_0^{k-j}}^{\hat{T}^{k-j}}\langle\varphi_0|\overset{s_k}{\hat{s}}|\varphi_0\rangle\overbrace{\lambda_0^{N-k}\langle\varphi_0|}^{\hat{T}^{N-k}})}{\lambda_0^N} + \frac{\lambda_0^j\langle\varphi_0|\hat{s}|\varphi_1\rangle\lambda_1^{k-j}\langle\varphi_1|\hat{s}|\varphi_0\rangle\lambda_0^{N-k}}{\lambda_0^N}$$

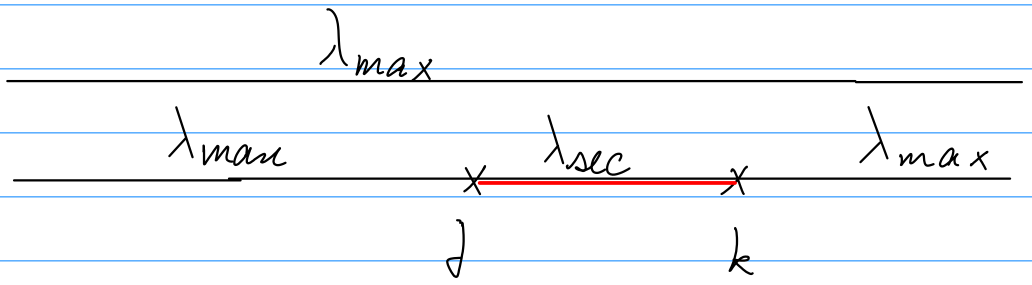
$(\langle\varphi_0|\hat{s}|\varphi_0\rangle)^2$ is cancelled by ↙

$$\langle s_j s_k\rangle - \langle s_j\rangle\langle s_k\rangle = \left(\frac{\lambda_1}{\lambda_0}\right)^{k-j}|\langle\varphi_0|\hat{s}|\varphi_1\rangle|^2 \sim e^{-|k-j|/\xi}$$

$$\xi = \underset{>0}{\ln \lambda_0/\lambda_1}$$

λ_0 : largest eigenvalue

λ_1 : next largest " .



Perron-Frobenius: Matrix $A = (a_{ij})$ $a_{ij} > 0$;

The largest (in absolute value) eigenvalue of A is non degenerate, i.e. $\lambda_1/\lambda_0 < 1$, $\xi < \infty$

⇒ For a 1D system with a finite nr of states/site: NO PHASE TRANSITION.

1 D Ising chain.

$F = \mathcal{E} - TS$. Suppose all spins $+$

$$+ \; + \; + \; + \; \ldots\ldots\ldots \; + \; + \; + \; +$$

$\mathcal{E} = -NK$ $\Omega = 1$ if all spins $+$

$\rightarrow F = -NK - k_B T \ln 1 = -NK$

$$+ \quad + \quad + \mid - \quad - \quad - \quad - \quad - \mid + \quad + \quad +$$

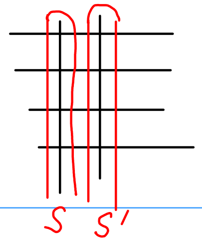
$\Omega = N(N-1)/2$

$F = -(N-4)K - k_B T \ln N^2/2$ 2 bonds cut

more stable than $+ + \ldots$ phase.

2D

$$\mathcal{Z} = \mathrm{Tr}\, \hat{T}^N =$$



$$\langle s | T | s' \rangle \underset{2D}{\Longleftrightarrow} \underset{2^N}{\left\langle \begin{matrix} s_1 \\ \vdots \\ s_N \end{matrix} \right|} \hat{T} \underset{2^N}{\left| \begin{matrix} s'_1 \\ \vdots \\ s'_N \end{matrix} \right\rangle} .$$

Relation with quantum mechanics.

$$\hat{T} = e^{-\beta \hat{H}} \qquad \hat{H} : \text{'hamiltonian'}$$

$$i\hbar \partial_t |\psi\rangle = \hat{H} |\psi\rangle$$

$$|\psi(t)\rangle = e^{-i \hat{H} t/\hbar} |\psi(0)\rangle .$$

$$i\hbar \partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

$$\beta \Leftrightarrow it$$

max eigenval of $\hat{T}$ → lowest eigenval of $\hat{H}$.

$\hat{H}$ is often a hamiltonian describing particles

vacuum state $|vac\rangle$ $\quad \mathcal{E}_{vac} = 0.$

next energy eigenval. $\quad mc^2.$

$$\lambda_0 = e^{-\beta \mathcal{E}_{vac}} = 1$$

$$\lambda_1 = e^{-\beta(\mathcal{E}_{vac} + mc^2)} = e^{-\boxed{\beta mc^2}} .$$

$1/\xi$