Math 5620 Lecture notes: Day 13

Summary:

We had a go at the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x) \frac{\partial u}{\partial x} \right) \qquad x \in (a,b),\ t > 0$$

$$u(x,0) = g(x) \qquad x \in (a,b),\ t = 0$$

$$u(a,t) = \alpha(t) \qquad x = \alpha,\ t > 0$$

$$u(b,t) = \beta(t) \qquad x = \beta,\ t > 0.$$

We decided a simplified problem might be better. So a reduced problem was selected

$(a,b) \longrightarrow (0,1)$

$k(x) \longrightarrow k$

take out t dependence

$\alpha(t) = \alpha$

$\beta(t) = \beta$

$\frac{\partial u}{\partial t} = 0$

The simplification resulted in

$$\begin{cases} \frac{d^2 u}{dx^2} = f(x) \\ u(0) = \alpha \\ u(1) = \beta \end{cases}$$

The next step is to discretize the ODE.

$$\Rightarrow \quad u'' \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Use an h to equally space points on $[0,1]$ with $x_j = j*h$ for $j = 0,1,2,\dots,m+1$. Set $x_0 = 0$, $x_{m+1} = 1$. Then interpret the problem on the mesh function.

$$\begin{bmatrix} -2 & 1 & 0 & & 0 \\ 1 & -2 & 1 & 0 \cdots & 0 \\ & & \ddots & & 1 \\ 0 & & & 1 & -2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix}$$

We can solve the system of equations but what does that imply.

If we add consistency and stability we show that the approximation as $h \to 0$ converges to the continuous solution.

⋮

Lots of Work

⋮

$$LTE \to 0 \quad + \quad \|A^{-1}\| \le C \quad \Rightarrow \quad \text{Global Error} \to 0\,!$$

The development of ways to approximately solve DEs. If a problem is complicated, the analysis may be too tough. So, what can we do!?

What can change?

→ Neumann boundary condition vs Dirichlet boundary conditions

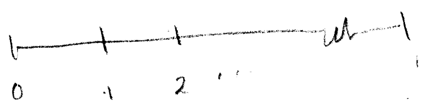
$$① \begin{cases} \dfrac{d^2u}{dx^2} = f \\ u(0) = \alpha \\ u(1) = \beta \end{cases} \qquad ② \begin{cases} \dfrac{d^2u}{dx^2} = f \\ u'(0) = V_0 \\ u'(1) = V_1 \end{cases}$$

Other versions.

$$\begin{cases} u'' = f \\ u'(0) = \alpha \\ u(1) = \beta \end{cases} \qquad \begin{cases} u'' = f \\ u(0) = \alpha \\ u'(1) = \beta \end{cases}$$

Mixed Conditions

Let's consider the left endpoint. $j=0$



At $j=0$. Suppose we are given $u'(0)=\alpha$. Then as a first attempt, set

$$\frac{U_1 - U_0}{h} = \alpha$$

$$\Rightarrow \quad U_1 = U_0 + h$$

Then continue: $j=1$

$$U_2 - 2U_1 + U_0$$

$$U_{j+1} - 2U_j + U_{j-1} = f$$

The structure

$$\frac{1}{h^2}\begin{bmatrix} -h & h & 0 & & & \\ 1 & -2 & 1 & 0 & & \\ 0 & 1 & -2 & 1 & 0 & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 0 & h^2 \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ \\ f(x_m) \\ \beta \end{bmatrix} \begin{matrix} \leftarrow \text{Neumann} \\ \\ \\ \\ \\ \leftarrow \text{Dirichlet} \end{matrix}$$

Problem: the error looks like

$$\boxed{\tau_0 = \frac{1}{2} h\, u''(x_0) + O(h^2)}$$

$\tau_1 \sim O(h^2)$

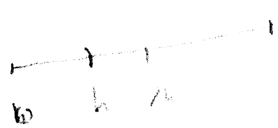
The problem is that τ_0 pollutes the rest of the approximation

Another way to approach $u'(x_0) = \alpha$

$$\frac{1}{h^2}\left(U_{-1} - 2U_0 + U_1\right) = f(x_0)$$

$$\underbrace{\frac{1}{2h}\left(U_1 - U_{-1}\right) = \alpha}_{\text{contained}}$$

Another way



one sheet

Ex: $\frac{1}{h}\left(\frac{3}{2}U_0 - 2U_1 + \frac{1}{2}U_2\right) = \alpha$

Structure:

$$\frac{1}{h^2}\begin{bmatrix} \frac{3h}{2} & -2h & \frac{h}{2} & & \\ 1 & -2 & 1 & & 0 \\ 0 & 1 & -2 & 1 & \\ & & & \ddots & \\ & 0 & & 1 & -2 \quad 1 \\ & & & & 0 \quad h^2 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ \\ \\ u_m \\ u_{m+1} \end{bmatrix} = \begin{bmatrix} \alpha \\ f_1 \\ f_2 \\ \\ f_m \\ \beta \end{bmatrix}$$

$$D x^{k+1} = b - (L + U + D - D) x^k$$

$$= (b - Ax) + Dx$$

$$x^{k+1} = D^{-1} r + x$$

$$y(x-h) = y(x) - h y'(x) + \tfrac{1}{2} h^2 y'' + \dots$$

$$y(x) = y(x)$$

$$y(x+h) = y(x) + h y'(x) + \tfrac{1}{2} h^2 y''$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = -$$

$f(x)$

$$\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 1 & \dots \\ 1 & 0 & 1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}$$

$\neq 0$?