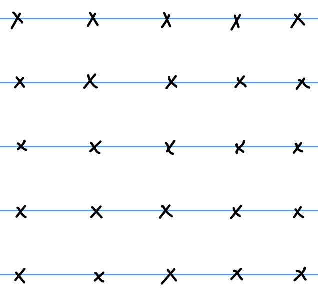
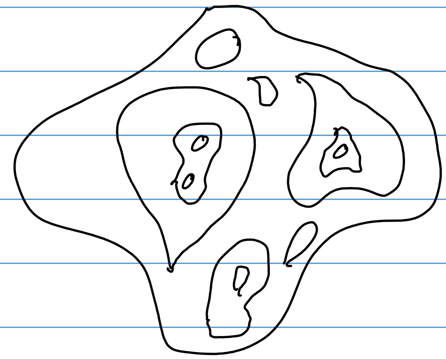
Renormalisation Theory for Critical Phase Transitions

Droplet picture





$m \sim (T-T_c)^{\beta} \quad T<T_c$

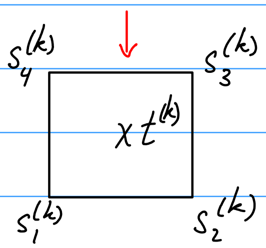
$c_V \sim |T-T_c|^{\alpha}$

$\xi \sim |T-T_c|^{-\nu}$

$\chi \sim |T-T_c|^{-\gamma}$

$m \sim h^{1/\delta}$

$g(r) \sim \dfrac{1}{r^{d-2+\eta}}$



$t^{(k)} = \pm 1$

If $s_1^{(k)} + s_2^{(k)} + s_3^{(k)} + s_4^{(k)} > 0$ then $t^{(k)} = 1$

. . . . . < 0 " $t^{(k)} = -1$

$= 0$ " $t^{(k)} = \pm 1$ equal probabilities.

$$W(t^{(k)}, s_1^{(k)}, s_2^{(k)}, s_3^{(k)}, s_4^{(k)}) = \begin{cases} 1 & \text{for } \sum_i s_i^{(k)} > 0, \; t^{(k)} = 1 \\ 1 & \text{"} \sum_i s_i^{(k)} < 0, \; t^{(k)} = -1 \\ 0.5 & \text{"} \sum_i s_i^{(k)} = 0, \; t^{(k)} = 1 \\ 0.5 & \text{"} \sum_i s_i^{(k)} = 0, \; t^{(k)} = -1 \end{cases}$$

$\Rightarrow \sum_{t^{(k)}} W(t^{(k)}; s_1^{(k)}, s_2^{(k)}, s_3^{(k)}, s_4^{(k)}) = 1$ for any set of $s_i^{(k)}$

We define a 'new' Hamiltonian, which governs the behaviour of the $t^{(k)}$

$$e^{-\beta \mathcal{H}(\{t^{(k)}\})} = \sum_{s_i^{(k)} = \pm 1} e^{-\beta \mathcal{H}(\{s_i\})} \, W(t^{(k)}; s_1^{(k)} \ldots s_4^{(k)})$$

Partition function for the new Hamiltonian:

$$Z' = \sum_{\{t^{(k)}\}} e^{-\beta \mathcal{H}(\{t^{(k)}\})} = \sum_{\{t^{(k)} = \pm 1\}} \sum_{\{s_i^{(k)} = \pm\}} e^{-\beta \mathcal{H}(\{s_i\})} \, W(t^{(k)}; s_1^{(k)} \ldots s_4^{(k)}).$$

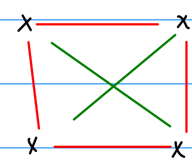
$$= \sum_{\{s_i^{(k)} = \pm 1\}} \underbrace{\sum_{\{t^{(k)} = \pm 1\}} W(t^{(k)}; s_1^{(k)} \ldots s_4^{(k)})}_{1} \, e^{-\beta \mathcal{H}(\{s_i\})} = Z \quad !$$

So, transforming from 'fine' to 'coarse' spins leaves the Hamiltonian invariant.

$\neq$

Suppose we carry out the coarsening procedure for the Ising model with next-nearest neighbour couplings.

$$-\beta H(\{s_i\}) = \underline{J} \sum_{\langle ij \rangle} s_i s_j + \underline{K} \sum_{\langle\langle ij \rangle\rangle} s_i s_j$$



$$-\beta H(\{s_i\}) = J' \sum_{\langle kk' \rangle} t^{(k)} t^{(k')} + K' \sum_{\langle\langle kk' \rangle\rangle} t^{(k)} t^{(k')} + \text{several other couplings.}$$

We see that J, K maps onto J' K'

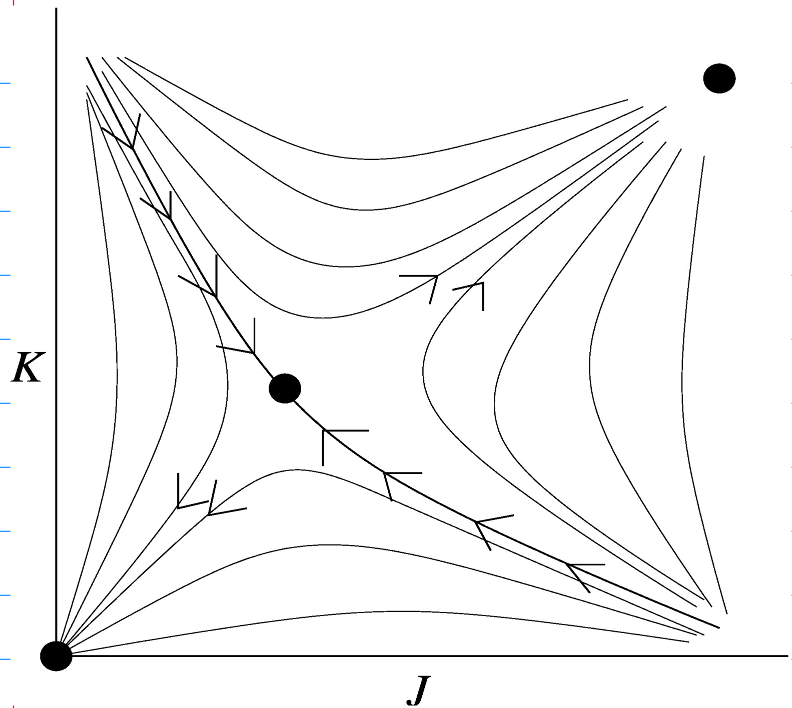
This mapping induces a <u>flow</u> in J, K space.

We take $B = 0$. Later we consider weak fields

New lattice constant $= l \times$ old lattice constant

$l \downarrow$ 2 in the example

l: <u>rescaling length</u>



$T_{low} \xrightarrow{\ell} T'_{lower} < T_{low}$

$T_{high} \xRightarrow{\ell} T'_{higher} > T_{high}$

Correlation length $\xi(J,K)$ = average size of region with single spin (+1 or −1).
We measure $\xi(J,K)$ in units of the lattice constant

Scaling by a factor ℓ therefore implies

$$\xi(J',K') = \xi(J,K)/\ell$$

The Free energy F is extensive: $F \sim N$ (nr of sites)
Free energy per site: $f = F/N$.

We have seen: $Z_{N/\ell^d}(J',K') = Z_N(J,K)$ → d: dimension.

Therefore: $F(J',K') = F(J,K)$

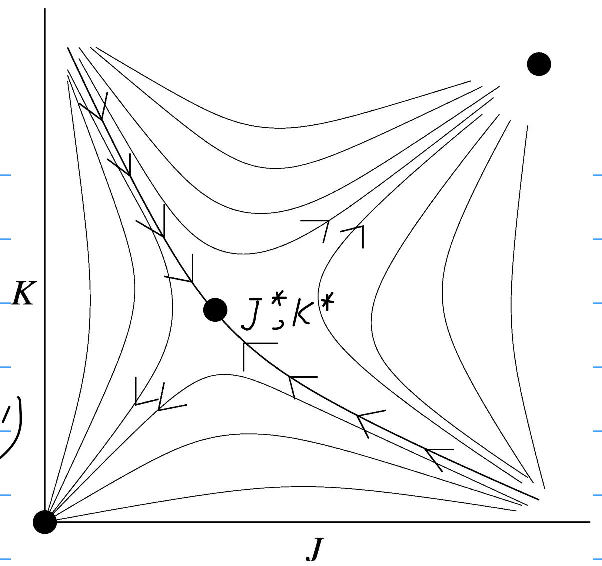
So: $$f(J',K') = \frac{F(J',K')}{N/\ell^d} = \ell^d \frac{F(J,K)}{N} = \ell^d f(J,K).$$

In summary: $$\xi(J',K') = \xi(J,K)/\ell$$
$$f(J',K') = \ell^d f(J,K).$$

J^*, K^*: Fixed point.

Small deviation:

$$J^* + \Delta J \,; \quad K^* + \Delta K$$



$$(J^* + \Delta J, K^* + \Delta K) \xrightarrow{\text{rescaling}} (J^* + \Delta J', K^* + \Delta K')$$

$$\begin{pmatrix} \Delta J' \\ \Delta K' \end{pmatrix} = \begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} \Delta J \\ \Delta K \end{pmatrix}$$

Diagonalize A. eigenvalues λ, μ.

From flow diagram: $\lambda < 1$; $\mu > 1$.

λ and μ depend on l

Apply rescaling twice: eigenvalues λ^2, μ^2

rescaling length l^2

So: $\lambda(l^2) = \lambda^2(l)$; $\mu(l^2) = \mu^2(l)$.

Then: $\lambda = l^y \quad \mu = l^z$.

y, z: scaling dimensions.

We can write any vector in the J, K plane as

$$\underline{v} = t\,\hat{\underline{t}} + s\,\hat{\underline{s}}, \qquad \text{so } (J, K) \Leftrightarrow (t, s)$$

where $\hat{\underline{s}}, \hat{\underline{t}}$ are the eigenvectors corresponding to λ and μ:

$$(A)\,\hat{\underline{s}} = \lambda\,\hat{\underline{s}}$$

$$(A)\,\hat{\underline{t}} = \mu\,\hat{\underline{t}}$$

So: $(A)(s\hat{\underline{s}} + t\hat{\underline{t}}) = s\lambda\hat{\underline{s}} + t\mu\hat{\underline{t}} = s\,l^y\hat{\underline{s}} + t\,l^z\hat{\underline{t}}$

$$(A^n)(s\hat{\underline{s}} + t\hat{\underline{t}}) = s\lambda^n\hat{\underline{s}} + t\mu^n\hat{\underline{t}} = s\,l^{yn}\hat{\underline{s}} + t\,l^{zn}\hat{\underline{t}}$$

So $\quad s^{(n)} = s\,l^{yn} \qquad y<0$

$\quad t^{(n)} = t\,l^{zn} \qquad z>0$

Recall $\xi(J', K') = \xi(J, K)/l$

$\rightarrow \quad \xi(s', t') = \xi(s, t)/l$

After n rescalings:

$$\xi\left(l^{ny} s, l^{nz} t\right) = \xi(s,t)/l^n$$

Choose t and n such that $l^{nz} t = 1 \rightarrow l^n = t^{-1/z}$

$$\xi\left(t^{-y/z} s, 1\right) = \xi(s,t)\, t^{1/z}$$

$$\rightarrow \quad \xi(s,t) = t^{-1/z}\, \xi(0,1)$$

We interpret t as $\sim |T-T_c|$

So $\quad \xi \sim |T-T_c|^{-1/z}$ $\quad \nu$ $\qquad \boxed{\nu = 1/z}$

$\uparrow$

Similar game with $f(s,t)$.

First: extend the model with a (small) magnetic field h:

$$-\beta H(\{s_i\}) = J \sum_{\langle ij \rangle} s_i s_j + K \sum_{\langle\langle ij \rangle\rangle} s_i s_j + h \sum_i s_i$$

matrix A is $3 \times 3 \rightarrow 3$ eigenvalues λ, μ, κ

$\lambda = l^y, \ \mu = t^z, \ \kappa = t^v \qquad v > 0$

$y<0 \qquad z>0$

$$f(s', t', h') = l^d f(s,t,h)$$

$$\rightarrow f\left(l^{ny} s, l^{nz} t, l^{nv} h\right) = l^{nd} f(s,t,h)$$

Choose $\underline{h = 0}$

$$f(s,t,0) = \ell^{-nd} f(\ell^{ny} s, \ell^{nz} t, 0)$$

Choose t (small) and n (large) such that

$\ell^{nz} t = 1$. Then $\ell^{ny} s \to 0$ (s small, $y<0$, n large).

$$\to f(s,t,0) = \ell^{-nd} f(0,1,0) = t^{d/z} f(0,1,0)$$

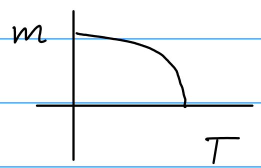
Recall $t \sim |T - T_c|$

Entropy: $S = \left.\frac{\partial F}{\partial T}\right)_{N,h}$ and $c_h = T \frac{\partial S}{\partial T} = T \frac{\partial^2 F}{\partial T^2}$

$\frac{\partial}{\partial t} \Leftrightarrow \frac{\partial}{\partial T}$ hence $c_v \sim t^{d/z - 2} = t^{-\alpha}$ (by definition)

$\nu = 1/z \Rightarrow \boxed{\alpha = 2 - d\nu}$ Scaling relation.

$m \sim t^{\beta}$



$m = \left.\frac{\partial f}{\partial h}\right)_{h=0}$ $\qquad f(s,t,h) = \ell^{-nd} f(\ell^{ny} s, \ell^{nz} t, \ell^{nv} h)$

$$= t^{d/z} f(0, 1, t^{-v/z} h)$$

$$\to \frac{\partial f}{\partial h} = t^{d/z - v/z} \frac{\partial f}{\partial h}(0, 1, t^{-v/z} h)$$

Recall $\nu = 1/z \to \beta = (d - v)\nu$

$$\chi = \left.\frac{\partial m}{\partial h}\right)_{h=0} \sim t^{d/z - 2v/z} \frac{\partial^2 f}{\partial h^2}(0,1,0) \sim t^{-\gamma}$$

$$\to -\gamma = (d - 2v)\nu$$

Note: $\beta = (d-v)\nu$ involves the scaling dimension v.
This depends on the scaling transf.

Combine $\beta = (d-v)\nu$ and $-\gamma = (d-2v)\nu$ to eliminate v

$$\gamma + 2\beta = d\nu \qquad \text{Combine with } \alpha = 2 - d\nu$$

$$\rightarrow \boxed{\alpha + 2\beta + \gamma = 2}$$

$$m \sim h^{1/\delta} \; (T = T_c). \qquad (t = 0)$$

Choose v (small) and n (large) such that $\ell^{nv} h = 0$

$$\ell^{n} = h^{1/v}$$

$$f(S, 0, h) = h^{d/v} f(0, 0, 1)$$

$$m = \frac{\partial f}{\partial h} \sim h^{d/v - 1} \qquad \rightarrow \frac{1}{\delta} = \frac{d}{v} - 1$$

$$\rightarrow \boxed{\beta(\delta - 1) = \gamma}$$

$$g(r) \sim \frac{1}{r^{d-2+\eta}}$$

We find a relation for η along different lines!

magnetization $\langle m_j \rangle = \dfrac{\sum_{\{S_i\}} e^{-\beta H_0 + h \sum_i S_i} S_j}{\sum_{\{S_i\}} e^{-\beta H_0 + h \sum_i S_i}} = \langle m \rangle$

$$\frac{\partial m}{\partial h} = \frac{\sum_{\{S_i\}} e^{-\beta H_0} \left(\sum_k S_k\right) S_j}{\sum_{\{S_i\}} e^{-\beta H_0}} - \frac{\sum_{\{S_i\}} e^{-\beta H_0} \left(\sum_k S_k\right) \sum_{\{S_i\}} e^{-\beta H_0} S_i}{\left(\sum_{\{S_i\}} e^{-\beta H_0}\right)^2}$$

$$= \sum_k \langle S_k S_j \rangle - \left(\sum_k \langle S_k \rangle\right) \langle S_j \rangle$$

$$= \sum_k \left(\langle S_k S_j \rangle - \langle S_k \rangle \langle S_j \rangle \right) = \sum_k g(k, j)$$

Continuum limit: $\dfrac{\partial m}{\partial h} \sim \displaystyle\int g(r)\, d^d r$

$$\sim \int_0^{\xi} \frac{1}{r^{d-2+\eta}} r^{d-1} dr = \xi^{2-\eta} = t^{-\nu(2-\eta)}$$

We also have $\left.\frac{\partial m}{\partial h}\right)_{h=0} = \chi \sim t^{-\gamma}$

So: $\boxed{\gamma = \nu(2-\eta)}$

Scaling relations:

$$\boxed{\begin{aligned} &\alpha + 2\beta + \gamma = 2 \\ &d\nu = 2 - \alpha \\ &\beta(\delta - 1) = \gamma \\ &\gamma = \nu(2-\eta) \end{aligned}}$$

Ising 2D

$\alpha = 0$

$\beta = 1/8$

$\delta = 15$

$\gamma = 7/4$

$\nu = \underline{1}$

$\eta = 1/4$