# Relational Solver Notes

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## 1 Roadmap

This document sketches the relational solver that accompanies the Mox type system notes. The goal is to develop a mode solver that is powerful enough to handle the mode constraints produced by the type checker.

At a high level, we will develop a solver for binary constraints between modes. We will first analyze a class of constraints that can be solved in polynomial time. Then we will develop a more powerful solver that can handle that class of constraints.

## 2 Constraint Language

As a first step, assume we have a finite domain $V$ of values, and a set of binary relations $R_i \subseteq V \times V$. Given a set of variables and asserted contraints between them, we want to determine if there exists a valuation of the variables that satisfies all the constraints.

In general this is an NP-complete problem: consider $V = \{0, 1, 2, \ldots, k\}$ and $R_1 = \{(a, b) \mid a \neq b\} \subseteq V \times V$. Given a graph we can use this constraints to encode the $k$-coloring problem: we want to assign a color to each vertex such that no two adjacent vertices have the same color. The $k$-coloring problem is NP-complete, so this problem is also NP-complete.

However, certain classes of constraints can be solved in polynomial time. Consider the set of constraints $R_i = \{(a, b) \mid b \geq a + i\} \subseteq V \times V$. Given a graph of variables and constraints, we can solve this problem in polynomial time using the Floyd-Warshall algorithm.

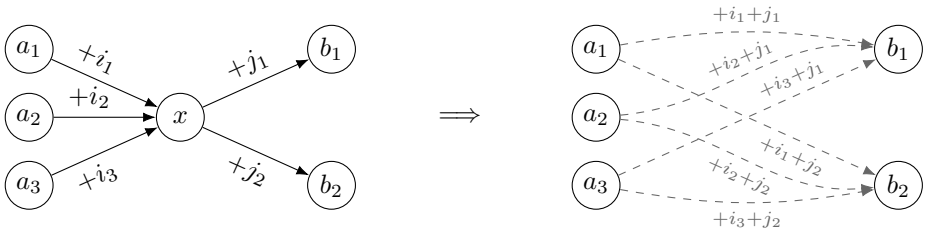
Equivalently, we can solve the problem by variable elimination: we take a variable $x$ and all adjacent constraints on it. We assert all transitive constraints (where $R_i$ composes with $R_j$ to produce $R_{i+j}$) and repeat until all variables are eliminated. If, during this process, we ever see a constraint between a variable and itself with $i \neq 0$, then the constraints are unsatisfiable.

Why does this elimination strategy work for this class of constraints, but not for the general case, and in particular for the $k$-coloring problem with inequality constraints?

Consider a variable $x$ with:

- a set of predecessors $a_1, a_2, a_3$ with $R_{i_1}(a_1, x), R_{i_2}(a_2, x), R_{i_3}(a_3, x)$ constraints on them,
- a set of successors $b_1, b_2$ with $R_{j_1}(x, b_1), R_{j_2}(x, b_2)$ constraints on them,

When eliminating $x$, we assert every transitive constraint $R_{i_p+j_q}(a_p, b_q)$ for $p \in \{1, 2, 3\}$ and $q \in \{1, 2\}$:

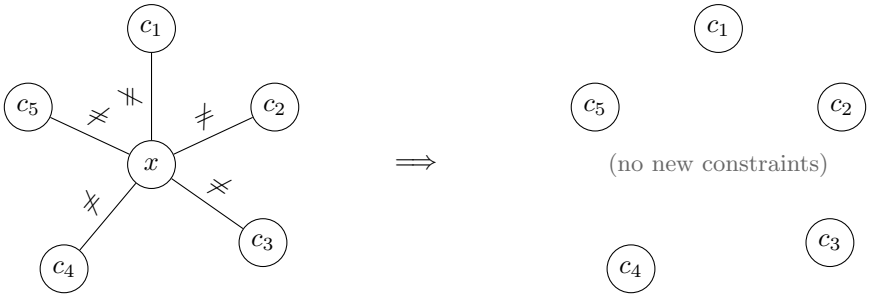



Before elimination After elimination

Figure 1: Difference constraints generate informative transitives.

I claim that if there is a solution for the neighboring variables that satisfies all of those transitive constraints, then there is a solution for $x$ that satisfies the original constraints: we can simply set $x$ to any value in the interval $\max(a_1 + i_1, a_2 + i_2, a_3 + i_3) \leq x \leq \min(b_1 + j_1, b_2 + j_2)$. This interval is guaranteed to be non-empty if the transitive constraints hold.

The key blocker for $k$-coloring is that this property does not hold for inequality constraints. Suppose for example we have a vertex $x$ and $k + 1$ neighbors with inequality constraints. The transitive constraints are trivial if $k \geq 3$, because if we have $a \neq x \neq b$, then for a given value of $a$, all values of $b$ are still possible, by choosing a particular value for $x$. Thus, the strategy of variable elimination does not work for $k$-coloring, for $k \geq 3$: for $\neq$ constraints, eliminating $x$ produces no useful transitives; every neighbour pair remains unconstrained:




Before elimination After elimination

Figure 2: Inequality constraints add no new relations between neighbours.

The OxCaml mode solver has constraints of the form $x \leq G(y)$ where $G$ are modalities with left adjoints. Like the interval constraints we considered above, these constraints can be solved in polynomial time using variable elimination, and for the same reason.

## 2.1 Which constraints can be solved by elimination?

For a set of constraints to be solvable by elimination, we need that the original problem has a solution iff the transformed problem has a solution. We will now analyze precisely when this property holds, and provide an easily checkable criterion for it.

Eliminating $x$ yields edges $R_{ij} = R_i^\top R_0^* R_j$ directed from $y_j$ to $y_i$ for all $i \geq j$, including self-loops $R_{ii}$. If there were existing relations between the $y_i$ then we intersect them with these. Figure 3 illustrates the local transformation on the "star" centered at $x$.

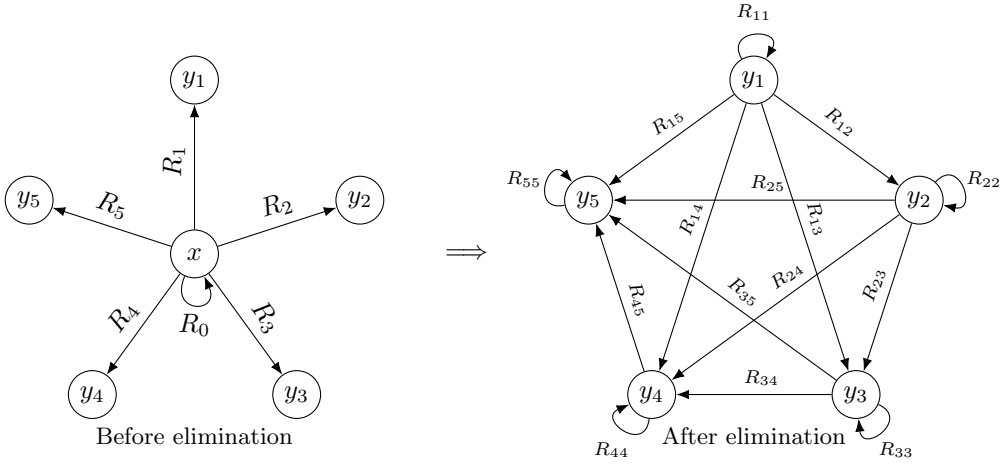



Figure 3: Eliminating $x$ in the radial layout yields edges $R_{ji}$ (with $R_{ji} \stackrel{\text{def}}{=} R_j^\top R_0^* R_i$) directed from $y_j$ to $y_i$ for all $i \geq j$, as well as self loops $R_{ii}$.

**The key question is: for which families of relations $\mathsf{Rel}$ does this algorithm work?**

**Answer.** Intuitively, the elimination step is sound if every time we eliminate a variable $x$, the constraints we generate between its neighbors are "complete": any solution of the projected constraints on the neighbors can be extended to a value of $x$. The obstruction is exactly the phenomenon we saw with $\neq$-constraints: locally consistent projections need not extend.

We now characterize when elimination is complete.

## 2.2 Closure and slices

Fix a finite domain $V$ and a finite family of binary relations $\mathsf{Rel} \subseteq \mathcal{P}(V \times V)$. The elimination algorithm only ever uses relations obtained from $\mathsf{Rel}$ by:

- composition: $RS = \{(a, c) \mid \exists b.\ (a, b) \in R \wedge (b, c) \in S\}$,
- intersection: $R \cap S$,
- converse: $R^\top = \{(b, a) \mid (a, b) \in R\}$,
- restriction to the diagonal (from self-loops): $R^* = \{(a, a) \mid (a, a) \in R\}$.

Let $\overline{\mathsf{Rel}}$ be the smallest set of relations containing $\mathsf{Rel}$ and closed under these four operations.

For a relation $R \in \overline{\mathsf{Rel}}$ and an element $b \in V$ we write:

$$R[\_, b] \;=\; \{a \in V \mid (a, b) \in R\} \qquad \text{and} \qquad R[a, \_] \;=\; \{b \in V \mid (a, b) \in R\}$$

for its *column* and *row* slices. These are exactly the unary constraints on $x$ induced by fixing the other endpoint.

Let $\mathcal{F}$ be the family of all such slices:

$$\mathcal{F} \;=\; \{\, R[\_, b], R[a, \_] \mid R \in \overline{\mathsf{Rel}},\ a, b \in V \,\} \;\subseteq\; \mathcal{P}(V).$$

Because $\overline{\mathsf{Rel}}$ is closed under intersection, so is $\mathcal{F}$: for any $A, B \in \mathcal{F}$ there exists $C \in \mathcal{F}$ with $C = A \cap B$. This intersection-closure is crucial below.

## 2.3 A Helly-style condition

The key combinatorial property is a Helly-type condition on $\mathcal{F}$.

**Definition 1** (Helly-2 for slices)**.** *We say that* $\mathsf{Rel}$ *has the* Helly-2 slice property *if the following holds:*

*For every finite subfamily* $\{F_1, \ldots, F_k\} \subseteq \mathcal{F}$*, if all pairwise intersections are non-empty,*

$$F_i \cap F_j \neq \varnothing \quad \textit{for all } i \neq j,$$

*then the total intersection is non-empty,*

$$\bigcap_{i=1}^{k} F_i \neq \varnothing.$$

Because $\mathcal{F}$ is intersection-closed, this is equivalent to the absence of a *bad triple*:

**Lemma 2** (Triples suffice for intersection-closed families)**.** *Assume* $\mathcal{F}$ *is closed under intersection. Then* $\mathcal{F}$ *has the Helly-2 slice property if and only if there do not exist* $A, B, C \in \mathcal{F}$ *such that*

$$A \cap B \neq \varnothing, \quad B \cap C \neq \varnothing, \quad C \cap A \neq \varnothing, \quad A \cap B \cap C = \varnothing.$$

*Proof.* ($\Rightarrow$) Immediate: such a triple would violate Helly-2.

($\Leftarrow$) Suppose Helly-2 fails. Pick a counterexample $\{F_1, \ldots, F_m\} \subseteq \mathcal{F}$ of minimal size: all pairs intersect, but $\bigcap_i F_i = \varnothing$. Since $\mathcal{F}$ is intersection-closed, for each $i \geq 2$ the set $A_i := F_1 \cap F_i$ lies in $\mathcal{F}$, is non-empty, and $\bigcap_{i=2}^{m} A_i = \bigcap_{i=1}^{m} F_i = \varnothing$. If some $A_i \cap A_j$ were empty, then $F_1, F_i, F_j$ would form a bad triple. Otherwise $\{A_2, \ldots, A_m\}$ is a smaller counterexample, contradicting minimality. □

Thus in our setting we can detect failure of the Helly-2 slice property by a finite search for such bad triples among slices in $\overline{\mathsf{Rel}}$.

## 2.4 Correctness of elimination

We now state the main correctness criterion.

**Theorem 3** (Characterization of admissible constraint families)**.** *Let $V$ be finite and $\mathsf{Rel} \subseteq \mathcal{P}(V \times V)$. Consider the following variable-elimination algorithm: at each step, pick a variable $x$, and for every two neighbors $y_i \xleftarrow{R_i} x \xrightarrow{R_j} y_j$ insert the composed edge $y_i \xleftarrow{R_i R_0^* R_j} y_j$ (where $R_0^*$ is the current self-loop on $x$, possibly* Id*), intersecting with any existing edge between $y_i$ and $y_j$, then delete $x$. If at any point a self-loop on some variable becomes empty on the diagonal (i.e. forbids all $(a, a)$), report unsatisfiable.*

*Then the following are equivalent:*

1. *For every finite constraint graph over* Rel*, this elimination algorithm decides satisfiability correctly: the final instance is satisfiable iff the original instance is satisfiable.*

2. Rel *has the Helly-2 slice property.*

*Proof sketch.* (2) $\Rightarrow$ (1) (*completeness of elimination*). It is enough to show that a *single* elimination step is complete, and then argue by induction on the number of variables.

Consider eliminating $x$ from its star. Fix an assignment to the neighbors $\{y_i\}$ that satisfies all constraints generated by the algorithm between the $y_i$. For each incident edge, this assignment induces a slice $S_i \in \mathcal{F}$ of admissible values for $x$ (e.g., for $y_i \xrightarrow{R_i} x$ with $y_i$ fixed to $b_i$ we get $S_i = R_i[\_, b_i]$). By construction, every binary constraint added between $y_i$ and $y_j$ enforces that $S_i$ and $S_j$ intersect: if they did not, the corresponding composed relation would have removed the chosen pair $(b_i, b_j)$. Hence the family $\{S_i\}$ is pairwise-intersecting. By the Helly-2 slice property, $\bigcap_i S_i \neq \varnothing$. Picking any $a$ in this intersection and setting $x := a$ extends the neighbor assignment to a full solution of the original constraints around $x$. Inductively, every model of the final instance lifts to a model of the original one.

Soundness of the algorithm (it never introduces spurious solutions) holds because every new edge relation is logically implied by the existence of an intermediate $x$ satisfying its incident constraints.

(1) $\Rightarrow$ (2) (*necessity*). Assume the Helly-2 slice property fails. By Lemma 2 and intersection-closure, there exist $S_1, S_2, S_3 \in \mathcal{F}$ with all pairwise intersections non-empty but empty triple intersection. By definition of $\mathcal{F}$, we can realize each $S_i$ as the admissible values for some variable $x$ given a fixed neighbor $y_i$ and a relation $R_i \in \overline{\mathsf{Rel}}$. We construct a constraint star with center $x$ and leaves $y_1, y_2, y_3$ so that, for the chosen values of the $y_i$, the corresponding slices are precisely $S_i$. By pairwise intersection, every pair of leaves admits a value of $x$; hence the elimination algorithm, which only asserts pairwise compositions, accepts this partial assignment to $y_1, y_2, y_3$. But by emptiness of $S_1 \cap S_2 \cap S_3$, no value of $x$ satisfies all three constraints simultaneously, so the original instance is unsatisfiable. Thus elimination is not complete, contradicting (1). □

**Corollary 4** (Finite check)**.** *For finite $V$, the Helly-2 slice property for* $\mathsf{Rel}$ *is decidable:*

1. *compute* $\overline{\mathsf{Rel}}$ *(finite);*
2. *form all slices $R[\_, b], R[a, \_]$;*
3. *check that no triple of slices has non-empty pairwise intersections and empty triple intersection.*

*If no such triple exists, variable elimination as above is sound and complete for all constraint graphs over* $\mathsf{Rel}$.

In particular, difference constraints and the OxCaml-style modal constraints $x \leq G(y)$ with adjoints fall into this framework: their induced slices are (discrete) intervals, which satisfy Helly-2, so elimination is complete. By contrast, $\neq$-constraints generate non-convex slices, violate Helly-2, and are correctly rejected by this criterion.

## 2.5 Many-sorted constraint families

We now generalize the previous discussion from a single carrier set $V$ to a many-sorted setting. This is the natural abstraction for type/mode systems where variables, modes, or resources live in different universes and constraints relate values across sorts.

**Sorted carriers and typed relations.** Let $\mathcal{O}$ be a finite set of sorts. For each $A \in \mathcal{O}$, fix a finite carrier $V_A$. A binary relation is *typed* by its source and target sorts, so a relation $R \in \mathsf{Rel}(A, B)$ is a subset of $V_A \times V_B$.

Given two typed relations $R \in \mathsf{Rel}(A, B)$ and $S \in \mathsf{Rel}(B, C)$ with matching middle sort $B$, their composition is the typed relation $(RS)_{A,C} = \{(a, c) \mid \exists b \in V_B.\ (a, b) \in R \wedge (b, c) \in S\}$. Intersection is defined only for relations with the same type: $R \cap S \in \mathsf{Rel}(A, B)$. Converse flips the type: $(R)^\top = R^\top \in \mathsf{Rel}(B, A)$. For a same-sort relation $R \in \mathsf{Rel}(A, A)$, diagonal restriction is $R^* = \{(a, a) \mid (a, a) \in R\} \in \mathsf{Rel}(A, A)$.

Let $\overline{\mathsf{Rel}}$ be the least family of typed relations containing $\mathsf{Rel}$ and closed under these typed operations (composition, intersection, converse, and diagonal restriction where defined).

**Slices by sort.** For a typed relation $R \in \overline{\mathsf{Rel}}(A, B)$ and element $b \in V_B$, define the slice

$$R[\_, b] \;=\; \{a \in V_A \mid (a, b) \in R\}$$

For each sort $A$, collect all slices into the sorted slice family

$$\mathcal{F}_A \;=\; \{R[\_, b] \mid R \in \overline{\mathsf{Rel}}(A, B),\; b \in V_B\}.$$

As in the single-sorted case, each $\mathcal{F}_A$ is closed under intersection because $\overline{\mathsf{Rel}}$ is closed under intersection of like-typed relations.

**Definition 5** (Helly-2 by sort)**.** *We say that a typed family* $\mathsf{Rel}$ *has the* Helly-2 slice property *if for every sort* $A \in \mathcal{O}$ *the slice family* $\mathcal{F}_A$ *satisfies Helly-2: for every finite subfamily* $\{F_1, \ldots, F_k\} \subseteq \mathcal{F}_A$*, pairwise non-empty intersections imply a non-empty total intersection.*

By intersection-closure, Lemma 2 applies sortwise: for each $A$ it is enough to rule out *bad triples* $A, B, C \in \mathcal{F}_A$ with all pairwise intersections non-empty but empty triple intersection.

**Typed elimination.** Constraint graphs are now many-sorted: each variable $x : A$ has sort $A \in \mathcal{O}$ and each edge $x \xrightarrow{R} y$ carries a typed relation $R \in \overline{\mathsf{Rel}}(A, B)$. Self-loops on $x$ carry same-sort relations $R \in \overline{\mathsf{Rel}}(A, A)$.

When eliminating a variable $x : A$, we first normalize the local star so that all incident non-loop edges are oriented *outward* from $x$: we replace any incoming edge $y \xrightarrow{S} x$ by the outgoing edge $x \xrightarrow{S^\top} y$. Let the outgoing edges be $x : A \xrightarrow{R_i} y_i : B_i$ and the self-loop be $R_0 \in \overline{\mathsf{Rel}}(A, A)$. For each pair $i \le j$ we insert the composed edge from $y_j$ to $y_i$ with typed relation

$$R_{ij} \;=\; (R_i)^\top\, R_0^*\, R_j \quad \in\; \overline{\mathsf{Rel}}(B_i, B_j),$$

intersecting with any existing relation between $y_i$ and $y_j$. If some same-sort self-loop becomes empty on the diagonal, report unsatisfiable.

**Theorem 6** (Many-sorted characterization)**.** *For finite carriers* $\{V_A\}_{A \in \mathcal{O}}$ *and a typed family* $\mathsf{Rel}$*, the typed variable-elimination algorithm above is sound and complete for all many-sorted constraint graphs over* $\mathsf{Rel}$ *if and only if* $\mathsf{Rel}$ *has the Helly-2 slice property by sort.*

*Proof sketch.* The argument is verbatim the single-sorted proof, carried out for each eliminated sort. □

# 3 Solver Architecture

## 3.1 Further optimizations

- Keep domain constraints (diagonally restricted relations) up to date.
- Remove constraints that are redundant given the domain constraints.
- Remove diagonal constraints by union-find on the variables that have to be equal.

## 3.2 Incremental solving

## 3.3 Generalization with levels

# 4 Open Questions