Landau theory

Ising Model $\quad Z = \sum_{\{s_i = \pm 1\}} e^{J \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i}$

$\beta = 1/k_B T, \quad J = \beta K, \quad B = \beta h$

$m = \frac{1}{N} \sum_i s_i \rightarrow$ magnetisation

$Z = \sum_m Z_{B=0}(m, J)\, e^{BmN}$

$F = -k_B T \ln Z.$

$F(m) = -k_B T \ln Z_{B=0}(m, J) - k_B T B m N.$

$\Rightarrow \frac{F}{N}(m) = -hm + f(m, J) \qquad m$ small.

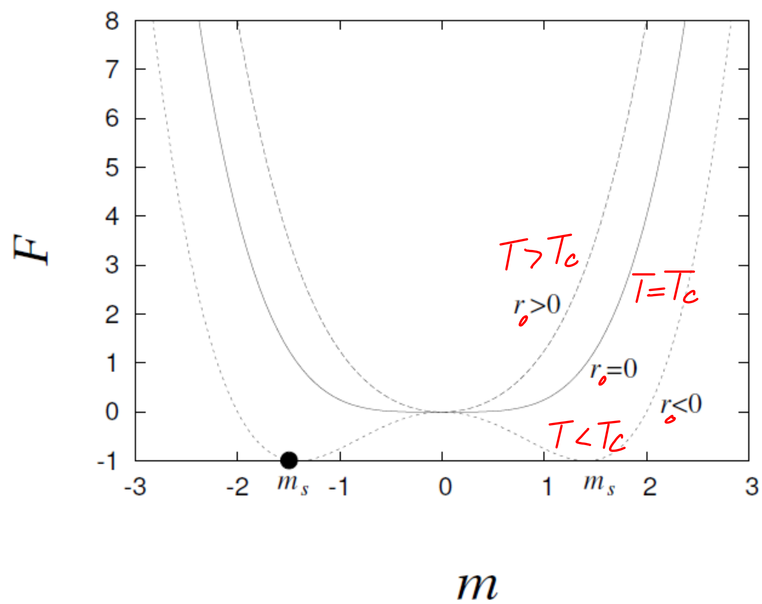
$\frac{F}{N}(m) = -hm + \cancel{g} + r_0 m^2 + u m^4$

$h = 0: \quad f(m) = r_0 m^2 + u m^4.$

$r_0 = r t$

$t = \frac{T - T_c}{T_c}$

t: reduced tempera-ture



$\frac{F}{N} = r_0 m^2 + u m^4; \quad \frac{dF}{dm} = 0 \rightarrow$

$2r_0 m = -4um^3 \to m^2 = \frac{-r_0}{2u}$ or $m=0$

$$m = \sqrt{\frac{-rt}{2(u_0+u_1t+\dots)}} \sim t^{1/2} \qquad \beta = 1/2$$

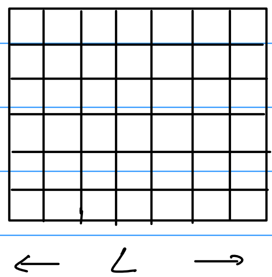
$c_h \sim \frac{\partial^2 F}{\partial T^2}$ $\quad r_0 m^2 \sim t^2$ hence $c_h \sim$ const $\quad \alpha = 0$

$\chi \sim |T-T_c|^{-\gamma} \quad \gamma = 1$

$m_{T=T_c} \sim h^{1/\delta} \qquad \delta = 3$

Landau theory $\Rightarrow$ mean field critical exponents.

Landau-Ginzburg Theory $\quad m \to m(\underline{r})$.



Each box: $\ll L$
contains many spins.

$$H = \int \left[k(\nabla m)^2 + h(\underline{r})m(\underline{r}) + r\,m^2(\underline{r}) + u\,m^4(\underline{r}) \right] d^d r$$

$\mathcal{H}_0(m(\underline{r}))$

$$g(\underline{r},\underline{r}') = \langle m(\underline{r})m(\underline{r}')\rangle_0 - \langle m(\underline{r})\rangle_0 \langle m(\underline{r}')\rangle_0$$

$$\langle m(\underline{r})\rangle = \frac{\int_{-\infty}^{\infty} e^{-\beta H} m(\underline{r}) \prod_{\underline{r}} dm(\underline{r})}{\int_{-\infty}^{\infty} e^{-\beta H} \underbrace{\boxed{\prod_{\underline{r}} dm(\underline{r})}}_{\text{‘}\mathcal{D}m\text{’}}}.$$

$$h(\underline{r}) = h\,\delta(\underline{r})$$

$$\langle m(\underline{r})\rangle_h = \frac{\int e^{-\beta(H_0 + h\,m(\underline{0}))} m(\underline{r})\,\mathcal{D}m}{\int e^{-\beta(H_0+h\,m(\underline{0}))}\mathcal{D}m}$$

$$\frac{d}{dh}\langle m(\underline{r})\rangle_{h=0} = -\beta \frac{\int e^{-\beta H_0} m(\underline{0})m(\underline{r})\,\mathcal{D}m}{\int e^{-\beta H_0}\mathcal{D}m} + \beta \frac{\int e^{\beta H_0} m(\underline{r})\mathcal{D}m \int e^{\beta H_0} m(\underline{0})\,\mathcal{D}m}{\left(\int e^{-\beta H_0}\mathcal{D}m\right)^2}$$

$$= -\beta g(\underline{r}) \qquad \text{Fluctuation-response.}$$

$$Z = \int e^{-\beta H} Dm$$ Which configurations give dominant contribution?

Saddle point: $\int e^{\Lambda f} d^d r \approx e^{\Lambda f_{max}}$, Λ large

Use $H(m(\underline{r}) + \delta m(\underline{r})) - H(m(\underline{r})) = 0$. (variational).
1st order in δm

$$\int \left[k(\nabla m + \nabla \delta m)^2 + h(\underline{r})(m+\delta m) + r(m+\delta m)^2 + u(m+\delta m)^4\right] d^d r$$

$$- \int \left(k(\nabla m)^2 + h(\underline{r})\, m + r m^2 + u m^4\right) d^d r$$

$$\simeq \int \left(2k\, \nabla m\, \nabla \delta m + h(\underline{r})\delta m + 2 r m \delta m + 4 u m^3 \delta m\right) d^d r$$

$$= \int \big(\underbrace{-2k\nabla^2 m + h(\underline{r}) + 2 r m + 4 u m^3}_{=0}\big)\delta m \; d^d r = 0$$

$$-2k\nabla^2 m + h\,\delta(\underline{r}) + 2 r m + 4 u m^3 = 0.$$

$h = 0$ and $T > T_c$ $(r > 0)$: $m_0 = 0$

$h = 0$ and $T < T_c$ $(r < 0)$: $m_0^2 = -r/2u$

$m(\underline{r}) = m_0 + h\varphi(\underline{r})$

$\underline{T > T_c}$ $(m_0 = 0) \rightarrow -2k\nabla^2\varphi + \delta(\underline{r}) + 2r\varphi = 0.$

$$\rightarrow \underline{\nabla^2\varphi - \frac{r}{k}\varphi = \frac{\delta(\underline{r})}{k}} \qquad T > T_c$$

$\underline{T < T_c}$ $\left(m_0 = \sqrt{\frac{-r}{2u}}\right) \rightarrow -2k\nabla^2\varphi + \delta(\underline{r}) + 2r\varphi + 3um_0^2\varphi = 0$

$$\rightarrow \underline{\nabla^2\varphi + \frac{2r}{k}\varphi = \frac{\delta(\underline{r})}{k}} \qquad T < T_c$$

$$\left(\nabla^2 - \frac{1}{\xi^2}\right)\varphi(\underline{r}) = A\delta(\underline{r}) \quad \text{Helmholtz eqn.}$$

Dimension d: $\varphi(\underline{r}) \sim |\underline{r}|^{\frac{1-D}{2}} e^{-|\underline{r}|/\xi} \qquad |\underline{r}| \gg \xi$

$\sim |\underline{r}|^{2-D} e^{-|\underline{r}|/\xi} \qquad |\underline{r}| \ll \xi$

ξ : correlation length. $\xi^2 \sim \frac{k}{r} \sim \frac{1}{t} \qquad \xi \sim \frac{1}{\sqrt{t}} \sim t^{-\nu}$

$\nu = 1/2$.

$m(\underline{r}) = m_0 + h\varphi(\underline{r})$; $\quad g(\underline{r}) = \frac{\partial m}{\partial h} \rightarrow g(\underline{r}) \sim \varphi(\underline{r})$

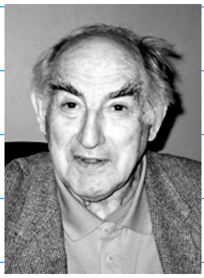
At T_c: $\xi \to \infty \qquad |\underline{r}| \lesssim \xi \qquad \frac{e^{-r/\xi}}{r^{d-2}} = g(r) = \frac{1}{r^{d-2+\eta}}$

$\eta = 0$

Summary

Landau-Ginzburg	Experiment 3D.
$\alpha = 0$	$\alpha = 0.1$
$\beta = 1/2$	$\beta = 0.3 - 0.4$
$\gamma = 1$	$\gamma \simeq 1.25$
$\nu = 1/2$	$\nu \simeq 0.7$.

Ginzburg Criterion



Vitaly Ginzburg

To what extent is it justified to replace $m(\underline{r})$ by m_0?

$$\int \left[\nabla(\delta m(\underline{r}))\right]^2 + r(\delta m(\underline{r}))^2 + b u m_0^2 (\delta m(\underline{r}))^2 \, d^d r$$

$$\ll \int \left(r m_0^2 + u m_0^4\right) d^d r$$

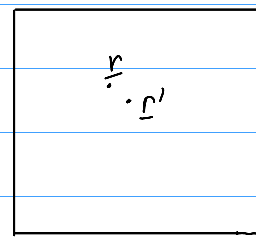
$$\langle (\delta m(r))^2 \rangle \ll m_0^2 \qquad \text{Ginzburg criterion}$$

$$M = \frac{1}{V}\int m(\underline{r})\, d^d r$$

$$V^2\langle \delta M^2\rangle = \left\langle \int m(\underline{r})\, d^d r \int m(\underline{r}')\, d^d r' \right\rangle - \left\langle \int m(\underline{r})\, d^d r \right\rangle \left\langle \int m(\underline{r}')\, d^d r' \right\rangle$$

$$= \int \underbrace{\left(\langle m(\underline{r})\, m(\underline{r}')\rangle - \langle m(\underline{r})\rangle \langle m(\underline{r}')\rangle \right)}_{g(\underline{r},\underline{r}')} d^3 r\, d^3 r'$$

$$= \int g(\underline{r},\underline{r}')\, d^d r\, d^d r'$$



$V \gg \xi^d$

$$\underline{R} = \frac{\underline{r} + \underline{r}'}{2} \qquad \underline{s} = \underline{r} - \underline{r}'$$

$$g(\underline{r},\underline{r}') \to g(\underline{s}) \qquad g(s) = \frac{e^{-s/\xi}}{s^{d-2}}$$

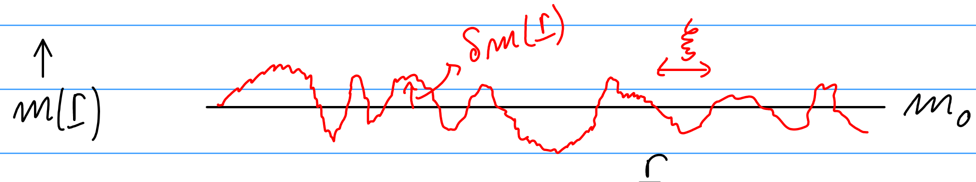
$$V \int g(s)\, d^d s = VC \int g(s)\, s^{d-1} ds = V \underset{\mathcal{O}(1).}{C} \int_0^{\xi} \frac{1}{s^{d-2}} s^{d-1} ds$$

$$= V\xi^2 = V^2 \langle \delta M^2 \rangle$$

$$\langle \delta M^2 \rangle = \frac{\xi^2}{V}$$

$$M = \frac{1}{V}\int m(\underline{r})\, d^d r$$

$$\delta m(\underline{r}) = m(\underline{r}) - m_0$$



$T < T_c.$

Patches ξ^d Number $\frac{V}{\xi^d} = N$

$$\delta M \sim \frac{\delta m}{\sqrt{N}} = \sqrt{\frac{\langle (\delta m(\underline{r}))^2\rangle}{V/\xi^d}}$$

$$\to \langle (\delta m(\underline{r}))^2 \rangle = \delta M^2 \frac{V}{\xi^d} = \xi^{2-d} = t^{\frac{d-2}{2}} \ll m_0^2 \sim t$$

$\hookrightarrow \xi^2/V = \delta M^2$

$$\frac{\langle (\delta m(\underline{r}))^2 \rangle}{m_0^2} = t^{\frac{d-4}{2}} \ll 1 \qquad d > 4$$