The Landau-Ginzburg Hamiltonian describes a system with a one-dimensional order parameter close to a phase transition

$$\beta \mathcal{H} = \int \left[(\underline{\nabla} x(\underline{r}))^2 + r_0 x^2(r) + u x^4(r) + h x(\underline{r}) \right] d^d r$$

$x(\underline{r})$ is a real field, e.g. a magnetization.

We simplify this by taking $u \equiv h = 0$.

We are then left with the Gaussian model:

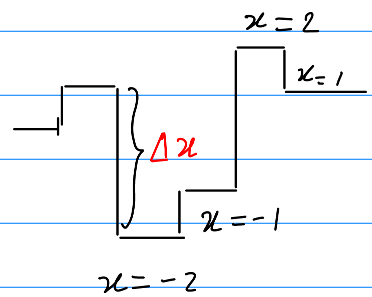
$$\beta \mathcal{H} = \int \left[(\underline{\nabla} x(\underline{r}))^2 + r_0 x^2(\underline{r}) \right] d^d r.$$

Let us now relate this form to a discrete Lattice model:

Consider a crystal surface, characterized by x_i.
$\underline{r}_i$: 2D vector $\rightarrow$ location on the surface
x_i : height of the surface at $\underline{r}_i$.

Energy: penalty for height differences:

1D cartoon:



$$\beta \mathcal{H} = J/2 \sum_{\langle i,j \rangle} (x_i - x_j)^2$$ Periodic boundary cond.

The values assumed by the x_i are discrete. This is not relevant at high T.

Fourier transform: $x_i = \frac{1}{\sqrt{N}} \sum_{\underline{q}} e^{i\underline{q}\cdot\underline{r}_i} x_{\underline{q}}$

$$x_{\underline{q}} = \sum_i e^{-i\underline{q}\cdot\underline{r}_i} x_i \; ; \quad x_{-\underline{q}} = \sum_i e^{i\underline{q}\cdot\underline{r}_i} x_i = x^*_{\underline{q}}$$

$$\beta H = J/2 \sum_i \left[\left(x_i - x_{i+\hat{\underline{e}}_x}\right)^2 + \left(x_i - x_{i+\underline{e}_y}\right)^2 \right]$$

$$= J/2 \sum_i \left[x_i^2 - x_i x_{i+\underline{e}_x} - x_{i+\underline{e}_x} x_i + x^2_{i+e_x} + x_i^2 - x_i x_{i+\underline{e}_y} - x_{i+\underline{e}_y} x_i + x^2_{i+\underline{e}_y} \right]$$

$$= J/2 \sum_i \left(4x_i^2 - x_i x_{i+\hat{\underline{e}}_x} - x_{i+\hat{\underline{e}}_x} x_i - x_i x_{i+\hat{\underline{e}}_y} - x_{i+\hat{\underline{e}}_y} x_i \right)$$

$$\sum_i x_i^2 = \frac{1}{N} \sum_i \sum_{\underline{q}} x_{\underline{q}} e^{i\underline{q}\cdot\underline{r}_i} \sum_{\underline{q}'} x_{\underline{q}'} e^{i\underline{q}'\cdot\underline{r}_i}$$

Use the fact that $\sum_i e^{i\underline{q}\cdot\underline{r}_i} = N\delta_{\underline{q},0}$

So: $\sum_i e^{i\underline{q}\cdot\underline{r}_i} e^{i\underline{q}'\cdot\underline{r}_i} = N\delta_{\underline{q}+\underline{q}',0} \rightarrow \underline{q} = -\underline{q}'$

$$\rightarrow \sum_i x_i^2 = \sum_{\underline{q}} x_{\underline{q}} x_{-\underline{q}}$$

$$\sum_i x_i x_{i+\underline{e}_x} = \frac{1}{N} \sum_i \sum_{\underline{q}} x_{\underline{q}} e^{i\underline{q}\cdot\underline{r}_i} \sum_{\underline{q}'} x_{\underline{q}'} e^{i\underline{q}'\cdot(\underline{r}_i + \underline{e}_x)}$$

Perform $\sum_i$: $\quad \sum_{\underline{q}} x_{\underline{q}} x_{\underline{q}'} e^{iq'_x}$

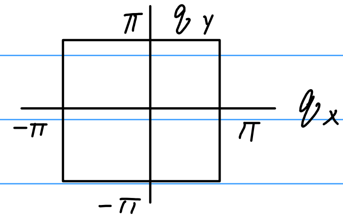
Do this for all terms, and take $J=1$ (see below)

$$\rightarrow \beta H = \frac{1}{2} \sum_{\underline{q}} x_{\underline{q}} x_{-\underline{q}} \left(4 - e^{iq_x} - e^{-iq_x} - e^{iq_y} - e^{-iq_y} \right)$$

$$\rightarrow \quad = \frac{1}{2} \sum_{\underline{q}} x_{\underline{q}} x_{-\underline{q}} \left(4 - 2\cos q_x - 2\cos q_y \right).$$

For $|\underline{q}| \ll 1$: $\beta H \approx \sum_{q} x_{\underline{q}} x_{-\underline{q}} \, q^2$

q_x lies inside $[-\pi, \pi]$ } Brillouin zone
q_y " " $[-\pi, \pi]$.



What is the meaning of the $r_0 \, x_{\underline{q}} x_{-\underline{q}}$ term?

In $\underline{r}$ space: $\int r_0 \, x^2(\underline{r}) \, d^3 r$

This gives an extra penalty for $x(\underline{r})$ to deviate from $x(\underline{r}) = 0$.

Gaussian model (continuum limit):

$$\beta H = \frac{1}{2} \sum_{\underline{q} \in BZ} \left(q^2 + r_0 \right) x_{\underline{q}} x_{-\underline{q}}$$

This model contains <u>one</u> parameter: r_0
The Brillouin zone is part of the definition of the model as it fixes the lattice constant to 1:

$$q_x, q_y \in [-\pi, \pi]$$

Let's calculate $\langle x(\underline{r}) \, x(\underline{r}') \rangle = \frac{1}{N} \sum_{\underline{q}} \langle x_{\underline{q}} x_{\underline{q}'} \rangle \, e^{i(\underline{q} \cdot \underline{r} + \underline{q}' \cdot \underline{r}')}$

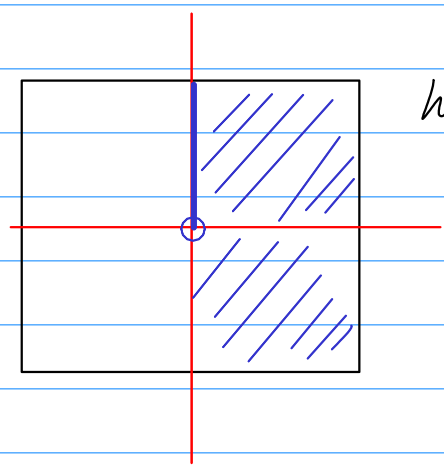
<u>Note</u> $\underline{q}$ lies in BZ, but $x_{\underline{q}}$ and $x_{-\underline{q}}$ are not independent ($x_{\underline{q}} = x^*_{-\underline{q}}$)

The Gaussian Hamiltonian:

$$\frac{1}{2}\sum_{\underline{q}} (r_0 + q^2) m_{\underline{q}} m_{-\underline{q}}$$

contains interaction terms that are symmetric under $\underline{q} \to -\underline{q}$, except for $\underline{q} = 0$

Therefore, we split the Brillouin zone into two parts, except for $\underline{q} = \underline{0}$



We call the blue, hashed part BZ^+

Then:

$$\mathcal{H}_{gauss} = \frac{1}{2} r_0 m_{\underline{0}}^2 + \sum_{\underline{q} \in BZ^+} (r_0 + \underline{q}^2) m_{\underline{q}} m_{-\underline{q}}$$

$$\mathcal{Z} = \int_{-\infty}^{\infty} dm_0 e^{r_0 m_{\underline{0}}/2} \int_{-\infty}^{\infty} \prod_{\underline{q} \in BZ^+} dm_{\underline{q}} dm_{-\underline{q}} e^{(r_0 + \underline{q}^2) m_{\underline{q}} m_{-\underline{q}}}$$

$$\sqrt{\frac{2\pi}{r_0}} \prod_{\underline{q} \in BZ^+} \left(\frac{2\pi}{r_0 + q^2}\right) = \prod_{\underline{q} \in BZ} \left(\frac{2\pi}{r_0 + q^2}\right)^{1/2}$$

$$\Rightarrow \ln \mathcal{Z} = \int_{BZ} \frac{1}{2} \frac{2\pi}{r_0 + q^2} \, dq$$

Without proof, we state

$$\langle x_{\underline{q}_1} x_{\underline{q}_2} \rangle = \frac{1}{q^2 + r_0} \delta(\underline{q}_1 + \underline{q}_2).$$

It is obvious that $\langle x_{\underline{q}_1} x_{\underline{q}_2} \rangle = 0$ for $\underline{q}_1 \neq \pm \underline{q}_2$ as the weight factor

$$\prod_{\underline{q}} e^{(r_0 + q^2) x_{\underline{q}} x_{-\underline{q}}}$$

factorizes, and $\langle x_{\underline{q}} \rangle$ vanishes due to antisymmetry

$$\langle x_{\underline{q}} x_{\underline{q}} \rangle = \underbrace{\langle (\mathrm{Re}\, x_{\underline{q}})^2 \rangle - \langle (\mathrm{Im}\, x_{\underline{q}})^2 \rangle}_{\text{Identical}}$$

This enables us to analyze the real space corr. fction:

$$\langle x(\underline{r}) x(\underline{r}') \rangle = \sum_{\underline{q}\underline{q}'} \underbrace{\langle x_{\underline{q}} x_{\underline{q}'} \rangle}_{\frac{1}{q^2 + r_0} \delta_{\underline{q}+\underline{q}';0}} e^{i(\underline{q}\cdot\underline{r} + \underline{q}'\cdot\underline{r}')}$$

$$= \sum_{\underline{q}} \frac{1}{q^2 + r_0} e^{i \underline{q}\cdot(\underline{r} - \underline{r}')}.$$

$|\underline{r} - \underline{r}'|$ large: $|\underline{q}|^2 \ll r_0$

$$\langle x(\underline{r}) x(\underline{r}') \rangle \simeq r^{\frac{1-d}{2}} e^{-r/\sqrt{r_0}} \qquad d: \text{dimension.}$$

Renormalization of the Gaussian model

$$\beta H = \frac{1}{2} \sum_{\underline{q} \in BZ} (q^2 + r_0) x_{\underline{q}} x_{-\underline{q}} = \sum_{q \in BZ^+} (q^2 + r_0) x_{\underline{q}} x_{-\underline{q}}$$

This model contains one parameter: r_0

The Brillouin zone is part of the definition of the model: $q_x, q_y \in [-\pi, \pi]$

Note: we can rescale $x_{\underline{q}}$: $\tilde{x}_{\underline{q}} = \lambda x_{\underline{q}}$

$$Z = \int \prod_{\underline{q} \in B_+} dx_{\underline{q}}\, dx_{-\underline{q}}\, e^{-\sum_{\underline{q}} x_{\underline{q}} x_{-\underline{q}} (r_0 + q^2)} =$$

$$\lambda^{-N} \int \prod_{\underline{q} \in B_+} d\tilde{x}_{\underline{q}}\, d\tilde{x}_{-\underline{q}}\, e^{-\lambda^{-2} \sum_{\underline{q}} \tilde{x}_{\underline{q}} \tilde{x}_{-\underline{q}} (r_0 + q^2)}$$

The prefactor λ^{-N} does not change the decay length of the correlation function $\Rightarrow$ critical behavior does not change by this. $\Rightarrow$ We can always rescale $x_{\underline{q}}$ in order to restore the prefactor 1 of the term $\underline{q}^2 x_{\underline{q}} x_{-\underline{q}}$

Renormalization in $\underline{q}$-space

1. Integrate over $x_{\underline{q}}$, $\frac{\pi}{b} \leq q_i \leq \pi \qquad i = 1, \ldots, d$

2. Rescale $\underline{q} \to b\underline{q}$ in order to restore the 'original' Brillouin zone

3. Rescale $x_{\underline{q}} \to \frac{\tilde{x}_{\underline{q}}}{\lambda}$ in order to restore the prefactor 1 in front of $q^2 x_{\underline{q}} x_{-\underline{q}}$.

① $$\int \prod_{\frac{\pi}{b}<q_i<\pi} dx_{q}\, e^{-\sum_{\underline{q}}(q^2+r_0)x_{\underline{q}}x_{-\underline{q}}}$$

This leaves us with a Hamiltonian of the same form, but now: $|q_i| < \frac{\pi}{b}$

② We rescale the new lattice constant b back to 1: $\quad \underline{q} \rightarrow b\,\underline{q}$

Then the $\underline{q}$ span the original BZ.

$\rightarrow \underline{q}' = b\,\underline{q}$;

$$H' = \sum_{|q_i|<\frac{\pi}{b}}' (q^2+r_0)x_{\underline{q}}x_{-\underline{q}} = \sum_{|q_i'|<\pi} \left[\left(\frac{q'}{b}\right)^2 + r_0\right] x_{\underline{q}'}x_{-\underline{q}'}$$

③ Restore the prefactor of the q'^2 term by rescaling: $\tilde{x}_{q'} = b\,x_{q'}$

$$\Rightarrow H' = \sum_{|q_i|<\pi} \big(q'^2 + \underbrace{b^2 r_0}_{r_0'}\big)\, \tilde{x}_{q'}\tilde{x}_{-q'}$$

We see that $r_0' = b^2 r_0 = b^z r_0$; $z = 2$

$\nu = \frac{1}{z}$ hence $\nu = \frac{1}{2}$: $\xi \sim \Big(\underbrace{\frac{T-T_c}{T_c}}_{\sim r_0}\Big)^{-1/2}$

Note: r_0 is a relevant parameter

Flow: $\quad \circ \rightarrow \rightarrow \rightarrow \quad r_0$

Note: model is only properly defined for $r_0 > 0$.