# A Proof of Gödel's First Incompleteness Theorem With Extra Unexpected Findings

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## Abstract

Gödel's First Incompleteness Theorem (FIT) is a landmark result in mathematical logic that reveals the inherent limitations of formal axiomatic systems capable of modelling basic arithmetic. This paper provides a detailed, step-by-step proof of the theorem, demonstrating that in any consistent formal system $F$, there exist *true* statements that cannot be proven within the system. The proof will be rigorously established using truth tables to ensure clarity and precision. As a consequence of the proof, the paper further presents extra unexpected findings about the size and topology of the Universe.

## Introduction

Gödel's First Incompleteness Theorem (hereby denoted as FIT), formulated by Kurt Gödel in 1931, is one of the most profound discoveries in the field of mathematical logic. The theorem states that in any consistent formal system that is capable of expressing basic arithmetic, there exist *true* statements that cannot be proven within the system. This result shattered the prevailing belief in the early 20th century that mathematics could be both complete and consistent, as envisioned by David Hilbert and other mathematicians who sought a complete and consistent set of axioms for all of mathematics.

Gödel's work emerged during a period of intense investigation into the foundations of mathematics. The early 20th century saw significant efforts to formalise mathematics, driven by the desire to eliminate paradoxes and inconsistencies. The development of set theory by Georg Cantor and the formalisation of logic by Gottlob Frege and Bertrand Russell were pivotal in this endeavour. However, paradoxes such as Russell's paradox highlighted the need for a more rigorous foundation.

In this context, Gödel's Incompleteness Theorems provided a groundbreaking insight. The First Incompleteness Theorem (FIT) demonstrated that no formal system, if it is consistent and capable of expressing elementary arithmetic, can be both complete and consistent. This means that there will always be true mathematical statements that cannot be derived from the system's axioms. Gödel's proof employed innovative techniques, including the arithmetization of syntax and the construction of self-referential statements, which have since become fundamental tools in mathematical logic.

The implications of Gödel's First Incompleteness Theorem extend beyond mathematics, influencing fields such as computer science, philosophy, and cognitive science. It has profound consequences for our understanding of the limits of formal systems and the nature of mathematical truth, challenging the notion that all mathematical truths can be captured by a finite set of axioms and rules.

Gödel's First Incompleteness Theorem (FIT) is not only a cornerstone of mathematical logic but also a pivotal result with far-reaching implications in philosophy. Its importance in mathematical logic lies in its revelation of the fundamental limitations of formal axiomatic systems. By demonstrating that any sufficiently powerful and consistent formal system cannot be both complete and consistent, Gödel's theorem fundamentally altered our understanding of the scope and limits of formal mathematical reasoning.

In mathematical logic, the theorem underscores the inherent limitations of formal systems in capturing all mathematical truths. This has profound implications for the development of formal theories and the pursuit of a complete and consistent set of axioms for mathematics. Gödel's work showed that no matter how comprehensive a formal system is, there will always be true statements that elude formal proof within that system. This insight has influenced subsequent research in logic, leading to the development of alternative logical frameworks and a deeper exploration of the foundations of mathematics.

Philosophically, Gödel's First Incompleteness Theorem (FIT) challenges the notion of mathematical certainty and the belief in the absolute completeness of mathematical knowledge. It raises fundamental questions about the nature of mathematical truth and the limits of human knowledge. The theorem suggests that there are truths that lie beyond formal proof, prompting philosophical inquiries into the nature of these truths and the means by which we can know them.

Moreover, Gödel's theorem has implications for the philosophy of mind and artificial intelligence. It has been interpreted as suggesting that human cognition cannot be fully captured by formal systems, implying that the human mind possesses capabilities that transcend formal logical reasoning. This has sparked debates about the nature of human intelligence, the possibility of artificial intelligence, and the limits of computational models of the mind.

In summary, Gödel's First Incompleteness Theorem (FIT) is a landmark result that has reshaped our understanding of formal systems, mathematical truth, and the limits of human knowledge. Its significance extends beyond mathematics, influencing philosophical thought and prompting ongoing exploration of the foundations of logic and the nature of intelligence.

# Preliminaries

## Formal Systems

A formal system is a structured framework used to derive conclusions from a set of axioms through the application of inference rules. It consists of the following components:

1. Alphabet: a finite set of symbols used to construct statements within the system. These symbols can include variables, constants, operators, and punctuation marks.
2. Syntax: a set of rules that define how symbols can be combined to form well-formed formulas (WFFs) or statements. Syntax rules ensure that the statements are constructed in a grammatically correct manner.
3. Axioms: a set of foundational statements or propositions that are assumed to be true without proof. Axioms serve as the starting point for deriving other statements within the system.
4. Inference Rules: a set of logical rules that dictate how new statements can be derived from existing ones. These rules are used to manipulate and combine axioms and previously derived statements to produce new conclusions.
5. Theorems: statements that can be derived from the axioms using the inference rules. A theorem is considered proven if it can be shown to follow logically from the axioms through a finite sequence of applications of the inference rules.

A formal system is designed to be precise and unambiguous, allowing for rigorous proofs and logical reasoning. The goal of a formal system is to provide a clear and systematic method for deriving truths within a specific domain of knowledge, such as arithmetic, geometry, or logic.

## (Arithmetic) Reasoning

Arithmetic is the branch of mathematics that deals with the properties and manipulation of numbers. In the context of Gödel's First Incompleteness Theorem (FIT), we focus on the basic arithmetic operations and properties that are essential for understanding the theorem.

- Peano Axioms: among the Peano axioms, we highlight the **Induction Principle** as we are going to use it to prove FIT to be true. If a property holds *true* for the base case and holds *true* for the successor *n* and also the *n+1* successor then it holds *true* for all cases.

Understanding these basics is crucial for grasping Gödel's First Incompleteness Theorem (FIT), as the theorem applies to formal systems that can express these fundamental concepts. The theorem shows that even in such systems, there are *true* statements that cannot be proven within the system, highlighting the limitations of formal (arithmetic) reasoning.

# Proof

In any **consistent formal system** that is capable of expressing basic arithmetic, there exist *true* statements about the natural numbers that cannot be proven within the system. This means that no matter how comprehensive the system is, there will always be some truths that lie beyond its reach.

Since we are talking about truth and falsehood we can use a truth table to describe the logical state of a boolean statement within a standard model of arithmetic. Gödel’s FIT was initially devised for numbers but in this proof we will consider the whole Measurable Universe ($U$) as a formal system that is capable of expressing basic arithmetic. We thereby assume all numbers can be expressed within $U$. That means there are *true* statements that can never be produced within the lifetime of $U$. We can define $U$ as a collection of *situations*, regardless of their position in time and space. Each situation is either *true* or *false*, exists or not, which allows us to consider it a statement. One may argue that a statement can also be simultaneously *true* and *false* if we consider $U$ at a quantum level. However, for this demonstration we want to prove Gödel's First Incompleteness Theorem on a **consistent formal system.** We know for a fact that when a situation is measured it cannot be an indeterminate, so it's either *true* or *false*; and this is also supported by quantum mechanics. Therefore, we can describe all events within $U$ as being *true* or *false*. For this demonstration, we will define an **event** as a sequence of 2 situations. Also, because an event can be measured by measuring 2 situations, the event can also be interpreted as a statement, which can either be *true* or *false*.

Using Logic, we can use the *logical consequence* to measure the truthness of an event (which is a causal transition from situation A to situation B). The definition of a truth table for the *logical consequence* is as follows:

| p | q | (p → q) |
|---|---|---|
| F | F | **T** |
| F | T | **T** |
| T | F | **F** |
| T | T | **T** |

- Each column is a situation.
- The truth table has 2 situations ($p$ and $q$) and the consequential outcome which is true ($T$) or false ($F$).
- Since $p \to q$ is a consequential outcome, it can be regarded as a third new situation, which can also be *true* or *false.*
- The truth table initially has 4 different types of events
    - **FF**: situation $p$ is *false*, situation $q$ is *false* and the outcome is *true.*
    - **FT**: situation $p$ is *false*, situation $q$ is *true* and the outcome is *true.*
    - **TF**: situation $p$ is *true*, situation $q$ is *false* and the outcome is *false.*
    - **TT**: situation $p$ is true, situation $q$ is true and the outcome is *true.*
- By definition, the $U$ is a set that contains all possible events $FF, FT, TF, TT$ as presented by the first 2 columns in the table above.
- On the right column, we see the list of outcomes.
- Since the outcome is a situation too, we can use the list of outcomes to feed a new logical consequence truth table. In other words, the outcomes ($p \to q$) of the previous truth table become the seeds ($p$) for the next truth table.

- So, within $U$, each truth table can be regarded as a *reality ($R$)*, where certain types of events can take place.
- $U$ can therefore be regarded as a sequence of *realities:*

$$R(0) \rightarrow R(1) \rightarrow R(2) \rightarrow R(3) \rightarrow R(4) \rightarrow \ldots \rightarrow R(n-1) \rightarrow R(n) \rightarrow R(n+1)$$

- In order for FIT to be proven *true* for system $U$, then it must be *true* that there are certain events measured as being *true* that never exist in the consistent part of $U$ (as we will demonstrate). Let's expand the sequence of realities for $U$, starting with $R(0)$ which we define as the *base case reality.*

The sequence of realities is defined as follows:

?

| p | q | (p → q) |
|---|---|---|
| F | F | **T** |
| F | T | **T** |
| T | F | **F** |
| T | T | **T** |

Reality(0)

| p | q | (p → q) |
|---|---|---|
| T | F | **F** |
| T | T | **T** |
| F | F | **T** |
| T | T | **T** |

Reality(N)

| p | q | (p → q) |
|---|---|---|
| F | F | **T** |
| T | T | **T** |
| T | F | **F** |
| T | T | **T** |

Reality(N+1)

$R(0)$ transitions into $R(1)$ which then transitions into $R(2)$ which then transitions into $R(3)$. What-is-more, we see that $R(3)$ is logically equivalent to R(1), so now we see that $U$ forms a loop of realities made of $R(N)$ and $R(N + 1)$.

We add a merely theoretical question mark (?) in the drawing which represents an indetermination to highlight that the base case must be reachable somehow from the reality loop. However, we note that the transitions are deterministic which means that in order to get out of the reality loop there must be an non-deterministic event that forces the "*reality Wagon*" to go to base reality.

We notice that $U$ is formed by a consistent part made of a set of realities $R(N)$ and $R(N + 1)$ (denoted as system $F$) and an inconsistent part (which we call system $I$). For the sake of this proof, we only need to consider system $F$ since FIT only applies to consistent systems.

A side note to highlight that a formal system is only considered consistent if it is impossible to derive both a statement $P$ and its negation from the axioms and inference rules of that system. We prove that $F$ is consistent because it forms a deterministic loop, which is another way to say that even as the "reality Wagon" jumps on and off to a certain reality, when it comes back there are no statements previously measured that have a different logical state.

In order to prove the truthfulness of FIT we must find a set of events with non-zero cardinality that are causally *true* outside of $F$. And we can do just that by pointing out that *FT-type* events ($p$ being *false* and $q$ being *true*) can never be measured inside of $F$. In order to measure *FT-type* events, one must be outside of $F$, therefore making it incomplete. We thereby consider the FIT proven ***TRUE***.

# Extra Unexpected Findings

This is the most important part of the paper. FIT has already been proven *true* by Kurt Gödel himself. However this is the first time a proof for FIT has been proposed using Truth Tables. The advantage of using Truth Tables to prove FIT is that it showcases the determinism of the transitions within formal system $F$. This determinism suggests that the Measurable Universe ($U$) is not finite in time but is finite in space. This is because since the reality loop set has cardinality greater than 1 then there must be a finite sequence of events that cause the "reality Wagon" to move to the next reality.

Also, an extra finding is that the outcome set from $R(0)$ is exactly the same as the outcome set $R(N + 1)$ which means that $R(0)$ might be a merely theoretical reality that is never reached, therefore making it not real and inexistent. This is because we artificially seeded $R(0)$ with all possible outcomes which might not be *true*. But then the Measurable Universe corrects itself (even if we seed it with impossible events) by entering into a loop of realities $N$ and $N + 1$.

As we proved that within each reality there is a finite sequence of events that cause the "reality Wagon" to move to the next reality, we also know that a finite set of outcomes will seed the next reality, which then in turn results in a finite and deterministic list of measurements. Therefore, it is apparent that a deterministic list of measured events in a looped $U$ must be **finite in space** and maybe **infinite in time** as the loop has no obvious terminal event.

A final finding from this proof is that if $U$ and $F$ are the same set of deterministic events, because $R(0)$ was an artificial seed, then it means the entire Universe $(F + I)$ must be deterministic as well as closed since it is a loop.

# Conclusion

In this paper we have introduced a novel approach to prove Gödel's First Incompletude Theorem using Truth Tables. This new approach seems to suggest that the Universe is looped which leads to unexpected conclusions. Apart from confirming the Universe’s incompleteness, we have also proven something new which is that it is closed and finite in space, which are statements that have not been proven or disproven up until now.

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