Math 5620 Lecture Notes Day 17.

Elliptic PDEs.

We began with the 5 point stencil applied to

$$\Delta u = f \quad \text{on } [0,1] \times [0,1] \qquad (\text{Cartesian Product})$$

$$u\big|_{\partial\Omega} = g \quad \text{on } \partial\Omega \leftarrow \text{perimeter}$$

Thus is the following.

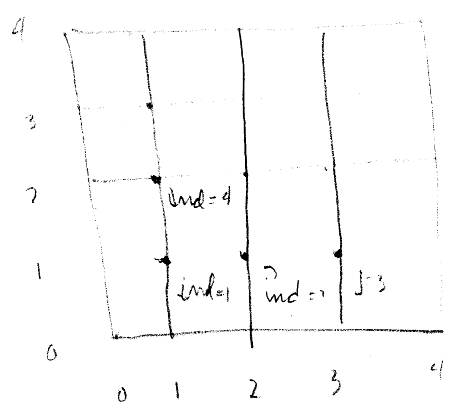
$$\Delta u \approx \frac{1}{h^2}\left(-4U_{i,j} + U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}\right) = f_{i,j}$$

$$\uparrow$$

$f(x,y)$ at (x_i, y_j)

So, we are in need of a way to compute solutions for this.

$$A^h = \frac{1}{h^2}\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ & & \end{bmatrix}$$



$\Rightarrow$

$$ind = i + (j-1) * m$$
$$= i + (j-1) * 3 \quad \underline{ex.}$$

$$ntot = m * m$$
$$\uparrow \quad \uparrow$$
$$x \quad y$$

From this we can transform back as:

$$i+(j-1)*m = \text{ind}$$

$\text{ind} = 4 \Rightarrow j = 1$

$\frac{\text{ind}}{3} = \frac{4}{3} \rightsquigarrow 1$

So in modular arithmetic we write

$$j = \text{ind}/m$$

then

$$\text{ind} = i + (j-1)*m$$

$$\Rightarrow i = \text{ind} - (j-1)*m$$

If these are executed in order we can compute (i,j) from ind.

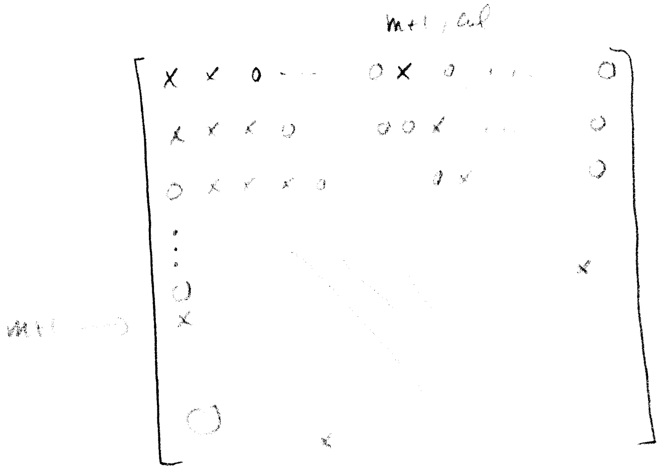
So, the system looks like

$$\begin{bmatrix} T & I & & 0 \\ I & T & I & \\ & I & & I \\ 0 & & I & T \end{bmatrix} = \text{block } m \times m$$

Ex: $m = 100 \Rightarrow$

$$A \in \mathbb{R}^{m^2 \times m^2}, \quad b \in \mathbb{R}^{n}$$

We could write a code to use Gaussian elim. But this will do the following.



As we eliminate below the main diagonal, the values between the main diagonal and the "out-rigger" diagonal "fill-in"! So, even though these values start out 0, they will be nonzero below and there must be storage for these

If we decide to LU, we will need to provide storage for both L & U!

Alternatives

1. Iterative solution methods
2. Approximations
   - ILU / MILU
   - neglect the terms that "fill in"
     - $\Rightarrow$ they should be small to neglect

Best to use Iterative Systems - Before this we probably need to know about convergence

Accuracy: Defn τ_{ij} = LTE

5pt - Stencil : $\tau_{ij} = \frac{1}{12} h^2 (u_{xxxx} + u_{yyyy}) + O(h^4)$

$\Rightarrow$ All this is easily extracted using 1-d results

Stability: Test eigenvalue

$$\lambda_{1,1} = -2\pi^2 + O(h^2) \qquad \lambda_{m,m} = -8/h^2$$

$$\Rightarrow K_2(A) = \|A\|_2 \|A^{-1}\|_2$$

$$\Rightarrow K_2(A) = \frac{8}{h^2}\cdot\frac{1}{2\pi^2} \to O(1/h^2)$$

As $h \to 0 \Rightarrow K_2(A) \longrightarrow +\infty$ bad!

9 point Stencil

$$\begin{matrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{matrix} \cdot \frac{1}{h^2}$$

Structure of matrix:

$\frac{1}{h^2}$

-20 4 0 0 ... 0 4 1 0 0 ... 0
4 -20 4 0 ... 1 4 1 0 ...
4 -20 4 0 ... 0 1 4 1 0
3 bands
4 1
1 4

These schemes are symmetric!

Iterative Schemes for Linear Systems

Jacobi Iteration

$$\underline{u^{(k+1)} = u^{(k)} + D^{-1} r^{(k)}}$$

Let's build a code.

Gauss-Seidel – uses values already computed

$$u^{(k+1)} = (\quad)^{-1}$$

解 $Ax=b$

$(L+D+U)x=b$

$(D+U)x = b-Lx$

$x = (D+U)^{-1}(b-Lx)$

$= (D+U)^{-1}\left(b-(L+D+U-(D+U))x\right)$

$= (D+U)^{-1}\left\{(b-Ax) + (D+U)x\right\}$

$= (D+U)^{-1}\underline{\underline{r}} + Ix$

$\Rightarrow \quad x^{(k+1)} = x^{(k)} + (D+U)^{-1}r^{(k)}$

$r^{k} = b - Ax^{(k)}$