For our work we analyze the error in approximations using Taylor series expansion at a point.

Ex: $D_+ u(\bar{x}) = \dfrac{u(\bar{x}+h) - u(\bar{x})}{h}$

$$= u'(\bar{x}) + \tfrac{1}{2} h u''(\bar{x}) + \tfrac{1}{6} h^2 u'''(\bar{x}) + O(h^3)$$

$O(h^4)$ is called "big-oh" notation.

$\Rightarrow$ the rest of the terms behaves in some fashion.

Appendix A: Measuring Error

In the scalar case, $x \in \mathbb{R}^1$, let's consider an IVP of the form

$$u'(t) = f(u(t)) \qquad u(0) = \eta$$

Suppose we are interested in $u(t)$ at a specific value $t = T$. We can use the notation

$$\tilde{z} = u(T) = \text{exact solution}$$

and let z be the computed solution. The error will be defined by

$$E = z - \tilde{z} = \text{signed error.}$$

Absolute Error.

A natural measure of error is the absolute value of E

$$|E| = |z - \tilde{z}| = \text{absolute error.}$$

Ex: $\tilde{z} = 2.2$, $z = 2.20345$

$$|z - \tilde{z}| = 0.00345 = 3.45 \times 10^{-3}$$

This seems very reasonable. The error is approximately 10^{-3}. We may have other measures of error that produce 10^{-6} or 10^{6} as estimates of the error.

Ex: Above we had $\tilde{z} = 2.2$ say in meters. Now, suppose we change units to nanometers. This means, $\tilde{z} = 2.2 \times 10^{9}$ nanometers, with

$$|E| = 3.45 \times 10^{6}$$

and if we change to units of kilometers, we see that

$$|E| = 3.45 \times 10^{-6}$$

Since $\tilde{z} = 2.2 \times 10^{-3}$ and $z = 2.20345 \times 10^{-3}$

This seems inconsistent.

$\Rightarrow$ the problem is in scaling.

Relative Error:

$$E_{abs} = |E| = |z - \hat{z}|$$

$$E_{rel} = \left|\frac{z - \hat{z}}{\hat{z}}\right| = \text{relative or \% error}$$

What we end up having is a "dimension-less" measure of error. Basically, the units divide out. Using relative error makes sure the measure of error is "properly scaled."

To avoid issues we will assume our DE's are properly scaled.

Big-Oh and little-oh notation

When discussing convergence of approximations to behave in certain ways.

Ex: $\begin{cases} \hat{u}'(t) = f(\hat{u}(t)) \\ \hat{u}(0) = \eta \end{cases} \Rightarrow \dfrac{u(t+h) - u(t)}{h} = f(u(t))$

$\Rightarrow u(t+h) = u(t) + h f(u(t))$

As $h \to 0$ we want $u \to \hat{u}$. So, we can write

$$\underbrace{|E(h)| = |\hat{u}(t) - u(t,h)|}_{\text{abs. error}}$$

or

$$|E_h| = \frac{|\hat{u}(t) - u(t,h)|}{|\hat{u}(t)|}$$

We will use E_{abs}!

At each point we will need to compute errors

Some notation.

If $f(h)$ and $g(h)$ are two functions of h, then we say

$$f(h) = O(g(h))$$

as $h \to 0$ if there is some constant C such that

$$\left|\frac{f(h)}{g(h)}\right| < C \quad \Rightarrow \quad |f(h)| \le C\,|g(h)|$$

for all h sufficiently small

$\Rightarrow$ "$f(h)$ converges to zero at least as fast as $g(h)$"

For finite differences we will usually see $g(h) = h^q$ for some q.

We will use

$$f(h) = o(g(h)) \quad \text{as } h \to 0$$

This means

$$\left|\frac{f(h)}{g(h)}\right| \to 0 \quad \text{as } h \to 0$$

"Little-oh" is stronger than "big-oh" convergence or estimates.

Examples:

1. $2h^3 = O(h^2)$ as $h \to 0$ since $\frac{2h^3}{h^2} = 2h < 1$ C $\frac{1}{2}$ $h < \frac{1}{2}$

2. $2h^3 = o(h^2)$ as $h \to 0$ since $\frac{2h^3}{h^2} = 2h \to 0$ as $h \to 0$

3. $\sin(h) = O(h)$ as $h \to 0$ since $\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} \cdots < h$ for all $h > 0$

4. $\sin(h) = h + o(h)$ as $h \to 0$ since $\frac{\sin(h) - h}{h} = O(h^2)$

5. $\sqrt{h} = O(1)$ as $h \to 0$

$\sqrt{h} = o(1)$

$\sqrt{h} \neq O(h)$ $\to +\infty$

6. ...

Vector Errors in A.3 → as we need!

So, when we compute

$$E_{ans} = |\hat{u}(t) - u(t,h)| \leq Ch^q \qquad q > 0$$

So that

$$E_{ans} = O(h^p) \qquad p > 1$$

Let's get the coefficients of difference quotients

Given $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and consider the value $u(x_i)$ can be written as

$$u(x_i) = u(\bar{x}) + (x_i - \bar{x})u'(\bar{x}) + \frac{1}{2}(x_i - \bar{x})^2 u''(\bar{x}) + \ldots + \frac{1}{k!}(x_i - \bar{x})^k u^{(k)}(\bar{x}) + \ldots$$

with $n \geq k+1$. We are interested in approximation for the k^{th} derivative of u at $\bar{x}$.

The difference quotient will be determined by

$$\bar{u}^{(k)}(\bar{x}) = c_1 u(x_1) + c_2 u(x_2) + \ldots + c_n u(x_n) - \underbrace{O(h^p)}_{\text{truncation error}}$$

~~dotum error~~

We use

$$\frac{1}{(i-1)!} \sum_{j=1}^{n} c_j (x_j - \bar{x})^{i-1} = \begin{cases} 1, & \text{if } i-1 = k \\ 0, & \text{otherwise} \end{cases}$$

For $i = 1, 2, \ldots, n$.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ (x_1 - \bar{x}) & (x_2 - \bar{x}) & \cdots & & (x_n - \bar{x}) \\ \vdots & & & & \vdots \\ (x_1 - \bar{x})^{n-1} & & & & (x_n - \bar{x})^{n-1} \end{bmatrix}}_{\text{Vander Monde!}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k! \\ \vdots \end{bmatrix}$$

Go through each

Ex: $\frac{dy}{dx} + 4y = 3y^3$

$$\Rightarrow \quad \frac{dy}{dx} = -4y + 3y^3 = (4+3y^2)y$$

Constant Solutions / Equilibrium Solution

Given an ODE

$$\frac{dy}{dx} = F(x,y)$$

What does this mean? No change in y!

$$\frac{dy}{dx} = 0 \Rightarrow \boxed{y(x) = C}$$

Ex: $\frac{dy}{dx} = 2xy^2 - 4xy$

$$= 2xy(y-2)$$

$y=0,\ y=2$

Ex: Malthusian vs Logistic model