Math 5620 Lecture Notes: Day 9

We are at a point where we need to solve a tridiagonal system of equations.

$$A\bar{U} = F \Rightarrow \bar{U} = \text{"}A^{-1}\text{"}F$$

The two ways are (1) a direct method (Gaussian Elim.) or an iterative method (Jacobi - iteration). As you are working towards solving the linear systems we want to determine whether or not the approximation of the linear systems provide an accurate approximation.

Using any of the errors we want to show that the results converge to the exact solution of the BVP. To establish Convergence we can use

- local truncation error (LTE) $\Rightarrow$ convergence.
- stability - we need a definition

Local Truncation Error: Given a finite difference approximation of the form

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j)$$

$j = 1, 2, \ldots, m$ with

$U_0 = \alpha$

$U_{m+1} = \beta$

the LTE is obtained by replacing the values U_j with $u(x_j)$

We know that $u(x_j)$ will not likely satisfy the discrete equation. We write

$$\tau_j = \frac{1}{h^2}\left(u(x_{j-1}) - 2u(x_j) + u(x_{j+1})\right) - f(x_j)$$

for $j = 1, 2, \dots, m$. In practice, we don't know $u(x_j)$, but if we assume that u is "smooth" enough to expand the function in Taylor series

$$\tau_j = \left[u''(x_j) + \frac{1}{12}h^2 u''''(x_j) + O(h^4)\right] - f(x_j)$$

The original ODE is

$$u''(x_j) = f(x_j) \Rightarrow u''(x_j) - f(x_j) = 0$$

$$\Rightarrow \quad \tau_j = \left[\frac{1}{12}h^2 u''''(x_j) + O(h^4)\right]$$

If $u''''(x)$ is not known it is independent of h. So $\tau_j = O(h^2)$ as $h \to 0$

Def

$$\tau = A\hat{u} - F = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_m \end{bmatrix}$$

where $\hat{u}$ is the vector of true solution and so

$$A\hat{u} = F + \tau$$

Global Error: To obtain the global error from τ we write

$$E = \underset{\text{approx}}{U} - \underset{\text{exact at } x_j}{\hat{U}}$$

Then we have

$$A\hat{u} - F = \tau$$

and
$$A\mathcal{U} - F = 0$$

Subtracting gives

$$E = \mathcal{U} - \hat{U} = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_m \end{bmatrix}$$

and

$$(A\mathcal{U} - \cancel{F}) - (A\hat{u} - \cancel{F}) = -\tau$$

$$\Rightarrow A(\mathcal{U} - \hat{u}) = -\tau$$

We can write this out as

$$\frac{1}{h^2}(E_{j-1} - 2E_j + E_{j+1}) = -\tau_j$$

for $j = 1, 2, \dots, m$ with $\underbrace{E_0 = E_{m+1} = 0}_{BCs}$

From this it will be easy to see why, if the LTE is $O(h^2)$ we expect that the global error is also $O(h^2)$.

$$\Rightarrow \begin{cases} e''(x) = -\tau(x) & \text{on } x \in (0,1) \\ e(0) = 0 \\ e(1) = 0 \end{cases}$$

So, $AE = -\tau$ is a discretization of the above ODE and since

$$\tau(x) \approx \tfrac{1}{12} h^2 u''''(x)$$

integrating twice gives

$$e(x) \approx -\tfrac{1}{12} h^2 u'' + \tfrac{1}{12} h^2 \left(u''(0) + x(u''(1) - u''(0)) \right)$$

$$O(h^2) + O(h^2) = O(h^2)$$

Stability: The Global Error analysis assumes the solution of the difference equation gives a decent approximation to the solution of the differential equation. In the above we have actually assumed that the solution of the ODE provides a decent approximation of the difference equation. — We cannot assume this.

Instead we can look directly at the discrete system for a given h.

$$A^h E^h = -\tau^h \qquad h = \text{mesh width}$$

Note $A^h \in \mathbb{R}^{m \times m}$ with $h = 1/m+1$. If we denote $(A^h)^{-1}$ as the inverse of A^h, then

$$E^h = -(A^h)^{-1} \tau^h$$

and

$$\| E^h \| \le \| (A^h)^{-1} \tau^h \|$$
$$\le \| (A^h)^{-1} \| \cdot \| \tau^h \|$$

Note: What does

$$\|(A^h)^{-1}\tau^h\|$$

mean, vs

$$\|(A^h)^{-1}\| \, \|\tau_h\|$$

mean?

For our purposes:

$$\|\tau_h\| \le C h^2$$

we want $\|E^h\| \le C h^2$

to match. This is what we need if we want the global error to be $O(h^2)$ when the local truncation error (LTE) is $O(h^2)$

Def: Suppose a finite difference method for a linear BVP gives a sequence of matrix equations of the form $A^h U^h = F^h$ where h is the mesh width. We say the method is stable if $(A^h)^{-1}$ exists for all h sufficiently small (for $h < h_0$) and if there is a constant $C > 0$ independent of h, such that

$$\|(A^h)^{-1}\| \le C$$

for all $h < h_0$.

Main results Consistency

Def: A finite difference method is said to be **consistent** with a BVP

if

$$\|\tau^h\| \to 0$$

as $h \to 0$

$$\Rightarrow \|\tau^h\| = O(h^p)$$

for $p > 0$

2.9 Convergence.

A method is said to be convergent if $\|E^h\| \to 0$ as $h \to 0$.

Combining the ideas given

Consistency + stability $\Longrightarrow$ Convergence

$$\Rightarrow \|E^h\| \le \|(A^h)^{-1}\tau^h\|$$

$$\le \|(A^h)^{-1}\| \cdot \|\tau^h\|$$

$$\le C\|\tau^h\|$$

as $h \to 0$.

$\Rightarrow$ Method is $O(h^p)$ + stability ($\|(A^h)^{-1}\| \le C$)

$\Rightarrow$ Convergence