Math 5620 Lecture Notes: Day 12.

Last time, we discussed several analogous problems

1. Constant Coefficient ODE

$$ay'' + by' + cy = 0, \quad a, b, c, \text{ constants}$$

Assume $y = e^{rx}$

$$\Rightarrow ar^2 + br + c = 0 \Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1: $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$ $\quad r_1 \neq r_2$

Case 2: $y_1 = e^{rx}$, $y_2 = xe^{rx}$ $\quad r_1 = r_2$

Case 3: $y_1 = e^{ax}\cos(bx)$, $y_2 = e^{ax}\sin(bx)$ $\quad r = a \pm ib$

2. As another example

$$ax^2y'' + bxy' + cy = 0 \qquad a, b, c, \text{ constant}$$

Assume $y = x^r$

$$\Rightarrow ax^2(r(r-1)x^{r-2}) + bx(rx^{r-1}) + cx^r = 0$$

$$\Rightarrow x^r(a(r(r-1)) + br + c) = 0 \qquad x \neq 0$$

$$\Rightarrow \underline{ar^2 + (b-a)r + c = 0}$$

indicial equation

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

$\vdots$

Note that this same idea can be used in PDEs!

So, suppose we have a difference formula for an equation of the form

$$\frac{du}{dt} = au \rightsquigarrow \frac{u(t+h) - u(t)}{h} = a \cdot u(t) + \text{error}$$

$$\Rightarrow \quad u(t+h) \doteq u(t) + a\,u(t) \cdot h$$

We have seen how this can be used to define $t_0, t_1 = t_0 + h, t_2 = t_0 + 2h, \ldots$

$$\Rightarrow \{t_0, t_1, \ldots, t_m, t_{m+1}\}$$

and $\quad U_0 =$ initial condition

with $\quad U_1 = U_0 + ah\,U_0 \qquad U_2 = U_1 + ah\,U_1$

$$\vdots$$

$$U_{k+1} = U_k + ah\,U_k = (1 + ah)\,U_k$$

$$\vdots$$

$$U_{m+1} = (1 + ah)\,U_m$$

Let's rewrite just a bit.

$$U_0 = \alpha$$

$$U_1 = (1 + ah)\,U_0 = (1 + ah)\,\alpha$$

$$U_2 = (1 + ah)\,U_1 = (1 + ah)(1 + ah)\,\alpha = (1 + ah)^2\,\alpha$$

$$\vdots$$

$$U_k = (1 + ah)^k\,U_0$$

$$\vdots$$

$$\boxed{U_{m+1} = (1 + ah)^{m+1}\,U_0}$$

What this means we can write

$$U_j = (1+\alpha h)^j \cdot \alpha$$

$$= \alpha\,(1+\alpha h)^j$$

$$= \alpha \cdot z^j$$

This gives the exact solution of the "discrete" equation

Another example: For,

$$a U_{j+1} + b U_j + c U_{j-1} = 0$$

Assume $U_j = z^j$ as per the last example:

$$a z^{j+1} + b z^j + c z^{j-1} = 0$$

$$\Rightarrow z^{j-1}\left(a z^2 + b z + c\right) = 0$$

$$\Rightarrow a z^2 + b z + c = 0 \quad \Rightarrow \quad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This gives two possible choices

$$z_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \Rightarrow U_j = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)^j$$

OR

$$z_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \Rightarrow U_j = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)^j$$

$$\Rightarrow U_j = \boxed{B z_1^j + C z_2^j}$$

general solution

The problem at hand is

$$\begin{cases} c v_{j-1} + (a - \lambda) v_j + b v_{j+1} \\ v_0 = v_{m+1} = 0 \end{cases}$$

This is analogous to the two pt. BVP for continuous problems

So, we will assume a solution for given by.

$$v_j = B z_1^j + C z_2^j$$

where B and C are arbitrary constants the roots are from

$$c + (a-\lambda)z + bz^2 = 0$$

We will show that the roots are distinct (a little later). We can use

$$v_0 = 0 \Rightarrow 0 = B z_1^0 + C z_2^0 = B + C$$

$$v_{m+1} = 0 \Rightarrow 0 = B z_1^{m+1} + C z_2^{m+1}$$

So, $C = -B$ and

$$v_j = B z_1^j - B z_2^j = B(z_1^j - z_2^j)$$

$$v_{m+1} = B(z_1^{m+1} - z_2^{m+1}) = 0 \Rightarrow z_1^{m+1} - z_2^{m+1} = 0$$

$$\Rightarrow z_1^{m+1} = z_2^{m+1}$$

$$\Rightarrow \left(\frac{z_1}{z_2}\right)^{m+1} = 1 = e^{i(2s\pi)} \qquad s = 1, 2, \dots m$$

Now take the $(m+1)$ root.

$$\frac{z_1}{z_2} = e^{i\left(\frac{2s\pi}{m+1}\right)} \qquad s = 1, 2, \dots, m$$

Next we can write

$$c + (a-\lambda)z + bz^2 = 0$$

$$\Rightarrow \frac{c}{b} + \frac{(a-\lambda)}{b} z + z^2 = 0$$

note

$$z_1 \cdot z_2 = c/b$$

So, far we have gotten to:

$$\begin{cases} \dfrac{z_1}{z_2} = e^{i\left(\frac{2s\pi}{m+1}\right)}, \\ z_1 \cdot z_2 = c/b \implies z_2 = \dfrac{c}{b} \cdot z_1 \end{cases}$$

Elimination of z_2 gives

$$\frac{z_1}{\frac{c}{b} z_1^{-1}} = \frac{z_1^2 b}{c} = e^{i\left(\frac{2s\pi}{m+1}\right)}$$

$$\implies z_1^2 = \frac{c}{b} e^{i\left(\frac{2s\pi}{m+1}\right)};$$

$$\implies z_1 = (c/b)^{1/2} e^{i(s\pi/m+1)}$$

$$z_2 = (c/b)^{1/2} e^{-is\pi/m+1} \longleftarrow z_1 \cdot z_2 = c/b$$

Now from the characteristic polynomial

$$z_1 + z_2 = \frac{(\lambda - a)}{b}$$

Solve for λ.

$$\begin{aligned} \lambda &= b(z_1 + z_2) + a \\ &= a + b\left(\left(\tfrac{c}{b}\right)^{1/2} e^{is\pi/m+1} + \left(\tfrac{c}{b}\right)^{1/2} e^{-is\pi/m+1}\right. \\ &= a + b \cdot (c/b)^{1/2} \left(e^{is\pi/m+1} + e^{-is\pi/m+1}\right) \\ &= a + 2b(c/b)^{1/2} \cos(s\pi/m+1) \end{aligned}$$

That is a lot of work!

$$v_j = B z_1^j + C z_2^j$$

$$= B\left(\frac{c}{b}\right)^{\frac{1}{2}j}\left(e^{ij s\pi/m+1} - e^{-ij s\pi/(m+1)}\right)$$

$$= B(c/b)^{\frac{1}{2}j}(2i)\left(\frac{e^{ij s\pi/m+1} - e^{-ij s\pi/m+1}}{2i}\right)$$

$$= 2iB\left(\frac{c}{b}\right)^{\frac{1}{2}j} \sin\left(sj\pi/m+1\right)$$

These are the components in the eigenvector associated w/ λ.

$$v_s^\lambda : \left\{\left(\frac{c}{b}\right)^{\frac{1}{2}} \sin\left(s\pi/m+1\right), \frac{c}{b}\sin\left(\frac{2s\pi}{m+1}\right), \left(\frac{c}{b}\right)^{3/2} \sin\left(\frac{3s\pi}{m+1}\right), \ldots, \left(\frac{c}{b}\right)^{m/2} \sin\left(\frac{ms\pi}{m+1}\right)\right.$$

So, ... that's enough of that!

So, back to our simple problem

$$\begin{cases} u'' = f \\ u(0) = \alpha \\ u(1) = \beta \end{cases} \Rightarrow AU = F$$

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & & 0 \\ & & \ddots & \\ & 0 & & -2 \end{bmatrix}$$

$a = -2, \quad b = 1, \quad c = 1$ ✓