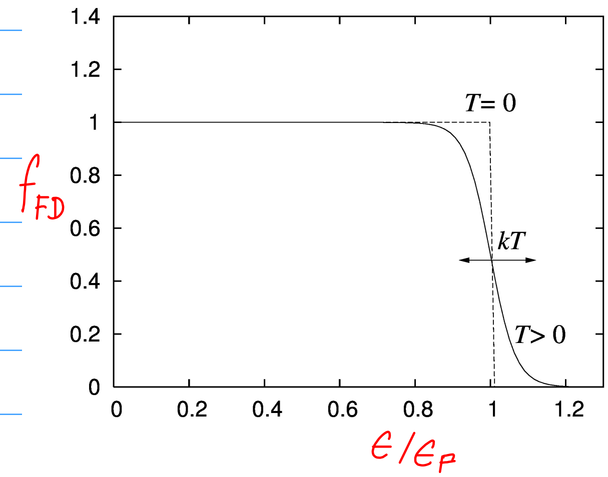
The specific heat of solids: electron contribution

Insulators : no significant contribution

Metals : continuous density of states across Fermi level

Degenerate Fermi gas: $k_B T \ll \epsilon_F$

Consider (nearly) free electrons.



$\epsilon_F = \mu(T=0)$.

$T=0$: filled Fermi sphere: $f_{FD} = \begin{cases} 1 & |\underline{k}| < k_F = \sqrt{\dfrac{2m\epsilon_F}{\hbar^2}} \\ 0 & |\underline{k}| > k_F \end{cases}$

$$N = \overset{\text{spin}}{2} \sum_{|\underline{k}|<k_F} = 2 \frac{V}{(2\pi)^3} \int_{|\underline{k}|<k_F} d^3k = \frac{2\,V}{(2\pi)^3} \frac{4}{3}\pi k_F^3 = \frac{V}{3\pi^2} k_F^3$$

$$\Rightarrow \quad k_F = (3n\pi^2)^{1/3}.$$

$$\sum_{\text{states}} \dots = 2\sum_{\underline{k}} \dots = \frac{V}{\pi^2} \int k^2 dk \dots \overset{\frac{\hbar^2 k^2}{2m} = \epsilon}{=} \frac{V}{\pi^2} \int 2m\epsilon \, d\sqrt{2m\epsilon} \dots =$$

$$= \frac{1}{2}\frac{V}{\pi^2}(2m)^{3/2} \int \sqrt{\epsilon} \dots d\epsilon \equiv \int_0^{\epsilon_F} \underbrace{g(\epsilon)}_{\text{DOS.}} \dots d\epsilon$$

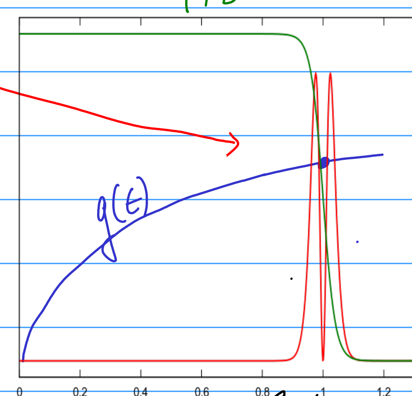
$$g(\epsilon) = \text{const}\,\sqrt{\epsilon}$$

$$T>0: \quad \mathcal{E} = 2\sum_{\underline{k}} \frac{\epsilon_{\underline{k}}}{e^{\beta\epsilon_{\underline{k}}}+1} = \int_0^\infty \frac{\epsilon}{e^{\beta(\epsilon-\epsilon_F)}+1}\, g(\epsilon)\, d\epsilon$$

$$= \underbrace{\int_0^\infty \frac{\epsilon_F\, g(\epsilon)}{e^{\beta(\epsilon-\epsilon_F)}+1}\, d\epsilon}_{N\epsilon_F} + \int_0^\infty \frac{(\epsilon-\epsilon_F)}{e^{\underset{\underset{T}{\uparrow}}{\beta}(\epsilon-\epsilon_F)}+1}\, g(\epsilon)\, d\epsilon$$

$$C_V = \left.\frac{\partial\mathcal{E}}{\partial T}\right)_{N,V} = \frac{1}{k_B T^2}\int_0^\infty \frac{(\epsilon-\epsilon_F)^2 e^{\beta(\epsilon-\epsilon_F)}}{\left(e^{\beta(\epsilon-\epsilon_F)}+1\right)^2}\, g(\epsilon)\, d\epsilon$$

$$= \frac{1}{k_B T^2}\int_0^\infty \frac{(\epsilon-\epsilon_F)^2\, g(\epsilon)\, d\epsilon}{\left(e^{\beta(\epsilon-\epsilon_F)/2}+e^{-\beta(\epsilon-\epsilon_F)/2}\right)^2}$$



$$\approx \frac{1}{k_B T^2}\, g(\epsilon_F)\int_0^\infty \frac{(\epsilon-\epsilon_F)^2\, d\epsilon}{\left(e^{\beta(\epsilon-\epsilon_F)/2}+e^{-\beta(\epsilon-\epsilon_F)/2}\right)^2}$$

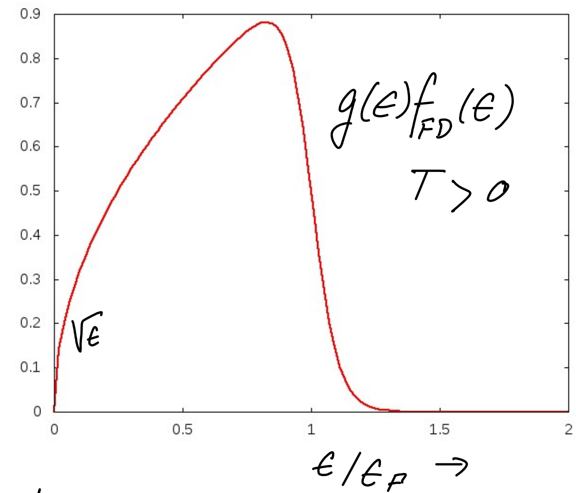
$$x = \beta(\epsilon-\epsilon_F) = (\epsilon-\epsilon_F)/k_B T$$

$$C_V = \frac{\partial\mathcal{E}}{\partial T} = k_B^2\, T\, g(\epsilon_F)\underbrace{\int_{x_F=-\beta\epsilon_F \to -\infty}^{\infty} \frac{x^2\, dx}{\left(e^{x/2}+e^{-x/2}\right)^2}}_{\pi^2/3}$$

$$C_V = \frac{\pi^2}{3}\, k_B^2\, T\, g(\epsilon_F) \quad \sim T$$

$\mu(T) = \epsilon_F$; N fixed.

$$N = \int_0^\infty \frac{\overbrace{g(\epsilon)}^{\sqrt{\epsilon}}}{e^{\beta(\epsilon-\mu)}+1}\, d\epsilon$$



$\Delta(\epsilon) \qquad \Delta'(\epsilon) = g(\epsilon)$.

$$N = \int_0^\infty \Delta(\epsilon) \frac{\beta}{\left(e^{\beta(\epsilon-\mu)/2} + e^{-\beta(\epsilon-\mu)/2}\right)^2}\, d\epsilon$$

$$\Delta(\epsilon) = \Delta(\mu) + g(\mu)(\epsilon-\mu) + \frac{1}{2} g'(\mu)(\epsilon-\mu)^2 + \dots$$

$$N = \int_0^\infty \left[\Delta(\mu) + \underbrace{(\epsilon-\mu) g(\mu)}_{0 \text{ antisymm}} + \frac{(\epsilon-\mu)^2}{2} g'(\mu)\right] \underbrace{\frac{\beta}{\left(e^{\beta(\epsilon-\mu)/2} + e^{-\beta(\epsilon-\mu)/2}\right)^2}}_{-\frac{df}{d\epsilon}}\, d\epsilon$$

1^{st} term: $-\Delta(\mu) \int_0^\infty \frac{df}{d\epsilon}\, d\epsilon = -\Delta(\mu)\Big(\underbrace{f(\infty)}_{0} - \underbrace{f(0)}_{1}\Big) = \Delta(\mu)$

$$\rightarrow N(T) = \Delta(\mu) + g'(\mu) \frac{\pi^2}{6} (k_B T)^2 = N(T=0)$$

$$\frac{dN}{dT} = 0 = g(\mu) \underset{\frac{d\mu}{dT}}{\underset{\parallel}{\mu'}} + g''(\mu)\mu' \frac{\pi^2}{6}(k_B T)^2 + g'(\mu)\frac{\pi^2}{3} k_B^2 T$$

$$g(\mu)\mu' = -g'(\mu)\frac{\pi^2}{3} k_B T \qquad \mu' \sim T$$

$$\mu = \mu(T=0) + C T^2$$