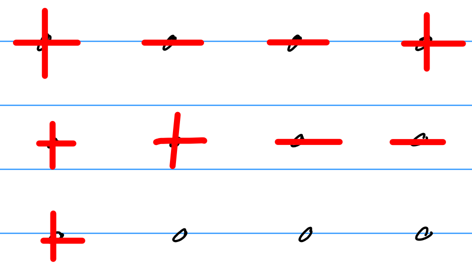
The Ising model

Spins $\quad s_i = \pm 1.$

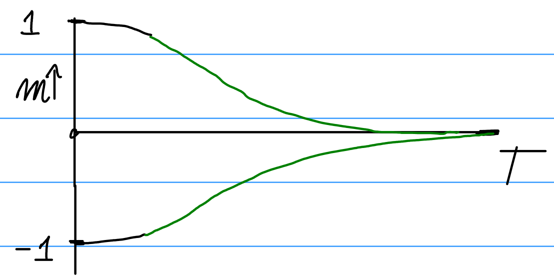


magnetic

$$H = -K\sum_{\langle ij\rangle} s_i s_j - h\sum_i s_i .$$

$\langle ij\rangle$ nearest neighbour.

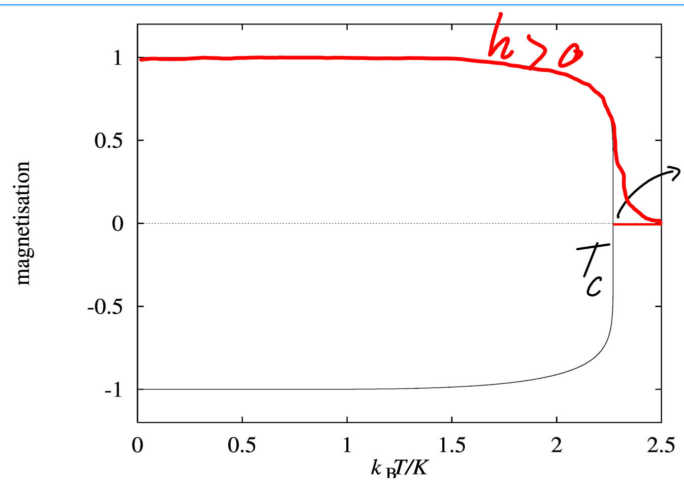
$h=0$



$$m = \frac{N_+ - N_-}{N}$$

$e^{-\beta H}$. $\quad T \to \infty \quad \beta = 0.$

L. Onsager 1944.



$\frac{K}{k_B T} \approx 0.44\ldots$

$T \to T_c$ from below $\quad m(T) \propto (T_c - T)^{\beta} \quad \beta = 1/8$

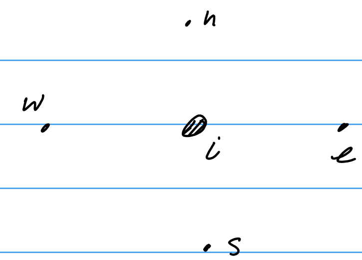
critical exponent

$$\chi_m = \frac{\partial m}{\partial h}\Big)_{T,\,h=0} \propto |T - T_c|^{-\gamma} \qquad \gamma = 7/4$$

$$c_h = |T - T_c|^{-\alpha} \qquad \alpha = 0 \qquad \text{(logarithmic)}.$$

$$T = T_c \qquad m = h^{1/\delta} \qquad \delta = 15.$$

Mean Field Analysis



$$H = -K\sum_{\langle ij\rangle} s_i s_j - h\sum_i s_i$$

$$= -\frac{K}{2}\sum_i \sum_{d=enws} s_i s_{i,d} - h\sum_i s_i$$

q : nr of nearest neighb.
2D square lattice $q=4$

$$= -\frac{K}{2} q m \sum_i s_i - h\sum_i s_i \qquad qm = \Big\langle \sum_{d=enws} s_{i,d} \Big\rangle$$

$$Z = 2^N \cosh^N\left(+\beta\left(\frac{Kqm}{2} + h\right)\right)$$

$$= -Kqm\sum_i s_i + \frac{KNq}{2} m^2 - h\sum_i s_i = H_{MFA}$$

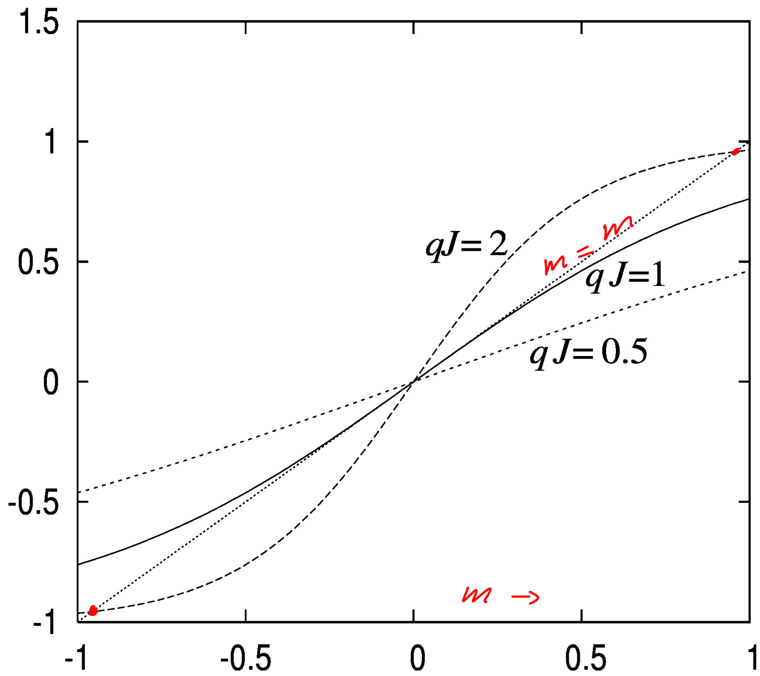
$$Z = \sum_{\{s_i=\pm\}} e^{-\beta H_{MFA}} = e^{-\beta\frac{KNqm^2}{2}} \prod_i \sum_{s_i=\pm 1} e^{(\beta Kqm+\beta h)s_i}$$

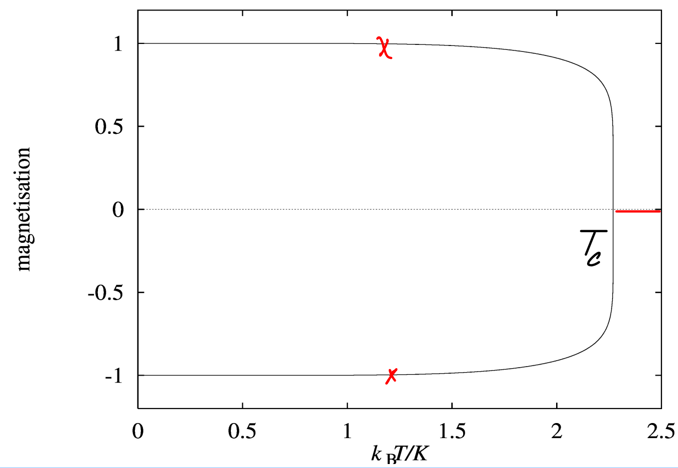
$$= 2^N e^{-\beta\frac{KN}{2}qm^2} \cosh^N \beta(Kqm+h)s_i$$

$$\frac{F}{N} = -\frac{k_B T}{N}\ln Z = \frac{Kqm^2}{2} - k_B T \ln\left(2\cosh\left(\frac{Kqm+h}{k_B T}\right)\right)$$

$$m = \langle s_i \rangle = \frac{\sum_{s_i=\pm 1} e^{\beta(Kqm+h)s_i}\, s_i}{\sum_{s_i=\pm 1} e^{\beta(Kqm+h)s_i}} = \tanh \beta(Kqm+h).$$

$$h=0;\ J=\beta K \Rightarrow m = \tanh(Jqm)$$





$J_c = \frac{1}{q} = \frac{1}{4}$ 2D square lattice

0.44 (exact).

$$m = \tanh(qJm) \qquad m = (T_c - T)^\beta \qquad \beta = 1/8 \text{ exact}$$

$$m = qJm - \frac{1}{3}(qJm)^3 \Rightarrow \frac{1}{3}(qJ)^3 m^3 = (qJ-1)m$$

$$\Rightarrow m^2 \sim qJ - 1 = qJ - qJ_c = \frac{qK}{k_B T} - \frac{qK}{k_B T_c}$$

$$m^2 \sim \frac{1}{T} - \frac{1}{T_c} \simeq \frac{T_c - T}{T T_c} \sim T_c - T$$

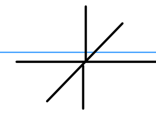
$$m \sim (T_c - T)^{1/2} \qquad \beta = 1/2 \qquad \text{exact: } 1/8$$

$$\chi \sim |T - T_c|^{-1} \qquad \gamma = 1 \qquad \text{exact: } 7/4$$

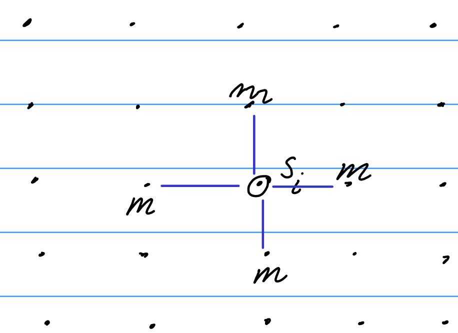
$$C_h \sim \text{const (jumps at } T_c) \qquad \alpha = 0 \qquad \text{exact: } 0$$

$m \sim h^{1/3}$ at $T = T_c$ $\quad \delta = 3$ $\quad$ exact : 15

$D = 3 \quad \beta = 0.324$



$$H = -K \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i$$



$$H_{MFA} = H_0 + H_i$$

$$H_i = -4 K s_i m = -q K s_i m$$

$$\langle s_i \rangle = \frac{\sum_{\{s_j = \pm 1\}} e^{-\beta(H_0 + H_i)} s_i}{\sum_{\{s_j = \pm 1\}} e^{-\beta(H_0 + H_i)}} = \frac{\sum_{s_i = \pm 1} e^{-\beta H_i} s_i}{\sum_{s_i = \pm 1} e^{-\beta H_i}}$$

$$= \frac{e^{+\beta q K m} - e^{-\beta q K m}}{e^{\beta q K m} + e^{-\beta q K m}} = \underline{\tanh(\beta q K m) = m}$$