Math 5620 Lecture Notes Day 8

We have taken

$$u_t = (k\,u_x)_x + \psi(x) \Rightarrow \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) + \psi(x)$$

$$\Downarrow$$

$$u''(x) = f(x) \qquad \frac{d^2u}{dx^2} = f(x)$$

$$\Downarrow$$

$$\frac{u(x_j+h) - 2u(x_j) + u(x_j-h)}{h^2} \approx f(x_j) \qquad j = 1, 2, \dots, m, \qquad h = 1/m+1$$

$$\Downarrow$$

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = F_j \qquad F_j = f(x_j)$$

$$\Downarrow$$

$$\begin{cases} AU = F \\ U_0 = \alpha \\ U_{m+1} = \beta \end{cases} \qquad A = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}, \quad U = \begin{bmatrix} U_1 \\ \vdots \\ U_m \end{bmatrix}$$

$$F = \begin{bmatrix} f_1 - \alpha/h^2 \\ f_2 \\ \vdots \\ f_m - \beta/h^2 \end{bmatrix}$$

$$\Downarrow$$

$$U = \text{"}A^{-1}\text{"}F$$

We need methods - efficient and effective

Thomas Algorithm ...

→ discuss Wikipedia page on Assignment #2.

Another method: Jacobi Iteration:

Suppose we have a system of the form

$$Ax = b, \quad A \in \mathbb{R}^{m \times m}, \quad x \in \mathbb{R}^m, \quad b \in \mathbb{R}^m$$

Now we can rewrite

$$A = (L + D + U)$$

$$L = \begin{bmatrix} 0 & \cdots & & 0 \\ x & \ddots & & \\ & & \ddots & \\ x & \cdots & x & 0 \end{bmatrix} \quad D = \begin{bmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{bmatrix} \quad U = \begin{bmatrix} 0 & x & x & \cdots & x \\ & \ddots & & & \vdots \\ & & \ddots & & x \\ & 0 & & \ddots & \\ & & & & 0 \end{bmatrix}$$

$$\Rightarrow \quad Ax = (L + D + U)x$$

$$= (L + U)x + Dx = b$$

$$\hookrightarrow \quad Dx = b - (L + U)x$$

$$\hookrightarrow \quad x = D^{-1}(b - (L + U)x)$$

Suggests an iteration.

$$x^{(k+1)} = D^{-1}(b - (L + U)\, x^{(k)})$$

So, we need to start with some $x^{(0)}$ as a guess and compute $x^{(1)}$ and so on..

So,

$$x^{(k+1)} = D^{-1}\left(b - (\underbrace{L+D-D+U}) x^{(k)}\right)$$

$$= D^{-1}\left(b - (A-D)x^{(k)}\right)$$

$$= D^{-1}\left(\underbrace{b - Ax^{(k)}} + Dx^{(k)}\right)$$

$$= D^{-1}(b - Ax^{(k)}) + x^{(k)}$$

$$= x^{(k)} + D^{-1}r^{(k)}$$

residual vector.

Def Suppose we have a linear system of equation with

$$Ax = b$$

Then for any vector $x \in \mathbb{R}^n$, the residual vector r is defined by

$$r = b - Ax$$

This gives a way to test a vector to see if x satisfies the linear system

$$r = 0 \Rightarrow b - Ax = 0 \Rightarrow Ax = b.$$

So, we have two linear system solvers.

1. Thomas Alg.
2. Jacobi Iteration

We already have seen how to make the Thomas algorithm more efficient.

1 store as vectors
2 use smart ways to eliminate

The same is true for the Jacobi iteration. We just need to be careful with matrix mult.

Code for matrix - vector product.

$A \cdot x \rightarrow y$

```
for i in range (m):                  <------  sum = 0.0
    y[i] = 0.0
    for j in range (m):
        y[i] = y[i] + A[i][j] * x[j]
    end                              <------  y[i] = sum.
end
```

Version to avoid access issues for arrays