Math 5620 Lecture Notes Day 3.

Last time:

Taylor Series on Difference Quotients:

Ex $D_+ f(\bar{x}) = \frac{f(\bar{x}+h) - f(\bar{x})}{h} \approx f'(\bar{x})$

$$\Rightarrow \text{error} = \left| f'(\bar{x}) - \frac{f(\bar{x}+h) - f(\bar{x})}{h} \right|$$

$$= \left| f'(\bar{x}) - \left\{ \frac{1}{h}\left(f(\bar{x}) + f'(\bar{x})h + \frac{1}{2} f''(\eta) h^2 \right) \right\} \right|$$

$$= \left| \frac{1}{2} f''(\eta) h^2 \right|$$

$$\leq C h \qquad (\text{Done})$$

As $h \to 0 \Rightarrow Ch \to 0$.

Implication: f must be a function that is twice continuously differentiable.

Ex $D_0 f(\bar{x}) = \frac{f(\bar{x}+h) - f(\bar{x}-h)}{2h} \approx f'(\bar{x})$

$$\Rightarrow \text{error} = \left| f'(\bar{x}) - \frac{1}{2h}\left\{ \left(\cancel{f(\bar{x})} + h f'(\bar{x}) + \tfrac{1}{2} h^2 \cancel{f''(\bar{x})} + \tfrac{1}{3!} h^3 f'''(\xi_1)\right) - \left(\cancel{f(\bar{x})} - h f'(\bar{x}) + \tfrac{1}{2} h^2 \cancel{f''(\bar{x})} - \tfrac{1}{3!} h^3 f'''(\xi_2)\right)\right\} \right|$$

$$= \left| f'(\bar{x}) - \frac{1}{2h}\left\{ 2h f'(\bar{x}) + \frac{1}{6} h^3 \left(f'''(\xi_1) + f'''(\xi_2) \right) \right\} \right|$$

$$= \left| -\frac{1}{12} h^2 \left(f'''(\xi_1) + f'''(\xi_2) \right) \right| \leq C h^2$$

Let's look at this a bit differently. Each of these difference quotients is a "linear" combination of function values.

Ex: $D_+ f(\bar{x}) = \frac{1}{h} f(\bar{x}+h) - \frac{1}{h} f(\bar{x})$

$= c_1 f(\bar{x}+h) + c_2 f(\bar{x})$

Ex: $D_0 f(\bar{x}) = \frac{1}{2h} f(\bar{x}+h) - \frac{1}{2h} f(\bar{x}-h)$

$= c_1 f(\bar{x}+h) + c_2 f(\bar{x}-h)$

Now, suppose we try the following tack!

$$f'(\bar{x}) \approx a_{-1} f(\bar{x}-h) + a_0 f(\bar{x}) + a_1 f(\bar{x}+h)$$

where a_{-1}, a_0, a_1 are unknown constants, possibly dependent on h. Why! Let's start by expansion in Taylor series.

$$f(\bar{x}-h) = f(\bar{x}) - h f'(\bar{x}) + \frac{1}{2} h^2 f''(\bar{x}) - \frac{1}{6} h^3 f'''(\bar{x}) + \dots$$

$$f(\bar{x}) = f(\bar{x})$$

$$f(\bar{x}+h) = f(\bar{x}) + h f'(\bar{x}) + \frac{1}{2} h^2 f''(\bar{x}) + \frac{1}{6} h^3 f'''(\bar{x}) + \dots$$

So, plug these into the form

$$\begin{aligned} f'(\bar{x}) = \; & a_{-1} \left(f(\bar{x}) + h f'(\bar{x}) + \tfrac{1}{2} h^2 f''(\bar{x}) + \tfrac{1}{3!} h^3 f'''(\xi_1) \right. \\ & + a_0 f(\bar{x}) \\ & + a_1 \left(f(\bar{x}) - h f'(\bar{x}) + \tfrac{1}{2} h^2 f''(\bar{x}) - \tfrac{1}{3!} h^3 f'''(\eta_2) \right) \end{aligned}$$

$$= f(\bar{x}) \cdot (a_1 + a_0 + a_{-1})$$
$$+ h f'(\bar{x}) (a_1 - a_{-1})$$
$$+ \frac{1}{2} h^2 f''(\bar{x}) (a_1 + a_{-1})$$
$$+ \frac{1}{6} h^3 (a_1 f'''(\xi_1) - a_{-1} f'''(\xi_2))$$

Let's interpret

1st term: $a_1 + a_0 + a_{-1} = 0$

2nd term: $h \cdot (a_1 - a_{-1}) = 1 \Rightarrow a_1 - a_{-1} = 1/h$

3rd term: $\frac{1}{2} h^2 (a_1 + a_{-1}) = 0 \Rightarrow a_1 + a_{-1} = 0$

We now have a system of 3 linear equations in 3-unknowns

$$a_1 + a_0 + a_{-1} = 0$$
$$a_1 - a_{-1} = 1/h$$
$$a_1 + a_{-1} = 0$$

add $\rightarrow$ $2a_1 = 1/h \Rightarrow a_1 = \frac{1}{2h}$

$\rightarrow a_1 + a_{-1} = \frac{1}{2h} + a_{-1} = 0$

$\Rightarrow a_{-1} = -1/2h$

$\Rightarrow a_1 + a_0 + a_{-1} = \frac{1}{2h} + a_0 - \frac{1}{2h}$

$= a_0 = 0 \rightarrow a_0 = 0$

$\Rightarrow$ $a_1 = 1/2h$

$a_0 = 0$ $\Rightarrow f'(\bar{x}) \approx \frac{1}{2h} \cdot f(\bar{x}+h) + (-1/2h) f(\bar{x}-h)$

$a_{-1} = -1/2h$

$$\approx \frac{f(\bar{x}+h) - f(\bar{x}-h)}{2h}$$ ✓

So, let's do a general case

Suppose we define a function at some number of points on the real line, $x_1, x_2, \ldots, x_n$. Now, let's approximate derivatives at a given point.

$$u(x_i) = u(\bar{x}) + (x_i - \bar{x})\, u'(\bar{x}) + \cdots + \frac{1}{k!}(x_i - \bar{x})^k u^{(k)}(\bar{x}) + \cdots$$

We can use this to approximate the derivatives we need. That is set

$$c_1 u(x_1) + c_2 u(x_2) + \cdots + c_n u(x_n) = u'(\bar{x}) \qquad \left(\text{or } u^{(k)}(\bar{x}) \right)$$

$$+ \, O(h^p)$$

much p as large as possible.

plugging in the expansions we see.

$$c_1 \left(u(\bar{x}) + u'(\bar{x})(x_1 - \bar{x}) + \tfrac{1}{2} u''(\bar{x})(x_1 - \bar{x})^2 + \cdots + \tfrac{1}{k!} u^{(k)}(\bar{x})(x_1 - \bar{x})^k \right)$$

$$+ c_2 \left(u(\bar{x}) + u'(\bar{x})(x_2 - \bar{x}) + \tfrac{1}{2} u''(\bar{x})(x_2 - \bar{x})^2 + \cdots + \tfrac{1}{k!} u^{(k)}(\bar{x})(x_2 - \bar{x})^k \right)$$

$$+ c_3 \left(u(\bar{x}) + u'(\bar{x})(x_3 - \bar{x}) + \tfrac{1}{2} u''(\bar{x})(x_3 - \bar{x})^2 + \cdots + \tfrac{1}{k!} u^{(k)}(\bar{x})(x_3 - \bar{x})^k \right)$$

$$+ \cdots +$$

$$+ c_k \left(u(\bar{x}) + u'(\bar{x})(x_k - \bar{x}) + \tfrac{1}{2} u(\bar{x})(x_k - \bar{x})^2 + \cdots + \tfrac{1}{k!} u^{(k)}(\bar{x})(x_k - \bar{x})^k \right)$$

Then,

$$= u(\bar{x})\left(c_1 + c_2 + \cdots + c_k\right) + u'(\bar{x})\left(c_1(x_1 - \bar{x}) + c_2(x_2 - \bar{x}) + \cdots + c_k(x_k - \bar{x}) \right)$$

$$+ \tfrac{1}{2} u''(\bar{x}) \left(c_1 (x_1 - \bar{x})^2 + c_2 (x_2 - \bar{x})^2 + \cdots + c_k (x_k - \bar{x})^k \right)$$

$$+\frac{1}{3!}u'''(\bar{x})\left(c_1(x_1-\bar{x})^3+c_2(x_2-\bar{x})^3+\dots+c_n(x_n-\bar{x})^3\right)$$

$$+\dots+$$

$$+\frac{1}{k!}u^{(k)}(\bar{x})\left(c_1(x_1-\bar{x})^k+c_2(x_2-\bar{x})^k+\dots+c_n(x_n-\bar{x})^k\right)$$

Recall we are finding $c_1, c_2, \dots, c_k, \dots, c_n$. So, suppose we want to approximate $u'(\bar{x})$. So

$$u(\bar{x}) \longrightarrow 0$$

$$\Rightarrow c_1+c_2+c_3+\dots+c_n=0$$

$$\Rightarrow c_1(x_1-\bar{x})+c_2(x_2-\bar{x})+\dots+c_n(x_n-\bar{x})=1 \quad \text{we approx } u'(\bar{x})$$

$$\Rightarrow c_1(x_1-\bar{x})^2+c_2(x_2-\bar{x})^2+\dots+c_n(x_n-\bar{x})^2=0$$

$$\vdots$$

$$\Rightarrow c_1(x_1-\bar{x})^k+c_2(x_2-\bar{x})^k+\dots+c_n(x_n-\bar{x})^k=0$$

matrix form $\Rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ (x_1-\bar{x}) & (x_2-\bar{x}) & \cdots & & (x_n-\bar{x}) \\ \vdots & & & & \\ (x_1-\bar{x})^k & \cdots & & & (x_n-\bar{x})^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This defines a linear system of equations, with a VanderMonde matrix.

Code on page 11. function c = fdcoeffV(k, xbar, x)

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A = ones(n,n);
⋮
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