So, we should define norms that make sense for our approximate solution

$$\|E^h\|_\infty = \max_{1\le i\le m} |u_i - \hat{u}_i| \quad \sim \quad \|e(x)\|_\infty = \max_{x} |u(x) - \hat{u}(x)|$$

$$\|E^h\|_1 = \sum_{j=1}^{m} |u_j - \hat{u}_j| \quad \sim \quad \|e(x)\|_1 = \int |u(x) - \hat{u}(x)|\, dx \quad \rightarrow dx = h$$

$$\|E^h\|_2 = \left(\sum_{j=1}^{m} |u_j - \hat{u}_j|^2\right)^{1/2} \quad \sim \quad \|e(x)\|_2 = \left(\int |u(x) - \hat{u}(x)|^2\, dx\right)^{1/2} \quad \rightarrow h^{1/2}$$

Now to elliptic PDEs

$$a_1(x,y)\,u_{xx} + a_2(x,y)\,u_{xy} + a_3(x,y)\,u_{yy} + a_4(x,y)\,u_x + a_5(x,y)\,u_y + a_6(x,y)\,u = f$$

So, if

$\rho = a_2^2 - 4a_1a_3 < 0 \Rightarrow$ Elliptic $\qquad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \ldots \qquad a_1 = a_3 = 1 \Rightarrow \rho = -4$

$\rho = a_2^2 - 4a_1a_3 = 0 \Rightarrow$ Parabolic $\qquad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial y^2} \qquad a_1 = 1,\ a_2 = 0,\ a_3 = 0 \Rightarrow \rho = 0$

$\rho = a_2^2 - 4a_1a_3 > 0 \Rightarrow$ Hyperbolic $\qquad \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 1 \Rightarrow \qquad a_1 = 1,\ a_3 = -1 \Rightarrow \rho = +4 > 0$

__Note:__

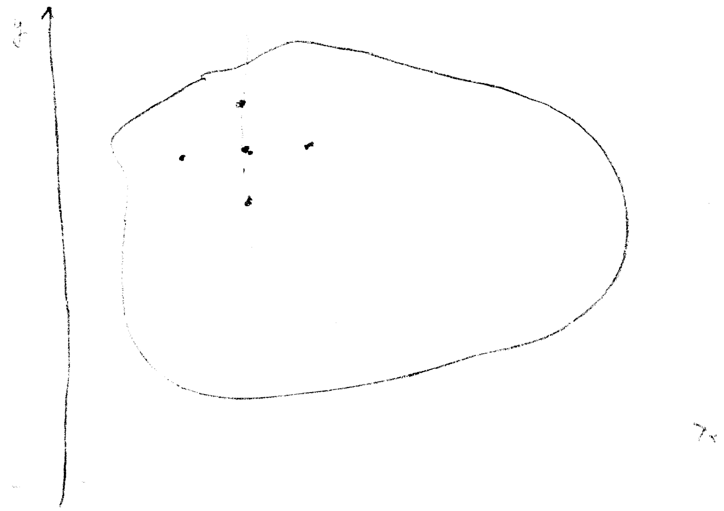
- Elliptic $\Rightarrow$ 2 imaginary characteristics $\Rightarrow$ ∞ speed of propagation
- Parabolic $\Rightarrow$ 1 real characteristic curve $\Rightarrow$
- Hyperbolic $\Rightarrow$ 2 real characteristics $\Rightarrow$ finite speed of propagation

Now to a canonical problem

$$\Delta u = f \quad \text{on } \Omega \subset \mathbb{R}^2$$

$$u|_{\partial\Omega} = g \quad \text{on } \partial\Omega \quad \text{boundary of } \Omega$$

Repeating the idea of finite difference method, define a set of points



$$\Delta u = \nabla \cdot \nabla u = \nabla^2 u = f$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

For PDEs we need

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+\Delta x, y) - 2u(x,y) + u(x-\Delta x, y)}{\Delta x^2} \qquad \Delta x \neq \nabla^2 x$$

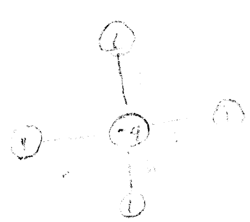
$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u(x, y+\Delta y) - 2u(x,y) + u(x, y-\Delta y)}{\Delta y^2}$$

$\Delta y = \Delta x = h$

$$= \frac{1}{h^2}\Big(\big(u(x+h) - 2u(x,y) + u(x-h,y)\big) + \big(u(x,y+h) - 2u(x,y) + u(x,y-h)\big)\Big)$$

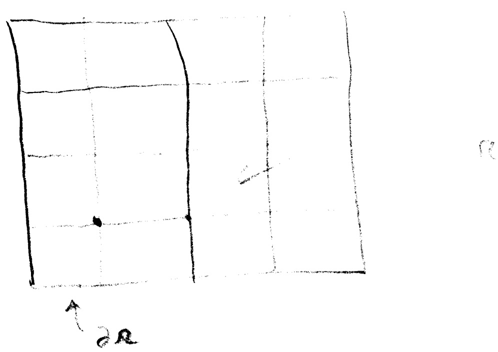
$$= \frac{1}{h^2}\big(-4u(x,y) + u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)\big)$$

Stencils : 5 pt



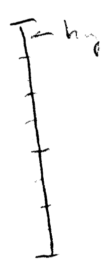
Let's simplify the problem

Ω = square/rectangle on $[0,1]$



Reference: Grid Generation

Take the intersection of parallel lines or create using a mesh in each direction.

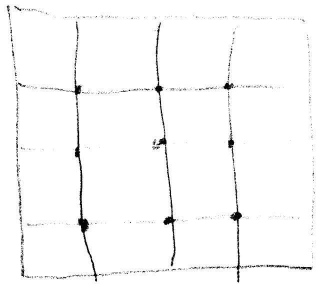


h, h_y

So, the best trick is to take Ω and transform this to a rectangular domain – picture in book

Reference
Grid

Let's assume the mesh/grid is defined on a unit square



3×3 points or 9 mesh pts

So we set

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h_x^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h_y^2}$$

$$= \frac{1}{h^2}\left(-4U_{i,j} + U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right) = f_{i,j}$$

Computing is essentially linear arrays. So, we would like to have a convenient matrix equation:

$(i,j)=(1,1) \longrightarrow 1$

$(i,j)=(2,1) \longrightarrow 2$

$(i,j)=(3,1) \longrightarrow 3$

$(i,j)=(1,2) \longrightarrow 4$

$\vdots$

$(i,j)=(3,3) \longrightarrow 9$

for $m+1$ points in each direction

$ind = i + (j-1)\,m \qquad i=1,2,3,\ j=1,2,3$

$$\begin{cases} j = 1 + ind/m \\ i = ind - (j-1)*m \end{cases}$$

Matrix:

$(i,j) = (1,1) \longrightarrow \text{ind} = 1$ $g(4)$

$$\frac{u_2 - 2u_1 + u_{Bc}}{h^2} + \frac{u_4 - 2u_1 + u_{Bc}}{h^2} = f_1$$

$$\hookrightarrow \frac{u_2 - 2u_1}{h^2} + \frac{u_4 - 2u_1}{h^2} = f_1 - \left((Bc)_x - (Bc)_y \right) / h^2$$

Interior

$(i,j) = (2,1) \longrightarrow \text{ind} = 2$

$$\frac{1}{h^2}\left(u_3 - 2u_2 + u_1\right) + \frac{1}{h^2}\left(u_5 - 2u_2 + u_{Bc_y}\right) = f_2$$

$$\Rightarrow \frac{1}{h^2}\left(u_3 - 2u_2 + u_1\right) + \frac{1}{h^2}\left(u_5 - 2u_2\right) = f_2 - \frac{1}{h^2}(Bc)$$

$(i,j) = (3,1) \rightarrow \text{ind} = 3$

$$\Rightarrow \frac{1}{h^2}\left(u_{Bc} - 2u_3 + u_2\right) + \frac{1}{h^2}\left(u_6 - 2u_3 + u_{Bc}\right) =$$

$$\hookrightarrow \frac{1}{h^2}\left(-2u_3 + u_4\right) + \frac{1}{h^2}\left(u_6 - 2u_3\right) = f_3 - \frac{1}{h^2}\left(u_{Bc_x} + u_{Bc_y}\right)$$

$(i,j) = (1,2) \longrightarrow \text{ind} = 4$

$$\Rightarrow \frac{1}{h^2}\left(u_5 - 2u_4 + Bc\right) + \frac{1}{h^2}\left(u_7 - 2u_4 + u_1\right) = 0$$

Structure:

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots\ 0 \\ 0 & 1 & -4 & & & & & & & 1 \\ & & & \ddots & & & & & & \\ & & & & & & & & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_{m \cdot m} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{m \times m} \end{bmatrix}$$

$$\begin{bmatrix} T & I & 0 \\ I & T & I \\ 0 & I & T \end{bmatrix} \begin{bmatrix} u \end{bmatrix} = F$$

Block tridiagonal matrix!

So,

$$A^h U^h = F^h$$

$$\Rightarrow U^h = (A^h)^{-1} F^h$$

Direct methods?

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \quad 0 & a_{2n} \\ & & & \\ a_{n1} & x & x & \end{bmatrix} \rightsquigarrow \begin{bmatrix} x & x & & 0 \\ & \ddots & \text{Fills in} & x \\ x & & & x \\ 0 & x & x & x \end{bmatrix}$$

LU or BS