→ Higher Derivatives of a f^n.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right)$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$$

The above result is true for a f^n which is differentiable n times.

→ If $f(x)$ and $g(x)$ are functions differentiable n times, then, for their linear combination

$$c_1 f(x) + c_2 (g(x)) \qquad (c_1, c_2 \in \text{const})$$

We have,

~~$c_1 d^n f(x)$~~

$$\frac{d^n}{dx^n}(c_1 f(x) + c_2 (g(x)))$$

$$= c_1 \frac{d^n f(x)}{dx^n} + c_2 \frac{d^n g(x)}{dx^n}$$

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$$\frac{d^n(uv)}{dx^n}$$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d^2(uv)}{dx^2} = \frac{d}{dx}\left(\frac{d(uv)}{dx}\right)$$

$$= \frac{d}{dx}\left\{v\frac{du}{dx} + u\frac{dv}{dx}\right\}$$

$$\Rightarrow \frac{d^2(uv)}{dx^2} = \frac{d}{dx}\left(v\frac{du}{dx}\right) + \frac{d}{dx}\left(u\frac{dv}{dx}\right)$$

$$= \frac{dv}{dx}\cdot\frac{du}{dx} + v\frac{d^2u}{dx^2} + \frac{du}{dx}\cdot\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Also,

$$\frac{d^3(uv)}{dx^3} = \frac{d}{dx}\left(\frac{d^2(uv)}{dx^2}\right)$$

$$= \underline{\frac{d^2u}{dx^2}\frac{dv}{dx}} + \frac{d^2v}{dx^2}\frac{du}{dx}$$

$$+ \underline{\frac{dv}{dx}\frac{d^2u}{dx^2}} + \frac{d^3u}{dx^3}v$$

$$+ \underline{\frac{d^2u}{dx^2}\frac{dv}{dx}} + \frac{d^2v}{dx^2}\frac{du}{dx}$$

$$+ \frac{du}{dx}\frac{d^2v}{dx^2} + u\cdot\frac{d^3v}{dx^3}$$

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Now,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d^2(uv)}{dx^2} = u\frac{d^2v}{dx^2} + v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx}$$

uv

$$\frac{d^3y}{dx^3} = u\frac{d^3v}{dx^3} + v\frac{d^3u}{dx^3} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{d^2v}{dx^2}\frac{du}{dx}$$

$$\frac{d^ny}{dx^n} = \frac{d^n(uv)}{dx^n}$$

$$= u\frac{d^nv}{dx} + v\frac{d^nu}{dx} + \cdots$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$$\frac{d^2(uv)}{dx^2} = 1.u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + 1.v\frac{d^2u}{dx^2}$$

$$\frac{d^3(uv)}{dx^3} = 1.u\frac{d^3v}{dx^3} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{d^2v}{dx^2}\frac{du}{dx} + 1.v\frac{d^3u}{dx^3}$$

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$$\frac{d^4(uv)}{dx^4}$$

$$= ① u \frac{d^4 v}{dx^4} + ④ \frac{d^3 u}{dx^3} \frac{dv}{dx} + ⑥ \frac{d^2 u}{dx^2} \frac{d^2 v}{dx^2}$$

$$+ ④ \frac{d^3 v}{dx^3} \frac{du}{dx} + ① v \frac{d^4 u}{dx^4}$$

—— (IV)

$$(x+y)^n = x^n + x^{n-1} y.$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

⋮

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y^r$$

$$= {}^nC_0 \, x^n y^0 + {}^nC_1 \, x^{n-1} y^1 + {}^nC_2 \, x^{n-2} y^2 + \dots + {}^nC_{n-1} \, x y^{n-1} + {}^nC_n \, x^0 y^n$$

for $n = 4$,

$$(x+y)^4 = \sum_{r=0}^{4} {}^4C_r \, x^{4-r} y^r$$

$$= {}^4C_0 \, x^4 y^0 + {}^4C_1 \, x^3 y^1 + {}^4C_2 \, x^2 y^2 + {}^4C_3 \, x y^3 + {}^4C_4 \, x^0 y^4$$

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Arranging eq^n - (4) A/c to the above binomial expansion, We have,

$$\cancel{\frac{d^n(uv)}{dx}} \quad \frac{d^4(uv)}{dx^4}$$

$$= {}^4C_0 \frac{d^4u}{dx^4} \cdot v + {}^4C_1 \frac{d^3u}{dx^3} \frac{d^1v}{dx^1}$$

$$+ {}^4C_2 \frac{d^2u}{dx^2} \frac{d^2v}{dx^2} + {}^4C_3 \frac{d^1u}{dx^1} \frac{d^3v}{dx^3}$$

$$+ {}^4C_4 \, u \cdot \frac{d^4v}{dx^4}$$

$${}^4C_0 = 1$$
$${}^4C_1 = 4$$
$${}^4C_2 = 6$$
$${}^4C_3 = 4$$
$${}^4C_4 = 1$$

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & {}^1C_0 & & {}^1C_1 & & & \\
 & & & 1 & & 1 & & & \\
 & & {}^2C_0 & & {}^2C_1 & & {}^2C_2 & & \\
 & & 1 & & 2 & & 1 & & \\
 & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 & \\
 & 1 & & 3 & & 3 & & 1 & \\
{}^4C_0 & & {}^4C_1 & & {}^4C_2 & & {}^4C_3 & & {}^4C_4 \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

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Hence, Leibnitz Theorem.

$$\boxed{\begin{aligned}\frac{d^n(uv)}{dx^n} &= {}^nC_0 \frac{d^n u}{dx^n} v + {}^nC_1 \frac{d^{n-1}u}{dx^{n-1}} \cdot \frac{dv}{dx} \\ &+ {}^nC_2 \frac{d^{n-2}u}{dx^{n-2}} \frac{d^2v}{dx^2} + \cdots\cdots + \cdots \\ &+ {}^nC_{n-1} \frac{du}{dx} \cdot \frac{d^{n-1}v}{dx^{n-1}} + {}^nC_n\, u \cdot \frac{d^n v}{dx^n}\end{aligned}}$$

Or

$$\begin{aligned}D^n(uv) = {}&{}^nC_0 (D^n u)\cdot(v) + {}^nC_1 (D^{n-1}u)(Dv) \\ &+ {}^nC_2 (D^{n-2}u)(D^2 v) + \cdots\cdots + \cdots \\ &+ {}^nC_r (D^{n-r}u)(D^r v) + \cdots\cdots \\ &+ {}^nC_{n-1}(Du)(D^{n-1}v) + {}^nC_n\, u(D^n v)\end{aligned}$$

Or In short,

$$\frac{d^n(uv)}{dx^n} = \sum_{r=0}^{n} {}^nC_r \frac{d^{n-r}u}{dx^{n-r}} \frac{d^r v}{dx^r}$$

Or

$$\boxed{D^n(uv) = \sum_{r=0}^{n} {}^nC_r (D^{n-r}u)(D^r v)}$$

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