Motive to find
the general expression for

$$\frac{d^n}{dx^n}\left(\prod_{i=1}^{n} f_i(x)\right) = ?$$

$$\frac{d}{dx}\left(\prod_{i=1}^{2} f_i(x)\right) = f_1(x)\frac{d}{dx}f_2(x) + f_2(x)\frac{d}{dx}f_1(x)$$

i.e $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\rightarrow \frac{d^2}{dx^2}(uv) = u\frac{d^2v}{dx^2} + \frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

$$\frac{d^3}{dx^3}(uv) = u\frac{d^3v}{dx^3} + 3\frac{d^2v}{dx^2}\frac{du}{dx} + 3\frac{dv}{dx}\frac{d^2u}{dx^2} + v\frac{d^3u}{dx^3}$$

$$\vdots$$

$$\frac{d^n(uv)}{dx^n} = {}^nC_0\, u\frac{d^nv}{dx^n} + {}^nC_1\frac{d^{n-1}v}{dx^{n-1}}\frac{du}{dx} + {}^nC_2\frac{d^{n-2}v}{dx^{n-2}}\frac{d^2u}{dx^2} + \cdots + {}^nC_r\frac{d^{n-r}v}{dx^{n-r}}\frac{d^ru}{dx^r} + \cdots + {}^nC_{n-1}\frac{dv}{dx}\frac{d^{n-1}u}{dx^{n-1}} + {}^nC_n\, v\frac{d^nu}{dx^n}$$

(check for the limitations on the value of n)

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$$\frac{d}{dx}\left(\prod_{i=1}^{3} f_i(x)\right)$$

$$= f_1(x)\, f_2(x) \frac{d}{dx} f_3(x) + f_1(x) \frac{d}{dx}(f_2(x)) f_3(x)$$

$$+ \frac{d}{dx}(f_1(x)) \cdot f_2(x) f_3(x)$$

i.e $\frac{d}{dx}(uvw) = \frac{du}{dx} vw + u \frac{dv}{dx} w + uv \frac{dw}{dx}$

$$\frac{d^2(uvw)}{dx^2}$$

$$= \frac{d}{dx}\left\{ \frac{du}{dx} vw + uw \frac{dv}{dx} + uv \frac{dw}{dx} \right\}$$

$$= \frac{d}{dx}\left(\frac{du}{dx} vw\right) + \frac{d}{dx}\left(uw \frac{dv}{dx}\right)$$

$$+ \frac{d}{dx}\left(uv \frac{dw}{dx}\right)$$

$$= \frac{d^2u}{dx^2} vw + \frac{du}{dx}\frac{dv}{dx} w + \frac{du}{dx}\frac{dw}{dx} v$$

$$+ \frac{d^2v}{dx^2} uw + \frac{du}{dx}\frac{dv}{dx} w + \frac{dv}{dx}\frac{dw}{dx} u$$

$$+ \frac{d^2w}{dx^2} uv + \frac{du}{dx}\frac{dw}{dx} v + \frac{dv}{dx}\frac{dw}{dx} u$$

$$\to \quad = \frac{d^2u}{dx^2} vw + \frac{d^2v}{dx^2} uw + \frac{d^2w}{dx^2} uv$$

$$+ 2\frac{du}{dx}\frac{dv}{dx} w + 2\frac{du}{dx}\frac{dw}{dx} v$$

$$+ 2\frac{dv}{dx}\frac{dw}{dx} u$$

$$\frac{d^3(uvw)}{dx^3}$$

$$= \frac{d}{dx}\left(\frac{d^2(uvw)}{dx^2}\right)$$

$$= \frac{d}{dx}\left\{\frac{d^2u}{dx^2}vw + \frac{d^2v}{dx^2}uw + \frac{d^2w}{dx^2}uv\right.$$

$$+ 2\frac{du}{dx}\frac{dv}{dx}w + 2\frac{dv}{dx}\frac{dw}{dx}u$$

$$\left. + 2\frac{du}{dx}\frac{dw}{dx}v\right\}$$

$$= \frac{d^3u}{dx^3}vw + \frac{d^2u}{dx^2}\frac{dv}{dx}w + \frac{d^2u}{dx^2}\frac{dw}{dx}v$$

$$+ \frac{d^3v}{dx^3}uw + \frac{d^2v}{dx^2}\frac{du}{dx}w + \frac{d^2v}{dx^2}\frac{dw}{dx}u$$

$$+ \frac{d^3w}{dx^3}uv + \frac{d^2w}{dx^2}\frac{du}{dx}v + \frac{d^2w}{dx^2}\frac{dv}{dx}u$$

$$+ 2\left\{\frac{d^2u}{dx^2}\frac{dv}{dx}w + \frac{du}{dx}\frac{d^2v}{dx^2}w + \frac{dv}{dx}\frac{dw}{dx}\frac{du}{dx}\right\}$$

$$+ 2\left\{\frac{d^2v}{dx^2}\frac{dw}{dx}u + \frac{dv}{dx}\frac{d^2w}{dx^2}u + \frac{du}{dx}\frac{dv}{dx}\frac{dw}{dx}\right\}$$

$$+ 2\left\{\frac{d^2u}{dx^2}\frac{dw}{dx}v + \frac{du}{dx}\frac{d^2w}{dx^2}v + \frac{du}{dx}\frac{dw}{dx}\frac{dv}{dx}\right\}$$

$$= \frac{d^3u}{dx^3}vw + \frac{d^3v}{dx^3}uw + \frac{d^3w}{dx^3}uv$$

$$+ 3\frac{d^2u}{dx^2}\frac{dw}{dx}w + 3\frac{d^2u}{dx^2}\frac{dw}{dx}v + 3\frac{d^2v}{dx^2}\frac{du}{dx}w$$

$$+ 3\frac{d^2v}{dx^2}\frac{dw}{dx}u + 3\frac{d^2w}{dx^2}\frac{du}{dx}v + 3\frac{d^2w}{dx^2}\frac{dv}{dx}u$$

$+ \textcircled{6}\ \frac{du}{dx} \cdot \frac{dv}{dx} \cdot \frac{dw}{dx}$

Now,

$\frac{d^2 (uvw)}{dx^2}$ is based on the

- expansion of $(a+b+c)^2$
$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

$\frac{d^3 (uvw)}{dx^3}$ is based on the expansion

of $(a+b+c)^3$
$= a^3 + b^3 + c^3 + 3a^2 b + 3ab^2 + 3b^2 c$
$+ 3bc^2 + 3a^2 c + 3ac^2 + 6abc$

Similarly,

~~a+b+c~~ $\frac{d^4 (uvw)}{dx^4}$. will be

based on the expansion of
$(a+b+c)^4$.

And, $\frac{d^n (uvw)}{dx^n}$ is ~~a~~based on the

expansion of $(a+b+c)^n$.

$$\frac{d}{dx}\left(\prod_{i=1}^{4} f_i(x)\right) = \frac{d}{dx}\{f_1(x) f_2(x) f_3(x) f_4(x)\}$$

$$= f_1'(x) f_2(x) f_3(x) f_4(x) + f_1(x) f_2'(x) f_3(x) f_4(x)$$

$$+ f_1(x) f_2(x) f_3'(x) f_4(x) + f_1(x) f_2(x) f_3(x) f_4'(x)$$

i.e $\frac{d}{dx}(abcd)$

$$= \frac{da}{dx} bcd + a \cdot \frac{db}{dx} cd + ab \frac{dc}{dx} d + abc \frac{dd}{dx}$$

$$\frac{d^2}{dx^2}(uvwx)$$

$$= \frac{d}{dx}\left\{ \frac{d}{dx}(uvwx) \right\}$$

$$= \frac{d}{dx}\left\{ \frac{du}{dx} vwx + u \frac{dv}{dx} wx + uv \frac{dw}{dx} x + uvw \frac{dx}{dx} \right\}$$

$$= \frac{d}{dx}\left\{ \frac{du}{dx} vwx \right\} + \frac{d}{dx}\left\{ u \frac{dv}{dx} wx \right\} + \frac{d}{dx}\left\{ uv \frac{dw}{dx} x \right\} + \frac{d}{dx}\left\{ uvw \frac{dx}{dx} \right\}$$

$$= \frac{d^2u}{dx^2} vwx + \frac{du}{dx}\frac{dv}{dx} wx + \frac{du}{dx}\frac{dw}{dx} vx$$

$$+ \frac{du}{dx}\frac{dx}{dx} wv + \frac{d^2v}{dx^2} uwx$$

$$+ \frac{du}{dx}\frac{dv}{dx} wx + \frac{dv}{dx}\frac{dw}{dx} ux + \frac{dv}{dx}\frac{dx}{dx} uv$$

$$+ \frac{d^2w}{dx^2} uvX + \frac{du}{dx}\frac{dw}{dx} vX$$

$$+ \frac{dv}{dx}\frac{dw}{dx} uX + \frac{dw}{dx}\frac{dX}{dx} uv$$

$$+ \frac{d^2X}{dx^2} uvw + \frac{dX}{dx}\frac{du}{dx} vw + \frac{dX}{dx}\frac{dv}{dx} uw$$

$$+ \frac{dX}{dx}\frac{dw}{dx} uv$$

$$= \frac{d^2u}{dx^2} vwX + \frac{d^2v}{dx^2} uwX + \frac{d^2w}{dx^2} uvX$$

$$+ \frac{d^2X}{dx^2} uvw + 2\frac{du}{dx}\frac{dv}{dx} wX$$

$$+ 2\frac{du}{dx}\frac{dw}{dx} vX + 2\frac{du}{dx}\frac{dX}{dx} vw$$

$$+ 2\frac{dv}{dx}\frac{dw}{dx} uX + 2\frac{dv}{dx}\frac{dX}{dx} uw$$

$$+ 2\frac{dw}{dx}\frac{dX}{dx} uv.$$

The above is based on the expansion of $(a+b+c+d)^2$

$$= a^2+b^2+c^2+d^2+2ab+2bc$$
$$+ 2cd + 2ad + 2bd + 2ac$$

Similarly, $(a+b+c+d)^3$ ←

$\frac{d^3abcd}{dx^3}$ is based on the exp. of

Hence,

~~(a+b+c+d~~

$\frac{d^n(abcd)}{dx^n}$ is based on the expansion of $(a+b+c+d)^n$.

→ Conclusion →

$\frac{d^n}{dx^n}\left(\prod_{i=1}^{2} f_i(x)\right)$ is based on binomial expansion of $(a+b)^n$ $\forall\ n \geq 1$

$\frac{d^n}{dx^n}\left(\prod_{i=1}^{3} f_i(x)\right)$ is based on the ~~binomial~~ expansion of $(a+b+c)^n$

$\frac{d^n}{dx^n}\left(\prod_{i=1}^{4} f_i(x)\right)$ is based on the expansion of $(a+b+c+d)^n$

⋮

* full generalisation

Similarly, $\frac{d^n}{dx^n}\left(\prod_{i=1}^{m} f_i(x)\right)$ is based on the expansion of $(a+b+c+d+\dots)^n$

Hence, we need to incorporate Multinomial Theorem.

(Going for Horizontal evaluation)

→ Also,

$\frac{d^2}{dx^2}\left(\prod_{i=1}^{2} f_i(x)\right)$ is based on the expansion of $(a+b)^2$

$\frac{d^2}{dx^2}\left(\prod_{i=1}^{3} f_i(x)\right)$ is based on the expansion of $(a+b+c)^2$

$\frac{d^2}{dx^2}\left(\prod_{i=1}^{4} f_i(x)\right)$ is based on the expansion of $(a+b+c+d)^2$

⋮

$\frac{d^2}{dx^2}\left(\prod_{i=1}^{n} f_i(x)\right)$ is based on the expansion of $[a+b+c+d+---]^2$

→ Similarly,

$\frac{d^3}{dx^3}\left(\prod_{i=1}^{2} f_i(x)\right)$ is based on the expansion of $(a+b)^3$

$\frac{d^3}{dx^3}\left(\prod_{i=1}^{3} f_i(x)\right)$ is based on the expansion of $(a+b+c)^3$.

$\frac{d^3}{dx^3}\left(\prod_{i=1}^{4} f_i(x)\right)$ is based on the expansion of $(a+b+c+d)^3$.

$\vdots$

$\frac{d^3}{dx^3}\left(\prod_{i=1}^{n} f_i(x)\right)$ is based on the expansion of $(a+b+c+d+\cdots)^3$

Also,

$\frac{d^4}{dx^4}\left(\prod_{i=1}^{2} f_i(x)\right)$ is based on $(a+b)^4$.

$\frac{d^4}{dx^4}\left(\prod_{i=1}^{3} f_i(x)\right)$ is based on $(a+b+c)^4$

$\frac{d^4}{dx^4}\left(\prod_{i=1}^{4} f_i(x)\right)$ is based on $(a+b+c+d)^4$

$\vdots$

$\frac{d^4}{dx^4}\left(\prod_{i=1}^{n} f_i(x)\right)$ is based on $(a+b+c+d+\cdots)^4$.

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and so on