Relational Solver Notes

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1 Roadmap

This document sketches the relational solver that accompanies the Mox type system notes. The goal is to develop a mode solver that is powerful enough to handle the mode constraints produced by the type checker.

At a high level, we will develop a solver for binary constraints between modes. We will first analyze a class of constraints that can be solved in polynomial time. Then we will develop a more powerful solver that can handle that class of constraints.

2 Constraint Language

As a first step, assume we have a finite domain V of values, and a set of binary relations $R_i \subseteq V \times V$. Given a set of variables and asserted contraints between them, we want to determine if there exists a valuation of the variables that satisfies all the constraints.

In general this is a NP-complete problem: consider $V = \{0, 1, 2, \ldots, k\}$ and $R_1 = \{(a, b) \mid a \neq b\} \subseteq V \times V$. Given a graph we can use this constraints to encode the k-coloring problem: we want to assign a color to each vertex such that no two adjacent vertices have the same color. The k-coloring problem is NP-complete, so this problem is also NP-complete.

However, certain classes of constraints can be solved in polynomial time. Consider the set of constraints $R_i = \{(a, b) \mid b \geq a + i\} \subseteq V \times V$. Given a graph of variables and constraints, we can solve this problem in polynomial time using the Floyd-Warshall algorithm.

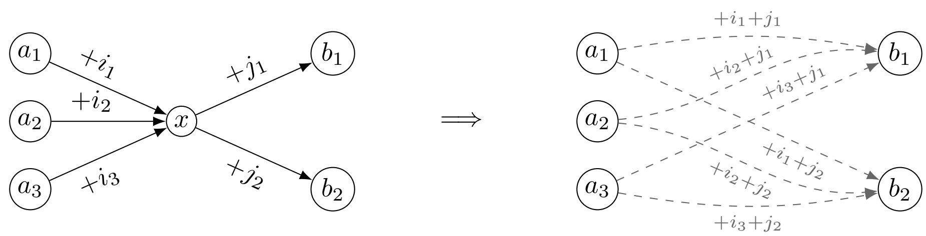
Equivalently, we can solve the problem by variable elimination: we take a variable x and all adjacent constraints on it. We assert all transitive constraints (where R_i composes with R_j to produce R_{i+j}) and repeat until all variables are eliminated. If, during this process, we ever see a constraint between a variable and itself with $i \neq 0$, then the constraints are unsatisfiable.

Why does this elimination strategy work for this class of constraints, but not for the general case, and in particular for the k-coloring problem with inequality constraints?

Consider a variable x with:

- a set of predecessors a_1, a_2, a_3 with $R_{i_1}(a_1, x), R_{i_2}(a_2, x), R_{i_3}(a_3, x)$ constraints on them,
- a set of successors b_1, b_2 with $R_{j_1}(x, b_1), R_{j_2}(x, b_2)$ constraints on them,

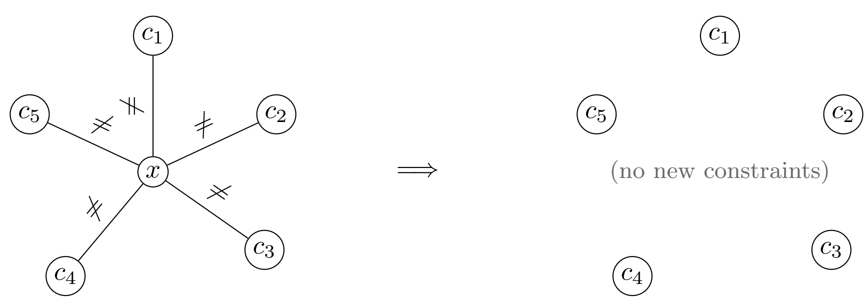
When eliminating x, we assert every transitive constraint $R_{i_p+j_q}(a_p, b_q)$ for $p \in \{1, 2, 3\}$ and $q \in \{1, 2\}$:



Before elimination After elimination

I claim that if there is a solution for the neighboring variables that satisfies all of those transitive constraints, then there is a solution for x that satisfies the original constraints: we can simply set x to any value in the interval $\max(a_1 + i_1, a_2 + i_2, a_3 + i_3) \leq x \leq \min(b_1 + j_1, b_2 + j_2)$. This interval is guaranteed to be non-empty if the transitive constraints hold.

The key blocker for k-coloring is that this property does not hold for inequality constraints. Suppose for example we have a vertex x and $k+1$ neighbors with inequality constraints. The transitive constraints are trivial if $k \geq 3$, because if we have $a \neq x \neq b$, then for a given value of a, all values of b are still possible, by choosing a particular value for x. Thus, the strategy of variable elimination does not work for k-coloring, for $k \geq 3$: for $\neq$ constraints, eliminating x produces no useful transitives; every neighbour pair remains unconstrained:



Before elimination After elimination

The OxCaml mode solver has constraints of the form $x \leq G(y)$ where G are modalities with left adjoints. Like the interval constraints we considered above, these constraints can be solved in polynomial time using variable elimination, and for the same reason.

3 Solver Architecture

4 Open Questions