Math 5620 Lecture Notes : Day ~~14~~ 15

Last time we started into PDE approximations. We need to do a bit more. But first we will cover how to do a computational convergence analysis. Suppose we have decided to use a central difference approximation for the BVP

$$\begin{cases} u'' = f \\ u(a) = \alpha \\ u(b) = \beta \end{cases}$$

So, for $j = 1, 2, \ldots, m$

$$\begin{cases} \frac{1}{h^2}\left(U_{j-1} - 2U_j + U_{j+1}\right) = f(x_j) \\ U_0 = \alpha \\ U_{m+1} = \beta \end{cases}$$

Now, we know this process results in an approximation of order h^2. This is determined by Taylor series and a bit of matrix theory. This does not mean that a computer program will show this.

- Roundoff error
- measurement error
- other errors
- as $h \to 0$.
- the real problem is more complicated.

So, we may need a computational verification of the convergence we expect.

Here is one way to proceed:

- Choose a decreasing sequence of values for h, say $\{h_0, h_0/2, h_0/4, h_0/8, \ldots\}$

- For each $h < h_0$, if an exact solution is known,

  - compute the approximate solution and store the results in a vector

$$\bar{U} = \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \vdots \\ \bar{U}_m \end{bmatrix}$$

  - evaluate the solution at each point in the mesh

$$\hat{U} = \begin{bmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_m) \end{bmatrix}$$

  - compute the norm of the error.

$$\|E\| = \|\hat{u} - \bar{U}\|$$

which norm

- Having computed the error for each h, fit the collected data to a linear.

Note: $\|E\| \leq C h^p$

$$\Rightarrow \underbrace{\log(\|E\|) \leq \log(C) + p \log h}_{\text{input}}$$

$\log C$ does not matter too much

To fit the data to the linear tabulating the following

h	E_h	$\log h$	$\log E$
h_0	E_{h_0}	$\log h_0$	$\log E_h$
h_1			
h_2			

← fit these to $\log h$

$y = a + C p$

So, if $u(x)$ is available, then will work.

If $u(x)$ is not available, use the finest resolution

$$\Rightarrow |u(x) - u_{h_k}|$$

↑ computed value

For

$$\begin{cases} u'' = f \\ u(0) = \alpha \\ u(1) = \beta \end{cases} \Rightarrow (p = 2)$$

Data set should be

h, $\|E_h\|$, $\log h$, $\log\|E_h\|$ $\Rightarrow$ p!

Back to the PDE deal

For a PDE of the form

$$a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u = f$$

if

$a_2^2 - 4a_1a_3 < 0 \Rightarrow$ elliptic   poisson

$a_2^2 - 4a_1a_3 = 0 \Rightarrow$ parabolic   heat

$a_2^2 - 4a_1a_3 > 0 \Rightarrow$ hyperbolic   wave

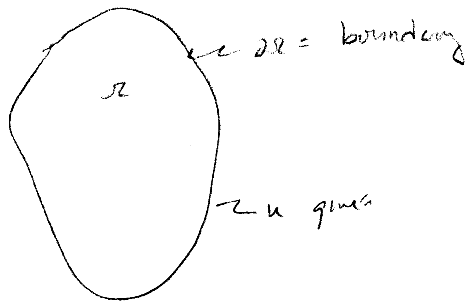
For $ax^2 + bx + c = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, let's begin with the elliptic problem.

Ex: $(k u_x)_x + (k u_y)_y = f$

$+$ BCs



If $f=0$, the equation is called Laplace's equation. For f some arbitrary the equation is referred to as the Poisson problem

Operator Notation: If k = const

$k u_{xx} + k u_{yy} = g \;\Rightarrow\; \boxed{u_{xx} + u_{yy} = f}$

and

$$u_{xx} + u_{yy} = \Delta u = \nabla^2 u = \nabla \cdot \nabla u = f$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \Rightarrow \nabla \cdot \nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Note: Δx = increment in x, not Laplacian

The 5-point stencil

For $0 < x, y < 1$

Since

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x-h, y)}{2h} \qquad \frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+h) - u(x, y-h)}{2h}$$

$$\approx \frac{u(x+\Delta x, y) - u(x-h, y)}{2h} \qquad \approx \frac{u(x, y+h) - u(x, y-h)}{2h}$$

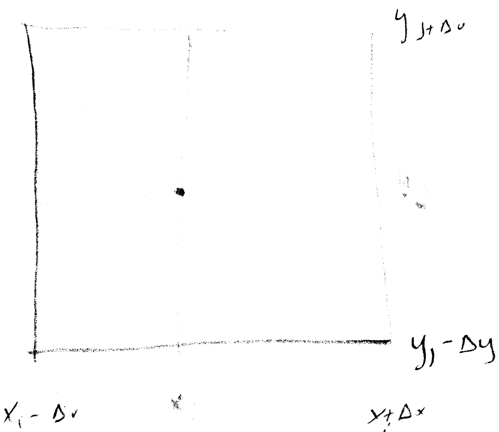
$\vdots$

So, we

$$\Delta u \approx \frac{1}{h^2}\left(u(x+h, y) - 2u(x, y) + u(x-h, y)\right)$$

$$+ \frac{1}{h^2}\left(u(x, y+h) - 2u(x, y) + u(x, y-h)\right) = f(x, y)$$

Mesh in 2D for F.D.



So, how to write in an appropriate matrix form

- Lexicographical in x, y order
- " y, x order
- Special cases