Math 5620 Lecture Notes: Day 1

→ Cover the syllabus and how to access the syllabus

→ All materials will be available via Github.

→ There is a list of projects and such problems. Most of these files are implementations of algorithms in Matlab. We will use Python and C to rewrite the code -

→ Organization of the book.

Part I: We will first study the approximate solution of steady state boundary value problems. We will need to develop iterative linear solvers for systems of linear equations that come from our work.

Part II: In the second part we will consider initial value problems for ODEs and then onto initial value problems in multiple dimensions that are partial differential equations.

Chapter 1: Finite Differences Approximations:

Def. An equation the involves the derivatives of some functions of one or more independent and dependent variables.

Some examples:

1. Exponential Models.

$$\begin{cases} \frac{dA}{dt} = kA \\ A(0) = A_0 \end{cases}$$ IVP/ODE

2. Logistic Model of Population Change.

$$\begin{cases} \frac{dP}{dt} = \alpha A - \beta A^2 \\ P(0) = P_0 \end{cases}$$ IVP / ODE /

3. Spring Mass Systems:

$$\begin{cases} m\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + ky = f(t) \\ y(0) = y_0 \\ \frac{dy}{dt}(0) = V_0 \end{cases}$$ IVP / ODE / 2nd Order

4. Stability of Structures

$$\begin{cases} y''(x) + \frac{P}{EI} y(x) = 0 \\ y(0) = 0 \\ y(L) = 0 \end{cases}$$ BVP / ODE / 2nd Order

5. Heat Equation

$$\begin{cases} \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \\ u(x,0) = f(x) \\ \frac{\partial u}{\partial x}(x,0) = \frac{\partial u}{\partial x} \end{cases}$$ IVP / PDE / 2nd Order

6. Potential Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \text{BVP / PDE / 2}^{\text{nd}}\text{ Order}$$

$u = g(x,y)$ on the boundary of some region

7. Conservation Laws

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad \text{IVP / PDE / 1}^{\text{st}}\text{ order}$$

$$u(x,0) = f(x) \qquad x \in \mathbb{R}$$

8. Cahn-Hilliard equation

$$\frac{\partial \phi}{\partial t} = \cdot \nabla \cdot M \nabla \mu \qquad \nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]$$

$$\mu = -F(\phi) + \nabla \cdot \varepsilon^2 \qquad \nabla \cdot \phi = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}$$

$$\nabla \phi = \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{bmatrix}$$

There are quite literally hundreds if not more ODEs and PDEs that are used in the modeling of physical systems.

Some Issues:

1. First, most simulations being used by scientists involve approximations even when we have a solution we must use approximations.

Ex: Suppose we solve

$$\begin{cases} \dfrac{dA}{dt} = kA \\ A(0) = A_0 \end{cases}$$

$$\Rightarrow \frac{1}{A}\, dA = k\, dt$$

$$\Rightarrow \ln|A| = kt + c_0$$

$$\Rightarrow A(t) = e^{kt + c_0} = e^{kt} \cdot e^{c_0}$$

$$= e^{kt} \cdot c_1 \longrightarrow c_1 = e^{c_0}$$

Then if, $A(0) = A_0$ we obtain a unique solution of the form,

$$A(0) = c_1 \cdot e^0 = c_1 = A_0,$$

So, we write

$$A(t) = A_0 e^{kt}$$

This is a nice form. However, if

$$k = 1,\ A_0 = 1$$

we have a form

$$A(t) = e^t$$

there is a problem: e is an irrational number. This requires an infinite number of digits to get an exact representation.

$\Rightarrow$ Round off errors must be accounted for.

Ex: Euler Buckling Load.

$$\frac{d^2y}{dx^2} + \frac{P}{EI}\, y = 0 \qquad \longrightarrow \ y = \text{distance from eq.}$$

$$\left.\begin{array}{l} y'(0) = 0 \\ y'(L) = 0 \end{array}\right\} \text{ Pinned BCs}$$

Rewrite: $\lambda^2 = \frac{P}{EI}$

$$\Rightarrow \frac{d^2y}{dx^2} + \lambda^2 y = 0$$

$$\Rightarrow y(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$$

$$\left\{\begin{array}{l} y(0) = 0 \Rightarrow C_1 + C_2 \cdot 0 = C_1 = 0 \ \checkmark \\ y(L) = 0 = \cancel{C_1 \cos(\lambda L)} + C_2 \sin(\lambda L) \end{array}\right.$$

$$= C_2 \sin(\lambda L) = 0$$

So $C_2 = 0$ __or__ $\sin(\lambda L) = 0 \Rightarrow \lambda L = n\pi \qquad n = 0, 1, 2, \ldots$

$$\Rightarrow y(x) = C_2 \sin\left(\frac{n\pi}{L} \cdot x\right)$$

There are ∞-many solutions.

With all this "cheery" news, let's start by stating some definitions.

__Def:__ Suppose $f(x)$ is some function of an independent variable x. Then we say the derivative of f exists at a pt

$$f'(a) \doteq \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Note that if f has a derivative at all points $a \in (x_0, x_1)$ we say that f is differentiable on (x_0, x_1).

So, the derivative requires the existance of a limit. Since we cannot guarantee infinite accuracy in numbers like e or π. We will not be too upset if we approximate derivatives in an ODE or PDE

So if

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ (Mathematically Speaking)

Then

$$f'(a) \simeq \frac{f(x) - f(a)}{x - a}$$

How good is this approximation?

Before we address the accuracy problem, let's define a couple of terms

Def. The rational expression

$$= \frac{f(x) - f(a)}{x - a} = D_+(a)$$

is called a difference quotient.

Def We can also write

$$D_+(\bar{x}) = \frac{u(\bar{x} + h) - u(\bar{x})}{h}$$

where $x - a = h \Rightarrow x = a + h$ and with $a = \bar{x}$.

This is called the incremental form of the difference quotient – h is called an increment.

Ex: $\frac{dA}{dt} = kA \Rightarrow \frac{A(t+\Delta t) - A(t)}{\Delta t} \approx k A(t)$

$$\Rightarrow A(t+\Delta t) \approx A(t) + k \cdot \Delta t \, A(t)$$

If we have $A_0 = A(0)$

increment $h = \Delta t$

$$\Rightarrow A(0+\Delta t) \approx A(0) + k\, A(0) \cdot \Delta t$$

$$\Rightarrow A(\Delta t) \approx A_0 + k A_0 \Delta t$$

$$= A_0 (1 + k \Delta t)$$

Our very first F.D. method!