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NOTES ON THE EIGENSYSTEM OF MAGNETOHYDRODYNAMICS*

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Abstract. The eigenstructure of the equations governing one-dimensional ideal magnetohydrodynamics is examined, motivated by the wish to exploit it for construction of high-resolution computational algorithms. The results are given in simple forms that avoid indeterminacy or degeneracy whenever possible. The unavoidable indeterminacy near the magnetosonic (or triple umbilic) state is analysed and shown to cause no difficulty in evaluating a numerical flux function. The structure of wave paths close to this singularity is obtained, and simple expressions are presented for the structure coefficients that govern wave steepening.

Key words. magnetohydrodynamics, eigenvectors, numerical flux function

AMS subject classifications. 35L65, 35L67, 35P10, 65M06, 76W05

1. Introduction. For any system of hyperbolic partial differential equations, expressed as

$$V_t + AV_x = 0, \tag{1}$$

with A a diagonalisable matrix, the eigensystem of A plays a prominent role. The eigenvalues of A are the wavespeeds, the right eigenvectors define the paths taken in phase space by simple waves, and the left eigenvectors define the characteristic equations.

For the system of compressible magnetohydrodynamics (referred to henceforward simply as MHD) the eigensystem was given by Jeffrey and Tanuiti [1] and is also available in many subsequent works. However, the form in which the eigenvectors were originally given is singular or indeterminate in a number of special cases. Brio and Wu [2] employed a variety of identities that occur in the algebra to provide alternative expressions that are well formed except close to the "triple umbilic," where the fast, slow, and Alfvén speeds coincide. These revised eigenvectors demonstrated that the MHD equations are always diagonalisable although not everywhere convex. Brio and Wu also created from their analysis a computational scheme based on solving linearized Riemann problems and have been followed in this by Zachary and Colella [3] and Woodward and Dai [4].

Balsara [5] (see also references therein) has compared the quality of numerical solutions based on a total variation diminishing interpolation of the field variables (see Harten [6]) with interpolation of the primitive variables and finds that the first procedure is noticeably superior. This stresses again the importance of the eigenstructure for numerical simulations. Bell, Colella, and Trangenstein [7] propose a strategy for computing fluxes due to nonconvex waves by first solving linearised Riemann problems and then inserting nonlinear structure. An understanding of the degeneracies in the linear problem is essential to assessing this approach.

In § 2 of this note, we further refine the analysis in [2]–[4], leading to additional simplifications, which permit a clarification of the local eigenstructure close to the

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triple umbilic T. In § 3 we ascertain the wave paths near T and relate them to the general classification of quadratic degeneracies in [8]. We then consider in § 4 the case where a Riemann problem is solved by linearising about a state S close to T and find that although the jumps across individual waves are then sensitive to the choice of S, the computed flux is not. Studies of the acceleration of MHD winds [9]–[13] often deal with states very close to T. The present analysis should increase confidence in the reliability of linearised Riemann solvers in such situations. In § 5, we give simple expressions for the "structure coefficients" that govern nonlinear wave steepening. Section 6 contains a remark on the relevance of one-dimensional analysis to higher-dimensional calculations.

2. Eigenstructure. The governing equations of MHD, written in the form (1), with the unknowns taken to be the primitive variables

$$V = (\rho, v_x, v_y, v_z, B_y, B_z, p)^T$$

and properties varying only in the x-direction, give rise to a matrix

$$A = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & \frac{B_y}{4\pi\rho} & \frac{B_z}{4\pi\rho} & \frac{1}{\rho} \\ 0 & 0 & v_x & 0 & -\frac{B_x}{4\pi\rho} & 0 & 0 \\ 0 & 0 & 0 & v_x & 0 & -\frac{B_x}{4\pi\rho} & 0 \\ 0 & B_y & -B_x & 0 & v_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & v_x & 0 \\ 0 & \rho a^2 & 0 & 0 & 0 & 0 & v_x \end{bmatrix},$$

where a, the regular acoustic wavespeed, is given by

$$a^2 = \gamma p/\rho,$$

for an ideal gas and for other substances in thermodynamic equilibrium by

$$a^2 = (\partial p/\partial \rho)_s.$$

In what follows, any formula not explicitly containing γ refers to the general case. Because of the one-dimensional assumption, B_x is a constant.

We define

$$b_{x,y,z} = B_{x,y,z}/\sqrt{4\pi\rho}, \quad b^2 = b_x^2 + b_y^2 + b_z^2, \quad b_\perp^2 = b_y^2 + b_z^2. \tag{2}$$

As is well known, these equations admit seven different types of wave motion whose speeds are, in increasing order,

$$\lambda_{1,2,3,4,5,6,7} = v_x - c_f,\ v_x - b_x,\ v_x - c_s,\ v_x,\ v_x + c_s,\ v_x + b_x,\ v_x + c_f.$$

The analytically less tractable cases, on which we concentrate here, are those with speeds $\lambda = v_x \pm c_{f,s}$; these are the magnetoacoustic waves. From $\det(A - \lambda I) = 0$ it is found that c_f^2, c_s^2 are, respectively, the larger and smaller values of c^2 satisfying

$$c^4 - (a^2 + b^2)c^2 + a^2 b_x^2 = 0. \tag{3}$$

From (3) the following useful identities follow at once:

$$\begin{aligned} &c_f c_s = a|b_x|, && c_f^2 + c_s^2 = a^2 + b^2, \\ &c_{f,s}^4 - a^2 b_x^2 = c_{f,s}^2 (c_{f,s}^2 - c_{s,f}^2), && (c^2 - a^2)(c^2 - b_x^2) = c^2 b_\perp^2. \end{aligned} \tag{4}$$

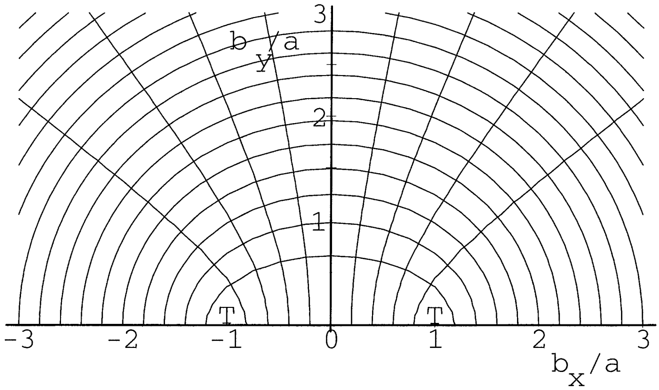

FIG. 1. *Contours of equal c_f (ellipses) and c_s (hyperbolae).*

It is also useful to note that c_f^2 is usually greater than either a^2 or b_x^2, and c_s^2 is usually less than either. There is, however, the possibility that $c_f^2 = c_s^2$ if $b_x^2 = a^2$ and $b_\perp = 0$. This is the "triple umbilic" case investigated in detail below.

We will later make use of a diagram whose coordinate axes represent $(b_x/a), (b_\perp/a)$. The loci of constant c_f, c_s in such a diagram are shown in Fig. 1. They are, respectively, confocal ellipses and hyperbolae, with foci lying at the triple points T.

The right and left eigenvectors of the fast magnetoacoustic waves can be found fairly straightforwardly as

$$r_f^\pm = \begin{pmatrix} \rho \\ \pm c_f \\ \pm\frac{-c_f}{c_f^2-b_x^2} b_x b_y \\ \pm\frac{-c_f}{c_f^2-b_x^2} b_x b_z \\ \frac{c_f^2}{c_f^2-b_x^2} b_y\sqrt{4\pi\rho} \\ \frac{c_f^2}{c_f^2-b_x^2} b_z\sqrt{4\pi\rho} \\ \rho a^2 \end{pmatrix}, \quad l_f^\pm = \begin{pmatrix} 0 \\ \pm c_f \\ \pm\frac{-c_f}{c_f^2-b_x^2} b_x b_y \\ \pm\frac{-c_f}{c_f^2-b_x^2} b_x b_z \\ \frac{c_f^2}{c_f^2-b_x^2} \frac{b_y}{\sqrt{4\pi\rho}} \\ \frac{c_f^2}{c_f^2-b_x^2} \frac{b_z}{\sqrt{4\pi\rho}} \\ 1/\rho \end{pmatrix}^T. \tag{5}$$

The eigenvectors of the slow magnetoacoustic waves can be obtained by replacing f with s throughout.

One may verify, using the identities (4), that the product $l_1 \cdot r_2$ vanishes if subscripts 1 and 2 refer to any distinct pair chosen from the four magnetoacoustic waves. When l and r refer to the same wave, we find, with c equal to c_f or c_s,

$$l \cdot r = \frac{2c^2(c_f^2 - c_s^2)}{|c^2 - b_x^2|}. \tag{6}$$

This expression is singular or indeterminate in a variety of cases. To obtain orthonormal eigenvectors we need to multiply the right and left eigenvectors by factors $k^{r,l}$ such that

$$k^r k^l = \frac{|c^2 - b_x^2|}{2c^2(c_f^2 - c_s^2)}.$$

The algebra is simplified by introducing positive parameters $\alpha_{f,s}$ defined by

$$\alpha_f^2 = \frac{a^2 - c_s^2}{c_f^2 - c_s^2}, \quad \alpha_s^2 = \frac{c_f^2 - a^2}{c_f^2 - c_s^2}. \tag{7}$$

Similar parameters were introduced by Brio and Wu [2] and utilised in [3] and [4].[1] In fact, α_s is defined here identically, but our α_f has an additional factor a/c_f. For our parameters, there is useful symmetry in the relationships

$$\alpha_s^2 + \alpha_f^2 = 1,$$

$$\alpha_s^2 c_s^2 + \alpha_f^2 c_f^2 = a^2,$$

and

$$\alpha_s \alpha_f = \frac{a b_\perp}{c_f^2 - c_s^2}.$$

Inspection of the eigenvectors below will reveal that the parameters $\alpha_{f,s}$ are in many senses a measure of how closely the fast/slow waves approximate the behaviour of acoustic waves. Thus, if $\alpha_f \simeq 1$, the fast wave is "almost an acoustic wave."

Appropriate normalisation factors turn out to be

$$k^r_{f,s} = \alpha_{f,s}, \quad k^l_{f,s} = \frac{\alpha_{f,s}}{2a^2}.$$

The a^2 has been placed in k^l to give r the same dimensions as V. Inserting these factors into (5) leads, after extensive use of the identities, to

$$(8) \qquad r_f^\pm = \begin{pmatrix} \alpha_f \rho \\ \pm \alpha_f c_f \\ \mp \alpha_s c_s \beta_y \mathrm{sgn} b_x \\ \mp \alpha_s c_s \beta_z \mathrm{sgn} b_x \\ \alpha_s \sqrt{4\pi\rho}\, a \beta_y \\ \alpha_s \sqrt{4\pi\rho}\, a \beta_z \\ \alpha_f \rho a^2 \end{pmatrix}, \quad l_f^\pm = \frac{1}{2a^2} \begin{pmatrix} 0 \\ \pm \alpha_f c_f \\ \mp \alpha_s c_s \beta_y \mathrm{sgn} b_x \\ \mp \alpha_s c_s \beta_z \mathrm{sgn} b_x \\ \alpha_s a \beta_y / \sqrt{4\pi\rho} \\ \alpha_s a \beta_z / \sqrt{4\pi\rho} \\ \alpha_f / \rho \end{pmatrix}^T,$$

$$(9) \qquad r_s^\pm = \begin{pmatrix} \alpha_s \rho \\ \pm \alpha_s c_s \\ \pm \alpha_f c_f \beta_y \mathrm{sgn} b_x \\ \pm \alpha_f c_f \beta_z \mathrm{sgn} b_x \\ -\alpha_f \sqrt{4\pi\rho}\, a \beta_y \\ -\alpha_f \sqrt{4\pi\rho}\, a \beta_z \\ \alpha_s \rho a^2 \end{pmatrix}, \quad l_s^\pm = \frac{1}{2a^2} \begin{pmatrix} 0 \\ \pm \alpha_s c_s \\ \pm \alpha_f c_f \beta_y \mathrm{sgn} b_x \\ \pm \alpha_f c_f \beta_z \mathrm{sgn} b_x \\ -\alpha_f a \beta_y / \sqrt{4\pi\rho} \\ -\alpha_f a \beta_z / \sqrt{4\pi\rho} \\ \alpha_s / \rho \end{pmatrix}^T.$$

Here we follow [2] in writing

$$\beta_y = \frac{b_y}{b_\perp}, \qquad \beta_z = \frac{b_z}{b_\perp}.$$

These quantities are indeterminate if $b_\perp \simeq 0$. It is suggested in [2] that the values they then assume may be given arbitrarily; for example, $\beta_y = \beta_z = 1/\sqrt{2}$. This preserves the orthonormality of the eigenvectors.

To make this paper self-contained we now list the six cases, indicated in Fig. 2, that can potentially cause trouble. In four of these the reader can easily verify that our expressions for the eigenvectors remain well formed and linearly independent.

[1] In the expression for l_s given in [3], the last component should be $\tau\alpha_s$. Also the normalising factors $R^\pm_{f,s}$ given there can be simplified considerably.

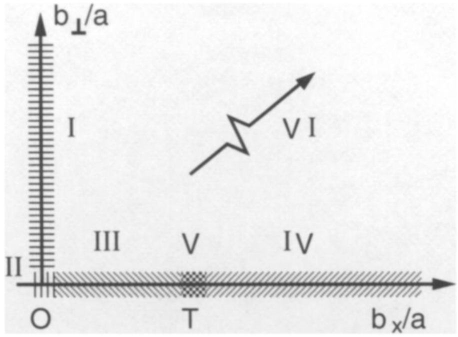

FIG. 2. *The six singular cases of linear MHD.*

Case I ($b_x \simeq 0$). In this case

$$c_f^2 \simeq a^2 + b_\perp^2, \quad c_s^2 \simeq \frac{a^2 b_x^2}{a^2+b_\perp^2}, \tag{10}$$
$$\alpha_f^2 \simeq \frac{a^2}{a^2+b_\perp^2}, \quad \alpha_s^2 \simeq \frac{b_\perp^2}{a^2+b_\perp^2}.$$

Case II ($b_x \simeq 0$ and $b_\perp \simeq 0$). This is the hydrodynamic limit; we have

$$c_f^2 \simeq a^2, \quad c_s^2 \simeq 0, \tag{11}$$
$$\alpha_f^2 \simeq 1, \quad \alpha_s^2 \simeq 0.$$

The fast waves become regular acoustic waves, and the slow waves combine with the Alfvén waves to produce shear waves.

Case III ($|b_x| < a - \epsilon$ and $b_\perp \simeq 0$). In this case

$$c_f^2 \simeq a^2, \quad c_s^2 \simeq b_x^2, \tag{12}$$
$$\alpha_f^2 \simeq 1, \quad \alpha_s^2 \simeq 0.$$

Again, the fast waves are regular acoustic waves.

Case IV ($|b_x| > a + \epsilon$ and $b_\perp \simeq 0$). In this case

$$c_f^2 \simeq b_x^2, \quad c_s^2 \simeq a^2, \tag{13}$$
$$\alpha_f^2 \simeq 0, \quad \alpha_s^2 \simeq 1.$$

This time, the slow waves are regular acoustic waves.

Case V ($|b_x| \simeq a$ and $b_\perp \simeq 0$). This is the magnetosonic case, which is the most interesting one, in which

$$c_f^2 \simeq a^2, \quad c_s^2 \simeq a^2, \tag{14}$$

but the quantities $\alpha_{f,s}$ become indeterminate. However, since $\alpha_f^2 + \alpha_s^2 = 1$, the eigenvectors cannot be singular. The forms taken by $\alpha_{f,s}$ close to the singularity have not been given previously, but an expansion

$$b_x = (a + \epsilon_1)\mathrm{sgn} B_x, \quad b_\perp = \epsilon_2,$$

can be inserted into their expressions, leading after some algebra to the simple results

$$\alpha_f = \sin\frac{\phi}{2} + \delta_f, \quad \alpha_s = \cos\frac{\phi}{2} + \delta_s, \tag{15}$$

where

$$\tan\phi = \frac{b_\perp}{|b_x| - a}.$$

It can be shown that

$$|\delta_{f,s}| \le \frac{b_\perp}{4a}.$$

Case VI ($|b_x| \gg a$ and $b_\perp \gg a$). This is the vacuum limit. In this case

$$\begin{aligned} c_f^2 &\simeq b^2, & c_s^2 &\simeq 0, \\ \alpha_f^2 &\simeq 0, & \alpha_s^2 &\simeq 1. \end{aligned} \tag{16}$$

It may be checked that both slow eigenvectors collapse onto the entropy eigenvector. This means that the eigenvector system is no longer complete and that the MHD equations are ill posed in a way that cannot be removed by any renormalization. (The same behaviour occurs of course in the Euler equations.) Therefore, all of the analysis above has been carried out on the assumption that a is suitably scaled to serve as a reference quantity.

For completeness, we now list the easily derived eigenvectors for the Alfvén waves,

$$r_A^{\pm} = \begin{pmatrix} 0 \\ 0 \\ \pm\beta_z \\ \mp\beta_y \\ -\sqrt{4\pi\rho}\beta_z \mathrm{sgn} b_x \\ \sqrt{4\pi\rho}\beta_y \mathrm{sgn} b_x \\ 0 \end{pmatrix}, \quad l_A^{\pm} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ \pm\beta_z \\ \mp\beta_y \\ -\beta_z \mathrm{sgn} b_x/\sqrt{4\pi\rho} \\ \beta_y \mathrm{sgn} b_x/\sqrt{4\pi\rho} \\ 0 \end{pmatrix}^T, \tag{17}$$

and for the contact, or entropy, wave,

$$r_e = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad l_e = \frac{1}{a^2}\begin{pmatrix} a^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}^T. \tag{18}$$

3. Wave paths near the triple point. A wave path is a path in phase space everywhere parallel to the right eigenvector of a given wave family, i.e., $dV \propto r_k$. It is the projection into phase space of a simple wave. In the section $X = b_x/a$, $Y = b_\perp/a$ of phase space the path of a wave is defined by

$$\frac{dY}{dX} = \frac{a\,db_\perp - b_\perp d\,a}{a\,db_x - b_x d\,a}. \tag{19}$$

Close to the triple point this simplifies to

$$\frac{dY}{dX} = -\frac{d\,b_\perp}{d\,a} = -\frac{\rho}{(1+\kappa)a\sqrt{4\pi\rho}}\frac{d\,B_\perp}{d\,\rho}.$$

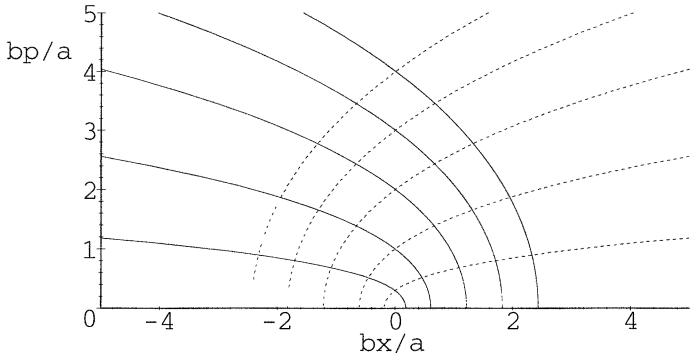

FIG. 3. *Wavepaths near the triple point T. Slow waves are shown with dotted curves and fast waves with continuous curves. The figure is self-similar, and the units are arbitrary.*

The last step uses the fact that magnetoacoustic waves are isentropic and introduces

$$\kappa = \frac{\rho}{a}\left(\frac{\partial a}{\partial \rho}\right)_s, \tag{20}$$

the value of which for an ideal gas is $(\gamma - 1)/2$. By inserting the appropriate components of r we find that for a fast wave, whether of the left- or right-going family,

$$\frac{dY}{dX}_f = -\frac{1}{1+\kappa}\frac{\alpha_s}{\alpha_f} = -\frac{1}{1+\kappa}\cot\frac{\phi}{2}.$$

Similarly, for either slow wave

$$\frac{dY}{dX}_s = \frac{1}{1+\kappa}\frac{\alpha_f}{\alpha_s} = \frac{1}{1+\kappa}\tan\frac{\phi}{2}.$$

For an ideal gas, these differential equations can be integrated in polar coordinates centered on the triple point. The general form of the resulting curves is independent of γ. For $\gamma = \frac{5}{3}$ the solution is

$$\begin{aligned} r(\phi) &= \frac{r(\pi/2)}{(1+\cos\phi)^{4/5}(1-\frac{1}{4}\cos\phi)^{1/5}} \qquad \text{(fast)} \\ &= \frac{r(\pi/2)}{(1-\cos\phi)^{4/5}(1+\frac{1}{4}\cos\phi)^{1/5}} \qquad \text{(slow)}. \end{aligned}$$

Figure 3 shows these trajectories.

In a general study of 2×2 systems of conservation laws with quadratic nonlinearity, Schaeffer and Shearer [8] obtain wavepaths with this topology for problems which they classify as type IV. The 3×3 set of model equations for MHD studied in [16] can also be shown to yield wave diagrams of the same type, provided that the parameter c occurring in that model is taken greater than 2.0 (R.-S. Myong, private communication). For this kind of nonlinearity, the Hugoniot curves were worked out by Isaacson et al. [17]. It can be shown that the Riemann problem is then well posed for problems without dissipation if compressive shocks are the only ones admitted.

These wave diagrams are easily extended to the three-dimensional space b_x/a, $b_y/a, b_z/a$ by rotating them round the b_x/a axis. Fast and slow trajectories are contained in a fixed plane through the axis. Alfvén wave trajectories are arcs of circles centered on this axis and normal to it.

4. Solution of linearised Riemann problems. Suppose that we are given Riemann initial data ($V = V_L$, $x < 0$; $V = V_R$, $x > 0$) and wish to solve for its evolution by linearising the governing equations around some average state V_0. The solution will be divided into eight uniform regions by seven discontinuities moving with speeds $\lambda_k(V_0)$. Across the kth wave the change of state is

$$l_k \cdot (V_R - V_L)\, r_k,$$

where l_k, r_k are also evaluated at V_0. Finite-volume schemes of the Godunov type often make use of such a solution to determine the flux passing between two cells, but if the solution should prove to be very sensitive to the choice of V_0, this would undermine confidence in their use.

Consider therefore a linearised Riemann problem with certain fixed data but various candidates for V_0 all lying close to the triple point T. The jumps across the Alfvén and contact discontinuities will not be sensitive to the choice of V_0 since these eigenvectors are well behaved near T. To understand what happens to the other jumps, observe that the right-going magnetoacoustic waves will produce a pair of jumps contained in the subspace spanned by r_f^+, r_s^+. But reference to (8), (9) shows that

$$\begin{aligned} r_f^+ &= \alpha_f r_a^+ + \alpha_s r_m^+, \\ r_s^+ &= \alpha_s r_a^+ - \alpha_f r_m^+, \end{aligned}$$

where r_a^+ (which produces an acoustic effect) and r_m^+ (which changes the tranverse velocity and the magnitude of the transverse field) are insensitive to the choice of V_0. Therefore the combined jump across both waves is the projection of $V_R - V_L$ into the stable subspace spanned by r_a^+, r_m^+, and a similar argument applies to the left-going pair. Any sensitivity will be the result of how the combined jump gets divided between r_f^+, r_s^+, but we now show that this will not affect the value of the flux at $x = 0$.

A formula for the flux F^* across $x = 0$, when the equations are written in conservation form

$$U_t + F(U)_x = 0,$$

is

$$F^*(U_L, U_R) = \frac{1}{2}\left[F_L + F_R\right] - \frac{1}{2}\sum_{k=1}^{k=7} a_k |\lambda|_k R_k, \tag{21}$$

where the wavestrengths a_k and speeds λ_k have already been defined and the R_k are the right eigenvectors for the conserved variables $R_k = U_V\, r_k$. Here U_V is the Jacobian matrix that relates changes in the primitive and conserved variables; it is not sensitive to the choice of V_0. The output from this formula is not sensitive to the choice of V_0 if the fast and slow waves both move in the same direction, because if they share the same sign for λ, it is the sum of their contributions that appears. Thus the critical case is where $c_s < |v_x| < c_f$. Suppose that $v_x > 0$ so that we have the

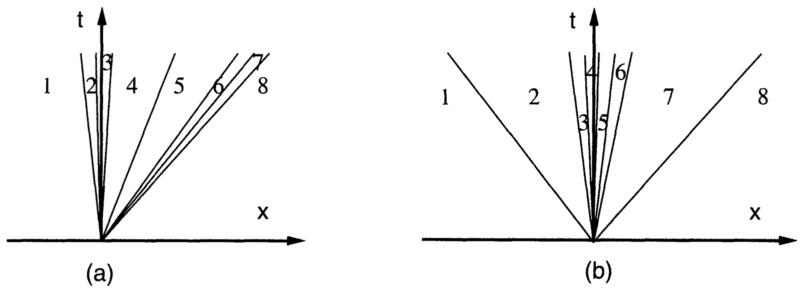

FIG. 4. *Critical cases for the sensitivity of numerical flux to the choice of mean state:* (a) $b_\perp \simeq 0$, $b_x \simeq a$, (b) $\mathbf{b} \simeq 0$.

situation depicted in Fig. 4(a), and consider then the terms arising from the fast and slow left-going magnetoacoustic waves. We have wavestrengths

$$a_f^- = (\alpha_f l_a^- + \alpha_s l_m^-) \cdot (V_R - V_L),$$
$$a_s^- = (\alpha_s l_a^- - \alpha_f l_m^-) \cdot (V_R - V_L).$$

If V_0 is chosen randomly within a circle of radius ϵ, centered on T, both a_f^- and a_s^- may change by order unity. In the formula, however, they are multiplied respectively by $|c_f - v_x|$ and by $|c_s - v_x|$. Since the wavespeeds can be shown to satisfy

$$c_f - c_s \leq \epsilon a,$$

both of these factors must be of order ϵ, and the arbitrariness in choosing V_0 has only an $\mathcal{O}(\epsilon)$ effect on the value of the computed flux.

In reality, if V_0 were very close to T, the spacing between the fast, slow, and Alfvén waves would be comparable to the width they acquire from dissipation and dispersion. Waves having a complicated internal structure then arise, and it is an open question whether they can be correctly captured by a numerical method that does not explicitly account for the dissipative mechanisms. We make no attempt to answer this question here but direct the interested reader to to a sampling of the recent literature [15]–[17], [18], [19], [20].

If V_0 lies very close to the origin ($\mathbf{b} \simeq 0$), then the slow and Alfvén wavespeeds are both very small and we have the situation in Fig. 4(b). The space spanned by these waves includes the regular Euler eigenvectors describing shear waves. Numerically, similar considerations will apply. The calculated flux will not be sensitive to the precise choice of V_0, even though the way the shear wave is built from the slow and Alfvén waves is sensitive. This provides a justification for assigning arbitrary values to β_y, β_z in this limit.

5. Structure coefficients. Structure coefficients measure the tendency of a simple wave to spread or steepen. They are defined through [21]

$$s_k = \mathrm{grad}_V\, \lambda_k \cdot r_k. \tag{22}$$

A wave for which $s_k = 0$ is said to be linearly degenerate; it does not change its own wavespeed. (This is always the case for the Alfvén and contact waves.) Clearly it is important to define s_k through properly normalised eigenvectors.

The magnetoacoustic wavespeeds depend on v_x, a, $b_\perp$, and b_x. Taking $\lambda_f^+ = v_x + c_f$ as an example, we have

$$\frac{\partial \lambda_f^+}{\partial v_x} = 1, \quad \frac{\partial \lambda_f^+}{\partial a} = \frac{c_f \alpha_f^2}{a}, \quad \frac{\partial \lambda_f^+}{\partial b_\perp} = \frac{c_f \alpha_f \alpha_s}{a}, \quad \frac{\partial \lambda_f^+}{\partial b_x} = \frac{c_s \alpha_s^2}{a} \operatorname{sgn} B_x,$$

where the last three results come from differentiating (3) and employing the identities. Then (using κ as defined in (20))

$$s_f^+ = \frac{\partial \lambda_f^+}{\partial v_x} dv_x + \frac{\partial \lambda_f^+}{\partial a} \frac{\kappa a d\rho}{\rho} + \frac{\partial \lambda_f^+}{\partial b_\perp} \left[\frac{dB_\perp}{\sqrt{4\pi\rho}} - \frac{1}{2} \frac{B_\perp d\rho}{\sqrt{4\pi\rho^3}} \right] + \frac{\partial \lambda_f^+}{\partial b_x} \left[-\frac{1}{2} \frac{B_x d\rho}{\sqrt{4\pi\rho^3}} \right].$$

After inserting the elements of r_f^+, considerable simplification ensues. The result can be expressed for either fast wave as

$$\frac{s_f^\pm}{\alpha_f c_f} = \frac{d\lambda_f}{dv_x} = (\kappa + 1)\alpha_f^2 + \frac{3}{2}\alpha_s^2 \tag{23}$$

and for either slow wave as

$$\frac{s_s^\pm}{\alpha_s c_s} = \frac{d\lambda_s}{dv_x} = (\kappa + 1)\alpha_s^2 + \frac{3}{2}\alpha_f^2. \tag{24}$$

In the special cases $\alpha_{f,s} = 1$, when the fast or slow wave assumes the guise of a regular acoustic wave, a classical result is recovered. These results can be confirmed by comparing them with equivalent but more complicated expressions given in [22].

Although it is clear from the above expressions that the ratios $d\lambda_{f,s}/dv_x$ are always positive, it does not follow that either $\lambda_{f,s}$ or v_x individually behaves monotonically through a simple wave. Indeed, if $b_\perp = 0$ so that either α_f or α_s vanishes, reference to the eigenvectors shows that $dv_x = 0$ for that wave. Thus v_x, and also λ, is stationary with respect to x at such a point, and the wave is neither expansive nor compressive. This is the "loss of convexity" first observed in [2].

Zachary and Colella [3] employ the structure coefficients (expressions for which they do not give) to add numerical dissipation in the style of Bell, Colella, and Trangenstein [7]. However, for the Euler equations, Roe [23] found that such additional dissipation was required only at stationary sonic points. Application of this simplified strategy to the MHD equations appears to be confirmed by numerical experiments reported in [24], but only extensive testing will decide the issue.

6. Remark on multidimensional problems. If the two- or three-dimensional MHD equations are written in the conservation form that is usual for discontinuity-capturing calculations, the eigenvectors are not simple extensions of their one-dimensional versions. Also, the constraint $B_x = 0$ gets replaced by the more complicated constraint $\operatorname{div} \mathbf{B} = 0$. Powell [25] has recently shown that a partially conservative form of the multidimensional equations, obtained by manoeuvring terms proportional to $\operatorname{div} \mathbf{B}$, retains the one-dimensional eigenstructure, with the addition of an eighth wave that convects $\operatorname{div} \mathbf{B}$ as a passive scalar. This observation has enabled the one-dimensional analysis to be applied almost directly to adaptive-grid calculations of a cometary atmosphere interacting with the solar wind [26].

7. Conclusions. The eigenvectors of the MHD equations have been given in forms that are more compact and illuminating than previously. The new expressions have been used to clarify a singularity in the wave structure, the behaviour of linearised Riemann problems, and the steepening of simple waves.

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