Math 5620 Lecture Notes : Day 10 ①

Summary:

LTE - Local Truncation Error

τ_j = the error that occurs when the exact solution $u(x)$ is substituted into the difference equation

for $j = 1, 2, \ldots, m$. For a given "discretization" we write

$$AU = F \Rightarrow AU - F = 0$$

and for LTE

$$A\hat{U} = F + \tau \Rightarrow A\hat{U} - F = \tau$$

<u>Subtract:</u>

$$(AU - F) - (A\hat{U} - F) = -\tau$$

$$\Rightarrow A(U - \hat{U}) = -\tau$$

<u>Global Error:</u>

$$\Rightarrow AE = -\tau$$

$$\Rightarrow E = -A^{-1}\tau$$

$$\Rightarrow \|E\| = \|A^{-1}\tau\|$$

$$\Rightarrow \|E\| \leq \|A^{-1}\| \underbrace{\|\tau\|}_{O(h^2)}$$

$$\Rightarrow \|E\| \leq \underbrace{\|A^{-1}\|} \cdot \|\tau\|$$

If this is bounded away from zero $\Rightarrow$ $\|E\| \leq Ch^2 \Rightarrow \|E\| \leq O(h^2)$

What we want is the error to go to zero as $h \to 0$, but more. If

$$A^h E^h = -\tau^h \Rightarrow E^h = -(A^h)^{-1}\tau^h$$

So

$$\|E^h\| \le \|(A^h)^{-1}\| \cdot \|\tau^h\|$$

as $h \to 0$ we need

$$\|(A^h)^{-1}\| \le C$$

where C does not depend on h.

Ex:

$$\underbrace{\frac{1}{h^2}}_{\text{harm}}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & 0 \\ & & \ddots & \\ & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} f_1 - \alpha/h^2 \\ \vdots \\ f_m - \beta/h^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = h^2\begin{bmatrix} f_1 - \alpha/h^2 \\ \\ \\ f_m - \beta/h^2 \end{bmatrix}$$

Note: A is invertible.

<u>Properties</u>:

Consistency — $\|\tau^h\| \to 0$ as $h \to 0$ $\Rightarrow$ if $\|\tau^h\| \le Ch^p$ — consistency

Stability $\|(A^h)^{-1}\| \le C$

Consistency + Stability $\Rightarrow$ Convergence

So, we need a

So, LTE can be small as long as $\|t\| \le C h^p$

$\Rightarrow$ Taylor series.

Aside: FE & FV methods use function representation

$$\begin{cases} u'' = f \\ u(0) = \alpha \\ u(1) = \beta \end{cases}$$

FEM. Assume

$$u(x) = \sum_{j=1}^{N} a_j \varphi_j(x)$$

- linear combination of basis function

i) φ_j = piecewise linear

ii) φ_j = trig / Fourier series.

LeVeque opts for the use of induced norms (page 250)

$$\|Ax\| \le C\|x\|$$

$$\Rightarrow \frac{\|Ax\|}{\|x\|} \le C$$

$$\Rightarrow \|A\| = \max_{\substack{x \in \mathbb{R}^m \\ x \neq 0}} \frac{\|Ax\|}{\|x\|} = \max_{\substack{x \in \mathbb{R}^m \\ \|x\| = 1}} \|Ax\|$$

So

Frobenius Norm:

$$\|A\| = \left(\sum_{j=1}^{m} \sum_{i=1}^{m} |a_{ij}|^2 \right)^{1/2} \qquad \text{Not consistent}$$

Special Cases

$$\|A\|_1 = \max_{1 \le j \le m} \sum_{i=1}^{m} |a_{ij}| \quad \text{– max. column sum}$$

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{m} |a_{ij}| \quad \text{– max row sum}$$

$$\|A\|_2 = \sqrt{\rho(A^T A)} \qquad \rho \text{ – spectral radius}$$

Condition #

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|$$

Back to stability in the 2-norm:

$$\|A^{-1}\|_2 = \rho(A^{-1}) = \max_{1 \le p \le m} |\lambda_p^{-1}| = \left(\min_{1 \le p \le m} |\lambda_p| \right)^{-1}$$

We just need to compute eigenvalues. The eigenvalues of A are

$$\lambda_p = \frac{2}{h^2}\left(\cos(p\pi h) - 1\right) \qquad p = 1, \ldots, m$$

We will prove the eigenvalues are these as follows...