Supplementary Material for 'Recursive Flow: A Generative Framework for MIMO Channel Estimation'

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Appendix A
Proof of Theorem 1

To invoke Brouwer's Fixed-Point Theorem, we consider the composite operator of the entire inner loop, denoted as $\mathcal{T} : \mathbb{C}^{N_r \times N_t} \to \mathbb{C}^{N_r \times N_t}$, which maps the input state $\mathbf{H}^{(k,0)}$ to the output $\mathbf{H}^{(k+1,0)}$ after N_2 recursive steps. We must identify a compact convex ball $\mathcal{K}_{R^\star}$ such that $\mathcal{T}(\mathcal{K}_{R^\star}) \subseteq \mathcal{K}_{R^\star}$.

A. Continuity

The neural network $\mathcal{D}$ (with continuous activation functions), the linear matrix operations in the projection $\mathcal{P}$, and the convex combination with the anchor are all continuous mappings. Since the matrix inverse $(\mathbf{M} + w^{-1}\mathbf{I})^{-1}$ exists and is continuous for $w^{-1} > 0$, the composite operator $\mathcal{T}$ is continuous on $\mathbb{C}^{N_r \times N_t}$.

B. Self-Mapping on a Compact Set

We establish a global upper bound on the projected states, independent of the iteration index. Based on the Bounded Denoiser Hypothesis (Assumption 1), the output is bounded by $\|\mathcal{D}(\mathbf{H})\|_F \le B_{\text{flow}}$. Consider the projection step at any inner index i:

$$\mathbf{H}_{\text{proj}}^{(i)} = (\mathbf{R} + w^{-1}\mathcal{D}(\mathbf{H}^{(k,i)}))(\mathbf{M} + w^{-1}\mathbf{I})^{-1}. \tag{1}$$

Applying the norm inequality and the spectral bound $\|w^{-1}(\mathbf{M} + w^{-1}\mathbf{I})^{-1}\|_2 \le 1$:

$$\begin{aligned}\|\mathbf{H}_{\text{proj}}^{(i)}\|_F &\le \|\mathbf{R}(\mathbf{M} + w^{-1}\mathbf{I})^{-1}\|_F + \|\mathcal{D}(\cdot)\|_F \cdot 1 \\ &\le C_{\text{obs}} + B_{\text{flow}} \triangleq R_{\text{limit}}.\end{aligned} \tag{2}$$

Crucially, R_{limit} is a constant determined solely by the observations and the physical prior manifold. It does not depend on the norm of the intermediate state $\mathbf{H}^{(k,i)}$ or the step index i.

Define the invariant ball $\mathcal{K}_{R_{\text{limit}}} \triangleq \{\mathbf{H} \mid \|\mathbf{H}\|_F \le R_{\text{limit}}\}$, which is inherently a convex set. Assume the initialization satisfies $\mathbf{H}^{(k,0)} \in \mathcal{K}_{R_{\text{limit}}}$, implying the anchor $\boldsymbol{\epsilon} \in \mathcal{K}_{R_{\text{limit}}}$. Since we have established that any projected state is bounded by R_{limit}, we also have $\mathbf{H}_{\text{proj}}^{(i)} \in \mathcal{K}_{R_{\text{limit}}}$. The update rule $\mathbf{H}^{(k,i+1)} = t'_i\boldsymbol{\epsilon} + (1 - t'_i)\mathbf{H}_{\text{proj}}^{(i)}$ constitutes a convex combination, where $t'_i = (1 - (i+1)/N_2)^\beta$. By the definition of convexity, the resulting state $\mathbf{H}^{(k,i+1)}$ must remain within $\mathcal{K}_{R_{\text{limit}}}$ for any $t'_i \in [0, 1)$. By induction, the final output satisfies $\mathbf{H}^{(k,N_2)} \in \mathcal{K}_{R_{\text{limit}}}$.

Therefore, we have proved that if the input $\mathbf{H}^{(k,0)} \in \mathcal{K}_{R^\star}$, then the output of the composite inner loop $\mathcal{T}(\mathbf{H}^{(k,0)}) \in \mathcal{K}_{R^\star}$. Thus, $\mathcal{T}$ is a self-mapping on the compact convex set $\mathcal{K}_{R^\star}$. By Brouwer's Fixed-Point Theorem, there exists at least one fixed point $\mathbf{H}^\star \in \mathcal{K}_{R^\star}$.

Appendix B
Numerical Validations of Assumption 2

Fig. 1 illustrates the evolution of the spectral radius as a function of the inner iteration numbers N_2 under various outer iteration settings N_1. Specifically, the first and second rows correspond to the initial and fully converged iterations, respectively. The simulation parameters are configured as $\beta = 16, \lambda = 2, \alpha = 0.6$ within a 16×64 MIMO system across three distinct SNR regimes. A stability threshold of 1.0 is established as the convergence criterion, where the spectral radius consistently below this boundary signifies algorithmic convergence. While only the initial and fully converged stages are presented, the spectral radius evolution across the intermediate progression of N_1 iterations exhibits qualitatively similar behavior.

The spectral radius of $\mathbf{J}_{\mathcal{D},i}$ consistently exceeds unity, yet exhibits a clear asymptotic trend toward stability threshold as the inner iterations progress. This behavior is analytically tied to the generative refinement term $t \cdot \mathbf{V}$ in the update formula; as the temporal variable t monotonically decreases toward zero across the flow trajectory, the effective influence of the generative prior is gradually attenuated. Consequently, the operator's expansive tendency is suppressed, causing its spectral radius to shrink toward the stability boundary. In contrast, $\mathbf{J}_{\mathcal{P},i}$ maintains a spectral radius strictly below unity. This stabilizing behavior can be analytically explained by the structure of the proximal solution. Specifically, the eigenvalues of the linear part of $\mathcal{P}$, denoted by μ_i, satisfy the relationship $\mu_i = 1/(w\lambda_i + 1)$, where λ_i represent the eigenvalues of matrix $\mathbf{M}$. Consequently, the spectral radius is expressed as

$$\rho(\mathbf{J}_{\mathcal{P},i}) = \max_i |\mu_i| = \max_i |1/(w\lambda_i + 1)|. \tag{3}$$

As t evolves from 1 toward 0, the variance annealing parameter w monotonically decreases, causing the denominator to approach unity. Consequently, the spectral radius increases from a highly contractive state and asymptotically approaches the stability boundary.

Intriguingly, $\rho(\mathbf{T}_i)$ is not only maintained below the stability boundary but is also consistently lower than $\rho(\mathbf{J}_{\mathcal{P},i})$ alone. This phenomenon suggests a synergy rooted in the complementary compression between the learned generative

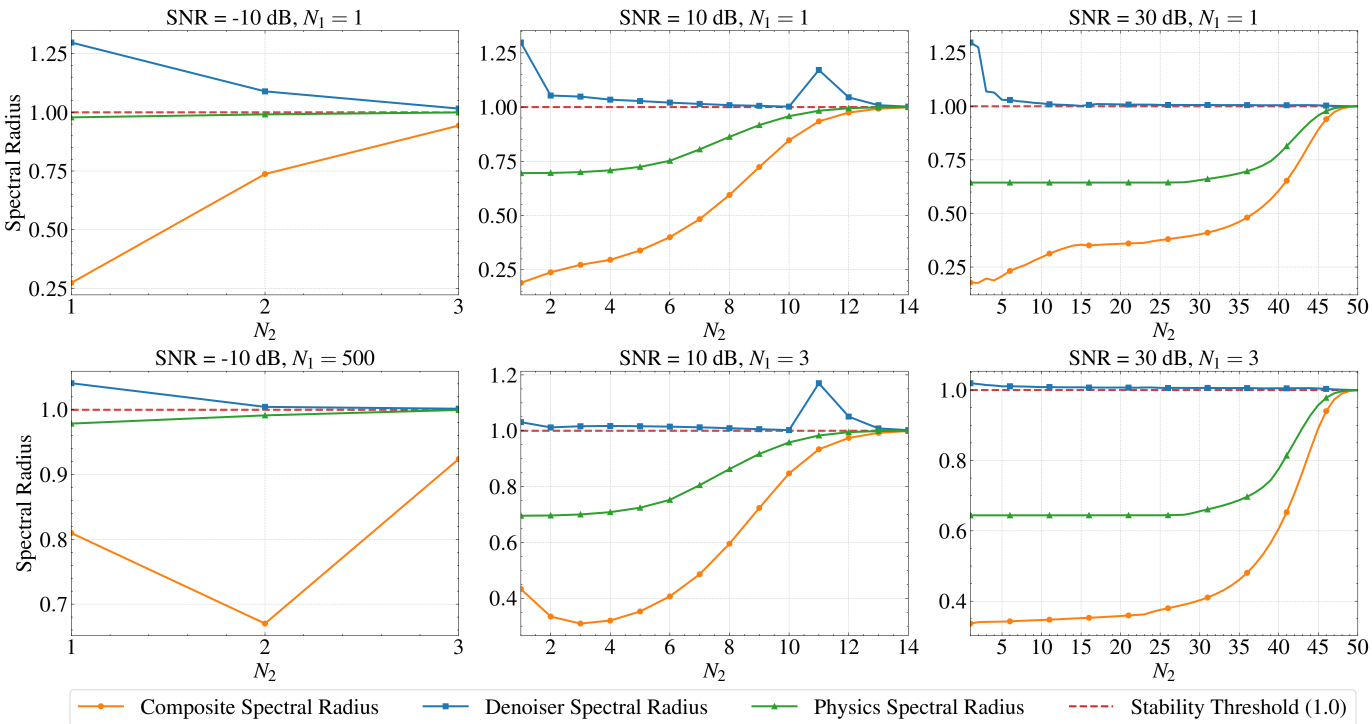

Fig. 1. The evolution of spectral radius for the composite operator $\mathcal{P} \circ \mathcal{D}$, the denoiser $\mathcal{D}$, and the physics-based operator $\mathcal{P}$ as a function of inner iteration numbers (N_2) under various SNR and outer iteration number.

prior and the physical manifold. The flow dynamics provide update directions aligned with the measurement likelihood, effectively reinforcing the proximal contraction to enhance the stability margin and accelerate convergence.

With increasing inner loop iterations N_2, all spectral radii asymptotically converge toward unity, signifying a quasi-equilibrium state where generative refinement and data consistency reach a dynamic balance. Crucially, the convergence of $\rho(\mathbf{T}_i)$ from below the threshold ensures that the algorithm maximizes informational utility while strictly adhering to the stability and contractivity requirements throughout the iterative evolution.

Appendix C
Proof of Theorem 2

In this appendix, we provide a rigorous derivation of the global asymptotic stability in a generalized metric space for the RC-Flow algorithm. The objective is to bound the spectral radius of the total Jacobian matrix $\mathbf{J}_{\mathcal{T}} \in \mathbb{C}^{N \times N}$ (where $N = N_t N_r$), which governs the sensitivity of the output of the k-th outer iteration, $\mathbf{H}^{(k+1,0)}$, with respect to its input, $\mathbf{H}^{(k,0)}$.

A. Stability Criterion via Induced Metric Space

We invoke Assumption 2, which postulates that the composite inner-loop operator $\mathbf{T}_i \triangleq \mathbf{J}_{\mathcal{P},i}\mathbf{J}_{\mathcal{D},i}$ satisfies the spectral contraction condition $\rho(\mathbf{T}_i) \leq \gamma < 1$. According to the Householder theorem, for any matrix $\mathbf{A}$ with $\rho(\mathbf{A}) < 1$ and any given $\eta > 0$, there exists an induced vector norm $\|\cdot\|_*$ such that:

$$\|\mathbf{A}\|_* \leq \rho(\mathbf{A}) + \eta. \tag{4}$$

However, since the operator $\mathbf{T}_i$ is time-varying, strictly establishing stability requires a common metric. Given the smooth evolution of the parameters, we analytically extend Assumption 2 to postulate the existence of a common induced norm $\|\cdot\|$ for the sequence $\{\mathbf{T}_i\}_{i=0}^{N_2-1}$, such that:

$$\|\mathbf{T}_i\|_* \leq \gamma' < 1, \quad \forall i. \tag{5}$$

Consequently, our proof strategy is to establish the contraction of the total Jacobian $\mathbf{J}_{\mathcal{T}}$ within this generalized metric space $(\mathbb{C}^N, \|\cdot\|_*)$.

B. Differential Dynamics of the Inner Loop

Consider the k-th outer loop iteration. Let the input state be $\mathbf{H}^{(k,0)}$. According to Algorithm 1, the anchor point is reset to the previous estimate, i.e., $\boldsymbol{\epsilon} = \mathbf{H}^{(k,0)}$. The inner loop generates a sequence of states $\{\mathbf{H}^{(k,i)}\}_{i=0}^{N_2}$, where the update rule at the i-th inner step is given by:

$$\mathbf{H}^{(k,i+1)} = t'_i\mathbf{H}^{(k,0)} + (1 - t'_i)\mathcal{P}(\mathcal{D}(\mathbf{H}^{(k,i)})), \tag{6}$$

where $\mathcal{P}(\cdot)$ and $\mathcal{D}(\cdot)$ denote the physics-aware projection and flow-matching denoiser operators, respectively, and $t' \in [0, 1)$ is the time-varying anchor coefficient.

We define the cumulative sensitivity matrix at inner step i as $\mathbf{J}^{(i)} \triangleq \frac{\partial \mathrm{vec}(\mathbf{H}^{(\mathrm{k,i})})}{\partial \mathrm{vec}(\mathbf{H}^{(\mathrm{k,0})})}$. By applying the chain rule to the recurrence relation, we obtain:

$$\frac{\partial \mathrm{vec}(\mathbf{H}^{(\mathrm{k,i+1})})}{\partial \mathrm{vec}(\mathbf{H}^{(\mathrm{k,0})})} = t'\mathbf{I} + (1 - t')\mathbf{J}_{\mathcal{P},i}\mathbf{J}_{\mathcal{D},i}\frac{\partial \mathrm{vec}(\mathbf{H}^{(\mathrm{k,i})})}{\partial \mathrm{vec}(\mathbf{H}^{(\mathrm{k,0})})}, \tag{7}$$

where $\mathbf{I}$ is the identity matrix resulting from differentiating the anchor term $t'\mathbf{H}^{(k,0)}$. Using the definition of the composite

Jacobian $\mathbf{T}_i$, the dynamics follow a non-homogeneous linear recurrence:

$$\mathbf{J}^{(i+1)} = t'_i\mathbf{I} + (1-t'_i)\mathbf{T}_i\mathbf{J}^{(i)}. \tag{8}$$

C. Recursive Expansion and Closed-Form Solution

The base case for the recurrence is $\mathbf{J}^{(0)} = \mathbf{I}$. To reveal the structure of the total Jacobian, let us define the path transition operator $\mathbf{\Phi}_{m,j}$, which represents the accumulated decay and rotation from step j to m:

$$\mathbf{\Phi}_{m,j} \triangleq \begin{cases} \prod_{n=j}^{m-1}(1-t'_n)\mathbf{T}_n, & \text{if } m > j, \\ \mathbf{I}, & \text{if } m = j. \end{cases} \tag{9}$$

Expanding the recurrence relation (8) iteratively:

$$\begin{aligned} \mathbf{J}^{(1)} &= t'_0\mathbf{\Phi}_{1,1} + \mathbf{\Phi}_{1,0}, \\ \mathbf{J}^{(2)} &= t'_1\mathbf{\Phi}_{2,2} + t'_0\mathbf{\Phi}_{2,1} + \mathbf{\Phi}_{2,0}. \end{aligned} \tag{10}$$

By induction, the total Jacobian $\mathbf{J}_{\mathcal{T}} \triangleq \mathbf{J}^{(N_2)}$ at the end of the inner loop can be expressed as a weighted sum of historical transition paths:

$$\mathbf{J}_{\mathcal{T}} = \underbrace{\mathbf{\Phi}_{N_2,0}}_{\text{Residual of Initialization}} + \sum_{i=0}^{N_2-1} \underbrace{t'_i\mathbf{\Phi}_{N_2,i+1}}_{\text{Accumulated Anchor Injections}}. \tag{11}$$

D. Strict Contraction Analysis via Partition of Unity

We now apply the induced norm $\|\cdot\|_*$ defined in (5) to the closed-form expansion (11). Using the triangle inequality and the sub-multiplicativity property of the induced norm, we obtain:

$$\|\mathbf{J}_{\mathcal{T}}\|_* \leq \|\mathbf{\Phi}_{N_2,0}\|_* + \sum_{i=0}^{N_2-1} t'_i\|\mathbf{\Phi}_{N_2,i+1}\|_*. \tag{12}$$

The norm of the transition operator is bounded by the product of individual operator norms. Using $\|\mathbf{T}_n\|_* \leq \gamma' < 1$, we have:

$$\|\mathbf{\Phi}_{N_2,j}\|_* \leq \prod_{n=j}^{N_2-1}(1-t'_n)\|\mathbf{T}_n\|_* \leq (\gamma')^{N_2-j}\prod_{n=j}^{N_2-1}(1-t'_n). \tag{13}$$

For brevity, let us define the cumulative scalar decay factor P_j:

$$P_j \triangleq \prod_{n=j}^{N_2-1}(1-t'_n), \quad \text{with } P_{N_2} \triangleq 1. \tag{14}$$

Substituting this bound into the inequality (12), we arrive at:

$$\|\mathbf{J}_{\mathcal{T}}\|_* \leq P_0(\gamma')^{N_2} + \sum_{i=0}^{N_2-1} t'_iP_{i+1}(\gamma')^{N_2-(i+1)}. \tag{15}$$

To prove strict contraction, we utilize the algebraic property of the weighting coefficients. Observe that the recurrence relation $P_i = (1-t'_i)P_{i+1}$ implies $t'_iP_{i+1} = P_{i+1} - P_i$. Consequently, the sum of the coefficients (excluding the contraction factor γ') forms a telescoping series:

$$P_0 + \sum_{i=0}^{N_2-1} t'_iP_{i+1} = P_0 + \sum_{i=0}^{N_2-1}(P_{i+1} - P_i) = P_{N_2} = 1. \tag{16}$$

This confirms that the weights form a partition of unity. Thus, the upper bound in (15) represents a convex combination of the terms $(\gamma')^k$.

Since $\gamma' < 1$ and $N_2 \geq 1$, the factor $(\gamma')^k$ is strictly less than 1 for all $k \geq 1$. Specifically, the exponents in (15) are $N_2, N_2-1, \ldots, 0$. Unless $t'_{N_2-1} = 1$, the sum contains terms scaled by $\gamma' < 1$. Thus, the strict inequality holds:

$$\|\mathbf{J}_{\mathcal{T}}\|_* < P_0 \cdot 1 + \sum_{i=0}^{N_2-1} t'_iP_{i+1} \cdot 1 = 1. \tag{17}$$

Since $\|\mathbf{J}_{\mathcal{T}}\|_* < 1$, the mapping is a strict contraction in the metric space $(\mathbb{C}^N, \|\cdot\|_*)$. Therefore, the proof is complete.