Q1

Prove, or give a counterexample to disprove, the following hold for all sets S and all binary relations $R_1, R_2 \subseteq S \times S$ on S:

1.

16 If R_1 and R_2 are reflexive then so is $R_1; R_2$.

2.

17 If R_1 and R_2 are symmetric then so is $R_1; R_2$.

3.

If R_1 and R_2 are antisymmetric then so is $R_1; R_2$.

Q2

Define the function $f : \mathbb{N} \to \mathbb{N}$ recursively as follows:

- $f(0) = 0$
- $f(1) = 1$
- $f(n) = 3f(n-1) - f(n-2)$ for $n \geq 2$

Consider the following code that could be used to compute $f(n)$:

```
myFunc(n):
  if n<2:
    return n
  else:
    x = 3*myFunc(n-1)
    y = myFunc(n-2)
    return x-y
```

Let $T(n)$ denote the running time required to execute **myFunc(n)**.

1.

19 Compute the values of $f(n)$ for $n \in \{2, 3, 4, 5\}$

2.

20 Prove that $f(n) \leq 3^n$ for all $n \in \mathbb{N}$

3.

21 With justification, give a **recursive equation** that gives an asymptotic upper bound for $T(n)$

4.

22 Give, with justification, an asymptotic upper bound for $T(n)$

5.

23 Prove, or disprove, that $T(n) \in O(f(n))$

Q3

Draw a graph that:

- has 8 vertices and 12 edges
- is planar
- has chromatic number 3
- has an Eulerian path

Justify that your graph has each of these properties.

1.

› draw your graph

2.

Justify that your graph is planar

3.

Justify that your graph has chromatic number 3

4.

Justify that your graph has chromatic number 3

5.

Justify that your graph has an Eulerian path

Q4

Let p, q, r be propositional variables, and consider the following formulas:

$$\begin{aligned} A &= ((p \to q) \land (q \to r)) \\ B &= (p \to r) \\ C &= ((p \to \bot) \to p) \\ D &= \neg(\top \to r) \end{aligned}$$

1.

28 Prove or disprove

$A \equiv B$

2.

29 Prove or disprove

$B \equiv C \to \neg D$

3.

30 Explain how you could use the transitivity of logical equivalence, and the logical equivalences presented in the lectures to establish if

$A, C \models \neg D$

4.

31 Prove or disprove

$A, C \models \neg D$

(You do not have to use the procedure outlined in 6(c))

5.

32 Using the results of 6(a), 6(b) and/or 6(d), prove or disprove:

$B \models A$

Partial marks are available for any approach that does not use the results of 6(a), 6(b) and 6(d).

Q5

Prove, or give a counterexample to disprove the following for all probability spaces (X, P), and events $A, B \subseteq X$:

1.

33 $P(A|B) \cdot P(B|A) \leq P(A \cap B)$

2.

34 If A and B are independent then $P(A|B) \cdot P(B|A) = P(A \cap B)$

3.

35 If $P(A|B) \cdot P(B|A) = P(A \cap B)$ then A and B are independent.