Math 5620 Lecture Notes : Day 11

We want a finite difference method to be

1. Consistent : $|\tau_j| \le C h^p$, $p>0$ for $j=1,2,\dots,m$

and

2. stability: $\|(A^h)^{-1}\| \le C$.

Let's look at 2-norm stability: $\|A^{-1}\|_2 \le C$.

- Note: $\|A\|_2 = \sqrt{\rho(A^T A)}$ $\Rightarrow$ we want the eigenvalues and eigenvectors for the differencing matrix.
- Note: $A \in \mathbb{R}^{m \times m}$ is symmetric. This implies $\|A\|_2 = \rho(A) = \max_{1\le p\le m} |\lambda_p|$

$\Rightarrow \|A^{-1}\| = \rho(A^{-1}) = \max_{1\le p\le m} |(\lambda_p)^{-1}| = \left(\min_{1\le p\le m} |\lambda_p|\right)^{-1}$

So, all we need to do is compute and show that the eigenvalues are bounded away from zero as $h \to 0$.

- Note the structure is so simple

To the problem. It can be shown that

$$\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right), \quad p = 1,2,\dots,m$$

The eigenvectors corresponding to λ_p has components u_j^p for $j=1,2,\dots,m$

$$u_j^p = \sin(p\pi j h)$$

We can verify this by checking

$$A u^p = \lambda_p u^p$$

(2)

The j^{th} component of Au^p is

$$(Au^p)_j = \frac{1}{h^2}\left(u^p_{j-1} - 2u^p_j + u^p_{j+1}\right)$$

$$= \frac{1}{h^2}\left(\sin(p\pi(j-1)h) - 2\sin(p\pi jh) + \sin(p\pi(j+1)h)\right)$$

$$= \frac{1}{h^2}\left(\sin(p\pi jh)\cdot\cos(p\pi h) - 2\sin(p\pi hj) + \sin(p\pi hj)\cos(p\pi h)\right)$$

$$= \frac{1}{h^2}\sin(p\pi jh)\cdot(2\cos(p\pi h) - 2)$$

$$= \underbrace{\frac{2}{h^2}(\cos(p\pi jh) - 1)}_{\lambda_p} \cdot \underbrace{\sin(p\pi jh)}_{u^p_j}$$

This works at the boundary if we assume $u^p_0 = u^p_{m+1} = 0$. The smallest eigenvalue in magnitude is

$$\lambda_1 = \frac{2}{h^2}(\cos(\pi h) - 1)$$

$$= \frac{2}{h^2}\left(\left(1 - \frac{(\pi h)^2}{2!} + \frac{(\pi h)^4}{4!} - \cdots\right) - 1\right)$$

$$= \frac{2}{h^2}\left(-\frac{\pi^2 h^2}{2} + \frac{\pi h^4}{4!} + \cdots\right)$$

$$= -\pi^2 + \frac{2}{4!}\pi^4 h^2 - \cdots$$

$$= -\pi^2 + O(h^2) \to 0$$

$$\approx -\pi^2$$

This is bounded away from zero.

So, back to the error,

$$\|E^h\|_2 \leq \|(A^h)^{-1}\|_2 \cdot \|\tau^h\|_2 \approx \frac{1}{\pi^2} \cdot \|\tau^h\|$$

and since

$$\tau_j \approx \frac{1}{12} h^2 u''''(x_j)$$

then

$$\|\tau^h\|_2 \approx \frac{1}{12} h^2 \|u''''\|_2 = \frac{1}{12} h^2 \underbrace{\|f''\|_2}_{RMS} \leq C \|f''\|_\infty$$

the "eigenfunctions" for this ODE are

$$u^p(x) = \sin(p\pi x) \qquad p = 1, 2, \ldots$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} u^p(x) = \mu\, u^p(x) \qquad \Rightarrow \mu_p = -p^2\pi^2$$

$$\vdots$$

The basic "How To"

Eigenvalues of a tridiagonal Toeplitz matrix

$$Av = \lambda v$$

for

$$A = \begin{bmatrix} a & b & & 0 \\ b & \ddots & \ddots & \\ & \ddots & \ddots & b \\ 0 & & b & a \end{bmatrix} \in \mathbb{R}^{m \times m}, \quad \lambda \in \mathbb{R}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \in \mathbb{R}^m$$

Starting from $Av = \lambda v$, we can write out $(Av - \lambda v) = (A - \lambda I)v$

$$
\begin{aligned}
(a-\lambda)v_1 + bv_2 &= 0 \\
cv_1 + (a-\lambda)v_2 + bv_3 &= 0 \\
+ cv_2 + (a-\lambda)v_3 + bv_4 &= 0 \\
&\ddots \\
cv_{m-1} + (a-\lambda)v_m &= 0
\end{aligned}
$$

If we define $v_0 = 0$ and $v_{m+1} = 0$. This gives

$$
\begin{cases}
cv_{j-1} + (a-\lambda)v_j + bv_{j+1} = 0 \\
v_0 = 0 \\
v_m = 0
\end{cases}
$$

Let's step back a bit - This is a two term recursion for the components of v

Aside: Suppose we have

$$\alpha u'' + \beta u' + \gamma u = 0$$

Assume $u = e^{rx}$

$$\Rightarrow \alpha r^2 e^{rx} + \beta r e^{rx} + \gamma e^{rx}$$

$$= \underbrace{e^{rx}}_{\neq 0}(\alpha r^2 + \beta r + \gamma) = 0$$

$$\Rightarrow \alpha r^2 + \beta r + \gamma = 0$$

$$\Rightarrow r = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \quad \text{Done}$$

Now consider:

$$\alpha v_{j+1} + \beta v_j + \gamma v_{j-1} = 0$$

$$\Rightarrow \quad v_j = z^j$$

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$$\Rightarrow \quad \alpha z^{j+1} + \beta z^j + \gamma z^{j-1} = 0$$

$$\Rightarrow \quad z^{j-1}\left(\alpha z^2 + \beta z + \gamma\right) = 0$$

$z \neq 0$

$$\Rightarrow \quad \alpha z^2 + \beta z + \gamma = 0$$

Same!

Try this for the problem at hand