Relations

Problem 1
True or false:

(a) If a set Σ is totally ordered, then the corresponding lexicographic partial order on Σ^* must also be totally ordered.

(b) If a set Σ is totally ordered, then the corresponding lenlex order on Σ^* must also be totally ordered.

(c) Every finite poset has a Hasse diagram.

(d) Every finite poset has a topological sorting.

(e) Every finite poset has a minimum element.

(f) Every finite totally ordered set has a maximum element.

(g) An infinite poset cannot have a maximum element.

Problem 2
Give an example of a relation which is:

(a) Symmetric, transitive, not reflexive, and not antireflexive

(b) Antisymmetric and antireflexive

(c) Reflexive, Antisymmetric, not transitive.

Problem 3
Let S be a set, and let $I = \{(x, x) : x \in S\}$ be the equality relation on S. Show that a binary relation $R \subseteq S \times S$ is both symmetric and antisymmetric if, and only if $\emptyset \subseteq R \subseteq I$.

Problem 4
Prove, or provide a counterexample to disprove, that for all sets S and all binary relations $R_1, R_2 \subseteq S \times S$:

(a) If R_1 and R_2 are transitive, then $R_1 \cup R_2$ is transitive.

(b)† If R_1 and R_2 are partial orders, then $R_1 \cap R_2$ is a partial order. (18S2)

(c)† If R_1 and R_2 are symmetric, then $R_1 \setminus R_2$ is symmetric. (19T3)

† indicates a previous exam question
* indicates a difficult/advanced question.

Problem 5† (20T2)

Let $(B, \preceq)$ be a partially ordered set, A be a set, and $f : A \to B$ a function from A to B. Define $R \subseteq A \times A$ as follows:

$$(a, b) \in R \quad \text{if and only if} \quad f(a) \preceq f(b)$$

(a) Give a counterexample to show that in general R is not a partial order.

(b) (i) State a restriction on f that will ensure R is a partial order, and

(ii) Prove that under that restriction R is a partial order.

(c) Prove that $R \cap R^{\leftarrow}$ is an equivalence relation.

Problem 6† (17S2)

Let $\Sigma = \{0, 1\}$. We define the *prefix relation*, $\preceq$, on Σ^* as follows: $w \preceq w'$ if, and only if, $w' = wv$ for some $v \in \Sigma^*$.

(a) Show $\preceq$ is a partial order on Σ^*.

(b) Draw the Hasse diagram for the poset $(\Sigma^{\leq 2}, \preceq)$.

(c) (i) What is $\mathsf{lub}(010101, 011011)$ (if it exists)?

(ii) What is $\mathsf{glb}(010101, 011011)$ (if it exists)?

(d) Is a sorting of Σ^* by the

(i) lexicographic order

(ii) lenlex order

a topological sort of $(\Sigma^*, \preceq)$?

Problem 7*

Consider the poset $(\mathbb{N}, |)$. Show that for all $x, y \in \mathbb{N}$:

(a) $\gcd(x, y) = \text{glb}(x, y)$

(b) $\text{lcm}(x, y) = \text{lub}(x, y)$