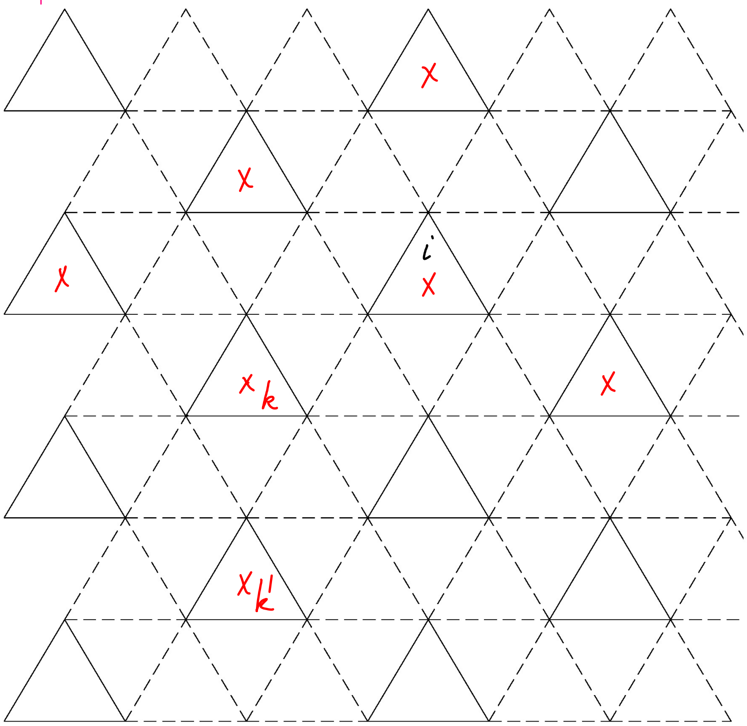
Real Space Renormalisation for the Ising Model on a Triangular Lattice



$s_i = \pm 1.$

$-\beta H(\{s_i\}) = J \sum_{\langle ij \rangle} s_i s_j \cancel{+ B \sum_i s_i}$

$t_k = \pm 1$

$\sum_i s_i^{(k)} > 0 \quad \rightarrow \quad t_k = 1$

$\qquad\qquad < 0 \qquad t_k = -1$

We carry out the standard renormalisation proce

$$A e^{J' t_k t_{k'}} = \sum_{\{s_i^{(k)}, s_j^{(k')}\}} e^{-\beta \mathcal{H}(s_i^{(k)}, s_j^{(k')})} W(t_k, s_i^{(k)}) W(t_{k'}, s_j^{(k')}) .$$

k, k' nearest neighbours; $J > 0$ Ferromagnetic

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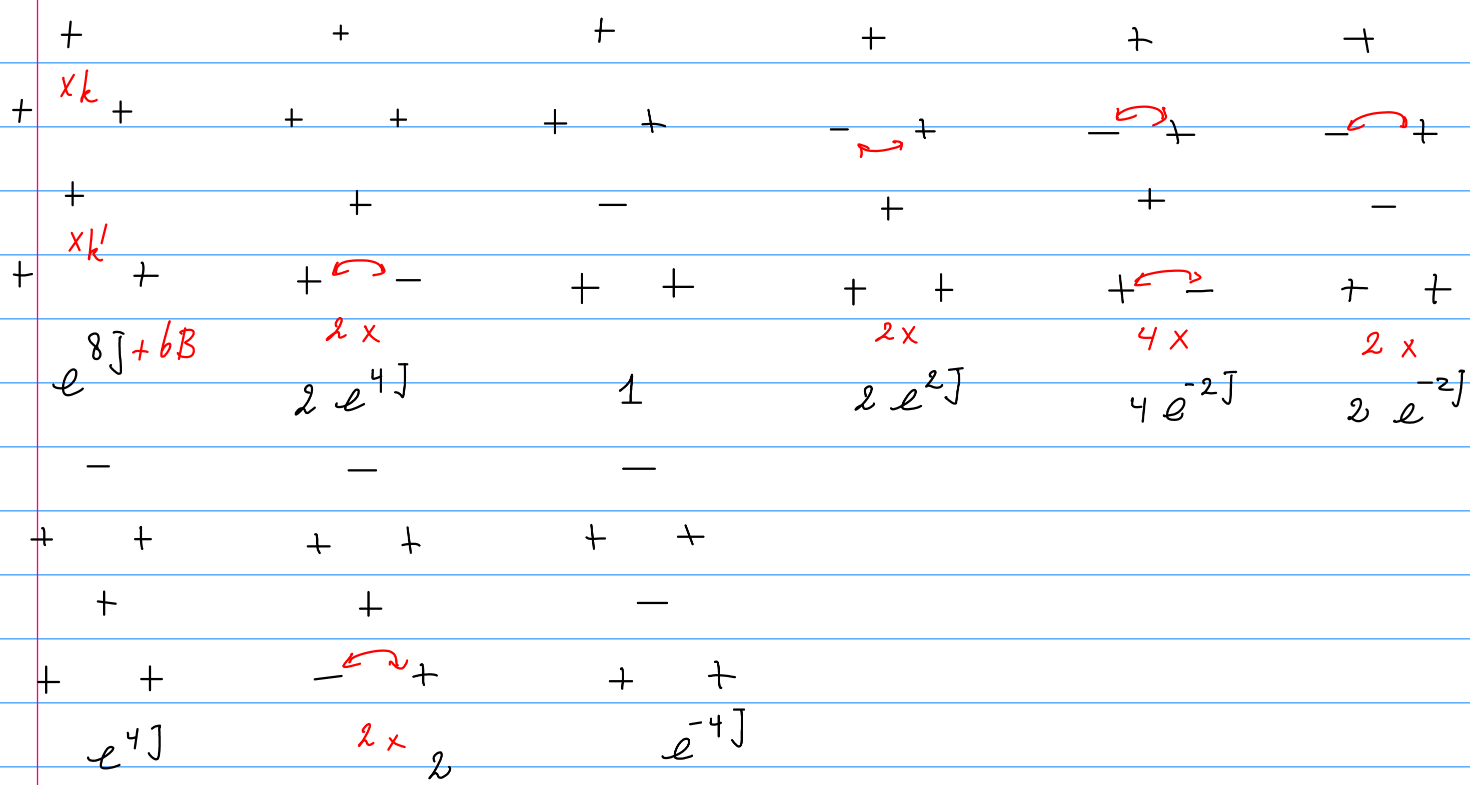
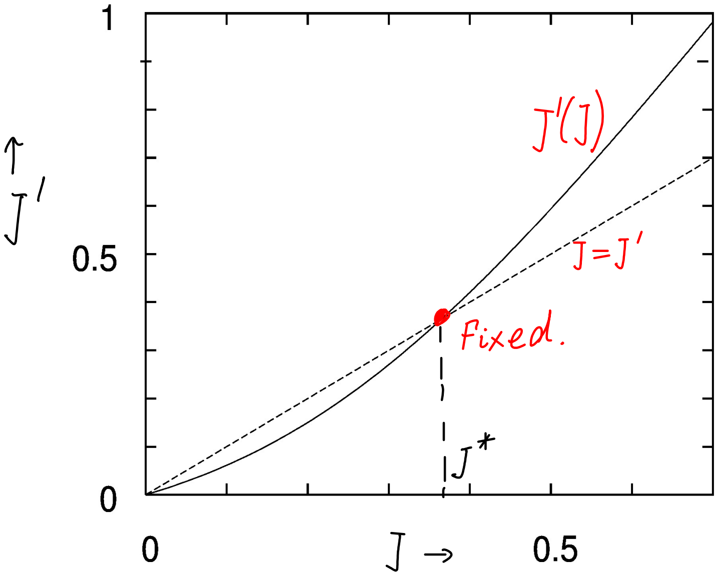


$$Ae^{J'} = e^{8J} + 3e^{4J} + 2e^{2J} + 3 + 6e^{-2J} + e^{-4J}.$$

We can do the same for opposite coarse spins, and find: $Ae^{-J'} = 2e^{4J} + 2e^{2J} + 4 + 6e^{-2J} + 2e^{-4J}$

$$\frac{Ae^{J'}}{Ae^{-J'}} = e^{2J'} = \frac{e^{8J} + 3e^{4J} + 2e^{2J} + 3 + 6e^{-2J} + e^{-4J}}{2e^{4J} + 2e^{2J} + 4 + 6e^{-2J} + 2e^{-4J}}.$$

$$J' = \frac{1}{2}\log\frac{Ae^{J'}}{Ae^{-J'}} = \frac{1}{2}\log\left(\frac{e^{8J} + 3e^{4J} + 2e^{2J} + 3 + 6e^{-2J} + e^{-4J}}{2e^{4J} + 2e^{2J} + 4 + 6e^{-2J} + 2e^{-4J}}\right)$$



We find $J^* = 0.365$

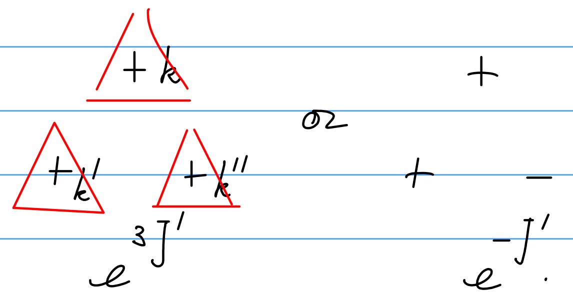
and $\dfrac{dJ'^*(J)}{dJ} = 1.544 = l^z$

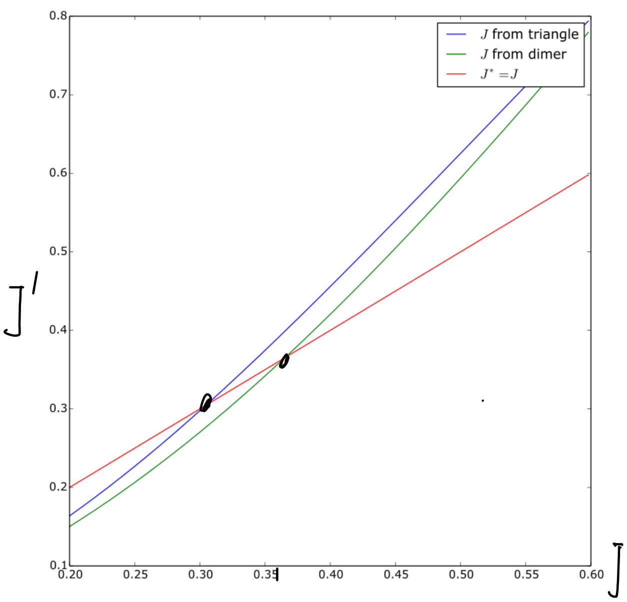
$l = \sqrt{3} \to z \approx 0.79$

$\nu = \dfrac{1}{z} \simeq 1.26$

$\nu_{exact} = 1$ (Houtappel 1950)

Add a magnetic field: $\nu = 2.02$ $\nu_{exact} = 1.875$.





There must be a

3-point interaction.

Qu: How to find better values for z and v?

Answer: cumulant expansion.

$$H = H_0 + V \qquad H_0 : \text{'simple'}$$
$$V : \text{small correction.}$$

$$Z = \sum_r e^{H_0 + V}$$

$$\langle e^V \rangle_0 = \frac{1}{Z_0} \sum_r e^{H_0} e^V = \frac{Z}{Z_0} \rightarrow Z = Z_0 \langle e^V \rangle_0$$

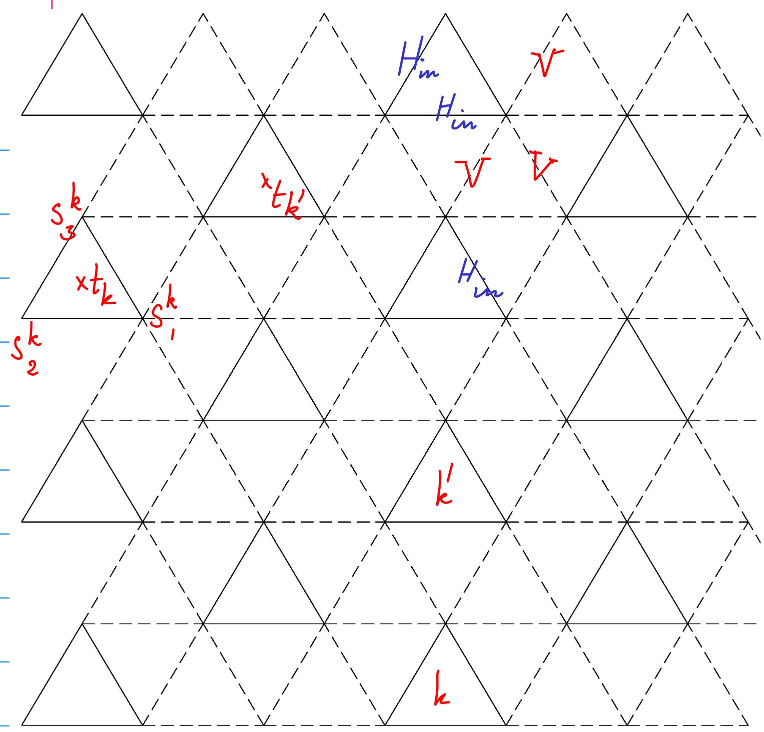
$$\langle e^V \rangle_0 = 1 + \langle V \rangle_0 + \frac{1}{2} \langle V^2 \rangle_0 + \frac{1}{3!} \langle V^3 \rangle_0 + \dots$$

$\Rightarrow$ Perturbation expansion of Z.

$$F = -k_B T \ln Z = -k_B T \ln Z_0 - k_B T \ln \Big(1 + \underbrace{\langle V \rangle_0 + \tfrac{1}{2} \langle V^2 \rangle_0 + \dots}_{x}\Big)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \dots$$

$$F = F_0 - k_B T \Big(\langle V \rangle_0 + \tfrac{1}{2} \big(\langle V^2 \rangle_0 - \langle V \rangle_0^2 \big) + \tfrac{1}{3} \dots \Big).$$ Cumulant expansion.



$H_0 = H_{in}$

H_{in}: couplings across solid lines

V: couplings across dashed lines.

Fine grained sites: ki.

$$H_{in} = J \sum_{\langle ki, ki' \rangle} s_i^k s_{i'}^k$$

$$V = J \sum_{\substack{\langle ki, k'i' \rangle \\ k \neq k'}} s_i^k s_{i'}^{k'}$$

Recall:

$$e^{\mathcal{H}'(\{t_k\})} = \sum_{\{s_i^k\}} e^{\mathcal{H}(\{s_i^k\})} \prod_k W(t_k, s_i^{(k)}) = \sum_{\{s_i^k\}} e^{\mathcal{H}_{in} + V} \prod_k W(t_k, s_i^{(k)})$$

We define $e^{\overline{\mathcal{H}}(\{t_k\}, \{s_i^{(k)}\})} = e^{\mathcal{H}_{in}(\{s_i^{(k)}\})} \prod_k W(t_k, s_i^{(k)})$,

So $e^{\mathcal{H}'(\{t_k\})} = \sum_{\{s_i^k\}} e^{\overline{\mathcal{H}}(\{t_k\}, \{s_i^{(k)}\})} e^{V}$

We have $e^{\mathcal{H}'(\{t_k\})} = Z_0 \langle e^{V} \rangle_0 \Rightarrow$

$$\mathcal{H}'(\{t_k\}) = A + \langle V \rangle_0 + \frac{1}{2}\left(\langle V^2 \rangle_0 - \langle V \rangle_0^2\right) + \frac{1}{3} \ldots$$

$$\langle V\rangle_0 = J\sum_{\langle ik,i'k'\rangle}\langle s_i^k s_{i'}^{k'}\rangle_0 = J\sum_{\langle ik,i'k'\rangle}\langle s_i^k\rangle_0\,\langle s_{i'}^{k'}\rangle_0$$

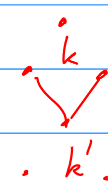
$$\bar{\mathcal{H}}$$

Take $t_k = +1$.

$$\begin{array}{cccc} + & + & - & + \\ +\;+ & -\;+ & +\;+ & +\;- \end{array}$$

$$\langle s_i^k\rangle = \frac{e^{3J}+e^{-J}-e^{-J}+e^{-J}}{e^{3J}+3e^{-J}} = \frac{e^{3J}+e^{-J}}{e^{3J}+3e^{-J}}.$$

$$t_k = -1:\quad \langle s_i^k\rangle = -\frac{e^{3J}+e^{-J}}{e^{3J}+3e^{-J}}.$$



$$\mathcal{H}'(t_k,t_{k'}) = J' t_k t_{k'} = 2J\left(\frac{e^{3J}+e^{-J}}{e^{3J}+3e^{-J}}\right)^2 t_k t_{k'}$$

Fixed point: $J'=J \Rightarrow \dfrac{e^{3J^*}+e^{-J^*}}{e^{3J^*}+3e^{-J^*}} = \dfrac{1}{\sqrt{2}} \rightarrow J^* = 0.3356$

$$\left.\frac{dJ'}{dJ}\right|_{J=J^*} = 2\left(\frac{e^{4J^*}+1}{e^{4J^*}+3}\right)^2 + 4J^*\left(\frac{e^{4J^*}+1}{e^{4J^*}+3}\right)\left(\frac{4e^{4J^*}}{e^{4J^*}+3} - \frac{e^{4J^*}+1}{(e^{4J^*}+3)^2}\,4e^{4J^*}\right)$$

$$= 1.624 = l^{z} = \sqrt{3}^{\,z} \rightarrow z = 0.883$$

Before $z = 0.79$

exact: $z = 1.0$

magnetic scaling dim..

$\nu = 1.37 < \nu_{exact} = 1.875.$

$\lambda_T = (\sqrt{3})^{z} \rightarrow$ converges towards 1.782 (5-cells)

Cumulant expansion

$\lambda_T = 1.784$