Math 5620 Lecture Day 2.

So, we have some examples of DEs to work on. We also wrote down a couple of useful definition

Def: A function is differentiable at a point, a, if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

exists.

Assumption: We must assume that f is differentiable on some interval (x_ℓ, x_r) with $a \in (x_\ell, x_r)$ to make mathematical sense.

: We will change variables for convenience

$$\frac{f(x) - f(a)}{x - a} = \frac{f(x+h) - f(x)}{h} \Rightarrow \text{Set } x = a + h \Rightarrow x - a = h$$

$$\Rightarrow f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

: Notation from books looks like

$$a = \bar{x} \Rightarrow f'(\bar{x}) \approx \frac{f(\bar{x}+h) - f(\bar{x})}{h} = D_+(\bar{x})$$

$\Rightarrow$ We will let $h > 0$ vary from 0 to some upper value

It seems we are fixing $h > 0$ which biases the approximation to $x + h$. We will come up with a fix in a few pages

: Notation – the expression

$$D_+(x) = \frac{u(\bar{x}+h) - u(\bar{x})}{h}$$

is called a difference quotient or finite difference.

Ex: Malthus-model for population density

$$\begin{cases} \dfrac{dA}{dt} = kA \\ A(0) = A_0 \end{cases}$$

To begin, let's approximate the derivative $A'(t)$

$$\frac{dA}{dt} \approx \frac{A(t+h) - A(t)}{h}$$

To be a bit more descriptive let $h = \Delta t$

$$\frac{dA}{dt} \approx \frac{A(t+\Delta t) - A(t)}{\Delta t}$$

At this point,

$$\begin{cases} \dfrac{A(t+\Delta t) - A(t)}{\Delta t} \approx k \cdot A(t^*) \\ A(0) = A_0 \end{cases} \qquad t^* = ?$$

So, what comes of this.

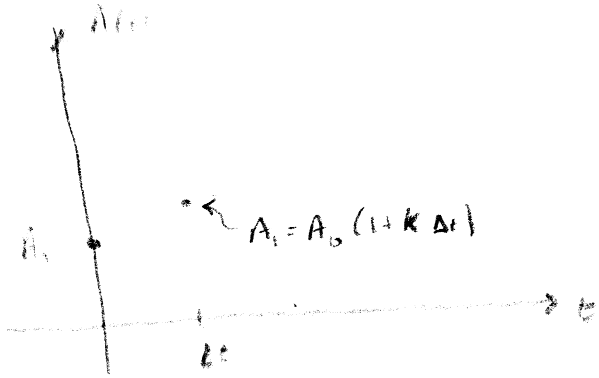
$$t = 0 \Rightarrow A(0) = A_0 \checkmark$$

$$t = \Delta t \Rightarrow \frac{A(\Delta t) - A(0)}{\Delta t} = k\, A(t^*) \Rightarrow A(\Delta t) \approx A(0) + k \cdot \Delta t\, A(0)$$

Set $t^* = 0$

$$\Rightarrow A(\Delta t) \approx A_0 (1 + \Delta t\, k)$$

This gives a way to compute $A(\Delta t)$ from known info. Once the difference (3)
quotient is selected, we have a way to compute approx.



For [illegible], t^*, given as Δt

$$\Rightarrow \quad A_1 = A(\Delta t) = A_0(1 + k\,\Delta t)$$

Now to some more content on difference quotients

<u>Define</u>: $\quad D_- f(\bar{x}) = \dfrac{f(\bar{x}) - f(\bar{x} - h)}{h}$

Note that

$$\lim_{h \to 0} \frac{f(a) - f(a-h)}{h} = f'(a)$$

So, this allows us to say

$$D_- f(a) \approx f'(a)$$

This gives us a second alternative to $D_+ f(a)$

Ex: Return to

$$\begin{cases} \dfrac{dA}{dt} = kA \\ A(0) = A_0 \end{cases}$$

Then

$$\frac{dA}{dt} \approx \frac{A(t) - A(t-h)}{h} = \frac{A(t) - A(t-\Delta t)}{\Delta t}$$

and

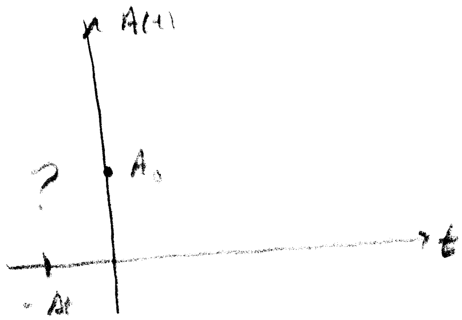
$$\frac{A(t) - A(t-\Delta t)}{\Delta t} \approx k\,A(t^*)$$

? what do we know? $\Rightarrow$ $t^* = 0$

Putting this all together, we can write

$$A(0) - A(-\Delta t) = \Delta t \cdot k\,A(0)$$

$\Rightarrow$ oops there is something wrong!



Back to the forward difference

$$A(t+\Delta t) = A(t) + \Delta t \cdot k\,A(t+\Delta t)$$

$$\Rightarrow A(t+\Delta t)\left(1 - \Delta t\,k\right) = A(t)$$

$$\Rightarrow A(t+\Delta t) = \frac{1}{1-\Delta t\,k} \cdot A(t)$$

take $\Delta t < 1/k$

hmm

Which is best?

As a further alternative consider

$$D_0 f(\bar{x}) = \frac{f(\bar{x}+h) - f(\bar{x}-h)}{2h} = \frac{D_+ f(\bar{x}) + D_- f(\bar{x})}{2}$$

we will eventually come to a point where we can write out general difference quotients

First, we need to assign properties to the differences.

Errors: Last time we talked about how the measure errors:

$$\text{error} = \left| f'(\bar{x}) - D_? f(\bar{x}) \right| = ?$$

Ex: $D_+ f(\bar{x})$

$$\text{error} = \left| f'(\bar{x}) - D_+ f(\bar{x}) \right|$$

$$= \left| f'(\bar{x}) - \frac{f(\bar{x}+h) - f(\bar{x})}{h} \right|$$

$$= \left| f'(\bar{x}) - \frac{1}{h}\left\{ \left(\cancel{f(\bar{x})} + f'(\bar{x})\, h + \tfrac{1}{2} f''(\xi)\, h^2 \right) - \cancel{f(\bar{x})} \right\} \right|$$

$$= \left| \cancel{f'(\bar{x})} - \frac{1}{h} \cdot \left(\cancel{f'(\bar{x}) \cdot h} \right) + \tfrac{1}{2} f''(\xi)\, h \right| = \left| \tfrac{1}{2} h f''(\xi) \right| =$$

$$= \underbrace{C}_{|\frac{1}{2} f''(\xi)|} h$$

Taylor Series w/ remainder

$$f(\bar{x}+h) = f(\bar{x}) + f'(\bar{x})(\bar{x}+h-\bar{x}) + \tfrac{1}{2} f''(\bar{x})(\bar{x}+h-\bar{x})^2 + \dots$$

$$= f(\bar{x}) + f'(\bar{x})\, h + \tfrac{1}{2} f''(\xi)\, h^2$$

So, if we work only on $f \in C^3(I)$.

We can do this for any function and diff. quotient

Ex: $D_0 f(x) = \dfrac{f(\bar{x}+h) - f(\bar{x}-h)}{2h}$

$$\text{error} = \left| f'(\bar{x}) - D_0 f(\bar{x}) \right|$$

$$= \left| f'(\bar{x}) - \frac{f(\bar{x}+h) - f(\bar{x}-h)}{2h} \right|$$

$$= \Big| f'(\bar{x}) - \frac{1}{2h} \cdot \Big\{ \big(\cancel{f(\bar{x})} + h f'(\bar{x}) + \cancel{\tfrac{1}{2} h^2 f''(\bar{x})} + \tfrac{1}{6} h^3 f'''(\xi_1) \big)$$
$$- \big(\cancel{f(\bar{x})} - f'(\bar{x}) h + \cancel{\tfrac{1}{2} h^2 f''(\bar{x})} - \tfrac{1}{6} h^3 f'''(\xi_2) \big) \Big\}$$

$$= \Big| \cancel{f'(\bar{x})} - \frac{1}{2h} \Big\{ \cancel{2h f'(\bar{x})} + \tfrac{1}{3} h^3 \big(f'''(\xi_1) + f'''(\xi_2) \big) \Big\}$$

$$= \left| \frac{1}{2h} \left(\frac{1}{3} h^3 \right) \left(f'''(\xi_1) + f'''(\xi_2) \right) \right|$$

$$= C h^2$$

This gives a way to compare.

$$D_+ f(\bar{x}) \longrightarrow C_1 h$$

$$D_0 f(\bar{x}) \longrightarrow C_2 h^2$$

$\to$ we need to discuss as $h \to 0$ and what C_1 and C_2 depend on.

$$C_1 = C_1 \left(f''(\xi) \right) \qquad C_2 = C_2 \left(f'''(\eta) \right)$$