Borrowing Notes

Jules Jacobs

November 11, 2025

Extend areality with borrowing:

- $0 = \textsc{global}$
- $1 = \textsc{regional}$
- $2 = \textsc{local}$
- $3 = \textsc{borrowed}$

where $0 \leq 1 \leq 2 \leq 3$, i.e. $\textsc{global} \leq \textsc{regional} \leq \textsc{local} \leq \textsc{borrowed}$.

Global modality The global modality maps all points to GLOBAL, except for BORROWED which is mapped to itself.

$$f(x) = \begin{cases} 0 & \text{if } x \in \{0, 1, 2\} \\ 3 & \text{if } x = 3 \end{cases}$$

It has the following left adjoint:

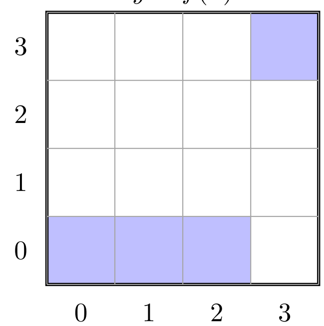
$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 3 & \text{if } x \in \{1, 2, 3\} \end{cases}$$

It has the following right adjoint:

$$h(x) = \begin{cases} 2 & \text{if } x \in \{0, 1, 2\} \\ 3 & \text{if } x = 3 \end{cases}$$

Pictorially:

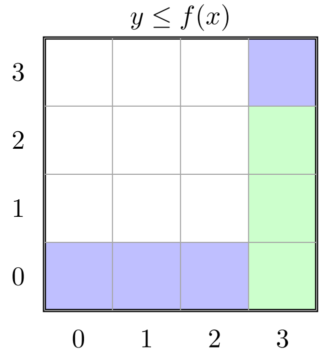
$y = f(x)$



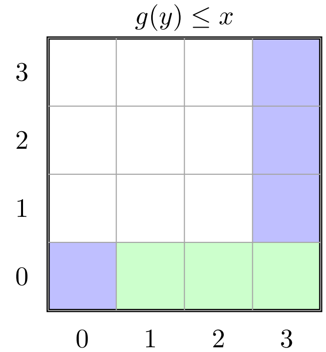
Because the lattice only has four elements, we can witness each adjunction by shading the pairs that satisfy the defining inequalities. Blue cells highlight the points where the inequality is tight, making the graphs of f, g, and h easy to spot.

Left adjoint $g \dashv f$ The condition $g(y) \leq x \iff y \leq f(x)$ means both relations pick out exactly the same lattice points:

$y \leq f(x)$

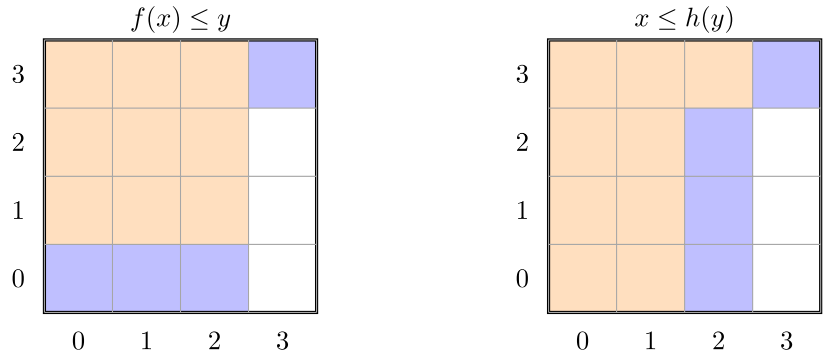


$g(y) \leq x$



Slogan: the points below f (left) are exactly the points to the right of g (right).

Right adjoint $f \dashv h$ Likewise, the right adjoint h is characterised by $f(x) \leq y \iff x \leq h(y)$:



Slogan: the points above f (left) are exactly the points to the left of h (right).

Clamping upward: $f_c(x) = \max(x, c)$ For any constant $c \in \{0, 1, 2, 3\}$ consider the inflationary map $f_c(x) = \max(x, c)$. It always has a left adjoint

$$g_c(y) = \begin{cases} 0 & \text{if } y \le c, \\ y & \text{if } y > c, \end{cases}$$

reflecting that any point below c "backs off" to the global mode while elements above c stay fixed. When $c = 0$ the map is the identity, so both adjoints coincide with f_c. For every $c > 0$ there is *no* right adjoint: if $y < c$ then no element x satisfies $\max(x, c) \le y$, so the defining implication for a right adjoint would fail.

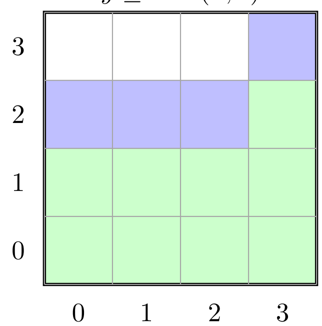
Clamping downward: $f_c(x) = \min(x, c)$ Dually, the deflationary map $f_c(x) = \min(x, c)$ always has a right adjoint

$$h_c(y) = \begin{cases} y & \text{if } y < c, \\ 3 & \text{if } y \ge c, \end{cases}$$

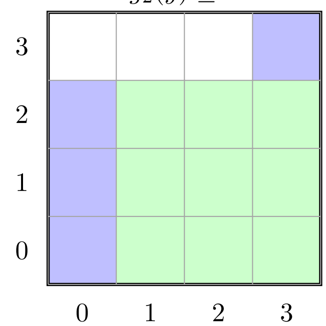
which keeps points strictly below c unchanged and collapses everything above c to the top mode 3. This map has a left adjoint only in the trivial case $c = 3$ (again the identity), because for any $c < 3$ and $y > c$ there is no x with $y \le \min(x, c)$.

Adjoint pictures (example $c = 2$) The adjunction laws become especially tangible on our four-point lattice. Below we spell out the non-trivial clamp $\max(x, 2)$ together with its left adjoint g_2, and $\min(x, 2)$ together with its right adjoint h_2.

$y \leq \max(x, 2)$

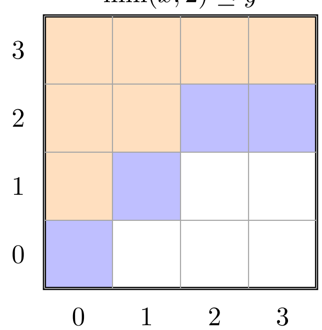


$g_2(y) \leq x$

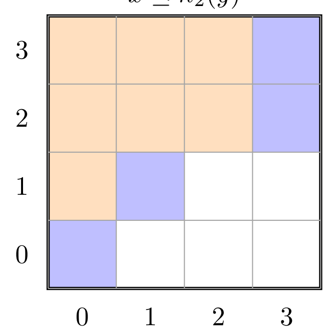


Slogan: the points below $\max(x, 2)$ *(left) are exactly the points to the right of* g_2 *(right).*

$\min(x, 2) \leq y$



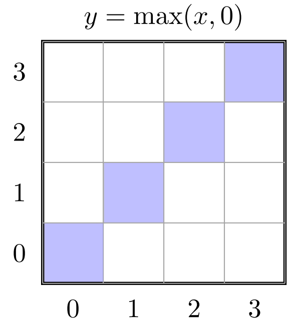
$x \leq h_2(y)$



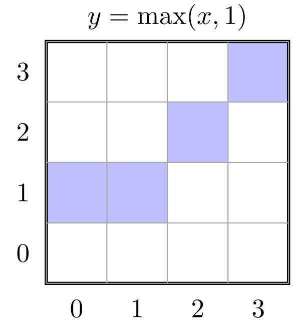
Slogan: the points above $\min(x, 2)$ *(left) are exactly the points to the left of* h_2 *(right).*

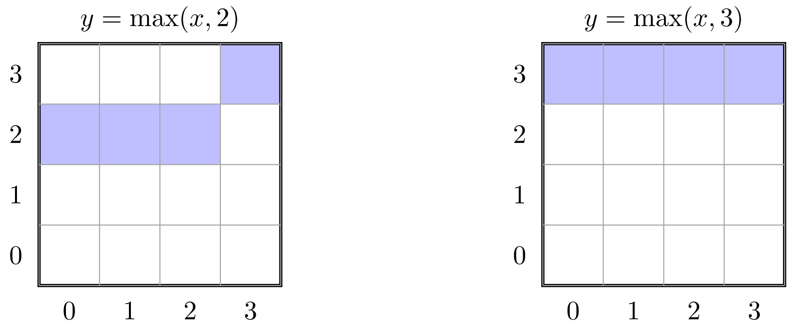
Pictures. The following grids show $y = f_c(x)$ for every clamp level. Blue squares highlight the equality points, just like in the earlier adjoint figures.

$y = \max(x, 0)$

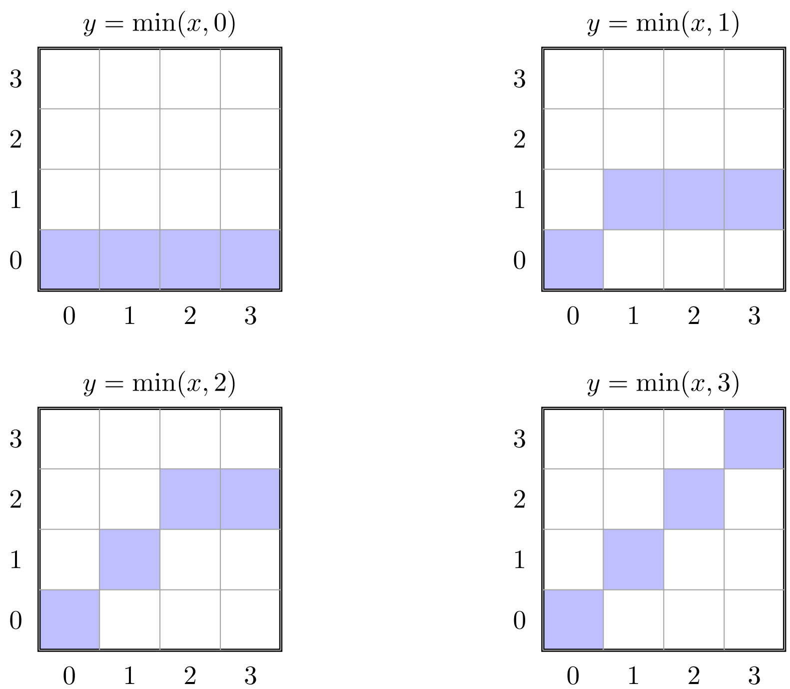


$y = \max(x, 1)$





Left to right: increasing the lower bound pushes more points upward while the left adjoint collapses the shaded columns back to 0 or fixes them.



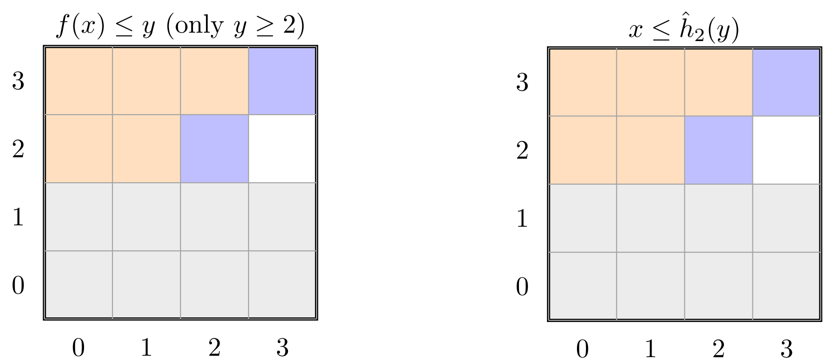
Left to right: lowering the ceiling flattens more of the graph, and the right adjoint h_c matches the columns where the clamp leaves inputs unchanged.

Partial adjoints on restricted fibres Although the clamps fail to admit total adjoints on the whole lattice, their diagonal points still behave adjointly when we restrict to the regions where the inequalities have witnesses.

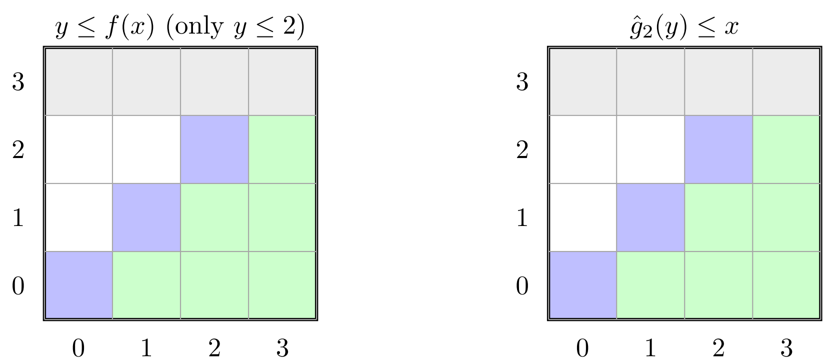
- For $f(x) = \max(x, 2)$ every point with $y \geq 2$ has preimages, so the upset $\{2, 3\}$ supports a partial right adjoint $\hat{h}_2(y) = y$. The equality points $(2, 2)$

and $(3,3)$ on the diagonal survive and certify the adjunction inside that upset.

- For $f(x) = \min(x,2)$ every point with $y \le 2$ injects into some x, so the downset $\{0,1,2\}$ supports a partial left adjoint $\hat{g}_2(y) = y$. Here the diagonal points $(0,0)$, $(1,1)$, and $(2,2)$ witness the restricted adjunction.



Slogan: once we ignore $y < 2$, the remaining points above f agree perfectly with the points below the partial adjoint $\hat{h}_2$.



Slogan: discarding $y > 2$ leaves a perfect match between the points below f and the points to the right of the partial left adjoint $\hat{g}_2$.