Number Theory

Problem 1
How many numbers are there between 100 and 1000 that are

(a) divisible by 3?

(b) divisible by 5?

(c) divisible by 15?

Problem 2

(a) What is:

  (i) $\gcd(420, 720)$?
  (ii) $\mathrm{lcm}(420, 720)$?
  (iii) $720 \text{ div } 42$?
  (iv) $5^{20} \% 7$?

(b) True or false:

  (i) $42|7$?
  (ii) $7|42$?
  (iii) $3 + 5|9 + 23$?
  (iv) $27 \equiv_{(6)} 33$?
  (v) $-1 \equiv_{(7)} 22$?

Problem 3† (2020 T2)
Prove, or give a counterexample to disprove:

(a) For all $x \in \mathbb{R}$:

$$|\lfloor x \rfloor| = \lfloor |x| \rfloor$$

(b) For all $x \in \mathbb{Z}$:

$$42 | x^7 - x$$

(c) For all $x, y, z \in \mathbb{Z}$, with $z > 1$ and $z \nmid y$:

$$(x \text{ div } y) \equiv_{(z)} ((x \% z) \text{ div } (y \% z))$$

† indicates a previous exam question
* indicates a difficult/advanced question.

Problem 4
Prove that for all $m, n, p \in \mathbb{Z}$ with $n \geq 1$:

(a) $0 \leq (m \,\%\, n) < n$

(b) $m \equiv_{(n)} p$ if, and only if $(m \,\%\, n) = (p \,\%\, n)$

Problem 5
Suppose $m \equiv_{(n)} m'$ and $p \equiv_{(n)} p'$. Prove that:

(a) $m + p \equiv_{(n)} m' + p'$

(b) $m \cdot p \equiv_{(n)} m' \cdot p'$

Problem 6

(a) Prove that the 4 digit number $n = abcd$ is:

  (i) divisible by 5 if and only if the last digit d is divisible by 5.

  (ii) divisible by 9 if and only if the digit sum $a + b + c + d$ is divisible by 9.

  (iii) divisible by 11 if and only if $a - b + c - d$ is divisible by 11.

(b) Find a similar rule to determine if a 4 digit number is divisible by 7.

Problem 7*
Prove that for all $n \in \mathbb{Z}$:

$$\gcd(n, n+1) = 1.$$

Problem 8*
Prove that for all $x, y, z \in \mathbb{Z}$:

$$\gcd(\gcd(x, y), z) = \gcd(x, \gcd(y, z)).$$