Math 5620 Lecture Notes: Day 11

Let's do a bit more on the ODE deal:

1. Coefficient generation:

In a case when we have $\bar{x} = x_J$ for some point the values in column J will be 0 except for the first entry.

$$A_{\cdot J} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

How does this affect the solution process. If we choose Jacobi Iteration there will be a problem

$$c^{(k+1)} = c^{(k)} + D^{-1} r^{(k)}$$

↑ residual

↑ inverse of $\text{diag}(a_{11}, a_{22}, \dots, a_{nn})$

Since $a_{JJ} = 0$.

$\Rightarrow$ Jacobi won't work.

B.T.W. This does not imply A^{-1} does not exist.

Ex: $n = 3 \rightarrow f'(\bar{x}) \approx f'(0)$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$a_{22} = 0 \Rightarrow$ problem for Jacobi

Reduction

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

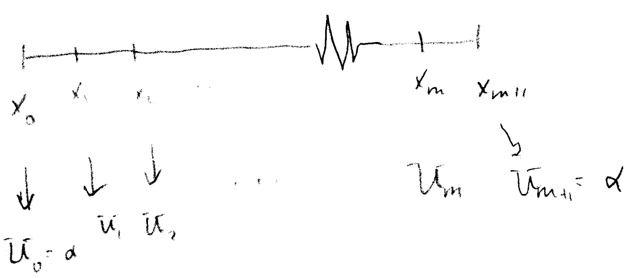
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] \begin{array}{l} \to c_1 = -\frac{1}{2} \\ \to C_2 + 2C_1 = C_2 + 1 = 1 \Rightarrow C_2 = 0 \\ \to 2c_3 = 1 \Rightarrow C_3 = \frac{1}{2} \end{array}$$

$$\Rightarrow \vec{c} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

So, ...

2. In some cases we can reorder the nodes - the discrete points. This can help in some problems.



$$A \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = F$$

Reorder : Odds and Evens

$\Rightarrow m =$ even integer

$j = 1$;

$$\frac{1}{h^2}(u_0 - 2u_1 + u_2) = f(x_1)$$

$j = 3$:

$$\frac{1}{h^2}(u_2 - 2u_3 + u_4) = f(x_3)$$

$j = m-1$

$$\frac{1}{h^2}(u_{m-2} - 2u_{m-1} + u_m) = f(x_{m-1})$$

$j = 2$

$$\frac{1}{h^2}(u_1 - 2u_2 + u_3) = f(x_2)$$

$j = 4$

$$\frac{1}{h^2}(u_3 - 2u_4 + u_5) = f(x_4)$$

$$\vdots$$

$j = m$

$$\frac{1}{h^2}(u_{m-1} - 2u_m + u_{m+1}) = f(x_m)$$

Now to the matrix performance

$$\frac{1}{h^2}\begin{bmatrix} -2 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & -2 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 0 \\ & & & & 0 & 1 & 1 & & & 0 \\ & & & -2 & & 0 & & & & 1 \\ 1 & 1 & 0 & & -2 & & 0 & & & \\ 0 & 1 & & & & & & & & \\ & & & 1 & & 0 & & & & -2 \\ & 0 & & 1 & & & & & & \end{bmatrix}\begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{m-1} \\ u_2 \\ u_4 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} f(x_1) - \alpha/h^2 \\ f(x_3) \\ \vdots \\ f(x_2) \\ \vdots \\ f(x_m) - \beta/h^2 \end{bmatrix}$$

This is something to consider for parallel computing

A bit more: General Linear Second Order ODEs

$$a(x)\cdot u''(x) + b(x)\,u'(x) + c(x)\,u(x) = f(x)$$

Ex: $(k(x)\cdot u'(x))' + c\,u = f(x)$

$\Rightarrow k'(x)\,u' + k(x)\,u''(x) + c(x)\,u(x) = f(x)$.

$\rightarrow k(x)\,u'' + k'(x)\,u' + cu = f$

Let's construct a scheme that is second order.

$$a(x_i)\cdot \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + b(x_i)\cdot\left(\frac{U_{i+1} - U_{i-1}}{2h}\right) + c(x_i)\,U_i = f(x_i)$$

The matrix form is given on page 35 in textbook

Ex: $(k(x)\,u'(x))' = f(x)$

$$\uparrow$$
$$\left(\frac{U_j - U_{j-1}}{h}\right)$$

$$\Rightarrow \left(k(x_{j+1/2})\cdot\frac{U_j - U_{j-1}}{h}\right)' = f(x)$$

$$\Rightarrow \frac{(k_{j+1/2})\left(\frac{U_{j+1} - U_j}{h}\right) - (k_{j-1/2})\left(\frac{U_j - U_{j-1}}{h}\right)}{h} = f(x)$$

$$\Rightarrow \frac{1}{h^2}\left(k_{j+1/2}\,(U_{j+1} - U_j) - k_{j-1/2}\,(U_j - U_{j-1})\right) = f$$

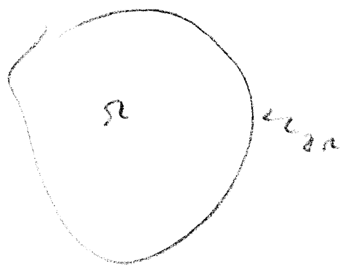
Elliptic Partial DEs

$$a_1 \frac{\partial^2 u}{\partial x^2} + a_2 \frac{\partial u}{\partial x \partial y} + a_3 \frac{\partial^2 u}{\partial y^2} + a_4 \frac{\partial u}{\partial x} + a_5 \frac{\partial u}{\partial y} + a_6 u = f$$

If a_1, a_2, a_3 satisfy

$$a_2^2 - 4a_1a_3 < 0$$

the equation is an elliptic PDE. This will be defined on an open region in 2-D space, say $\Omega \subset \mathbb{R}^2$. Also, boundary conditions will be specified



Laplacian: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in 2-D

Gradient: $\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$

Ex:

$$\Delta u = \nabla \cdot \nabla u = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Canonical Examples

$\Delta u = 0$ — Laplace's equation $\Rightarrow a_1 = 1, a_2 = 0, a_3 = 1 \Rightarrow a_2^2 - 4a_1a_3 < 0$

$\Delta u = f$ — Poisson equation $\Rightarrow$ same