Math 5620 Lecture Notes: Day 18

So, for a 5-point stencil

$$A^h = \begin{bmatrix} -4 & 1 & & 1 & & \\ 1 & -4 & \ddots & & \ddots & \\ & \ddots & \ddots & & 0 & \\ 1 & & & \ddots & & \\ & \ddots & & & \ddots & 1 \\ & & 1 & & 1 & -4 \end{bmatrix}$$

Our choices are:

1. Direct Methods
   - Gaussian Elimination
   - LU factorization

2. Iterative Methods
   - Jacobi Iteration
   - Gauss-Seidel
   - SOR
   - Other matrix splitting methods
   - Gradient based method
   - Descent Methods
   - Conjugate Gradient

Then we can consider "preconditioning"

Ex: (5 point stencil)

$$-4u_{i,j} + u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - f_{i,j} = 0$$

$$\Rightarrow \quad 4u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - f_{i,j}\left(h^2/4\right)$$

This suggests an explicit iteration formula.

$$u_{i,j}^{(k+1)} = \frac{1}{4}\underbrace{\left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)}\right)}_{\text{needs an initial approximation}} - \frac{h^2}{4} f_{i,j}$$

```
for i in range(m+1)
    for j in range(m+1)
        u_ij = 0.25 * (u_i-1,j + u_i+1,j + u_i,j-1 + u_i,j+1) - h*h * f_ij
```

iterate.

This way of generating iteration schemes is not efficient. So, lets look at matrix splitting methods. For

$$Ax = b$$

we define the splitting

$$A = M - N$$

Both M, N are mxm matrices

Then

$$Mu - Nu = f$$

$$\Rightarrow Mu = Nu + f$$

This suggest that given $u^{(0)}$ we can define a sequence of approximations to u by

$$Mu^{(k+1)} = Nu^{(k)} + f$$

$$\Rightarrow u^{(k+1)} = M^{-1}Nu^{(k)} + M^{-1}f$$

Ex. For the system

$$Au = f \qquad \rightarrow \qquad A = (-L + D - U)$$

Choose $\quad M = D, \quad N = L + D$

We should at least know that there are some other stencils / methods

9-point stencil

$$\frac{1}{h^2} \cdot \begin{matrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{matrix} \cdot u$$

We can treat this as follows ...

A.

LTE:

$$-\tau_{ij} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} - f_{ij}$$

Taylor Series:

$$u_{i+1,j} = u_{ij} + h(u_x)_{ij} + \tfrac{1}{2}h^2(u_{xx})_{ij} + \tfrac{1}{6}h^3(u_{xxx})_{ij} + \tfrac{1}{24}h^4(u_{xxxx})_{ij} + \cdots$$

$$u_{i-1,j} = u_{ij} - h(u_x)_{ij} + \tfrac{1}{2}h^2(u_{xx})_{ij} - \tfrac{1}{6}h^3(u_{xxx})_{ij} + \tfrac{1}{24}h^4(u_{xxxx})_{ij} + \cdots$$

$$u_{i,j+1} = u_{ij} + h(u_x)_{ij} + \tfrac{1}{2}h^2(u_{xx})_{ij} + \tfrac{1}{6}h^3(u_{xxx})_{ij} + \tfrac{1}{24}h^4(u_{xxxx})_{ij} + \cdots$$

$$u_{i,j-1} = u_{ij} - h(u_x)_{ij} + \tfrac{1}{2}h^2(u_{xx})_{ij} - \tfrac{1}{6}h^3(u_{xxx})_{ij} + \tfrac{1}{24}h^4(u_{xxxx})_{ij} + \cdots$$

$$u_{ij} = u_{ij}$$

So

$$-\tau_{ij} = \Big((4-4)u_{ij} + h\cdot(1-1+0+1-1)(u_x)_{ij} + \tfrac{1}{2}h^2(\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2})(u_{xx})_{ij} + \tfrac{1}{6}h^3(1-1+1-1)(u_{xxx})_{ij} + \tfrac{1}{24}h^4(1+1+1+1)(u_{xxxx})_{ij}\Big)\tfrac{1}{h^2} - f_{ij}$$

$$= (u_{xx})_{ij} + O(h^4)\cdot\frac{1}{h^2} - \cancel{f_{ij}}$$

$$= O(h^2)$$

So, $|\tau_{ij}| \to 0$ as $h \to 0$.

Stability requires the spectral radius of A^h as before since

$$A^h E^h = -\tau^h$$

$$\Rightarrow E^h = \text{global error} = -\text{"}(A^h)^{-1}\text{"} \cdot \tau^h$$

So,

$$\|E^n\| = \|(A^h)^{-1} \cdot \tau^n\|$$

$$\leq \|(A^h)^{-1}\| \cdot \|\tau_n\|$$

With similar, but more tedious work, we find the components of the eigenvectors to be

$$u_{i,j}^{p,q} = \sin(p\pi i h)\sin(q\pi j h)$$ ← due to Fourier analysis.

$$\Rightarrow \lambda_{p,q} = \frac{2}{h^2}\left((\cos(p\pi h) - 1) + (\cos(q\pi h) - 1)\right)$$

The closest to the origin is

$$\lambda_{1,1} = -2\pi^2 + O(h^2)$$

So, in the 2-norm

$$\rho((A^h)^{-1}) = 1/\lambda_{1,1} \approx -1/2\pi^2$$

So, the matrix / method is stable ⇒ theoretically we should expect a limit exists as $h \to 0$.

The condition # of A^h is

$$K_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

Since $\lambda_{m,m} \approx -8/h^2$

$$\Rightarrow K_2(A^h) \approx 4/(\pi^2 h^2) = O(1/h^2)$$

as $h \to 0$. So $K_2(A^h) \to +\infty$ which is very bad