Let's start into a simple test problem. For this problem we should derive at least one difference quotient for the second derivative.

Ex $$u''(\bar{x}) = \frac{u(\bar{x}+h) - 2u(\bar{x}) + u(\bar{x}-h)}{h^2} + \underbrace{\text{error}}_{O(h^p)}$$

$$\left| u''(\bar{x}) - \frac{1}{h^2}\left\{ \left(\cancel{u(\bar{x})} + \cancel{h u'(\bar{x})} + \frac{h^2}{2}u''(\bar{x}) + \cancel{\frac{h^3}{6}u'''(\bar{x})} + \frac{h^4}{24}u^{iv}(\bar{x}) + \cdots\right) \right.\right.$$
$$- 2u(\bar{x})$$
$$\left.\left. + \left(\cancel{u(\bar{x})} - \cancel{h u'(\bar{x})} + \frac{h^2}{2}u''(\bar{x}) - \cancel{\frac{h^3}{6}u'''(\bar{x})} + \frac{h^4}{24}u''''(\bar{x}) + \cdots\right)\right\} \right|$$

$$= \left| u''(\bar{x}) - \frac{1}{h^2}\left\{ h^2 u''(\bar{x}) + \frac{h^4}{12}u''''(\bar{x}) + \cdots \right\} \right|$$

$$= \left| \frac{h^2}{12} f''''(\xi) \right|$$

$$= O(h^2)$$

Now, let's head into the problems we face.

Heat Equation:

$$u_t(x,t) = K u_{xx}(x,t) + \psi(x,t)$$

and we will impose initial conditions of the form

$$u(x,0) = u_0(x)$$

with boundary conditions

$$u(0,t) = \alpha, \quad u(L,t) = \beta$$

The idea is that in many problems we will see a transient behavior in problems. However the system will be expected to evolve into a steady state or equilibrium. This would imply

$$\frac{\partial u}{\partial t} = 0 \Rightarrow 0 = k\, u_{xx} + \psi$$

$$\Rightarrow u_{xx} = \frac{1}{k}\psi$$

for $u = u(x)$. So we can write

$$u''(x) = f(x)$$

where $f(x) = \frac{1}{k}\psi(x)$. The BCs can be translated to

$$u(0) = \alpha \qquad \Rightarrow [a,b] \to [0,1]$$
$$u(1) = \beta$$

This is a 2pt BVP.

Now, suppose we want to approximate the solution at discrete points in the domain of u. This means we will look for values $U_0, U_1, \ldots, U_m, U_{m+1}$ where $U_j \approx u(x_j)$. We have

1. $x_j = j * h$
2. $h = 1/(m+1)$
3. $U_0 = \alpha$, $U_{m+1} = \beta \Rightarrow$ the solution is exact on the boundary $x = 0, x = 1$

We can substitute

$$\text{1) } D^2 U_j = \frac{1}{h^2}(U_{j-1} - 2U_j + U_{j+1})$$

Then

$$\frac{1}{h^2}\left(U_{j-1} - 2U_j + U_{j+1}\right) = f(x_j)$$

for $j = 1, 2, \ldots, m$

So,

known

$j=0$, $\quad U_0 = \alpha$

$j=1$, $\quad \dfrac{U_0 - 2U_1 + U_2}{h^2} = f(x_1)$

$j=2$ $\quad \dfrac{U_1 - 2U_2 + U_3}{h^2} = f(x_2)$

$\vdots$

$j=m$ $\quad \dfrac{U_{m-1} - 2U_m + U_{m+1}}{h^2} = f(x_m)$

$j=m+1$ $\quad U_{m+1} = \beta.$

We will resume the matrix form for this

$$\frac{1}{h^2}\begin{bmatrix} 1 & 0 & 0 & & & & 0 \\ 1 & -2 & 1 & 0 & & & \\ 0 & 1 & -2 & 1 & 0 & & 0 \\ & 0 & 1 & -2 & 1 & 0 & \\ & & & \ddots & & & \\ & & & & & 0 & 1 \\ 0 & \cdots & & & & & \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_{m-1} \\ U_m \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} \alpha \\ f(x_1) \\ f(x_2) \\ \\ \\ f(x_m) \\ \beta \end{bmatrix}$$

or, we can eliminate U_0 and U_{m+1} since they are known. If we set

$$U_0 = \alpha, \quad U_{m+1} = \beta$$

then

The equation becomes

$$\begin{bmatrix} \frac{\alpha}{h^2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\beta}{h^2} \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & & 1 \\ & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \\ \\ u_m \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \\ f(x_m) \end{bmatrix}$$

$$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & & 1 \\ & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ \\ \\ u_m \end{bmatrix} = \begin{bmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{bmatrix}$$

$$\Rightarrow A U = F$$

What we want to do is figure out how much error exists. We know that approximating errors in u'' by D^2 makes some error and we know to analyze that with Taylor series. This is more complicated than what we have done.

Suppose we define

$$\hat{U} = \begin{bmatrix} u(x_1) \\ \\ \\ u(x_m) \end{bmatrix} = \text{vector of exact values}$$

then define

$$E = U - \hat{U} = \text{signed error of components}$$

Then what will be the measure. We use $|E|$ before but this was for scalar values

We will use the norm of the vector as a measure. There are a number of errors we can choose

1. $\|E\|_\infty = \max_{1\le j\le m} |E_j| = \max_{1\le j\le m} |U_j - u(x_j)|$ = ∞- norm or sup norm

2. $\|E\|_1 = h \cdot \sum_{j=1}^{m} |E_j|$ = 1-norm of error

hmm? ← we must have this to scale errors

3. $\|E\|_2 = \left(h \sum_{j=1}^{m} |E_j|^2 \right)^{1/2}$

If any of these is small, they will all be small.

Appendix A.

Errors in vectors will use norms for vectors

Def: A vector norm for $v \in \mathbb{R}^m$ is a function $\|v\| \in \mathbb{R}$ where

1. $\|x\| \ge 0$ for any $x \in \mathbb{R}^m$ and $\|v\| = 0$ iff $x = 0$.
2. If a is any scalar (number), then $\|ax\| = |a| \, \|x\|$
3. If x and y are vectors, then $\|x+y\| \le \|x\| + \|y\|$.

The definitions we will use are given above.

Important: Norm Equivalence

Question: Suppose we have

$$\|error\| \le C h^p$$

in one norm. Can we find a different norm that gives a different value of p? The answer is No!

Norms defined on any vector space that is finite dimensional are equivalent. So for $\|x\|_a$ and $\|x\|_b$ on the same vector space there exist positive constants C_1 and C_2 such that

$$C_1 \|x\|_a \le \|x\|_b \le C_2 \|x\|_a$$

Ex:

$$\|x\|_\infty \le \|x\|_1 \le m\|x\|_\infty$$

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{m}\, \|x\|_\infty$$

$$\|x\|_2 \le \|x\|_1 \le \sqrt{m}\, \|x\|_2$$

For our case we will use

$$\|e(h)\|_\beta \le C_2 \|e(h)\|_\alpha \le C_2 C h^p = O(h^p)$$

~ mesh size or mesh width

Errors in Functions: $U(x) - u(x)$

1. $\|e\|_1 = \int_a^b |e(x)|\,dx$
2. $\|e\|_\infty = \max_{a \le x \le b} |e(x)|$
3. $\|e\|_2 = \left(\int_a^b |e(x)|^2\,dx\right)^{1/2}$
4. $\|e\|_q = \left(\int_a^b |e(x)|^q\,dx\right)^{1/q}$