Logic

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Introduction

Logic

Logic (from the Greek λογική, *logikē*)[1] refers to both the study of modes of reasoning (which are valid and which are fallacious)[2][3] and the use of valid reasoning. In the latter sense, logic is used in most intellectual activities, including philosophy and science, but in the first sense, is primarily studied in the disciplines of philosophy, mathematics, semantics, and computer science. It examines general forms that arguments may take. In mathematics, it is the study of valid inferences within some formal language.[4] Logic is also studied in argumentation theory.[5]

Logic was studied in several ancient civilizations, including India,[6] China,[7] and Greece. In the west, logic was established as a formal discipline by Aristotle, who gave it a fundamental place in philosophy. The study of logic was part of the classical trivium, which also included grammar and rhetoric.

Logic is often divided into three parts, inductive reasoning, abductive reasoning, and deductive reasoning.

The study of logic

> “Upon this first, and in one sense this sole, rule of reason, that in order to learn you must desire to learn, and in so desiring not be satisfied with what you already incline to think, there follows one corollary which itself deserves to be inscribed upon every wall of the city of philosophy: Do not block the way of inquiry.”
>
> —Charles Sanders Peirce, "First Rule of Logic"

The concept of logical form is central to logic, it being held that the validity of an argument is determined by its logical form, not by its content. Traditional Aristotelian syllogistic logic and modern symbolic logic are examples of formal logics.

- **Informal logic** is the study of natural language arguments. The study of fallacies is an especially important branch of informal logic. The dialogues of Plato[8] are good examples of informal logic.
- **Formal logic** is the study of inference with purely formal content. An inference possesses a *purely formal content* if it can be expressed as a particular application of a wholly abstract rule, that is, a rule that is not about any particular thing or property. The works of Aristotle contain the earliest known formal study of logic. Modern formal logic follows and expands on Aristotle.[9] In many definitions of logic, logical inference and inference with purely formal content are the same. This does not render the notion of informal logic vacuous, because no formal logic captures all of the nuance of natural language.
- **Symbolic logic** is the study of symbolic abstractions that capture the formal features of logical inference.[10][11] Symbolic logic is often divided into two branches: propositional logic and predicate logic.
- **Mathematical logic** is an extension of symbolic logic into other areas, in particular to the study of model theory, proof theory, set theory, and recursion theory.

Logical form

Logic is generally accepted to be **formal**; when it aims to analyze and represent the *form* of any valid argument type. The form of an argument is displayed by representing its sentences in the formal grammar and symbolism of a logical language to make its content usable in formal inference. If one considers the notion of form to be too philosophically loaded, one could say that formalizing is nothing else than translating English sentences into the language of logic.

This is known as showing the *logical form* of the argument. It is necessary because indicative sentences of ordinary language show a considerable variety of form and complexity that makes their use in inference impractical. It requires, first, ignoring those grammatical features, which are irrelevant to logic (such as gender and declension, if the argument is in Latin), replacing conjunctions which are not relevant to logic (such as 'but') with logical conjunctions like 'and' and replacing ambiguous, or alternative logical expressions ('any', 'every', etc.) with expressions of a standard type (such as 'all', or the universal quantifier $\forall$).

Second, certain parts of the sentence must be replaced with schematic letters. Thus, for example, the expression 'all As are Bs' shows the logical form which is common to the sentences 'all men are mortals', 'all cats are carnivores', 'all Greeks are philosophers' and so on.

That the concept of form is fundamental to logic was already recognized in ancient times. Aristotle uses variable letters to represent valid inferences in *Prior Analytics*, leading Jan Łukasiewicz to say that the introduction of variables was 'one of Aristotle's greatest inventions'[12]. According to the followers of Aristotle (such as Ammonius), only the logical principles stated in schematic terms belong to logic, not those given in concrete terms. The concrete terms 'man', 'mortal', etc., are analogous to the substitution values of the schematic placeholders 'A', 'B', 'C', which were called the 'matter' (Greek 'hyle') of the inference.

The fundamental difference between modern formal logic and traditional, or Aristotelian logic, lies in their differing analysis of the logical form of the sentences they treat.

- In the traditional view, the form of the sentence consists of (1) a subject (e.g. 'man') plus a sign of quantity ('all' or 'some' or 'no'); (2) the copula which is of the form 'is' or 'is not'; (3) a predicate (e.g. 'mortal'). Thus: all men are mortal. The logical constants such as 'all', 'no' and so on, plus sentential connectives such as 'and' and 'or' were called 'syncategorematic' terms (from the Greek 'kategorei' – to predicate, and 'syn' – together with). This is a fixed scheme, where each judgment has an identified quantity and copula, determining the logical form of the sentence.
- According to the modern view, the fundamental form of a simple sentence is given by a recursive schema, involving logical connectives, such as a quantifier with its bound variable, which are joined to by juxtaposition to other sentences, which in turn may have logical structure.
- The modern view is more complex, since a single judgement of Aristotle's system will involve two or more logical connectives. For example, the sentence "All men are mortal" involves, in term logic, two non-logical terms "is a man" (here *M*) and "is mortal" (here *D*): the sentence is given by the judgement A(M,D). In predicate logic, the sentence involves the same two non-logical concepts, here analyzed as $m(x)$ and $d(x)$, and the sentence is given by $\forall x.(m(x) \to d(x))$, involving the logical connectives for universal quantification and implication.
- But equally, the modern view is more powerful. Medieval logicians recognized the problem of multiple generality, where Aristotelian logic is unable to satisfactorily render such sentences as "Some guys have all the luck", because both quantities "all" and "some" may be relevant in an inference, but the fixed scheme that Aristotle used allows only one to govern the inference. Just as linguists recognize recursive structure in natural languages, it appears that logic needs recursive structure.

Deductive and inductive reasoning, and retroductive inference

Deductive reasoning concerns what follows necessarily from given premises (if a, then b). However, inductive reasoning—the process of deriving a reliable generalization from observations—has sometimes been included in the study of logic. Similarly, it is important to distinguish deductive validity and inductive validity (called "cogency"). An inference is deductively valid if and only if there is no possible situation in which all the premises are true but the conclusion false. An inductive argument can be neither valid nor invalid; its premises give only some degree of probability, but not certainty, to its conclusion.

The notion of deductive validity can be rigorously stated for systems of formal logic in terms of the well-understood notions of semantics. Inductive validity on the other hand requires us to define a reliable generalization of some set of observations. The task of providing this definition may be approached in various ways, some less formal than others; some of these definitions may use mathematical models of probability. For the most part this discussion of logic deals only with deductive logic.

Retroductive inference is a mode of reasoning that Peirce proposed as operating over and above induction and deduction to "open up new ground" in processes of theorizing (1911, p. 2). He defines retroduction as a logical inference that allows us to "render comprehensible" some observations/events which we perceive, by relating these back to a posited state of affairs that would help to shed light on the observations (Peirce, 1911, p. 2). He remarks that the "characteristic formula" of reasoning that he calls retroduction is that it involves reasoning from a consequent (any observed/experienced phenomena that confront us) to an antecedent (that is, a posited state of things that helps us to render comprehensible the observed phenomena). Or, as he otherwise puts it, it can be considered as "regressing from a consequent to a hypothetical antecedent" (1911, p. 4). See for instance, the discussion at: http://www.helsinki.fi/science/commens/dictionary.html

Some authors have suggested that this mode of inference can be used within social theorizing to postulate social structures/mechanisms that explain the way that social outcomes arise in social life and that in turn also indicate that these structures/mechanisms are alterable with sufficient social will (and visioning of alternatives). In other words, this logic is specifically liberative in that it can be used to point to transformative potential in our way of organizing our social existence by our re-examining/exploring the deep structures that generate outcomes (and life chances for people). In her book on New Racism (2010) Norma Romm offers an account of various interpretations of what can be said to be involved in retroduction as a form of inference and how this can also be seen to be linked to a style of theorizing (and caring) where processes of knowing (which she sees as dialogically rooted) are linked to social justice projects (http://www.springer.com/978-90-481-8727-0)

Consistency, validity, soundness, and completeness

Among the important properties that logical systems can have:

- **Consistency**, which means that no theorem of the system contradicts another.[13]
- **Validity**, which means that the system's rules of proof will never allow a false inference from true premises. A logical system has the property of soundness when the logical system has the property of validity and uses only premises that prove true (or, in the case of axioms, are true by definition).[13]
- **Completeness**, of a logical system, which means that if a formula is true, it can be proven (if it is true, it is a theorem of the system).
- **Soundness**, the term soundness has multiple separate meanings, which creates a bit of confusion throughout the literature. Most commonly, soundness refers to logical systems, which means that if some formula can be proven in a system, then it is true in the relevant model/structure (if A is a theorem, it is true). This is the converse of completeness. A distinct, peripheral use of soundness refers to arguments, which means that the premises of a valid argument are true in the actual world.

Some logical systems do not have all four properties. As an example, Kurt Gödel's incompleteness theorems show that sufficiently complex formal systems of arithmetic cannot be consistent and complete;[11] however, first-order predicate logics not extended by specific axioms to be arithmetic formal systems with equality can be complete and consistent.[14]

Rival conceptions of logic

Logic arose (see below) from a concern with correctness of argumentation. Modern logicians usually wish to ensure that logic studies just those arguments that arise from appropriately general forms of inference. For example, Thomas Hofweber writes in the Stanford Encyclopedia of Philosophy that logic "does not, however, cover good reasoning as a whole. That is the job of the theory of rationality. Rather it deals with inferences whose validity can be traced back to the formal features of the representations that are involved in that inference, be they linguistic, mental, or other representations".[4]

By contrast, Immanuel Kant argued that logic should be conceived as the science of judgment, an idea taken up in Gottlob Frege's logical and philosophical work, where thought (German: *Gedanke*) is substituted for judgment (German: *Urteil*). On this conception, the valid inferences of logic follow from the structural features of judgments or thoughts.

History

The earliest sustained work on the subject of logic is that of Aristotle.[15] Aristotelian logic became widely accepted in science and mathematics and remained in wide use in the West until the early 19th century.[16] Aristotle's system of logic was responsible for the introduction of hypothetical syllogism,[17] temporal modal logic,[18][19] and inductive logic[20], as well as influential terms such as terms, predicables, syllogisms and propositions. In Europe during the later medieval period, major efforts were made to show that Aristotle's ideas were compatible with Christian faith. During the High Middle Ages, logic became a main focus of philosophers, who would engage in critical logical analyses of philosophical arguments, often using variations of the methodology of scholasticism. In 1323, William of Ockham's influential *Summa Logicae* was released. By the 18th century, the structured approach to arguments had degenerated and fallen out of favour, as depicted in Holberg's satirical play *Erasmus Montanus*.


Aristotle, 384–322 BC.

The Chinese logical philosopher Gongsun Long (ca. 325–250 BC) proposed the paradox "One and one cannot become two, since neither becomes two."[21] In China, the tradition of scholarly investigation into logic, however, was repressed by the Qin dynasty following the legalist philosophy of Han Feizi.

In India, innovations in the scholastic school, called Nyaya, continued from ancient times into the early 18th century with the Navya-Nyaya school. By the 16th century, it developed theories resembling modern logic, such as Gottlob Frege's "distinction between sense and reference of proper names" and his "definition of number," as well as the theory of "restrictive conditions for universals" anticipating some of the developments in modern set theory.[22] Since 1824, Indian logic attracted the attention of many Western scholars, and has had an influence on important 19th-century logicians such as Charles Babbage, Augustus De Morgan, and George Boole.[23] In the 20th century, Western philosophers like Stanislaw Schayer and Klaus Glashoff have explored Indian logic more extensively.

The syllogistic logic developed by Aristotle predominated in the West until the mid-19th century, when interest in the foundations of mathematics stimulated the development of symbolic logic (now called mathematical logic). In 1854, George Boole published *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*, introducing symbolic logic and the principles of what is now known as Boolean logic. In 1879, Gottlob Frege published *Begriffsschrift* which inaugurated modern logic with the invention of quantifier notation. From 1910 to 1913, Alfred North Whitehead and Bertrand Russell published *Principia Mathematica*[10] on the foundations of mathematics, attempting to derive mathematical truths from axioms and inference rules in symbolic logic. In 1931, Gödel raised serious problems with the foundationalist program and logic ceased to focus on such issues.

The development of logic since Frege, Russell, and Wittgenstein had a profound influence on the practice of philosophy and the perceived nature of philosophical problems (see Analytic philosophy), and Philosophy of mathematics. Logic, especially sentential logic, is implemented in computer logic circuits and is fundamental to computer science. Logic is commonly taught by university philosophy departments, often as a compulsory discipline.

Topics in logic

Syllogistic logic

The *Organon* was Aristotle's body of work on logic, with the *Prior Analytics* constituting the first explicit work in formal logic, introducing the syllogistic.[24] The parts of syllogistic logic, also known by the name term logic, are the analysis of the judgements into propositions consisting of two terms that are related by one of a fixed number of relations, and the expression of inferences by means of syllogisms that consist of two propositions sharing a common term as premise, and a conclusion which is a proposition involving the two unrelated terms from the premises.

Aristotle's work was regarded in classical times and from medieval times in Europe and the Middle East as the very picture of a fully worked out system. However, it was not alone: the Stoics proposed a system of propositional logic that was studied by medieval logicians. Also, the problem of multiple generality was recognised in medieval times. Nonetheless, problems with syllogistic logic were not seen as being in need of revolutionary solutions.

Today, some academics claim that Aristotle's system is generally seen as having little more than historical value (though there is some current interest in extending term logics), regarded as made obsolete by the advent of propositional logic and the predicate calculus. Others use Aristotle in argumentation theory to help develop and critically question argumentation schemes that are used in artificial intelligence and legal arguments.

Propositional logic (sentential logic)

A propositional calculus or logic (also a sentential calculus) is a formal system in which formulae representing propositions can be formed by combining atomic propositions using logical connectives, and in which a system of formal proof rules allows certain formulae to be established as "theorems".

Predicate logic

Predicate logic is the generic term for symbolic formal systems such as first-order logic, second-order logic, many-sorted logic, and infinitary logic.

Predicate logic provides an account of quantifiers general enough to express a wide set of arguments occurring in natural language. Aristotelian syllogistic logic specifies a small number of forms that the relevant part of the involved judgements may take. Predicate logic allows sentences to be analysed into subject and argument in several additional ways, thus allowing predicate logic to solve the problem of multiple generality that had perplexed medieval logicians.

The development of predicate logic is usually attributed to Gottlob Frege, who is also credited as one of the founders of analytical philosophy, but the formulation of predicate logic most often used today is the first-order logic presented in Principles of Mathematical Logic by David Hilbert and Wilhelm Ackermann in 1928. The analytical generality of predicate logic allowed the formalisation of mathematics, drove the investigation of set theory, and allowed the development of Alfred Tarski's approach to model theory. It provides the foundation of modern mathematical logic.

Frege's original system of predicate logic was second-order, rather than first-order. Second-order logic is most prominently defended (against the criticism of Willard Van Orman Quine and others) by George Boolos and Stewart Shapiro.

Modal logic

In languages, modality deals with the phenomenon that sub-parts of a sentence may have their semantics modified by special verbs or modal particles. For example, "*We go to the games*" can be modified to give "*We should go to the games*", and "*We can go to the games*"" and perhaps "*We will go to the games*". More abstractly, we might say that modality affects the circumstances in which we take an assertion to be satisfied.

The logical study of modality dates back to Aristotle,[25] who was concerned with the alethic modalities of necessity and possibility, which he observed to be dual in the sense of De Morgan duality. While the study of necessity and possibility remained important to philosophers, little logical innovation happened until the landmark investigations of Clarence Irving Lewis in 1918, who formulated a family of rival axiomatizations of the alethic modalities. His work unleashed a torrent of new work on the topic, expanding the kinds of modality treated to include deontic logic and epistemic logic. The seminal work of Arthur Prior applied the same formal language to treat temporal logic and paved the way for the marriage of the two subjects. Saul Kripke discovered (contemporaneously with rivals) his theory of frame semantics which revolutionised the formal technology available to modal logicians and gave a new graph-theoretic way of looking at modality that has driven many applications in computational linguistics and computer science, such as dynamic logic.

Informal reasoning

The motivation for the study of logic in ancient times was clear: it is so that one may learn to distinguish good from bad arguments, and so become more effective in argument and oratory, and perhaps also to become a better person. Half of the works of Aristotle's Organon treat inference as it occurs in an informal setting, side by side with the development of the syllogistic, and in the Aristotelian school, these informal works on logic were seen as complementary to Aristotle's treatment of rhetoric.

This ancient motivation is still alive, although it no longer takes centre stage in the picture of logic; typically dialectical logic will form the heart of a course in critical thinking, a compulsory course at many universities.

Argumentation theory is the study and research of informal logic, fallacies, and critical questions as they relate to every day and practical situations. Specific types of dialogue can be analyzed and questioned to reveal premises, conclusions, and fallacies. Argumentation theory is now applied in artificial intelligence and law.

Mathematical logic

Mathematical logic really refers to two distinct areas of research: the first is the application of the techniques of formal logic to mathematics and mathematical reasoning, and the second, in the other direction, the application of mathematical techniques to the representation and analysis of formal logic.[26]

The earliest use of mathematics and geometry in relation to logic and philosophy goes back to the ancient Greeks such as Euclid, Plato, and Aristotle.[27] Many other ancient and medieval philosophers applied mathematical ideas and methods to their philosophical claims.[28]


Kurt Gödel

One of the boldest attempts to apply logic to mathematics was undoubtedly the logicism pioneered by philosopher-logicians such as Gottlob Frege and Bertrand Russell: the idea was that mathematical theories were logical tautologies, and the programme was to show this by means to a reduction of mathematics to logic.[10] The various attempts to carry this out met with a series of failures, from the crippling of Frege's project in his *Grundgesetze* by Russell's paradox, to the defeat of Hilbert's program by Gödel's incompleteness theorems.

Both the statement of Hilbert's program and its refutation by Gödel depended upon their work establishing the second area of mathematical logic, the application of mathematics to logic in the form of proof theory.[29] Despite the negative nature of the incompleteness theorems, Gödel's completeness theorem, a result in model theory and another application of mathematics to logic, can be understood as showing how close logicism came to being true: every rigorously defined mathematical theory can be exactly captured by a first-order logical theory; Frege's proof calculus is enough to *describe* the whole of mathematics, though not *equivalent* to it. Thus we see how complementary the two areas of mathematical logic have been.

If proof theory and model theory have been the foundation of mathematical logic, they have been but two of the four pillars of the subject. Set theory originated in the study of the infinite by Georg Cantor, and it has been the source of many of the most challenging and important issues in mathematical logic, from Cantor's theorem, through the status of the Axiom of Choice and the question of the independence of the continuum hypothesis, to the modern debate on large cardinal axioms.

Recursion theory captures the idea of computation in logical and arithmetic terms; its most classical achievements are the undecidability of the Entscheidungsproblem by Alan Turing, and his presentation of the Church-Turing thesis.[30] Today recursion theory is mostly concerned with the more refined problem of complexity classes — when is a problem efficiently solvable? — and the classification of degrees of unsolvability.[31]

Philosophical logic

Philosophical logic deals with formal descriptions of natural language. Most philosophers assume that the bulk of "normal" proper reasoning can be captured by logic, if one can find the right method for translating ordinary language into that logic. Philosophical logic is essentially a continuation of the traditional discipline that was called "Logic" before the invention of mathematical logic. Philosophical logic has a much greater concern with the connection between natural language and logic. As a result, philosophical logicians have contributed a great deal to the development of non-standard logics (e.g., free logics, tense logics) as well as various extensions of classical logic (e.g., modal logics), and non-standard semantics for such logics (e.g., Kripke's technique of supervaluations in the semantics of logic).

Logic and the philosophy of language are closely related. Philosophy of language has to do with the study of how our language engages and interacts with our thinking. Logic has an immediate impact on other areas of study. Studying logic and the relationship between logic and ordinary speech can help a person better structure his own arguments and critique the arguments of others. Many popular arguments are filled with errors because so many people are untrained in logic and unaware of how to formulate an argument correctly.

Computational Logic

Logic cut to the heart of computer science as it emerged as a discipline: Alan Turing's work on the Entscheidungsproblem followed from Kurt Gödel's work on the incompleteness theorems, and the notion of general purpose computers that came from this work was of fundamental importance to the designers of the computer machinery in the 1940s.

In the 1950s and 1960s, researchers predicted that when human knowledge could be expressed using logic with mathematical notation, it would be possible to create a machine that reasons, or artificial intelligence. This turned out to be more difficult than expected because of the complexity of human reasoning. In logic programming, a program consists of a set of axioms and rules. Logic programming systems such as Prolog compute the consequences of the axioms and rules in order to answer a query.

Today, logic is extensively applied in the fields of Artificial Intelligence, and Computer Science, and these fields provide a rich source of problems in formal and informal logic. Argumentation theory is one good example of how logic is being applied to artificial intelligence. The ACM Computing Classification System in particular regards:

- Section F.3 on Logics and meanings of programs and F.4 on Mathematical logic and formal languages as part of the theory of computer science: this work covers formal semantics of programming languages, as well as work of formal methods such as Hoare logic
- Boolean logic as fundamental to computer hardware: particularly, the system's section B.2 on Arithmetic and logic structures, relating to operatives AND, NOT, and OR;
- Many fundamental logical formalisms are essential to section I.2 on artificial intelligence, for example modal logic and default logic in Knowledge representation formalisms and methods, Horn clauses in logic programming, and description logic.

Furthermore, computers can be used as tools for logicians. For example, in symbolic logic and mathematical logic, proofs by humans can be computer-assisted. Using automated theorem proving the machines can find and check proofs, as well as work with proofs too lengthy to be written out by hand.

Bivalence and the law of the excluded middle

The logics discussed above are all "bivalent" or "two-valued"; that is, they are most naturally understood as dividing propositions into true and false propositions. Non-classical logics are those systems which reject bivalence.

Hegel developed his own dialectic logic that extended Kant's transcendental logic but also brought it back to ground by assuring us that "neither in heaven nor in earth, neither in the world of mind nor of nature, is there anywhere such an abstract 'either–or' as the understanding maintains. Whatever exists is concrete, with difference and opposition in itself".[32]

In 1910 Nicolai A. Vasiliev extended the law of excluded middle and the law of contradiction and proposed the law of excluded fourth and logic tolerant to contradiction.[33] In the early 20th century Jan Łukasiewicz investigated the extension of the traditional true/false values to include a third value, "possible", so inventing ternary logic, the first multi-valued logic.

Logics such as fuzzy logic have since been devised with an infinite number of "degrees of truth", represented by a real number between 0 and 1.[34]

Intuitionistic logic was proposed by L.E.J. Brouwer as the correct logic for reasoning about mathematics, based upon his rejection of the law of the excluded middle as part of his intuitionism. Brouwer rejected formalisation in mathematics, but his student Arend Heyting studied intuitionistic logic formally, as did Gerhard Gentzen. Intuitionistic logic has come to be of great interest to computer scientists, as it is a constructive logic and can be applied for extracting verified programs from proofs.

Modal logic is not truth conditional, and so it has often been proposed as a non-classical logic. However, modal logic is normally formalised with the principle of the excluded middle, and its relational semantics is bivalent, so this inclusion is disputable.

"Is logic empirical?"

What is the epistemological status of the laws of logic? What sort of argument is appropriate for criticizing purported principles of logic? In an influential paper entitled "Is logic empirical?"[35] Hilary Putnam, building on a suggestion of W.V. Quine, argued that in general the facts of propositional logic have a similar epistemological status as facts about the physical universe, for example as the laws of mechanics or of general relativity, and in particular that what physicists have learned about quantum mechanics provides a compelling case for abandoning certain familiar principles of classical logic: if we want to be realists about the physical phenomena described by quantum theory, then we should abandon the principle of distributivity, substituting for classical logic the quantum logic proposed by Garrett Birkhoff and John von Neumann.[36]

Another paper by the same name by Sir Michael Dummett argues that Putnam's desire for realism mandates the law of distributivity.[37] Distributivity of logic is essential for the realist's understanding of how propositions are true of the world in just the same way as he has argued the principle of bivalence is. In this way, the question, "Is logic empirical?" can be seen to lead naturally into the fundamental controversy in metaphysics on realism versus anti-realism.

Implication: strict or material?

It is obvious that the notion of implication formalised in classical logic does not comfortably translate into natural language by means of "if… then…", due to a number of problems called the *paradoxes of material implication*.

The first class of paradoxes involves counterfactuals, such as "If the moon is made of green cheese, then 2+2=5", which are puzzling because natural language does not support the principle of explosion. Eliminating this class of paradoxes was the reason for C. I. Lewis's formulation of strict implication, which eventually led to more radically revisionist logics such as relevance logic.

The second class of paradoxes involves redundant premises, falsely suggesting that we know the succedent because of the antecedent: thus "if that man gets elected, granny will die" is materially true since granny is mortal, regardless of the man's election prospects. Such sentences violate the Gricean maxim of relevance, and can be modelled by logics that reject the principle of monotonicity of entailment, such as relevance logic.

Tolerating the impossible

Hegel was deeply critical of any simplified notion of the Law of Non-Contradiction. It was based on Leibniz's idea that this law of logic also requires a sufficient ground to specify from what point of view (or time) one says that something cannot contradict itself. A building, for example, both moves and does not move; the ground for the first is our solar system and for the second the earth. In Hegelian dialectic, the law of non-contradiction, of identity, itself relies upon difference and so is not independently assertable.

Closely related to questions arising from the paradoxes of implication comes the suggestion that logic ought to tolerate inconsistency. Relevance logic and paraconsistent logic are the most important approaches here, though the concerns are different: a key consequence of classical logic and some of its rivals, such as intuitionistic logic, is that they respect the principle of explosion, which means that the logic collapses if it is capable of deriving a contradiction. Graham Priest, the main proponent of dialetheism, has argued for paraconsistency on the grounds that there are in fact, true contradictions.[38]

Rejection of logical truth

The philosophical vein of various kinds of skepticism contains many kinds of doubt and rejection of the various bases upon which logic rests, such as the idea of logical form, correct inference, or meaning, typically leading to the conclusion that there are no logical truths. Observe that this is opposite to the usual views in philosophical skepticism, where logic directs skeptical enquiry to doubt received wisdoms, as in the work of Sextus Empiricus.

Friedrich Nietzsche provides a strong example of the rejection of the usual basis of logic: his radical rejection of idealisation led him to reject truth as a "mobile army of metaphors, metonyms, and anthropomorphisms—in short ... metaphors which are worn out and without sensuous power; coins which have lost their pictures and now matter only as metal, no longer as coins". [39] His rejection of truth did not lead him to reject the idea of either inference or logic completely, but rather suggested that "logic [came] into existence in man's head [out] of illogic, whose realm originally must have been immense. Innumerable beings who made inferences in a way different from ours perished". [40] Thus there is the idea that logical inference has a use as a tool for human survival, but that its existence does not support the existence of truth, nor does it have a reality beyond the instrumental: "Logic, too, also rests on assumptions that do not correspond to anything in the real world".[41]

This position held by Nietzsche however, has come under extreme scrutiny for several reasons. He fails to demonstrate the validity of his claims and merely asserts them rhetorically. Furthermore, his position has been claimed to be self-refuting by philosophers, such as Jürgen Habermas, who have accused Nietszche of not even having a coherent perspective let alone a theory of knowledge.[42] Georg Lukacs in his book *The Destruction of Reason* has asserted that "Were we to study Nietzsche's statements in this area from a logico-philosophical angle, we would be confronted by a dizzy chaos of the most lurid assertions, arbitrary and violently incompatible".[43] Bertrand Russell referred to Nietzsche's claims as "empty words" in his book *A History of Western Philosophy*.[44]

Notes

[1] "possessed of reason, intellectual, dialectical, argumentative", also related to wiktionary:λόγος (logos), "word, thought, idea, argument, account, reason, or principle" (Liddell & Scott 1999; Online Etymology Dictionary 2001).

[2] Richard Henry Popkin; Avrum Stroll (1 July 1993). *Philosophy Made Simple* (http://books.google.com/books?id=TWNo-4euyesC&pg=PR7). Random House Digital, Inc. p. 238. ISBN 978-0-385-42533-9. . Retrieved 5 March 2012.

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[5] Cox, J. Robert; Willard, Charles Arthur, eds. (1983). *Advances in Argumentation Theory and Research*. Southern Illinois University Press. ISBN 978-0-8093-1050-0.

[6] For example, Nyaya (syllogistic recursion) dates back 1900 years.

[7] Mohists and the school of Names date back at 2200 years.

[8] Plato (1976). Buchanan, Scott. ed. *The Portable Plato*. Penguin. ISBN 0-14-015040-4.

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[11] For a more modern treatment, see Hamilton, A. G. (1980). *Logic for Mathematicians*. Cambridge University Press. ISBN 0-521-29291-3.

[12] Łukasiewicz, Jan (1957). *Aristotle's syllogistic from the standpoint of modern formal logic* (2nd ed.). Oxford University Press. p. 7. ISBN 978-0-19-824144-7.

[13] Bergmann, Merrie, James Moor, and Jack Nelson. The Logic Book fifth edition. New York, NY: McGraw-Hill, 2009.

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[15] E.g., Kline (1972, p.53) wrote "A major achievement of Aristotle was the founding of the science of logic".

[16] " Aristotle (http://chemistry.mtu.edu/~pcharles/SCIHISTORY/aristotle.html)", MTU Department of Chemistry.

[17] Jonathan Lear (1986). " *Aristotle and Logical Theory* (http://books.google.com/books?id=lXI7AAAAIAAJ&pg=PA34&dq&hl=en#v=onepage&q=&f=false)". Cambridge University Press. p.34. ISBN 0-521-31178-0

[18] Simo Knuuttila (1981). " *Reforging the great chain of being: studies of the history of modal theories* (http://books.google.com/books?id=iCCUF_OtA8AC&pg=PA71&dq&hl=en#v=onepage&q=&f=false)". Springer Science & Business. p.71. ISBN 90-277-1125-9

[19] Michael Fisher, Dov M. Gabbay, Lluís Vila (2005). " *Handbook of temporal reasoning in artificial intelligence* (http://books.google.com/books?id=Ajsvo6jWNhgC&pg=PA119&dq&hl=en#v=onepage&q=&f=false)". Elsevier. p.119. ISBN 0-444-51493-7

[20] Harold Joseph Berman (1983). " *Law and revolution: the formation of the Western legal tradition* (http://books.google.com/books?id=9-8fIBVgCQYC&pg=PA133&dq&hl=en#v=onepage&q=&f=false)". Harvard University Press. p.133. ISBN 0-674-51776-8

[21] The four Catuṣkoṭi logical divisions are formally very close to the four opposed propositions of the Greek tetralemma, which in turn are analogous to the four truth values of modern relevance logic Cf. Belnap (1977); Jayatilleke, K. N., (1967, The logic of four alternatives, in *Philosophy East and West*, University of Hawaii Press).

[22] Kisor Kumar Chakrabarti (June 1976). "Some Comparisons Between Frege's Logic and Navya-Nyaya Logic". *Philosophy and Phenomenological Research* (International Phenomenological Society) **36** (4): 554–563. doi:10.2307/2106873. JSTOR 2106873. "This paper consists of three parts. The first part deals with Frege's distinction between sense and reference of proper names and a similar distinction in Navya-Nyaya logic. In the second part we have compared Frege's definition of number to the Navya-Nyaya definition of number. In the third part we have shown how the study of the so-called 'restrictive conditions for universals' in Navya-Nyaya logic anticipated some of the developments of modern set theory."

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[39] Nietzsche, 1873, On Truth and Lies in a Nonmoral Sense.

[40] Nietzsche, 1882, *The Gay Science*.

[41] Nietzsche, 1878, *Human, All Too Human*

[42] Babette Babich, Habermas, Nietzsche, and Critical Theory

[43] http://www.marxists.org/archive/lukacs/works/destruction-reason/ch03.htm

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External links and further readings

- Logic (http://philpapers.org/browse/logic-and-philosophy-of-logic) at PhilPapers
- Logic (https://inpho.cogs.indiana.edu/taxonomy/2245) at the Indiana Philosophy Ontology Project
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- An Outline for Verbal Logic (http://logic-law.com/index.php?title=Verbal_Logic)
- Introductions and tutorials
  - An Introduction to Philosophical Logic (http://www.galilean-library.org/manuscript.php?postid=43782), by Paul Newall, aimed at beginners.
  - forall x: an introduction to formal logic (http://www.fecundity.com/logic/), by P.D. Magnus, covers sentential and quantified logic.
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  - Math & Logic: The history of formal mathematical, logical, linguistic and methodological ideas. (http://etext.lib.virginia.edu/DicHist/analytic/anaVII.html) In *The Dictionary of the History of Ideas.*
- Online Tools

- Interactive Syllogistic Machine (http://thefirstscience.org/syllogistic-machine/) A web based syllogistic machine for exploring fallacies, figures, terms, and modes of syllogisms.
- Reference material
  - Translation Tips (http://www.earlham.edu/~peters/courses/log/transtip.htm), by Peter Suber, for translating from English into logical notation.
  - Ontology and History of Logic. An Introduction (http://www.ontology.co/history-of-logic.htm) with an annotated bibliography.
- Reading lists
  - The London Philosophy Study Guide (http://www.ucl.ac.uk/philosophy/LPSG/) offers many suggestions on what to read, depending on the student's familiarity with the subject:
    - Logic & Metaphysics (http://www.ucl.ac.uk/philosophy/LPSG/L&M.htm)
    - Set Theory and Further Logic (http://www.ucl.ac.uk/philosophy/LPSG/SetTheory.htm)
    - Mathematical Logic (http://www.ucl.ac.uk/philosophy/LPSG/MathLogic.htm)

History of logic

The **history of logic** is the study of the development of the science of valid inference (logic). Formal logic was developed in ancient times in China, India, and Greece. Greek logic, particularly Aristotelian logic, found wide application and acceptance in science and mathematics.

Aristotle's logic was further developed by Islamic and Christian philosophers in the Middle Ages, reaching a high point in the mid-fourteenth century. The period between the fourteenth century and the beginning of the nineteenth century was largely one of decline and neglect, and is regarded as barren by at least one historian of logic.[1]

Logic was revived in the mid-nineteenth century, at the beginning of a revolutionary period when the subject developed into a rigorous and formalistic discipline whose exemplar was the exact method of proof used in mathematics. The development of the modern so-called "symbolic" or "mathematical" logic during this period is the most significant in the two-thousand-year history of logic, and is arguably one of the most important and remarkable events in human intellectual history.[2]

Progress in mathematical logic in the first few decades of the twentieth century, particularly arising from the work of Gödel and Tarski, had a significant impact on analytic philosophy and philosophical logic, particularly from the 1950s onwards, in subjects such as modal logic, temporal logic, deontic logic, and relevance logic.

Prehistory of logic

The pyramids of Egypt were built using geometry

Valid reasoning has been employed in all periods of human history. However, logic studies the *principles* of valid reasoning, inference and demonstration. It is probable that the idea of demonstrating a conclusion first arose in connection with geometry, which originally meant the same as "land measurement".[3] In particular, the ancient Egyptians had empirically discovered some truths of geometry, such as the formula for the volume of a truncated pyramid.[4]

Another origin can be seen in Babylonia. Esagil-kin-apli's medical *Diagnostic Handbook* in the 11th century BC was based on a logical set of axioms and assumptions,[5] while Babylonian astronomers in the 8th and 7th centuries BC employed an internal logic within their predictive planetary systems, an important contribution to the philosophy of science.[6]

Logic in Greek philosophy

Before Plato

While the ancient Egyptians empirically discovered some truths of geometry, the great achievement of the ancient Greeks was to replace empirical methods by demonstrative science.The systematic study of this seems to have begun with the school of Pythagoras in the late sixth century BC.[4] The three basic principles of geometry are that certain propositions must be accepted as true without demonstration, that all other propositions of the system are derived from these, and that the derivation must be *formal*, that is, independent of the particular subject matter in question.[4] Fragments of early proofs are preserved in the works of Plato and Aristotle,[7] and the idea of a deductive system was probably known in the Pythagorean school and the Platonic Academy.[4]

Separately from geometry, the idea of a standard argument pattern is found in the *Reductio ad absurdum* used by Zeno of Elea, a pre-Socratic philosopher of the fifth century BC. This is the technique of drawing an obviously false, absurd or impossible conclusion from an assumption, thus demonstrating that the assumption is false.[8] Plato's Parmenides portrays Zeno as claiming to have written a book defending the monism of Parmenides by demonstrating the absurd consequence of assuming that there is plurality. Other philosophers who practised such *dialectic* reasoning were the so-called minor Socratics, including Euclid of Megara, who were probably followers of Parmenides and Zeno. The members of this school were called "dialecticians" (from a Greek word meaning "to discuss").

Further evidence that pre-Aristotelian thinkers were concerned with the principles of reasoning is found in the fragment called *Dissoi Logoi*, probably written at the beginning of the fourth century BC. This is part of a protracted debate about truth and falsity.[9]

Plato's logic

None of the surviving works of the great fourth-century philosopher Plato (428–347) include any formal logic,[10] but they include important contributions to the field of philosophical logic. Plato raises three questions:

- What is it that can properly be called true or false?
- What is the nature of the connection between the assumptions of a valid argument and its conclusion?
- What is the nature of definition?

Plato's academy

The first question arises in the dialogue Theaetetus, where Plato identifies thought or opinion with talk or discourse (*logos*).[11] The second question is a result of Plato's theory of Forms. Forms are not things in the ordinary sense, nor strictly ideas in the mind, but they correspond to what philosophers later called universals, namely an abstract entity common to each set of things that have the same name. In both The Republic and The Sophist, Plato suggests that the necessary connection between the premisses and the conclusion of an argument corresponds to a necessary connection between "forms".[12] The third question is about definition. Many of Plato's dialogues concern the search for a definition of some important concept (justice, truth, the Good), and it is likely that Plato was impressed by the importance of definition in mathematics.[13] What underlies every definition is a Platonic Form, the common nature present in different particular things. Thus a definition reflects the ultimate object of our understanding, and is the foundation of all valid inference. This had a great influence on Aristotle, in particular Aristotle's notion of the essence of a thing, the "what it is to be" a particular thing of a certain kind.[14]

Aristotle's logic

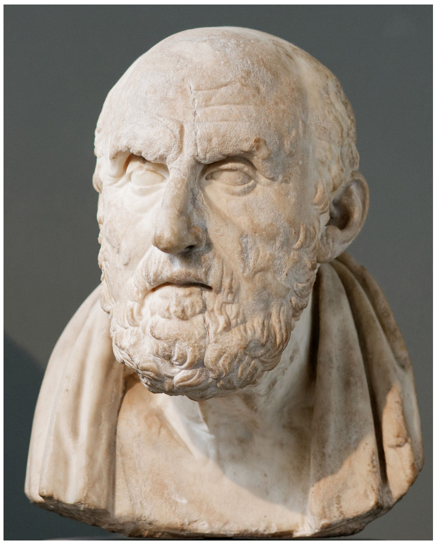
The logic of Aristotle, and particularly his theory of the syllogism, has had an enormous influence in Western thought.[15] His logical works, called the *Organon*, are the earliest formal study of logic that have come down to modern times. Though it is difficult to determine the dates, the probable order of writing of Aristotle's logical works is:

Aristotle's logic was still influential in the Renaissance

- *The Categories*, a study of the ten kinds of primitive term.
- *The Topics* (with an appendix called *On Sophistical Refutations*), a discussion of dialectics.
- *On Interpretation*, an analysis of simple categorical propositions, into simple terms, negation, and signs of quantity; and a comprehensive treatment of the notions of opposition and conversion. Chapter 7 is at the origin of the square of opposition (or logical square elaborated by Apuleius to be replaced, according to some scholars, by the logical hexagon of Robert Blanché presented in *Structures intellectuelles* (Vrin 1966). Chapter 9 contains the beginning of modal logic.
- *The Prior Analytics*, a formal analysis of valid argument or "syllogism".
- *The Posterior Analytics*, a study of scientific demonstration, containing Aristotle's mature views on logic.

These works are of outstanding importance in the history of logic. Aristotle was the first logician to attempt a systematic analysis of logical syntax, into noun (or *term*), and verb. In the *Categories*, he attempted to all the possible things that a term can refer to. This idea underpins his philosophical work, the *Metaphysics*, which also had a profound influence on Western thought. He was the first to deal with the principles of contradiction and excluded middle in a systematic way. He was the first *formal logician* (i.e. he gave the principles of reasoning using variables to show the underlying logical form of arguments). He was looking for relations of dependence which characterise necessary inference, and distinguished the validity of these relations, from the truth of the premises (the soundness of the argument). The *Prior Analytics* contains his exposition of the "syllogistic", where three important principles are applied for the first time in history: the use of variables, a purely formal treatment, and the use of an axiomatic system. In the *Topics* and *Sophistical Refutations* he also developed a theory of non-formal logic (e.g. the theory of fallacies).[16]

Stoic logic


Chrysippus of Soli

The other great school of Greek logic is that of the Stoics.[17] Stoic logic traces its roots back to the late 5th century BC philosopher, Euclid of Megara, a pupil of Socrates and slightly older contemporary of Plato. His pupils and successors were called "Megarians", or "Eristics", and later the "Dialecticians". The two most important dialecticians of the Megarian school were Diodorus Cronus and Philo who were active in the late 4th century BC. The Stoics adopted the Megarian logic and systemized it. The most important member of the school was Chrysippus (c. 278–c. 206 BC), who was its third head, and who formalized much of Stoic doctrine. He is supposed to have written over 700 works, including at least 300 on logic, almost none of which survive.[18][19] Unlike with Aristotle, we have no complete works by the Megarians or the early Stoics, and have to rely mostly on accounts (sometimes hostile) by later sources, including prominently Diogenes Laertius, Sextus Empiricus, Galen, Aulus Gellius, Alexander of Aphrodisias and Cicero.[20]

Three significant contributions of the Stoic school were (i) their account of modality, (ii) their theory of the Material conditional, and (iii) their account of meaning and truth.[21]

- *Modality*. According to Aristotle, the Megarians of his day claimed there was no distinction between potentiality and actuality.[22] Diodorus Cronus defined the possible as that which either is or will be, the impossible as what will not be true, and the contingent as that which either is already, or will be false.[23] Diodorus is also famous for his so-called Master argument, that the three propositions "everything that is past is true and necessary", "the impossible does not follow from the possible", and "What neither is nor will be is possible" are inconsistent. Diodorus used the plausibility of the first two to prove that nothing is possible if it neither is nor will be true.[24] Chrysippus, by contrast, denied the second premiss and said that the impossible could follow from the possible.[25]
- *Conditional statements*. The first logicians to debate conditional statements were Diodorus and his pupil Philo of Megara. Sextus Empiricus refers three times to a debate between Diodorus and Philo. Philo argued that a true conditional is one that does not begin with a truth and end with a falsehood. such as "if it is day, then I am talking". But Diodorus argued that a true conditional is what could not possibly begin with a truth and end with falsehood – thus the conditional quoted above could be false if it were day and I became silent. Philo's criterion of truth is what would now be called a truth-functional definition of "if ... then". In a second reference, Sextus says "According to him there are three ways in which a conditional may be true, and one in which it may be false."[26]
- *Meaning and truth*. The most important and striking difference between Megarian-Stoic logic and Aristotelian logic is that it concerns propositions, not terms, and is thus closer to modern propositional logic.[27] The Stoics distinguished between utterance (*phone*), which may be noise, speech (*lexis*), which is articulate but which may be meaningless, and discourse (*logos*), which is meaningful utterance. The most original part of their theory is the idea that what is expressed by a sentence, called a *lekton*, is something real. This corresponds to what is now called a *proposition*. Sextus says that according to the Stoics, three things are linked together, that which is signified, that which signifies, and the object. For example, what signifies is the word *Dion*, what is signified is what Greeks understand but barbarians do not, and the object is Dion himself.[28]

Logic in Asia

Logic in India

Formal logic began independently in ancient India and continued to develop through to early modern times, without any known influence from Greek logic.[29] Medhatithi Gautama (c. 6th century BCE) founded the *anviksiki* school of logic.[30] The *Mahabharata* (12.173.45), around the 5th century BCE, refers to the *anviksiki* and *tarka* schools of logic. Pāṇini (c. 5th century BCE) developed a form of logic (to which Boolean logic has some similarities) for his formulation of Sanskrit grammar. Logic is described by Chanakya (c. 350-283 BCE) in his *Arthashastra* as an independent field of inquiry *anviksiki*.[31]

Two of the six Indian schools of thought deal with logic: Nyaya and Vaisheshika. The Nyaya Sutras of Aksapada Gautama (c. 2nd century CE) constitute the core texts of the Nyaya school, one of the six orthodox schools of Hindu philosophy. This realist school developed a rigid five-member schema of inference involving an initial premise, a reason, an example, an application and a conclusion.[32] The idealist Buddhist philosophy became the chief opponent to the Naiyayikas. Nagarjuna (c. 150-250 CE), the founder of the Madhyamika ("Middle Way") developed an analysis known as the catuskoti (Sanskrit). This four-cornered argumentation systematically examined and rejected the affirmation of a proposition, its denial, the joint affirmation and denial, and finally, the rejection of its affirmation and denial. But it was with Dignaga (c 480-540 CE), who developed a formal syllogistic,[33] and his successor Dharmakirti that Buddhist logic reached its height. Their analysis centered on the definition of necessary logical entailment, "vyapti", also known as invariable concomitance or pervasion.[34] To this end a doctrine known as "apoha" or differentiation was developed.[35] This involved what might be called inclusion and exclusion of defining properties.

The difficulties involved in this enterprise, in part, stimulated the neo-scholastic school of Navya-Nyāya, which developed a formal analysis of inference in the sixteenth century. This later school began around eastern India and Bengal, and developed theories resembling modern logic, such as Gottlob Frege's "distinction between sense and reference of proper names" and his "definition of number," as well as the Navya-Nyaya theory of "restrictive conditions for universals" anticipating some of the developments in modern set theory.[36] Since 1824, Indian logic attracted the attention of many Western scholars, and has had an influence on important 19th-century logicians such as Charles Babbage, Augustus De Morgan, and particularly George Boole, as confirmed by his wife Mary Everest Boole who wrote in an "open letter to Dr Bose" titled "Indian Thought and Western Science in the Nineteenth Century" written in 1901:[37][38] "Think what must have been the effect of the intense Hinduizing of three such men as Babbage, De Morgan and George Boole on the mathematical atmosphere of 1830-1865"

In *La Logique et son histoire d' Aristote à Russell* (Armand Colin 1970) Robert Blanché, the author of *Structures intellectuelles* (Vrin, 1966) mentions that Józef Maria Bocheński speaks of a sort of Indian logical triangle to be compared with the square of Aristotle (or square of Apuleius), in other words with the square of opposition. This logical triangle announces the logical hexagon of Blanché. With this logical triangle, Indian logic proposes an interesting approach to the problem raised by the particular propositions of natural language. If Robert Blanché's logical hexagon is something more complete and therefore more powerful as regards the understanding of the relationship between logic and natural language, it may be that on a highly important point, Indian logic is superior to Aristotle's logic.

Logic in China

In China, a contemporary of Confucius, Mozi, "Master Mo", is credited with founding the Mohist school, whose canons dealt with issues relating to valid inference and the conditions of correct conclusions. In particular, one of the schools that grew out of Mohism, the Logicians, are credited by some scholars for their early investigation of formal logic. Due to the harsh rule of Legalism in the subsequent Qin Dynasty, this line of investigation disappeared in China until the introduction of Indian philosophy by Buddhists.

Medieval logic

Logic in Islamic philosophy

The works of Al-Farabi, Avicenna, Al-Ghazali, Averroes and other Muslim logicians both criticized and developed Aristotelian logic and were important in communicating the ideas of the ancient world to the medieval West.[39] Al-Farabi (Alfarabi) (873–950) was an Aristotelian logician who discussed the topics of future contingents, the number and relation of the categories, the relation between logic and grammar, and non-Aristotelian forms of inference.[40] Al-Farabi also considered the theories of conditional syllogisms and analogical inference, which were part of the Stoic tradition of logic rather than the Aristotelian.[41]

A text by Avicenna, founder of Avicennian logic

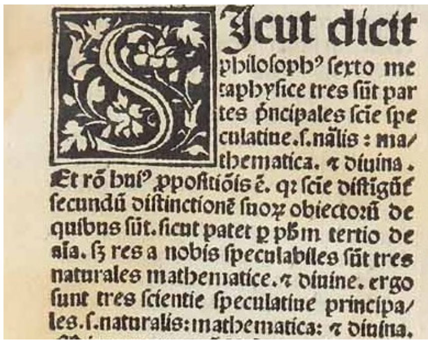
Ibn Sina (Avicenna) (980–1037) was the founder of Avicennian logic, which replaced Aristotelian logic as the domininant system of logic in the Islamic world,[42] and also had an important influence on Western medieval writers such as Albertus Magnus.[43] Avicenna wrote on the hypothetical syllogism[44] and on the propositional calculus, which were both part of the Stoic logical tradition.[45] He developed an original theory of "temporally modalized" syllogistic[40] and made use of inductive logic, such as the methods of agreement, difference and concomitant variation which are critical to the scientific method.[44] One of Avicenna's ideas had a particularly important influence on Western logicians such as William of Ockham. Avicenna's word for a meaning or notion (*ma'na*), was translated by the scholastic logicians as the Latin *intentio*. In medieval logic and epistemology, this is a sign in the mind that naturally represents a thing.[46] This was crucial to the development of Ockham's conceptualism. A universal term (e.g. "man") does not signify a thing existing in reality, but rather a sign in the mind (*intentio in intellectu*) which represents many things in reality. Ockham cites Avicenna's commentary on *Metaphysics* V in support of this view.[47]

Fakhr al-Din al-Razi (b. 1149) criticised Aristotle's "first figure" and formulated an early system of inductive logic, foreshadowing the system of inductive logic developed by John Stuart Mill (1806–1873).[48] Al-Razi's work was seen by later Islamic scholars as marking a new direction for Islamic logic, towards a Post-Avicennian logic. This was further elaborated by his student Afdaladdîn al-Khûnajî (d. 1249), who developed a form of logic revolving around the subject matter of conceptions and assents. In response to this tradition, Nasir al-Din al-Tusi (1201–1274) began a tradition of Neo-Avicennian logic which remained faithful to Avicenna's work and existed as an alternative to the more dominant Post-Avicennian school over the following centuries.[49]

Systematic refutations of Greek logic were written by the Illuminationist school, founded by Shahab al-Din Suhrawardi (1155–1191), who developed the idea of "decisive necessity", which refers to the reduction of all modalities (necessity, possibility, contingency and impossibility) to the single mode of necessity.[50] Ibn al-Nafis (1213–1288) wrote a book on Avicennian logic, which was a commentary of Avicenna's *Al-Isharat* (*The Signs*) and *Al-Hidayah* (*The Guidance*).[51] Another systematic refutation of Greek logic was written by Ibn Taymiyyah (1263–1328), the *Ar-Radd 'ala al-Mantiqiyyin* (*Refutation of Greek Logicians*), where he argued against the usefulness, though not the validity, of the syllogism[52] and in favour of inductive reasoning.[48] Ibn Taymiyyah also argued against the certainty of syllogistic arguments and in favour of analogy. His argument is that concepts founded on induction are themselves not certain but only probable, and thus a syllogism based on such concepts is no more certain than an argument based on analogy. He further claimed that induction itself is founded on a process of analogy. His model of analogical reasoning was based on that of juridical arguments.[53][54] This model of analogy has been used in the recent work of John F. Sowa.[54]

The *Sharh al-takmil fi'l-mantiq* written by Muhammad ibn Fayd Allah ibn Muhammad Amin al-Sharwani in the 15th century is the last major Arabic work on logic that has been studied.[55] However, "thousands upon thousands of pages" on logic were written between the 14th and 19th centuries, though only a fraction of the texts written during this period have been studied by historians, hence little is known about the original work on Islamic logic produced during this later period.[49]

Logic in medieval Europe



Brito's questions on the *Old Logic*

"Medieval logic" (also known as "Scholastic logic") generally means the form of Aristotelian logic developed in medieval Europe throughout the period c 1200–1600.[56] For centuries after Stoic logic had been formulated, it was the dominant system of logic in the classical world. When the study of logic resumed after the Dark Ages, the main source was the work of the Christian philosopher Boethius, who was familiar with some of Aristotle's logic, but almost none of the work of the Stoics.[57] Until the twelfth century the only works of Aristotle available in the West were the *Categories*, *On Interpretation* and Boethius' translation of the Isagoge of Porphyry (a commentary on the Categories). These works were known as the "Old Logic" (*Logica Vetus* or *Ars Vetus*). An important work in this tradition was the *Logica Ingredientibus* of Peter Abelard (1079–1142). His direct influence was small,[58] but his influence through pupils such as John of Salisbury was great, and his method of applying rigorous logical analysis to theology shaped the way that theological criticism developed in the period that followed.[59]

By the early thirteenth century the remaining works of Aristotle's *Organon* (including the *Prior Analytics*, *Posterior Analytics* and the *Sophistical Refutations*) had been recovered in the West and was revived by Saint Thomas Aquinas.[60] Logical work until then was mostly paraphrasis or commentary on the work of Aristotle.[61] The period from the middle of the thirteenth to the middle of the fourteenth century was one of significant developments in logic, particularly in three areas which were original, with little foundation in the Aristotelian tradition that came before. These were:[62]

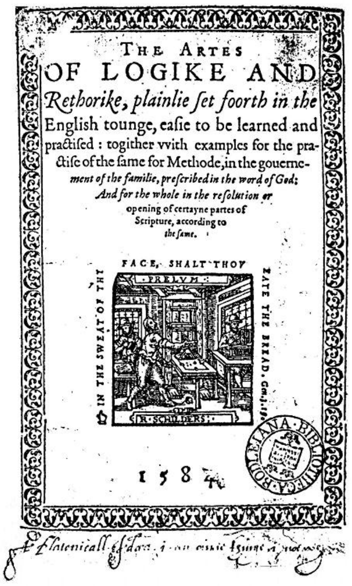
- The theory of supposition. Supposition theory deals with the way that predicates (e.g. 'man') range over a domain of individuals (e.g. all men).[63] In the proposition 'every man is an animal', does the term 'man' range over or 'supposit for' men existing in the present? Or does the range include past and future men? Can a term supposit for non-existing individuals? Some medievalists have argued that this idea was a precursor of modern first order logic.[64] "The theory of supposition with the associated theories of *copulatio* (sign-capacity of adjectival terms), *ampliatio* (widening of referential domain), and *distributio* constitute one of the most original achievements of Western medieval logic".[65]
- The theory of syncategoremata. Syncategoremata are terms which are necessary for logic, but which, unlike *categorematic* terms, do not signify on their own behalf, but 'co-signify' with other words. Examples of syncategoremata are 'and', 'not', 'every', 'if', and so on.
- The theory of consequences. A consequence is a hypothetical, conditional proposition: two propositions joined by the terms 'if ... then'. For example 'if a man runs, then God exists' (*Si homo currit, Deus est*).[66] A fully developed theory of consequences is given in Book III of William of Ockham's work Summa Logicae. There, Ockham distinguishes between 'material' and 'formal' consequences, which are roughly equivalent to the modern material implication and logical implication respectively. Similar accounts are given by Jean Buridan and Albert of Saxony.

The last great works in this tradition are the *Logic* of John Poinsot (1589–1644, known as John of St Thomas), the *Metaphysical Disputations* of Francisco Suarez (1548–1617), and the *Logica Demonstrativa* of Giovanni Girolamo Saccheri (1667–1733).

Traditional logic

The textbook tradition

Traditional logic generally means the textbook tradition that begins with Antoine Arnauld and Pierre Nicole's *Logic, or the Art of Thinking*, better known as the *Port-Royal Logic*.[67] Published in 1662, it was the most influential work on logic in England until the nineteenth century.[68] The book presents a loosely Cartesian doctrine (that the proposition is a combining of ideas rather than terms, for example) within a framework that is broadly derived from Aristotelian and medieval term logic. Between 1664 and 1700 there were eight editions, and the book had considerable influence after that.[68] The account of propositions that Locke gives in the *Essay* is essentially that of Port-Royal: "Verbal propositions, which are words, [are] the signs of our ideas, put together or separated in affirmative or negative sentences. So that proposition consists in the putting together or separating these signs, according as the things which they stand for agree or disagree." (Locke, *An Essay Concerning Human Understanding*, IV. 5. 6)



Dudley Fenner's *Art of Logic* (1584)

Another influential work was the *Novum Organum* by Francis Bacon, published in 1620. The title translates as "new instrument". This is a reference to Aristotle's work *Organon*. In this work, Bacon rejected the syllogistic method of Aristotle in favour of an alternative procedure "which by slow and faithful toil gathers information from things and brings it into understanding".[69] This method is known as inductive reasoning. The inductive method starts from empirical observation and proceeds to lower axioms or propositions. From the lower axioms more general ones can be derived (by induction). In finding the cause of a *phenomenal nature* such as heat, one must list all of the situations where heat is found. Then another list should be drawn up, listing situations that are similar to those of the first list except for the lack of heat. A third table lists situations where heat can vary. The *form nature*, or cause, of heat must be that which is common to all instances in the first table, is lacking from all instances of the second table and varies by degree in instances of the third table.

Other works in the textbook tradition include Isaac Watts' *Logick: Or, the Right Use of Reason* (1725), Richard Whately's *Logic* (1826), and John Stuart Mill's *A System of Logic* (1843). Although the latter was one of the last great works in the tradition, Mill's view that the foundations of logic lay in introspection[70] influenced the view that logic is best understood as a branch of psychology, an approach to the subject which dominated the next fifty years of its development, especially in Germany.[71]

Logic in Hegel's philosophy

Georg Wilhelm Friedrich Hegel

G.W.F. Hegel indicated the importance of logic to his philosophical system when he condensed his extensive *Science of Logic* into a shorter work published in 1817 as the first volume of his *Encyclopaedia of the Philosophical Sciences.* The "Shorter" or "Encyclopaedia" *Logic*, as it is often known, lays out a series of transitions which leads from the most empty and abstract of categories: Hegel begins with "Pure Being" and "Pure Nothing"—to the "Absolute"—the category which contains and resolves all the categories which preceded it. Despite the title, Hegel's *Logic* is not really a contribution to the science of valid inference. Rather than deriving conclusions about concepts through valid inference from premises, Hegel seeks to show that thinking about one concept compels thinking about another concept (one cannot, he argues, possess the concept of "Quality" without the concept of "Quantity"); and the compulsion here is not a matter of individual psychology, but arises almost organically from the content of the concepts themselves. His purpose is to show the rational structure of the "Absolute"—indeed of rationality itself. The method by which thought is driven from one concept to its contrary, and then to further concepts, is known as the Hegelian dialectic.

Although Hegel's *Logic* has had little impact on mainstream logical studies, its influence can be seen in Carl von Prantl's *Geschichte der Logik in Abendland* (1855–1867),[72] and in the work of the British Idealists—for example in F.H. Bradley's *Principles of Logic* (1883)—and in the economic, political and philosophical studies of Karl Marx and the various schools of Marxism.

Logic and psychology

Between the work of Mill and Frege stretched half a century during which logic was widely treated as a descriptive science, an empirical study of the structure of reasoning, and thus essentially as a branch of psychology.[73] The German psychologist Wilhelm Wundt, for example, discussed deriving "the logical from the psychological laws of thought", emphasizing that "psychological thinking is always the more comprehensive form of thinking."[74] This view was widespread among German philosophers of the period: Theodor Lipps described logic as "a specific discipline of psychology";[75] Christoph von Sigwart understood logical necessity as grounded in the individual's compulsion to think in a certain way;[76] and Benno Erdmann argued that "logical laws only hold within the limits of our thinking"[77] Such was the dominant view of logic in the years following Mill's work.[78] This psychological approach to logic was rejected by Gottlob Frege. It was also subjected to an extended and destructive critique by Edmund Husserl in the first volume of his *Logical Investigations* (1900), an assault which has been described as "overwhelming".[79] Husserl argued forcefully that grounding logic in psychological observations implied that all logical truths remained unproven, and that skepticism and relativism were unavoidable consequences.

Such criticisms did not immediately extirpate so-called "psychologism". For example, the American philosopher Josiah Royce, while acknowledging the force of Husserl's critique, remained "unable to doubt" that progress in psychology would be accompanied by progress in logic, and vice versa.[80]

Rise of modern logic

The period between the fourteenth century and the beginning of the nineteenth century had been largely one of decline and neglect, and is generally regarded as barren by historians of logic.[1] The revival of logic occurred in the mid-nineteenth century, at the beginning of a revolutionary period where the subject developed into a rigorous and formalistic discipline whose exemplar was the exact method of proof used in mathematics. The development of the modern so-called "symbolic" or "mathematical" logic during this period is the most significant in the 2,000-year history of logic, and is arguably one of the most important and remarkable events in human intellectual history.[2]

A number of features distinguish modern logic from the old Aristotelian or traditional logic, the most important of which are as follows:[81] Modern logic is fundamentally a *calculus* whose rules of operation are determined only by the *shape* and not by the *meaning* of the symbols it employs, as in mathematics. Many logicians were impressed by the "success" of mathematics, in that there had been no prolonged dispute about any truly mathematical result. C.S. Peirce noted[82] that even though a mistake in the evaluation of a definite integral by Laplace led to an error concerning the moon's orbit that persisted for nearly 50 years, the mistake, once spotted, was corrected without any serious dispute. Peirce contrasted this with the disputation and uncertainty surrounding traditional logic, and especially reasoning in metaphysics. He argued that a truly "exact" logic would depend upon mathematical, i.e., "diagrammatic" or "iconic" thought. "Those who follow such methods will ... escape all error except such as will be speedily corrected after it is once suspected". Modern logic is also "constructive" rather than "abstractive"; i.e., rather than abstracting and formalising theorems derived from ordinary language (or from psychological intuitions about validity), it constructs theorems by formal methods, then looks for an interpretation in ordinary language. It is entirely symbolic, meaning that even the logical constants (which the medieval logicians called "syncategoremata") and the categoric terms are expressed in symbols.

Periods of modern logic

The development of modern logic falls into roughly five periods:[83]

- The **embryonic period** from Leibniz to 1847, when the notion of a logical calculus was discussed and developed, particularly by Leibniz, but no schools were formed, and isolated periodic attempts were abandoned or went unnoticed.
- The **algebraic period** from Boole's Analysis to Schröder's *Vorlesungen*. In this period there were more practitioners, and a greater continuity of development.
- The **logicist period** from the Begriffsschrift of Frege to the *Principia Mathematica* of Russell and Whitehead. This was dominated by the "logicist school", whose aim was to incorporate the logic of all mathematical and scientific discourse in a single unified system, and which, taking as a fundamental principle that all mathematical truths are logical, did not accept any non-logical terminology. The major logicists were Frege, Russell, and the early Wittgenstein.[84] It culminates with the *Principia*, an important work which includes a thorough examination and attempted solution of the antinomies which had been an obstacle to earlier progress.
- The **metamathematical period** from 1910 to the 1930s, which saw the development of metalogic, in the finitist system of Hilbert, and the non-finitist system of Löwenheim and Skolem, the combination of logic and metalogic in the work of Gödel and Tarski. Gödel's incompleteness theorem of 1931 was one of the greatest achievements in the history of logic. Later in the 1930s Gödel developed the notion of set-theoretic constructibility.
- The **period after World War II**, when mathematical logic branched into four inter-related but separate areas of research: model theory, proof theory, computability theory, and set theory, and its ideas and methods began to influence philosophy.

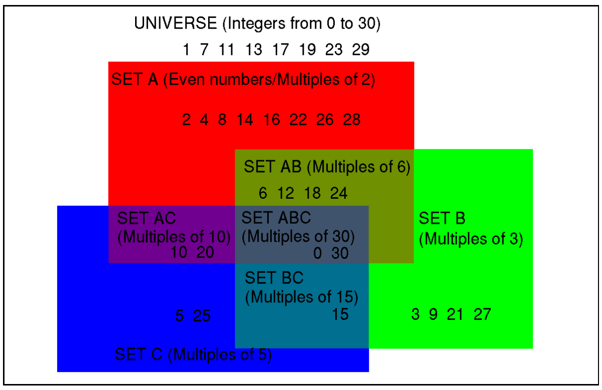
Embryonic period


Life of Raymond Lull. 14th-century manuscript.

The idea that inference could be represented by a purely mechanical process is found as early as Raymond Llull, who proposed a (somewhat eccentric) method of drawing conclusions by a system of concentric rings. The work of logicians such as the Oxford Calculators[85] led to a method of using letters instead of writing out logical calculations (*calculationes*) in words, a method used, for instance, in the *Logica magna* of Paul of Venice. Three hundred years after Llull, the English philosopher and logician Thomas Hobbes suggested that all logic and reasoning could be reduced to the mathematical operations of addition and subtraction.[86] The same idea is found in the work of Leibniz, who had read both Llull and Hobbes, and who argued that logic can be represented through a combinatorial process or calculus. But, like Llull and Hobbes, he failed to develop a detailed or comprehensive system, and his work on this topic was not published until long after his death. Leibniz says that ordinary languages are subject to "countless ambiguities" and are unsuited for a calculus, whose task is to expose mistakes in inference arising from the forms and structures of words;[87] hence, he proposed to identify an alphabet of human thought comprising fundamental concepts which could be composed to express complex ideas,[88] and create a *calculus ratiocinator* which would make reasoning "as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate."[89]

Gergonne (1816) said that reasoning does not have to be about objects about which we have perfectly clear ideas, since algebraic operations can be carried out without our having any idea of the meaning of the symbols involved.[90] Bolzano anticipated a fundamental idea of modern proof theory when he defined logical consequence or "deducibility" in terms of variables: a set of propositions *n*, *o*, *p* ... are *deducible* from propositions *a*, *b*, *c* ... in respect of the variables *i*, *j*, ... if any substitution for *i*, *j* that have the effect of making *a*, *b*, *c* ... true, simultaneously make the propositions *n*, *o*, *p* ... also.[91] This is now known as semantic validity.

Algebraic period



Boolean multiples

Modern logic begins with the so-called "algebraic school", originating with Boole and including Peirce, Jevons, Schröder and Venn.[92] Their objective was to develop a calculus to formalise reasoning in the area of classes, propositions and probabilities. The school begins with Boole's seminal work *Mathematical Analysis of Logic* which appeared in 1847, although De Morgan (1847) is its immediate precursor.[93] The fundamental idea of Boole's system is that algebraic formulae can be used to express logical relations. This idea occurred to Boole in his teenage years, working as an usher in a private school in Lincoln, Lincolnshire.[94] For example, let x and y stand for classes let the symbol = signify that the classes have the same members, xy stand for the class containing all and only the members of x and y and so on. Boole calls these *elective symbols*, i.e. symbols which select certain objects for consideration.[95] An expression in which elective symbols are used is called an *elective function*, and an equation of which the members are elective functions, is an *elective equation*.[96] The theory of elective functions and their "development" is essentially the modern idea of truth-functions and their expression in disjunctive normal form.[95]

Boole's system admits of two interpretations, in class logic, and propositional logic. Boole distinguished between "primary propositions" which are the subject of syllogistic theory, and "secondary propositions", which are the subject of propositional logic, and showed how under different "interpretations" the same algebraic system could represent both. An example of a primary proposition is "All inhabitants are either Europeans or Asiatics." An example of a secondary proposition is "Either all inhabitants are Europeans or they are all Asiatics."[97] These are easily distinguished in modern propositional calculus, where it is also possible to show that the first follows from the second, but it is a significant disadvantage that there is no way of representing this in the Boolean system.[98]

In his *Symbolic Logic* (1881), John Venn used diagrams of overlapping areas to express Boolean relations between classes or truth-conditions of propositions. In 1869 Jevons realised that Boole's methods could be mechanised, and constructed a "logical machine" which he showed to the Royal Society the following year.[95] In 1885 Allan Marquand proposed an electrical version of the machine that is still extant (picture at the Firestone Library [99]).

The defects in Boole's system (such as the use of the letter *v* for existential propositions) were all remedied by his followers. Jevons published *Pure Logic, or the Logic of Quality apart from Quantity* in 1864, where he suggested a symbol to signify exclusive or, which allowed Boole's system to be greatly simplified.[100] This was usefully exploited by Schröder when he set out theorems in parallel columns in his *Vorlesungen* (1890–1905). Peirce (1880) showed how all the Boolean elective functions could be expressed by the use of a single primitive binary operation, "neither ... nor ..." and equally well "not both ... and ...",[101] however, like many of Peirce's innovations, this remained unknown or unnoticed until Sheffer rediscovered it in 1913.[102] Boole's early work also lacks the idea of the logical sum which originates in Peirce (1867), Schröder (1877) and Jevons (1890),[103] and the concept of inclusion, first suggested by Gergonne (1816) and clearly articulated by Peirce (1870).

The success of Boole's algebraic system suggested that all logic must be capable of algebraic representation, and there were attempts to express a logic of relations in such form, of which the most ambitious was Schröder's monumental *Vorlesungen über die Algebra der Logik* ("Lectures on the Algebra of Logic", vol iii 1895), although the original idea was again anticipated by Peirce.[104]

Logicist period

Frege's "Concept Script"

After Boole, the next great advances were made by the German mathematician Gottlob Frege. Frege's objective was the program of Logicism, i.e. demonstrating that arithmetic is identical with logic.[105] Frege went much further than any of his predecessors in his rigorous and formal approach to logic, and his calculus or Begriffsschrift is important.[105] Frege also tried to show that the concept of number can be defined by purely logical means, so that (if he was right) logic includes arithmetic and all branches of mathematics that are reducible to arithmetic. He was not the first writer to suggest this. In his pioneering work *Die Grundlagen der Arithmetik* (The Foundations of Arithmetic), sections 15–17, he acknowledges the efforts of Leibniz, J.S. Mill as well as Jevons, citing the latter's claim that "algebra is a highly developed logic, and number but logical discrimination."[106]

Frege's first work, the *Begriffsschrift* ("concept script") is a rigorously axiomatised system of propositional logic, relying on just two connectives (negational and conditional), two rules of inference (*modus ponens* and substitution), and six axioms. Frege referred to the "completeness" of this system, but was unable to prove this.[107] The most significant innovation, however, was his explanation of the quantifier in terms of mathematical functions. Traditional logic regards the sentence "Caesar is a man" as of fundamentally the same form as "all men are mortal." Sentences with a proper name subject were regarded as universal in character, interpretable as "every Caesar is a man".[108] Frege argued that the quantifier expression "all men" does not have the same logical or semantic form as "all men", and that the universal proposition "every A is B" is a complex proposition involving two *functions*, namely ' – is A' and ' – is B' such that whatever satisfies the first, also satisfies the second. In modern notation, this would be expressed as

(x) Ax -> Bx

In English, "for all x, if Ax then Bx". Thus only singular propositions are of subject-predicate form, and they are irreducibly singular, i.e. not reducible to a general proposition. Universal and particular propositions, by contrast, are not of simple subject-predicate form at all. If "all mammals" were the logical subject of the sentence "all mammals are land-dwellers", then to negate the whole sentence we would have to negate the predicate to give "all mammals are *not* land-dwellers". But this is not the case.[109] This functional analysis of ordinary-language sentences later had a great impact on philosophy and linguistics.

This means that in Frege's calculus, Boole's "primary" propositions can be represented in a different way from "secondary" propositions. "All inhabitants are either Europeans or Asiatics" is

(x) [I(x) -> (E(x) v A(x))]

whereas "All the inhabitants are Europeans or all the inhabitants are Asiatics" is

(x) (I(x) -> E(x)) v (x) (I(x) -> A(x))

As Frege remarked in a critique of Boole's calculus:

> "The real difference is that I avoid [the Boolean] division into two parts ... and give a homogeneous presentation of the lot. In Boole the two parts run alongside one another, so that one is like the mirror image of the other, but for that very reason stands in no organic relation to it'[110]

As well as providing a unified and comprehensive system of logic, Frege's calculus also resolved the ancient problem of multiple generality. The ambiguity of "every girl kissed a boy" is difficult to express in traditional logic, but Frege's logic captures this through the different scope of the quantifiers. Thus

(x) [girl(x) -> E(y) (boy(y) & kissed(x,y))]

means that to every girl there corresponds some boy (any one will do) who the girl kissed. But

E(x) [boy(x) & (y) (girl(y) -> kissed(y,x))]

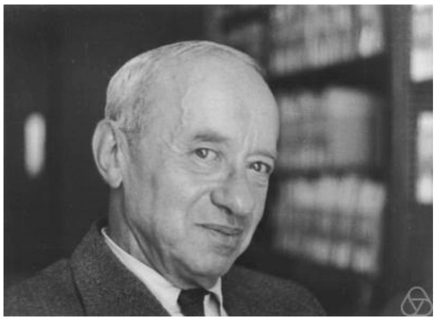
means that there is some particular boy whom every girl kissed. Without this device, the project of logicism would have been doubtful or impossible. Using it, Frege provided a definition of the ancestral relation, of the many-to-one relation, and of mathematical induction.[111]

This period overlaps with the work of the so-called "mathematical school", which included Dedekind, Pasch, Peano, Hilbert, Zermelo, Huntington, Veblen and Heyting. Their objective was the axiomatisation of branches of mathematics like geometry, arithmetic, analysis and set theory.

The logicist project received a near-fatal setback with the discovery of a paradox in 1901 by Bertrand Russell. This proved that the Frege's naive set theory led to a contradiction. Frege's theory is that for any formal criterion, there is a set of all objects that meet the criterion. Russell showed that a set containing exactly the sets that are not members of themselves would contradict its own definition (if it is not a member of itself, it is a member of itself, and if it is a member of itself, it is not).[112] This contradiction is now known as Russell's paradox. One important method of resolving this paradox was proposed by Ernst Zermelo.[113] Zermelo set theory was the first axiomatic set theory. It was developed into the now-canonical Zermelo–Fraenkel set theory (ZF).

The monumental Principia Mathematica, a three-volume work on the foundations of mathematics, written by Russell and Alfred North Whitehead and published 1910–13 also included an attempt to resolve the paradox, by means of an elaborate system of types: a set of elements is of a different type than is each of its elements (set is not the element; one element is not the set) and one cannot speak of the "set of all sets". The *Principia* was an attempt to derive all mathematical truths from a well-defined set of axioms and inference rules in symbolic logic.

Metamathematical period



Alfred Tarski

The names of Gödel and Tarski dominate the 1930s,[114] a crucial period in the development of metamathematics – the study of mathematics using mathematical methods to produce metatheories, or mathematical theories about other mathematical theories. Early investigations into metamathematics had been driven by Hilbert's program. which sought to resolve the ongoing crisis in the foundations of mathematics by grounding all of mathematics to a finite set of axioms, proving its consistency by "finitistic" means and providing a procedure which would decide the truth or falsity of any mathematical statement. Work on metamathematics culminated in the work of Gödel, who in 1929 showed that a given first-order sentence is deducible if and only if it is logically valid – i.e. it is true in every structure for its language. This is known as Gödel's completeness theorem. A year later, he proved two important theorems, which showed Hibert's program to be unattainable in its original form. The first is that no consistent system of axioms whose theorems can be listed by an effective procedure such as an algorithm or computer program is capable of proving all facts about the natural numbers. For any such system, there will always be statements about the natural numbers that are true, but that are unprovable within the system. The second is that if such a system is also capable of proving certain basic facts about the natural numbers, then the system cannot prove the consistency of the system itself. These two results are known as Gödel's incompleteness theorems, or simply *Gödel's Theorem*. Later in the decade, Gödel developed the concept of set-theoretic constructibility, as part of his proof that the axiom of choice and the continuum hypothesis are consistent with Zermelo–Fraenkel set theory.

In proof theory, Gerhard Gentzen developed natural deduction and the sequent calculus. The former attempts to model logical reasoning as it 'naturally' occurs in practice and is most easily applied to intuitionistic logic, while the latter was devised to clarify the derivation of logical proofs in any formal system. Since Gentzen's work, natural deduction and sequent calculi have been widely applied in the fields of proof theory, mathematical logic and computer science. Gentzen also proved normalization and cut-elimination theorems for intuitionistic and classical logic which could be used to reduce logical proofs to a normal form.[115][116]

Alfred Tarski, a pupil of Łukasiewicz, is best known for his definition of truth and logical consequence, and the semantic concept of logical satisfaction. In 1933, he published (in Polish) *The concept of truth in formalized languages*, in which he proposed his semantic theory of truth: a sentence such as "snow is white" is true if and only if snow is white. Tarski's theory separated the metalanguage, which makes the statement about truth, from the object language, which contains the sentence whose truth is being asserted, and gave a correspondence (the T-schema) between phrases in the object language and elements of an interpretation. Tarski's approach to the difficult idea of explaining truth has been enduringly influential in logic and philosophy, especially in the development of model theory.[117] Tarski also produced important work on the methodology of deductive systems, and on fundamental principles such as completeness, decidability, consistency and definability. According to Anita Feferman, Tarski "changed the face of logic in the twentieth century".[118]

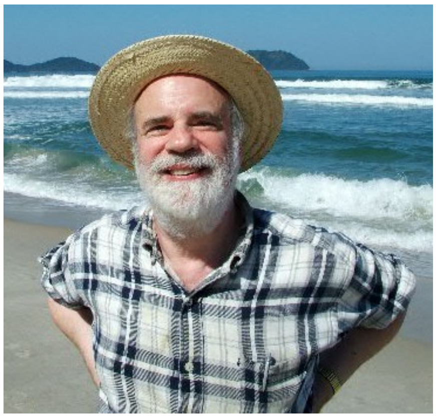
Alonzo Church and Alan Turing proposed formal models of computability, giving independent negative solutions to Hilbert's *Entscheidungsproblem* in 1936 and 1937, respectively. The *Entscheidungsproblem* asked for a procedure that, given any formal mathematical statement, would algorithmically determine whether the statement is true. Church and Turing proved there is no such procedure; Turing's paper introduced the halting problem as a key example of a mathematical problem without an algorithmic solution.

Church's system for computation developed into the modern λ-calculus, while the Turing machine became a standard model for a general-purpose computing device. It was soon shown that many other proposed models of computation were equivalent in power to those proposed by Church and Turing. These results led to the Church–Turing thesis that any deterministic algorithm that can be carried out by a human can be carried out by a Turing machine. Church proved additional undecidability results, showing that both Peano arithmetic and first-order logic are undecidable. Later work by Emil Post and Stephen Cole Kleene in the 1940s extended the scope of computability theory and introduced the concept of degrees of unsolvability.

The results of the first few decades of the twentieth century also had an impact upon analytic philosophy and philosophical logic, particularly from the 1950s onwards, in subjects such as modal logic, temporal logic, deontic logic, and relevance logic.

Logic after WWII

After World War II, mathematical logic branched into four inter-related but separate areas of research: model theory, proof theory, computability theory, and set theory.[119]



Saul Kripke

In set theory, the method of forcing revolutionized the field by providing a robust method for constructing models and obtaining independence results. Paul Cohen introduced this method in 1962 to prove the independence of the continuum hypothesis and the axiom of choice from Zermelo–Fraenkel set theory.[120] His technique, which was simplified and extended soon after its introduction, has since been applied to many other problems in all areas of mathematical logic.

Computability theory had its roots in the work of Turing, Church, Kleene, and Post in the 1930s and 40s. It developed into a study of abstract computability, which became known as recursion theory.[121] The priority method, discovered independently by Albert Muchnik and Richard Friedberg in the 1950s, led to major advances in the understanding of the degrees of unsolvability and related structures. Research into higher-order computability theory demonstrated its connections to set theory. The fields of constructive analysis and computable analysis were developed to study the effective content of classical mathematical theorems; these in turn inspired the program of reverse mathematics. A separate branch of computability theory, computational complexity theory, was also characterized in logical terms as a result of investigations into descriptive complexity.

Abraham Robinson

Model theory applies the methods of mathematical logic to study models of particular mathematical theories. Alfred Tarski published much pioneering work in the field, which is named after a series of papers he published under the title *Contributions to the theory of models*. In the 1960s, Abraham Robinson used model-theoretic techniques to develop calculus and analysis based on infinitesimals, a problem that first had been proposed by Leibniz.

In proof theory, the relationship between classical mathematics and intuitionistic mathematics was clarified via tools such as the realizability method invented by Georg Kreisel and Gödel's *Dialectica* interpretation. This work inspired the contemporary area of proof mining. The Curry-Howard correspondence emerged as a deep analogy between logic and computation, including a correspondence between systems of natural deduction and typed lambda calculi used in computer science. As a result, research into this class of formal systems began to address both logical and computational aspects; this area of research came to be known as modern type theory. Advances were also made in ordinal analysis and the study of independence results in arithmetic such as the Paris–Harrington theorem.

This was also a period, particularly in the 1950s and afterwards, when the ideas of mathematical logic begin to influence philosophical thinking. For example, tense logic is a formalised system for representing, and reasoning about, propositions qualified in terms of time. The philosopher Arthur Prior played a significant role in its development in the 1960s. Modal logics extend the scope of formal logic to include the elements of modality (for example, possibility and necessity). The ideas of Saul Kripke, particularly about possible worlds, and the formal system now called Kripke semantics have had a profound impact on analytic philosophy.[122] His best known and most influential work is *Naming and Necessity* (1980).[123] Deontic logics are closely related to modal logics: they attempt to capture the logical features of obligation, permission and related concepts. Ernst Mally, a pupil of Alexius Meinong, was the first to propose a formal deontic system in his *Grundgesetze des Sollens*, based on the syntax of Whitehead's and Russell's propositional calculus. Another logical system founded after World War II was fuzzy logic by Iranian mathematician Lotfi Asker Zadeh in 1965.

Notes

[1] Oxford Companion p. 498; Bochenski, Part I Introduction, *passim*

[2] Oxford Companion p. 500

[3] Kneale, p. 2

[4] Kneale p. 3

[5] H. F. J. Horstmanshoff, Marten Stol, Cornelis Tilburg (2004), *Magic and Rationality in Ancient Near Eastern and Graeco-Roman Medicine*, p. 99, Brill Publishers, ISBN 90-04-13666-5.

[6] D. Brown (2000), *Mesopotamian Planetary Astronomy-Astrology* , Styx Publications, ISBN 90-5693-036-2.

[7] Heath, *Mathematics in Aristotle*, cited in Kneale, p. 5

[8] Kneale p. 15

[9] Kneale, p. 16

[10] Kneale p. 17

[11] "forming an opinion is talking, and opinion is speech that is held not with someone else or aloud but in silence with oneself" *Theaetetus* 189E–190A

[12] Kneale p. 20. For example, the proof given in the *Meno* that the square on the diagonal is double the area of the original square presumably involves the forms of the square and the triangle, and the necessary relation between them

[13] Kneale p. 21

[14] Zalta, Edward N. " Aristotle's Logic (http://plato.stanford.edu/entries/aristotle-logic/#Def)". Stanford University, 18 March 2000. Retrieved 13 March 2010.

[15] See e.g. Aristotle's logic (http://plato.stanford.edu/entries/aristotle-logic/), Stanford Encyclopedia of Philosophy

[16] Bochenski p. 63

[17] "Throughout later antiquity two great schools of logic were distinguished, the Peripatetic which was derived from Aristotle, and the Stoic which was developed by Chrysippus from the teachings of the Megarians" – Kneale p. 113

[18] *Oxford Companion*, article "Chrysippus", p. 134

[19] (http://plato.stanford.edu/entries/logic-ancient/) Stanford Encyclopedia of Philosophy: Susanne Bobzien, *Ancient Logic*

[20] K. Huelser, Die Fragmente zur Dialektik der Stoiker, 4 vols, Stuttgart 1986-7

[21] Kneale 117–158

[22] *Metaphysics* Eta 3, 1046b 29

[23] Boethius, *Commentary on the Perihermenias*, Meiser p. 234

[24] Epictetus, *Dissertationes* ed. Schenkel ii. 19. I.

[25] Alexander p. 177

[26] Sextus, *Adv. Math.* Bk viii, Section 113

[27] See e.g. Lukasiewicz p. 21

[28] Sextus Bk viii., Sections 11, 12

[29] Bochenski p. 446

[30] S. C. Vidyabhusana (1971). *A History of Indian Logic: Ancient, Mediaeval, and Modern Schools*.

[31] R. P. Kangle (1986). *The Kautiliya Arthashastra* (1.2.11). Motilal Banarsidass.

[32] Bochenski p. 417 and *passim*

[33] Bochenski pp. 431–7

[34] Bochenski p. 438

[35] Bochenksi p. 441

[36] Kisor Kumar Chakrabarti (June 1976). "Some Comparisons Between Frege's Logic and Navya-Nyaya Logic". *Philosophy and Phenomenological Research* (International Phenomenological Society) **36** (4): 554–563. doi:10.2307/2106873. JSTOR 2106873. "This paper

consists of three parts. The first part deals with Frege's distinction between sense and reference of proper names and a similar distinction in Navya-Nyaya logic. In the second part we have compared Frege's definition of number to the Navya-Nyaya definition of number. In the third part we have shown how the study of the so-called 'restrictive conditions for universals' in Navya-Nyaya logic anticipated some of the developments of modern set theory."

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[45] Goodman, Lenn Evan (1992); *Avicenna*, p. 188, Routledge, ISBN 0-415-01929-X.

[46] Kneale p. 229

[47] Kneale: p. 266; Ockham: Summa Logicae i. 14; Avicenna: *Avicennae Opera* Venice 1508 f87rb

[48] Muhammad Iqbal, *The Reconstruction of Religious Thought in Islam*, "The Spirit of Muslim Culture" (cf. (http://www.allamaiqbal.com/works/prose/english/reconstruction) and (http://www.witness-pioneer.org/vil/Books/MI_RRTI/chapter_05.htm))

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[57] Kneale p. 198

[58] Stephen Dumont, article "Peter Abelard" in Gracia and Noone p. 492

[59] Kneale, pp. 202–3

[60] See e.g. Kneale p. 225

[61] Boehner p. 1

[62] Boehner pp. 19–76

[63] Boehner p. 29

[64] Boehner p. 30

[65] Ebbesen 1981

[66] Boehner pp. 54–5

[67] *Oxford Companion* p. 504, article "Traditional logic"

[68] Buroker xxiii

[69] Farrington, 1964, 89

[70] N. Abbagnano, "Psychologism" in P. Edwards (ed) *The Encyclopaedia of Philosophy*, MacMillan, 1967

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[73] See e.g. Psychologism (http://plato.stanford.edu/entries/psychologism), Stanford Encyclopedia of Philosophy

[74] Wilhelm Wundt, *Logik* (1880–1883); quoted in Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, pp. 115–116.

[75] Theodor Lipps, *Grundzüge der Logik* (1893); quoted in Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, p. 40

[76] Christoph von Sigwart, *Logik* (1873–78); quoted in Edmund Husserl, *Logical Investigations,* translated J.N. Findlay, Routledge, 2008, Volume 1, p. 51

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[80] Josiah Royce, "Recent Logical Enquiries and their Psychological Bearings" (1902) in John J. McDermott (ed) *The Basic Writings of Josiah Royce* Volume 2, Fordham University Press, 2005, p. 661

[81] Bochenski, p. 266

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[83] See Bochenski p. 269

[84] *Oxford Companion* p. 499

[85] Edith Sylla (1999), "Oxford Calculators", in *The Cambridge Dictionary of Philosophy*, Cambridge, Cambridgeshire: Cambridge.

[86] El. philos. sect. I de corp 1.1.2.

[87] Bochenski p. 274

[88] Rutherford, Donald, 1995, "Philosophy and language" in Jolley, N., ed., *The Cambridge Companion to Leibniz*. Cambridge Univ. Press.

[89] Wiener, Philip, 1951. *Leibniz: Selections*. Scribner.

[90] *Essai de dialectique rationelle*, 211n, quoted in Bochenski p. 277.

[91] *Wissenschaftslehre II* 198ff, quoted in Bochenski 280; see *Oxford 'Companion* p. 498.

[92] See e.g. Bochenski p. 296 and *passim*

[93] Before publishing, he wrote to De Morgan, who was just finishing his work *Formal Logic*. De Morgan suggested they should publish first, and thus the two books appeared at the same time, possibly even reaching the bookshops on the same day. cf. Kneale p. 404

[94] Kneale p. 404

[95] Kneale p. 407

[96] Boole (1847) p. 16

[97] Boole 1847 pp. 58–9

[98] Beaney p. 11

[99] http://finelib.princeton.edu/instruction/wri172_demonstration.php

[100] Kneale p. 422

[101] Peirce, "A Boolean Algebra with One Constant", 1880 MS, *Collected Papers* v. 4, paragraphs 12–20, reprinted *Writings* v. 4, pp. 218-21. Google Preview (http://books.google.com/books?id=E7ZUnx3FqrcC&q=378+Winter).

[102] *Trans. Amer. Math. Soc., xiv (1913)*, pp. 481–8. This is now known as the Sheffer stroke

[103] Bochenski 296

[104] See CP III

[105] Kneale p. 435

[106] Jevons, *The Principles of Science*, London 1879, p. 156, quoted in *Grundlagen* 15

[107] Beaney p. 10 – the completeness of Frege's system was eventually proved by Jan Łukasiewicz in 1934

[108] See for example the argument by the medieval logician William of Ockham that singular propositions are universal, in Summa Logicae III. 8 (??)

[109] "On concept and object" p. 198; Geach p. 48

[110] BLC p. 14, quoted in Beaney p. 12

[111] See e.g. The Internet Encyclopedia of Philosophy (http://www.utm.edu/research/iep/f/frege.htm), article "Frege"

[112] See e.g. Potter 2004

[113] Zermelo 1908

[114] Feferman 1999 p. 1

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[116] Alex Sakharov, " Cut Elimination Theorem (http://mathworld.wolfram.com/CutEliminationTheorem.html)" from MathWorld.

[117] Feferman and Feferman 2004, p. 122, discussing "The Impact of Tarski's Theory of Truth".

[118] Feferman 1999, p. 1

[119] See e.g. Barwise, *Handbook of Mathematical Logic*

[120] The Independence of the Continuum Hypothesis, II (http://links.jstor.org/sici?sici=0027-8424(19640115)51:1<105:TIOTCH>2.0.CO;2-U) Paul J. Cohen Proceedings of the National Academy of Sciences of the United States of America, Vol. 51, No. 1. (Jan. 15, 1964), pp. 105-110.

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[122] Jerry Fodor, " Water's water everywhere (http://www.lrb.co.uk/v26/n20/jerry-fodor/waters-water-everywhere)", *London Review of Books*, 21 October 2004

[123] See *Philosophical Analysis in the Twentieth Century: Volume 2: The Age of Meaning*, Scott Soames: "*Naming and Necessity* is among the most important works ever, ranking with the classical work of Frege in the late nineteenth century, and of Russell, Tarski and Wittgenstein in the first half of the twentieth century". Cited in Byrne, Alex and Hall, Ned. 2004. 'Necessary Truths'. *Boston Review* October/November 2004

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External links

- History of Logic in Relationship to Ontology (http://www.ontology.co/history-of-logic.htm) Annotated bibliography on the history of logic
- Ancient Logic (http://plato.stanford.edu/entries/logic-ancient) entry by Susanne Bobzien in the *Stanford Encyclopedia of Philosophy*
- Peter of Spain (http://plato.stanford.edu/entries/peter-spain) entry by Joke Spruyt in the *Stanford Encyclopedia of Philosophy*
- Paul Spade's "Thoughts Words and Things" (http://pvspade.com/Logic/docs/thoughts1_1a.pdf)
- John of St Thomas (http://www.newadvent.org/cathen/08479b.htm)
- Insights, Images, and Bios of 116 logicians (http://humbox.ac.uk/3682/)

The study of logic

Logical form

In logic the **logical form** of a sentence (or proposition or statement or truthbearer) or set of sentences is the form obtained by abstracting from the subject matter of its content terms or by regarding the content terms as mere placeholders or blanks on a form. In an ideal logical language, the logical form can be determined from syntax alone; formal languages used in formal sciences are examples of such languages. Logical form however should not be confused with the mere syntax used to represent it; there may be more than one string that represents the same logical form in a given language.[1]

The logical form of an argument is called the **argument form** or *test form* of the argument.

History

That the concept of form is fundamental to logic was already recognized in ancient times. Aristotle was probably the first to employ variable letters to represent valid inferences (in the Prior analytics). (For which reason Łukasiewicz says that the introduction of variables was 'one of Aristotle's greatest inventions').

According to the followers of Aristotle (such as Ammonius), only the logical principles stated in schematic terms belong to logic, and not those given in concrete terms. The concrete terms *man*, *mortal*, etc., are analogous to the substitution values of the schematic placeholders 'A', 'B', 'C', which were called the 'matter' (Greek *hyle*, Latin *materia*) of the argument.

The term "logical form" was introduced by Bertrand Russell in 1914, in the context of his program to formalize natural language and reasoning, which he called philosophical logic. Russell wrote: "Some kind of knowledge of logical forms, though with most people it is not explicit, is involved in in all understanding of discourse. It is the business of philosophical logic to extract this knowledge from its concrete integuments, and to render it explicit and pure." [2][3]

Example of argument form

To demonstrate the important notion of the *form* of an argument, substitute letters for similar items throughout the sentences in the original argument.

Original argument

> All humans are mortal.
>
> Socrates is human.
>
> Therefore, Socrates is mortal.

Argument Form

> All *H* are *M*.
>
> *S* is *H*.
>
> Therefore, *S* is *M*.

All we have done in the *Argument form* is to put 'H' for 'human' and 'humans', 'M' for 'mortal', and 'S' for 'Socrates'; what results is the *form* of the original argument. Moreover, each individual sentence of the *Argument form* is the *sentence form* of its respective sentence in the original argument.[4]

Importance of argument form

Attention is given to argument and sentence form, because *form is what makes an argument valid or cogent*. Some examples of valid argument forms are modus ponens, modus tollens, disjunctive syllogism, hypothetical syllogism and dilemma. Two invalid argument forms are affirming the consequent and denying the antecedent.

A logical argument, seen as an ordered set of sentences, has a logical form that derives from the form of its constituent sentences; the logical form of an argument is sometimes called argument form.[5] Some authors only define logical form with respect to whole arguments, as the schemata or inferential structure of the argument.[6] In argumentation theory or informal logic, an argument form is sometimes seen as a broader notion than the logical form.[7]

It consists of stripping out all spurious grammatical features from the sentence (such as gender, and passive forms), and replacing all the expressions specific to *the subject matter* of the argument by schematic variables. Thus, for example, the expression 'all A's are B's' shows the logical form which is common to the sentences 'all men are mortals', 'all cats are carnivores', 'all Greeks are philosophers' and so on.

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[5] J. C. Beall (2009). *Logic: the Basics* (http://books.google.com/books?id=FLnTavNvIqYC&pg=PA18). Taylor & Francis. p. 18. ISBN 978-0-415-77498-7. .

[6] Paul Tomassi (1999). *Logic* (http://books.google.com/books?id=TUVQr6InyNYC&pg=PA386). Routledge. pp. 386. ISBN 978-0-415-16696-6. .

[7] Robert C. Pinto (2001). *Argument, inference and dialectic: collected papers on informal logic* (http://books.google.com/books?id=eK0a5CgyV7kC&pg=PA84). Springer. p. 84. ISBN 978-0-7923-7005-5. .

Further reading

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- Gerhard Preyer, Georg Peter, ed. (2002). *Logical form and language*. Clarendon Press. ISBN 978-0-19-924555-0.
- Gila Sher (1991). *The bounds of logic: a generalized viewpoint*. MIT Press. ISBN 978-0-262-19311-5.

External links

- Logical form (http://philpapers.org/browse/logical-form) at PhilPapers
- Logical Form (http://plato.stanford.edu/entries/logical-form) entry by Paul Pietroski in the *Stanford Encyclopedia of Philosophy*
- Logical form (https://inpho.cogs.indiana.edu/taxonomy/2236) at the Indiana Philosophy Ontology Project
- Beaney, Michael, "Analysis", The Stanford Encyclopedia of Philosophy (Summer 2009 Edition), Edward N. Zalta (ed.) (http://plato.stanford.edu/entries/analysis/#6)
- IEP, Validity and Soundness (http://www.iep.utm.edu/val-snd/)

Informal logic

Informal logic, intuitively, refers to the principles of logic and logical thought outside of a formal setting. However, perhaps because of the **informal** in the title, the precise definition of **informal logic** is a matter of some dispute.[1] Ralph H. Johnson and J. Anthony Blair define informal logic as "a branch of logic whose task is to develop non-formal standards, criteria, procedures for the analysis, interpretation, evaluation, criticism and construction of argumentation."[2] This definition reflects what had been implicit in their practice and what others[3][4][5] were doing in their informal logic texts.

Informal logic is associated with (informal) fallacies, critical thinking, the Thinking Skills Movement[6] and the interdisciplinary inquiry known as argumentation theory. Frans H. van Eemeren writes that the label "informal logic" covers a "collection of normative approaches to the study of reasoning in ordinary language that remain closer to the practice of argumentation than formal logic."[7]

History

Informal logic as a distinguished enterprise under this name emerged roughly in the late 1970s as a sub-field of philosophy. The naming of the field was preceded by the appearance of a number of textbooks that rejected the symbolic approach to logic on pedagogical grounds as inappropriate and unhelpful for introductory textbooks on logic for a general audience, for example Howard Kahane's *Logic and Contemporary Rhetoric*, subtitled "The Use of Reason in Everyday Life", first published in 1971. Kahane's textbook was described on the notice of his death in the *Proceedings And Addresses of the American Philosophical Association* (2002) as "a text in informal logic, [that] was intended to enable students to cope with the misleading rhetoric one frequently finds in the media and in political discourse. It was organized around a discussion of fallacies, and was meant to be a practical instrument for dealing with the problems of everyday life. [It has] ... gone through many editions; [it is] ... still in print; and the thousands upon thousands of students who have taken courses in which his text [was] ... used can thank Howard for contributing to their ability to dissect arguments and avoid the deceptions of deceitful rhetoric. He tried to put into practice the ideal of discourse that aims at truth rather than merely at persuasion. (Hausman et al. 2002)"[8][9] Other textbooks from the era taking this approach were Michael Scriven's *Reasoning* (Edgepress, 1976) and *Logical Self-Defense* by Ralph Johnson and J. Anthony Blair, first published in 1977.[8] Earlier precursors in this tradition can be considered Monroe Beardsley's *Practical Logic* (1950) and Stephen Toulmin's *The Uses of Argument* (1958).[10]

The field perhaps became recognized under its current name with the *First International Symposium on Informal Logic* held in 1978. Although initially motivated by a new pedagogical approach to undergraduate logic textbooks, the scope of the field was basically defined by a list of 13 problems and issues which Blair and Johnson included as an appendix to their keynote address at this symposium:[8][11]

- the theory of logical criticism
- the theory of argument
- the theory of fallacy
- the fallacy approach vs. the critical thinking approach
- the viability of the inductive/deductive dichotomy
- the ethics of argumentation and logical criticism
- the problem of assumptions and missing premises
- the problem of context
- methods of extracting arguments from context
- methods of displaying arguments
- the problem of pedagogy
- the nature, division and scope of informal logic

- the relationship of informal logic to other inquiries

David Hitchcock argues that the naming of the field was unfortunate, and that *philosophy of argument* would have been more appropriate. He argues that more undergraduate students in North America study informal logic than any other branch of philosophy, but that as of 2003 informal logic (or philosophy of argument) was not recognized as separate sub-field by the World Congress of Philosophy.[8] Frans H. van Eemeren wrote that "informal logic" is mainly an approach to argumentation advanced by a group of US and Canadian philosophers and largely based on the previous works of Stephen Toulmin and to a lesser extent those of Chaïm Perelman.[7]

Alongside the symposia, since 1983 the journal *Informal Logic* has been the publication of record of the field, with Blair and Johnson as initial editors, with the editorial board now including two other colleagues from the University of Windsor—Christopher Tindale and Hans V. Hansen.[12] Other journals that regularly publish articles on informal logic include *Argumentation* (founded in 1986), *Philosophy and Rhetoric*, *Argumentation and Advocacy* (the journal of the American Forensic Association), and *Inquiry: Critical Thinking Across the Disciplines* (founded in 1988).[13]

Proposed definitions

Johnson and Blair (2000) proposed the following definition: "Informal logic designates that branch of logic whose task is to develop non-formal^2 standards, criteria, procedures for the analysis, interpretation, evaluation, critique and construction of argumentation in everyday discourse." Their meaning of non-formal^2 is taken from Barth and Krabbe (1982), which is explained below.

To understand the definition above, one must understand "informal" which takes its meaning in contrast to its counterpart "formal." (This point was not made for a very long time, hence the nature of informal logic remained opaque, even to those involved in it, for a period of time.) Here it is helpful to have recourse[14] to Barth and Krabbe (1982:14f) where they distinguish three senses of the term "form." By "form^1," Barth and Krabbe mean the sense of the term which derives from the Platonic idea of form—the ultimate metaphysical unit. Barth and Krabbe claim that most traditional logic is formal in this sense. That is, syllogistic logic is a logic of terms where the terms could naturally be understood as place-holders for Platonic (or Aristotelian) forms. In this first sense of "form," almost all logic is informal (not-formal). Understanding informal logic this way would be much too broad to be useful.

By "form^2," Barth and Krabbe mean the form of sentences and statements as these are understood in modern systems of logic. Here validity is the focus: if the premises are true, the conclusion must then also be true. Now validity has to do with the logical form of the statement that makes up the argument. In this sense of "formal," most modern and contemporary logic is "formal." That is, such logics canonize the notion of logical form, and the notion of validity plays the central normative role. In this second sense of form, informal logic is not-formal, because it abandons the notion of logical form as the key to understanding the structure of arguments, and likewise retires validity as normative for the purposes of the evaluation of argument. It seems to many that validity is too stringent a requirement, that there are good arguments in which the conclusion is supported by the premises even though it does not follow necessarily from them (as validity requires). An argument in which the conclusion is thought to be "beyond reasonable doubt, given the premises" is sufficient in law to cause a person to be sentenced to death, even though it does not meet the standard of logical validity. This type of argument, based on accumulation of evidence rather than pure deduction, is called a conductive argument.

By "form^3," Barth and Krabbe mean to refer to "procedures which are somehow regulated or regimented, which take place according to some set of rules." Barth and Krabbe say that "we do not defend formality^3 of all kinds and under all circumstances." Rather "we defend the thesis that verbal dialectics must have a certain form (i.e., must proceed according to certain rules) in order that one can speak of the discussion as being won or lost" (19). In this third sense of "form", informal logic can be formal, for there is nothing in the informal logic enterprise that stands opposed to the idea that argumentative discourse should be subject to norms, i.e., subject to rules, criteria, standards or procedures. Informal logic does present standards for the evaluation of argument, procedures for detecting missing premises etc.

Johnson and Blair (2000) noticed a limitation of their own definition, particularly with respect to "everyday discourse", which could indicate that it does not seek to understand specialized, domain-specific arguments made in natural languages. Consequently, they have argued that the crucial divide is between arguments made in formal languages and those made in natural languages.

Fisher and Scriven (1997) proposed a more encompassing definition, seeing informal logic as "the discipline which studies the practice of critical thinking and provides its intellectual spine". By "critical thinking" they understand "skilled and active interpretation and evaluation of observations and communications, information and argumentation."[15]

Criticisms

Some hold the view that informal logic is not a branch or subdiscipline of logic, or even the view that there cannot be such a thing as informal logic.[16][17][18] Massey criticizes informal logic on the grounds that it has no theory underpinning it. Informal logic, he says, requires detailed classification schemes to organize it, which in other disciplines is provided by the underlying theory. He maintains that there is no method of establishing the invalidity of an argument aside from the formal method, and that the study of fallacies may be of more interest to other disciplines, like psychology, than to philosophy and logic.[16]

Relation to critical thinking

Since the 1980s, informal logic has been partnered and even equated,[19] in the minds of many, with critical thinking. The precise definition of "critical thinking" is a subject of much dispute.[20] Critical thinking, as defined by Johnson, is the evaluation of an intellectual product (an argument, an explanation, a theory) in terms of its strengths and weaknesses.[20] While critical thinking will include evaluation of arguments and hence require skills of argumentation including informal logic, critical thinking requires additional abilities not supplied by informal logic, such as the ability to obtain and assess information and to clarify meaning. Also, many believe that critical thinking requires certain dispositions.[21] Understood in this way, "critical thinking" is a broad term for the attitudes and skills that are involved in analyzing and evaluating arguments. The critical thinking movement promotes critical thinking as an educational ideal. The movement emerged with great force in the 80s in North America as part of an ongoing critique of education as regards the thinking skills not being taught.

Relation to argumentation theory

The social, communicative practice of argumentation can and should be distinguished from implication (or entailment)—a relationship between propositions; and from inference—a mental activity typically thought of as the drawing of a conclusion from premises. Informal logic may thus be said to be a logic of argumentation, as distinguished from implication and inference.[22]

Argumentation theory (or the theory of argumentation) has come to be the term that designates the theoretical study of argumentation. This study is interdisciplinary in the sense that no one discipline will be able to provide a complete account. A full appreciation of argumentation requires insights from logic (both formal and informal), rhetoric, communication theory, linguistics, psychology, and, increasingly, computer science. Since the 1970s, there has been significant agreement that there are three basic approaches to argumentation theory: the logical, the rhetorical and the dialectical. According to Wenzel,[23] the logical approach deals with the product, the dialectical with the process, and the rhetorical with the procedure. Thus, informal logic is one contributor to this inquiry, being most especially concerned with the norms of argument.

Footnotes

[1] See Johnson 1999 for a survey of definitions.
[2] Johnson, Ralph H., and Blair, J. Anthony (1987), "The Current State of Informal Logic", *Informal Logic*, 9(2–3), 147–151. Johnson & Blair added "... in everyday discourse" but in (2000), modified their definition, and broadened the focus now to include the sorts of argument that occurs not just in everyday discourse but also disciplined inquiry—what Weinstein (1990) calls "stylized discourse."
[3] Scriven, 1976
[4] Munson, 1976
[5] Fogelin, 1978
[6] Resnick, 1989
[7] Frans H. van Eemeren (2009). "The Study of Argumentation" (http://books.google.com/books?id=RYRf2JACLGkC&pg=PA117). In Andrea A. Lunsford, Kirt H. Wilson, Rosa A. Eberly. *The SAGE handbook of rhetorical studies*. SAGE. p. 117. ISBN 978-1-4129-0950-1. .
[8] David Hitchcock, Informal logic 25 years later (http://www.humanities.mcmaster.ca/~hitchckd/25.pdf) in *Informal Logic at 25: Proceedings of the Windsor Conference (OSSA 2003)*
[9] JSTOR 3218569
[10] Fisher (2004) p. vii
[11] J. Anthony Blair and Ralph H. Johnson (eds.), Informal Logic: The First International Symposium, 3-28. Pt. Reyes, CA: Edgepress
[12] http://ojs.uwindsor.ca/ojs/leddy/index.php/informal_logic/about/editorialTeam
[13] Johnson and Blair (2000), p. 100
[14] As Johnson (1999) does.
[15] Johnson and Blair (2000), p. 95
[16] Massey, 1981
[17] Woods, 1980
[18] Woods, 2000
[19] Johnson (2000) takes the conflation to be part of the Network Problem and holds that settling the issue will require a theory of reasoning.
[20] Johnson, 1992
[21] Ennis, 1987
[22] Johnson, 1999
[23] Wenzel (1990)

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Special journal issue

The open access issue 20(2) (http://www.phaenex.uwindsor.ca/ojs/leddy/index.php/informal_logic/issue/view/277) of *Informal Logic* from year 2000 groups a number of papers addressing foundational issues, based on the Panel on Informal Logic that was held at the 1998 World Congress of Philosophy, including:

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External links

- Informal Logic (http://plato.stanford.edu/entries/logic-informal) entry by Leo Groarke in the *Stanford Encyclopedia of Philosophy*

Natural language

In the philosophy of language, a **natural language** (or **ordinary language**) is any language which arises in an unpremeditated fashion as the result of the innate facility for language possessed by the human intellect. A natural language is typically used for communication, and may be spoken, signed, or written. Natural language is distinguished from constructed languages and formal languages such as computer-programming languages or the "languages" used in the study of formal logic, especially mathematical logic.

Defining natural language

Though the exact definition varies between scholars, natural language can broadly be defined in contrast on the one hand to artificial or constructed languages, such as **computer programming languages** like Python and international auxiliary languages like Esperanto, and on the other hand to other communication systems in nature, such as the waggle dance of bees. Although there are a variety of natural languages, any cognitively normal human infant is able to learn any natural language. By comparing the different natural languages, scholars hope to learn something about the nature of human intelligence and the innate biases and constraints that shape natural language, which are sometimes called universal grammar.

Linguists have an incomplete understanding of all aspects of the rules underlying natural languages, and these rules are therefore objects of study. The understanding of natural languages reveals much about not only how language works (in terms of syntax, semantics, phonetics, phonology, etc.), but also about how the human mind and the human brain process language. In linguistic terms, *natural language* only applies to a language that has developed naturally, and the study of natural language primarily involves native (first language) speakers.[1]

While grammarians, writers of dictionaries, and language policy-makers all have a certain influence on the evolution of language, their ability to influence what people think they *ought* to say is distinct from what people actually say. The term *natural language* refers to actual linguistic behavior, and is aligned with descriptive linguistics rather than linguistic prescription. Thus non-standard language varieties (such as African American Vernacular English) are considered to be natural while standard language varieties (such as Standard American English) which are more prescribed can be considered to be at least somewhat artificial or constructed.[2]

Native language learning

The learning of one's own native language, typically that of one's parents, normally occurs spontaneously in early human childhood and is biologically, socially and ecologically driven. A crucial role of this process is the ability of humans from an early age to engage in speech repetition and so quickly acquire a spoken vocabulary from the pronunciation of words spoken around them. This together with other aspects of speech involves the neural activity of parts of the human brain such as the Wernicke's and Broca's areas.[3]

There are approximately 7,000 current human languages, and many, if not most seem to share certain properties, leading to the hypothesis of Universal Grammar, as argued by the generative grammar studies of Noam Chomsky and his followers. Recently, it has been demonstrated that a dedicated network in the human brain (crucially involving Broca's area, a portion of the left inferior frontal gyrus), is selectively activated by complex verbal structures (but not simple ones) of those languages that meet the Universal Grammar requirements.[4][5]

While it is clear that there are innate mechanisms that enable the learning of language and define the range of languages that can be learned, it is not clear that these mechanisms in anyway resemble a human language or universal grammar. The study of language acquisition is the domain of psycholinguistics and Chomsky always declined to engage in questions of how his putative language organ, the Language Acquisition Device or Universal Grammar, might have evolved.[6] During a period (the 1970s and 80s) when nativist Transformational Generative Grammar was becoming dominant in Linguistics, and called "Standard Theory", linguists who questioned these tenets were disenfranchised and Cognitive Linguistics and Computational Psycholinguistics were born and the more general term Emergentism developed for the anti-nativist view that language is emergent from more fundamental cognitive processes that are not specifically linguistic in nature.

Origins of natural language

There is disagreement among anthropologists on when language was first used by humans (or their ancestors). Estimates range from about two million (2,000,000) years ago, during the time of *Homo habilis*, to as recently as forty thousand (40,000) years ago, during the time of Cro-Magnon man. However recent evidence suggests modern human language was invented or evolved in Africa prior to the dispersal of humans from Africa around 50,000 years ago. Since all people including the most isolated indigenous groups such as the Andamanese or the Tasmanian aboriginals possess language, then it was presumedly present in the ancestral populations in Africa before the human population split into various groups to inhabit the rest of the world.[7][8]

Controlled languages

Controlled natural languages are subsets of natural languages whose grammars and dictionaries have been restricted in order to reduce or eliminate both ambiguity and complexity (for instance, by cutting down on rarely used superlative or adverbial forms or irregular verbs). The purpose behind the development and implementation of a controlled natural language typically is to aid non-native speakers of a natural language in understanding it, or to ease computer processing of a natural language. An example of a widely used controlled natural language is Simplified English, which was originally developed for aerospace industry maintenance manuals.

Constructed languages and international auxiliary languages

Constructed international auxiliary languages such as Esperanto and Interlingua (even those that have native speakers) are not generally considered natural languages.[9] The problem is that other languages have been used to communicate and evolve in a natural way, while Esperanto was selectively designed by L.L. Zamenhof from natural languages, not grown from the natural fluctuations in vocabulary and syntax. Some natural languages have become naturally "standardized" by children's natural tendency to correct for illogical grammar structures in their parents' language, which can be seen in the development of pidgin languages into creole languages (as explained by Steven Pinker in The Language Instinct), but this is not the case in many languages, including constructed languages such as Esperanto, where strict rules are in place as an attempt to consciously remove such irregularities. The possible exception to this are true native speakers of such languages.[10] More substantive basis for this designation is that the vocabulary, grammar, and orthography of Interlingua are natural; they have been standardized and presented by a linguistic research body, but they predated it and are not themselves considered a product of human invention.[11] Most experts, however, consider Interlingua to be naturalistic rather than natural.[9] Latino Sine Flexione, a second naturalistic auxiliary language, is also naturalistic in content but is no longer widely spoken.[12]

Modalities

Natural language manifests itself in modalities other than speech.

Sign languages

A sign language is a language which conveys meaning through visual rather than acoustic patterns—simultaneously combining hand shapes, orientation and movement of the hands, arms or body, and facial expressions to express a speaker's thoughts. Sign languages are natural languages which have developed in Deaf communities, which can include interpreters and friends and families of deaf people as well as people who are deaf or hard of hearing themselves.

In contrast, a manually coded language (or signed oral language) is a constructed sign system combining elements of a sign language and an oral language. For example, Signed Exact English (SEE) did not develop naturally in any population, but was "created by a committee of individuals".[13]

Written languages

In a sense, written language should be distinguished from natural language. Until recently in the developed world, it was common for many people to be fluent in spoken and yet remain illiterate; this is still the case in poor countries today. Furthermore, natural language acquisition during childhood is largely spontaneous, while literacy must usually be intentionally acquired.[14]

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Fallacy

In informal logic and rhetoric, a **fallacy** is usually an error in reasoning often due to a misconception or a presumption. Some so-called fallacies are not rhetorically intended to appeal to reason but rather to emotion, or a more nuanced disposition. An informal analysis of rhetorical patterns in fallacies should not be confused with rigorously formal arguments in logic, because rationally persuasive arguments require neither to be successful.

Though often used unintentionally, so-called fallacies can be used purposefully to win arguments. Such rhetorical devices, discussed in more detail below, are: "ignoring the question" to divert argument to unrelated issues using a red herring; making the argument personal (*argumentum ad hominem*) and discrediting the opposition's character, "begging the question" (*petitio principi*), the use of the non-sequitur, false cause and effect (*post hoc ergo propter hoc*), bandwagoning (everyone says so), the "false dilemma" or "either-or fallacy" in which the situation is oversimplified, "card-stacking" or selective use of facts, "false equivalence", and "false analogy". Another common device is the "false generalization", an abstraction of the argument that shifts discussion to platitudes where the facts of the matter are lost. There are many more tricks to divert attention from careful exploration of a subject.[1]

A well defined formal fallacy, logical fallacy or deductive fallacy, is typically called an invalid argument. An informal fallacy is an argument that may fail to be rationally persuasive.

Deductive fallacy

In philosophy, the term **logical fallacy** properly refers to a *formal fallacy*: a flaw in the structure of a deductive argument which renders the argument invalid.

However, the same terms are used in informal discourse to mean an argument which is problematic for any reason.

A logical form such as *A* and *B* is independent of any particular conjunction of meaningful propositions. Logical form alone can guarantee that given true premises, a true conclusion must follow. However, formal logic makes no such guarantee if any premise is false, the conclusion can be either true or false. Any formal mistake or logical fallacy similarly invalidates the deductive guarantee. The so-called fallacy fallacy is a failure to understand that all bets are off unless the argument is formally flawless and all premises are true.

Material fallacies

The taxonomy of verbal and material fallacies is based on the "Sophistical Refutations" (*De Sophisticis Elenchis*) of Aristotle's logical works Organon, in which he identified thirteen fallacies found in argument, which he groups into *material fallacies* and *verbal fallacies*. This taxonomy of material fallacies is as follows:

Fallacy of accident or sweeping generalization

- Fallacy of accident or sweeping generalization: a generalization that disregards exceptions.
  - Example

    Argument: *Cutting people is a crime. Surgeons cut people, therefore, surgeons are criminals.*

    Problem: Cutting people is not a crime in certain situations.

    Argument: *It is illegal for a stranger to enter someone's home uninvited. Firefighters enter people's homes uninvited, therefore firefighters are breaking the law.*

Problem: The exception does not break nor define the rule; *a dicto simpliciter ad dictum secundum quid* (where an accountable exception is ignored).

Converse fallacy of accident or hasty generalization

- Converse fallacy of accident or hasty generalization: argues from a special case to a general rule.
  - Example

    Argument: *Every person I've met has ten fingers, therefore, all people have ten fingers*.

    Problem: Those who have been met are not a representative subset of the entire set.
  - Also called reverse accident, destroying the exception, *a dicto secundum quid ad dictum simpliciter*

Irrelevant conclusion

- Irrelevant conclusion: diverts attention away from a fact in dispute rather than addressing it directly.
  - Example

    Argument: *Oliver believes that humans can fly, therefore humans can fly.*

    Problem: Oliver can be wrong. (In particular this is an appeal to authority.)
  - Special cases:
    - purely personal considerations (*argumentum ad hominem*),
    - popular sentiment (*argumentum ad populum*—appeal to the majority; *appeal to loyalty*.),
    - fear (*argumentum ad baculum*),
    - conventional propriety (*argumentum ad verecundiam*—appeal to authority)
    - to arouse pity for getting one's conclusion accepted (*argumentum ad misericordiam*)
    - forwarding the proposition under dispute without any certain proof (*argumentum ad ignorantiam*)
    - assuming a perceived defect in the origin of a claim discredits the claim itself (genetic fallacy)
  - Also called *Ignoratio Elenchi*, a "red herring"

Affirming the consequent/Denying the antecedent

- Affirming the consequent and Denying the antecedent: draws a conclusion from premises that do not support that conclusion by confusing necessary and sufficient conditions.
  - Affirming the consequent Example:

    Argument: *If people have the flu, they cough. Dax is coughing. Therefore, Dax has the flu.*

    Problem: Other things, such as asthma, can cause someone to cough. The argument treats having the flu as a necessary condition of coughing; in fact, having the flu is a sufficient condition of coughing, but it is not necessary to have the flu for one to cough.

    Argument: *If it rains, the ground gets wet. The ground is wet, therefore it rained.*

    Problem: There are other ways by which the ground could get wet (e.g. someone spilled water).
- Denying the antecedent Example

    Argument: *If it is raining outside, it must be cloudy. It is not raining outside. Therefore, it is not cloudy.*

    Problem: Rain is a sufficient condition of cloudiness, but cloudy conditions do not necessarily imply rain.

Begging the question

- Begging the question: demonstrates a conclusion by means of premises that assume that conclusion.
  - Example

    Argument: *Aspirin users are at risk of becoming dependent on the drug, because aspirin is an addictive substance.*

    Problem: The premise and the conclusion have the same meaning. If one has already accepted the premise, there is no need to reason to the conclusion. Obviously the premise is not logically irrelevant to the conclusion, for if the premise is true the conclusion must also be true. It is, however, logically irrelevant in *proving* the conclusion.
  - Also called *Petitio Principii*, or assuming the answer. Begging the question does not preclude the possibility that the statement is incorrect, and it is not sufficient proof in and of itself.
  - A related fallacy is *Circulus in Probando*, arguing in a circle, or circular reasoning. This is when two (or more) conclusions are used as premises to support each other, but unless one accepts one of them as true at the outset, there is no reason to accept the conclusions.

Fallacy of false cause

- Fallacy of false cause or *non sequitur*: incorrectly assumes one thing is the cause of another. *Non Sequitur* is Latin for "It does not follow."
  - Example

    Argument: *I hear the rain falling outside my window; therefore, the sun is not shining.*

    Problem: The conclusion is false because the sun can shine while it is raining.
  - Special cases
    - *post hoc ergo propter hoc*: believing that temporal succession implies a causal relation.
      - Example

    Argument: *It rained just before the car broke down. The rain caused the car to break down.*

    Problem: There may be no connection between the two events. Two events co-occurring is not an indication of causation.
    - *cum hoc ergo propter hoc*: believing that correlation implies a causal relation.
      - Example

    Argument: *Ice cream sales increase in summer, the rate of drowning deaths increases in summer. Therefore, ice cream consumption causes drowning.*

    Problem: No premise suggests the ice cream consumption is causing the deaths. The deaths and ice cream consumption could be unrelated, or something else could be causing both, such as summer heat.

    Also called *causation versus correlation.*

Fallacy of many questions

- Fallacy of many questions or loaded question: groups more than one question in the form of a single question.
  - Example

    Argument: *Have you stopped beating your wife?*

    Problem: Either a yes or no answer is an admission of guilt to beating your wife. (See also Mu.)
  - Also called *Plurium Interrogationum* and other terms

Straw man

- Straw man: A straw man argument is an informal fallacy based on misrepresenting an opponent's position so as to more easily refute it.[2]
  - Examples
    - Person A: *Sunny days are good.*
      Person B: *If all days were sunny, we'd never have rain, and without rain, we'd have famine and death. Therefore, you are wrong.*
      Problem: B falsely suggested that A said it would be good to have *only* sunny days, then refuted this misrepresentation rather than refuting A's actual assertion.
    - Mom: *You have been playing video games for too long these past few days. You should focus on your school work.*
      Son: *You think I play video games for 20 hours?*
      Problem: The son has exaggerated what Mom said.

No true Scotsman

- No true Scotsman: When faced with a counterexample to a universal claim, rather than denying the counterexample or rejecting the original universal claim, this fallacy modifies the subject of the assertion to exclude the specific case or others like it by rhetoric, without reference to any specific objective rule.
  - Example

    Person A: *All Scotsmen enjoy haggis.*

    Person B: *My uncle is a Scotsman, and he doesn't like haggis!*

    Person A: *Well, all **true** Scotsmen like haggis.*

Verbal fallacies

Verbal fallacies are those in which a conclusion is obtained by improper or ambiguous use of words. They are generally classified as follows.

Equivocation

- Equivocation consists in employing the same word in two or more senses, e.g. in a syllogism, the middle term being used in one sense in the major and another in the minor premise, so that in fact there are four not three terms. Often this happens when the two meanings are similar despite being distinctly different.

  Example Argument: *All heavy things have a great mass; Jim has a "heavy heart"; therefore Jim's heart has a great mass.*

  Problem: *Heavy* describes more than just weight. (Jim is sad.)

Connotation fallacies

- Connotation fallacies occur when a dysphemistic word is substituted for the speaker's actual quote and used to discredit the argument. It is a form of attribution fallacy.

Apophasis and argument by innuendo

- Argument by innuendo involves implicitly suggesting a conclusion without stating it outright. For example, a job reference that says a former employee "was never caught taking money from the cash box." In this example the overly specific nature of the innuendo implies that the employee was a thief, even though it does not make (or justify) a direct statement of accusation.[3]

Amphiboly

- Amphiboly is the result of ambiguity of grammatical structure.

  Example: The position of the adverb "only" in a sentence starting with "He only said that" results in a sentence in which it is uncertain as to which of the other three words the speaker is intending to modify with the adverb.

Fallacy of composition

- Fallacy of composition "From each to all". Arguing from some property of constituent parts, to the conclusion that the composite item has that property. This can be acceptable (i.e., not a fallacy) with certain arguments such as spatial arguments (e.g. "all the parts of the car are in the garage, therefore the car is in the garage").

  Example Argument: *All the musicians in a band (constituent parts) are highly skilled, therefore the band itself (composite item) is highly skilled.*

  Problem: The band members may be skilled musicians but may lack the ability to function properly as a group.

Division

- Division, the converse of the preceding, arguing from a property of the whole, to each constituent part.

  Example Argument: "The university (the whole) is 700 years old, therefore, all the staff (each part) are 700 years old".

  Problem: Each and every person currently on staff is younger than 700 years. The university continues to exist even when, one by one, each and every person on the original staff leaves and is replaced by a younger person. See Theseus' Ship paradox.

  Example Argument: "This liquid is part of a nutritious breakfast therefore the liquid is nutritious."

  Problem: Simply because the breakfast taken as a whole is nutritious does not necessarily mean that each part of that breakfast is nutritious (unless the definition of a nutritious breakfast requires all parts to be nutritious).

Proof by verbosity

- Proof by verbosity, sometimes colloquially referred to as *argumentum verbosum* - a rhetorical technique that tries to persuade by overwhelming those considering an argument with such a volume of material that the argument sounds plausible, superficially appears to be well-researched, and it is so laborious to untangle and check supporting facts that the argument might be allowed to slide by unchallenged.

Accent

- Accent, which occurs only in speaking and consists of emphasizing the wrong word in a sentence. e.g., "He is a fairly good pianist", according to the emphasis on the words, may imply praise of a beginner's progress or insult of an expert pianist.

- "*He* is a fairly good pianist." This argument places emphasis on the fact that "He", as opposed to anyone else, is a fairly good pianist.
- "He *is* a fairly good pianist." This is an assertion that he "is" a good pianist, as opposed to a poor one.
- "He is *a* fairly good pianist." The emphasis on "a" makes this an assertion that while he is a good pianist, there are other good pianists as well.
- "He is a *fairly* good pianist." This is an assertion that his ability as a pianist is fair, perhaps in need of improvement.
- "He is a fairly *good* pianist." This is an assertion that he is most certainly a good pianist, perhaps even impressively so.
- "He is a fairly good *pianist*." This is isolating his ability as only being good in the field of musical instruments, namely, the piano, and possibly excludes the idea that he is good at anything else.
- "*I* killed my wife?" in response to a police officer asking if he killed his wife. In court, the police officer states his reply to his question was "I killed my wife." This use of accent is demonstrated in the courtroom comedy My Cousin Vinny.

Figure of Speech

- Figure of Speech, the confusion between the metaphorical or figurative use of a word or phrase and the ordinary or literal use of a word or phrase.

  Example: The sailor was at home on the sea.

  Problem: The expression 'to be at home' does not literally mean that one's domicile is in that location.

This can happen in conjunction with Equivocation, whereby word or phrase is used literally in one part of an argument but figuratively in another part of the argument.

> Example: John lives on a house-boat on the sea. He feels at home on his house-boat. Therefore he is like a sailor because he is at home on the sea.

Fallacy of misplaced concreteness

- Fallacy of misplaced concreteness, identified by Whitehead in his discussion of metaphysics, this refers to the reification of concepts which exist only in discussion.

Example 1

Timmy argues:

1. Billy is a good tennis player.
2. Therefore, Billy is 'good', that is to say a morally good person.

Here the problem is that the word *good* has different meanings, which is to say that it is an *ambiguous* word. In the premise, Timmy says that Billy is good at a particular activity, in this case tennis. In the conclusion, Timmy states that Billy is a morally good person. These are clearly two different senses of the word "good". The premise might be true but the conclusion can still be false: Billy might be the best tennis player in the world but a rotten person morally. However, it is not legitimate to infer he is a bad person on the ground there has been a fallacious argument on the part of Timmy. Nothing concerning Billy's moral qualities is to be inferred from the premise. Appropriately, since it plays on an ambiguity, this sort of fallacy is called the fallacy of equivocation, that is, equating two incompatible terms or claims.

Example 2

One posits the argument:

1. Nothing is better than eternal happiness.
2. Eating a sandwich is better than nothing.
3. Therefore, eating a sandwich is better than eternal happiness.

This argument has the appearance of an inference that applies transitivity of the two-placed relation *is better than*, which in this critique we grant is a valid property. The argument is an example of *syntactic ambiguity*. In fact, the first premise semantically does not predicate an attribute of the subject, as would for instance the assertion

> Nothing is better than eternal happiness.

In fact it is semantically equivalent to the following universal quantification:

> Everything fails to be better than eternal happiness.

So instantiating this fact with *eating a sandwich*, it logically follows that

> Eating a sandwich fails to be better than eternal happiness.

Note that the premise *A sandwich is better than nothing* does not provide anything to this argument. This fact really means something such as

> Eating a sandwich is better than eating nothing at all.

Thus this is a fallacy of equivocation.

Other systems of classification

Of other classifications of fallacies in general the most famous are those of Francis Bacon and J. S. Mill. Bacon (*Novum Organum*, Aph. 33, 38 sqq.) divided fallacies into four Idola (Idols, i.e. False Appearances), which summarize the various kinds of mistakes to which the human intellect is prone. With these should be compared the Offendicula of Roger Bacon, contained in the Opus maius, pt. i. J. S. Mill discussed the subject in book v. of his Logic, and Jeremy Bentham's Book of Fallacies (1824) contains valuable remarks. See Rd. Whateley's Logic, bk. v.; A. de Morgan, Formal Logic (1847) ; A. Sidgwick, Fallacies (1883) and other textbooks.

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- Frans van Eemeren; Bart Garssen; Bert Meuffels (2009). *Fallacies and Judgments of Reasonableness: Empirical Research Concerning the Pragma-Dialectical Discussion*. Springer. ISBN 978-90-481-2613-2.

Historical texts

- Aristotle, On Sophistical Refutations (http://etext.library.adelaide.edu.au/a/aristotle/sophistical/), *De Sophistici Elenchi*. library.adelaide.edu.au
- William of Ockham, *Summa of Logic* (ca. 1323) Part III.4.
- John Buridan, *Summulae de dialectica* Book VII.
- Francis Bacon, the doctrine of the idols in *Novum Organum Scientiarum*, Aphorisms concerning The Interpretation of Nature and the Kingdom of Man, XXIIIff (http://fly.hiwaay.net/~paul/bacon/organum/aphorisms1.html). fly.hiwaay.net
- Arthur Schopenhauer, The Art of Controversy (http://www.gutenberg.net/1/0/7/3/10731/10731-8.txt) | *Die Kunst, Recht zu behalten - The Art Of Controversy* (bilingual) (http://coolhaus.de/art-of-controversy/), (also known as "Schopenhauers 38 stratagems"). gutenberg.net
- John Stuart Mill, A System of Logic - Raciocinative and Inductive (http://www.la.utexas.edu/research/poltheory/mill/sol/). Book 5, Chapter 7, Fallacies of Confusion (http://www.la.utexas.edu/research/poltheory/mill/sol/sol.b05.c07.html). la.utexas.edu

External links

- Fallacy (http://philpapers.org/browse/fallacies) at PhilPapers
- Fallacy (https://inpho.cogs.indiana.edu/idea/1980) at the Indiana Philosophy Ontology Project
- Informal logic (http://plato.stanford.edu/entries/logic-informal/#Fal) entry in the *Stanford Encyclopedia of Philosophy*
- Fallacy (http://www.iep.utm.edu/fallacy) entry in the *Internet Encyclopedia of Philosophy*
- Infographic poster of common logical fallacies (http://yourlogicalfallacyis.com/poster)
- Appeal to Authority (http://www.appealtoauthority.info) Appeal to Authority Logical Fallacy
- FallacyFiles.org (http://www.fallacyfiles.org) contains categorization (http://www.fallacyfiles.org/taxonomy.html) of fallacies with examples.
- 42 informal logical fallacies explained by Dr. Michael C. Labossiere (including examples) (http://www.nizkor.org/features/fallacies/), nizkor.org
- Humbug! The skeptic's field guide to spotting fallacies in thinking (http://www.scribd.com/doc/8009498/HUMBUG-eBook-by-Jef-Clark-and-Theo-Clark) – textbook on fallacies. scribd.com
- List of fallacies with clear examples (http://www.infidels.org/library/modern/mathew/logic.html), infidels.org
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- Stephen Downes Guide to the Logical Fallacies (http://onegoodmove.org/fallacy/welcome.htm), onegoodmove.org
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Mathematical logic

Mathematical logic (also known as **symbolic logic**) is a subfield of mathematics with close connections to the foundations of mathematics, theoretical computer science and philosophical logic.[1] The field includes both the mathematical study of logic and the applications of formal logic to other areas of mathematics. The unifying themes in mathematical logic include the study of the expressive power of formal systems and the deductive power of formal proof systems.

Mathematical logic is often divided into the fields of set theory, model theory, recursion theory, and proof theory. These areas share basic results on logic, particularly first-order logic, and definability. In computer science (particularly in the ACM Classification) mathematical logic encompasses additional topics not detailed in this article; see logic in computer science for those.

Since its inception, mathematical logic has both contributed to, and has been motivated by, the study of foundations of mathematics. This study began in the late 19th century with the development of axiomatic frameworks for geometry, arithmetic, and analysis. In the early 20th century it was shaped by David Hilbert's program to prove the consistency of foundational theories. Results of Kurt Gödel, Gerhard Gentzen, and others provided partial resolution to the program, and clarified the issues involved in proving consistency. Work in set theory showed that almost all ordinary mathematics can be formalized in terms of sets, although there are some theorems that cannot be proven in common axiom systems for set theory. Contemporary work in the foundations of mathematics often focuses on establishing which parts of mathematics can be formalized in particular formal systems (as in reverse mathematics) rather than trying to find theories in which all of mathematics can be developed.

Subfields and scope

The *Handbook of Mathematical Logic* makes a rough division of contemporary mathematical logic into four areas:

1. set theory
2. model theory
3. recursion theory, and
4. proof theory and constructive mathematics (considered as parts of a single area).

Each area has a distinct focus, although many techniques and results are shared among multiple areas. The borderlines amongst these fields, and the lines separating mathematical logic and other fields of mathematics, are not always sharp. Gödel's incompleteness theorem marks not only a milestone in recursion theory and proof theory, but has also led to Löb's theorem in modal logic. The method of forcing is employed in set theory, model theory, and recursion theory, as well as in the study of intuitionistic mathematics.

The mathematical field of category theory uses many formal axiomatic methods, and includes the study of categorical logic, but category theory is not ordinarily considered a subfield of mathematical logic. Because of its applicability in diverse fields of mathematics, mathematicians including Saunders Mac Lane have proposed category theory as a foundational system for mathematics, independent of set theory. These foundations use toposes, which resemble generalized models of set theory that may employ classical or nonclassical logic.

History

Mathematical logic emerged in the mid-19th century as a subfield of mathematics independent of the traditional study of logic (Ferreirós 2001, p. 443). Before this emergence, logic was studied with rhetoric, through the syllogism, and with philosophy. The first half of the 20th century saw an explosion of fundamental results, accompanied by vigorous debate over the foundations of mathematics.

Early history

Theories of logic were developed in many cultures in history, including China, India, Greece and the Islamic world. In 18th century Europe, attempts to treat the operations of formal logic in a symbolic or algebraic way had been made by philosophical mathematicians including Leibniz and Lambert, but their labors remained isolated and little known.

19th century

In the middle of the nineteenth century, George Boole and then Augustus De Morgan presented systematic mathematical treatments of logic. Their work, building on work by algebraists such as George Peacock, extended the traditional Aristotelian doctrine of logic into a sufficient framework for the study of foundations of mathematics (Katz 1998, p. 686).

Charles Sanders Peirce built upon the work of Boole to develop a logical system for relations and quantifiers, which he published in several papers from 1870 to 1885. Gottlob Frege presented an independent development of logic with quantifiers in his *Begriffsschrift*, published in 1879, a work generally considered as marking a turning point in the history of logic. Frege's work remained obscure, however, until Bertrand Russell began to promote it near the turn of the century. The two-dimensional notation Frege developed was never widely adopted and is unused in contemporary texts.

From 1890 to 1905, Ernst Schröder published *Vorlesungen über die Algebra der Logik* in three volumes. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic as it was understood at the end of the 19th century.

Foundational theories

Concerns that mathematics had not been built on a proper foundation led to the development of axiomatic systems for fundamental areas of mathematics such as arithmetic, analysis, and geometry.

In logic, the term *arithmetic* refers to the theory of the natural numbers. Giuseppe Peano (1888) published a set of axioms for arithmetic that came to bear his name (Peano axioms), using a variation of the logical system of Boole and Schröder but adding quantifiers. Peano was unaware of Frege's work at the time. Around the same time Richard Dedekind showed that the natural numbers are uniquely characterized by their induction properties. Dedekind (1888) proposed a different characterization, which lacked the formal logical character of Peano's axioms. Dedekind's work, however, proved theorems inaccessible in Peano's system, including the uniqueness of the set of natural numbers (up to isomorphism) and the recursive definitions of addition and multiplication from the successor function and mathematical induction.

In the mid-19th century, flaws in Euclid's axioms for geometry became known (Katz 1998, p. 774). In addition to the independence of the parallel postulate, established by Nikolai Lobachevsky in 1826 (Lobachevsky 1840), mathematicians discovered that certain theorems taken for granted by Euclid were not in fact provable from his axioms. Among these is the theorem that a line contains at least two points, or that circles of the same radius whose centers are separated by that radius must intersect. Hilbert (1899) developed a complete set of axioms for geometry, building on previous work by Pasch (1882). The success in axiomatizing geometry motivated Hilbert to seek complete axiomatizations of other areas of mathematics, such as the natural numbers and the real line. This would prove to be a major area of research in the first half of the 20th century.

The 19th century saw great advances in the theory of real analysis, including theories of convergence of functions and Fourier series. Mathematicians such as Karl Weierstrass began to construct functions that stretched intuition, such as nowhere-differentiable continuous functions. Previous conceptions of a function as a rule for computation, or a smooth graph, were no longer adequate. Weierstrass began to advocate the arithmetization of analysis, which sought to axiomatize analysis using properties of the natural numbers. The modern (ε, δ)-definition of limit and continuous functions was already developed by Bolzano in 1817 (Felscher 2000), but remained relatively unknown. Cauchy in 1821 defined continuity in terms of infinitesimals (see Cours d'Analyse, page 34). In 1858, Dedekind proposed a definition of the real numbers in terms of Dedekind cuts of rational numbers (Dedekind 1872), a definition still employed in contemporary texts.

Georg Cantor developed the fundamental concepts of infinite set theory. His early results developed the theory of cardinality and proved that the reals and the natural numbers have different cardinalities (Cantor 1874). Over the next twenty years, Cantor developed a theory of transfinite numbers in a series of publications. In 1891, he published a new proof of the uncountability of the real numbers that introduced the diagonal argument, and used this method to prove Cantor's theorem that no set can have the same cardinality as its powerset. Cantor believed that every set could be well-ordered, but was unable to produce a proof for this result, leaving it as an open problem in 1895 (Katz 1998, p. 807).

20th century

In the early decades of the 20th century, the main areas of study were set theory and formal logic. The discovery of paradoxes in informal set theory caused some to wonder whether mathematics itself is inconsistent, and to look for proofs of consistency.

In 1900, Hilbert posed a famous list of 23 problems for the next century. The first two of these were to resolve the continuum hypothesis and prove the consistency of elementary arithmetic, respectively; the tenth was to produce a method that could decide whether a multivariate polynomial equation over the integers has a solution. Subsequent work to resolve these problems shaped the direction of mathematical logic, as did the effort to resolve Hilbert's *Entscheidungsproblem*, posed in 1928. This problem asked for a procedure that would decide, given a formalized mathematical statement, whether the statement is true or false.

Set theory and paradoxes

Ernst Zermelo (1904) gave a proof that every set could be well-ordered, a result Georg Cantor had been unable to obtain. To achieve the proof, Zermelo introduced the axiom of choice, which drew heated debate and research among mathematicians and the pioneers of set theory. The immediate criticism of the method led Zermelo to publish a second exposition of his result, directly addressing criticisms of his proof (Zermelo 1908a). This paper led to the general acceptance of the axiom of choice in the mathematics community.

Skepticism about the axiom of choice was reinforced by recently discovered paradoxes in naive set theory. Cesare Burali-Forti (1897) was the first to state a paradox: the Burali-Forti paradox shows that the collection of all ordinal numbers cannot form a set. Very soon thereafter, Bertrand Russell discovered Russell's paradox in 1901, and Jules Richard (1905) discovered Richard's paradox.

Zermelo (1908b) provided the first set of axioms for set theory. These axioms, together with the additional axiom of replacement proposed by Abraham Fraenkel, are now called Zermelo–Fraenkel set theory (ZF). Zermelo's axioms incorporated the principle of limitation of size to avoid Russell's paradox.

In 1910, the first volume of *Principia Mathematica* by Russell and Alfred North Whitehead was published. This seminal work developed the theory of functions and cardinality in a completely formal framework of type theory, which Russell and Whitehead developed in an effort to avoid the paradoxes. *Principia Mathematica* is considered one of the most influential works of the 20th century, although the framework of type theory did not prove popular as a foundational theory for mathematics (Ferreirós 2001, p. 445).

Fraenkel (1922) proved that the axiom of choice cannot be proved from the remaining axioms of Zermelo's set theory with urelements. Later work by Paul Cohen (1966) showed that the addition of urelements is not needed, and the axiom of choice is unprovable in ZF. Cohen's proof developed the method of forcing, which is now an important tool for establishing independence results in set theory.

Symbolic logic

Leopold Löwenheim (1915) and Thoralf Skolem (1920) obtained the Löwenheim–Skolem theorem, which says that first-order logic cannot control the cardinalities of infinite structures. Skolem realized that this theorem would apply to first-order formalizations of set theory, and that it implies any such formalization has a countable model. This counterintuitive fact became known as Skolem's paradox.

In his doctoral thesis, Kurt Gödel (1929) proved the completeness theorem, which establishes a correspondence between syntax and semantics in first-order logic. Gödel used the completeness theorem to prove the compactness theorem, demonstrating the finitary nature of first-order logical consequence. These results helped establish first-order logic as the dominant logic used by mathematicians.

In 1931, Gödel published *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, which proved the incompleteness (in a different meaning of the word) of all sufficiently strong, effective first-order theories. This result, known as Gödel's incompleteness theorem, establishes severe limitations on axiomatic foundations for mathematics, striking a strong blow to Hilbert's program. It showed the impossibility of providing a consistency proof of arithmetic within any formal theory of arithmetic. Hilbert, however, did not acknowledge the importance of the incompleteness theorem for some time.

Gödel's theorem shows that a consistency proof of any sufficiently strong, effective axiom system cannot be obtained in the system itself, if the system is consistent, nor in any weaker system. This leaves open the possibility of consistency proofs that cannot be formalized within the system they consider. Gentzen (1936) proved the consistency of arithmetic using a finitistic system together with a principle of transfinite induction. Gentzen's result introduced the ideas of cut elimination and proof-theoretic ordinals, which became key tools in proof theory. Gödel (1958) gave a different consistency proof, which reduces the consistency of classical arithmetic to that of intutitionistic arithmetic in higher types.

Beginnings of the other branches

Alfred Tarski developed the basics of model theory.

Beginning in 1935, a group of prominent mathematicians collaborated under the pseudonym Nicolas Bourbaki to publish a series of encyclopedic mathematics texts. These texts, written in an austere and axiomatic style, emphasized rigorous presentation and set-theoretic foundations. Terminology coined by these texts, such as the words *bijection*, *injection*, and *surjection*, and the set-theoretic foundations the texts employed, were widely adopted throughout mathematics.

The study of computability came to be known as recursion theory, because early formalizations by Gödel and Kleene relied on recursive definitions of functions.[2] When these definitions were shown equivalent to Turing's formalization involving Turing machines, it became clear that a new concept – the computable function – had been discovered, and that this definition was robust enough to admit numerous independent characterizations. In his work on the incompleteness theorems in 1931, Gödel lacked a rigorous concept of an effective formal system; he immediately realized that the new definitions of computability could be used for this purpose, allowing him to state the incompleteness theorems in generality that could only be implied in the original paper.

Numerous results in recursion theory were obtained in the 1940s by Stephen Cole Kleene and Emil Leon Post. Kleene (1943) introduced the concepts of relative computability, foreshadowed by Turing (1939), and the arithmetical hierarchy. Kleene later generalized recursion theory to higher-order functionals. Kleene and Kreisel studied formal versions of intuitionistic mathematics, particularly in the context of proof theory.

Formal logical systems

At its core, mathematical logic deals with mathematical concepts expressed using formal logical systems. These systems, though they differ in many details, share the common property of considering only expressions in a fixed formal language, or signature. The systems of propositional logic and first-order logic are the most widely studied today, because of their applicability to foundations of mathematics and because of their desirable proof-theoretic properties.[3] Stronger classical logics such as second-order logic or infinitary logic are also studied, along with nonclassical logics such as intuitionistic logic.

First-order logic

First-order logic is a particular formal system of logic. Its syntax involves only finite expressions as well-formed formulas, while its semantics are characterized by the limitation of all quantifiers to a fixed domain of discourse.

Early results about formal logic established limitations of first-order logic. The Löwenheim–Skolem theorem (1919) showed that if a set of sentences in a countable first-order language has an infinite model then it has at least one model of each infinite cardinality. This shows that it is impossible for a set of first-order axioms to characterize the natural numbers, the real numbers, or any other infinite structure up to isomorphism. As the goal of early foundational studies was to produce axiomatic theories for all parts of mathematics, this limitation was particularly stark.

Gödel's completeness theorem (Gödel 1929) established the equivalence between semantic and syntactic definitions of logical consequence in first-order logic. It shows that if a particular sentence is true in every model that satisfies a particular set of axioms, then there must be a finite deduction of the sentence from the axioms. The compactness theorem first appeared as a lemma in Gödel's proof of the completeness theorem, and it took many years before logicians grasped its significance and began to apply it routinely. It says that a set of sentences has a model if and only if every finite subset has a model, or in other words that an inconsistent set of formulas must have a finite inconsistent subset. The completeness and compactness theorems allow for sophisticated analysis of logical consequence in first-order logic and the development of model theory, and they are a key reason for the prominence of first-order logic in mathematics.

Gödel's incompleteness theorems (Gödel 1931) establish additional limits on first-order axiomatizations. The **first incompleteness theorem** states that for any sufficiently strong, effectively given logical system there exists a statement which is true but not provable within that system. Here a logical system is effectively given if it is possible to decide, given any formula in the language of the system, whether the formula is an axiom. A logical system is sufficiently strong if it can express the Peano axioms. When applied to first-order logic, the first incompleteness theorem implies that any sufficiently strong, consistent, effective first-order theory has models that are not elementarily equivalent, a stronger limitation than the one established by the Löwenheim–Skolem theorem. The **second incompleteness theorem** states that no sufficiently strong, consistent, effective axiom system for arithmetic can prove its own consistency, which has been interpreted to show that Hilbert's program cannot be completed.

Other classical logics

Many logics besides first-order logic are studied. These include infinitary logics, which allow for formulas to provide an infinite amount of information, and higher-order logics, which include a portion of set theory directly in their semantics.

The most well studied infinitary logic is $L_{\omega_1,\omega}$. In this logic, quantifiers may only be nested to finite depths, as in first-order logic, but formulas may have finite or countably infinite conjunctions and disjunctions within them. Thus, for example, it is possible to say that an object is a whole number using a formula of $L_{\omega_1,\omega}$ such as

$$(x = 0) \lor (x = 1) \lor (x = 2) \lor \cdots.$$

Higher-order logics allow for quantification not only of elements of the domain of discourse, but subsets of the domain of discourse, sets of such subsets, and other objects of higher type. The semantics are defined so that, rather than having a separate domain for each higher-type quantifier to range over, the quantifiers instead range over all objects of the appropriate type. The logics studied before the development of first-order logic, for example Frege's logic, had similar set-theoretic aspects. Although higher-order logics are more expressive, allowing complete axiomatizations of structures such as the natural numbers, they do not satisfy analogues of the completeness and compactness theorems from first-order logic, and are thus less amenable to proof-theoretic analysis.

Another type of logics are fixed-point logics that allow inductive definitions, like one writes for primitive recursive functions.

One can formally define an extension of first-order logic — a notion which encompasses all logics in this section because they behave like first-order logic in certain fundamental ways, but does not encompass all logics in general, e.g. it does not encompass intuitionistic, modal or fuzzy logic. Lindström's theorem implies that the only extension of first-order logic satisfying both the compactness theorem and the Downward Löwenheim–Skolem theorem is first-order logic.

Nonclassical and modal logic

Modal logics include additional modal operators, such as an operator which states that a particular formula is not only true, but necessarily true. Although modal logic is not often used to axiomatize mathematics, it has been used to study the properties of first-order provability (Solovay 1976) and set-theoretic forcing (Hamkins and Löwe 2007).

Intuitionistic logic was developed by Heyting to study Brouwer's program of intuitionism, in which Brouwer himself avoided formalization. Intuitionistic logic specifically does not include the law of the excluded middle, which states that each sentence is either true or its negation is true. Kleene's work with the proof theory of intuitionistic logic showed that constructive information can be recovered from intuitionistic proofs. For example, any provably total function in intuitionistic arithmetic is computable; this is not true in classical theories of arithmetic such as Peano arithmetic.

Algebraic logic

Algebraic logic uses the methods of abstract algebra to study the semantics of formal logics. A fundamental example is the use of Boolean algebras to represent truth values in classical propositional logic, and the use of Heyting algebras to represent truth values in intuitionistic propositional logic. Stronger logics, such as first-order logic and higher-order logic, are studied using more complicated algebraic structures such as cylindric algebras.

Set theory

Set theory is the study of sets, which are abstract collections of objects. Many of the basic notions, such as ordinal and cardinal numbers, were developed informally by Cantor before formal axiomatizations of set theory were developed. The first such axiomatization, due to Zermelo (1908b), was extended slightly to become Zermelo–Fraenkel set theory (ZF), which is now the most widely used foundational theory for mathematics.

Other formalizations of set theory have been proposed, including von Neumann–Bernays–Gödel set theory (NBG), Morse–Kelley set theory (MK), and New Foundations (NF). Of these, ZF, NBG, and MK are similar in describing a cumulative hierarchy of sets. New Foundations takes a different approach; it allows objects such as the set of all sets at the cost of restrictions on its set-existence axioms. The system of Kripke–Platek set theory is closely related to generalized recursion theory.

Two famous statements in set theory are the axiom of choice and the continuum hypothesis. The axiom of choice, first stated by Zermelo (1904), was proved independent of ZF by Fraenkel (1922), but has come to be widely accepted by mathematicians. It states that given a collection of nonempty sets there is a single set *C* that contains exactly one element from each set in the collection. The set *C* is said to "choose" one element from each set in the collection. While the ability to make such a choice is considered obvious by some, since each set in the collection is nonempty, the lack of a general, concrete rule by which the choice can be made renders the axiom nonconstructive. Stefan Banach and Alfred Tarski (1924) showed that the axiom of choice can be used to decompose a solid ball into a finite number of pieces which can then be rearranged, with no scaling, to make two solid balls of the original size. This theorem, known as the Banach–Tarski paradox, is one of many counterintuitive results of the axiom of choice.

The continuum hypothesis, first proposed as a conjecture by Cantor, was listed by David Hilbert as one of his 23 problems in 1900. Gödel showed that the continuum hypothesis cannot be disproven from the axioms of Zermelo–Fraenkel set theory (with or without the axiom of choice), by developing the constructible universe of set theory in which the continuum hypothesis must hold. In 1963, Paul Cohen showed that the continuum hypothesis cannot be proven from the axioms of Zermelo–Fraenkel set theory (Cohen 1966). This independence result did not completely settle Hilbert's question, however, as it is possible that new axioms for set theory could resolve the hypothesis. Recent work along these lines has been conducted by W. Hugh Woodin, although its importance is not yet clear (Woodin 2001).

Contemporary research in set theory includes the study of large cardinals and determinacy. Large cardinals are cardinal numbers with particular properties so strong that the existence of such cardinals cannot be proved in ZFC. The existence of the smallest large cardinal typically studied, an inaccessible cardinal, already implies the consistency of ZFC. Despite the fact that large cardinals have extremely high cardinality, their existence has many ramifications for the structure of the real line. *Determinacy* refers to the possible existence of winning strategies for certain two-player games (the games are said to be *determined*). The existence of these strategies implies structural properties of the real line and other Polish spaces.

Model theory

Model theory studies the models of various formal theories. Here a theory is a set of formulas in a particular formal logic and signature, while a model is a structure that gives a concrete interpretation of the theory. Model theory is closely related to universal algebra and algebraic geometry, although the methods of model theory focus more on logical considerations than those fields.

The set of all models of a particular theory is called an elementary class; classical model theory seeks to determine the properties of models in a particular elementary class, or determine whether certain classes of structures form elementary classes.

The method of quantifier elimination can be used to show that definable sets in particular theories cannot be too complicated. Tarski (1948) established quantifier elimination for real-closed fields, a result which also shows the theory of the field of real numbers is decidable. (He also noted that his methods were equally applicable to algebraically closed fields of arbitrary characteristic.) A modern subfield developing from this is concerned with o-minimal structures.

Morley's categoricity theorem, proved by Michael D. Morley (1965), states that if a first-order theory in a countable language is categorical in some uncountable cardinality, i.e. all models of this cardinality are isomorphic, then it is categorical in all uncountable cardinalities.

A trivial consequence of the continuum hypothesis is that a complete theory with less than continuum many nonisomorphic countable models can have only countably many. Vaught's conjecture, named after Robert Lawson Vaught, says that this is true even independently of the continuum hypothesis. Many special cases of this conjecture have been established.

Recursion theory

Recursion theory, also called **computability theory**, studies the properties of computable functions and the Turing degrees, which divide the uncomputable functions into sets which have the same level of uncomputability. Recursion theory also includes the study of generalized computability and definability. Recursion theory grew from the work of Alonzo Church and Alan Turing in the 1930s, which was greatly extended by Kleene and Post in the 1940s.

Classical recursion theory focuses on the computability of functions from the natural numbers to the natural numbers. The fundamental results establish a robust, canonical class of computable functions with numerous independent, equivalent characterizations using Turing machines, λ calculus, and other systems. More advanced results concern the structure of the Turing degrees and the lattice of recursively enumerable sets.

Generalized recursion theory extends the ideas of recursion theory to computations that are no longer necessarily finite. It includes the study of computability in higher types as well as areas such as hyperarithmetical theory and α-recursion theory.

Contemporary research in recursion theory includes the study of applications such as algorithmic randomness, computable model theory, and reverse mathematics, as well as new results in pure recursion theory.

Algorithmically unsolvable problems

An important subfield of recursion theory studies algorithmic unsolvability; a decision problem or function problem is **algorithmically unsolvable** if there is no possible computable algorithm which returns the correct answer for all legal inputs to the problem. The first results about unsolvability, obtained independently by Church and Turing in 1936, showed that the Entscheidungsproblem is algorithmically unsolvable. Turing proved this by establishing the unsolvability of the halting problem, a result with far-ranging implications in both recursion theory and computer science.

There are many known examples of undecidable problems from ordinary mathematics. The word problem for groups was proved algorithmically unsolvable by Pyotr Novikov in 1955 and independently by W. Boone in 1959. The busy beaver problem, developed by Tibor Radó in 1962, is another well-known example.

Hilbert's tenth problem asked for an algorithm to determine whether a multivariate polynomial equation with integer coefficients has a solution in the integers. Partial progress was made by Julia Robinson, Martin Davis and Hilary Putnam. The algorithmic unsolvability of the problem was proved by Yuri Matiyasevich in 1970 (Davis 1973).

Proof theory and constructive mathematics

Proof theory is the study of formal proofs in various logical deduction systems. These proofs are represented as formal mathematical objects, facilitating their analysis by mathematical techniques. Several deduction systems are commonly considered, including Hilbert-style deduction systems, systems of natural deduction, and the sequent calculus developed by Gentzen.

The study of **constructive mathematics**, in the context of mathematical logic, includes the study of systems in non-classical logic such as intuitionistic logic, as well as the study of predicative systems. An early proponent of predicativism was Hermann Weyl, who showed it is possible to develop a large part of real analysis using only predicative methods (Weyl 1918).

Because proofs are entirely finitary, whereas truth in a structure is not, it is common for work in constructive mathematics to emphasize provability. The relationship between provability in classical (or nonconstructive) systems and provability in intuitionistic (or constructive, respectively) systems is of particular interest. Results such as the Gödel–Gentzen negative translation show that it is possible to embed (or *translate*) classical logic into intuitionistic logic, allowing some properties about intuitionistic proofs to be transferred back to classical proofs.

Recent developments in proof theory include the study of proof mining by Ulrich Kohlenbach and the study of proof-theoretic ordinals by Michael Rathjen.

Connections with computer science

The study of computability theory in computer science is closely related to the study of computability in mathematical logic. There is a difference of emphasis, however. Computer scientists often focus on concrete programming languages and feasible computability, while researchers in mathematical logic often focus on computability as a theoretical concept and on noncomputability.

The theory of semantics of programming languages is related to model theory, as is program verification (in particular, model checking). The Curry–Howard isomorphism between proofs and programs relates to proof theory, especially intuitionistic logic. Formal calculi such as the lambda calculus and combinatory logic are now studied as idealized programming languages.

Computer science also contributes to mathematics by developing techniques for the automatic checking or even finding of proofs, such as automated theorem proving and logic programming.

Descriptive complexity theory relates logics to computational complexity. The first significant result in this area, Fagin's theorem (1974) established that NP is precisely the set of languages expressible by sentences of existential second-order logic.

Foundations of mathematics

In the 19th century, mathematicians became aware of logical gaps and inconsistencies in their field. It was shown that Euclid's axioms for geometry, which had been taught for centuries as an example of the axiomatic method, were incomplete. The use of infinitesimals, and the very definition of function, came into question in analysis, as pathological examples such as Weierstrass' nowhere-differentiable continuous function were discovered.

Cantor's study of arbitrary infinite sets also drew criticism. Leopold Kronecker famously stated "God made the integers; all else is the work of man," endorsing a return to the study of finite, concrete objects in mathematics. Although Kronecker's argument was carried forward by constructivists in the 20th century, the mathematical community as a whole rejected them. David Hilbert argued in favor of the study of the infinite, saying "No one shall expel us from the Paradise that Cantor has created."

Mathematicians began to search for axiom systems that could be used to formalize large parts of mathematics. In addition to removing ambiguity from previously-naive terms such as function, it was hoped that this axiomatization would allow for consistency proofs. In the 19th century, the main method of proving the consistency of a set of axioms was to provide a model for it. Thus, for example, non-Euclidean geometry can be proved consistent by defining *point* to mean a point on a fixed sphere and *line* to mean a great circle on the sphere. The resulting structure, a model of elliptic geometry, satisfies the axioms of plane geometry except the parallel postulate.

With the development of formal logic, Hilbert asked whether it would be possible to prove that an axiom system is consistent by analyzing the structure of possible proofs in the system, and showing through this analysis that it is impossible to prove a contradiction. This idea led to the study of proof theory. Moreover, Hilbert proposed that the analysis should be entirely concrete, using the term *finitary* to refer to the methods he would allow but not precisely defining them. This project, known as Hilbert's program, was seriously affected by Gödel's incompleteness theorems, which show that the consistency of formal theories of arithmetic cannot be established using methods formalizable in those theories. Gentzen showed that it is possible to produce a proof of the consistency of arithmetic in a finitary system augmented with axioms of transfinite induction, and the techniques he developed to so do were seminal in proof theory.

A second thread in the history of foundations of mathematics involves nonclassical logics and constructive mathematics. The study of constructive mathematics includes many different programs with various definitions of *constructive*. At the most accommodating end, proofs in ZF set theory that do not use the axiom of choice are called constructive by many mathematicians. More limited versions of constructivism limit themselves to natural numbers, number-theoretic functions, and sets of natural numbers (which can be used to represent real numbers, facilitating the study of mathematical analysis). A common idea is that a concrete means of computing the values of the function must be known before the function itself can be said to exist.

In the early 20th century, Luitzen Egbertus Jan Brouwer founded intuitionism as a philosophy of mathematics. This philosophy, poorly understood at first, stated that in order for a mathematical statement to be true to a mathematician, that person must be able to *intuit* the statement, to not only believe its truth but understand the reason for its truth. A consequence of this definition of truth was the rejection of the law of the excluded middle, for there are statements that, according to Brouwer, could not be claimed to be true while their negations also could not be claimed true. Brouwer's philosophy was influential, and the cause of bitter disputes among prominent mathematicians. Later, Kleene and Kreisel would study formalized versions of intuitionistic logic (Brouwer rejected formalization, and presented his work in unformalized natural language). With the advent of the BHK interpretation and Kripke models, intuitionism became easier to reconcile with classical mathematics.

Notes

[1] Undergraduate texts include Boolos, Burgess, and Jeffrey (2002), Enderton (2001), and Mendelson (1997). A classic graduate text by Shoenfield (2001) first appeared in 1967.

[2] A detailed study of this terminology is given by Soare (1996).

[3] Ferreirós (2001) surveys the rise of first-order logic over other formal logics in the early 20th century.

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- Stanford Encyclopedia of Philosophy: First-order Model Theory (http://plato.stanford.edu/entries/modeltheory-fo/) – by Wilfrid Hodges.

- The London Philosophy Study Guide (http://www.ucl.ac.uk/philosophy/LPSG/) offers many suggestions on what to read, depending on the student's familiarity with the subject:
  - Mathematical Logic (http://www.ucl.ac.uk/philosophy/LPSG/MathLogic.htm)
  - Set Theory & Further Logic (http://www.ucl.ac.uk/philosophy/LPSG/SetTheory.htm)
  - Philosophy of Mathematics (http://www.ucl.ac.uk/philosophy/LPSG/PhilMath.htm)

Inference

Inference is the act or process of deriving logical conclusions from premises known or assumed to be true.[1] The conclusion drawn is also called an idiomatic. The laws of valid inference are studied in the field of logic.

Human inference (i.e. how humans draw conclusions) is traditionally studied within the field of cognitive psychology; artificial intelligence researchers develop automated inference systems to emulate human inference. Statistical inference allows for inference from quantitative data.

Definition of inference

The process by which a conclusion is inferred from multiple observations is called inductive reasoning. The conclusion may be correct or incorrect, or correct to within a certain degree of accuracy, or correct in certain situations. Conclusions inferred from multiple observations may be tested by additional observations.

This definition is disputable (due to its lack of clarity. Ref: Oxford English dictionary: "induction ... 3. Logic the inference of a general law from particular instances.") The definition given thus applies only when the "conclusion" is general.

1. A conclusion reached on the basis of evidence and reasoning. 2. The process of reaching such a conclusion: "order, health, and by inference cleanliness".

Examples of inference

Greek philosophers defined a number of syllogisms, correct three part inferences, that can be used as building blocks for more complex reasoning. We begin with the most famous of them all:

1. All men are mortal
2. Socrates is a man
3. Therefore, Socrates is mortal.

The reader can check that the premises and conclusion are true, but Logic is concerned with inference: does the truth of the conclusion follow from that of the premises?

The validity of an inference depends on the form of the inference. That is, the word "valid" does not refer to the truth of the premises or the conclusion, but rather to the form of the inference. An inference can be valid even if the parts are false, and can be invalid even if the parts are true. But a valid form with true premises will always have a true conclusion.

For example, consider the form of the following symbological track:

1. All fruits are sweet.
2. A banana is a fruit.
3. Therefore, a banana is sweet.

For the conclusion to be necessarily true, the premises need to be true.

Now we turn to an invalid form.

1. All A are B.

2. C is a B.
3. Therefore, C is an A.

To show that this form is invalid, we demonstrate how it can lead from true premises to a false conclusion.

1. All apples are fruit. (Correct)
2. Bananas are fruit. (Correct)
3. Therefore, bananas are apples. (Wrong)

A valid argument with false premises may lead to a false conclusion:

1. All tall people are Greek.
2. John Lennon was tall.
3. Therefore, John Lennon was Greek.

When a valid argument is used to derive a false conclusion from false premises, the inference is valid because it follows the form of a correct inference.

A valid argument can also be used to derive a true conclusion from false premises:

1. All tall people are musicians
2. John Lennon was tall
3. Therefore, John Lennon was a musician

In this case we have two false premises that imply a true conclusion.

Incorrect inference

An incorrect inference is known as a fallacy. Philosophers who study informal logic have compiled large lists of them, and cognitive psychologists have documented many biases in human reasoning that favor incorrect reasoning.

Automatic logical inference

AI systems first provided automated logical inference and these were once extremely popular research topics, leading to industrial applications under the form of expert systems and later business rule engines.

An inference system's job is to extend a knowledge base automatically. The knowledge base (KB) is a set of propositions that represent what the system knows about the world. Several techniques can be used by that system to extend KB by means of valid inferences. An additional requirement is that the conclusions the system arrives at are relevant to its task.

Example using Prolog

Prolog (for "Programming in Logic") is a programming language based on a subset of predicate calculus. Its main job is to check whether a certain proposition can be inferred from a KB (knowledge base) using an algorithm called backward chaining.

Let us return to our Socrates syllogism. We enter into our Knowledge Base the following piece of code:

```
mortal(X) :-       man(X).
man(socrates).
```

(Here ***:-*** can be read as "if". Generally, if $P \rightarrow Q$ (if P then Q) then in Prolog we would code ***Q:-P*** (Q if P).)
This states that all men are mortal and that Socrates is a man. Now we can ask the Prolog system about Socrates:

```
?- mortal(socrates).
```

(where *?-* signifies a query: Can *mortal(socrates).* be deduced from the KB using the rules) gives the answer "Yes".

On the other hand, asking the Prolog system the following:

```
?- mortal(plato).
```

gives the answer "No".

This is because Prolog does not know anything about Plato, and hence defaults to any property about Plato being false (the so-called closed world assumption). Finally ?- mortal(X) (Is anything mortal) would result in "Yes" (and in some implementations: "Yes": X=socrates)
Prolog can be used for vastly more complicated inference tasks. See the corresponding article for further examples.

Use with the semantic web

Recently automatic reasoners found in semantic web a new field of application. Being based upon first-order logic, knowledge expressed using one variant of OWL can be logically processed, i.e., inferences can be made upon it.

Bayesian statistics and probability logic

Philosophers and scientists who follow the Bayesian framework for inference use the mathematical rules of probability to find this best explanation. The Bayesian view has a number of desirable features—one of them is that it embeds deductive (certain) logic as a subset (this prompts some writers to call Bayesian probability "probability logic", following E. T. Jaynes).

Bayesians identify probabilities with degrees of beliefs, with certainly true propositions having probability 1, and certainly false propositions having probability 0. To say that "it's going to rain tomorrow" has a 0.9 probability is to say that you consider the possibility of rain tomorrow as extremely likely.

Through the rules of probability, the probability of a conclusion and of alternatives can be calculated. The best explanation is most often identified with the most probable (see Bayesian decision theory). A central rule of Bayesian inference is Bayes' theorem.

See Bayesian inference for examples.

Nonmonotonic logic[2]

A relation of inference is monotonic if the addition of premises does not undermine previously reached conclusions; otherwise the relation is nonmonotonic. Deductive inference, is monotonic: if a conclusion is reached on the basis of a certain set of premises, then that conclusion still holds if more premises are added.

By contrast, everyday reasoning is mostly nonmonotonic because it involves risk: we jump to conclusions from deductively insufficient premises. We know when it is worth or even necessary (e.g. in medical diagnosis) to take the risk. Yet we are also aware that such inference is defeasible—that new information may undermine old conclusions. Various kinds of defeasible but remarkably successful inference have traditionally captured the attention of philosophers (theories of induction, Peirce's theory of abduction, inference to the best explanation, etc.). More recently logicians have begun to approach the phenomenon from a formal point of view. The result is a large body of theories at the interface of philosophy, logic and artificial intelligence.

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External links

- Inference (http://philpapers.org/browse/inference) at PhilPapers
- Inference (https://inpho.cogs.indiana.edu/taxonomy/2397) at the Indiana Philosophy Ontology Project

Logical consequence

Logical consequence (also **entailment**) is one of the most fundamental concepts in logic. It is the relationship between statements that holds true when one logically "follows from" one or more others. Valid logical arguments are ones in which the conclusions follow from its premises, and its conclusions are consequences of its premises. The philosophical analysis of logical consequence involves asking, 'in what sense does a conclusion follow from its premises?' and 'what does it mean for a conclusion to be a consequence of premises?'[1] All of philosophical logic can be thought of as providing accounts of the nature of logical consequence, as well as logical truth.[2]

Logical consequence is taken to be both necessary and formal with examples explicated using models and proofs.[1] A sentence is said to be a logical consequence of a set of sentences, for a given language, if and only if, using logic alone (i.e. without regard to any interpretations of the sentences) the sentence must be true if every sentence in the set were to be true.[3]

Logicians make precise accounts of logical consequence with respect to a given language $\mathcal{L}$ by constructing a deductive system for $\mathcal{L}$, or by formalizing the intended semantics for $\mathcal{L}$. Alfred Tarski highlighted three salient features for which any adequate characterization of logical consequence needs to account: 1) that the logical consequence relation relies on the logical form of the sentences involved, 2) that the relation is a priori, i.e. it can be determined whether or not it holds without regard to sense experience, and 3) that the relation has a modal component.[3]

Formal accounts of logical consequence

The most widely prevailing view on how to best account for logical consequence is to appeal to formality. This is to say that whether or not statements follow from one another logically depends on the structure or logical form of the statements without regard to the contents of that form.

Syntactic accounts of logical consequence rely on schemes using inference rules. For instance, we can express the logical form of a valid argument as "All A are B. All C are A. Therefore, All C are B." This argument is formally valid, because every instance of arguments constructed using this scheme are valid.

This is in contrast to an argument like "Fred is Mike's brother's son. Therefore Fred is Mike's nephew." As this argument depends on the meanings of the words "Fred," "Mike," "brother," "son," and "nephew." The statement "Fred is Mike's nephew." is a material consequence of "Fred is Mike's brother's son," not a formal consequence. However, this is still an incomplete account of formal consequence, as the argument is still an instance of " P is Q 's brother's son. Therefore, P is Q 's nephew." Every instance of which is valid.[1]

Apriority of logical consequence

If you know that Q follows logically from P no information about the possible interpretations of P or Q will affect that knowledge. Our knowledge that Q is a logical consequence of P cannot be influenced by empirical knowledge.[1] Deductively valid arguments can be known to be so without recourse to experience, so they must be knowable a priori.[1] However, formality alone does not guarantee that logical consequence is not influenced by empirical knowledge. So the apriority of logical consequence is considered to be independent of formality.[1]

Proofs and models

The two prevailing techniques for providing accounts of logical consequence involve expressing the concept in terms of proofs and the other via models. The study of the syntactic consequence (of a logic) is called (its) proof theory whereas the study of (its) semantic consequence is called (its) model theory.[4]

Syntactic consequence

A formula A is a **syntactic consequence**[5][6][7][8] within some formal system $\mathcal{FS}$ of a set Γ of formulas if there is a formal proof in $\mathcal{FS}$ of A from the set Γ.

$$\Gamma \vdash_{\mathcal{FS}} A$$

Syntactic consequence does not depend on any interpretation of the formal system.[9]

Semantic consequence

A formula A is a **semantic consequence** within some formal system $\mathcal{FS}$ of a set of statements Γ

$$\Gamma \models_{\mathcal{FS}} A,$$

if and only if there is no model $\mathcal{I}$ in which all members of Γ are true and A is false.[10] Or, in other words, the set of the interpretations that make all members of Γ true is a subset of the set of the interpretations that make A true.

Modal accounts

Modal accounts of logical consequence are variations on the following basic idea:

$\Gamma \vdash A$ is true if and only if it is *necessary* that if all of the elements of Γ are true, then A is true.

Alternatively (and, most would say, equivalently):

$\Gamma \vdash A$ is true if and only if it is *impossible* for all of the elements of Γ to be true and A false.

Such accounts are called "modal" because they appeal to the modal notions of logical necessity and logical possibility. 'It is necessary that' is often expressed as a universal quantifier over possible worlds, so that the accounts above translate as:

> $\Gamma \vdash A$ is true if and only if there is no possible world at which all of the elements of Γ are true and A is false (untrue).

Consider the modal account in terms of the argument given as an example above:

> All frogs are green.
>
> Kermit is a frog.
>
> Therefore, Kermit is green.

The conclusion is a logical consequence of the premises because we can't imagine a possible world where (a) all frogs are green; (b) Kermit is a frog; and (c) Kermit is not green.

Modal-formal accounts

Modal-formal accounts of logical consequence combine the modal and formal accounts above, yielding variations on the following basic idea:

> $\Gamma \vdash A$ if and only if it is impossible for an argument with the same logical form as Γ / A to have true premises and a false conclusion.

Warrant-based accounts

The accounts considered above are all "truth-preservational," in that they all assume that the characteristic feature of a good inference is that it never allows one to move from true premises to an untrue conclusion. As an alternative, some have proposed "warrant-preservational" accounts, according to which the characteristic feature of a good inference is that it never allows one to move from justifiably assertible premises to a conclusion that is not justifiably assertible. This is (roughly) the account favored by intuitionists such as Michael Dummett.

Non-monotonic logical consequence

The accounts discussed above all yield monotonic consequence relations, i.e. ones such that if A is a consequence of Γ, then A is a consequence of any superset of Γ. It is also possible to specify non-monotonic consequence relations to capture the idea that, e.g., 'Tweety can fly' is a logical consequence of

> {Birds can typically fly, Tweety is a bird}

but not of

> {Birds can typically fly, Tweety is a bird, Tweety is a penguin}.

For more on this, see belief revision#Non-monotonic inference relation.

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Propositional calculus

In mathematical logic, a **propositional calculus** or **logic** (also called **sentential calculus** or **sentential logic**) is a formal system in which formulas of a formal language may be interpreted as representing propositions. A system of inference rules and axioms allows certain formulas to be derived, called theorems; which may be interpreted as true propositions. The series of formulas which is constructed within such a system is called a derivation and the last formula of the series is a theorem, whose derivation may be interpreted as a proof of the truth of the proposition represented by the theorem.

Truth-functional propositional logic is a propositional logic whose interpretation limits the truth values of its propositions to two, usually *true* and *false*. Truth-functional propositional logic and systems isomorphic to it are considered to be **zeroth-order logic**.

History

Although propositional logic (which is interchangeable with propositional calculus) had been hinted by earlier philosophers, it was developed into a formal logic by Chrysippus[1] and expanded by the Stoics. The logic was focused on propositions. This advancement was different from the traditional syllogistic logic which was focused on terms. However, later in antiquity, the propositional logic developed by the Stoics was no longer understood. As a result, the system was essentially reinvented by Peter Abelard.[2]

Propositional logic was eventually refined using symbolic logic. Gottfried Leibniz has been credited with being the founder of symbolic logic for his work with the calculus ratiocinator. Although his work was the first of its kind, it was unknown to the larger logical community. As a result, many of the advances achieved by Leibniz were reachieved by logicians like George Boole and Augustus De Morgan completely independent of Leibniz.[3]

Just as propositional logic can be seen as an advancement from the earlier syllogistic logic, Gottlob Frege's predicate logic was an advancement from the earlier propositional logic. Predicate logic has been described as combining "the distinctive features of syllogistic logic and propositional logic."[4] As a result, it ushered a new era in the history of logic. However, advances in propositional logic were still made after Frege. These include Natural Deduction, Truth-Trees and Truth-Tables. Natural deduction was invented by Jan Łukasiewicz. Truth-Trees were invented by Evert Willem Beth.[5] The invention of truth-tables, however, is of controversial attribution.

The ideas preceding truth tables have been found in both Frege[6] and Bertrand Russell[7] whereas the actual 'tabular structure' (i.e. being formed as a table) is generally credited to either Ludwig Wittgenstein, Emil Post or both (independently of one another).[6] Besides Frege and Russell, others credited for having preceding ideas of truth-tables include Philo, Boole, Charles Sanders Peirce, and Ernst Schröder. And besides Post and Wittgenstein, others credited with the tabular structure include Łukasiewicz, Schröder, Alfred North Whitehead, William Stanley Jevons, John Venn, and Clarence Irving Lewis.[7] Ultimately, some, like John Shosky, have concluded "It is far from clear that any one person should be given the title of 'inventor' of truth-tables.".[7]

Terminology

In general terms, a calculus is a formal system that consists of a set of syntactic expressions (*well-formed formulæ* or *wffs*), a distinguished subset of these expressions (axioms), plus a set of formal rules that define a specific binary relation, intended to be interpreted as logical equivalence, on the space of expressions.

When the formal system is intended to be a logical system, the expressions are meant to be interpreted as statements, and the rules, known as *inference rules*, are typically intended to be truth-preserving. In this setting, the rules (which may include axioms) can then be used to derive ("infer") formulæ representing true statements from given formulæ representing true statements.

The set of axioms may be empty, a nonempty finite set, a countably infinite set, or be given by axiom schemata. A formal grammar recursively defines the expressions and well-formed formulæ (wffs) of the language. In addition a semantics may be given which defines truth and valuations (or interpretations).

The language of a propositional calculus consists of

1. a set of primitive symbols, variously referred to as *atomic formulae*, *placeholders*, *proposition letters*, or *variables*, and
2. a set of operator symbols, variously interpreted as *logical operators* or *logical connectives*.

A *well-formed formula* (*wff*) is any atomic formula, or any formula that can be built up from atomic formulæ by means of operator symbols according to the rules of the grammar.

Mathematicians sometimes distinguish between propositional constants, propositional variables, and schemata. Propositional constants represent some particular proposition, while propositional variables range over the set of all atomic propositions. Schemata, however, range over all propositions. It is common to represent propositional constants by A , B , and C , propositional variables by P, Q , and R , and schematic letters are often Greek letters, most often φ , ψ , and χ .

Basic concepts

The following outlines a standard propositional calculus. Many different formulations exist which are all more or less equivalent but differ in the details of

1. their language, that is, the particular collection of primitive symbols and operator symbols,
2. the set of axioms, or distinguished formulæ, and
3. the set of inference rules.

We may represent any given proposition with a letter which we call a propositional constant, analogous to representing a number by a letter in mathematics, for instance, $a = 5$. We require that all propositions have exactly one of two truth-values: true or false. To take an example, let P be the proposition that it is raining outside. This will be true if it is raining outside and false otherwise.

- We then define truth-functional operators, beginning with negation. We write $\neg P$ to represent the negation of P , which can be thought of as the denial of P. In the example above, $\neg P$ expresses that it is not raining outside, or by a more standard reading: "It is not the case that it is raining outside." When P is true, $\neg P$ is false; and when P is false, $\neg P$ is true. $\neg\neg P$ always has the same truth-value as P.
- Conjunction is a truth-functional connective which forms a proposition out of two simpler propositions, for example, P and Q . The conjunction of P and Q is written $P \wedge Q$, and expresses that each are true. We read $P \wedge Q$ as " P and Q ". For any two propositions, there are four possible assignments of truth values:
    1. P is true and Q is true
    2. P is true and Q is false
    3. P is false and Q is true
    4. P is false and Q is false

The conjunction of P and Q is true in case 1 and is false otherwise. Where P is the proposition that it is raining outside and Q is the proposition that a cold-front is over Kansas, $P \land Q$ is true when it is raining outside and there is a cold-front over Kansas. If it is not raining outside, then $P \land Q$ is false; and if there is no cold-front over Kansas, then $P \land Q$ is false.

- Disjunction resembles conjunction in that it forms a proposition out of two simpler propositions. We write it $P \lor Q$, and it is read " P or Q ". It expresses that either P or Q is true. Thus, in the cases listed above, the disjunction of P and Q is true in all cases except 4. Using the example above, the disjunction expresses that it is either raining outside or there is a cold front over Kansas. (Note, this use of disjunction is supposed to resemble the use of the English word "or". However, it is most like the English inclusive "or", which can be used to express the truth of at least one of two propositions. It is not like the English exclusive "or", which expresses the truth of exactly one of two propositions. That is to say, the exclusive "or" is false when both P and Q are true (case 1). An example of the exclusive or is: You may have a bagel or a pastry, but not both. Often in natural language, given the appropriate context, the addendum "but not both" is omitted but implied. In mathematics, however, "or" is always used as inclusive or; if exclusive or is meant it will be specified, possibly by "xor".)
- Material conditional also joins two simpler propositions, and we write $P \rightarrow Q$, which is read "if P then Q ". The proposition to the left of the arrow is called the antecedent and the proposition to the right is called the consequent. (There is no such designation for conjunction or disjunction, since they are commutative operations.) It expresses that Q is true whenever P is true. Thus it is true in every case above except case 2, because this is the only case when P is true but Q is not. Using the example, if P then Q expresses that if it is raining outside then there is a cold-front over Kansas. The material conditional is often confused with physical causation. The material conditional, however, only relates two propositions by their truth-values—which is not the relation of cause and effect. It is contentious in the literature whether the material implication represents logical causation.
- Biconditional joins two simpler propositions, and we write $P \leftrightarrow Q$, which is read " P if and only if Q ". It expresses that P and Q have the same truth-value, thus P if and only if Q is true in cases 1 and 4, and false otherwise.

It is extremely helpful to look at the truth tables for these different operators, as well as the method of analytic tableaux.

Closure under operations

Propositional logic is closed under truth-functional connectives. That is to say, for any proposition φ , $\neg\varphi$ is also a proposition. Likewise, for any propositions φ and ψ , $\varphi \land \psi$ is a proposition, and similarly for disjunction, conditional, and biconditional. This implies that, for instance, $P \land Q$ is a proposition, and so it can be conjoined with another proposition. In order to represent this, we need to use parentheses to indicate which proposition is conjoined with which. For instance, $P \land Q \land R$ is not a well-formed formula, because we do not know if we are conjoining $P \land Q$ with R or if we are conjoining P with $Q \land R$. Thus we must write either $(P \land Q) \land R$ to represent the former, or $P \land (Q \land R)$ to represent the latter. By evaluating the truth conditions, we see that both expressions have the same truth conditions (will be true in the same cases), and moreover that any proposition formed by arbitrary conjunctions will have the same truth conditions, regardless of the location of the parentheses. This means that conjunction is associative, however, one should not assume that parentheses never serve a purpose. For instance, the sentence $P \land (Q \lor R)$ does not have the same truth conditions as $(P \land Q) \lor R$, so they are different sentences distinguished only by the parentheses. One can verify this by the truth-table method referenced above.

Note: For any arbitrary number of propositional constants, we can form a finite number of cases which list their possible truth-values. A simple way to generate this is by truth-tables, in which one writes P, Q , ..., Z for any list of k propositional constants—that is to say, any list of propositional constants with k entries. Below this list, one writes 2^k rows, and below P one fills in the first half of the rows with true (or T) and the second half with false (or F). Below Q one fills in one-quarter of the rows with T, then one-quarter with F, then one-quarter with T and the last quarter with F. The next column alternates between true and false for each eighth of the rows, then sixteenths, and so on, until the last propositional constant varies between T and F for each row. This will give a complete listing of cases or truth-value assignments possible for those propositional constants.

Argument

The propositional calculus then defines an *argument* as a set of propositions. A valid argument is a set of propositions, the last of which follows from—or is implied by—the rest. All other arguments are invalid. The simplest valid argument is modus ponens, one instance of which is the following set of propositions:

$$\begin{array}{ll} 1. & P \to Q \\ 2. & P \\ \hline \therefore & Q \end{array}$$

This is a set of three propositions, each line is a proposition, and the last follows from the rest. The first two lines are called premises, and the last line the conclusion. We say that any proposition C follows from any set of propositions $(P_1, ..., P_n)$, if C must be true whenever every member of the set $(P_1, ..., P_n)$ is true. In the argument above, for any P and Q, whenever $P \to Q$ and P are true, necessarily Q is true. Notice that, when P is true, we cannot consider cases 3 and 4 (from the truth table). When $P \to Q$ is true, we cannot consider case 2. This leaves only case 1, in which Q is also true. Thus Q is implied by the premises.

This generalizes schematically. Thus, where φ and ψ may be any propositions at all,

$$\begin{array}{ll} 1. & \varphi \to \psi \\ 2. & \varphi \\ \hline \therefore & \psi \end{array}$$

Other argument forms are convenient, but not necessary. Given a complete set of axioms (see below for one such set), modus ponens is sufficient to prove all other argument forms in propositional logic, and so we may think of them as derivative. Note, this is not true of the extension of propositional logic to other logics like first-order logic. First-order logic requires at least one additional rule of inference in order to obtain completeness.

The significance of argument in formal logic is that one may obtain new truths from established truths. In the first example above, given the two premises, the truth of Q is not yet known or stated. After the argument is made, Q is deduced. In this way, we define a deduction system as a set of all propositions that may be deduced from another set of propositions. For instance, given the set of propositions $A = \{P \vee Q, \neg Q \wedge R, (P \vee Q) \to R\}$, we can define a deduction system, Γ, which is the set of all propositions which follow from A. Reiteration is always assumed, so $P \vee Q, \neg Q \wedge R, (P \vee Q) \to R \in \Gamma$. Also, from the first element of A, last element, as well as modus ponens, R is a consequence, and so $R \in \Gamma$. Because we have not included sufficiently complete axioms, though, nothing else may be deduced. Thus, even though most deduction systems studied in propositional logic are able to deduce $(P \vee Q) \leftrightarrow (\neg P \to Q)$, this one is too weak to prove such a proposition.

Generic description of a propositional calculus

A **propositional calculus** is a formal system $\mathcal{L} = \mathcal{L}\left(\mathrm{A},\ \Omega,\ \mathrm{Z},\ \mathrm{I}\right)$, where:

- The *alpha set* A is a finite set of elements called *proposition symbols* or *propositional variables*. Syntactically speaking, these are the most basic elements of the formal language $\mathcal{L}$, otherwise referred to as *atomic formulæ* or *terminal elements*. In the examples to follow, the elements of A are typically the letters p, q, r, and so on.
- The *omega set* Ω is a finite set of elements called *operator symbols* or *logical connectives*. The set Ω is partitioned into disjoint subsets as follows:

$$\Omega = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_j \cup \ldots \cup \Omega_m.$$

  In this partition, Ω_j is the set of operator symbols of *arity* j.

In the more familiar propositional calculi, Ω is typically partitioned as follows:

$$\Omega_1 = \{\neg\},$$
$$\Omega_2 \subseteq \{\land, \lor, \to, \leftrightarrow\}.$$

A frequently adopted convention treats the constant logical values as operators of arity zero, thus:

$$\Omega_0 = \{0, 1\}.$$

Some writers use the tilde (~), or N, instead of $\neg$; and some use the ampersand (&), the prefixed K, or $\cdot$ instead of $\land$. Notation varies even more for the set of logical values, with symbols like {false, true}, {F, T}, or $\{\bot, \top\}$ all being seen in various contexts instead of {0, 1}.

- The *zeta set* Z is a finite set of *transformation rules* that are called *inference rules* when they acquire logical applications.
- The *iota set* I is a finite set of *initial points* that are called *axioms* when they receive logical interpretations.

The *language* of $\mathcal{L}$, also known as its set of *formulæ*, *well-formed formulas* or *wffs*, is inductively defined by the following rules:

1. Base: Any element of the alpha set A is a formula of $\mathcal{L}$.
2. If $p_1, p_2, \ldots, p_j$ are formulæ and f is in Ω_j, then $(f(p_1, p_2, \ldots, p_j))$ is a formula.
3. Closed: Nothing else is a formula of $\mathcal{L}$.

Repeated applications of these rules permits the construction of complex formulæ. For example:

1. By rule 1, p is a formula.
2. By rule 2, $\neg p$ is a formula.
3. By rule 1, q is a formula.
4. By rule 2, $(\neg p \lor q)$ is a formula.

Example 1. Simple axiom system

Let $\mathcal{L}_1 = \mathcal{L}(\mathrm{A}, \Omega, \mathrm{Z}, \mathrm{I})$, where A, Ω, Z, I are defined as follows:

- The alpha set A, is a finite set of symbols that is large enough to supply the needs of a given discussion, for example:

$$\mathrm{A} = \{p, q, r, s, t, u\}.$$

- Of the three connectives for conjunction, disjunction, and implication ($\land$, $\lor$, and $\to$), one can be taken as primitive and the other two can be defined in terms of it and negation ($\neg$). Indeed, all of the logical connectives can be defined in terms of a sole sufficient operator. The biconditional ($\leftrightarrow$) can of course be defined in terms of conjunction and implication, with $a \leftrightarrow b$ defined as $(a \to b) \land (b \to a)$.

  Adopting negation and implication as the two primitive operations of a propositional calculus is tantamount to having the omega set $\Omega = \Omega_1 \cup \Omega_2$ partition as follows:

$$\Omega_1 = \{\neg\},$$
$$\Omega_2 = \{\to\}.$$

- An axiom system discovered by Jan Łukasiewicz formulates a propositional calculus in this language as follows. The axioms are all substitution instances of:
  - $(p \to (q \to p))$
  - $((p \to (q \to r)) \to ((p \to q) \to (p \to r)))$
  - $((\neg p \to \neg q) \to (q \to p))$
- The rule of inference is modus ponens (i.e., from p and $(p \to q)$, infer q). Then $a \lor b$ is defined as $\neg a \to b$, and $a \land b$ is defined as $\neg(a \to \neg b)$.

Example 2. Natural deduction system

Let $\mathcal{L}_2 = \mathcal{L}(\mathrm{A}, \Omega, \mathrm{Z}, \mathrm{I})$, where A, Ω, Z, I are defined as follows:

- The alpha set A, is a finite set of symbols that is large enough to supply the needs of a given discussion, for example:

  $$\mathrm{A} = \{p, q, r, s, t, u\}.$$
- The omega set $\Omega = \Omega_1 \cup \Omega_2$ partitions as follows:

  $$\Omega_1 = \{\neg\},$$

  $$\Omega_2 = \{\land, \lor, \to, \leftrightarrow\}.$$

In the following example of a propositional calculus, the transformation rules are intended to be interpreted as the inference rules of a so-called *natural deduction system*. The particular system presented here has no initial points, which means that its interpretation for logical applications derives its theorems from an empty axiom set.

- The set of initial points is empty, that is, $\mathrm{I} = \varnothing$.
- The set of transformation rules, Z, is described as follows:

Our propositional calculus has ten inference rules. These rules allow us to derive other true formulae given a set of formulae that are assumed to be true. The first nine simply state that we can infer certain wffs from other wffs. The last rule however uses hypothetical reasoning in the sense that in the premise of the rule we temporarily assume an (unproven) hypothesis to be part of the set of inferred formulae to see if we can infer a certain other formula. Since the first nine rules don't do this they are usually described as *non-hypothetical* rules, and the last one as a *hypothetical* rule.

In describing the transformation rules, we may introduce a metalanguage symbol $\vdash$. It is basically a convenient shorthand for saying "infer that". The format is $\Gamma \vdash \psi$, in which Γ is a (possibly empty) set of formulae called premises, and ψ is a formula called conclusion. The transformation rule $\Gamma \vdash \psi$ means that if every proposition in Γ is a theorem (or has the same truth value as the axioms), then ψ is also a theorem. Note that considering the following rule Conjunction introduction, we will know whenever Γ has more than one formula, we can always safely reduce it into one formula using conjunction. So for short, from that time on we may represent Γ as one formula instead of a set. Another omission for convenience is when Γ is an empty set, in which case Γ may not appear.

Reductio ad absurdum (negation introduction)

From $(p \to q)$ and $\neg q$, infer $\neg p$.

That is, $\{p \to q, \neg q\} \vdash \neg p$.

Double negative elimination

From $\neg\neg p$, infer p.

That is, $\neg\neg p \vdash p$.

Conjunction introduction

From p and q, infer $(p \land q)$.

That is, $\{p, q\} \vdash (p \land q)$.

Conjunction elimination

From $(p \land q)$, infer p.

From $(p \land q)$, infer q.

That is, $(p \land q) \vdash p$ and $(p \land q) \vdash q$.

Disjunction introduction

From p, infer $(p \lor q)$.

From q, infer $(p \vee q)$.

That is, $p \vdash (p \vee q)$ and $q \vdash (p \vee q)$.

Disjunction elimination

From $(p \vee q)$ and $(p \rightarrow r)$ and $(q \rightarrow r)$, infer r.

That is, $\{p \vee q, p \rightarrow r, q \rightarrow r\} \vdash r$.

Biconditional introduction

From $(p \rightarrow q)$ and $(q \rightarrow p)$, infer $(p \leftrightarrow q)$.

That is, $\{p \rightarrow q, q \rightarrow p\} \vdash (p \leftrightarrow q)$.

Biconditional elimination

From $(p \leftrightarrow q)$, infer $(p \rightarrow q)$.

From $(p \leftrightarrow q)$, infer $(q \rightarrow p)$.

That is, $(p \leftrightarrow q) \vdash (p \rightarrow q)$ and $(p \leftrightarrow q) \vdash (q \rightarrow p)$.

Modus ponens (conditional elimination)

From p and $(p \rightarrow q)$, infer q.

That is, $\{p, p \rightarrow q\} \vdash q$.

Conditional proof (conditional introduction)

From [accepting p allows a proof of q], infer $(p \rightarrow q)$.

That is, $(p \vdash q) \vdash (\vdash (p \rightarrow q))$.

Basic and derived argument forms

Basic and Derived Argument Forms		
Name	**Sequent**	**Description**
Modus Ponens	$((p \rightarrow q) \wedge p) \vdash q$	If p then q; p; therefore q
Modus Tollens	$((p \rightarrow q) \wedge \neg q) \vdash \neg p$	If p then q; not q; therefore not p
Hypothetical Syllogism	$((p \rightarrow q) \wedge (q \rightarrow r)) \vdash (p \rightarrow r)$	If p then q; if q then r; therefore, if p then r
Disjunctive Syllogism	$((p \vee q) \wedge \neg p) \vdash q$	Either p or q, or both; not p; therefore, q
Constructive Dilemma	$((p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)) \vdash (q \vee s)$	If p then q; and if r then s; but p or r; therefore q or s
Destructive Dilemma	$((p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)) \vdash (\neg p \vee \neg r)$	If p then q; and if r then s; but not q or not s; therefore not p or not r
Bidirectional Dilemma	$((p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee \neg s)) \vdash (q \vee \neg r)$	If p then q; and if r then s; but p or not s; therefore q or not r
Simplification	$(p \wedge q) \vdash p$	p and q are true; therefore p is true
Conjunction	$p, q \vdash (p \wedge q)$	p and q are true separately; therefore they are true conjointly
Addition	$p \vdash (p \vee q)$	p is true; therefore the disjunction (p or q) is true
Composition	$((p \rightarrow q) \wedge (p \rightarrow r)) \vdash (p \rightarrow (q \wedge r))$	If p then q; and if p then r; therefore if p is true then q and r are true
De Morgan's Theorem (1)	$\neg(p \wedge q) \vdash (\neg p \vee \neg q)$	The negation of (p and q) is equiv. to (not p or not q)

De Morgan's Theorem (2)	$\neg(p \vee q) \vdash (\neg p \wedge \neg q)$	The negation of (p or q) is equiv. to (not p and not q)
Commutation (1)	$(p \vee q) \vdash (q \vee p)$	(p or q) is equiv. to (q or p)
Commutation (2)	$(p \wedge q) \vdash (q \wedge p)$	(p and q) is equiv. to (q and p)
Commutation (3)	$(p \leftrightarrow q) \vdash (q \leftrightarrow p)$	(p is equiv. to q) is equiv. to (q is equiv. to p)
Association (1)	$(p \vee (q \vee r)) \vdash ((p \vee q) \vee r)$	p or (q or r) is equiv. to (p or q) or r
Association (2)	$(p \wedge (q \wedge r)) \vdash ((p \wedge q) \wedge r)$	p and (q and r) is equiv. to (p and q) and r
Distribution (1)	$(p \wedge (q \vee r)) \vdash ((p \wedge q) \vee (p \wedge r))$	p and (q or r) is equiv. to (p and q) or (p and r)
Distribution (2)	$(p \vee (q \wedge r)) \vdash ((p \vee q) \wedge (p \vee r))$	p or (q and r) is equiv. to (p or q) and (p or r)
Double Negation	$p \vdash \neg\neg p$	p is equivalent to the negation of not p
Transposition	$(p \rightarrow q) \vdash (\neg q \rightarrow \neg p)$	If p then q is equiv. to if not q then not p
Material Implication	$(p \rightarrow q) \vdash (\neg p \vee q)$	If p then q is equiv. to not p or q
Material Equivalence (1)	$(p \leftrightarrow q) \vdash ((p \rightarrow q) \wedge (q \rightarrow p))$	(p is equiv. to q) means (if p is true then q is true) and (if q is true then p is true)
Material Equivalence (2)	$(p \leftrightarrow q) \vdash ((p \wedge q) \vee (\neg p \wedge \neg q))$	(p is equiv. to q) means either (p and q are true) or (both p and q are false)
Material Equivalence (3)	$(p \leftrightarrow q) \vdash ((p \vee \neg q) \wedge (\neg p \vee q))$	(p is equiv. to q) means, both (p or not q is true) and (not p or q is true)
Exportation[8]	$((p \wedge q) \rightarrow r) \vdash (p \rightarrow (q \rightarrow r))$	from (if p and q are true then r is true) we can prove (if q is true then r is true, if p is true)
Importation	$(p \rightarrow (q \rightarrow r)) \vdash ((p \wedge q) \rightarrow r)$	If p then (if q then r) is equivalent to if p and q then r
Tautology (1)	$p \vdash (p \vee p)$	p is true is equiv. to p is true or p is true
Tautology (2)	$p \vdash (p \wedge p)$	p is true is equiv. to p is true and p is true
Tertium non datur (Law of Excluded Middle)	$\vdash (p \vee \neg p)$	p or not p is true
Law of Non-Contradiction	$\vdash \neg(p \wedge \neg p)$	p and not p is false, is a true statement

Proofs in propositional calculus

One of the main uses of a propositional calculus, when interpreted for logical applications, is to determine relations of logical equivalence between propositional formulæ. These relationships are determined by means of the available transformation rules, sequences of which are called *derivations* or *proofs*.

In the discussion to follow, a proof is presented as a sequence of numbered lines, with each line consisting of a single formula followed by a *reason* or *justification* for introducing that formula. Each premise of the argument, that is, an assumption introduced as an hypothesis of the argument, is listed at the beginning of the sequence and is marked as a "premise" in lieu of other justification. The conclusion is listed on the last line. A proof is complete if every line follows from the previous ones by the correct application of a transformation rule. (For a contrasting approach, see proof-trees).

Example of a proof

- To be shown that $A \to A$.
- One possible proof of this (which, though valid, happens to contain more steps than are necessary) may be arranged as follows:

Example of a Proof		
Number	**Formula**	**Reason**
1	A	premise
2	$A \vee A$	From (1) by disjunction introduction
3	$(A \vee A) \wedge A$	From (1) and (2) by conjunction introduction
4	A	From (3) by conjunction elimination
5	$A \vdash A$	Summary of (1) through (4)
6	$\vdash A \to A$	From (5) by conditional proof

Interpret $A \vdash A$ as "Assuming A, infer A". Read $\vdash A \to A$ as "Assuming nothing, infer that A implies A", or "It is a tautology that A implies A", or "It is always true that A implies A".

Soundness and completeness of the rules

The crucial properties of this set of rules are that they are *sound* and *complete*. Informally this means that the rules are correct and that no other rules are required. These claims can be made more formal as follows.

We define a *truth assignment* as a function that maps propositional variables to **true** or **false**. Informally such a truth assignment can be understood as the description of a possible state of affairs (or possible world) where certain statements are true and others are not. The semantics of formulae can then be formalized by defining for which "state of affairs" they are considered to be true, which is what is done by the following definition.

We define when such a truth assignment A satisfies a certain wff with the following rules:

- A satisfies the propositional variable P if and only if $A(P) = \text{true}$
- A satisfies $\neg\phi$ if and only if A does not satisfy ϕ
- A satisfies $(\phi \wedge \psi)$ if and only if A satisfies both ϕ and ψ
- A satisfies $(\phi \vee \psi)$ if and only if A satisfies at least one of either ϕ or ψ
- A satisfies $(\phi \to \psi)$ if and only if it is not the case that A satisfies ϕ but not ψ
- A satisfies $(\phi \leftrightarrow \psi)$ if and only if A satisfies both ϕ and ψ or satisfies neither one of them

With this definition we can now formalize what it means for a formula ϕ to be implied by a certain set S of formulae. Informally this is true if in all worlds that are possible given the set of formulae S the formula ϕ also holds. This leads to the following formal definition: We say that a set S of wffs *semantically entails* (or *implies*) a certain wff ϕ if all truth assignments that satisfy all the formulae in S also satisfy ϕ.

Finally we define *syntactical entailment* such that ϕ is syntactically entailed by S if and only if we can derive it with the inference rules that were presented above in a finite number of steps. This allows us to formulate exactly what it means for the set of inference rules to be sound and complete:

Soundness

If the set of wffs S syntactically entails wff ϕ then S semantically entails ϕ

Completeness

If the set of wffs S semantically entails wff ϕ then S syntactically entails ϕ

For the above set of rules this is indeed the case.

Sketch of a soundness proof

(For most logical systems, this is the comparatively "simple" direction of proof)

Notational conventions: Let G be a variable ranging over sets of sentences. Let A , B , and C range over sentences. For " G syntactically entails A " we write " G proves A ". For " G semantically entails A " we write " G implies A ".

We want to show: (A)(G)(if G proves A , then G implies A).

We note that " G proves A " has an inductive definition, and that gives us the immediate resources for demonstrating claims of the form "If G proves A , then ...". So our proof proceeds by induction.

I. Basis. Show: If A is a member of G , then G implies A .
II. Basis. Show: If A is an axiom, then G implies A .
III. Inductive step (induction on n , the length of the proof):
   a. Assume for arbitrary G and A that if G proves A in n or fewer steps, then G implies A .
   b. For each possible application of a rule of inference at step $n+1$, leading to a new theorem B , show that G implies B .

Notice that Basis Step II can be omitted for natural deduction systems because they have no axioms. When used, Step II involves showing that each of the axioms is a (semantic) logical truth.

The Basis step(s) demonstrate(s) that the simplest provable sentences from G are also implied by G , for any G . (The is simple, since the semantic fact that a set implies any of its members, is also trivial.) The Inductive step will systematically cover all the further sentences that might be provable—by considering each case where we might reach a logical conclusion using an inference rule—and shows that if a new sentence is provable, it is also logically implied. (For example, we might have a rule telling us that from " A " we can derive " A or B ". In III.a We assume that if A is provable it is implied. We also know that if A is provable then " A or B " is provable. We have to show that then " A or B " too is implied. We do so by appeal to the semantic definition and the assumption we just made. A is provable from G , we assume. So it is also implied by G . So any semantic valuation making all of G true makes A true. But any valuation making A true makes " A or B " true, by the defined semantics for "or". So any valuation which makes all of G true makes " A or B " true. So " A or B " is implied.) Generally, the Inductive step will consist of a lengthy but simple case-by-case analysis of all the rules of inference, showing that each "preserves" semantic implication.

By the definition of provability, there are no sentences provable other than by being a member of G , an axiom, or following by a rule; so if all of those are semantically implied, the deduction calculus is sound.

Sketch of completeness proof

(This is usually the much harder direction of proof.)

We adopt the same notational conventions as above.

We want to show: If G implies A , then G proves A . We proceed by contraposition: We show instead that if G does **not** prove A then G does **not** imply A .

I. G does not prove A . (Assumption)
II. If G does not prove A , then we can construct an (infinite) "Maximal Set", G^* , which is a superset of G and which also does not prove A .
   1. Place an "ordering" on all the sentences in the language (e.g., shortest first, and equally long ones in extended alphabetical ordering), and number them $E_1, E_2, \ldots$
   2. Define a series G_n of sets ($G_0, G_1, \ldots$) inductively:
      i. $G_0 = G$
      ii. If $G_k \cup \{E_{k+1}\}$ proves A , then $G_{k+1} = G_k$

iii. If $G_k \cup \{E_{k+1}\}$ does **not** prove A , then $G_{k+1} = G_k \cup \{E_{k+1}\}$

3. Define G^* as the union of all the G_n . (That is, G^* is the set of all the sentences that are in any G_n .)
4. It can be easily shown that

i. G^* contains (is a superset of) G (by (b.i));

ii. G^* does not prove A (because if it proves A then some sentence was added to some G_n which caused it to prove ' A ; but this was ruled out by definition); and

iii. G^* is a "Maximal Set" (with respect to A): If *any* more sentences whatever were added to G^* , it *would* prove A . (Because if it were possible to add any more sentences, they should have been added when they were encountered during the construction of the G_n , again by definition)

III. If G^* is a Maximal Set (wrt A), then it is "truth-like". This means that it contains the sentence " C " only if it does *not* contain the sentence not- C ; If it contains " C " and contains "If C then B " then it also contains " B "; and so forth.

IV. If G^* is truth-like there is a " G^* -Canonical" valuation of the language: one that makes every sentence in G^* true and everything outside G^* false while still obeying the laws of semantic composition in the language.

V. A G^* -canonical valuation will make our original set G all true, and make A false.

VI. If there is a valuation on which G are true and A is false, then G does not (semantically) imply A .

QED

Another outline for a completeness proof

If a formula is a tautology, then there is a truth table for it which shows that each valuation yields the value true for the formula. Consider such a valuation. By mathematical induction on the length of the subformulae, show that the truth or falsity of the subformula follows from the truth or falsity (as appropriate for the valuation) of each propositional variable in the subformula. Then combine the lines of the truth table together two at a time by using "(P is true implies S) implies ((P is false implies S) implies S)". Keep repeating this until all dependencies on propositional variables have been eliminated. The result is that we have proved the given tautology. Since every tautology is provable, the logic is complete.

Interpretation of a truth-functional propositional calculus

An **interpretation of a truth-functional propositional calculus** $\mathcal{P}$ is an assignment to each propositional symbol of $\mathcal{P}$ of one or the other (but not both) of the truth values truth (**T**) and falsity (**F**), and an assignment to the connective symbols of $\mathcal{P}$ of their usual truth-functional meanings. An interpretation of a truth-functional propositional calculus may also be expressed in terms of truth tables.[9]

For n distinct propositional symbols there are 2^n distinct possible interpretations. For any particular symbol a , for example, there are $2^1 = 2$ possible interpretations:

1. a is assigned **T**, or
2. a is assigned **F**.

For the pair a , b there are $2^2 = 4$ possible interpretations:

1. both are assigned **T**,
2. both are assigned **F**,
3. a is assigned **T** and b is assigned **F**, or
4. a is assigned **F** and b is assigned **T**.[9]

Since $\mathcal{P}$ has $\aleph_0$, that is, denumerably many propositional symbols, there are $2^{\aleph_0} = \mathfrak{c}$, and therefore uncountably many distinct possible interpretations of $\mathcal{P}$.[9]

Interpretation of a sentence of truth-functional propositional logic

If ϕ and ψ are formulas of $\mathcal{P}$ and $\mathcal{I}$ is an interpretation of $\mathcal{P}$ then:

- A sentence of propositional logic is *true under an interpretation* $\mathcal{I}$ iff $\mathcal{I}$ assigns the truth value **T** to that sentence. If a sentence is true under an interpretation, then that interpretation is called a *model* of that sentence.
- ϕ is *false under an interpretation* $\mathcal{I}$ iff ϕ is not true under $\mathcal{I}$.[9]
- A sentence of propositional logic is *logically valid* iff it is true under every interpretation

  $\models \phi$ means that ϕ is logically valid

- A sentence ψ of propositional logic is a *semantic consequence* of a sentence ϕ iff there is no interpretation under which ϕ is true and ψ is false.
- A sentence of propositional logic is *consistent* iff it is true under at least one interpretation. It is inconsistent if it is not consistent.

Some consequences of these definitions:

- For any given interpretation a given formula is either true or false.[9]
- No formula is both true and false under the same interpretation.[9]
- ϕ is false for a given interpretation iff $\neg\phi$ is true for that interpretation; and ϕ is true under an interpretation iff $\neg\phi$ is false under that interpretation.[9]
- If ϕ and $(\phi \to \psi)$ are both true under a given interpretation, then ψ is true under that interpretation.[9]
- If $\models_{\mathrm{P}} \phi$ and $\models_{\mathrm{P}} (\phi \to \psi)$, then $\models_{\mathrm{P}} \psi$.[9]
- $\neg\phi$ is true under $\mathcal{I}$ iff ϕ is not true under $\mathcal{I}$.
- $(\phi \to \psi)$ is true under $\mathcal{I}$ iff either ϕ is not true under $\mathcal{I}$ or ψ is true under $\mathcal{I}$.[9]
- A sentence ψ of propositional logic is a semantic consequence of a sentence ϕ iff $(\phi \to \psi)$ is logically valid, that is, $\phi \models_{\mathrm{P}} \psi$ iff $\models_{\mathrm{P}} (\phi \to \psi)$.[9]

Alternative calculus

It is possible to define another version of propositional calculus, which defines most of the syntax of the logical operators by means of axioms, and which uses only one inference rule.

Axioms

Let ϕ, χ and ψ stand for well-formed formulæ. (The wffs themselves would not contain any Greek letters, but only capital Roman letters, connective operators, and parentheses.) Then the axioms are as follows:

Axioms		
Name	**Axiom Schema**	**Description**
THEN-1	$\phi \to (\chi \to \phi)$	Add hypothesis χ, implication introduction
THEN-2	$(\phi \to (\chi \to \psi)) \to ((\phi \to \chi) \to (\phi \to \psi))$	Distribute hypothesis ϕ over implication
AND-1	$\phi \land \chi \to \phi$	Eliminate conjunction
AND-2	$\phi \land \chi \to \chi$	
AND-3	$\phi \to (\chi \to (\phi \land \chi))$	Introduce conjunction
OR-1	$\phi \to \phi \lor \chi$	Introduce disjunction
OR-2	$\chi \to \phi \lor \chi$	
OR-3	$(\phi \to \psi) \to ((\chi \to \psi) \to (\phi \lor \chi \to \psi))$	Eliminate disjunction
NOT-1	$(\phi \to \chi) \to ((\phi \to \neg\chi) \to \neg\phi)$	Introduce negation
NOT-2	$\phi \to (\neg\phi \to \chi)$	Eliminate negation

NOT-3	$\phi \vee \neg\phi$	Excluded middle, classical logic
IFF-1	$(\phi \leftrightarrow \chi) \to (\phi \to \chi)$	Eliminate equivalence
IFF-2	$(\phi \leftrightarrow \chi) \to (\chi \to \phi)$	
IFF-3	$(\phi \to \chi) \to ((\chi \to \phi) \to (\phi \leftrightarrow \chi))$	Introduce equivalence

- Axiom THEN-2 may be considered to be a "distributive property of implication with respect to implication."
- Axioms AND-1 and AND-2 correspond to "conjunction elimination". The relation between AND-1 and AND-2 reflects the commutativity of the conjunction operator.
- Axiom AND-3 corresponds to "conjunction introduction."
- Axioms OR-1 and OR-2 correspond to "disjunction introduction." The relation between OR-1 and OR-2 reflects the commutativity of the disjunction operator.
- Axiom NOT-1 corresponds to "reductio ad absurdum."
- Axiom NOT-2 says that "anything can be deduced from a contradiction."
- Axiom NOT-3 is called "tertium non datur" (Latin: "a third is not given") and reflects the semantic valuation of propositional formulae: a formula can have a truth-value of either true or false. There is no third truth-value, at least not in classical logic. Intuitionistic logicians do not accept the axiom NOT-3.

Inference rule

The inference rule is modus ponens:

$$\phi,\ \phi \to \chi \vdash \chi .$$

Meta-inference rule

Let a demonstration be represented by a sequence, with hypotheses to the left of the turnstile and the conclusion to the right of the turnstile. Then the deduction theorem can be stated as follows:

If the sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n,\ \chi \vdash \psi$$

has been demonstrated, then it is also possible to demonstrate the sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n \vdash \chi \to \psi .$$

This deduction theorem (DT) is not itself formulated with propositional calculus: it is not a theorem of propositional calculus, but a theorem about propositional calculus. In this sense, it is a meta-theorem, comparable to theorems about the soundness or completeness of propositional calculus.

On the other hand, DT is so useful for simplifying the syntactical proof process that it can be considered and used as another inference rule, accompanying modus ponens. In this sense, DT corresponds to the natural conditional proof inference rule which is part of the first version of propositional calculus introduced in this article.

The converse of DT is also valid:

If the sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n \vdash \chi \to \psi$$

has been demonstrated, then it is also possible to demonstrate the sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n,\ \chi \vdash \psi$$

in fact, the validity of the converse of DT is almost trivial compared to that of DT:

If

$$\phi_1,\ \ldots,\ \phi_n \vdash \chi \to \psi$$

then

1: $\phi_1, \ \ldots, \ \phi_n, \ \chi \vdash \chi \to \psi$

2: $\phi_1, \ \ldots, \ \phi_n, \ \chi \vdash \chi$

and from (1) and (2) can be deduced

3: $\phi_1, \ \ldots, \ \phi_n, \ \chi \vdash \psi$

by means of modus ponens, Q.E.D.

The converse of DT has powerful implications: it can be used to convert an axiom into an inference rule. For example, the axiom AND-1,

$\vdash \phi \land \chi \to \phi$

can be transformed by means of the converse of the deduction theorem into the inference rule

$\phi \land \chi \vdash \phi$

which is conjunction elimination, one of the ten inference rules used in the first version (in this article) of the propositional calculus.

Example of a proof

The following is an example of a (syntactical) demonstration, involving only axioms THEN-1 and THEN-2:

Prove: $A \to A$ (Reflexivity of implication).

Proof:

1. $(A \to ((B \to A) \to A)) \to ((A \to (B \to A)) \to (A \to A))$

   Axiom THEN-2 with $\phi = A$, $\chi = B \to A$, $\psi = A$
2. $A \to ((B \to A) \to A)$

   Axiom THEN-1 with $\phi = A$, $\chi = B \to A$
3. $(A \to (B \to A)) \to (A \to A)$

   From (1) and (2) by modus ponens.
4. $A \to (B \to A)$

   Axiom THEN-1 with $\phi = A$, $\chi = B$
5. $A \to A$

   From (3) and (4) by modus ponens.

Equivalence to equational logics

The preceding alternative calculus is an example of a Hilbert-style deduction system. In the case of propositional systems the axioms are terms built with logical connectives and the only inference rule is modus ponens. Equational logic as standardly used informally in high school algebra is a different kind of calculus from Hilbert systems. Its theorems are equations and its inference rules express the properties of equality, namely that it is a congruence on terms that admits substitution.

Classical propositional calculus as described above is equivalent to Boolean algebra, while intuitionistic propositional calculus is equivalent to Heyting algebra. The equivalence is shown by translation in each direction of the theorems of the respective systems. Theorems ϕ of classical or intuitionistic propositional calculus are translated as equations $\phi = 1$ of Boolean or Heyting algebra respectively. Conversely theorems $x = y$ of Boolean or Heyting algebra are translated as theorems $(x \to y) \land (y \to x)$ of classical or propositional calculus respectively, for which $x \equiv y$ is a standard abbreviation. In the case of Boolean algebra $x = y$ can also be translated as $(x \land y) \lor (\neg x \land \neg y)$, but this translation is incorrect intuitionistically.

In both Boolean and Heyting algebra, inequality $x \leq y$ can be used in place of equality. The equality $x = y$ is expressible as a pair of inequalities $x \leq y$ and $y \leq x$. Conversely the inequality $x \leq y$ is expressible as the equality $x \wedge y = x$, or as $x \vee y = y$. The significance of inequality for Hilbert-style systems is that it corresponds to the latter's or entailment symbol $\vdash$. An entailment

$$\phi_1, \ \phi_2, \ \ldots, \ \phi_n \vdash \psi$$

is translated in the inequality version of the algebraic framework as

$$\phi_1 \ \wedge \ \phi_2 \ \wedge \ \ldots \ \wedge \ \phi_n \quad \leq \quad \psi$$

Conversely the algebraic inequality $x \leq y$ is translated as the entailment

$$x \ \vdash \ y.$$

The difference between implication $x \to y$ and inequality or entailment $x \leq y$ or $x \ \vdash \ y$ is that the former is internal to the logic while the latter is external. Internal implication between two terms is another term of the same kind. Entailment as external implication between two terms expresses a metatruth outside the language of the logic, and is considered part of the metalanguage. Even when the logic under study is intuitionistic, entailment is ordinarily understood classically as two-valued: either the left side entails, or is less-or-equal to, the right side, or it is not.

Similar but more complex translations to and from algebraic logics are possible for natural deduction systems as described above and for the sequent calculus. The entailments of the latter can be interpreted as two-valued, but a more insightful interpretation is as a set, the elements of which can be understood as abstract proofs organized as the morphisms of a category. In this interpretation the cut rule of the sequent calculus corresponds to composition in the category. Boolean and Heyting algebras enter this picture as special categories having at most one morphism per homset, i.e., one proof per entailment, corresponding to the idea that existence of proofs is all that matters: any proof will do and there is no point in distinguishing them.

Graphical calculi

It is possible to generalize the definition of a formal language from a set of finite sequences over a finite basis to include many other sets of mathematical structures, so long as they are built up by finitary means from finite materials. What's more, many of these families of formal structures are especially well-suited for use in logic.

For example, there are many families of graphs that are close enough analogues of formal languages that the concept of a calculus is quite easily and naturally extended to them. Indeed, many species of graphs arise as *parse graphs* in the syntactic analysis of the corresponding families of text structures. The exigencies of practical computation on formal languages frequently demand that text strings be converted into pointer structure renditions of parse graphs, simply as a matter of checking whether strings are wffs or not. Once this is done, there are many advantages to be gained from developing the graphical analogue of the calculus on strings. The mapping from strings to parse graphs is called *parsing* and the inverse mapping from parse graphs to strings is achieved by an operation that is called *traversing* the graph.

Other logical calculi

Propositional calculus is about the simplest kind of logical calculus in current use. It can be extended in several ways. (Aristotelian "syllogistic" calculus, which is largely supplanted in modern logic, is in *some* ways simpler – but in other ways more complex – than propositional calculus.) The most immediate way to develop a more complex logical calculus is to introduce rules that are sensitive to more fine-grained details of the sentences being used.

First-order logic (aka first-order predicate logic) results when the "atomic sentences" of propositional logic are broken up into terms, variables, predicates, and quantifiers, all keeping the rules of propositional logic with some new ones introduced. (For example, from "All dogs are mammals" we may infer "If Rover is a dog then Rover is a mammal".) With the tools of first-order logic it is possible to formulate a number of theories, either with explicit axioms or by rules of inference, that can themselves be treated as logical calculi. Arithmetic is the best known of these; others include set theory and mereology. Second-order logic and other higher-order logics are formal extensions of first-order logic. Thus, it makes sense to refer to propositional logic as *"zeroth-order logic"*, when comparing it with these logics.

Modal logic also offers a variety of inferences that cannot be captured in propositional calculus. For example, from "Necessarily p" we may infer that p. From p we may infer "It is possible that p". The translation between modal logics and algebraic logics is as for classical and intuitionistic logics but with the introduction of a unary operator on Boolean or Heyting algebras, different from the Boolean operations, interpreting the possibility modality, and in the case of Heyting algebra a second operator interpreting necessity (for Boolean algebra this is redundant since necessity is the De Morgan dual of possibility). The first operator preserves 0 and disjunction while the second preserves 1 and conjunction.

Many-valued logics are those allowing sentences to have values other than *true* and *false*. (For example, *neither* and *both* are standard "extra values"; "continuum logic" allows each sentence to have any of an infinite number of "degrees of truth" between *true* and *false*.) These logics often require calculational devices quite distinct from propositional calculus. When the values form a Boolean algebra (which may have more than two or even infinitely many values), many-valued logic reduces to classical logic; many-valued logics are therefore only of independent interest when the values form an algebra that is not Boolean.

Solvers

Finding solutions to propositional logic formulas is an NP-complete problem. However, practical methods exist (e.g., DPLL algorithm, 1962; Chaff algorithm, 2001) that are very fast for many useful cases. Recent work has extended the SAT solver algorithms to work with propositions containing arithmetic expressions; these are the SMT solvers.

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- Introduction to Mathematical Logic (http://www.ltn.lv/~podnieks/mlog/ml2.htm) by V. Detlovs and K. Podnieks
- Formal Predicate Calculus (http://www.qedeq.org/current/doc/math/qedeq_formal_logic_v1_en.pdf), contains a systematic formal development along the lines of Alternative calculus
- *forall x: an introduction to formal logic* (http://www.fecundity.com/logic/), by P.D. Magnus, covers formal semantics and proof theory for sentential logic.
- Propositional Logic on PlanetMath (GFDLed)
- Category:Propositional Calculus (http://www.proofwiki.org/wiki/Category:Propositional_Calculus) on ProofWiki (GFDLed)

Predicate logic

In mathematical logic, **predicate logic** is the generic term for symbolic formal systems like first-order logic, second-order logic, many-sorted logic, or infinitary logic. This formal system is distinguished from other systems in that its formulae contain variables which can be quantified. Two common quantifiers are the existential $\exists$ ("there exists") and universal $\forall$ ("for all") quantifiers. The variables could be elements in the universe under discussion, or perhaps relations or functions over that universe. For instance, an existential quantifier over a function symbol would be interpreted as modifier "there is a function". The foundations of predicate logic were developed independently by Gottlob Frege and Charles Peirce.[1]

In informal usage, the term "predicate logic" occasionally refers to first-order logic. Some authors consider the **predicate calculus** to be an axiomatized form of predicate logic, and the predicate logic to be derived from an informal, more intuitive development.[2]

Predicate logics also include logics mixing modal operators and quantifiers. See Modal logic, Saul Kripke, Barcan Marcus formulae, A. N. Prior, and Nicholas Rescher.

Syntax

Predicate calculus symbols may represent either variables, constants, functions or predicates.

1. **Constants** name specific objects or properties in the domain of discourse. Thus George, tree, tall and blue are examples of well formed constant symbols. The constants $\top$ (true) and $\perp$ (false) are sometimes included.
2. **Variable symbols** are used to designate general classes or objects or properties in the domain of discourse.
3. **Functions** denote a mapping of one or more elements in a set (the *domain* of the function) into a unique element of another set (the *range* of the function). Elements of the domain and range are objects in the world of discourse. Every function symbol has an associated *arity*, indicating the number of elements in the domain mapped onto each element of range.

A *function expression* is a function symbol followed by its arguments. The arguments are elements from the domain of the function; the number of arguments is equal to the arity of the function. The arguments are enclosed in parentheses and separated by commas. e.g.:

- f(X,Y)
- father(david)
- price(apple)

are all well-formed function expressions.

Predicate logics may be viewed syntactically as Chomsky grammars. As such, predicate logics (as well as modal logics and mixed modal predicate logics) may be viewed as context-sensitive, or more typically as context-free, grammars. As each one of the four Chomsky-type grammars have equivalent automata, these logics can be viewed as automata just as well.

Footnotes

[1] Eric M. Hammer: Semantics for Existential Graphs, *Journal of Philosophical Logic*, Volume 27, Issue 5 (Oktober 1998), page 489: "Development of first-order logic independently of Frege, anticipating prenex and Skolem normal forms"

[2] Among these authors is Stolyar, p. 166. Hamilton considers both to be calculi but divides them into an informal calculus and a formal calculus.

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Model theory

In mathematics, **model theory** is the study of (classes of) mathematical structures (e.g. groups, fields, graphs, universes of set theory) using tools from mathematical logic. It has close ties to abstract algebra, particularly universal algebra.

Objects of study in model theory are models for formal languages which are structures that give meaning to the sentences of these formal languages. If a model for a language moreover satisfies a particular sentence or theory (set of sentences satisfying special conditions), it is called a model *of* the sentence or theory.

This article focuses on finitary first order model theory of infinite structures. Finite model theory, which concentrates on finite structures, diverges significantly from the study of infinite structures in both the problems studied and the techniques used. Model theory in higher-order logics or infinitary logics is hampered by the fact that completeness does not in general hold for these logics. However, a great deal of study has also been done in such languages.

Model theory recognises and is intimately concerned with a duality: It examines semantical elements by means of syntactical elements of a corresponding language. To quote the first page of Chang and Keisler (1990):[1]

> universal algebra + logic = **model theory**.

Model theory developed rapidly during the 1990s, and a more modern definition is provided by Wilfrid Hodges (1997):

> **model theory** = algebraic geometry − fields.

In a similar way to proof theory, model theory is situated in an area of interdisciplinarity between mathematics, philosophy, and computer science. The most important professional organization in the field of model theory is the Association for Symbolic Logic.

An incomplete and somewhat arbitrary subdivision of model theory is into classical model theory, model theory applied to groups and fields, and geometric model theory. A missing subdivision is computable model theory, but this can arguably be viewed as an independent subfield of logic. Examples of early theorems from classical model theory include Gödel's completeness theorem, the upward and downward Löwenheim–Skolem theorems, Vaught's two-cardinal theorem, Scott's isomorphism theorem, the omitting types theorem, and the Ryll-Nardzewski theorem. Examples of early results from model theory applied to fields are Tarski's elimination of quantifiers for real closed fields, Ax's theorem on pseudo-finite fields, and Robinson's development of non-standard analysis. An important step in the evolution of classical model theory occurred with the birth of stability theory (through Morley's theorem on uncountably categorical theories and Shelah's classification program), which developed a calculus of independence and rank based on syntactical conditions satisfied by theories. During the last several decades applied model theory has repeatedly merged with the more pure stability theory. The result of this synthesis is called geometric model theory in this article (which is taken to include o-minimality, for example, as well as classical geometric stability theory). An example of a theorem from geometric model theory is Hrushovski's proof of the Mordell–Lang conjecture for function fields. The ambition of geometric model theory is to provide a *geography of mathematics* by embarking on a detailed study of definable sets in various mathematical structures, aided by the substantial tools developed in the study of pure model theory.

Example

To illustrate the basic relationship involving syntax and semantics in the context of a non-trivial model, one can start, on the syntactic side, with suitable axioms for the natural numbers such as Peano axioms, and the associated theory. Going on to the semantic side, one has the usual counting numbers as a model. In the 1930s, Skolem developed alternative models satisfying the axioms. This illustrates what is meant by interpreting a language or theory in a particular model. A more traditional example is interpreting the axioms of a particular algebraic system such as a group, in the context of a model provided by a specific group.

Universal algebra

Fundamental concepts in universal algebra are signatures σ and σ-algebras. Since these concepts are formally defined in the article on structures, the present article can content itself with an informal introduction which consists in examples of how these terms are used.

> The standard signature of rings is $\sigma_{ring} = \{\times,+,-,0,1\}$, where × and + are binary, − is unary, and 0 and 1 are nullary.
>
> The standard signature of semirings is $\sigma_{smr} = \{\times,+,0,1\}$, where the arities are as above.
>
> The standard signature of groups (with multiplicative notation) is $\sigma_{grp} = \{\times,^{-1},1\}$, where × is binary, $^{-1}$ is unary and 1 is nullary.
>
> The standard signature of monoids is $\sigma_{mnd} = \{\times,1\}$.
>
> A ring is a σ_{ring}-structure which satisfies the identities $u + (v + w) = (u + v) + w$, $u + v = v + u$, $u + 0 = u$, $u + (-u) = 0$, $u \times (v \times w) = (u \times v) \times w$, $u \times 1 = u$, $1 \times u = u$, $u \times (v + w) = (u \times v) + (u \times w)$ and $(v + w) \times u = (v \times u) + (w \times u)$.
>
> A group is a σ_{grp}-structure which satisfies the identities $u \times (v \times w) = (u \times v) \times w$, $u \times 1 = u$, $1 \times u = u$, $u \times u^{-1} = 1$ and $u^{-1} \times u = 1$.
>
> A monoid is a σ_{mnd}-structure which satisfies the identities $u \times (v \times w) = (u \times v) \times w$, $u \times 1 = u$ and $1 \times u = u$.
>
> A semigroup is a $\{\times\}$-structure which satisfies the identity $u \times (v \times w) = (u \times v) \times w$.
>
> A magma is just a $\{\times\}$-structure.

This is a very efficient way to define most classes of algebraic structures, because there is also the concept of σ-homomorphism, which correctly specializes to the usual notions of homomorphism for groups, semigroups, magmas and rings. For this to work, the signature must be chosen well.

Terms such as the σ_{ring}-term $t(u,v,w)$ given by $(u + (v \times w)) + (-1)$ are used to define identities $t = t'$, but also to construct free algebras. An equational class is a class of structures which, like the examples above and many others, is defined as the class of all σ-structures which satisfy a certain set of identities. Birkhoff's theorem states:

> A class of σ-structures is an equational class if and only if it is not empty and closed under subalgebras, homomorphic images, and direct products.

An important non-trivial tool in universal algebra are ultraproducts $\Pi_{i\in I}A_i/U$, where I is an infinite set indexing a system of σ-structures A_i, and U is an ultrafilter on I.

While model theory is generally considered a part of mathematical logic, universal algebra, which grew out of Alfred North Whitehead's (1898) work on abstract algebra, is part of algebra. This is reflected by their respective MSC classifications. Nevertheless model theory can be seen as an extension of universal algebra.

Finite model theory

Finite model theory is the area of model theory which has the closest ties to universal algebra. Like some parts of universal algebra, and in contrast with the other areas of model theory, it is mainly concerned with finite algebras, or more generally, with finite σ-structures for signatures σ which may contain relation symbols as in the following example:

> The standard signature for graphs is $\sigma_{grph}=\{E\}$, where E is a binary relation symbol.
>
> A graph is a σ_{grph}-structure satisfying the sentences $\forall u\forall v(uEv \rightarrow vEu)$ and $\forall u\neg(uEu)$.

A σ-homomorphism is a map that commutes with the operations and preserves the relations in σ. This definition gives rise to the usual notion of graph homomorphism, which has the interesting property that a bijective homomorphism need not be invertible. Structures are also a part of universal algebra; after all, some algebraic structures such as ordered groups have a binary relation <. What distinguishes finite model theory from universal algebra is its use of more general logical sentences (as in the example above) in place of identities. (In a model-theoretic context an identity *t=t'* is written as a sentence $\forall u_1 u_2 \ldots u_n (t = t')$.)

The logics employed in finite model theory are often substantially more expressive than first-order logic, the standard logic for model theory of infinite structures.

First-order logic

Whereas universal algebra provides the semantics for a signature, logic provides the syntax. With terms, identities and quasi-identities, even universal algebra has some limited syntactic tools; first-order logic is the result of making quantification explicit and adding negation into the picture.

A first-order **formula** is built out of atomic formulas such as $R(f(x,y),z)$ or $y = x + 1$ by means of the Boolean connectives $\neg, \wedge, \vee, \rightarrow$ and prefixing of quantifiers $\forall v$ or $\exists v$. A sentence is a formula in which each occurrence of a variable is in the scope of a corresponding quantifier. Examples for formulas are φ (or φ(x) to mark the fact that at most x is an unbound variable in φ) and ψ defined as follows:

$$\begin{aligned} \varphi &= \forall u \forall v (\exists w (x \times w = u \times v) \rightarrow (\exists w (x \times w = u) \vee \exists w (x \times w = v))) \wedge x \neq 0 \wedge x \neq 1, \\ \psi &= \forall u \forall v ((u \times v = x) \rightarrow (u = x) \vee (v = x)) \wedge x \neq 0 \wedge x \neq 1. \end{aligned}$$

(Note that the equality symbol has a double meaning here.) It is intuitively clear how to translate such formulas into mathematical meaning. In the σ_{smr}-structure $\mathcal{N}$ of the natural numbers, for example, an element *n* **satisfies** the formula φ if and only if *n* is a prime number. The formula ψ similarly defines irreducibility. Tarski gave a rigorous definition, sometimes called "Tarski's definition of truth", for the satisfaction relation $\models$, so that one easily proves:

$\mathcal{N} \models \phi(n) \iff n$ is a prime number.

$\mathcal{N} \models \psi(n) \iff n$ is irreducible.

A set *T* of sentences is called a (first-order) theory. A theory is **satisfiable** if it has a **model** $\mathcal{M} \models T$, i.e. a structure (of the appropriate signature) which satisfies all the sentences in the set *T*. Consistency of a theory is usually defined in a syntactical way, but in first-order logic by the completeness theorem there is no need to distinguish between satisfiability and consistency. Therefore model theorists often use "consistent" as a synonym for "satisfiable".

A theory is called **categorical** if it determines a structure up to isomorphism, but it turns out that this definition is not useful, due to serious restrictions in the expressivity of first-order logic. The Löwenheim–Skolem theorem implies that for every theory *T*[2] which has an infinite model and for every infinite cardinal number κ, there is a model $\mathcal{M} \models T$ such that the number of elements of $\mathcal{M}$ is exactly κ. Therefore only finite structures can be described by a categorical theory.

Lack of expressivity (when compared to higher logics such as second-order logic) has its advantages, though. For model theorists, the Löwenheim–Skolem theorem is an important practical tool rather than the source of Skolem's paradox. In a certain sense made precise by Lindström's theorem, first-order logic is the most expressive logic for which both the Löwenheim–Skolem theorem and the compactness theorem hold.

Due to Gödel, the compactness theorem says that every unsatisfiable first-order theory has a finite unsatisfiable subset. This theorem is of central importance in infinite model theory, where the words "by compactness" are commonplace. One way to prove it is by means of ultraproducts. An alternative proof uses the completeness theorem, which is otherwise reduced to a marginal role in most of modern model theory.

Axiomatizability, elimination of quantifiers, and model-completeness

The first step, often trivial, for applying the methods of model theory to a class of mathematical objects such as groups, or trees in the sense of graph theory, is to choose a signature σ and represent the objects as σ-structures. The next step is to show that the class is an elementary class, i.e. axiomatizable in first-order logic (i.e. there is a theory *T* such that a σ-structure is in the class if and only if it satisfies *T*). E.g. this step fails for the trees, since connectedness cannot be expressed in first-order logic. Axiomatizability ensures that model theory can speak about the right objects. Quantifier elimination can be seen as a condition which ensures that model theory does not say too much about the objects.

A theory *T* has quantifier elimination if every first-order formula $\varphi(x_1,...,x_n)$ over its signature is equivalent modulo *T* to a first-order formula $\psi(x_1,...,x_n)$ without quantifiers, i.e. $\forall x_1 \ldots \forall x_n(\phi(x_1,\ldots,x_n) \leftrightarrow \psi(x_1,\ldots,x_n))$ holds in all models of *T*. For example the theory of algebraically closed fields in the signature $\sigma_{ring}=(\times,+,-,0,1)$ has quantifier elimination because every formula is equivalent to a Boolean combination of equations between polynomials.

A substructure of a σ-structure is a subset of its domain, closed under all functions in its signature σ, which is regarded as a σ-structure by restricting all functions and relations in σ to the subset. An embedding of a σ-structure $\mathcal{A}$ into another σ-structure $\mathcal{B}$ is a map f: A → B between the domains which can be written as an isomorphism of $\mathcal{A}$ with a substructure of $\mathcal{B}$. Every embedding is an injective homomorphism, but the converse holds only if the signature contains no relation symbols.

If a theory does not have quantifier elimination, one can add additional symbols to its signature so that it does. Early model theory spent much effort on proving axiomatizability and quantifier elimination results for specific theories, especially in algebra. But often instead of quantifier elimination a weaker property suffices:

A theory *T* is called model-complete if every substructure of a model of *T* which is itself a model of *T* is an elementary substructure. There is a useful criterion for testing whether a substructure is an elementary substructure, called the Tarski–Vaught test. It follows from this criterion that a theory *T* is model-complete if and only if every first-order formula $\varphi(x_1,...,x_n)$ over its signature is equivalent modulo *T* to an existential first-order formula, i.e. a formula of the following form:

$$\exists v_1 \ldots \exists v_m \psi(x_1,\ldots,x_n,v_1,\ldots,v_m),$$

where ψ is quantifier free. A theory that is not model-complete may or may not have a **model completion**, which is a related model-complete theory that is not, in general, an extension of the original theory. A more general notion is that of **model companions**.

Categoricity

As observed in the section on first-order logic, first-order theories cannot be categorical, i.e. they cannot describe a unique model up to isomorphism, unless that model is finite. But two famous model-theoretic theorems deal with the weaker notion of κ-categoricity for a cardinal κ. A theory *T* is called **κ-categorical** if any two models of *T* that are of cardinality κ are isomorphic. It turns out that the question of κ-categoricity depends critically on whether κ is bigger than the cardinality of the language (i.e. $\aleph_0$ + |σ|, where |σ| is the cardinality of the signature). For finite or countable signatures this means that there is a fundamental difference between $\aleph_0$-cardinality and κ-cardinality for uncountable κ.

A few characterizations of $\aleph_0$-categoricity include:

For a complete first-order theory *T* in a finite or countable signature the following conditions are equivalent:

1. *T* is $\aleph_0$-categorical.
2. For every natural number *n*, the Stone space $S_n(T)$ is finite.

3. For every natural number n, the number of formulas $\varphi(x_1, ..., x_n)$ in n free variables, up to equivalence modulo T, is finite.

This result, due independently to Engeler, Ryll-Nardzewski and Svenonius, is sometimes referred to as the Ryll-Nardzewski theorem.

Further, $\aleph_0$-categorical theories and their countable models have strong ties with oligomorphic groups. They are often constructed as Fraïssé limits.

Michael Morley's highly non-trivial result that (for countable languages) there is only *one* notion of uncountable categoricity was the starting point for modern model theory, and in particular classification theory and stability theory:

> Morley's categoricity theorem
>
> If a first-order theory T in a finite or countable signature is κ-categorical for some uncountable cardinal κ, then T is κ-categorical for all uncountable cardinals κ.

Uncountably categorical (i.e. κ-categorical for all uncountable cardinals κ) theories are from many points of view the most well-behaved theories. A theory that is both $\aleph_0$-categorical and uncountably categorical is called **totally categorical**.

Model theory and set theory

Set theory (which is expressed in a countable language), if it is consistent, has a countable model; this is known as Skolem's paradox, since there are sentences in set theory which postulate the existence of uncountable sets and yet these sentences are true in our countable model. Particularly the proof of the independence of the continuum hypothesis requires considering sets in models which appear to be uncountable when viewed from *within* the model, but are countable to someone *outside* the model.

The model-theoretic viewpoint has been useful in set theory; for example in Kurt Gödel's work on the constructible universe, which, along with the method of forcing developed by Paul Cohen can be shown to prove the (again philosophically interesting) independence of the axiom of choice and the continuum hypothesis from the other axioms of set theory.

In the other direction, model theory itself can be formalized within ZFC set theory. The development of the fundamentals of model theory (such as the compactness theorem) rely on the axiom of choice, or more exactly the Boolean prime ideal theorem. Other results in model theory depend on set-theoretic axioms beyond the standard ZFC framework. For example, if the Continuum Hypothesis holds then every countable model has an ultrapower which is saturated (in its own cardinality). Similarly, if the Generalized Continuum Hypothesis holds then every model has a saturated elementary extension. Neither of these results are provable in ZFC alone. Finally, some questions arising from model theory (such as compactness for infinitary logics) have been shown to be equivalent to large cardinal axioms.

Other basic notions of model theory

Reducts and expansions

A field or a vector space can be regarded as a (commutative) group by simply ignoring some of its structure. The corresponding notion in model theory is that of a **reduct** of a structure to a subset of the original signature. The opposite relation is called an *expansion* - e.g. the (additive) group of the rational numbers, regarded as a structure in the signature $\{+,0\}$ can be expanded to a field with the signature $\{\times,+,1,0\}$ or to an ordered group with the signature $\{+,0,<\}$.

Similarly, if σ' is a signature that extends another signature σ, then a complete σ'-theory can be restricted to σ by intersecting the set of its sentences with the set of σ-formulas. Conversely, a complete σ-theory can be regarded as a σ'-theory, and one can extend it (in more than one way) to a complete σ'-theory. The terms reduct and expansion are sometimes applied to this relation as well.

Interpretability

Given a mathematical structure, there are very often associated structures which can be constructed as a quotient of part of the original structure via an equivalence relation. An important example is a quotient group of a group.

One might say that to understand the full structure one must understand these quotients. When the equivalence relation is definable, we can give the previous sentence a precise meaning. We say that these structures are **interpretable**.

A key fact is that one can translate sentences from the language of the interpreted structures to the language of the original structure. Thus one can show that if a structure *M* interprets another whose theory is undecidable, then *M* itself is undecidable.

Using the compactness and completeness theorems

Gödel's completeness theorem (not to be confused with his incompleteness theorems) says that a theory has a model if and only if it is consistent, i.e. no contradiction is proved by the theory. This is the heart of model theory as it lets us answer questions about theories by looking at models and vice-versa. One should not confuse the completeness theorem with the notion of a complete theory. A complete theory is a theory that contains every sentence or its negation. Importantly, one can find a complete consistent theory extending any consistent theory. However, as shown by Gödel's incompleteness theorems only in relatively simple cases will it be possible to have a complete consistent theory that is also recursive, i.e. that can be described by a recursively enumerable set of axioms. In particular, the theory of natural numbers has no recursive complete and consistent theory. Non-recursive theories are of little practical use, since it is undecidable if a proposed axiom is indeed an axiom, making proof-checking a supertask.

The compactness theorem states that a set of sentences S is satisfiable if every finite subset of S is satisfiable. In the context of proof theory the analogous statement is trivial, since every proof can have only a finite number of antecedents used in the proof. In the context of model theory, however, this proof is somewhat more difficult. There are two well known proofs, one by Gödel (which goes via proofs) and one by Malcev (which is more direct and allows us to restrict the cardinality of the resulting model).

Model theory is usually concerned with first-order logic, and many important results (such as the completeness and compactness theorems) fail in second-order logic or other alternatives. In first-order logic all infinite cardinals look the same to a language which is countable. This is expressed in the Löwenheim–Skolem theorems, which state that any countable theory with an infinite model $\mathfrak{A}$ has models of all infinite cardinalities (at least that of the language) which agree with $\mathfrak{A}$ on all sentences, i.e. they are 'elementarily equivalent'.

Types

Fix an L -structure M, and a natural number n. The set of definable subsets of M^n over some parameters A is a Boolean algebra. By Stone's representation theorem for Boolean algebras there is a natural dual notion to this. One can consider this to be the topological space consisting of maximal consistent sets of formulae over A. We call this the space of (complete) n -types over A, and write $S_n(A)$.

Now consider an element $m \in M^n$. Then the set of all formulae ϕ with parameters in A in free variables $x_1, \ldots, x_n$ so that $M \models \phi(m)$ is consistent and maximal such. It is called the *type* of m over A.

One can show that for any n -type p, there exists some elementary extension N of M and some $a \in N^n$ so that p is the type of a over A.

Many important properties in model theory can be expressed with types. Further many proofs go via constructing models with elements that contain elements with certain types and then using these elements.

Illustrative Example: Suppose M is an algebraically closed field. The theory has quantifier elimination . This allows us to show that a type is determined exactly by the polynomial equations it contains. Thus the space of n -types over a subfield A is bijective with the set of prime ideals of the polynomial ring $A[x_1, \ldots, x_n]$. This is the same set as the spectrum of $A[x_1, \ldots, x_n]$. Note however that the topology considered on the type space is the constructible topology: a set of types is basic open iff it is of the form $\{p : f(x) = 0 \in p\}$ or of the form $\{p : f(x) \neq 0 \in p\}$. This is finer than the Zariski topology.

History

Model theory as a subject has existed since approximately the middle of the 20th century. However some earlier research, especially in mathematical logic, is often regarded as being of a model-theoretical nature in retrospect. The first significant result in what is now model theory was a special case of the downward Löwenheim–Skolem theorem, published by Leopold Löwenheim in 1915. The compactness theorem was implicit in work by Thoralf Skolem,[3] but it was first published in 1930, as a lemma in Kurt Gödel's proof of his completeness theorem. The Löwenheim–Skolem theorem and the compactness theorem received their respective general forms in 1936 and 1941 from Anatoly Maltsev.

Notes

[1] Chang and Keisler, p. 1 (http://books.google.com/books?id=uiHq0EmaFp0C&pg=PA1).

[2] In a countable signature. The theorem has a straightforward generalization to uncountable signatures.

[3] *All three commentators [i.e. Vaught, van Heijenoort and Dreben] agree that both the completeness and compactness theorems were implicit in Skolem 1923 [...],* Dawson (1993).

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Proof theory

Proof theory is a branch of mathematical logic that represents proofs as formal mathematical objects, facilitating their analysis by mathematical techniques. Proofs are typically presented as inductively-defined data structures such as plain lists, boxed lists, or trees, which are constructed according to the axioms and rules of inference of the logical system. As such, proof theory is syntactic in nature, in contrast to model theory, which is semantic in nature. Together with model theory, axiomatic set theory, and recursion theory, proof theory is one of the so-called *four pillars* of the foundations of mathematics.[1]

Proof theory is important in philosophical logic, where the primary interest is in the idea of a proof-theoretic semantics, an idea which depends upon technical ideas in structural proof theory to be feasible.

History

Although the formalisation of logic was much advanced by the work of such figures as Gottlob Frege, Giuseppe Peano, Bertrand Russell, and Richard Dedekind, the story of modern proof theory is often seen as being established by David Hilbert, who initiated what is called Hilbert's program in the foundations of mathematics. Kurt Gödel's seminal work on proof theory first advanced, then refuted this program: his completeness theorem initially seemed to bode well for Hilbert's aim of reducing all mathematics to a finitist formal system; then his incompleteness theorems showed that this is unattainable. All of this work was carried out with the proof calculi called the Hilbert systems.

In parallel, the foundations of structural proof theory were being founded. Jan Łukasiewicz suggested in 1926 that one could improve on Hilbert systems as a basis for the axiomatic presentation of logic if one allowed the drawing of conclusions from assumptions in the inference rules of the logic. In response to this Stanisław Jaśkowski (1929) and Gerhard Gentzen (1934) independently provided such systems, called calculi of natural deduction, with Gentzen's approach introducing the idea of symmetry between the grounds for asserting propositions, expressed in introduction rules, and the consequences of accepting propositions in the elimination rules, an idea that has proved very important in proof theory.[2] Gentzen (1934) further introduced the idea of the sequent calculus, a calculus advanced in a similar spirit that better expressed the duality of the logical connectives,[3] and went on to make fundamental advances in the formalisation of intuitionistic logic, and provide the first combinatorial proof of the consistency of Peano arithmetic. Together, the presentation of natural deduction and the sequent calculus introduced the fundamental idea of analytic proof to proof theory,

Formal and informal proof

The *informal* proofs of everyday mathematical practice are unlike the *formal* proofs of proof theory. They are rather like high-level sketches that would allow an expert to reconstruct a formal proof at least in principle, given enough time and patience. For most mathematicians, writing a fully formal proof is too pedantic and long-winded to be in common use.

Formal proofs are constructed with the help of computers in interactive theorem proving. Significantly, these proofs can be checked automatically, also by computer. (Checking formal proofs is usually simple, whereas *finding* proofs (automated theorem proving) is generally hard.) An informal proof in the mathematics literature, by contrast, requires weeks of peer review to be checked, and may still contain errors.

Kinds of proof calculi

The three most well-known styles of proof calculi are:

- The Hilbert calculi
- The natural deduction calculi
- The sequent calculi

Each of these can give a complete and axiomatic formalization of propositional or predicate logic of either the classical or intuitionistic flavour, almost any modal logic, and many substructural logics, such as relevance logic or linear logic. Indeed it is unusual to find a logic that resists being represented in one of these calculi.

Consistency proofs

As previously mentioned, the spur for the mathematical investigation of proofs in formal theories was Hilbert's program. The central idea of this program was that if we could give finitary proofs of consistency for all the sophisticated formal theories needed by mathematicians, then we could ground these theories by means of a metamathematical argument, which shows that all of their purely universal assertions (more technically their provable Π^0_1 sentences) are finitarily true; once so grounded we do not care about the non-finitary meaning of their existential theorems, regarding these as pseudo-meaningful stipulations of the existence of ideal entities.

The failure of the program was induced by Kurt Gödel's incompleteness theorems, which showed that any ω-consistent theory that is sufficiently strong to express certain simple arithmetic truths, cannot prove its own consistency, which on Gödel's formulation is a Π^0_1 sentence.

Much investigation has been carried out on this topic since, which has in particular led to:

- Refinement of Gödel's result, particularly J. Barkley Rosser's refinement, weakening the above requirement of ω-consistency to simple consistency;
- Axiomatisation of the core of Gödel's result in terms of a modal language, provability logic;
- Transfinite iteration of theories, due to Alan Turing and Solomon Feferman;
- The recent discovery of self-verifying theories, systems strong enough to talk about themselves, but too weak to carry out the diagonal argument that is the key to Gödel's unprovability argument.

See also Mathematical logic

Structural proof theory

Structural proof theory is the subdiscipline of proof theory that studies proof calculi that support a notion of analytic proof. The notion of analytic proof was introduced by Gentzen for the sequent calculus; there the analytic proofs are those that are cut-free. His natural deduction calculus also supports a notion of analytic proof, as shown by Dag Prawitz. The definition is slightly more complex: we say the analytic proofs are the normal forms, which are related to the notion of normal form in term rewriting. More exotic proof calculi such as Jean-Yves Girard's proof nets also support a notion of analytic proof.

Structural proof theory is connected to type theory by means of the Curry-Howard correspondence, which observes a structural analogy between the process of normalisation in the natural deduction calculus and beta reduction in the typed lambda calculus. This provides the foundation for the intuitionistic type theory developed by Per Martin-Löf, and is often extended to a three way correspondence, the third leg of which are the cartesian closed categories.

Proof-theoretic semantics

In linguistics, type-logical grammar, categorial grammar and Montague grammar apply formalisms based on structural proof theory to give a formal natural language semantics.

Tableau systems

Analytic tableaux apply the central idea of analytic proof from structural proof theory to provide decision procedures and semi-decision procedures for a wide range of logics.

Ordinal analysis

Ordinal analysis is a powerful technique for providing combinatorial consistency proofs for theories formalising arithmetic and analysis.

Logics from proof analysis

Several important logics have come from insights into logical structure arising in structural proof theory.

Notes

[1] E.g., Wang (1981), pp. 3–4, and Barwise (1978).
[2] Prawitz (1965).
[3] Girard, Lafont, and Taylor (1988).

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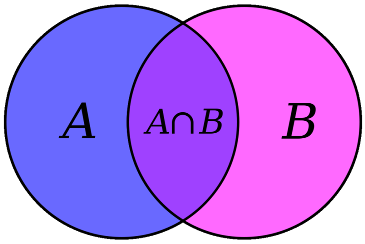
Set theory

Set theory is the branch of mathematics that studies sets, which are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used in the definitions of nearly all mathematical objects.



A Venn diagram illustrating the intersection of two sets.

The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with the axiom of choice, are the best-known.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice. Beyond its foundational role, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

History

Mathematical topics typically emerge and evolve through interactions among many researchers. Set theory, however, was founded by a single paper in 1874 by Georg Cantor: "On a Characteristic Property of All Real Algebraic Numbers".[1][2]

Georg Cantor

Since the 5th century BC, beginning with Greek mathematician Zeno of Elea in the West and early Indian mathematicians in the East, mathematicians had struggled with the concept of infinity. Especially notable is the work of Bernard Bolzano in the first half of the 19th century.[3] Modern understanding of infinity began in 1867–71, with Cantor's work on number theory. An 1872 meeting between Cantor and Richard Dedekind influenced Cantor's thinking and culminated in Cantor's 1874 paper.

Cantor's work initially polarized the mathematicians of his day. While Karl Weierstrass and Dedekind supported Cantor, Leopold Kronecker, now seen as a founder of mathematical constructivism, did not. Cantorian set theory eventually became widespread, due to the utility of Cantorian concepts, such as one-to-one correspondence among sets, his proof that there are more real numbers than integers, and the "infinity of infinities" ("Cantor's paradise") resulting from the power set operation. This utility of set theory led to the article "Mengenlehre" contributed in 1898 by Arthur Schoenflies to Klein's encyclopedia.

The next wave of excitement in set theory came around 1900, when it was discovered that Cantorian set theory gave rise to several contradictions, called antinomies or paradoxes. Bertrand Russell and Ernst Zermelo independently found the simplest and best known paradox, now called Russell's paradox: consider "the set of all sets that are not members of themselves", which leads to a contradiction since it must be a member of itself, and not a member of itself. In 1899 Cantor had himself posed the question "What is the cardinal number of the set of all sets?", and obtained a related paradox. Russell used his paradox as a theme in his 1903 review of continental mathematics in his

The Principles of Mathematics.

The momentum of set theory was such that debate on the paradoxes did not lead to its abandonment. The work of Zermelo in 1908 and Abraham Fraenkel in 1922 resulted in the set of axioms ZFC, which became the canonical axioms for set theory. The work of analysts such as Henri Lebesgue demonstrated the great mathematical utility of set theory, which has since become woven into the fabric of modern mathematics. Set theory is commonly used as a foundational system, although in some areas category theory is thought to be a preferred foundation.

Basic concepts

Set theory begins with a fundamental binary relation between an object o and a set A. If o is a **member** (or **element**) of A, write $o \in A$. Since sets are objects, the membership relation can relate sets as well.

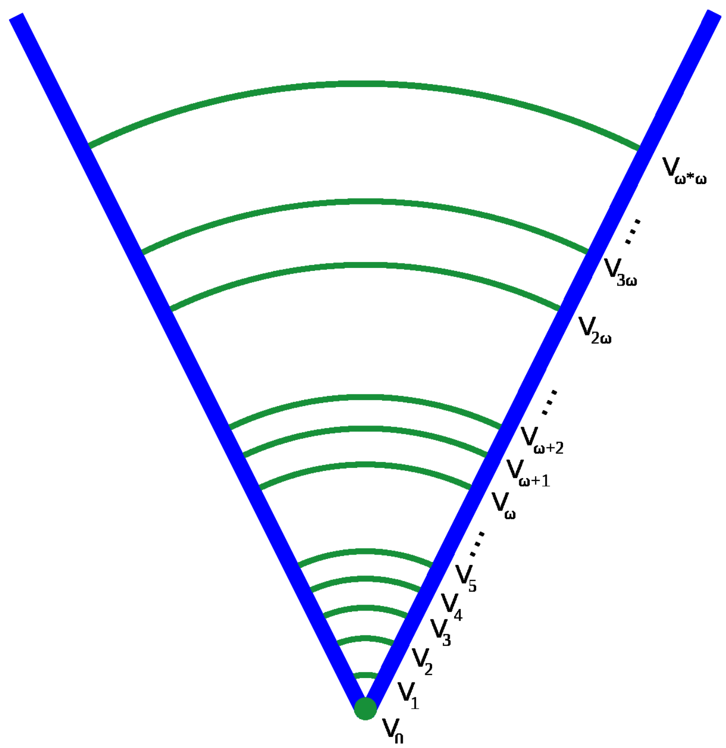
A derived binary relation between two sets is the subset relation, also called **set inclusion**. If all the members of set A are also members of set B, then A is a **subset** of B, denoted $A \subseteq B$. For example, $\{1,2\}$ is a subset of $\{1,2,3\}$, but $\{1,4\}$ is not. From this definition, it is clear that a set is a subset of itself; for cases where one wishes to rule out this, the term **proper subset** is defined. A is called a **proper subset** of B if and only if A is a subset of B, but B is **not** a subset of A.

Just as arithmetic features binary operations on numbers, set theory features binary operations on sets. The:

- **Union** of the sets A and B, denoted $A \cup B$, is the set of all objects that are a member of A, or B, or both. The union of $\{1, 2, 3\}$ and $\{2, 3, 4\}$ is the set $\{1, 2, 3, 4\}$.
- **Intersection** of the sets A and B, denoted $A \cap B$, is the set of all objects that are members of both A and B. The intersection of $\{1, 2, 3\}$ and $\{2, 3, 4\}$ is the set $\{2, 3\}$.
- **Set difference** of U and A, denoted $U \setminus A$, is the set of all members of U that are not members of A. The set difference $\{1,2,3\} \setminus \{2,3,4\}$ is $\{1\}$, while, conversely, the set difference $\{2,3,4\} \setminus \{1,2,3\}$ is $\{4\}$. When A is a subset of U, the set difference $U \setminus A$ is also called the **complement** of A in U. In this case, if the choice of U is clear from the context, the notation A^c is sometimes used instead of $U \setminus A$, particularly if U is a universal set as in the study of Venn diagrams.
- **Symmetric difference** of sets A and B is the set of all objects that are a member of exactly one of A and B (elements which are in one of the sets, but not in both). For instance, for the sets $\{1,2,3\}$ and $\{2,3,4\}$, the symmetric difference set is $\{1,4\}$. It is the set difference of the union and the intersection, $(A \cup B) \setminus (A \cap B)$ or $(A \setminus B) \cup (B \setminus A)$.
- **Cartesian product** of A and B, denoted $A \times B$, is the set whose members are all possible ordered pairs (a,b) where a is a member of A and b is a member of B. The cartesian product of $\{1, 2\}$ and {red, white} is {(1, red), (1, white), (2, red), (2, white)}.
- **Power set** of a set A is the set whose members are all possible subsets of A. For example, the power set of $\{1, 2\}$ is $\{ \{\}, \{1\}, \{2\}, \{1,2\} \}$.

Some basic sets of central importance are the empty set (the unique set containing no elements), the set of natural numbers, and the set of real numbers.

Some ontology



An initial segment of the von Neumann hierarchy.

A set is pure if all of its members are sets, all members of its members are sets, and so on. For example, the set containing only the empty set is a nonempty pure set. In modern set theory, it is common to restrict attention to the **von Neumann universe** of pure sets, and many systems of axiomatic set theory are designed to axiomatize the pure sets only. There are many technical advantages to this restriction, and little generality is lost, since essentially all mathematical concepts can be modeled by pure sets. Sets in the von Neumann universe are organized into a cumulative hierarchy, based on how deeply their members, members of members, etc. are nested. Each set in this hierarchy is assigned (by transfinite recursion) an ordinal number α, known as its **rank**. The rank of a pure set X is defined to be the least upper bound of all successors of ranks of members of X. For example, the empty set is assigned rank 0, while the set containing only the empty set is assigned rank 1. For each ordinal α, the set V_α is defined to consist of all pure sets with rank less than α. The entire von Neumann universe is denoted V.

Axiomatic set theory

Elementary set theory can be studied informally and intuitively, and so can be taught in primary schools using Venn diagrams. The intuitive approach tacitly assumes that a set may be formed from the class of all objects satisfying any particular defining condition. This assumption gives rise to paradoxes, the simplest and best known of which are Russell's paradox and the Burali-Forti paradox. Axiomatic set theory was originally devised to rid set theory of such paradoxes.[4]

The most widely studied systems of axiomatic set theory imply that all sets form a cumulative hierarchy. Such systems come in two flavors, those whose ontology consists of:

- *Sets alone*. This includes the most common axiomatic set theory, **Zermelo–Fraenkel set theory (ZFC)**, which includes the axiom of choice. Fragments of ZFC include:
  - Zermelo set theory, which replaces the axiom schema of replacement with that of separation;
  - General set theory, a small fragment of Zermelo set theory sufficient for the Peano axioms and finite sets;
  - Kripke-Platek set theory, which omits the axioms of infinity, powerset, and choice, and weakens the axiom schemata of separation and replacement.
- *Sets and proper classes*. This includes Von Neumann-Bernays-Gödel set theory, which has the same strength as ZFC for theorems about sets alone, and Morse-Kelley set theory, which is stronger than ZFC.

The above systems can be modified to allow **urelements**, objects that can be members of sets but that are not themselves sets and do not have any members.

The systems of **New Foundations NFU** (allowing urelements) and **NF** (lacking them) are not based on a cumulative hierarchy. NF and NFU include a "set of everything," relative to which every set has a complement. In these systems urelements matter, because NF, but not NFU, produces sets for which the axiom of choice does not hold.

Systems of constructive set theory, such as CST, CZF, and IZF, embed their set axioms in intuitionistic logic instead of first order logic. Yet other systems accept standard first order logic but feature a nonstandard membership relation. These include rough set theory and fuzzy set theory, in which the value of an atomic formula embodying the membership relation is not simply **True** or **False**. The Boolean-valued models of ZFC are a related subject.

An enrichment of ZFC called Internal Set Theory was proposed by Edward Nelson in 1977.

Applications

Many mathematical concepts can be defined precisely using only set theoretic concepts. For example, mathematical structures as diverse as graphs, manifolds, rings, and vector spaces can all be defined as sets satisfying various (axiomatic) properties. Equivalence and order relations are ubiquitous in mathematics, and the theory of mathematical relations can be described in set theory.

Set theory is also a promising foundational system for much of mathematics. Since the publication of the first volume of *Principia Mathematica*, it has been claimed that most or even all mathematical theorems can be derived using an aptly designed set of axioms for set theory, augmented with many definitions, using first or second order logic. For example, properties of the natural and real numbers can be derived within set theory, as each number system can be identified with a set of equivalence classes under a suitable equivalence relation whose field is some infinite set.

Set theory as a foundation for mathematical analysis, topology, abstract algebra, and discrete mathematics is likewise uncontroversial; mathematicians accept that (in principle) theorems in these areas can be derived from the relevant definitions and the axioms of set theory. Few full derivations of complex mathematical theorems from set theory have been formally verified, however, because such formal derivations are often much longer than the natural language proofs mathematicians commonly present. One verification project, Metamath, includes derivations of more than 10,000 theorems starting from the ZFC axioms and using first order logic.

Areas of study

Set theory is a major area of research in mathematics, with many interrelated subfields.

Combinatorial set theory

Combinatorial set theory concerns extensions of finite combinatorics to infinite sets. This includes the study of cardinal arithmetic and the study of extensions of Ramsey's theorem such as the Erdős–Rado theorem.

Descriptive set theory

Descriptive set theory is the study of subsets of the real line and, more generally, subsets of Polish spaces. It begins with the study of pointclasses in the Borel hierarchy and extends to the study of more complex hierarchies such as the projective hierarchy and the Wadge hierarchy. Many properties of Borel sets can be established in ZFC, but proving these properties hold for more complicated sets requires additional axioms related to determinacy and large cardinals.

The field of effective descriptive set theory is between set theory and recursion theory. It includes the study of lightface pointclasses, and is closely related to hyperarithmetical theory. In many cases, results of classical descriptive set theory have effective versions; in some cases, new results are obtained by proving the effective version first and then extending ("relativizing") it to make it more broadly applicable.

A recent area of research concerns Borel equivalence relations and more complicated definable equivalence relations. This has important applications to the study of invariants in many fields of mathematics.

Fuzzy set theory

In set theory as Cantor defined and Zermelo and Fraenkel axiomatized, an object is either a member of a set or not. In fuzzy set theory this condition was relaxed by Lotfi A. Zadeh so an object has a *degree of membership* in a set, as number between 0 and 1. For example, the degree of membership of a person in the set of "tall people" is more flexible than a simple yes or no answer and can be a real number such as 0.75.

Inner model theory

An **inner model** of Zermelo–Fraenkel set theory (ZF) is a transitive class that includes all the ordinals and satisfies all the axioms of ZF. The canonical example is the constructible universe *L* developed by Gödel. One reason that the study of inner models is of interest is that it can be used to prove consistency results. For example, it can be shown that regardless of whether a model *V* of ZF satisfies the continuum hypothesis or the axiom of choice, the inner model *L* constructed inside the original model will satisfy both the generalized continuum hypothesis and the axiom of choice. Thus the assumption that ZF is consistent (has at least one model) implies that ZF together with these two principles is consistent.

The study of inner models is common in the study of determinacy and large cardinals, especially when considering axioms such as the axiom of determinacy that contradict the axiom of choice. Even if a fixed model of set theory satisfies the axiom of choice, it is possible for an inner model to fail to satisfy the axiom of choice. For example, the existence of sufficiently large cardinals implies that there is an inner model satisfying the axiom of determinacy (and thus not satisfying the axiom of choice).[5]

Large cardinals

A **large cardinal** is a cardinal number with an extra property. Many such properties are studied, including inaccessible cardinals, measurable cardinals, and many more. These properties typically imply the cardinal number must be very large, with the existence of a cardinal with the specified property unprovable in Zermelo-Fraenkel set theory.

Determinacy

Determinacy refers to the fact that, under appropriate assumptions, certain two-player games of perfect information are determined from the start in the sense that one player must have a winning strategy. The existence of these strategies has important consequences in descriptive set theory, as the assumption that a broader class of games is determined often implies that a broader class of sets will have a topological property. The axiom of determinacy (AD) is an important object of study; although incompatible with the axiom of choice, AD implies that all subsets of the real line are well behaved (in particular, measurable and with the perfect set property). AD can be used to prove that the Wadge degrees have an elegant structure.

Forcing

Paul Cohen invented the method of forcing while searching for a model of ZFC in which the axiom of choice or the continuum hypothesis fails. Forcing adjoins to some given model of set theory additional sets in order to create a larger model with properties determined (i.e. "forced") by the construction and the original model. For example, Cohen's construction adjoins additional subsets of the natural numbers without changing any of the cardinal numbers of the original model. Forcing is also one of two methods for proving relative consistency by finitistic methods, the other method being Boolean-valued models.

Cardinal invariants

A **cardinal invariant** is a property of the real line measured by a cardinal number. For example, a well-studied invariant is the smallest cardinality of a collection of meagre sets of reals whose union is the entire real line. These are invariants in the sense that any two isomorphic models of set theory must give the same cardinal for each invariant. Many cardinal invariants have been studied, and the relationships between them are often complex and related to axioms of set theory.

Set-theoretic topology

Set-theoretic topology studies questions of general topology that are set-theoretic in nature or that require advanced methods of set theory for their solution. Many of these theorems are independent of ZFC, requiring stronger axioms for their proof. A famous problem is the normal Moore space question, a question in general topology that was the subject of intense research. The answer to the normal Moore space question was eventually proved to be independent of ZFC.

Objections to set theory as a foundation for mathematics

From set theory's inception, some mathematicians have objected to it as a foundation for mathematics. The most common objection to set theory, one Kronecker voiced in set theory's earliest years, starts from the constructivist view that mathematics is loosely related to computation. If this view is granted, then the treatment of infinite sets, both in naive and in axiomatic set theory, introduces into mathematics methods and objects that are not computable even in principle. Ludwig Wittgenstein questioned the way Zermelo–Fraenkel set theory handled infinities. Wittgenstein's views about the foundations of mathematics were later criticised by Georg Kreisel and Paul Bernays, and investigated by Crispin Wright, among others.

Category theorists have proposed topos theory as an alternative to traditional axiomatic set theory. Topos theory can interpret various alternatives to that theory, such as constructivism, finite set theory, and computable set theory.[6]

Notes

[1] Cantor, Georg (1874), "Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen", *J. Reine Angew. Math.* **77**: 258–262, doi:10.1515/crll.1874.77.258

[2] Johnson, Philip (1972), *A History of Set Theory*, Prindle, Weber & Schmidt, ISBN 0-87150-154-6

[3] Bolzano, Bernard (1975), Berg, Jan, ed., *Einleitung zur Größenlehre und erste Begriffe der allgemeinen Größenlehre*, Bernard-Bolzano-Gesamtausgabe, edited by Eduard Winter et al., **Vol. II, A, 7**, Stuttgart, Bad Cannstatt: Friedrich Frommann Verlag, p. 152, ISBN 3-7728-0466-7

[4] In his 1925, John von Neumann observed that "set theory in its first, "naive" version, due to Cantor, led to contradictions. These are the well-known antinomies of the set of all sets that do not contain themselves (Russell), of the set of all transfinte ordinal numbers (Burali-Forti), and the set of all finitely definable real numbers (Richard)." He goes on to observe that two "tendencies" were attempting to "rehabilitate" set theory. Of the first effort, exemplified by Bertrand Russell, Julius König, Hermann Weyl and L. E. J. Brouwer, von Neumann called the "overall effect of their activity . . . devastating". With regards to the axiomatic method employed by second group composed of Zermelo, Abraham Fraenkel and Arthur Moritz Schoenflies, von Neumann worried that "We see only that the known modes of inference leading to the antinomies fail, but who knows where there are not others?" and he set to the task, "in the spirit of the second group", to "produce, by means of a finite number of purely formal operations . . . all the sets that we want to see formed" but not allow for the antinomies. (All quotes from von Neumann 1925 reprinted in van Heijenoort, Jean (1967, third printing 1976), "From Frege to Gödel: A Source Book in Mathematical Logic, 1979–1931", Harvard University Press, Cambridge MA, ISBN 0-674-32449-8 (pbk). A synopsis of the history, written by van Heijenoort, can be found in the comments that precede von Neumann's 1925.

[5] Jech, Thomas (2003), *Set Theory: Third Millennium Edition*, Springer Monographs in Mathematics, Berlin, New York: Springer-Verlag, ISBN 978-3-540-44085-7, p. 642.

[6] Ferro, A.; Omodeo, E. G.; Schwartz, J. T. (1980), "Decision procedures for elementary sublanguages of set theory. I. Multi-level syllogistic and some extensions", *Comm. Pure Appl. Math.* **33** (5): 599–608, doi:10.1002/cpa.3160330503

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External links

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- Arthur Schoenflies (1898) Mengenlehre (http://www.archive.org/stream/encyklomath101encyrich#page/n229) in Klein's encyclopedia.

Computability theory

Computability theory, also called **recursion theory**, is a branch of mathematical logic, of computer science, and of the theory of computation that originated in the 1930s with the study of computable functions and Turing degrees. The field has since grown to include the study of generalized computability and definability. In these areas, recursion theory overlaps with proof theory and effective descriptive set theory.

The basic questions addressed by recursion theory are "What does it mean for a function from the natural numbers to themselves to be computable?" and "How can noncomputable functions be classified into a hierarchy based on their level of noncomputability?". The answers to these questions have led to a rich theory that is still being actively researched.

Recursion theorists in mathematical logic often study the theory of relative computability, reducibility notions and degree structures described in this article. This contrasts with the theory of subrecursive hierarchies, formal methods and formal languages that is common in the study of computability theory in computer science. There is considerable overlap in knowledge and methods between these two research communities, however, and no firm line can be drawn between them.

Computable and uncomputable sets

Recursion theory originated in the 1930s, with work of Kurt Gödel, Alonzo Church, Alan Turing, Stephen Kleene and Emil Post.[1]

The fundamental results the researchers obtained established Turing computability as the correct formalization of the informal idea of effective calculation. These results led Stephen Kleene (1952) to coin the two names "Church's thesis" (Kleene 1952:300) and "Turing's Thesis" (p. 376). Nowadays these are often considered as a single hypothesis, the **Church–Turing thesis**, which states that any function that is computable by an algorithm is a computable function. Although initially skeptical, by 1946 Gödel argued in favor of this thesis:

"Tarski has stressed in his lecture (and I think justly) the great importance of the concept of general recursiveness (or Turing's computability). It seems to me that this importance is largely due to the fact that with this concept one has for the first time succeeded in giving an absolute notion to an interesting epistemological notion, i.e., one not depending on the formalism chosen.*"(Gödel 1946 in Davis 1965:84).[2]

With a definition of effective calculation came the first proofs that there are problems in mathematics that cannot be effectively decided. Church (1936a, 1936b) and Turing (1936), inspired by techniques used by Gödel (1931) to prove his incompleteness theorems, independently demonstrated that the Entscheidungsproblem is not effectively decidable. This result showed that there is no algorithmic procedure that can correctly decide whether arbitrary mathematical propositions are true or false.

Many problems of mathematics have been shown to be undecidable after these initial examples were established. In 1947, Markov and Post published independent papers showing that the word problem for semigroups cannot be effectively decided. Extending this result, Pyotr Novikov and William Boone showed independently in the 1950s that the word problem for groups is not effectively solvable: there is no effective procedure that, given a word in a finitely presented group, will decide whether the element represented by the word is the identity element of the group. In 1970, Yuri Matiyasevich proved (using results of Julia Robinson) Matiyasevich's theorem, which implies that Hilbert's tenth problem has no effective solution; this problem asked whether there is an effective procedure to decide whether a Diophantine equation over the integers has a solution in the integers. The list of undecidable problems gives additional examples of problems with no computable solution.

The study of which mathematical constructions can be effectively performed is sometimes called **recursive mathematics**; the *Handbook of Recursive Mathematics* (Ershov *et al.* 1998) covers many of the known results in this field.

Turing computability

The main form of computability studied in recursion theory was introduced by Turing (1936). A set of natural numbers is said to be a **computable set** (also called a **decidable**, **recursive**, or **Turing computable** set) if there is a Turing machine that, given a number *n*, halts with output 1 if *n* is in the set and halts with output 0 if *n* is not in the set. A function *f* from the natural numbers to themselves is a **recursive** or **(Turing) computable function** if there is a Turing machine that, on input *n*, halts and returns output *f*(*n*). The use of Turing machines here is not necessary; there are many other models of computation that have the same computing power as Turing machines; for example the μ-recursive functions obtained from primitive recursion and the μ operator.

The terminology for recursive functions and sets is not completely standardized. The definition in terms of μ-recursive functions as well as a different definition of *rekursiv* functions by Gödel led to the traditional name *recursive* for sets and functions computable by a Turing machine. The word **decidable** stems from the German word **Entscheidungsproblem** which was used in the original papers of Turing and others. In contemporary use, the term "computable function" has various definitions: according to Cutland (1980), it is a partial recursive function (which can be undefined for some inputs), while according to Soare (1987) it is a total recursive (equivalently, general recursive) function. This article follows the second of these conventions. Soare (1996) gives additional comments about the terminology.

Not every set of natural numbers is computable. The halting problem, which is the set of (descriptions of) Turing machines that halt on input 0, is a well known example of a noncomputable set. The existence of many noncomputable sets follows from the facts that there are only countably many Turing machines, and thus only countably many computable sets, but there are uncountably many sets of natural numbers.

Although the Halting problem is not computable, it is possible to simulate program execution and produce an infinite list of the programs that do halt. Thus the halting problem is an example of a **recursively enumerable set**, which is a set that can be enumerated by a Turing machine (other terms for recursively enumerable include **computably enumerable** and **semidecidable**). Equivalently, a set is recursively enumerable if and only if it is the range of some computable function. The recursively enumerable sets, although not decidable in general, have been studied in detail in recursion theory.

Areas of research

Beginning with the theory of recursive sets and functions described above, the field of recursion theory has grown to include the study of many closely related topics. These are not independent areas of research: each of these areas draws ideas and results from the others, and most recursion theorists are familiar with the majority of them.

Relative computability and the Turing degrees

Recursion theory in mathematical logic has traditionally focused on **relative computability**, a generalization of Turing computability defined using oracle Turing machines, introduced by Turing (1939). An oracle Turing machine is a hypothetical device which, in addition to performing the actions of a regular Turing machine, is able to ask questions of an **oracle**, which is a particular set of natural numbers. The oracle machine may only ask questions of the form "Is n in the oracle set?". Each question will be immediately answered correctly, even if the oracle set is not computable. Thus an oracle machine with a noncomputable oracle will be able to compute sets that are not computable without an oracle.

Informally, a set of natural numbers A is **Turing reducible** to a set B if there is an oracle machine that correctly tells whether numbers are in A when run with B as the oracle set (in this case, the set A is also said to be (**relatively**) **computable from** B and **recursive in** B). If a set A is Turing reducible to a set B and B is Turing reducible to A then the sets are said to have the same **Turing degree** (also called **degree of unsolvability**). The Turing degree of a set gives a precise measure of how uncomputable the set is.

The natural examples of sets that are not computable, including many different sets that encode variants of the halting problem, have two properties in common:

1. They are recursively enumerable, and
2. Each can be translated into any other via a many-one reduction. That is, given such sets A and B, there is a total computable function f such that $A = \{x : f(x) \in B\}$. These sets are said to be **many-one equivalent** (or **m-equivalent**).

Many-one reductions are "stronger" than Turing reductions: if a set A is many-one reducible to a set B, then A is Turing reducible to B, but the converse does not always hold. Although the natural examples of noncomputable sets are all many-one equivalent, it is possible to construct recursively enumerable sets A and B such that A is Turing reducible to B but not many-one reducible to B. It can be shown that every recursively enumerable set is many-one reducible to the halting problem, and thus the halting problem is the most complicated recursively enumerable set with respect to many-one reducibility and with respect to Turing reducibility. Post (1944) asked whether *every* recursively enumerable set is either computable or Turing equivalent to the halting problem, that is, whether there is no recursively enumerable set with a Turing degree intermediate between those two.

As intermediate results, Post defined natural types of recursively enumerable sets like the simple, hypersimple and hyperhypersimple sets. Post showed that these sets are strictly between the computable sets and the halting problem with respect to many-one reducibility. Post also showed that some of them are strictly intermediate under other reducibility notions stronger than Turing reducibility. But Post left open the main problem of the existence of recursively enumerable sets of intermediate Turing degree; this problem became known as **Post's problem**. After ten years, Kleene and Post showed in 1954 that there are intermediate Turing degrees between those of the computable sets and the halting problem, but they failed to show that any of these degrees contains a recursively enumerable set. Very soon after this, Friedberg and Muchnik independently solved Post's problem by establishing the existence of recursively enumerable sets of intermediate degree. This groundbreaking result opened a wide study of the Turing degrees of the recursively enumerable sets which turned out to possess a very complicated and non-trivial structure.

There are uncountably many sets that are not recursively enumerable, and the investigation of the Turing degrees of all sets is as central in recursion theory as the investigation of the recursively enumerable Turing degrees. Many degrees with special properties were constructed: **hyperimmune-free degrees** where every function computable relative to that degree is majorized by a (unrelativized) computable function; **high degrees** relative to which one can compute a function f which dominates every computable function g in the sense that there is a constant c depending on g such that $g(x) < f(x)$ for all $x > c$; **random degrees** containing algorithmically random sets; **1-generic** degrees of 1-generic sets; and the degrees below the halting problem of limit-recursive sets.

The study of arbitrary (not necessarily recursively enumerable) Turing degrees involves the study of the Turing jump. Given a set A, the **Turing jump** of A is a set of natural numbers encoding a solution to the halting problem for oracle Turing machines running with oracle A. The Turing jump of any set is always of higher Turing degree than the original set, and a theorem of Friedburg shows that any set that computes the Halting problem can be obtained as the Turing jump of another set. Post's theorem establishes a close relationship between the Turing jump operation and the arithmetical hierarchy, which is a classification of certain subsets of the natural numbers based on their definability in arithmetic.

Much recent research on Turing degrees has focused on the overall structure of the set of Turing degrees and the set of Turing degrees containing recursively enumerable sets. A deep theorem of Shore and Slaman (1999) states that the function mapping a degree x to the degree of its Turing jump is definable in the partial order of the Turing degrees. A recent survey by Ambos-Spies and Fejer (2006) gives an overview of this research and its historical progression.

Other reducibilities

An ongoing area of research in recursion theory studies reducibility relations other than Turing reducibility. Post (1944) introduced several **strong reducibilities**, so named because they imply truth-table reducibility. A Turing machine implementing a strong reducibility will compute a total function regardless of which oracle it is presented with. **Weak reducibilities** are those where a reduction process may not terminate for all oracles; Turing reducibility is one example.

The strong reducibilities include:

One-one reducibility

: A is **one-one reducible** (or **1-reducible**) to B if there is a total computable injective function f such that each n is in A if and only if $f(n)$ is in B.

Many-one reducibility

: This is essentially one-one reducibility without the constraint that f be injective. A is **many-one reducible** (or **m-reducible**) to B if there is a total computable function f such that each n is in A if and only if $f(n)$ is in B.

Truth-table reducibility

: A is truth-table reducible to B if A is Turing reducible to B via an oracle Turing machine that computes a total function regardless of the oracle it is given. Because of compactness of Cantor space, this is equivalent to saying that the reduction presents a single list of questions (depending only on the input) to the oracle simultaneously, and then having seen their answers is able to produce an output without asking additional questions regardless of the oracle's answer to the initial queries. Many variants of truth-table reducibility have also been studied.

Further reducibilities (positive, disjunctive, conjunctive, linear and their weak and bounded versions) are discussed in the article Reduction (recursion theory).

The major research on strong reducibilities has been to compare their theories, both for the class of all recursively enumerable sets as well as for the class of all subsets of the natural numbers. Furthermore, the relations between the reducibilities has been studied. For example, it is known that every Turing degree is either a truth-table degree or is the union of infinitely many truth-table degrees.

Reducibilities weaker than Turing reducibility (that is, reducibilities that are implied by Turing reducibility) have also been studied. The most well known are arithmetical reducibility and hyperarithmetical reducibility. These reducibilities are closely connected to definability over the standard model of arithmetic.

Rice's theorem and the arithmetical hierarchy

Rice showed that for every nontrivial class C (which contains some but not all r.e. sets) the index set $E = \{e$: the eth r.e. set W_e is in $C\}$ has the property that either the halting problem or its complement is many-one reducible to E, that is, can be mapped using a many-one reduction to E (see Rice's theorem for more detail). But, many of these index sets are even more complicated than the halting problem. These type of sets can be classified using the arithmetical hierarchy. For example, the index set FIN of class of all finite sets is on the level Σ_2, the index set REC of the class of all recursive sets is on the level Σ_3, the index set COFIN of all cofinite sets is also on the level Σ_3 and the index set COMP of the class of all Turing-complete sets Σ_4. These hierarchy levels are defined inductively, Σ_{n+1} contains just all sets which are recursively enumerable relative to Σ_n; Σ_1 contains the recursively enumerable sets. The index sets given here are even complete for their levels, that is, all the sets in these levels can be many-one reduced to the given index sets.

Reverse mathematics

The program of **reverse mathematics** asks which set-existence axioms are necessary to prove particular theorems of mathematics in subsystems of second-order arithmetic. This study was initiated by Harvey Friedman and was studied in detail by Stephen Simpson and others; Simpson (1999) gives a detailed discussion of the program. The set-existence axioms in question correspond informally to axioms saying that the powerset of the natural numbers is closed under various reducibility notions. The weakest such axiom studied in reverse mathematics is **recursive comprehension**, which states that the powerset of the naturals is closed under Turing reducibility.

Numberings

A numbering is an enumeration of functions; it has two parameters, e and x and outputs the value of the e-th function in the numbering on the input x. Numberings can be partial-recursive although some of its members are total recursive, that is, computable functions. Admissible numberings are those into which all others can be translated. A Friedberg numbering (named after its discoverer) is a one-one numbering of all partial-recursive functions; it is necessarily not an admissible numbering. Later research dealt also with numberings of other classes like classes of recursively enumerable sets. Goncharov discovered for example a class of recursively enumerable sets for which the numberings fall into exactly two classes with respect to recursive isomorphisms.

The priority method

For further explanation, see the section Post's problem and the priority method *in the article* Turing degree.

Post's problem was solved with a method called the **priority method**; a proof using this method is called a **priority argument**. This method is primarily used to construct recursively enumerable sets with particular properties. To use this method, the desired properties of the set to be constructed are broken up into an infinite list of goals, known as **requirements**, so that satisfying all the requirements will cause the set constructed to have the desired properties. Each requirement is assigned to a natural number representing the priority of the requirement; so 0 is assigned to the most important priority, 1 to the second most important, and so on. The set is then constructed in stages, each stage attempting to satisfy one of more of the requirements by either adding numbers to the set or banning numbers from the set so that the final set will satisfy the requirement. It may happen that satisfying one requirement will cause another to become unsatisfied; the priority order is used to decide what to do in such an event.

Priority arguments have been employed to solve many problems in recursion theory, and have been classified into a hierarchy based on their complexity (Soare 1987). Because complex priority arguments can be technical and difficult to follow, it has traditionally been considered desirable to prove results without priority arguments, or to see if results proved with priority arguments can also be proved without them. For example, Kummer published a paper on a proof for the existence of Friedberg numberings without using the priority method.

The lattice of recursively enumerable sets

When Post defined the notion of a simple set as an r.e. set with an infinite complement not containing any infinite r.e. set, he started to study the structure of the recursively enumerable sets under inclusion. This lattice became a well-studied structure. Recursive sets can be defined in this structure by the basic result that a set is recursive if and only if the set and its complement are both recursively enumerable. Infinite r.e. sets have always infinite recursive subsets; but on the other hand, simple sets exist but do not have a coinfinite recursive superset. Post (1944) introduced already hypersimple and hyperhypersimple sets; later maximal sets were constructed which are r.e. sets such that every r.e. superset is either a finite variant of the given maximal set or is co-finite. Post's original motivation in the study of this lattice was to find a structural notion such that every set which satisfies this property is neither in the Turing degree of the recursive sets nor in the Turing degree of the halting problem. Post did not find such a property and the solution to his problem applied priority methods instead; Harrington and Soare (1991) found eventually such a property.

Automorphism problems

Another important question is the existence of automorphisms in recursion-theoretic structures. One of these structures is that one of recursively enumerable sets under inclusion modulo finite difference; in this structure, A is below B if and only if the set difference $B - A$ is finite. Maximal sets (as defined in the previous paragraph) have the property that they cannot be automorphic to non-maximal sets, that is, if there is an automorphism of the recursive enumerable sets under the structure just mentioned, then every maximal set is mapped to another maximal set. Soare (1974) showed that also the converse holds, that is, every two maximal sets are automorphic. So the maximal sets form an orbit, that is, every automorphism preserves maximality and any two maximal sets are transformed into each other by some automorphism. Harrington gave a further example of an automorphic property: that of the creative sets, the sets which are many-one equivalent to the halting problem.

Besides the lattice of recursively enumerable sets, automorphisms are also studied for the structure of the Turing degrees of all sets as well as for the structure of the Turing degrees of r.e. sets. In both cases, Cooper claims to have constructed nontrivial automorphisms which map some degrees to other degrees; this construction has, however, not been verified and some colleagues believe that the construction contains errors and that the question of whether there is a nontrivial automorphism of the Turing degrees is still one of the main unsolved questions in this area (Slaman and Woodin 1986, Ambos-Spies and Fejer 2006).

Kolmogorov complexity

The field of Kolmogorov complexity and algorithmic randomness was developed during the 1960s and 1970s by Chaitin, Kolmogorov, Levin, Martin-Löf and Solomonoff (the names are given here in alphabetical order; much of the research was independent, and the unity of the concept of randomness was not understood at the time). The main idea is to consider a universal Turing machine U and to measure the complexity of a number (or string) x as the length of the shortest input p such that $U(p)$ outputs x. This approach revolutionized earlier ways to determine when an infinite sequence (equivalently, characteristic function of a subset of the natural numbers) is random or not by invoking a notion of randomness for finite objects. Kolmogorov complexity became not only a subject of independent study but is also applied to other subjects as a tool for obtaining proofs. There are still many open problems in this area. For that reason, a recent research conference in this area was held in January 2007[3] and a list of open problems[4] is maintained by Joseph Miller and Andre Nies.

Frequency computation

This branch of recursion theory analyzed the following question: For fixed m and n with $0 < m < n$, for which functions A is it possible to compute for any different n inputs $x_1, x_2, ..., x_n$ a tuple of n numbers $y_1, y_2, ..., y_n$ such that at least m of the equations $A(x_k) = y_k$ are true. Such sets are known as (m, n)-recursive sets. The first major result in this branch of Recursion Theory is Trakhtenbrot's result that a set is computable if it is (m, n)-recursive for some m, n with $2m > n$. On the other hand, Jockusch's semirecursive sets (which were already known informally before Jockusch introduced them 1968) are examples of a set which is (m, n)-recursive if and only if $2m < n + 1$. There are uncountably many of these sets and also some recursively enumerable but noncomputable sets of this type. Later, Degtev established a hierarchy of recursively enumerable sets that are $(1, n + 1)$-recursive but not $(1, n)$-recursive. After a long phase of research by Russian scientists, this subject became repopularized in the west by Beigel's thesis on bounded queries, which linked frequency computation to the above mentioned bounded reducibilities and other related notions. One of the major results was Kummer's Cardinality Theory which states that a set A is computable if and only if there is an n such that some algorithm enumerates for each tuple of n different numbers up to n many possible choices of the cardinality of this set of n numbers intersected with A; these choices must contain the true cardinality but leave out at least one false one.

Inductive inference

This is the recursion-theoretic branch of learning theory. It is based on Gold's model of learning in the limit from 1967 and has developed since then more and more models of learning. The general scenario is the following: Given a class S of computable functions, is there a learner (that is, recursive functional) which outputs for any input of the form $(f(0),f(1),...,f(n))$ a hypothesis. A learner M learns a function f if almost all hypotheses are the same index e of f with respect to a previously agreed on acceptable numbering of all computable functions; M learns S if M learns every f in S. Basic results are that all recursively enumerable classes of functions are learnable while the class REC of all computable functions is not learnable. Many related models have been considered and also the learning of classes of recursively enumerable sets from positive data is a topic studied from Gold's pioneering paper in 1967 onwards.

Generalizations of Turing computability

Recursion theory includes the study of generalized notions of this field such as arithmetic reducibility, hyperarithmetical reducibility and α-recursion theory, as described by Sacks (1990). These generalized notions include reducibilities that cannot be executed by Turing machines but are nevertheless natural generalizations of Turing reducibility. These studies include approaches to investigate the analytical hierarchy which differs from the arithmetical hierarchy by permitting quantification over sets of natural numbers in addition to quantification over individual numbers. These areas are linked to the theories of well-orderings and trees; for example the set of all indices of recursive (nonbinary) trees without infinite branches is complete for level Π^1_1 of the analytical hierarchy. Both Turing reducibility and hyperarithmetical reducibility are important in the field of effective descriptive set theory. The even more general notion of degrees of constructibility is studied in set theory.

Continuous computability theory

Computability theory for digital computation is well developed. Computability theory is less well developed for analog computation that occurs in analog computers, analog signal processing, analog electronics, neural networks and continuous-time control theory, modelled by differential equations and continuous dynamical systems.[5][6]

Relationships between definability, proof and computability

There are close relationships between the Turing degree of a set of natural numbers and the difficulty (in terms of the arithmetical hierarchy) of defining that set using a first-order formula. One such relationship is made precise by Post's theorem. A weaker relationship was demonstrated by Kurt Gödel in the proofs of his completeness theorem and incompleteness theorems. Gödel's proofs show that the set of logical consequences of an effective first-order theory is a recursively enumerable set, and that if the theory is strong enough this set will be uncomputable. Similarly, Tarski's indefinability theorem can be interpreted both in terms of definability and in terms of computability.

Recursion theory is also linked to second order arithmetic, a formal theory of natural numbers and sets of natural numbers. The fact that certain sets are computable or relatively computable often implies that these sets can be defined in weak subsystems of second order arithmetic. The program of reverse mathematics uses these subsystems to measure the noncomputability inherent in well known mathematical theorems. Simpson (1999) discusses many aspects of second-order arithmetic and reverse mathematics.

The field of proof theory includes the study of second-order arithmetic and Peano arithmetic, as well as formal theories of the natural numbers weaker than Peano arithmetic. One method of classifying the strength of these weak systems is by characterizing which computable functions the system can prove to be total (see Fairtlough and Wainer (1998)). For example, in primitive recursive arithmetic any computable function that is provably total is actually primitive recursive, while Peano arithmetic proves that functions like the Ackerman function, which are not primitive recursive, are total. Not every total computable function is provably total in Peano arithmetic, however; an example of such a function is provided by Goodstein's theorem.

Name of the subject

The field of mathematical logic dealing with computability and its generalizations has been called "recursion theory" since its early days. Robert I. Soare, a prominent researcher in the field, has proposed (Soare 1996) that the field should be called "computability theory" instead. He argues that Turing's terminology using the word "computable" is more natural and more widely understood than the terminology using the word "recursive" introduced by Kleene. Many contemporary researchers have begun to use this alternate terminology.[7] These researchers also use terminology such as *partial computable function* and *computably enumerable* (*c.e.*) *set* instead of *partial recursive function* and *recursively enumerable* (*r.e.*) *set*. Not all researchers have been convinced, however, as explained by Fortnow[8] and Simpson.[9] Some commentators argue that both the names *recursion theory* and *computability theory* fail to convey the fact that most of the objects studied in recursion theory are not computable.[10]

Rogers (1967) has suggested that a key property of recursion theory is that its results and structures should be invariant under computable bijections on the natural numbers (this suggestion draws on the ideas of the Erlangen program in geometry). The idea is that a computable bijection merely renames numbers in a set, rather than indicating any structure in the set, much as a rotation of the Euclidean plane does not change any geometric aspect of lines drawn on it. Since any two infinite computable sets are linked by a computable bijection, this proposal identifies all the infinite computable sets (the finite computable sets are viewed as trivial). According to Rogers, the sets of interest in recursion theory are the noncomputable sets, partitioned into equivalence classes by computable bijections of the natural numbers.

Professional organizations

The main professional organization for recursion theory is the **Association for Symbolic Logic**, which holds several research conferences each year. The interdisciplinary research Association **Computability in Europe** (**CiE**) also organizes a series of annual conferences. *CiE 2012* will be the Turing Centenary Conference, held in Cambridge as part of the **Alan Turing Year**.

Notes

[1] Many of these foundational papers are collected in *The Undecidable* (1965) edited by Martin Davis.

[2] The full paper can also be found at pages 150ff (with commentary by Charles Parsons at 144ff) in Feferman et. al. editors 1990 *Kurt Gödel Volume II Publications 1938-1974*, Oxford University Press, New York, ISBN 978-0-19-514721-6. Both reprintings have the following footnote * added to the Davis volume by Gödel in 1965: "To be more precise: a function of integers is computable in any formal system containing arithmetic if and only if it is computable in arithmetic, where a function *f* is called computable in *S* if there is in *S* a computable term representing *f* (p. 150).

[3] Conference on Logic, Computability and Randomness (http://www-2.dc.uba.ar/logic2007/), January 10–13, 2007.

[4] The homepage (http://www.cs.auckland.ac.nz/~nies/) of Andre Nies has a list of open problems in Kolmogorov complexity

[5] Orponen, P. (1997). "A survey of continuous-time computation theory". *Advances in algorithms, languages, and complexity*. CiteSeerX: 10.1.1.53.1991 (http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.53.1991).

[6] Moore, C. (1996). "Recursion theory on the reals and continuous-time computation". *Theoretical Computer Science*. CiteSeerX: 10.1.1.6.5519 (http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.6.5519).

[7] Mathscinet searches for the titles like "computably enumerable" and "c.e." show that many papers have been published with this terminology as well as with the other one.

[8] Lance Fortnow, " Is it Recursive, Computable or Decidable? (http://weblog.fortnow.com/2004/02/is-it-recursive-computable-or.html)," 2004-2-15, accessed 2006-1-9.

[9] Stephen G. Simpson, " What is computability theory? (http://www.cs.nyu.edu/pipermail/fom/1998-August/001993.html)," FOM email list, 1998-8-24, accessed 2006-1-9.

[10] Harvey Friedman, " Renaming recursion theory (http://www.cs.nyu.edu/pipermail/fom/1998-August/002017.html)," FOM email list, 1998-8-28, accessed 2006-1-9.

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External links

- Association for Symbolic Logic homepage (http://www.aslonline.org/)
- Computability in Europe homepage (http://www.maths.leeds.ac.uk/cie/)
- Webpage on Recursion Theory Course at Graduate Level with approximately 100 pages of lecture notes (http://www.comp.nus.edu.sg/~fstephan/recursiontheory.html)
- German language lecture notes on inductive inference (http://www.comp.nus.edu.sg/~fstephan/learning.ps)

Deductive and inductive reasoning

Deductive reasoning

Deductive reasoning, also **deductive logic** or **logical deduction** or, informally, **"top-down" logic**,[1] is the process of reasoning from one or more general statements (premises) to reach a logically certain conclusion.[2]

Deductive reasoning links premises with conclusions. If all premises are true, the terms are clear, and the rules of deductive logic are followed, then the conclusion reached is necessarily true.

Deductive reasoning (top-down logic) contrasts with inductive reasoning (bottom-up logic) in the following way: In deductive reasoning, a conclusion is reached from general statements, but in inductive reasoning the conclusion is reached from specific examples. Note, however, that the inductive reasoning mentioned here is not the same as induction used in mathematical proofs. That mathematical induction is actually a form of deductive reasoning.

Simple Example

An example of a deductive argument:

1. All men are mortal.
2. John is a man.
3. Therefore, John is mortal.

The first premise states that all objects classified as "men" have the attribute "mortal". The second premise states that "John" is classified as a "man" – a member of the set "men". The conclusion then states that "John" must be "mortal" because he inherits this attribute from his classification as a "man".

Law of Detachment

The law of detachment (also known as **affirming the antecedent** and **Modus ponens**) is the first form of deductive reasoning. A single conditional statement is made, and a hypothesis (P) is stated. The conclusion (Q) is then deduced from the statement and the hypothesis. The most basic form is listed below:

1. $P \rightarrow Q$ (conditional statement)
2. P (hypothesis stated)
3. Q (conclusion deduced)

In deductive reasoning, we can conclude Q from P by using the law of detachment.[3] However, if the conclusion (Q) is given instead of the hypothesis (P) then there is no valid conclusion.

The following is an example of an argument using the law of detachment in the form of an if-then statement:

1. If an angle $A>90°$, then A is an obtuse angle.
2. $A=120°$
3. A is an obtuse angle.

Since the measurement of angle A is greater than 90°, we can deduce that A is an obtuse angle.

Law of Syllogism

The law of syllogism takes two conditional statements and forms a conclusion by combining the hypothesis of one statement with the conclusion of another. Here is the general form, with the true premise P:

1. $P \to Q$
2. $Q \to R$
3. Therefore, $P \to R$.

The following is an example:

1. If Larry is sick, then he will be absent from school.
2. If Larry is absent, then he will miss his classwork.
3. If Larry is sick, then he will miss his classwork.

We deduced the final statement by combining the hypothesis of the first statement with the conclusion of the second statement. We also conclude that this could be a false statement.

Deductive Logic: Validity and Soundness

Deductive arguments are evaluated in terms of their *validity* and *soundness*. It is possible to have a deductive argument that is logically valid but is not sound.

An argument is valid if it is impossible for its premises to be true while its conclusion is false. In other words, the conclusion must be true if the premises, whatever they may be, are true. An argument can be valid even though the premises are false.

An argument is sound if it is valid and the premises are true.

The following is an example of an argument that is *valid*, but not *sound*:

1. Everyone who eats steak is a quarterback.
2. John eats steak.
3. Therefore, John is a quarterback.

The example's first premise is false – there are people who eat steak and are not quarterbacks – but the conclusion must be true, so long as the premises are true (i.e. it is impossible for the premises to be true and the conclusion false). Therefore the argument is valid, but not sound.

In this example, the first statement uses categorical reasoning, saying that all steak-eaters are definitely quarterbacks. This theory of deductive reasoning – also known as term logic – was developed by Aristotle, but was superseded by propositional (sentential) logic and predicate logic.

Deductive reasoning can be contrasted with inductive reasoning, in regards to validity and soundness. In cases of inductive reasoning, even though the premises are true and the argument is "valid", it is possible for the conclusion to be false (determined to be false with a counterexample or other means).

Hume's Skepticism

Philosopher David Hume presented grounds to doubt deduction by questioning induction. Hume's problem of induction starts by suggesting that the use of even the simplest forms of *induction* simply cannot be justified by inductive reasoning itself. Moreover, induction cannot be justified by deduction either. Therefore, induction cannot be justified rationally. Consequently, if induction is not yet justified, then deduction seems to be left to rationally justify itself – an objectionable conclusion to Hume.

Deductive reasoning and Education

Deductive reasoning is generally thought of as a skill that develops without any formal teaching or training. As a result of this belief, deductive reasoning skills are not taught in secondary schools, where students are expected to use reasoning more often and at a higher level.[4] It is in high school, for example, that students have an abrupt introduction to mathematical proofs – which rely heavily on deductive reasoning.[4]

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- Deductive reasoning (https://inpho.cogs.indiana.edu/idea/636) at the Indiana Philosophy Ontology Project
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Inductive reasoning

Inductive reasoning, also known as **induction** or informally **"bottom-up" logic**,[1] is a kind of reasoning that constructs or evaluates general propositions that are derived from specific examples. Inductive reasoning contrasts with deductive reasoning, in which specific examples are derived from general propositions.

Definition

The philosophical definition of inductive reasoning is much more nuanced than simple progression from particular/individual instances to broader generalizations. Rather, the premises of an inductive logical argument indicate some degree of support (inductive probability) for the conclusion but do not entail it; that is, they suggest truth but do not ensure it. In this manner, there is the possibility of moving from generalizations to individual instances. Inductive reasoning consists of inferring general principles or rules from specific facts. A well-known laboratory example of inductive reasoning works like a guessing game. The participants are shown cards that contain figures differing in several ways, such as shape, number, and color. On each trial, they are given two cards and asked to choose the one that represents a particular concept. After they choose a card, the researcher says "right" or "wrong."[2]

Though many dictionaries define inductive reasoning as reasoning that derives general principles from specific observations, this usage is outdated.[3]

Description

Inductive reasoning is probabilistic; it only states that, given the premises, the conclusion is *probable*.

A statistical syllogism is an example of inductive reasoning:

1. 90% of humans are right-handed.
2. Joe is a human.
3. Therefore, the probability that Joe is right-handed is 90% (therefore, if we are required to guess we will choose "right-handed" in the absence of any other evidence).

As a stronger example:

> 100% of life forms that we know of depend on liquid water to exist.
>
> Therefore, if we discover a new life form it will probably depend on liquid water to exist.

This argument could have been made every time a new life form was found, and would have been correct every time. While it is possible that in the future a life form that does not require water will be discovered, in the absence of other factors (e.g. if it were from another planet) then the conclusion is probably correct as it has been in the past.

As a result, the argument may be stated less formally as:

> All life forms that we know of depend on liquid water to exist.
>
> All life depends on liquid water to exist.

Inductive vs. deductive reasoning

Unlike deductive arguments, inductive reasoning allows for the possibility that the conclusion is false, even if all of the premises are true.[4] Instead of being valid or invalid, inductive arguments are either *strong* or *weak*, which describes how *probable* it is that the conclusion is true.[5]

A classical example of an incorrect inductive argument was presented by John Vickers:

> All of the swans we have seen are white.
>
> Therefore, all swans are white.

Note that this definition of *inductive* reasoning excludes mathematical induction, which is a form of *deductive* reasoning.

Induction

Inductive reasoning has been criticized by thinkers as diverse as Sextus Empiricus[6] and Karl Popper.[7]

The classic philosophical treatment of the problem of induction was given by the Scottish philosopher David Hume. Hume highlighted the fact that our everyday habits of mind depend on drawing uncertain conclusions from our relatively limited experiences rather than on deductively valid arguments. For example, we believe that bread will nourish us because it has done so in the past, despite no guarantee that it will do so. Hume argued that it is impossible to justify inductive reasoning: specifically, that it cannot be justified deductively, so our only option is to justify it inductively. Since this is circular he concluded that it is impossible to justify induction.[8]

However, Hume then stated that even if induction were proved unreliable, we would still have to rely on it. So instead of a position of severe skepticism, Hume advocated a practical skepticism based on common sense, where the inevitability of induction is accepted.[9]

Bias

Inductive reasoning is also known as hypothesis construction because any conclusions made are based on current knowledge and predictions. As with deductive arguments, biases can distort the proper application of inductive argument, thereby preventing the reasoner from forming the most logical conclusion based on the clues. Examples of these biases include the availability heuristic, confirmation bias, and the predictable-world bias.

The availability heuristic causes the reasoner to depend primarily upon information that is readily available to him/her. People have a tendency to rely on information that is easily accessible in the world around them. For example, in surveys, when people are asked to estimate the percentage of people who died from various causes, most respondents would choose the causes that have been most prevalent in the media such as terrorism, and murders, and airplane accidents rather than causes such as disease and traffic accidents, which have been technically "less accessible" to the individual since they are not emphasized as heavily in the world around him/her.

The confirmation bias is based on the natural tendency to confirm rather than to deny a current hypothesis. Research has demonstrated that people are inclined to seek solutions to problems that are more consistent with known hypotheses rather than attempt to refute those hypotheses. Often, in experiments, subjects will ask questions that seek answers that fit established hypotheses, thus confirming these hypotheses. For example, if it is hypothesized that Sally is a sociable individual, subjects will naturally seek to confirm the premise by asking questions that would produce answers confirming that Sally is in fact a sociable individual.

The predictable-world bias revolves around the inclination to perceive order where it has not been proved to exist. A major aspect of this bias is superstition, which is derived from the inability to acknowledge that coincidences are merely coincidences. Gambling, for example, is one of the most obvious forms of predictable-world bias. Gamblers often begin to think that they see patterns in the outcomes and, therefore, believe that they are able to predict outcomes based upon what they have witnessed. In reality, however, the outcomes of these games are difficult, if not impossible to predict. The perception of order arises from wishful thinking. Since people constantly seek some type of order to explain or justify their beliefs and experiences, it is difficult for them to acknowledge that the perceived or assumed order may be entirely different from that they believe they are experiencing.[10]

Types

Generalization

A generalization (more accurately, an *inductive generalization*) proceeds from a premise about a sample to a conclusion about the population.

> The proportion Q of the sample has attribute A.
>
> Therefore:
>
> The proportion Q of the population has attribute A.

Example

There are 20 balls—either black or white—in an urn. To estimate their respective numbers, you draw a sample of four balls and find that three are black and one is white. A good inductive generalization would be that there are 15 black, and five white, balls in the urn.

How much the premises support the conclusion depends upon (a) the number in the sample group, (b) the number in the population, and (c) the degree to which the sample represents the population (which may be achieved by taking a random sample). The hasty generalization and the biased sample are generalization fallacies.

Statistical syllogism

A statistical syllogism proceeds from a generalization to a conclusion about an individual.

A proportion Q of population P has attribute A.

An individual X is a member of P.

Therefore:

There is a probability which corresponds to Q that X has A.

The proportion in the first premise would be something like "3/5ths of", "all", "few", etc. Two dicto simpliciter fallacies can occur in statistical syllogisms: "accident" and "converse accident".

Simple induction

Simple induction proceeds from a premise about a sample group to a conclusion about another individual.

Proportion Q of the known instances of population P has attribute A.

Individual I is another member of P.

Therefore:

There is a probability corresponding to Q that I has A.

This is a combination of a generalization and a statistical syllogism, where the conclusion of the generalization is also the first premise of the statistical syllogism.

Argument from analogy

The process of analogical inference involves noting the shared properties of two or more things, and from this basis inferring that they also share some further property:[11]

P and Q are similar in respect to properties a, b, and c.

Object P has been observed to have further property x.

Therefore, Q probably has property x also.

Analogical reasoning is very frequent in common sense, science, philosophy and the humanities, but sometimes it is accepted only as an auxiliary method. A refined approach is case-based reasoning. For more information on inferences by analogy, see Juthe, 2005 [12].

Causal inference

A causal inference draws a conclusion about a causal connection based on the conditions of the occurrence of an effect. Premises about the correlation of two things can indicate a causal relationship between them, but additional factors must be confirmed to establish the exact form of the causal relationship.

Prediction

A prediction draws a conclusion about a future individual from a past sample.

Proportion Q of observed members of group G have had attribute A.

Therefore:

There is a probability corresponding to Q that other members of group G will have attribute A when next observed.

Bayesian inference

As a logic of induction rather than a theory of belief, Bayesian inference does not determine which beliefs are *a priori* rational, but rather determines how we should rationally change the beliefs we have when presented with evidence. We begin by committing to a prior probability for a hypothesis based on logic or previous experience, and when faced with evidence, we adjust the strength of our belief in that hypothesis in a precise manner using Bayesian logic.

Inductive inference

Around 1960, Ray Solomonoff founded the theory of universal inductive inference, the theory of prediction based on observations; for example, predicting the next symbol based upon a given series of symbols. This is a formal inductive framework that combines algorithmic information theory with the Bayesian framework. Universal inductive inference is based on solid philosophical foundations[13] and can be considered as a mathematically formalized Occam's razor. Fundamental ingredients of the theory are the concepts of algorithmic probability and Kolmogorov complexity.

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- *The Mind, Limber* (http://dudespaper.com/the-mind-limber.html) An article which employs the film The Big Lebowski to explain the value of inductive reasoning.

Cogency

Wikipedia does not have an encyclopedic article for **Cogency** *(search results* [1]*)*.

You may want to read Wiktionary's entry on "**cogency**" instead.

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Semantics

Semantics (from Greek: sēmantikós)[1][2] is the study of meaning. It focuses on the relation between *signifiers*, such as words, phrases, signs, and symbols, and what they stand for, their denotata.

Linguistic semantics is the study of meaning that is used to understand human expression through language. Other forms of semantics include the semantics of programming languages, formal logics, and semiotics.

The word *semantics* itself denotes a range of ideas, from the popular to the highly technical. It is often used in ordinary language to denote a problem of understanding that comes down to word selection or connotation. This problem of understanding has been the subject of many formal enquiries, over a long period of time, most notably in the field of formal semantics. In linguistics, it is the study of interpretation of signs or symbols as used by agents or communities within particular circumstances and contexts.[3] Within this view, sounds, facial expressions, body language, and proxemics have semantic (meaningful) content, and each has several branches of study. In written language, such things as paragraph structure and punctuation have semantic content; in other forms of language, there is other semantic content.[3]

The formal study of semantics intersects with many other fields of inquiry, including lexicology, syntax, pragmatics, etymology and others, although semantics is a well-defined field in its own right, often with synthetic properties.[4] In philosophy of language, semantics and reference are closely connected. Further related fields include philology, communication, and semiotics. The formal study of semantics is therefore complex.

Semantics contrasts with syntax, the study of the combinatorics of units of a language (without reference to their meaning), and pragmatics, the study of the relationships between the symbols of a language, their meaning, and the users of the language.[5]

In international scientific vocabulary semantics is also called *semasiology*.

Linguistics

In linguistics, **semantics** is the subfield that is devoted to the study of meaning, as inherent at the levels of words, phrases, sentences, and larger units of discourse (termed *texts*). The basic area of study is the meaning of signs, and the study of relations between different linguistic units and compounds: homonymy, synonymy, antonymy, hypernymy, hyponymy, meronymy, metonymy, holonymy, paronyms. A key concern is how meaning attaches to larger chunks of text, possibly as a result of the composition from smaller units of meaning. Traditionally, semantics has included the study of *sense* and denotative *reference*, truth conditions, argument structure, thematic roles, discourse analysis, and the linkage of all of these to syntax.

Montague grammar

In the late 1960s, Richard Montague proposed a system for defining semantic entries in the lexicon in terms of the lambda calculus. In these terms, the syntactic parse of the sentence *John ate every bagel* would consist of a subject (*John*) and a predicate (*ate every bagel*); Montague showed that the meaning of the sentence as a whole could be decomposed into the meanings of its parts and relatively few rules of combination. The logical predicate thus obtained would be elaborated further, e.g. using truth theory models, which ultimately relate meanings to a set of Tarskiian universals, which may lie outside the logic. The notion of such meaning atoms or primitives is basic to the language of thought hypothesis from the 1970s.

Despite its elegance, Montague grammar was limited by the context-dependent variability in word sense, and led to several attempts at incorporating context, such as:

- Situation semantics (1980s): truth-values are incomplete, they get assigned based on context
- Generative lexicon (1990s): categories (types) are incomplete, and get assigned based on context

Dynamic turn in semantics

In Chomskyan linguistics there was no mechanism for the learning of semantic relations, and the nativist view considered all semantic notions as inborn. Thus, even novel concepts were proposed to have been dormant in some sense. This view was also thought unable to address many issues such as metaphor or associative meanings, and semantic change, where meanings within a linguistic community change over time, and qualia or subjective experience. Another issue not addressed by the nativist model was how perceptual cues are combined in thought, e.g. in mental rotation.[6]

This view of semantics, as an innate finite meaning inherent in a lexical unit that can be composed to generate meanings for larger chunks of discourse, is now being fiercely debated in the emerging domain of cognitive linguistics[7] and also in the non-Fodorian camp in philosophy of language.[8] The challenge is motivated by:

- factors internal to language, such as the problem of resolving indexical or anaphora (e.g. *this x*, *him*, *last week*). In these situations *context* serves as the input, but the interpreted utterance also modifies the context, so it is also the output. Thus, the interpretation is necessarily dynamic and the meaning of sentences is viewed as context change potentials instead of propositions.
- factors external to language, i.e. language is not a set of labels stuck on things, but "a toolbox, the importance of whose elements lie in the way they function rather than their attachments to things."[8] This view reflects the position of the later Wittgenstein and his famous *game* example, and is related to the positions of Quine, Davidson, and others.

A concrete example of the latter phenomenon is semantic underspecification – meanings are not complete without some elements of context. To take an example of one word, *red*, its meaning in a phrase such as *red book* is similar to many other usages, and can be viewed as compositional.[9] However, the colours implied in phrases such as *red wine* (very dark), and *red hair* (coppery), or *red soil*, or *red skin* are very different. Indeed, these colours by themselves would not be called *red* by native speakers. These instances are contrastive, so *red wine* is so called only in comparison with the other kind of wine (which also is not *white* for the same reasons). This view goes back to de Saussure:

> Each of a set of synonyms like *redouter* ('to dread'), *craindre* ('to fear'), *avoir peur* ('to be afraid') has its particular value only because they stand in contrast with one another. No word has a value that can be identified independently of what else is in its vicinity.[10]

and may go back to earlier Indian views on language, especially the Nyaya view of words as indicators and not carriers of meaning.[11]

An attempt to defend a system based on propositional meaning for semantic underspecification can be found in the generative lexicon model of James Pustejovsky, who extends contextual operations (based on type shifting) into the lexicon. Thus meanings are generated on the fly based on finite context.

Prototype theory

Another set of concepts related to fuzziness in semantics is based on prototypes. The work of Eleanor Rosch in the 1970s led to a view that natural categories are not characterizable in terms of necessary and sufficient conditions, but are graded (fuzzy at their boundaries) and inconsistent as to the status of their constituent members. One may compare it with Jung's archetype, though the concept of archetype sticks to static concept. Some post-structuralists are against the fixed or static meaning of the words. Derrida, following Nietzsche, talked about slippages in fixed meanings. Here are some examples from Bangla fuzzy words [12][13]

Systems of categories are not objectively *out there* in the world but are rooted in people's experience. These categories evolve as learned concepts of the world – meaning is not an objective truth, but a subjective construct, learned from experience, and language arises out of the "grounding of our conceptual systems in shared embodiment and bodily experience".[14] A corollary of this is that the conceptual categories (i.e. the lexicon) will not be identical for different cultures, or indeed, for every individual in the same culture. This leads to another debate (see the Sapir–Whorf hypothesis or Eskimo words for snow).

Theories in semantics

Model theoretic semantics

Originates from Montague's work (see above). A highly formalized theory of natural language semantics in which expressions are assigned denotations (meanings) such as individuals, truth values, or functions from one of these to another. The truth of a sentence, and more interestingly, its logical relation to other sentences, is then evaluated relative to a model.

Formal (or truth-conditional) semantics

Pioneered by the philosopher Donald Davidson, another formalized theory, which aims to associate each natural language sentence with a meta-language description of the conditions under which it is true, for example: `Snow is white' is true if and only if snow is white. The challenge is to arrive at the truth conditions for any sentences from fixed meanings assigned to the individual words and fixed rules for how to combine them. In practice, truth-conditional semantics is similar to model-theoretic semantics; conceptually, however, they differ in that truth-conditional semantics seeks to connect language with statements about the real world (in the form of meta-language statements), rather than with abstract models.

Lexical and conceptual semantics

This theory is an effort to explain properties of argument structure. The assumption behind this theory is that syntactic properties of phrases reflect the meanings of the words that head them.[15] With this theory, linguists can better deal with the fact that subtle differences in word meaning correlate with other differences in the syntactic structure that the word appears in.[15] The way this is gone about is by looking at the internal structure of words.[16] These small parts that make up the internal structure of words are termed *semantic primitives*.[16]

Lexical semantics

A linguistic theory that investigates word meaning. This theory understands that the meaning of a word is fully reflected by its context. Here, the meaning of a word is constituted by its contextual relations.[17] Therefore, a distinction between degrees of participation as well as modes of participation are made.[17] In order to accomplish this distinction any part of a sentence that bears a meaning and combines with the meanings of other constituents is labeled as a semantic constituent. Semantic constituents that cannot be broken down into more elementary constituents are labeled minimal semantic constituents.[17]

Computational semantics

Computational semantics is focused on the processing of linguistic meaning. In order to do this concrete algorithms and architectures are described. Within this framework the algorithms and architectures are also analyzed in terms of decidability, time/space complexity, data structures they require and communication protocols.[18]

Computer science

In computer science, the term *semantics* refers to the meaning of languages, as opposed to their form (syntax). According to Euzenat, semantics "provides the rules for interpreting the syntax which do not provide the meaning directly but constrains the possible interpretations of what is declared."[19] In other words, semantics is about interpretation of an expression. Additionally, the term is applied to certain types of data structures specifically designed and used for representing information content.

Programming languages

The semantics of programming languages and other languages is an important issue and area of study in computer science. Like the syntax of a language, its semantics can be defined exactly.

For instance, the following statements use different syntaxes, but cause the same instructions to be executed:

Statement	Programming languages
`x += y`	C, C++, C#, Java, Perl, Python, Ruby, PHP, etc.
`x := x + y`	ALGOL, BCPL, Simula, ALGOL 68, SETL, Pascal, Smalltalk, Modula-2, Ada, Standard ML, OCaml, Eiffel, Object Pascal (Delphi), Oberon, Dylan, VHDL, etc.
`ADD x, y`	Assembly languages: Intel 8086
`LET X = X + Y`	BASIC: early
`x = x + y`	BASIC: most dialects; Fortran, MATLAB
`Set x = x + y`	Caché ObjectScript
`ADD Y TO X GIVING X`	COBOL
`(incf x y)`	Common Lisp

Generally these operations would all perform an arithmetical addition of 'y' to 'x' and store the result in a variable called 'x'.

Various ways have been developed to describe the semantics of programming languages formally, building on mathematical logic:[20]

- Operational semantics: The meaning of a construct is specified by the computation it induces when it is executed on a machine. In particular, it is of interest *how* the effect of a computation is produced.
- Denotational semantics: Meanings are modelled by mathematical objects that represent the effect of executing the constructs. Thus *only* the effect is of interest, not how it is obtained.
- Axiomatic semantics: Specific properties of the effect of executing the constructs are expressed as *assertions*. Thus there may be aspects of the executions that are ignored.

Semantic models

Terms such as *semantic network* and *semantic data model* are used to describe particular types of data models characterized by the use of directed graphs in which the vertices denote concepts or entities in the world, and the arcs denote relationships between them.

The Semantic Web refers to the extension of the World Wide Web via embedding added semantic metadata, using semantic data modelling techniques such as Resource Description Framework (RDF) and Web Ontology Language (OWL).

Psychology

In psychology, *semantic memory* is memory for meaning – in other words, the aspect of memory that preserves only the *gist*, the general significance, of remembered experience – while episodic memory is memory for the ephemeral details – the individual features, or the unique particulars of experience. Word meaning is measured by the company they keep, i.e. the relationships among words themselves in a semantic network. The memories may be transferred intergenerationally or isolated in one generation due to a cultural disruption. Different generations may have different experiences at similar points in their own time-lines. This may then create a vertically heterogeneous semantic net for certain words in an otherwise homogeneous culture.[21] In a network created by people analyzing their understanding of the word (such as Wordnet) the links and decomposition structures of the network are few in number and kind, and include *part of*, *kind of*, and similar links. In automated ontologies the links are computed vectors without explicit meaning. Various automated technologies are being developed to compute the meaning of words: latent semantic indexing and support vector machines as well as natural language processing, neural networks and predicate calculus techniques.

Ideasthesia is a rare psychological phenomenon that in certain individuals associates semantic and sensory representations. Activation of a concept (e.g., that of the letter *A*) evokes sensory-like experiences (e.g., of red color).

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Mathematical model

A **mathematical model** is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed **mathematical modelling**. Mathematical models are used not only in the natural sciences (such as physics, biology, earth science, meteorology) and engineering disciplines (e.g. computer science, artificial intelligence), but also in the social sciences (such as economics, psychology, sociology and political science); physicists, engineers, statisticians, operations research analysts and economists use mathematical models most extensively. A model may help to explain a system and to study the effects of different components, and to make predictions about behaviour.

Mathematical models can take many forms, including but not limited to dynamical systems, statistical models, differential equations, or game theoretic models. These and other types of models can overlap, with a given model involving a variety of abstract structures. In general, mathematical models may include logical models, as far as logic is taken as a part of mathematics. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed.

Examples of mathematical models

- Many everyday activities carried out without a thought are uses of mathematical models. A geographical map projection of a region of the earth onto a small, plane surface is a model[1] which can be used for many purposes such as planning travel.
- Another simple activity is predicting the position of a vehicle from its initial position, direction and speed of travel, using the equation that distance travelled is the product of time and speed. This is known as dead reckoning when used more formally. Mathematical modelling in this way does not necessarily require formal mathematics; animals have been shown to use dead reckoning.[2][3]
- *Population Growth*. A simple (though approximate) model of population growth is the Malthusian growth model. A slightly more realistic and largely used population growth model is the logistic function, and its extensions.
- *Model of a particle in a potential-field*. In this model we consider a particle as being a point of mass which describes a trajectory in space which is modeled by a function giving its coordinates in space as a function of time. The potential field is given by a function $V : \mathbf{R}^3 \to \mathbf{R}$ and the trajectory is a solution of the differential equation

  $$m\frac{d^2}{dt^2}x(t) = -\nabla V x(t)$$

  Note this model assumes the particle is a point mass, which is certainly known to be false in many cases in which we use this model; for example, as a model of planetary motion.

- *Model of rational behavior for a consumer*. In this model we assume a consumer faces a choice of n commodities labeled 1,2,...,n each with a market price $p_1, p_2,..., p_n$. The consumer is assumed to have a *cardinal* utility function U (cardinal in the sense that it assigns numerical values to utilities), depending on the amounts of commodities $x_1, x_2,..., x_n$ consumed. The model further assumes that the consumer has a budget M which is used to purchase a vector $x_1, x_2,..., x_n$ in such a way as to maximize $U(x_1, x_2,..., x_n)$. The problem of rational behavior in this model then becomes an optimization problem, that is:

  $$\max U(x_1, x_2, \ldots, x_n)$$

  subject to:

  $$\sum_{i=1}^{n} p_i x_i \leq M.$$

$$x_i \geq 0 \quad \forall i \in \{1, 2, \ldots, n\}$$

This model has been used in general equilibrium theory, particularly to show existence and Pareto efficiency of economic equilibria. However, the fact that this particular formulation assigns *numerical values* to levels of satisfaction is the source of criticism (and even ridicule). However, it is not an essential ingredient of the theory and again this is an idealization.

- *Neighbour-sensing model* explains the mushroom formation from the initially chaotic fungal network.
- *Computer Science*: models in Computer Networks, data models, surface model,...
- *Mechanics*: movement of rocket model,...

Modeling requires selecting and identifying relevant aspects of a situation in the real world.

Some applications

Since prehistorical times simple models such as maps and pre-designed diagrams have been used.

Often when engineers analyze a system to be controlled or optimized, they use a mathematical model. In analysis, engineers can build a descriptive model of the system as a hypothesis of how the system could work, or try to estimate how an unforeseeable event could affect the system. Similarly, in control of a system, engineers can try out different control approaches in simulations.

A mathematical model usually describes a system by a set of variables and a set of equations that establish relationships between the variables. Variables may be of many types; real or integer numbers, boolean values or strings, for example. The variables represent some properties of the system, for example, measured system outputs often in the form of signals, timing data, counters, and event occurrence (yes/no). The actual model is the set of functions that describe the relations between the different variables.

Building blocks

There are six basic groups of variables namely: decision variables, input variables, state variables, exogenous variables, random variables, and output variables. Since there can be many variables of each type, the variables are generally represented by vectors.

Decision variables are sometimes known as independent variables. Exogenous variables are sometimes known as parameters or constants. The variables are not independent of each other as the state variables are dependent on the decision, input, random, and exogenous variables. Furthermore, the output variables are dependent on the state of the system (represented by the state variables).

Objectives and constraints of the system and its users can be represented as functions of the output variables or state variables. The objective functions will depend on the perspective of the model's user. Depending on the context, an objective function is also known as an index of performance, as it is some measure of interest to the user. Although there is no limit to the number of objective functions and constraints a model can have, using or optimizing the model becomes more involved (computationally) as the number increases.

Classifying mathematical models

Many mathematical models can be classified in some of the following ways:

1. **Linear vs. nonlinear:** Mathematical models are usually composed by variables, which are abstractions of quantities of interest in the described systems, and operators that act on these variables, which can be algebraic operators, functions, differential operators, etc. If all the operators in a mathematical model exhibit linearity, the resulting mathematical model is defined as linear. A model is considered to be nonlinear otherwise.
   The question of linearity and nonlinearity is dependent on context, and linear models may have nonlinear expressions in them. For example, in a statistical linear model, it is assumed that a relationship is linear in the

parameters, but it may be nonlinear in the predictor variables. Similarly, a differential equation is said to be linear if it can be written with linear differential operators, but it can still have nonlinear expressions in it. In a mathematical programming model, if the objective functions and constraints are represented entirely by linear equations, then the model is regarded as a linear model. If one or more of the objective functions or constraints are represented with a nonlinear equation, then the model is known as a nonlinear model.
Nonlinearity, even in fairly simple systems, is often associated with phenomena such as chaos and irreversibility. Although there are exceptions, nonlinear systems and models tend to be more difficult to study than linear ones. A common approach to nonlinear problems is linearization, but this can be problematic if one is trying to study aspects such as irreversibility, which are strongly tied to nonlinearity.
2. **Deterministic vs. probabilistic (stochastic):** A deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables. Therefore, deterministic models perform the same way for a given set of initial conditions. Conversely, in a stochastic model, randomness is present, and variable states are not described by unique values, but rather by probability distributions.
3. **Static vs. dynamic:** A static model does not account for the element of time, while a dynamic model does. Dynamic models typically are represented with difference equations or differential equations.
4. **Discrete vs. Continuous:** A discrete model does not take into account the function of time and usually uses time-advance methods, while a Continuous model does. Continuous models typically are represented with f(t) and the changes are reflected over continuous time intervals.
5. **Deductive, inductive, or floating:** A deductive model is a logical structure based on a theory. An inductive model arises from empirical findings and generalization from them. The floating model rests on neither theory nor observation, but is merely the invocation of expected structure. Application of mathematics in social sciences outside of economics has been criticized for unfounded models.[4] Application of catastrophe theory in science has been characterized as a floating model.[5]

A priori information

Mathematical modeling problems are used often classified into black box or white box models, according to how much a priori information is available of the system. A black-box model is a system of which there is no a priori information available. A white-box model (also called glass box or clear box) is a system where all necessary information is available. Practically all systems are somewhere between the black-box and white-box models, so this concept is useful only as an intuitive guide for deciding which approach to take.

Usually it is preferable to use as much a priori information as possible to make the model more accurate. Therefore the white-box models are usually considered easier, because if you have used the information correctly, then the model will behave correctly. Often the a priori information comes in forms of knowing the type of functions relating different variables. For example, if we make a model of how a medicine works in a human system, we know that usually the amount of medicine in the blood is an exponentially decaying function. But we are still left with several unknown parameters; how rapidly does the medicine amount decay, and what is the initial amount of medicine in blood? This example is therefore not a completely white-box model. These parameters have to be estimated through some means before one can use the model.

In black-box models one tries to estimate both the functional form of relations between variables and the numerical parameters in those functions. Using a priori information we could end up, for example, with a set of functions that probably could describe the system adequately. If there is no a priori information we would try to use functions as general as possible to cover all different models. An often used approach for black-box models are neural networks which usually do not make assumptions about incoming data. The problem with using a large set of functions to describe a system is that estimating the parameters becomes increasingly difficult when the amount of parameters (and different types of functions) increases.

Subjective information

Sometimes it is useful to incorporate subjective information into a mathematical model. This can be done based on intuition, experience, or expert opinion, or based on convenience of mathematical form. Bayesian statistics provides a theoretical framework for incorporating such subjectivity into a rigorous analysis: one specifies a prior probability distribution (which can be subjective) and then updates this distribution based on empirical data. An example of when such approach would be necessary is a situation in which an experimenter bends a coin slightly and tosses it once, recording whether it comes up heads, and is then given the task of predicting the probability that the next flip comes up heads. After bending the coin, the true probability that the coin will come up heads is unknown, so the experimenter would need to make an arbitrary decision (perhaps by looking at the shape of the coin) about what prior distribution to use. Incorporation of the subjective information is necessary in this case to get an accurate prediction of the probability, since otherwise one would guess 1 or 0 as the probability of the next flip being heads, which would be almost certainly wrong.[6]

Complexity

In general, model complexity involves a trade-off between simplicity and accuracy of the model. Occam's razor is a principle particularly relevant to modeling; the essential idea being that among models with roughly equal predictive power, the simplest one is the most desirable. While added complexity usually improves the realism of a model, it can make the model difficult to understand and analyze, and can also pose computational problems, including numerical instability. Thomas Kuhn argues that as science progresses, explanations tend to become more complex before a Paradigm shift offers radical simplification.

For example, when modeling the flight of an aircraft, we could embed each mechanical part of the aircraft into our model and would thus acquire an almost white-box model of the system. However, the computational cost of adding such a huge amount of detail would effectively inhibit the usage of such a model. Additionally, the uncertainty would increase due to an overly complex system, because each separate part induces some amount of variance into the model. It is therefore usually appropriate to make some approximations to reduce the model to a sensible size. Engineers often can accept some approximations in order to get a more robust and simple model. For example Newton's classical mechanics is an approximated model of the real world. Still, Newton's model is quite sufficient for most ordinary-life situations, that is, as long as particle speeds are well below the speed of light, and we study macro-particles only.

Training

Any model which is not pure white-box contains some parameters that can be used to fit the model to the system it is intended to describe. If the modeling is done by a neural network, the optimization of parameters is called *training*. In more conventional modeling through explicitly given mathematical functions, parameters are determined by curve fitting.

Model evaluation

A crucial part of the modeling process is the evaluation of whether or not a given mathematical model describes a system accurately. This question can be difficult to answer as it involves several different types of evaluation.

Fit to empirical data

Usually the easiest part of model evaluation is checking whether a model fits experimental measurements or other empirical data. In models with parameters, a common approach to test this fit is to split the data into two disjoint subsets: training data and verification data. The training data are used to estimate the model parameters. An accurate model will closely match the verification data even though these data were not used to set the model's parameters. This practice is referred to as cross-validation in statistics.

Defining a metric to measure distances between observed and predicted data is a useful tool of assessing model fit. In statistics, decision theory, and some economic models, a loss function plays a similar role.

While it is rather straightforward to test the appropriateness of parameters, it can be more difficult to test the validity of the general mathematical form of a model. In general, more mathematical tools have been developed to test the fit of statistical models than models involving differential equations. Tools from non-parametric statistics can sometimes be used to evaluate how well the data fit a known distribution or to come up with a general model that makes only minimal assumptions about the model's mathematical form.

Scope of the model

Assessing the scope of a model, that is, determining what situations the model is applicable to, can be less straightforward. If the model was constructed based on a set of data, one must determine for which systems or situations the known data is a "typical" set of data.

The question of whether the model describes well the properties of the system between data points is called interpolation, and the same question for events or data points outside the observed data is called extrapolation.

As an example of the typical limitations of the scope of a model, in evaluating Newtonian classical mechanics, we can note that Newton made his measurements without advanced equipment, so he could not measure properties of particles travelling at speeds close to the speed of light. Likewise, he did not measure the movements of molecules and other small particles, but macro particles only. It is then not surprising that his model does not extrapolate well into these domains, even though his model is quite sufficient for ordinary life physics.

Philosophical considerations

Many types of modeling implicitly involve claims about causality. This is usually (but not always) true of models involving differential equations. As the purpose of modeling is to increase our understanding of the world, the validity of a model rests not only on its fit to empirical observations, but also on its ability to extrapolate to situations or data beyond those originally described in the model. One can argue that a model is worthless unless it provides some insight which goes beyond what is already known from direct investigation of the phenomenon being studied.

An example of such criticism is the argument that the mathematical models of Optimal foraging theory do not offer insight that goes beyond the common-sense conclusions of evolution and other basic principles of ecology.[7]

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Consequent

A **consequent** is the second half of a hypothetical proposition. In the standard form of such a proposition, it is the part that follows "then". In an implication, if ϕ implies ψ then ϕ is called the antecedent and ψ is called the **consequent**.[1]

Examples:

- If P, then Q.

Q is the consequent of this hypothetical proposition.

- If X is a mammal, then X is an animal.

Here, "X is an animal" is the consequent.

- If computers can think, then they are alive.

"They are alive" is the consequent.

The consequent in a hypothetical proposition is not necessarily a consequence of the antecedent.

- If monkeys are purple, then fish speak Klingon.

"Fish speak Klingon" is the consequent here, but intuitively is not a consequence of (nor does it have anything to do with) the claim made in the antecedent that "monkeys are purple".

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Antecedent (logic)

An **antecedent** is the first half of a hypothetical proposition.

Examples:

- If **P**, then **Q**.

This is a nonlogical formulation of a hypothetical proposition. In this case, the antecedent is **P**, and the consequent is **Q**. In an implication, if ϕ implies ψ then ϕ is called the **antecedent** and ψ is called the consequent.[1]

- If X is a man, then X is mortal.

"X is a man" is the antecedent for this proposition.

- If men have walked on the moon, then I am the king of France.

Here, "men have walked on the moon" is the antecedent.

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Consistency, validity, soundness, and completeness

Formal system

A **formal system** is, broadly defined as any well-defined system of abstract thought based on the model of mathematics. Euclid's *Elements* is often held to be the first formal system and displays the characteristic of a formal system. The entailment of the system by its logical foundation is what distinguishes a formal system from others which may have some basis in an abstract model. Often the formal system will be the basis for or even identified with a larger theory or field (e.g. Euclidean geometry) consistent with the usage in modern mathematics such as model theory. A formal system need not be mathematical as such, Spinoza's Ethics for example imitates the form of Euclid's Elements.

Each formal system has a formal language, which is composed by primitive symbols. These symbols act on certain rules of formation and are developed by inference from a set of axioms. The system thus consists of any number of formulas built up through finite combinations of the primitive symbols—combinations that are formed from the axioms in accordance with the stated rules.[1]

Formal systems in mathematics consist of the following elements:

1. A finite set of symbols (i.e. the alphabet), that can be used for constructing formulas (i.e. finite strings of symbols).
2. A grammar, which tells how well-formed formulas (abbreviated *wff*) are constructed out of the symbols in the alphabet. It is usually required that there be a decision procedure for deciding whether a formula is well formed or not.
3. A set of axioms or axiom schemata: each axiom must be a wff.
4. A set of inference rules.

A formal system is said to be recursive (i.e. effective) if the set of axioms and the set of inference rules are decidable sets or semidecidable sets, according to context.

Some theorists use the term *formalism* as a rough synonym for *formal system*, but the term is also used to refer to a particular style of *notation*, for example, Paul Dirac's bra-ket notation.

Related subjects

Logical system

A *logical system* or, for short, *logic*, is a formal system together with a form of semantics, usually in the form of model-theoretic interpretation, which assigns truth values to sentences of the formal language, that is, formulae that contain no free variables. A logic is sound if all sentences that can be derived are true in the interpretation, and complete if, conversely, all true sentences can be derived.

Formal proofs

Formal proofs are sequences of wffs. For a wff to qualify as part of a proof, it might either be an axiom or be the product of applying an inference rule on previous wffs in the proof sequence. The last wff in the sequence is recognized as a theorem.

The point of view that generating formal proofs is all there is to mathematics is often called *formalism*. David Hilbert founded metamathematics as a discipline for discussing formal systems. Any language that one uses to talk about a formal system is called a *metalanguage*. The metalanguage may be nothing more than ordinary natural language, or it may be partially formalized itself, but it is generally less completely formalized than the formal language component of the formal system under examination, which is then called the *object language*, that is, the object of the discussion in question.

Once a formal system is given, one can define the set of theorems which can be proved inside the formal system. This set consists of all wffs for which there is a proof. Thus all axioms are considered theorems. Unlike the grammar for wffs, there is no guarantee that there will be a decision procedure for deciding whether a given wff is a theorem or not. The notion of *theorem* just defined should not be confused with *theorems about the formal system*, which, in order to avoid confusion, are usually called metatheorems.

Formal language

In mathematics, logic, and computer science, a formal language is a language that is defined by precise mathematical or machine processable formulas. Like languages in linguistics, formal languages generally have two aspects:

- the syntax of a language is what the language looks like (more formally: the set of possible expressions that are valid utterances in the language)
- the semantics of a language are what the utterances of the language mean (which is formalized in various ways, depending on the type of language in question)

A special branch of mathematics and computer science exists that is devoted exclusively to the theory of language syntax: formal language theory. In formal language theory, a language is nothing more than its syntax; questions of semantics are not included in this specialty.

Formal grammar

In computer science and linguistics a formal grammar is a precise description of a formal language: a set of strings. The two main categories of formal grammar are that of generative grammars, which are sets of rules for how strings in a language can be generated, and that of analytic grammars, which are sets of rules for how a string can be analyzed to determine whether it is a member of the language. In short, an analytic grammar describes how to *recognize* when strings are members in the set, whereas a generative grammar describes how to *write* only those strings in the set.

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Consistency

In logic, a **consistent** theory is one that does not contain a contradiction.[1] The lack of contradiction can be defined in either semantic or syntactic terms. The semantic definition states that a theory is consistent if and only if it has a model, i.e. there exists an interpretation under which all formulas in the theory are true. This is the sense used in traditional Aristotelian logic, although in contemporary mathematical logic the term **satisfiable** is used instead. The syntactic definition states that a theory is consistent if and only if there is no formula P such that both P and its negation are provable from the axioms of the theory under its associated deductive system.

If these semantic and syntactic definitions are equivalent for a particular logic, the logic is **complete**. The completeness of sentential calculus was proved by Paul Bernays in 1918[2] and Emil Post in 1921,[3] while the completeness of predicate calculus was proved by Kurt Gödel in 1930,[4] and consistency proofs for arithmetics restricted with respect to the induction axiom schema were proved by Ackermann (1924), von Neumann (1927) and Herbrand (1931).[5] Stronger logics, such as second-order logic, are not complete.

A **consistency proof** is a mathematical proof that a particular theory is consistent. The early development of mathematical proof theory was driven by the desire to provide finitary consistency proofs for all of mathematics as part of Hilbert's program. Hilbert's program was strongly impacted by incompleteness theorems, which showed that sufficiently strong proof theories cannot prove their own consistency (provided that they are in fact consistent).

Although consistency can be proved by means of model theory, it is often done in a purely syntactical way, without any need to reference some model of the logic. The cut-elimination (or equivalently the normalization of the underlying calculus if there is one) implies the consistency of the calculus: since there is obviously no cut-free proof of falsity, there is no contradiction in general.

Consistency and completeness in arithmetic and set theory

In theories of arithmetic, such as Peano arithmetic, there is an intricate relationship between the consistency of the theory and its completeness. A theory is complete if, for every formula φ in its language, at least one of φ or $\neg \varphi$ is a logical consequence of the theory.

Presburger arithmetic is an axiom system for the natural numbers under addition. It is both consistent and complete.

Gödel's incompleteness theorems show that any sufficiently strong effective theory of arithmetic cannot be both complete and consistent. Gödel's theorem applies to the theories of Peano arithmetic (PA) and Primitive recursive arithmetic (PRA), but not to Presburger arithmetic.

Moreover, Gödel's second incompleteness theorem shows that the consistency of sufficiently strong effective theories of arithmetic can be tested in a particular way. Such a theory is consistent if and only if it does *not* prove a particular sentence, called the Gödel sentence of the theory, which is a formalized statement of the claim that the theory is indeed consistent. Thus the consistency of a sufficiently strong, effective, consistent theory of arithmetic can never be proven in that system itself. The same result is true for effective theories that can describe a strong enough fragment of arithmetic – including set theories such as Zermelo–Fraenkel set theory. These set theories cannot prove their own Gödel sentences – provided that they are consistent, which is generally believed.

Because consistency of ZF is not provable in ZF, the weaker notion **relative consistency** is interesting in set theory (and in other sufficiently expressive axiomatic systems). If *T* is a theory and *A* is an additional axiom, *T* + *A* is said to be consistent relative to *T* (or simply that *A* is consistent with *T*) if it can be proved that if *T* is consistent then *T* + *A* is consistent. If both *A* and ¬*A* are consistent with *T*, then *A* is said to be independent of *T*.

First-order logic

A set of formulas Φ in first-order logic is **consistent** (written Con Φ) if and only if there is no formula ϕ such that $\Phi \vdash \phi$ and $\Phi \vdash \neg\phi$. Otherwise Φ is **inconsistent** and is written Inc Φ.

Φ is said to be **simply consistent** if and only if for no formula ϕ of Φ, both ϕ and the negation of ϕ are theorems of Φ.

Φ is said to be **absolutely consistent** or **Post consistent** if and only if at least one formula of Φ is not a theorem of Φ.

Φ is said to be **maximally consistent** if and only if for every formula ϕ , if Con ($\Phi \cup \phi$) then $\phi \in \Phi$.

Φ is said to **contain witnesses** if and only if for every formula of the form $\exists x\phi$ there exists a term t such that $(\exists x\phi \to \phi\frac{t}{x}) \in \Phi$. See First-order logic.

Basic results

1. The following are equivalent:
   1. Inc Φ
   2. For all $\phi,\ \Phi \vdash \phi.$
2. Every satisfiable set of formulas is consistent, where a set of formulas Φ is satisfiable if and only if there exists a model $\mathfrak{I}$ such that $\mathfrak{I} \models \Phi$.
3. For all Φ and ϕ :
   1. if not $\Phi \vdash \phi$, then Con $(\Phi \cup \{\neg\phi\})$;
   2. if Con Φ and $\Phi \vdash \phi$, then Con $(\Phi \cup \{\phi\})$;
   3. if Con Φ, then Con $(\Phi \cup \{\phi\})$ or Con $(\Phi \cup \{\neg\phi\})$.
4. Let Φ be a maximally consistent set of formulas and contain witnesses. For all ϕ and ψ :
   1. if $\Phi \vdash \phi$, then $\phi \in \Phi$,
   2. either $\phi \in \Phi$ or $\neg\phi \in \Phi$,
   3. $(\phi \vee \psi) \in \Phi$ if and only if $\phi \in \Phi$ or $\psi \in \Phi$,
   4. if $(\phi \to \psi) \in \Phi$ and $\phi \in \Phi$, then $\psi \in \Phi$,
   5. $\exists x\phi \in \Phi$ if and only if there is a term t such that $\phi\frac{t}{x} \in \Phi$.

Henkin's theorem

Let Φ be a maximally consistent set of S -formulas containing witnesses.

Define a binary relation $\sim$ on the set of S -terms such that $t_0 \sim t_1$ if and only if $t_0 \equiv t_1 \in \Phi$; and let $\bar{t}$ denote the equivalence class of terms containing t ; and let $T_\Phi := \{\ \bar{t} \mid t \in T^S\}$ where T^S is the set of terms based on the symbol set S .

Define the S -structure $\mathfrak{T}_\Phi$ over T_Φ the **term-structure** corresponding to Φ by:

1. for n -ary $R \in S$, $R^{\mathfrak{T}_\Phi}\overline{t_0}\ldots\overline{t_{n-1}}$ if and only if $Rt_0\ldots t_{n-1} \in \Phi$;
2. for n -ary $f \in S$, $f^{\mathfrak{T}_\Phi}(\overline{t_0}\ldots\overline{t_{n-1}}) := \overline{ft_0\ldots t_{n-1}}$;
3. for $c \in S$, $c^{\mathfrak{T}_\Phi} := \bar{c}$.

Let $\mathfrak{I}_\Phi := (\mathfrak{T}_\Phi, \beta_\Phi)$ be the **term interpretation** associated with Φ, where $\beta_\Phi(x) := \bar{x}$.

For all ϕ , $\mathfrak{I}_\Phi \models \phi$ if and only if $\phi \in \Phi$.

Sketch of proof

There are several things to verify. First, that $\sim$ is an equivalence relation. Then, it needs to be verified that (1), (2), and (3) are well defined. This falls out of the fact that $\sim$ is an equivalence relation and also requires a proof that (1) and (2) are independent of the choice of $t_0, \ldots, t_{n-1}$ class representatives. Finally, $\mathfrak{I}_\Phi \models \Phi$ can be verified by induction on formulas.

Footnotes

[1] Tarski 1946 states it this way: "A deductive theory is called CONSISTENT or NON-CONTRADICTORY if no two asserted statements of this theory contradict each other, or in other words, if of any two contradictory sentences . . . at least one cannot be proved," (p. 135) where Tarski defines *contradictory* as follows: "With the help of the word *not* one forms the NEGATION of any sentence; two sentences, of which the first is a negation of the second, are called CONTRADICTORY SENTENCES" (p. 20). This definition requires a notion of "proof". Gödel in his 1931 defines the notion this way: "The class of *provable formulas* is defined to be the smallest class of formulas that contains the axioms and is closed under the relation "immediate consequence", i.e. formula *c* of *a* and *b* is defined as an *immediate consequence* in terms of *modus ponens* or substitution; cf Gödel 1931 van Heijenoort 1967:601. Tarski defines "proof" informally as "statements follow one another in a definite order according to certain principles . . . and accompanied by considerations intended to establish their validity[true conclusion for all true premises -- Reichenbach 1947:68]" cf Tarski 1946:3. Kleene 1952 defines the notion with respect to either an induction or as to paraphrase) a finite sequence of formulas such that each formula in the sequence is either an axiom or an "immediate consequence" of the preceding formulas; "A *proof is said to be a proof* of *its last formula, and this formula is said to be* (formally) provable *or be a* (formal) theorem" cf Kleene 1952:83.

[2] van Heijenoort 1967:265 states that Bernays determined the *independence* of the axioms of *Principia Mathematica*, a result not published until 1926, but he says nothing about Bernays proving their *consistency*.

[3] Post proves both consistency and completeness of the propositional calculus of PM, cf van Heijenoort's commentary and Post's 1931 *Introduction to a general theory of elementary propositons* in van Heijenoort 1967:264ff. Also Tarski 1946:134ff.

[4] cf van Heijenoort's commentary and Gödel's 1930 *The completeness of the axioms of the functional calculus of logic* in van Heijenoort 1967:582ff

[5] cf van Heijenoort's commentary and Herbrand's 1930 *On the consistency of arithmetic* in van Heijenoort 1967:618ff.

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External links

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Validity

In logic, an argument is **valid** if and only if its conclusion is logically entailed by its premises and each step in the argument is logical. A formula is valid if and only if it is true under every interpretation, and an argument form (or schema) is valid if and only if every argument of that logical form is valid.

Validity of arguments

An argument is valid if and only if the truth of its premises entails the truth of its conclusion and each step, sub-argument, or logical operation in the argument is valid. Under such conditions it would be self-contradictory to affirm the premises and deny the conclusion. The corresponding conditional of a valid argument is a logical truth and the negation of its corresponding conditional is a contradiction. The conclusion is a logical consequence of its premises.

An argument that is not valid is said to be "invalid".

An example of a valid argument is given by the following well-known syllogism (also known as modus ponens):

All men are mortal.

Socrates is a man.

Therefore, Socrates is mortal.

What makes this a valid argument is not that it has true premises and a true conclusion, but the logical necessity of the conclusion, given the two premises. The argument would be just as valid were the premises and conclusion false. The following argument is of the same logical form but with false premises and a false conclusion, and it is equally valid:

All cups are green.

Socrates is a cup.

Therefore, Socrates is green.

No matter how the universe might be constructed, it could never be the case that these arguments should turn out to have simultaneously true premises but a false conclusion. The above arguments may be contrasted with the following invalid one:

All men are mortal.

Socrates is mortal.

Therefore, Socrates is a man.

In this case, the conclusion does not follow inescapably from the premises. All men are mortal, but not all mortals are men. Every living creature is mortal; therefore, even though both premises are true and the conclusion happens to be true in this instance, the argument is invalid because it depends on an incorrect operation of implication. Such fallacious arguments have much in common with what are known as howlers in mathematics.

A standard view is that whether an argument is valid is a matter of the argument's logical form. Many techniques are employed by logicians to represent an argument's logical form. A simple example, applied to two of the above illustrations, is the following: Let the letters 'P', 'Q', and 'S' stand, respectively, for the set of men, the set of mortals, and Socrates. Using these symbols, the first argument may be abbreviated as:

All P are Q.

S is a P.

Therefore, S is a Q.

Similarly, the third argument becomes:

All P are Q.

S is a Q.

Therefore, S is a P.

An argument is **formally valid** if its form is one such that for each interpretation under which the premises are all true, the conclusion is also true. As already seen, the interpretation given above (for the third argument) does cause the second argument form to have true premises and false conclusion (if P is a not human creature), hence demonstrating its invalidity.

Valid formula

A formula of a formal language is a valid formula if and only if it is true under every possible interpretation of the language. In propositional logic, they are tautologies.

Validity of statements

A statement can be called valid, i.e. logical truth, if it is true in all interpretations.

Validity and soundness

Validity of deduction is not affected by the truth of the premise or the truth of the conclusion. The following deduction is perfectly valid:

All animals live on Mars.

All humans are animals.

Therefore, all humans live on Mars.

The problem with the argument is that it is not *sound*. In order for a deductive argument to be sound, the deduction must be valid and **all** the premises true.

Satisfiability and validity

Model theory analyzes formulae with respect to particular classes of interpretation in suitable mathematical structures. On this reading, formula is valid if all such interpretations make it true. An inference is valid if all interpretations that validate the premises validate the conclusion. This is known as *semantic validity*.[1]

Preservation

In **truth-preserving** validity, the interpretation under which all variables are assigned a truth value of 'true' produces a truth value of 'true'.

In a **false-preserving** validity, the interpretation under which all variables are assigned a truth value of 'false' produces a truth value of 'false'.[2]

Preservation properties	Logical connective sentences
True and false preserving:	Logical conjunction (AND, $\wedge$) • Logical disjunction (OR, $\vee$)
True preserving only:	Tautology ($\top$) • Biconditional (XNOR, $\leftrightarrow$) • Implication ($\rightarrow$) • Converse implication ($\leftarrow$)
False preserving only:	Contradiction ($\perp$) • Exclusive disjunction (XOR, $\oplus$) • Nonimplication ($\nrightarrow$) • Converse nonimplication ($\nleftarrow$)
Non-preserving:	Proposition • Negation ($\neg$) • Alternative denial (NAND, $\uparrow$) • Joint denial (NOR, $\downarrow$)

n-Validity

A formula **A** of a first order language Q is n-valid iff it is true for every interpretation of Q that has a domain of exactly n members.

ω-Validity

A formula of a first order language is ω-valid iff it is true for every interpretation of the language and it has a domain with an infinite number of members.

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Completeness

In general, an object is **complete** if nothing needs to be added to it. This notion is made more specific in various fields.

Logical completeness

In logic, semantic completeness is the converse of soundness for formal systems. A formal system is "semantically complete" when all its tautologies are theorems, whereas a formal system is "sound" when all theorems are tautologies (that is, they are semantically valid formulas: formulas that are true under every interpretation of the language of the system that is consistent with the rules of the system). Kurt Gödel, Leon Henkin, and Emil Post all published proofs of completeness. (See History of the Church–Turing thesis.) A formal system is consistent if for all formulas φ of the system, the formulas φ and ¬φ (the negation of φ) are not both theorems of the system (that is, they cannot be both proved with the rules of the system).

- A formal system S is **semantically complete** or simply **complete**, if and only if every tautology of S is a theorem of S. That is, $\models_{\mathcal S} \varphi \ \to \vdash_{\mathcal S} \varphi$.[1]
- A formal system S is **strongly complete** or **complete in the strong sense** if and only if for every set of premises Γ, any formula which semantically follows from Γ is derivable from Γ. That is, $\Gamma \models_{\mathcal S} \varphi \ \to \ \Gamma \vdash_{\mathcal S} \varphi$.
- A formal system S is **syntactically complete** or **deductively complete** or **maximally complete** or simply **complete** if and only if for each formula φ of the language of the system either φ or ¬φ is a theorem of S. This is also called **negation completeness**. In another sense, a formal system is **syntactically complete** if and only if no unprovable axiom can be added to it as an axiom without introducing an inconsistency. Truth-functional propositional logic and first-order predicate logic are semantically complete, but not syntactically complete (for example, the propositional logic statement consisting of a single variable "*a*" is not a theorem, and neither is its negation, but these are not tautologies). Gödel's incompleteness theorem shows that any recursive system that is sufficiently powerful, such as Peano arithmetic, cannot be both consistent and complete.
- A formal system is **inconsistent** if and only if every sentence is a theorem.[2]
- A system of logical connectives is functionally complete if and only if it can express all propositional functions.
- A language is **expressively complete** if it can express the subject matter for which it is intended.
- A formal system is **complete with respect to a property** if and only if every sentence that has the property is a theorem.

Mathematical completeness

In mathematics, "complete" is a term that takes on specific meanings in specific situations, and not every situation in which a type of "completion" occurs is called a "completion". See, for example, algebraically closed field or compactification.

- The completeness of the real numbers is one of the defining properties of the real number system. It may be described equivalently as either the completeness of **R** as metric space or as a partially ordered set (see below).
- A metric space is *complete* if every Cauchy sequence in it converges. See Complete metric space.
- A uniform space is *complete* if every Cauchy net in it converges (or equivalently every Cauchy filter in it converges).
- In functional analysis, a subset *S* of a topological vector space *V* is *complete* if its span is dense in *V*. In the particular case of Hilbert spaces (or more generally, inner product spaces), an orthonormal basis is a set that is both complete and orthonormal.

- A measure space is *complete* if every subset of every null set is measurable. See complete measure.
- In commutative algebra, a commutative ring can be completed at an ideal (in the topology defined by the powers of the ideal). See Completion (ring theory).
- More generally, any topological group can be completed at a decreasing sequence of open subgroups.
- In statistics, a statistic is called *complete* if it does not allow an unbiased estimator of zero. See completeness (statistics).
- In graph theory, a *complete graph* is an undirected graph in which every pair of vertices has exactly one edge connecting them.
- In category theory, a category *C* is *complete* if every diagram from a small category to *C* has a limit; it is *cocomplete* if every such functor has a colimit.
- In order theory and related fields such as lattice and domain theory, *completeness* generally refers to the existence of certain suprema or infima of some partially ordered set. Notable special usages of the term include the concepts of complete Boolean algebra, complete lattice, and complete partial order (cpo). Furthermore, an ordered field is *complete* if every non-empty subset of it that has an upper bound within the field has a least upper bound within the field, which should be compared to the (slightly different) order-theoretical notion of bounded completeness. Up to isomorphism there is only one complete ordered field: the field of real numbers (but note that this complete ordered field, which is also a lattice, is not a complete lattice).
- In algebraic geometry, an algebraic variety is *complete* if it satisfies an analog of compactness. See complete algebraic variety.
- In quantum mechanics, a complete set of commuting operators (or CSCO) is one whose eigenvalues are sufficient to specify the physical state of a system.

Computing

- In algorithms, the notion of completeness refers to the ability of the algorithm to find a solution if one exists, and if not, to report that no solution is possible.
- In computational complexity theory, a problem *P* is **complete** for a complexity class **C**, under a given type of reduction, if *P* is in **C**, and every problem in **C** reduces to *P* using that reduction.
  For example, each problem in the class **NP-complete** is complete for the class **NP**, under polynomial-time, many-one reduction.
- In computing, a data-entry field can autocomplete the entered data based on the prefix typed into the field; that capability is known as *autocompletion*.
- In software testing, completeness has for goal the functional verification of call graph (between software item) and control graph (inside each software item).
- The concept of completeness is found in knowledge base theory.

Economics, finance, and industry

- Complete markets versus incomplete markets
- In auditing, completeness is one of the financial statement assertions that have to be ensured. For example, auditing classes of transactions. Rental expense which includes 12-month or 52-week payments should be all booked according to the terms agreed in the tenancy agreement.
- Oil or gas well completion, the process of making a well ready for production.

Botany

- A **complete** flower is a flower with both male and female reproductive structures as well as petals and sepals. See Sexual reproduction in plants.

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Soundness

In mathematical logic, a logical system has the **soundness** property if and only if its inference rules prove only formulas that are valid with respect to its semantics. In most cases, this comes down to its rules having the property of preserving *truth*, but this is not the case in general.

Of arguments

An argument is **sound** if and only if

1. The argument is valid.
2. All of its premises are true.

For instance,

> All men are mortal.
>
> Socrates is a man.
>
> Therefore, Socrates is mortal.

The argument is valid (because the conclusion is true based on the premises, that is, that the conclusion follows the premises) and since the premises are in fact true, the argument is sound.

The following argument is valid but not sound:

> All organisms with wings can fly.
>
> Penguins have wings.
>
> Therefore, penguins can fly.

Since the first premise is actually false, the argument, though valid, is not sound.

Logical systems

Soundness is among the most fundamental properties of mathematical logic. A soundness property provides the initial reason for counting a logical system as desirable. The completeness property means that every validity (truth) is provable. Together they imply that all and only validities are provable.

Most proofs of soundness are trivial. For example, in an axiomatic system, proof of soundness amounts to verifying the validity of the axioms and that the rules of inference preserve validity (or the weaker property, truth). Most axiomatic systems have only the rule of modus ponens (and sometimes substitution), so it requires only verifying the validity of the axioms and one rule of inference.

Soundness properties come in two main varieties: weak and strong soundness, of which the former is a restricted form of the latter.

Soundness

Soundness of a deductive system is the property that any sentence that is provable in that deductive system is also true on all interpretations or structures of the semantic theory for the language upon which that theory is based. In symbols, where S is the deductive system, L the language together with its semantic theory, and P a sentence of L: if $\vdash_S P$, then also $\vDash_L P$.

Strong soundness

Strong soundness of a deductive system is the property that any sentence P of the language upon which the deductive system is based that is derivable from a set Γ of sentences of that language is also a logical consequence of that set, in the sense that any model that makes all members of Γ true will also make P true. In symbols where Γ is a set of sentences of L: if $\Gamma \vdash_S P$, then also $\Gamma \vDash_L P$. Notice that in the statement of strong soundness, when Γ is empty, we have the statement of weak soundness.

Arithmetic soundness

If T is a theory whose objects of discourse can be interpreted as natural numbers, we say T is *arithmetically sound* if all theorems of T are actually true about the standard mathematical integers. For further information, see ω-consistent theory.

Relation to completeness

The converse of the soundness property is the semantic completeness property. A deductive system with a semantic theory is strongly complete if every sentence P that is a semantic consequence of a set of sentences Γ can be derived in the deduction system from that set. In symbols: whenever $\Gamma \vDash P$, then also $\Gamma \vdash P$. Completeness of first-order logic was first explicitly established by Gödel, though some of the main results were contained in earlier work of Skolem.

Informally, a soundness theorem for a deductive system expresses that all provable sentences are true. Completeness states that all true sentences are provable.

Gödel's first incompleteness theorem shows that for languages sufficient for doing a certain amount of arithmetic, there can be no effective deductive system that is complete with respect to the intended interpretation of the symbolism of that language. Thus, not all sound deductive systems are complete in this special sense of completeness, in which the class of models (up to isomorphism) is restricted to the intended one. The original completeness proof applies to *all* classical models, not some special proper subclass of intended ones.

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External links

- Validity and Soundness (http://www.iep.utm.edu/val-snd/) in the *Internet Encyclopedia of Philosophy*.

Rival conceptions of logic

Definitions of logic

Many treatises on logic begin with a discursion on the difficulty of defining the subject, many do not even attempt to provide a definition. Nevertheless, many definitions have been offered

This article divides the definitions into two classes: first are the simple definitions, that consist of a pithy sentence characterising the topic; second are theoretical definitions, where the definition of logic turns on an analysis the definer provides.

Simple definitions of logic

Arranged in approximate chronological order.

- The tool for distinguishing between the true and the false (Averroes).
- The science of reasoning, teaching the way of investigating unknown truth in connection with a thesis (Robert Kilwardby).
- The art whose function is to direct the reason lest it err in the manner of inferring or knowing (John Poinsot).
- The art of conducting reason well in knowing things (Antoine Arnauld).
- The right use of reason in the inquiry after truth (Isaac Watts).
- The Science, as well as the Art, of reasoning (Richard Whately).
- The science of the operations of the understanding which are subservient to the estimation of evidence (John Stuart Mill).
- The science of the laws of discursive thought (James McCosh).
- The science of the most general laws of truth (Gottlob Frege).
- The science which directs the operations of the mind in the attainment of truth (George Hayward Joyce).
- The analysis and appraisal of arguments (Harry J. Gensler).
- The branch of philosophy concerned with analysing the patterns of reasoning by which a conclusion is drawn from a set of premisses (Collins English Dictionary)
- The formal systematic study of the principles of valid inference and correct reasoning (Penguin Encyclopedia).

Theoretical definitions of logic

Quine (1940, pp. 2–3) defines logic in terms of a logical vocabulary, which in turn is identified by an argument that the many particular vocabularies —Quine mentions geological vocabulary— are used in their particular discourses together with a common, topic-independent kernel of terms.[1] These terms, then, constitute the logical vocabulary, and the logical truths are those truths common to all particular topics.

Hofweber (2004) lists several definitions of logic, and goes on to claim that all definitions of logic are of one of four sorts. These are that logic is the study of: (i) artificial formal structures, (ii) sound inference (e.g., Poinsot), (iii) tautologies (e.g., Watts), or (iv) general features of thought (e.g., Frege). He argues then that these definitions are related to each other, but do not exhaust each other, and that an examination of formal ontology shows that these mismatches between rival definitions is due to tricky issues in ontology.

Notes

[1] Cf. Ferreiros, 2001

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Argumentation theory

Argumentation theory, or **argumentation**, is the interdisciplinary study of how conclusions can be reached through logical reasoning; that is, claims based, soundly or not, on premises. It includes the arts and sciences of civil debate, dialogue, conversation, and persuasion. It studies rules of inference, logic, and procedural rules in both artificial and real world settings.

Argumentation includes debate and negotiation which are concerned with reaching mutually acceptable conclusions. It also encompasses eristic dialog, the branch of social debate in which victory over an opponent is the primary goal. This art and science is often the means by which people protect their beliefs or self-interests in rational dialogue, in common parlance, and during the process of arguing.

Argumentation is used in law, for example in trials, in preparing an argument to be presented to a court, and in testing the validity of certain kinds of evidence. Also, argumentation scholars study the post hoc rationalizations by which organizational actors try to justify decisions they have made irrationally.

Key components of argumentation

- Understanding and identifying arguments, either explicit or implied, and the goals of the participants in the different types of dialogue.
- Identifying the premises from which conclusions are derived
- Establishing the "burden of proof" — determining who made the initial claim and is thus responsible for providing evidence why his/her position merits acceptance
- For the one carrying the "burden of proof", the advocate, to marshal evidence for his/her position in order to convince or force the opponent's acceptance. The method by which this is accomplished is producing valid, sound, and cogent arguments, devoid of weaknesses, and not easily attacked.
- In a debate, fulfillment of the burden of proof creates a burden of rejoinder. One must try to identify faulty reasoning in the opponent's argument, to attack the reasons/premises of the argument, to provide counterexamples

if possible, to identify any fallacies, and to show why a valid conclusion cannot be derived from the reasons provided for his/her argument.

Internal structure of arguments

Typically an argument has an internal structure, comprising the following

1. a set of assumptions or premises
2. a method of reasoning or deduction and
3. a conclusion or point.

An argument must have at least one premise and one conclusion.

Often classical logic is used as the method of reasoning so that the conclusion follows logically from the assumptions or support. One challenge is that if the set of assumptions is inconsistent then anything can follow logically from inconsistency. Therefore it is common to insist that the set of assumptions be consistent. It is also good practice to require the set of assumptions to be the minimal set, with respect to set inclusion, necessary to infer the consequent. Such arguments are called MINCON arguments, short for minimal consistent. Such argumentation has been applied to the fields of law and medicine. A second school of argumentation investigates abstract arguments, where 'argument' is considered a primitive term, so no internal structure of arguments is taken on account.

In its most common form, argumentation involves an individual and an interlocutor/or opponent engaged in dialogue, each contending differing positions and trying to persuade each other. Other types of dialogue in addition to persuasion are eristic, information seeking, inquiry, negotiation, deliberation, and the dialectical method (Douglas Walton). The dialectical method was made famous by Plato and his use of Socrates critically questioning various characters and historical figures.

Argumentation and the grounds of knowledge

Argumentation theory had its origins in foundationalism, a theory of knowledge (epistemology) in the field of philosophy. It sought to find the grounds for claims in the forms (logic) and materials (factual laws) of a universal system of knowledge. But argument scholars gradually rejected Aristotle's systematic philosophy and the idealism in Plato and Kant. They questioned and ultimately discarded the idea that argument premises take their soundness from formal philosophical systems. The field thus broadened.[1]

Karl R. Wallace's seminal essay, "The Substance of Rhetoric: Good Reasons" in the *Quarterly Journal of Speech* (1963) 44, led many scholars to study "marketplace argumentation" - the ordinary arguments of ordinary people. The seminal essay on marketplace argumentation is Ray Lynn Anderson and C. David Mortensen,"Logic and Marketplace Argumentation" *Quarterly Journal of Speech* 53 (1967): 143-150.[2][3] This line of thinking led to a natural alliance with late developments in the sociology of knowledge.[4] Some scholars drew connections with recent developments in philosophy, namely the pragmatism of John Dewey and Richard Rorty. Rorty has called this shift in emphasis "the linguistic turn".

In this new hybrid approach argumentation is used with or without empirical evidence to establish convincing conclusions about issues which are moral, scientific, epistemic, or of a nature in which science alone cannot answer. Out of pragmatism and many intellectual developments in the humanities and social sciences, "non-philosophical" argumentation theories grew which located the formal and material grounds of arguments in particular intellectual fields. These theories include informal logic, social epistemology, ethnomethodology, speech acts, the sociology of knowledge, the sociology of science, and social psychology. These new theories are not non-logical or anti-logical. They find logical coherence in most communities of discourse. These theories are thus often labeled "sociological" in that they focus on the social grounds of knowledge.

Approaches to argumentation in communication and informal logic

In general, the label "argumentation" is used by communication scholars such as (to name only a few: Wayne E. Brockriede, Douglas Ehninger, Joseph W. Wenzel, Richard Rieke, Gordon Mitchell, Carol Winkler, Eric Gander, Dennis S. Gouran, Daniel J. O'Keefe, Mark Aakhus, Bruce Gronbeck, James Klumpp, G. Thomas Goodnight, Robin Rowland, Dale Hample, C. Scott Jacobs, Sally Jackson, David Zarefsky, and Charles Arthur Willard) while the term "informal logic" is preferred by philosophers, stemming from University of Windsor philosophers Ralph H. Johnson and J. Anthony Blair. Harald Wohlrapp developed a criterion for *validness* (Geltung, Gültigkeit) as *freedom of objections*.

Trudy Govier, Douglas Walton, Michael Gilbert, Harvey Seigal, Michael Scriven, and John Woods (to name only a few) are other prominent authors in this tradition. Over the past thirty years, however, scholars from several disciplines have co-mingled at international conferences such as that hosted by the University of Amsterdam (the Netherlands) and the International Society for the Study of Argumentation (ISSA). Other international conferences are the biannual conference held at Alta, Utah sponsored by the (US) National Communication Association and American Forensics Association and conferences sponsored by the Ontario Society for the Study of Argumentation (OSSA).

Some scholars (such as Ralph H. Johnson) construe the term "argument" narrowly, as exclusively written discourse or even discourse in which all premises are explicit. Others (such as Michael Gilbert) construe the term "argument" broadly, to include spoken and even nonverbal discourse, for instance the degree to which a war memorial or propaganda poster can be said to argue or "make arguments." The philosopher Stephen E. Toulmin has said that an argument is a claim on our attention and belief, a view that would seem to authorize treating, say, propaganda posters as arguments. The dispute between broad and narrow theorists is of long standing and is unlikely to be settled. The views of the majority of argumentation theorists and analysts fall somewhere between these two extremes.

Kinds of argumentation

Conversational argumentation

The study of naturally-occurring conversation arose from the field of sociolinguistics. It is usually called *conversation analysis.* Inspired by ethnomethodology, it was developed in the late 1960s and early 1970s principally by the sociologist Harvey Sacks and, among others, his close associates Emanuel Schegloff and Gail Jefferson. Sacks died early in his career, but his work was championed by others in his field, and CA has now become an established force in sociology, anthropology, linguistics, speech-communication and psychology.[5] It is particularly influential in interactional sociolinguistics, discourse analysis and discursive psychology, as well as being a coherent discipline in its own right. Recently CA techniques of sequential analysis have been employed by phoneticians to explore the fine phonetic details of speech.

Empirical studies and theoretical formulations by Sally Jackson and Scott Jacobs, and several generations of their students, have described argumentation as a form of managing conversational disagreement within communication contexts and systems that naturally prefer agreement.

Mathematical argumentation

The basis of mathematical truth has been the subject of long debate. Frege in particular sought to demonstrate (see Gottlob Frege, The Foundations of Arithemetic, 1884, and Logicism in Philosophy of mathematics) that arithmetical truths can be derived from purely logical axioms and therefore are, in the end, logical truths. The project was developed by Russell and Whitehead in their Principia Mathematica. If an argument can be cast in the form of sentences in Symbolic Logic, then it can be tested by the application of accepted proof procedures. This has been carried out for Arithmetic using Peano axioms. Be that as it may, an argument in Mathematics, as in any other discipline, can be considered valid only if it can be shown that it cannot have true premises and a false conclusion.

Scientific argumentation

Perhaps the most radical statement of the social grounds of scientific knowledge appears in Alan G.Gross's *The Rhetoric of Science* (Cambridge: Harvard University Press, 1990). Gross holds that science is rhetorical "without remainder," meaning that scientific knowledge itself cannot be seen as an idealized ground of knowledge. Scientific knowledge is produced rhetorically, meaning that it has special epistemic authority only insofar as its communal methods of verification are trustworthy. This thinking represents an almost complete rejection of the foundationalism on which argumentation was first based.

Legal argumentation

Legal arguments are spoken presentations to a judge or appellate court by a lawyer, or parties when representing themselves of the legal reasons why they should prevail. Oral argument at the appellate level accompanies written briefs, which also advance the argument of each party in the legal dispute. A closing argument, or summation, is the concluding statement of each party's counsel reiterating the important arguments for the trier of fact, often the jury, in a court case. A closing argument occurs after the presentation of evidence.

Political argumentation

Political arguments are used by academics, media pundits, candidates for political office and government officials. Political arguments are also used by citizens in ordinary interactions to comment about and understand political events.[6] The rationality of the public is a major question in this line of research. Political scientist Samuel L. Popkin coined the expression "low information voters" to describe most voters who know very little about politics or the world in general.

In practice, a "low information voter" may not be aware of legislation that their representative has sponsored in Congress. A low-information voter may base their ballot box decision on a media sound-bite, or a flier received in the mail. It is possible for a media sound-bite or campaign flier to present a political position for the incumbent candidate that completely contradicts the legislative action taken in Washington D.C. on behalf of the constituents. It may only take a small percentage of the overall voting group who base their decision on the inaccurate information, a voter block of 10 to 12%, to swing an overall election result. When this happens, the constituency at large may have been duped or fooled. Nevertheless, the election result is legal and confirmed. Savvy Political consultants will take advantage of low-information voters and sway their votes with disinformation because it can be easier and sufficiently effective. Institutions such as factcheck.org [7] have come about in recent years to help counter the effects of such campaign tactics. Factcheck.org's stated goal is "We aim to reduce the level of deception and confusion in U.S. politics, for voters".[8]

Psychological aspects

Psychology has long studied the non-logical aspects of argumentation. For example, studies have shown that simple repetition of an idea is often a more effective method of argumentation than appeals to reason. Propaganda often utilizes repetition.[9] Nazi rhetoric has been studied extensively as, inter alia, a repetition campaign.

Empirical studies of communicator credibility and attractiveness, sometimes labeled *charisma,* have also been tied closely to empirically-occurring arguments. Such studies bring argumentation within the ambit of persuasion theory and practice.

Some psychologists such as William J. McGuire believe that the syllogism is the basic unit of human reasoning. They have produced a large body of empirical work around McGuire's famous title "A Syllogistic Analysis of Cognitive Relationships." A central line of this way of thinking is that logic is contaminated by psychological variables such as "wishful thinking," in which subjects confound the likelihood of predictions with the desirability of the predictions. People hear what they want to hear and see what they expect to see. If planners want something to happen they see it as likely to happen. Thus planners ignore possible problems, as in the American experiment with prohibition. If they hope something will not happen, they see it as unlikely to happen. Thus smokers think that they personally will avoid cancer. Promiscuous people practice unsafe sex. Teenagers drive recklessly.

Theories

Argument fields

Stephen E. Toulmin and Charles Arthur Willard have championed the idea of argument fields, the former drawing upon Ludwig Wittgenstein's notion of language games, (Sprachspiel) the latter drawing from communication and argumentation theory, sociology, political science, and social epistemology. For Toulmin, the term "field" designates discourses within which arguments and factual claims are grounded.[10] For Willard, the term "field" is interchangeable with "community," "audience," or "readership."[11] Along similar lines, G. Thomas Goodnight has studied "spheres" of argument and sparked a large literature created by younger scholars responding to or using his ideas.[12] The general tenor of these field theories is that the premises of arguments take their meaning from social communities.[13]

Field studies might focus on social movements, issue-centered publics (for instance, pro-life versus pro-choice in the abortion dispute), small activist groups, corporate public relations campaigns and issue management, scientific communities and disputes, political campaigns, and intellectual traditions.[14] In the manner of a sociologist, ethnographer, anthropologist, participant-observer, and journalist, the field theorist gathers and reports on real-world human discourses, gathering case studies that might eventually be combined to produce high-order explanations of argumentation processes. This is not a quest for some master language or master theory covering all specifics of human activity. Field theorists are agnostic about the possibility of a single grand theory and skeptical about the usefulness of such a theory. Theirs is a more modest quest for "mid-range" theories that might permit generalizations about families of discourses.

Stephen E. Toulmin's Contributions

By far, the most influential theorist has been the late Stephen E. Toulmin, the Cambridge educated philosopher and student of Wittgenstein.[15] What follows below is a sketch of his ideas.

An Alternative to Absolutism and Relativism

Toulmin has argued that absolutism (represented by theoretical or analytic arguments) has limited practical value. Absolutism is derived from Plato's idealized formal logic, which advocates universal truth; thus absolutists believe that moral issues can be resolved by adhering to a standard set of moral principles, regardless of context. By contrast, Toulmin asserts that many of these so-called standard principles are irrelevant to real situations encountered by human beings in daily life.

To describe his vision of daily life, Toulmin introduced the concept of argument fields; in *The Uses of Argument* (1958), Toulmin states that some aspects of arguments vary from field to field, and are hence called "field-dependent," while other aspects of argument are the same throughout all fields, and are hence called "field-invariant." The flaw of absolutism, Toulmin believes, lies in its unawareness of the field-dependent aspect of argument; absolutism assumes that all aspects of argument are field invariant.

Toulmin's theories resolve to avoid the defects of absolutism without resorting to relativism: relativism, Toulmin asserted, provides no basis for distinguishing between a moral or immoral argument. In *Human Understanding* (1972), Toulmin suggests that anthropologists have been tempted to side with relativists because they have noticed the influence of cultural variations on rational arguments; in other words, the anthropologist or relativist overemphasizes the importance of the "field-dependent" aspect of arguments, and becomes unaware of the "field-invariant" elements. In an attempt to provide solutions to the problems of absolutism and relativism, Toulmin attempts throughout his work to develop standards that are neither absolutist nor relativist for assessing the worth of ideas.

Toulmin believes that a good argument can succeed in providing good justification to a claim, which will stand up to criticism and earn a favourable verdict.

Components of argument

In *The Uses of Argument* (1958), Toulmin proposed a layout containing six interrelated components for analyzing arguments:

1. Claim: Conclusions whose merit must be established. For example, if a person tries to convince a listener that he is a British citizen, the claim would be "I am a British citizen." (1)
2. Data: The facts we appeal to as a foundation for the claim. For example, the person introduced in 1 can support his claim with the supporting data "I was born in Bermuda." (2)
3. Warrant: The statement authorizing our movement from the data to the claim. In order to move from the data established in 2, "I was born in Bermuda," to the claim in 1, "I am a British citizen," the person must supply a warrant to bridge the gap between 1 & 2 with the statement "A man born in Bermuda will legally be a British Citizen." (3)
4. Backing: Credentials designed to certify the statement expressed in the warrant; backing must be introduced when the warrant itself is not convincing enough to the readers or the listeners. For example, if the listener does not deem the warrant in 3 as credible, the speaker will supply the legal provisions as backing statement to show that it is true that "A man born in Bermuda will legally be a British Citizen."
5. Rebuttal: Statements recognizing the restrictions to which the claim may legitimately be applied. The rebuttal is exemplified as follows, "A man born in Bermuda will legally be a British citizen, unless he has betrayed Britain and has become a spy of another country."
6. Qualifier: Words or phrases expressing the speaker's degree of force or certainty concerning the claim. Such words or phrases include "possible," "probably," "impossible," "certainly," "presumably," "as far as the evidence goes," or "necessarily." The claim "I am definitely a British citizen" has a greater degree of force than the claim "I am a British citizen, presumably."

The first three elements "claim," "data," and "warrant" are considered as the essential components of practical arguments, while the second triad "qualifier," "backing," and "rebuttal" may not be needed in some arguments.

When first proposed, this layout of argumentation is based on legal arguments and intended to be used to analyze the rationality of arguments typically found in the courtroom; in fact, Toulmin did not realize that this layout would be applicable to the field of rhetoric and communication until his works were introduced to rhetoricians by Wayne Brockriede and Douglas Ehninger. Only after he published *Introduction to Reasoning* (1979) were the rhetorical applications of this layout mentioned in his works.

The Evolution of Knowledge

Toulmin's *Human Understanding* (1972) asserts that conceptual change is evolutionary. This book attacks Thomas Kuhn's explanation of conceptual change in *The Structure of Scientific Revolutions.* Kuhn held that conceptual change is a revolutionary (as opposed to an evolutionary) process in which mutually exclusive paradigms compete to replace one another. Toulmin criticizes the relativist elements in Kuhn's thesis, as he points out that the mutually exclusive paradigms provide no ground for comparison; in other words, Kuhn's thesis has made the relativists' error of overemphasizing the "field variant" while ignoring the "field invariant," or commonality shared by all argumentation or scientific paradigms.

Toulmin proposes an evolutionary model of conceptual change comparable to Darwin's model of biological evolution. On this reasoning, conceptual change involves innovation and selection. Innovation accounts for the appearance of conceptual variations, while selection accounts for the survival and perpetuation of the soundest conceptions. Innovation occurs when the professionals of a particular discipline come to view things differently from their predecessors; selection subjects the innovative concepts to a process of debate and inquiry in what Toulmin considers as a "forum of competitions." The soundest concepts will survive the forum of competition as replacements or revisions of the traditional conceptions.

From the absolutists' point of view, concepts are either valid or invalid regardless of contexts; from a relativists' perspective, one concept is neither better nor worse than a rival concept from a different cultural context. From Toulmin's perspective, the evaluation depends on a process of comparison, which determines whether or not one concept will provide improvement to our explanatory power more so than its rival concepts.

Rejection of Certainty

In *Cosmopolis* (1990), Toulmin traces the Quest for Certainty back to Descartes and Hobbes, and lauds Dewey, Wittgenstein, Heidegger and Rorty for abandoning that tradition.

Pragma-dialectics

Scholars at the University of Amsterdam in the Netherlands have pioneered a rigorous modern version of dialectic under the name *pragma-dialectics*. The intuitive idea is to formulate clearcut rules that, if followed, will yield rational discussion and sound conclusions. Frans H. van Eemeren, the late Rob Grootendorst, and many of their students have produced a large body of work expounding this idea.

The dialectical conception of reasonableness is given by ten rules for critical discussion, all being instrumental for achieving a resolution of the difference of opinion (from Van Eemeren, Grootendorst, & Snoeck Henkemans, 2002, p. 182-183). The theory postulates this as an ideal model, and not something one expects to find as an empirical fact. The model can however serve as an important heuristic and critical tool for testing how reality approximates this ideal and point to where discourse goes wrong, that is, when the rules are violated. Any such violation will constitute a fallacy. Albeit not primarily focused on fallacies, pragma-dialectics provides a systematic approach to deal with them in a coherent way.

Artificial intelligence

Efforts have been made within the field of artificial intelligence to perform and analyze the act of argumentation with computers. Argumentation has been used to provide a proof-theoretic semantics for non-monotonic logic, starting with the influential work of Dung (1995). Computational argumentation systems have found particular application in domains where formal logic and classical decision theory are unable to capture the richness of reasoning, domains such as law and medicine. In Elements of Argumentation, Philippe Besnard and Anthony Hunter introduce techniques for formalizing deductive argumentation in artificial intelligence, emphasizing emerging formalizations for practical argumentation.[16] A comprehensive overview of this area can be found in a recent book edited by Iyad Rahwan and Guillermo R. Simari.[17]

Within Computer Science, the ArgMAS workshop series (Argumentation in Multi-Agent Systems), the CMNA workshop series,[18] and now the COMMA Conference,[19] are regular annual events attracting participants from every continent. The journal Argument & Computation[20] is dedicated to exploring the intersection between argumentation and computer science.

Notes

[1] Bruce Gronbeck. "From Argument to Argumentation: Fifteen Years of Identity Crisis." Jack Rhodes and Sara Newell, ed.s *Proceedings of the Summer Conference on Argumentation*. 1980.

[2] See Joseph W. Wenzel "Perspectives on Argument." Jack Rhodes and Sara Newell, ed.s Proceedings of the Summer Conference on Argumentation. 1980.

[3] David Zarefsky. "Product, Process, or Point of View? Jack Rhodes and Sara Newell, ed.s *Proceedings of the Summer Conference on Argumentation*. 1980.

[4] See Ray E. McKerrow. "Argument Communities: A Quest for Distinctions."

[5] Psathas, George (1995): Conversation Analysis, Thousand Oaks: Sage Sacks, Harvey. (1995). Lectures on Conversation. Blackwell Publishing. ISBN 1-55786-705-4. Sacks, Harvey, Schegloff, Emanuel A., & Jefferson, Gail (1974). A simple systematic for the organization of turn-taking for conversation. Language, 50, 696-735. Schegloff, Emanuel A. (2007). Sequence Organization in Interaction: A Primer in Conversation Analysis, Volume 1, Cambridge: Cambridge University Press. Ten Have, Paul (1999): Doing Conversation Analysis. A Practical Guide, Thousand Oaks: Sage.

[6] Michael McGee. "The 'Ideograph' as a Unit of Analysis in Political Argument." Jack Rhodes and Sara Newell, eds. Proceedings of the Summer Conference on Argumentation. 1980.

[7] http://factcheck.org/

[8] http://factcheck.org/about/

[9] Jacques Ellul, *Propaganda*, Vintage, 1973, ISBN 0-394-71874-7 ISBN 978-0394718743.

[10] Stephen E. Toulmin. *The uses of argument.* 1959.

[11] Charles Arthur Willard. "Some Questions About Toulmin's View of Argument Fields." Jack Rhodes and Sara Newell, eds. *Proceedings of the Summer Conference on Argumentation*. 1980. "Field Theory: A Cartesian Meditation." George Ziegelmueller and Jack Rhodes, eds. *Dimensions of Argument: Proceedings of the Second Summer Conference on Argumentation.*

[12] G. T. Goodnight, "The Personal, Technical, and Public Spheres of Argument." *Journal of the American Forensics Association.* (1982) 18:214-227. See also: http://en.wikipedia.org/wiki/Public_sphere

[13] Bruce E. Gronbeck. "Sociocultural Notions of Argument Fields: A Primer." George Ziegelmueller and Jack Rhodes, eds. *Dimensions of Argument: Proceedings of the Second Summer Conference on Argumentation.* (1981) 1-20.

[14] Robert Rowland, "Purpose, Argument Fields, and Theoretical Justification." *Argumentation.* vol. 22 Number 2 (2008) 235-250.

[15] Loui, Ronald P. (2006). "A Citation-Based Reflection on Toulmin and Argument" (http://books.google.com/books?id=3xE5ichwr5MC&lpg=PA31&ots=gGspeEoAs-&dq=A Citation-Based Reflection on Toulmin and Argument&pg=PA31#v=onepage&q=A Citation-Based Reflection on Toulmin and Argument&f=false). In Hitchcock, David; Verheij, Bart. *Arguing on the Toulmin Model: New Essays in Argument Analysis and Evaluation*. Springer Netherlands. pp. 31–38. doi:10.1007/978-1-4020-4938-5_3. ISBN 978-1-4020-4937-8. . Retrieved 2010-06-25. "Toulmin's 1958 work is essential in the field of argumentation"

[16] P. Besnard & A. Hunter, "Elements of Argumentation." MIT Press, 2008. See also: http://mitpress.mit.edu/catalog/item/default.asp?ttype=2&tid=11482

[17] I. Rahwan & G. R. Simari (Eds.), "Argumentation in Artificial Intelligence." Springer, 2009. See also: http://www.springer.com/computer/artificial/book/978-0-387-98196-3

[18] Computational Models of Natural Argument (http://www.cmna.info)

[19] Computational Models of Argument (http://www.csc.liv.ac.uk/~comma/)

[20] Journal of Argument & Computation (http://www.tandf.co.uk/journals/tarc)

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Further reading

Flagship journals:

- *Argumentation*
- *Informal Logic*
- *Argumentation and Advocacy* (formerly *Journal of the American Forensic Association*)
- *Social Epistemology*
- *Episteme: A Journal of Social Epistemology*
- *Journal of Argument and Computation*

External links

- Universiteit Utrecht (http://www.cs.uu.nl/people/henry/research/argtheory.html) **(Dutch)**
- Universiteit Twente (http://www.tcw.utwente.nl/theorieenoverzicht/Levels of theories/micro/Argumentation Theory.doc/) **(Dutch)**
- L'Argumentation: Introduction à l'étude du discours (http://ebooks.unibuc.ro/lls/MarianaTutescu-Argumentation/sommaire.htm) **(French)** Free on-line book by Mariana Tutescu previously published in 1998 as ISBN 973-575-248-4
- Argumentum.ch (http://www.argumentum.ch), E-course of Argumentation Theory for the Human and Social Sciences
- Interview with Stephen Toulmin in JAC (http://www.jacweb.org/Archived_volumes/Text_articles/V13_I2_Olson_Toulmin.htm)

Rationality

In philosophy, **rationality** is the characteristic of any action, belief, or desire, that makes their choice a necessity.[1] It is a normative concept of reasoning in the sense that rational people should derive conclusions in a consistent way given the information at disposal. It refers to the conformity of one's beliefs with one's reasons to believe, or with one's actions with one's reasons for action. However, the term "rationality" tends to be used differently in different disciplines, including specialized discussions of economics, sociology, psychology, and political science. A **rational** decision is one that is not just reasoned, but is also optimal for achieving a goal or solving a problem.

Determining optimality for rational behavior requires a quantifiable formulation of the problem, and the making of several key assumptions. When the goal or problem involves making a decision, rationality factors in how much information is available (e.g. complete or incomplete knowledge). Collectively, the formulation and background assumptions are the model within which rationality applies. Illustrating the relativity of rationality: if one accepts a model in which benefiting oneself is optimal, then rationality is equated with behavior that is self-interested to the point of being selfish; whereas if one accepts a model in which benefiting the group is optimal, then purely selfish behavior is deemed irrational. It is thus meaningless to assert rationality without also specifying the background model assumptions describing how the problem is framed and formulated.

Theories of rationality

The German sociologist Max Weber proposed an interpretation of social action that distinguished between four different idealized types of rationality. The first, which he called *Zweckrational* or purposive/instrumental rationality, is related to the expectations about the behavior of other human beings or objects in the environment. These expectations serve as means for a particular actor to attain ends, ends which Weber noted were "rationally pursued and calculated." The second type, Weber called *Wertrational* or value/belief-oriented. Here the action is undertaken for what one might call reasons intrinsic to the actor: some ethical, aesthetic, religious or other motive, independent of whether it will lead to success. The third type was affectual, determined by an actor's specific affect, feeling, or emotion – to which Weber himself said that this was a kind of rationality that was on the borderline of what he considered "meaningfully oriented." The fourth was traditional, determined by ingrained habituation. Weber emphasized that it was very unusual to find only one of these orientations: combinations were the norm. His usage also makes clear that he considered the first two as more significant than the others, and it is arguable that the third and fourth are subtypes of the first two.

The advantage in this interpretation is that it avoids a value-laden assessment, say, that certain kinds of beliefs are irrational. Instead, Weber suggests that a ground or motive can be given – for religious or affect reasons, for example — that may meet the criterion of explanation or justification even if it is not an explanation that fits the *Zweckrational* orientation of means and ends. The opposite is therefore also true: some means-ends explanations will not satisfy those whose grounds for action are **'Wertrational'**.

Weber's constructions of rationality have been critiqued both from a Habermasian (1984) perspective (as devoid of social context and under-theorised in terms of social power)[2] and also from a feminist perspective (Eagleton, 2003) whereby Weber's rationality constructs are viewed as imbued with masculine values and oriented toward the maintenance of male power.[3] An alternative position on rationality (which includes both bounded rationality (Simons and Hawkins, 1949),[4] as well as the affective and value-based arguments of Weber) can be found in the critique of Etzioni (1988),[5] who reframes thought on decision-making to argue for a reversal of the position put forward by Weber. Etzioni illustrates how purposive/instrumental reasoning is subordinated by normative considerations (ideas on how people 'ought' to behave) and affective considerations (as a support system for the development of human relationships).

In the psychology of reasoning, psychologists and cognitive scientists have defended different positions on human rationality. One prominent view, due to Philip Johnson-Laird and Ruth M.J. Byrne among others is that humans are rational in principle but they err in practice, that is, humans have the competence to be rational but their performance is limited by various factors.[6] However, it has been argued that many standard tests of reasoning, such as those on the conjunction fallacy, on the Wason selection task, or the base rate fallacy suffer from methodological and conceptual problems. This has led to disputes in psychology over whether researchers should (only) use standard rules of logic, probability theory and statistics, or rational choice theory as norms of good reasoning. Opponents of this view, such as Gerd Gigerenzer, favor a conception of bounded rationality, especially for tasks under high uncertainty.[7]

Richard Brandt proposed a 'reforming definition' of rationality, arguing someone is rational if their notions survive a form of cognitive-psychotherapy.[8]

Quality of rationality

It is believed by some philosophers (notably A.C. Grayling) that a good rationale must be independent of emotions, personal feelings or any kind of instincts. Any process of evaluation or analysis, that may be called rational, is expected to be highly objective, logical and "mechanical". If these minimum requirements are not satisfied i.e. if a person has been, even slightly, influenced by personal emotions, feelings, instincts or culturally specific, moral codes and norms, then the analysis may be termed irrational, due to the injection of subjective bias.

Modern cognitive science and neuroscience show that studying the role of emotion in mental function (including topics ranging from flashes of scientific insight to making future plans), that no human has ever satisfied this criterion, except perhaps a person with no affective feelings, for example an individual with a massively damaged amygdala or severe psychopathy. Thus, such an idealized form of rationality is best exemplified by computers, and not people. However, scholars may productively appeal to the idealization as a point of reference.

Theoretical and practical rationality

Kant had distinguished theoretical from practical reason. Rationality theorist Jesús Mosterín makes a parallel distinction between theoretical and practical rationality, although, according to him, reason and rationality are not the same: reason would be a psychological faculty, whereas rationality is an optimizing strategy.[9] Humans are not rational by definition, but they can think and behave rationally or not, depending on whether they apply, explicitly or implicitly, the strategy of theoretical and practical rationality to the thoughts they accept and to the actions they perform. Theoretical rationality has a formal component that reduces to logical consistency and a material component that reduces to empirical support, relying on our inborn mechanisms of signal detection and interpretation. Mosterín distinguishes between involuntary and implicit belief, on the one hand, and voluntary and explicit acceptance, on the other.[10] Theoretical rationality can more properly be said to regulate our acceptances than our beliefs. Practical rationality is the strategy for living one’s best possible life, achieving your most important goals and your own preferences in as far as possible. Practical rationality has also a formal component, that reduces to Bayesian decision theory, and a material component, rooted in human nature (lastly, in our genome).

Examples of Rationality Applied to Different Fields

Individuals or organizations are called rational if they make optimal decisions in pursuit of their goals. It is in these terms that one speaks, for example, of a rational allocation of resources, or of a rational corporate strategy. For such "rationality", the decision maker's goals are taken as part of the model, and not made subject to criticism, ethical or otherwise.

Debates arise in these four fields about whether or not people or organizations are "really" rational, as well as whether it make sense to model them as such in formal models. Some have argued that a kind of bounded rationality makes more sense for such models.

Others think that any kind of rationality along the lines of rational choice theory is a useless concept for understanding human behavior; the term *homo economicus* (economic man: the imaginary man being assumed in economic models who is logically consistent but amoral) was coined largely in honor of this view.

Artificial Intelligence

Within artificial intelligence, a *rational agent* is one that maximizes its expected *utility*, given its current knowledge. Utility is the usefulness of the consequences of its actions. The utility function is arbitrarily defined by the designer, but should be a function of *performance*, which is the directly measurable consequences, such as winning or losing money. In order to make a safe agent that plays defensively, a nonlinear function of performance is often desired, so that the reward for winning is lower than the punishment for losing. An agent might be rational within its own problem area, but finding the rational decision for arbitrarily complex problems is not practically possible. The rationality of human thought is a key problem in the psychology of reasoning.[11]

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- Stanford Encyclopedia of Philosophy entry on Historicist Theories of Rationality (http://plato.stanford.edu/entries/rationality-historicist/)
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Topics in logic

Term logic

In philosophy, **term logic**, also known as **traditional logic** or **Aristotelian logic**, is a loose name for the way of doing logic that began with Aristotle and that was dominant until the advent of modern predicate logic in the late nineteenth century. This entry is an introduction to the term logic needed to understand philosophy texts written before predicate logic came to be seen as the only formal logic of interest. Readers lacking a grasp of the basic terminology and ideas of term logic can have difficulty understanding such texts, because their authors typically assumed an acquaintance with term logic.

Aristotle's system

Aristotle's logical work is collected in the six texts that are collectively known as the *Organon*. Two of these texts in particular, namely the *Prior Analytics* and *De Interpretatione* contain the heart of Aristotle's treatment of judgements and formal inference, and it is principally this part of Aristotle's works that is about term logic.

The basics

The fundamental assumption behind the theory is that propositions are composed of two terms – hence the name "two-term theory" or "term logic" – and that the reasoning process is in turn built from propositions:

- **The term** is a part of speech representing something, but which is not true or false in its own right, such as "man" or "mortal".
- **The proposition** consists of two terms, in which one term (the "predicate") is "affirmed" or "denied" of the other (the "subject"), and which is capable of truth or falsity.
- **The syllogism** is an inference in which one proposition (the "conclusion") follows of necessity from two others (the "premises").

A proposition may be universal or particular, and it may be affirmative or negative. Traditionally, the four kinds of propositions are:

- A-type: Universal and affirmative ("Every philosopher is mortal")
- I-type: Particular and affirmative ("Some philosopher is mortal")
- E-type: Universal and negative ("Every philosopher is immortal")
- O-type: Particular and negative ("Some philosopher is immortal")

This was called the *fourfold scheme* of propositions (see types of syllogism for an explanation of the letters A, I, E, and O in the traditional square). Aristotle's *original* square of opposition, however, does not lack existential import:

- A-type: Universal and affirmative ("Every philosopher is mortal")
- I-type: Particular and affirmative ("Some philosopher is mortal")
- E-type: Universal and negative ("Not every philosopher is mortal")
- O-type: Particular and negative ("No philosopher is mortal")

In the Stanford Encyclopedia of Philosophy article, "The Traditional Square of Opposition", Terence Parsons explains:

> One central concern of the Aristotelian tradition in logic is the theory of the categorical syllogism. This is the theory of two-premised arguments in which the premises and conclusion share three terms among them, with each proposition containing two of them. It is distinctive of this enterprise that everybody agrees on which

syllogisms are valid. The theory of the syllogism partly constrains the interpretation of the forms. For example, it determines that the ***A*** form has existential import, at least if the ***I*** form does. For one of the valid patterns (Darapti) is:

Every *C is* B

Every *C is* A

So, some *A* is *B*

This is invalid if the ***A*** form lacks existential import, and valid if it has existential import. It is held to be valid, and so we know how the ***A*** form is to be interpreted. One then naturally asks about the ***O*** form; what do the syllogisms tell us about it? The answer is that they tell us nothing. This is because Aristotle did not discuss weakened forms of syllogisms, in which one concludes a particular proposition when one could already conclude the coresponding universal. For example, he does not mention the form:

No *C* is *B*

Every *A* is *C*

So, some *A* is not *B*

If people had thoughtfully taken sides for or against the validity of this form, that would clearly be relevant to the understanding of the O form. But the weakened forms were typically ignored...

One other piece of subject-matter bears on the interpretation of the ***O*** form. People were interested in Aristotle's discussion of “infinite” negation, which is the use of negation to form a term from a term instead of a proposition from a proposition. In modern English we use “non” for this; we make “non-horse,” which is true of exactly those things that are not horses. In medieval Latin “non” and “not” are the same word, and so the distinction required special discussion. It became common to use infinite negation, and logicians pondered its logic. Some writers in the twelfth and thirteenth centuries adopted a principle called “conversion by contraposition.” It states that

- ‘Every *S* is *P*’ is equivalent to ‘Every non-*P* is non-*S*’
- ‘Some *S* is not *P*’ is equivalent to ‘Some non-*P* is not non-*S*’

Unfortunately, this principle (which is not endorsed by Aristotle) conflicts with the idea that there may be empty or universal terms. For in the universal case it leads directly from the truth:

Every man is a being

to the falsehood:

Every non-being is a non-man

(which is false because the universal affirmative has existential import, and there are no non-beings). And in the particular case it leads from the truth (remember that the ***O*** form has no existential import):

A chimera is not a man

to the falsehood:

A non-man is not a non-chimera

These are [Jean] Buridan's examples, used in the fourteenth century to show the invalidity of contraposition. Unfortunately, by Buridan's time the principle of contraposition had been advocated by a number of authors. The doctrine is already present in several twelfth century tracts, and it is endorsed in the thirteenth century by Peter of Spain, whose work was republished for centuries, by William Sherwood, and by Roger Bacon. By the fourteenth century, problems associated with contraposition seem to be well-known, and authors generally cite the principle and note that it is not valid, but that it becomes valid with an additional assumption of existence of things falling under the subject term. For example, Paul of Venice in his eclectic and widely published *Logica Parva* from the end of the fourteenth century gives the traditional square with simple conversion but rejects conversion by contraposition, essentially for Buridan's reason.[1]

—Terence Parsons, The Stanford Encyclopedia of Philosophy

The term

A term (Greek *horos*) is the basic component of the proposition. The original meaning of the *horos* (and also of the Latin *terminus*) is "extreme" or "boundary". The two terms lie on the outside of the proposition, joined by the act of affirmation or denial. For early modern logicians like Arnauld (whose *Port-Royal Logic* was the best-known text of his day), it is a psychological entity like an "idea" or "concept". Mill considers it a word. To assert "all Greeks are men" is not to say that the concept of Greeks is the concept of men, or that word "Greeks" is the word "men". A proposition cannot be built from real things or ideas, but it is not just meaningless words either.

The proposition

In term logic, a "proposition" is simply a *form of language*: a particular kind of sentence, in which the subject and predicate are combined, so as to assert something true or false. It is not a thought, or an abstract entity. The word *"propositio"* is from the Latin, meaning the first premise of a syllogism. Aristotle uses the word premise (*protasis*) as a sentence affirming or denying one thing of another (*Posterior Analytics* 1. 1 24a 16), so a premise is also a form of words. However, as in modern philosophical logic, it means that which is asserted by the sentence. Writers before Frege and Russell, such as Bradley, sometimes spoke of the "judgment" as something distinct from a sentence, but this is not quite the same. As a further confusion the word "sentence" derives from the Latin, meaning an opinion or judgment, and so is equivalent to "proposition". The *logical quality* of a proposition is whether it is affirmative (the predicate is affirmed of the subject) or negative (the predicate is denied of the subject). Thus *every philosopher is mortal* is affirmative, since the mortality of philosophers is affirmed universally, whereas *no philosopher is mortal* is negative by denying such mortality in particular. The *quantity* of a proposition is whether it is universal (the predicate is affirmed or denied of all subjects or of "the whole") or particular (the predicate is affirmed or denied of some subject or a "part" thereof). In case where existential import is assumed, quantification implies the existence of at least one subject, unless disclaimed.

Singular terms

For Aristotle, the distinction between singular and universal is a fundamental metaphysical one, and not merely grammatical. A singular term for Aristotle is primary substance, which can only be predicated of itself: (this) "Callias" or (this) "Socrates" are not predicable of any other thing, thus one does not say *every Socrates* one says *every human* (*De Int.* 7; *Meta.* Δ9, 1018a4). It may feature as a grammatical predicate, as in the sentence "the person coming this way is Callias". But it is still a *logical* subject.

He contrasts "universal" (*katholou*, "whole") secondary substance, genera, with primary substance, particular specimens. The formal nature of universals, in so far as they can be generalized "always, or for the most part", are the subject matter of both scientific study and formal logic.[2]

The essential feature of the syllogistic is that, of the four terms in the two premises, one must occur twice. Thus

All Greeks are **men**

All **men** are mortal.

The subject of one premise, must be the predicate of the other, and so it is necessary to eliminate from the logic any terms which cannot function both as subject and predicate, namely singular terms.

However, in a popular 17th century version of the syllogistic, Port-Royal Logic, singular terms were treated as universals:[3]

All men are mortals

All Socrates are men

All Socrates are mortals

This is clearly awkward, a weakness exploited by Frege in his devastating attack on the system (from which, ultimately, it never recovered, see concept and object).

The famous syllogism "Socrates is a man ...", is frequently quoted as though from Aristotle,[4] but fact, it is nowhere in the *Organon*. It is first mentioned by Sextus Empiricus in his *Hyp. Pyrrh*. ii. 164.

Decline of term logic

Term logic began to decline in Europe during the Renaissance, when logicians like Rodolphus Agricola Phrisius (1444–1485) and Ramus began to promote place logics. The logical tradition called Port-Royal Logic, or sometimes "traditional logic", claimed that a proposition was a combination of ideas rather than terms, but otherwise followed many of the conventions of term logic. It was influential, especially in England, until the 19th century. Leibniz created a distinctive logical calculus, but nearly all of his work on logic was unpublished and unremarked until Louis Couturat went through the Leibniz *Nachlass* around 1900, publishing his pioneering studies in logic.

19th century attempts to algebratize logic, such as the work of Boole and Venn, typically yielded systems highly influenced by the term logic tradition. The first predicate logic was that of Frege's landmark *Begriffsschrift*, little read before 1950, in part because of its eccentric notation. Modern predicate logic as we know it began in the 1880s with the writings of Charles Sanders Peirce, who influenced Peano and even more, Ernst Schröder. It reached fruition in the hands of Bertrand Russell and A. N. Whitehead, whose *Principia Mathematica* (1910–13) made splendid use of a variant of Peano's predicate logic.

Term logic also survived to some extent in traditional Roman Catholic education, especially in seminaries. Medieval Catholic theology, especially the writings of Thomas Aquinas, had a powerfully Aristotelean cast, and thus term logic became a part of Catholic theological reasoning. For example, Joyce (1949), written for use in Catholic seminaries, made no mention of Frege or Bertrand Russell.[5]

A revival

Some philosophers have complained that predicate logic:

- Is unnatural in a sense, in that its syntax does not follow the syntax of the sentences that figure in our everyday reasoning. It is, as Quine acknowledged, "Procrustean," employing an artificial language of function and argument, quantifier and bound variable.
- Suffers from theoretical problems, probably the most serious being empty names and identity statements.

Even academic philosophers entirely in the mainstream, such as Gareth Evans, have written as follows:

> "I come to semantic investigations with a preference for *homophonic* theories; theories which try to take serious account of the syntactic and semantic devices which actually exist in the language ...I would prefer [such] a theory ... over a theory which is only able to deal with [sentences of the form "all A's are B's"] by "discovering" hidden logical constants ... The objection would not be that such [Fregean] truth conditions are not correct, but that, in a sense which we would all dearly love to have more exactly explained, the syntactic shape of the sentence is treated as so much misleading surface structure" (Evans 1977)

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External links

- Term logic (http://philpapers.org/browse/aristotelian-logic) at PhilPapers
- Aristotle's Logic (http://plato.stanford.edu/entries/aristotle-logic) entry by Robin Smith in the *Stanford Encyclopedia of Philosophy*
- Term logic (http://www.iep.utm.edu/aris-log) entry in the *Internet Encyclopedia of Philosophy*
- Aristotle's term logic online (http://aristotelianlogic.glashoff.net) -- This online program provides a platform for experimentation and research on Aristotelian logic.
- Annotated bibliographies of writings by:
  - Fred Sommers. (http://www.ontology.co/sommersf.htm)
  - George Englebretsen. (http://www.ontology.co/biblio/englebretseng.htm)
- PlanetMath: Aristotelian Logic.

- Interactive Syllogistic Machine for Term Logic (http://thefirstscience.org/syllogistic-machine/) A web based syllogistic machine for exploring fallacies, figures, terms, and modes of syllogisms.

Prior Analytics

The ***Prior Analytics*** is Aristotle's work on deductive reasoning, specifically the syllogism. It is also part of his *Organon*, which is the *instrument* or *manual* of logical and scientific methods.

Analytics comes from the Greek word "analutos" meaning "solvable" and the Greek verb "analuein" meaning "to solve". However, in Aristotle's corpus, there are distinguishable differences in the meaning of "analuein" and its cognates. There is also the possibility that Aristotle may have borrowed his use of the word "analysis" from his teacher Plato. On the other hand, the meaning that best fits the *Analytics* is one derived from the study of Geometry and this meaning is very close to what Aristotle calls ἐπιστήμη "episteme", knowing the reasoned facts. Therefore, Analysis is the process of finding the reasoned facts.[1]

Of the entire Aristotelian corpus, Aristotle gives priority to the study of his treatises on Logic. However, he never gave a general name to his treatises on Logic nor did he coin the word Logic. Aristotle's *Prior Analytics* represents the first time in history when Logic is scientifically investigated. On those grounds alone, Aristotle could be considered the Father of Logic for as he himself says in *Sophistical Refutations*, "... When it comes to this subject, it is not the case that part had been worked out before in advance and part had not; instead, nothing existed at all." [2]

A problem in meaning arises in the study of *Prior Analytics* for the word "syllogism" as used by Aristotle in general does not carry the same narrow connotation as it does at present; Aristotle defines this term in a way that would apply to a wide range of valid arguments. Some scholars prefer to use the word "deduction" instead as the meaning given by Aristotle to the Greek word συλλογισμός "sullogismos". At present, "syllogism" is used exclusively as the method used to reach a conclusion which is really the narrow sense in which it is used in the Prior Analytics dealing as it does with a much narrower class of arguments closely resembling the "syllogisms" of traditional logic texts: two premises followed by a conclusion each of which is a categorial sentence containing all together three terms, two extremes which appear in the conclusion and one middle term which appears in both premises but not in the conclusion. In the *Analytics* then, *Prior Analytics* is the first theoretical part dealing with the science of deduction and the *Posterior Analytics* is the second demonstratively practical part. *Prior Analytics* gives an account of deductions in general narrowed down to three basic syllogisms while *Posterior Analytics* deals with demonstration.[3]

In the Prior Analytics, Aristotle defines syllogism as "... A deduction in a discourse in which, certain things being supposed, something different from the things supposed results of necessity because these things are so." In modern times, this definition has led to a debate as to how the word "syllogism" should be interpreted. Scholars Jan Lukasiewicz, Józef Maria Bocheński and Günther Patzig have sided with the Protasis-Apodosis dichotomy while John Corcoran prefers to consider a syllogism as simply a deduction.[4]

In the third century AD, Alexander of Aphrodisias's commentary on the Prior Analytics is the oldest extant and one of the best of the ancient tradition and is presently available in the English language.[5]

In the sixth century, the first translation of Prior Analytics by Boethius appeared in Latin. No Westerner between Boethius and Abelard is known to have read the Prior Analytics. *Anonymus Aurelianensis III* from the second half of the twelfth century is the first extant Latin commentary.[6]

The Syllogism

The Prior Analytics represents the first formal study of logic which is the study of arguments; argument being in logic a series of true or false statements which lead to a true or false conclusion.[7] In the Prior Analytics, Aristotle identifies valid and invalid forms of arguments called syllogisms. A syllogism is an argument consisting of three sentences: two premises and a conclusion. Although Aristotles does not call them "categorical sentences," tradition does; he deals with them briefly in the *Analytics* and more extensively in *On Interpretation*.[8] Each proposition (statement that is a thought of the kind expressible by a declarative sentence)[9] of a syllogism is a categorical sentence which has a subject and a predicate connected by a verb. The usual way of connecting the subject and predicate of a categorical sentence as Aristotle does in *On Interpretation* is by using a linking verb e.g. P is S. However, in the Prior Analytics Aristotle rejects the usual form in favor of three of his inventions: 1) P belongs to S, 2) P is predicated of S and 3) P is said of S. Aristotle does not explain why he introduces these innovative expressions but scholars conjecture that the reason may have been that it facilitates the use of letters instead of terms avoiding the ambiguity that results in Greek when letters are used with the linking verb.[10] In his formulation of syllogistic propositions, instead of the copula ("All/some... are/are not..."), Aristotle uses the expression, "... belongs to/does not belong to all/some..." or "... is said/is not said of all/some..."[11] There are four different types of categorical sentences: universal affirmative (A), particular affirmative (I), universal negative (E) and particular negative (O).

- A - A belongs to every B
- E — A belongs to no B
- I - A belongs to some B
- O - A does not belong to some B

A method of symbolization that originated and was used in the Middle Ages greatly simplifies the study of the Prior Analytics. Following this tradition then, let:

a = belongs to every

e = belongs to no

i = belongs to some

o = does not belong to some

Categorical sentences may then be abbreviated as follows:

AaB = A belongs to every B (Every B is A)

AeB = A belongs to no B (No B is A)

AiB = A belongs to some B (Some B is A)

AoB = A does not belong to some B (Some B is not A)

From the viewpoint of modern logic, only a few sentences may be represented in this way.[12]

The Three Figures

Depending on the position of the middle term, Aristotle divides the syllogism into three kinds: Syllogism in the first, second and third figure.[13] If the Middle Term is subject of one premise and predicate of the other, the premises are in the First Figure. If the Middle Term is predicate of both premises, the premises are in the Second Figure. If the Middle Term is subject of both premises, the premises are in the Third Figure.[14]

Symbolically, the Three Figures may be represented as follows:

	First Figure	**Second Figure**	**Third Figure**
	Predicate — Subject	Predicate — Subject	Predicate — Subject
Major Premise	A ------------ B	B ------------ A	A ------------ B
Minor Premise	B ------------ C	B ------------ C	C ------------ B
Conclusion	A ********** C	A ********** C	A ********** C

[15]

Syllogism in the first figure

In the Prior Analytics translated by A. J. Jenkins as it appears in volume 8 of the Great Books of the Western World, Aristotle says of the First Figure: "... If A is predicated of all B, and B of all C, A must be predicated of all C."[16] In the Prior Analytics translated by Robin Smith, Aristotle says of the first figure: "... For if A is predicated of every B and B of every C, it is necessary for A to be predicated of every C."[17]

Taking a = is predicated of all = is predicated of every, and using the symbolical method used in the Middle Ages, then the first figure is simplified to:

If AaB

and BaC

then AaC.

Or what amounts to the same thing:

AaB, BaC; $therefore$ AaC [18]

When the four syllogistic propositions, a, e, i, o are placed in the first figure, Aristotle comes up with the following valid forms of deduction for the first figure:

AaB, BaC; therefore, AaC

AeB, BaC; therefore, AeC

AaB, BiC; therefore, AiC

AeB, BiC; therefore, AoC

In the Middle Ages, for mnemonic reasons they were called respectively "Barbara", "Celarent", "Darii" and "Ferio".[19]

The difference between the first figure and the other two figures is that the syllogism of the first figure is complete while that of the second and fourth is not. This is important in Aristotle's theory of the syllogism for the first figure is axiomatic while the second and third require proof. The proof of the second and third figure always leads back to the first figure.[20]

Syllogism in the second figure

This is what Robin Smith says in English that Aristotle said in Ancient Greek: "... If M belongs to every N but to no X, then neither will N belong to any X. For if M belongs to no X, neither does X belong to any M; but M belonged to every N; therefore, X will belong to no N (for the first figure has again come about)."[21]

The above statement can be simplified by using the symbolical method used in the Middle Ages:

If MaN

but MeX

then NeX.

For if MeX

then XeM

but MaN

therefore XeN.

When the four syllogistic propositions, a, e, i, o are placed in the second figure, Aristotle comes up with the following valid forms of deduction for the second figure:

MaN, MeX; therefore NeX

MeN, MaX; therefore NeX

MeN, MiX; therefore NoX

MaN, MoX; therefore NoX

In the Middle Ages, for mnemonic resons they were called respectively "Camestres", "Cesare", "Festino" and "Baroco".[22]

Syllogism in the third figure

Aristotle says in the Prior Analytics, "... If one term belongs to all and another to none of the same thing, or if they both belong to all or none of it, I call such figure the third." Referring to universal terms, "... then when both P and R belongs to every S, it results of necessity that P will belong to some R."[23]

Simplifying:

If PaS

and RaS

then PiR.

When the four syllogistic propositions, a, e, i, o are placed in the third figure, Aristotle develops six more valid forms of deduction:

PaS, RaS; therefore PiR

PeS, RaS; therefore PoR

PiS, RaS; therefore PiR

PaS, RiS; therefore PiR

PoS, RaS; therefore PoR

PeS, RiS; therefore PoR

In the Middle Ages, for mnemonic reasons, these six forms were called respectively: "Darapti", "Felapton", "Disamis", "Datisi", "Bocardo"and "Ferison".[24]

Table of syllogisms

Figure	Major Premise	Minor Premise	Conclusion	Mnemonic Name
First Figure	AaB	BaC	AaC	Barbara
	AeB	BaC	AeC	Celarent
	AaB	BiC	AiC	Darii
	AeB	BiC	AoC	Ferio
Second Figure	MaN	MeX	NeX	Camestres
	MeN	MaX	NeX	Cesare
	MeN	MiX	NoX	Festino
	MaN	MoX	NoX	Baroco
Third Figure	PaS	RaS	PiR	Darapti
	PeS	RaS	PoR	Felapton
	PiS	RaS	PiR	Disamis
	PaS	RiS	PiR	Datisi
	PoS	RaS	PoR	Bocardo
	PeS	RiS	PoR	Ferison

[25]

The Fourth Figure

"In Aristotelian syllogistic (*Prior Analytics*, Bk I Caps 4-7), syllogisms are divided into three figures according to the position of the middle term in the two premisses. The fourth figure, in which the middle term is the predicate in the major premiss and the subject in the minor, was added by Aristotle's pupil Theophrastus and does not occur in Aristotle's work, although there is evidence that Aristotle knew of fourth-figure syllogisms."[26]

Notes

[1] Patrick Hugh Byrne (1997). *Analysis and Science in Aristotle*. SUNY Press. p. 3. ISBN 0-7914-3321-8. "... while "decompose" - the most prevalent connotation of "analyze" in the modern period — is among Aristotle's meanings, it is neither the sole meaning nor the principal meaning nor the meaning which best characterizes the work, Analytics."

[2] Jonathan Barnes, ed. (1995). *The Cambridge Companion to Aristotle*. Cambridge University Press. p. 27. ISBN 0-521-42294-9. "History's first logic has also been the most influential..."

[3] Smith, Robin (1989). *Aristotle: Prior Analytics*. Hackett Publishing Co.. pp. XIII-XVI. ISBN 0-87220-064-7. "... This leads him to what I would regard as the most original and brilliant insight in the entire work."

[4] Lagerlund, Henrik (2000). *Modal Syllogistics in the Middle Ages*. BRILL. pp. 3–4. ISBN 90-04-11626-9 . "In the *Prior Analytics* Aristotle presents the first logical system, i.e., the theory of the syllogisms."

[5] Striker, Gisela (2009). *Aristotle: Prior Analytics, Book 1*. Oxford University Press. p. xx. ISBN 0-19-925041-7 .

[6] Ebbesen, Sten (2008). *Greek-Latin philosophical interaction*. Ashgate Publishing Ltd.. pp. 171–173. ISBN 0-7546-5837-5 . "Authoritative texts beget commentaries. Boethus of Sidon (late first century BC?) may have been one of the first to write one on *Prior Analytics*."

[7] Nolt, John; Rohatyn, Dennis (1988). *Logic: Schaum's outline of theory and problems*. McGraw Hill. p. 1. ISBN 0-07-053628-7.

[8] Robin Smith. *Aristotle: Prior Analytics*. p. XVII.

[9] John Nolt/Dennis Rohatyn. *Logic: Schaum's Outline of Theory and Problems*. pp. 274–275.

[10] Anagnostopoulos, Georgios (2009). *A Companion to Aristotle*. Wiley-Blackwell. p. 33. ISBN 1-4051-2223-8 .

[11] Patzig, Günther (1969). *Aristotle's theory of the syllogism*. Springer. p. 49. ISBN 90-277-0030-8 .

[12] *The Cambridge Companion to Aristotle*. pp. 34–35.

[13] *The Cambridge Companion to Aristotle*. p. 35. "At the foundation of Aristotle's syllogistic is a theory of a specific class of arguments: arguments having as premises exactly two categorical sentences with one term in common."

[14] Robin Smith. *Aristotle: Prior Analytics*. p. XVIII.

[15] Henrik Legerlund. *Modal Syllogistics in the Middle Ages*. p. 4.
[16] *Great Books of the Western World*. **8**. p. 40.
[17] Robin Smith. *Aristotle: Prior Analytics*. p. 4.
[18] *The Cambridge Companion to Aristotle*. p. 41.
[19] *The Cambridge Companion to Aristotle*. p. 41.
[20] Henrik Legerlund. *Modal Syllogistics in the Middle Ages*. p. 6.
[21] Robin Smith. *Aristotle: Prior Analytics*. p. 7.
[22] *The Cambridge Companion to Aristotle*. p. 41.
[23] Robin Smith. *Aristotle: Prior Analytics*. p. 9.
[24] *The Cambridge Companion to Aristotle*. p. 41.
[25] *The Cambridge Companion to Aristotle*. p. 41.
[26] Russell, Bertrand; Blackwell, Kenneth (1983). *Cambridge essays, 1888-99*. Routledge. p. 411. ISBN 0-04-920067-8 .

External links

- The text of the Prior Analytics is available from the MIT classics archive (http://classics.mit.edu/Aristotle/prior.html).
- **Prior Analytics**, trans. by A. J. Jenkinson
  - http://etext.library.adelaide.edu.au/a/a8pra/
- A Public Domain Audio Book Version of the Prior Analytics is available from Librivox.org (http://librivox.org/prior-analytics-by-aristotle/).
- Aristotle's Logic (http://plato.stanford.edu/entries/aristotle-logic) entry by Robin Smith in the *Stanford Encyclopedia of Philosophy*
- Aristotle's Prior Analytics: the Theory of Categorical Syllogism (http://www.ontology.co/aristotle-syllogism-categorical.htm) an annotated bibliography on Aristotle's syllogistic

Syllogism

A **syllogism** (Greek: συλλογισμός – *syllogismos* – "conclusion," "inference") is a kind of logical argument in which one proposition (the conclusion) is inferred from two or more others (the premises) of a specific form. In antiquity, two rival theories of the syllogism existed: Aristotelian syllogistic and Stoic syllogistic.[1]

Aristotle defines the syllogism as "a discourse in which certain (specific) things having been supposed, something different from the things supposed results of necessity because these things are so." [2]

Despite this very general definition, in the *Prior Analytics* Aristotle limits himself to categorical syllogisms, which consist of three categorical propositions.[3] These included categorical modal syllogisms.[4] From the Middle Ages onwards, "categorical syllogism" and "syllogism" were mostly used interchangeably, and the present article is concerned with this traditional use of "syllogism" only. The syllogism was at the core of traditional deductive reasoning, where facts are determined by combining existing statements, in contrast to inductive reasoning where facts are determined by repeated observations.

Within academic contexts, the syllogism was superseded by first-order predicate logic following the work of Gottlob Frege, in particular his *Begriffsschrift* (*Concept Script*) (1879), but syllogisms remain useful in some circumstances, and for general-audience introductions to logic.[5][6]

Basic structure

A categorical syllogism consists of three parts:

- Major premise
- Minor premise
- Conclusion

Each part is a categorical proposition, and each categorical proposition contains two categorical terms.[7] In Aristotle, each of the premises is in the form "All A are B," "Some A are B", "No A are B" or "Some A are not B", where "A" is one term and "B" is another. "All A are B," and "No A are B" are termed *universal* propositions; "Some A are B" and "Some A are not B" are termed *particular* propositions. More modern logicians allow some variation. Each of the premises has one term in common with the conclusion: in a major premise, this is the *major term* (*i.e.,* the predicate of the conclusion); in a minor premise, it is the *minor term* (the subject) of the conclusion. For example:

> Major premise: All humans are mortal.
>
> Minor premise: All Greeks are humans.
>
> Conclusion: All Greeks are mortal.

Each of the three distinct terms represents a category. In the above example, *humans*, *mortal*, and *Greeks*. *Mortal* is the major term, *Greeks* the minor term. The premises also have one term in common with each other, which is known as the *middle term*; in this example, *humans*. Both of the premises are universal, as is the conclusion.

> Major premise: All mortals die.
>
> Minor premise: Some mortals are men.
>
> Conclusion: Some men die.

Here, the major term is *die*, the minor term is *men*, and the middle term is *mortals*. The major premise is universal; the minor premise and the conclusion are particular.

A sorites is a form of argument in which a series of incomplete syllogisms is so arranged that the predicate of each premise forms the subject of the next until the subject of the first is joined with the predicate of the last in the conclusion. For example, if one argues that a given number of grains of sand does not make a heap and that an additional grain does not either, then to conclude that no additional amount of sand would make a heap is to construct a sorites argument.

Types of syllogism

There are infinitely many possible syllogisms, but only a finite number of logically distinct types, which we classify and enumerate below. Note that the syllogism above has the abstract form:

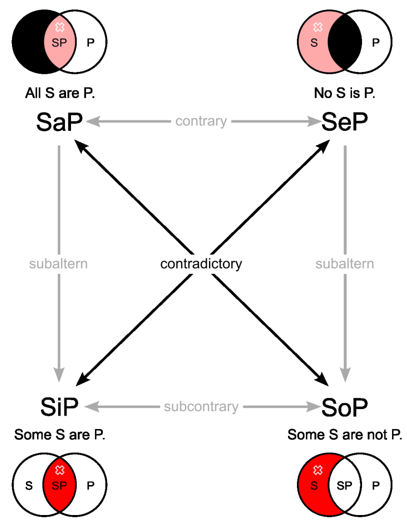
Major premise: All M are P.

Minor premise: All S are M.

Conclusion: All S are P.

(Note: M – Middle, S – subject, P – predicate. See below for more detailed explanation.)

The premises and conclusion of a syllogism can be any of four types, which are labeled by letters[8] as follows. The meaning of the letters is given by the table:



Relationships between the four types of propositions in the square of opposition (Black areas are empty, red areas are nonempty.)

code	*quantifier*	*subject*	*copula*	*predicate*	*type*	*example*
a	All	S	are	P	universal affirmatives	All humans are mortal.
e	No	S	are	P	universal negatives	No humans are perfect.
i	Some	S	are	P	particular affirmatives	Some humans are healthy.
o	Some	S	are not	P	particular negatives	Some humans are not clever.

In Analytics, Aristotle mostly uses the letters A, B and C (actually, the Greek letters alpha, beta and gamma) as term place holders, rather than giving concrete examples, an innovation at the time. It is traditional to use *is* rather than *are* as the copula, hence *All A is B* rather than *All As are Bs*. It is traditional and convenient practice to use a, e, i, o as infix operators to enable the categorical statements to be written succinctly thus:

Form	Shorthand
All A is B	AaB
No A is B	AeB
Some A is B	AiB
Some A is not B	AoB

The letter S is the subject of the conclusion, P is the predicate of the conclusion, and M is the middle term. The major premise links M with P and the minor premise links M with S. However, the middle term can be either the subject or the predicate of each premise where it appears. The differing positions of the major, minor, and middle terms gives rise to another classification of syllogisms known as the *figure*. Given that in each case the conclusion is S-P, the four figures are:

	Figure 1	*Figure 2*	*Figure 3*	*Figure 4*
Major premise:	M–P	P–M	M–P	P–M
Minor premise:	S–M	S–M	M–S	M–S

(Note, however, that, following Aristotle's treatment of the figures, some logicians—e.g., Peter Abelard and John Buridan—reject the fourth figure as a figure distinct from the first. See entry on the Prior Analytics.)

Putting it all together, there are 256 possible types of syllogisms (or 512 if the order of the major and minor premises is changed, though this makes no difference logically). Each premise and the conclusion can be of type A, E, I or O, and the syllogism can be any of the four figures. A syllogism can be described briefly by giving the letters for the premises and conclusion followed by the number for the figure. For example, the syllogism BARBARA above is AAA-1, or "A-A-A in the first figure".

The vast majority of the 256 possible forms of syllogism are invalid (the conclusion does not follow logically from the premises). The table below shows the valid forms. Even some of these are sometimes considered to commit the existential fallacy, meaning they are invalid if they mention an empty category. These controversial patterns are marked in *italics*.

Figure 1	*Figure 2*	*Figure 3*	*Figure 4*
Barbara	Cesare	Datisi	Calemes
Celarent	Camestres	Disamis	Dimatis
Darii	Festino	Ferison	Fresison
Ferio	Baroco	Bocardo	*Calemos*
Barbari	*Cesaro*	*Felapton*	*Fesapo*
Celaront	*Camestros*	*Darapti*	*Bamalip*

The letters A, E, I, O have been used since the medieval Schools to form mnemonic names for the forms as follows: 'Barbara' stands for AAA, 'Celarent' for EAE, etc.

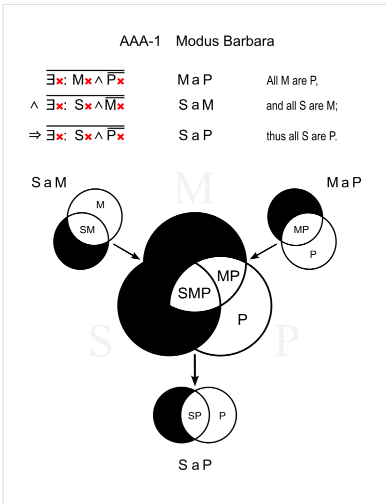
Next to each premise and conclusion is a shorthand description of the sentence. So in AAI-3, the premise "All squares are rectangles" becomes "MaP"; the symbols mean that the first term ("square") is the middle term, the second term ("rectangle") is the predicate of the conclusion, and the relationship between the two terms is labeled "a" (All M are P).

The following table shows all syllogisms that are essentially different. The similar syllogisms share actually the same premises, just written in a different way. For example "Some pets are kittens" (SiM in Darii) could also be written as "Some kittens are pets" (MiS is Datisi).

In the Venn diagrams, the black areas indicate no elements, and the red areas indicate at least one element.

Examples

Barbara (AAA-1)

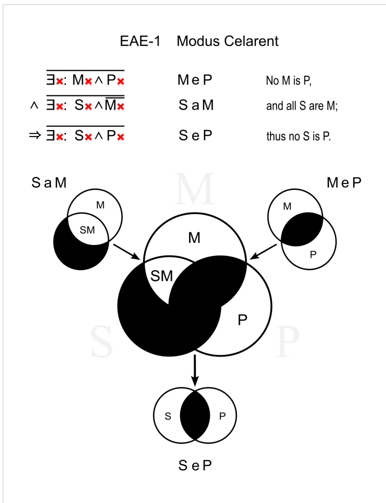


All men are mortal. (MaP)

All Greeks are men. (SaM)

∴ All Greeks are mortal. (SaP)

Celarent (EAE-1)



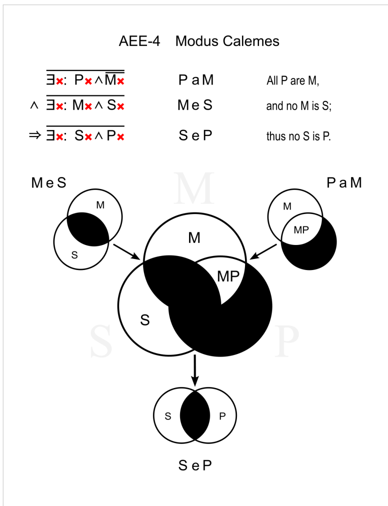
Similar: Cesare (EAE-2)

No reptiles have fur. (MeP)

All snakes are reptiles. (SaM)

∴ No snakes have fur. (SeP)

Calemes (AEE-4)



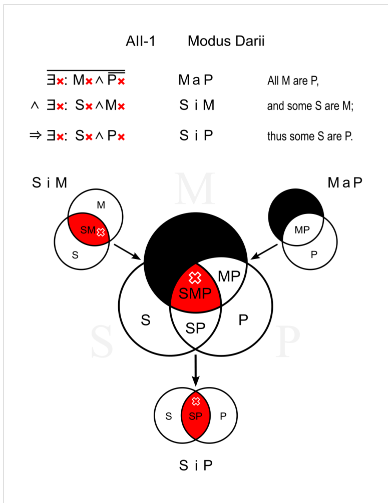
Calemes is like Celarent with S and P exchanged.
Similar: Camestres (AEE-2)

All snakes are reptiles. (PaM)

No reptiles have fur. (MeS)

∴ No fur bearing animal is a snake. (SeP)

Darii (AII-1)



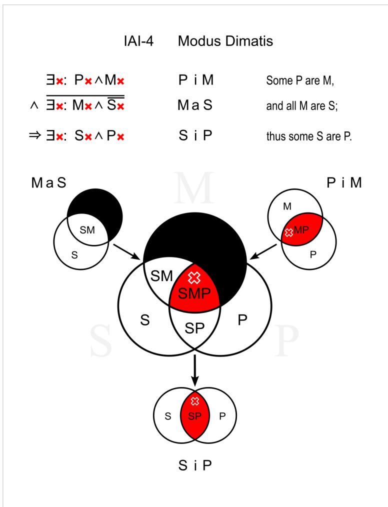
Similar: Datisi (AII-3)

All rabbits have fur. (MaP)

Some pets are rabbits. (SiM)

∴ Some pets have fur. (SiP)

Dimatis (IAI-4)



Dimatis is like Darii with S and P exchanged.

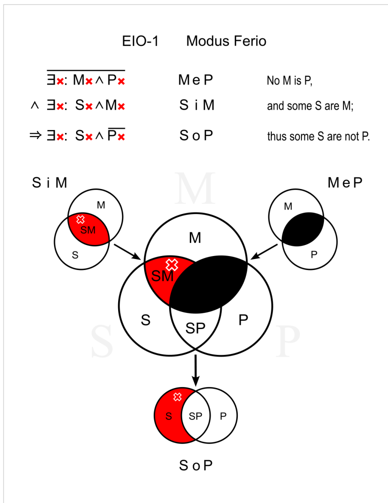
Similar: Disamis (IAI-3)

Some pets are rabbits. (PiM)

All rabbits have fur. (MaS)

∴ Some fur bearing animals are pets. (SiP)

Ferio (EIO-1)



Similar: Festino (EIO-2), Ferison (EIO-3), Fresison (EIO-4)

No homework is fun. (MeP)

Some reading is homework. (SiM)

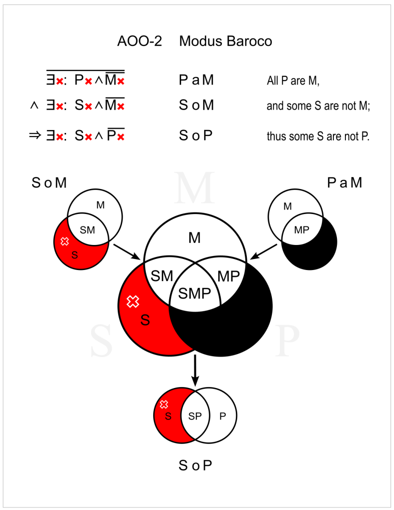
∴ Some reading is not fun. (SoP)

Baroco (AOO-2)

All informative things are useful. (PaM)

Some websites are not useful. (SoM)

∴ Some websites are not informative. (SoP)

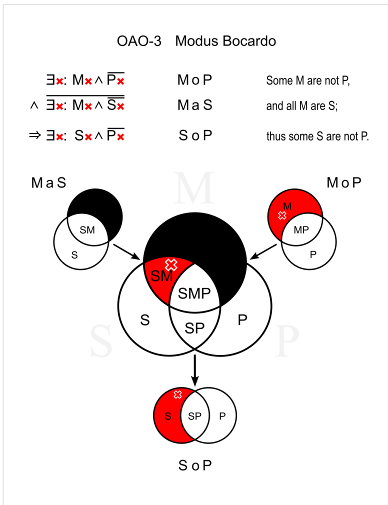


Bocardo (OAO-3)

Some cats have no tails. (MoP)

All cats are mammals. (MaS)

∴ Some mammals have no tails. (SoP)

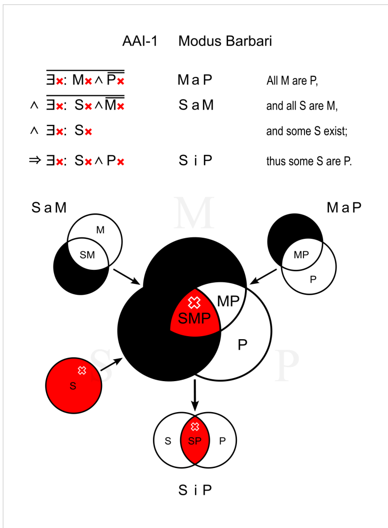


Barbari (AAI-1)

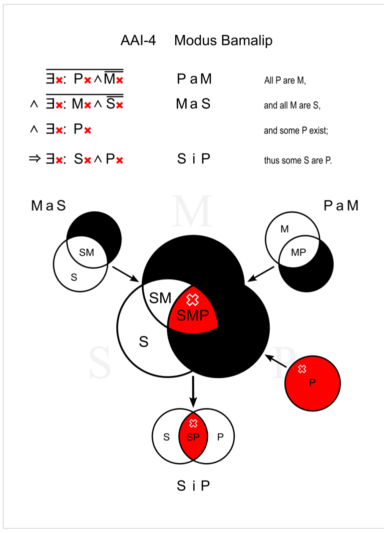
All men are mortal. (MaP)

All Greeks are men. (SaM)

∴ Some Greeks are mortal. (SiP)



Bamalip (AAI-4)



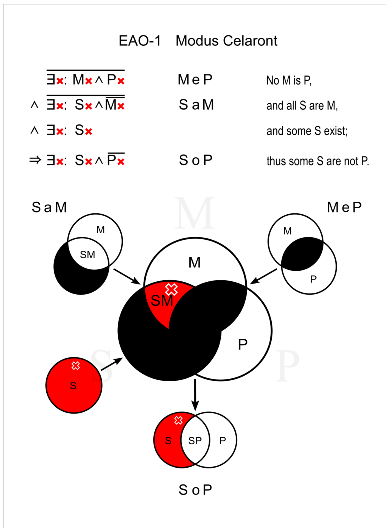
Bamalip is like *Barbari* with S and P exchanged:

All Greeks are men. (PaM)

All men are mortal. (MaS)

∴ Some mortals are Greek. (SiP)

Celaront (EAO-1)



Similar: *Cesaro (EAO-2)*

No reptiles have fur. (MeP)

All snakes are reptiles. (SaM)

∴ Some snakes have no fur. (SoP)

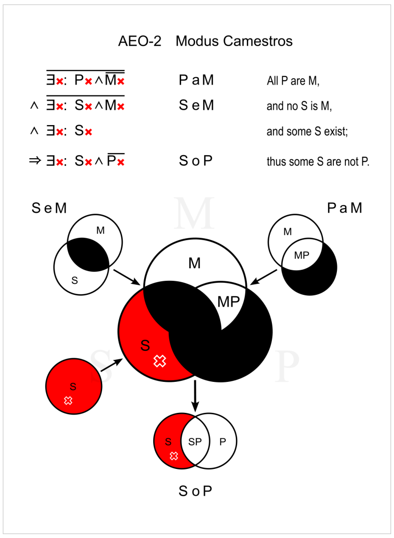
Camestros (AEO-2)

Similar: *Calemos (AEO-4)*

All horses have hooves. (PaM)

No humans have hooves. (SeM)

∴ Some humans are not horses. (SoP)



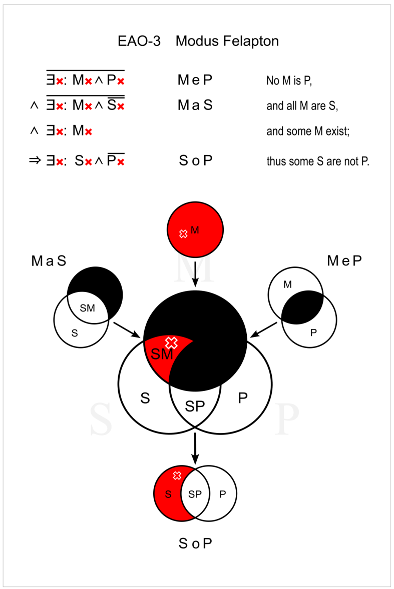
Felapton (EAO-3)

Similar: *Fesapo (EAO-4)*

No flowers are animals. (MeP)

All flowers are plants. (MaS)

∴ Some plants are not animals. (SoP)



Darapti (AAI-3)

All squares are rectangles. (MaP)

All squares are rhombs. (MaS)

∴ Some rhombs are rectangles. (SiP)

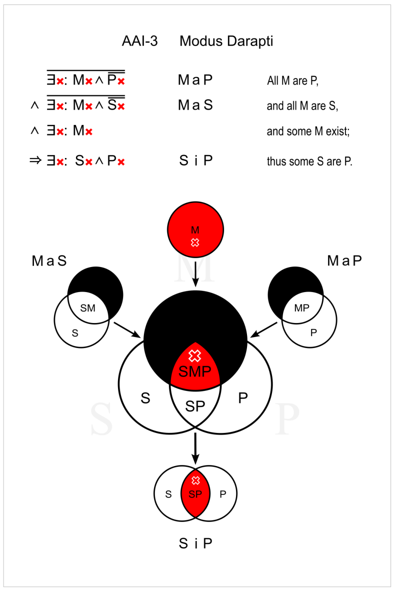


Table of all syllogisms

This table shows all 24 valid syllogisms, represented by Venn diagrams.
(9 of them, on the right side of the table, require that one category must not be empty.)
Syllogisms of the same type are in the same row, and very similar syllogisms are in the same column.

1	Barbara	Celarent		Darii		Ferio			Barbari	Celaront				
2		Cesare	Camestres			Festino	Baroco			Cesaro	Camestros			
3				Datisi	Disamis	Ferison		Bocardo				Felapton	Darapti	
4			Calemes		Dimatis	Fresison					Calemos	Fesapo		Bamalip

Terms in syllogism

We may, with Aristotle, distinguish **singular** terms such as *Socrates* and **general** terms such as *Greeks*. Aristotle further distinguished (a) terms that could be the subject of predication, and (b) terms that could be predicated of others by the use of the copula (is are). (Such a predication is known as a distributive as opposed to non-distributive as in *Greeks are numerous*. It is clear that Aristotle's syllogism works only for distributive predication for we cannot reason *All Greeks are animals, animals are numerous, therefore All Greeks are numerous*.) In Aristotle's view singular terms were of type (a) and general terms of type (b). Thus *Men* can be predicated of *Socrates* but *Socrates* cannot be predicated of anything. Therefore to enable a term to be interchangeable — that is to be either in the subject or predicate position of a proposition in a syllogism — the terms must be general terms, or categorical terms as they came to be called. Consequently the propositions of a syllogism should be categorical propositions (both terms general) and syllogisms employing just categorical terms came to be called categorical syllogisms.

It is clear that nothing would prevent a singular term occurring in a syllogism — so long as it was always in the subject position — however such a syllogism, even if valid, would not be a categorical syllogism. An example of such would be *Socrates is a man, All men are mortal, therefore Socrates is mortal.* Intuitively this is as valid as *All Greeks are men, all men are mortal therefore all Greeks are mortals*. To argue that its validity can be explained by the theory of syllogism it would be necessary to show that *Socrates is a man* is the equivalent of a categorical proposition. It can be argued *Socrates is a man* is equivalent to *All that are identical to Socrates are men*, so our non-categorical syllogism can be justified by use of the equivalence above and then citing BARBARA.

Existential import

If a statement includes a term so that the statement is false if the term has no instances (is not instantiated) then the statement is said to entail existential import with respect to that term. In particular, a universal statement of the form *All A is B* has existential import with respect to A if *All A is B* is false if there are no As.

The following problems arise:

(a) In natural language and normal use, which statements of the forms All A is B, No A is B, Some A is B and Some A is not B have existential import and with respect to which terms?

(b) In the four forms of categorical statements used in syllogism, which statements of the form AaB, AeB, AiB and AoB have existential import and with respect to which terms?

(c) What existential imports must the forms AaB, AeB, AiB and AoB have for the square of opposition be valid?

(d) What existential imports must the forms AaB, AeB, AiB and AoB have to preserve the validity of the traditionally valid forms of syllogisms?

(e) Are the existential imports required to satisfy (d) above such that the normal uses in natural languages of the forms All A is B, No A is B, Some A is B and Some A is not B are intuitively and fairly reflected by the categorical statements of forms Ahab, Abe, Ail and Alb?

For example, if it is accepted that AiB is false if there are no As and AaB entails AiB, then AiB has existential import with respect to A, and so does AaB. Further, if it is accepted that AiB entails BiA, then AiB and AaB have existential import with respect to B as well. Similarly, if AoB is false if there are no As, and AeB entails AoB, and AeB entails BeA (which in turn entails BoA) then both AeB and AoB have existential import with respect to both A and B. It follows immediately that all universal categorical statements have existential import with respect to both terms. If AaB and AeB is a fair representation of the use of statements in normal natural language of All A is B and No A is B respectively, then the following example consequences arise:

"All flying horses are mythological" is false if there are not flying horses.

If "No men are fire-eating rabbits" is true, then "There are fire-eating rabbits" is false.

and so on.

If it is ruled that no universal statement has existential import then the square of opposition fails in several respects (e.g. AaB does not entail AiB) and a number of syllogisms are no longer valid (e.g. BaC,AaB->AiC).

These problems and paradoxes arise in both natural language statements and statements in syllogism form because of ambiguity, in particular ambiguity with respect to All. If "Fred claims all his books were Pulitzer Prize winners", is Fred claiming that he wrote any books? If not, then is what he claims true? Suppose Jane says none of her friends are poor; is that true if she has no friends? The first-order predicate calculus avoids the problems of such ambiguity by using formulae that carry no existential import with respect to universal statements; existential claims have to be explicitly stated. Thus natural language statements of the forms All A is B, No A is B, Some A is B and Some A is not B can be exactly represented in first order predicate calculus in which any existential import with respect to terms A and/or B is made explicitly or not made at all. Consequently the four forms AaB, AeB, AiB and AoB can be represented in first order predicate in every combination of existential import, so that it can establish which construal, if any, preserves the square of opposition and the validly of the traditionally valid syllogism. Strawson claims that such a construal is possible, but the results are such that, in his view, the answer to question (e) above is *no*.

Syllogism in the history of logic

The Aristotelian syllogism dominated Western philosophical thought from the 3rd Century to the 17th Century. At that time, Sir Francis Bacon rejected the idea of syllogism and deductive reasoning by asserting that it was fallible and illogical.[9] Bacon offered a more inductive approach to logic in which experiments were conducted and axioms were drawn from the observations discovered in them.

In the 19th Century, modifications to syllogism were incorporated to deal with disjunctive ("A or B") and conditional ("if A then B") statements. Kant famously claimed, in *Logic* (1800), that logic was the one completed science, and that Aristotelian logic more or less included everything about logic there was to know. (This work is not necessarily representative of Kant's mature philosophy, which is often regarded as an innovation to logic itself.) Though there were alternative systems of logic such as Avicennian logic or Indian logic elsewhere, Kant's opinion stood unchallenged in the West until 1879 when Frege published his *Begriffsschrift* (*Concept Script*). This introduced a calculus, a method of representing categorical statements — and statements that are not provided for in syllogism as well — by the use of quantifiers and variables.

This led to the rapid development of sentential logic and first-order predicate logic, subsuming syllogistic reasoning, which was, therefore, after 2000 years, suddenly considered obsolete by many. The Aristotelian system is explicated in modern fora of academia primarily in introductory material and historical study.

One notable exception to this modern relegation is the continued application of Aristotelian logic by officials of the Congregation for the Doctrine of the Faith, and the Apostolic Tribunal of the Roman Rota, which still requires that arguments crafted by Advocates be presented in syllogistic format.

Syllogistic fallacies

People often make mistakes when reasoning syllogistically.[10]

For instance, from the premises some A are B, some B are C, people tend to come to a definitive conclusion that therefore some A are C.[11][12] However, this does not follow according to the rules of classical logic. For instance, while some cats (A) are black things (B), and some black things (B) are televisions (C), it does not follow from the parameters that some cats (A) are televisions (C). This is because first, the mood of the syllogism invoked is illicit (III), and second, the supposition of the middle term is variable between that of the middle term in the major premise, and that of the middle term in the minor premise (not all "some" cats are by necessity of logic the same "some black things").

Determining the validity of a syllogism involves determining the distribution of each term in each statement, meaning whether all members of that term are accounted for.

In simple syllogistic patterns, the fallacies of invalid patterns are:

- Undistributed middle: Neither of the premises accounts for all members of the middle term, which consequently fails to link the major and minor term.
- Illicit treatment of the major term: The conclusion implicates all members of the major term (P — meaning the proposition is negative); however, the major premise does not account for them all (i.e., P is either an affirmative predicate or a particular subject there).
- Illicit treatment of the minor term: Same as above, but for the minor term (S — meaning the proposition is universal) and minor premise (where S is either a particular subject or an affirmative predicate).
- Exclusive premises: Both premises are negative, meaning no link is established between the major and minor terms.
- Affirmative conclusion from a negative premise: If either premise is negative, the conclusion must also be.

- Negative conclusion from affirmative premises: If both premises are affirmative, the conclusion must also be.
- Existential fallacy: This is a more controversial one. If both premises are universal, i.e. "All" or "No" statements, one school of thought says they do not imply the existence of any members of the terms. In this case, the conclusion cannot be existential; i.e. beginning with "Some". Another school of thought says that affirmative statements (universal or particular) do imply the subject's existence, but negatives do not. A third school of thought says that the *any* type of proposition may or may not involve the subject's existence, and though this may condition the conclusion, it does not affect the form of the syllogism.

Notes

[1] Michael Frede, "Stoic vs. Peripatetic Syllogistic", Archive for the History of Philosophy 56, 1975, 99-124.

[2] Aristotle, "Prior Analytics", 24b18–20

[3] (http://plato.stanford.edu/entries/logic-ancient/#SynSemSen) Stanford Encyclopedia of Philosophy: *Ancient Logic* Aristotle Non-Modal Syllogistic

[4] (http://plato.stanford.edu/entries/logic-ancient/#ModLog) Stanford Encyclopedia of Philosophy: *Ancient Logic* Aristotle Modal Logic

[5] Hurley, Patrick J (2011). *A Concise Introduction to Logic*, Cengage Learning, ISBN 9780840034175

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[7] "Philosophical Dictionary: Caird-Catharsis" (http://www.philosophypages.com/dy/c.htm#capro). Philosophypages.com. 2002-08-08. . Retrieved 2009-12-14.

[8] According to Copi, p. 127: 'The letter names are presumed to come from the Latin words "***A***ff***I***rmo" and "n***E***g***O***," which mean "I affirm" and "I deny," respectively; the first capitalized letter of each word is for universal, the second for particular'

[9] Bacon, Francis. *The Great Instauration*, 1620

[10] See, e.g., Evans, J. St. B. T (1989). *Bias in human reasoning*. London: LEA.

[11] See the meta-analysis by Khemlani, S. & Johnson-Laird, P.N. (2012). Theories of the syllogism: A meta-analysis. *Psychological Bulletin*, *138*, 427-457.

[12] See the meta-analysis by Chater, N. & Oaksford, M. (1999). The Probability Heuristics Model of Syllogistic Reasoning. *Cognitive Psychology*, *38*, 191–258.

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- Smith Robin, 1986. "Immediate propositions and Aristotle's proof theory," Ancient Philosophy 6: 47–68.

External links

- Aristotle's Logic (http://plato.stanford.edu/entries/aristotle-logic) entry by Robin Smith in the *Stanford Encyclopedia of Philosophy*
- The Traditional Square of Opposition (http://plato.stanford.edu/entries/square) entry by Terence Parsons in the *Stanford Encyclopedia of Philosophy*
- Medieval Theories of the Syllogism (http://plato.stanford.edu/entries/medieval-syllogism) entry by Henrik Lagerlund in the *Stanford Encyclopedia of Philosophy*
- Aristotle's Prior Analytics: the Theory of Categorical Syllogism (http://www.ontology.co/aristotle-syllogism-categorical.htm) an annotated bibliography on Aristotle's syllogistic
- Abbreviatio Montana (http://www.humanities.mq.edu.au/Ockham/x52t06.html) article by Prof. R. J. Kilcullen of Macquarie University on the medieval classification of syllogisms.
- The Figures of the Syllogism (http://www.multicians.org/thvv/petrus-hispanius.html) is a brief table listing the forms of the syllogism.
- Interactive Syllogistic Machine (http://thefirstscience.org/syllogistic-machine/) A web based syllogistic machine for exploring fallacies, figures, and modes of syllogisms.
- Syllogistic Reasoning in Buddhism – Example & Worksheet (http://www.treasuryofwisdom.org/syllogism.pdf)
- Fuzzy Syllogistic System (http://books.google.com.tr/books?id=yfUysMS31PIC&pg=PA418&lpg=PA418&dq=syllogisms+huseyin+cakir&source=bl&ots=h1_c5DHt6_&sig=KJMQP6hJ4IaCHf8BvJu6vl5zCVk&hl=tr&sa=X&ei=mZxlT4HHJoOl0AWws_S4CA&ved=0CFYQ6AEwBw#v=onepage&q=syllogisms huseyin cakir&f=false)

Problem of multiple generality

The **problem of multiple generality** names a failure in traditional logic to describe certain intuitively valid inferences. For example, it is intuitively clear that if:

Some cat is feared by every mouse

then it follows logically that:

All mice are afraid of at least one cat

The syntax of traditional logic (TL) permits exactly four sentence types: "All As are Bs", "No As are Bs", "Some As are Bs" and "Some As are not Bs". Each type is a quantified sentence containing exactly one quantifier. Since the sentences above each contain two quantifiers ('some' and 'every' in the first sentence and 'all' and 'at least one' in the second sentence), they cannot be adequately represented in TL. The best TL can do is to incorporate the second quantifier from each sentence into the second term, thus rendering the artificial-sounding terms 'feared-by-every-mouse' and 'afraid-of-at-least-one-cat'. This in effect "buries" these quantifiers, which are essential to the inference's validity, within the hyphenated terms. Hence the sentence "Some cat is feared by every mouse" is alloted the same logical form as the sentence "Some cat is hungry". And so the logical form in TL is:

Some As are Bs

All Cs are Ds

which is clearly invalid.

The first logical calculus capable of dealing with such inferences was Gottlob Frege's *Begriffsschrift,* the ancestor of modern predicate logic, which dealt with quantifiers by means of variable bindings. Modestly, Frege did not argue that his logic was more expressive than extant logical calculi, but commentators on Frege's logic regard this as one of his key achievements.

Using modern predicate calculus, we quickly discover that the statement is ambiguous.

Some cat is feared by every mouse

could mean *(Some cat is feared) by every mouse*, i.e.

For every mouse m, there exists a cat c, such that c is feared by m,

$$\forall m.\,(\,\mathrm{Mouse}(m) \rightarrow \exists c.\,(\mathrm{Cat}(c) \wedge \mathrm{Fears}(m,c))\,)$$

in which case the conclusion is trivial.

But it could also mean *Some cat is (feared by every mouse)*, i.e.

There exists one cat c, such that for every mouse m, c is feared by m.

$$\exists c.\,(\,\mathrm{Cat}(c) \wedge \forall m.\,(\mathrm{Mouse}(m) \rightarrow \mathrm{Fears}(m,c))\,)$$

This example illustrates the importance of specifying the scope of quantifiers as *for all* and *there exists*.

Further reading

- Patrick Suppes, *Introduction to Logic*, D. Van Nostrand, 1957, ISBN 0-422-08072-7 .
- A. G. Hamilton, *Logic for Mathematicians*, Cambridge University Press, 1978, ISBN 0-521-29291-3.
- Paul Halmos and Steven Givant, *Logic as Algebra*, MAA, 1998, ISBN 0-88385-327-2.

Atomic sentence

In logic, an **atomic sentence** is a type of declarative sentence which is either true or false (may also be referred to as a proposition, statement or truthbearer) and which cannot be broken down into other simpler sentences. For example "The dog ran" is an atomic sentence in natural language, whereas "The dog ran and the cat hid." is a **molecular sentence** in natural language.

From a logical analysis, the truth or falsity of sentences in general is determined by only two things: the logical form of the sentence and the truth or falsity of its simple sentences. This is to say, for example, that the truth of the sentence "John is Greek and John is happy" is a function of the meaning of "and", and the truth values of the atomic sentences "John is Greek" and "John is happy". However, the truth or falsity of an atomic sentence is not a matter that is within the scope of logic itself, but rather whatever art or science the content of the atomic sentence happens to be talking about.[1]

Logic has developed artificial languages, for example sentential calculus and predicate calculus partly with the purpose of revealing the underlying logic of natural languages statements, the surface grammar of which may conceal the underlying logical structure; see Analytic Philosophy. In these artificial languages an Atomic Sentence is a string of symbols which can represent an elementary sentence in a natural language, and it can be defined as follows.

In a formal language, a well-formed formula (or wff) is a string of symbols constituted in accordance with the rules of syntax of the language. A term is a variable, an individual constant or a n-place function letter followed by n terms. An atomic formula is a wff consisting of either a sentential letter or an n-place predicate letter followed by n terms. A sentence is a wff in which any variables are bound. An **atomic sentence** is an atomic formula containing no variables. It follows that an atomic sentence contains no logical connectives, variables or quantifiers. A sentence consisting of one or more sentences and a logical connective is a compound (or molecular sentence). See *vocabulary* in First-order logic

Examples

Assumptions

In the following examples:

```
* let F, G, H be predicate letters;
* let a, b, c be individual constants;
* let x, y, z be variables.
```

Atomic Sentences

These wffs are atomic sentences; they contain no variables or conjunctions:

- $F(a)$
- $H(b, a, c)$

Atomic Formulae

These wffs are atomic formulae, but are not sentences (atomic or otherwise) because they include free variables:

- $F(x)$
- $G(a, z)$
- $H(x, y, z)$

Compound Sentences

These wffs are compound sentences. They are sentences, but are not atomic sentences because they are not atomic formulae:

- $\forall x\ (F(x))$
- $\exists z\ (G(a, z))$
- $\exists x\ \forall y\ \exists z\ (H(x, y, x))$
- $\forall x\ \exists z\ (F(x) \land G(a, z))$
- $\exists x\ \forall y \exists z\ (G(a, z) \lor H(x, y, z))$

Compound Formulae

These wffs are compound formulae. They are not atomic formulae but are built up from atomic formulae using logical connectives. They are also not sentences because they contain free variables:

- $F(x) \land G(a, z)$
- $G(a, z) \lor H(x, y, z)$

Interpretations

A sentence is either **true** or **false** under an **interpretation** which assigns values to the logical variables. We might for example make the following assignments:

Individual Constants

- a: Socrates
- b: Plato
- c: Aristotle

Predicates:

- Fα: α is sleeping

- $G\alpha\beta$: α hates β
- $H\alpha\beta\gamma$: α made β hit γ

Sentential variables:

- *p*: It is raining.

Under this interpretation the sentences discussed above would represent the following English statements:

- p: "It is raining."
- $F(a)$: "Socrates is sleeping."
- $H(b, a, c)$: "Plato made Socrates hit Aristotle."
- $\forall x\ (F(x))$: "Everybody is sleeping."
- $\exists z\ (G(a, z))$: "Socrates hates somebody."
- $\exists x\ \forall y\ \exists z\ (H(x, y, z))$: "Somebody made everybody hit somebody." (They may not have all hit the same person z, but they all did so *because* of the same person x.)
- $\forall x\ \exists z\ (F(x) \wedge G(a, z))$: "Everybody is sleeping and Socrates hates somebody."
- $\exists x\ \forall y\ \exists z\ (G(a, z) \vee H(x, y, z))$: "Either Socrates hates somebody or somebody made everybody hit somebody."

Translating sentences from a natural language into an artificial language

Sentences in natural languages can be ambiguous, whereas the languages of the sentential logic and predicate logics are precise. Translation can reveal such ambiguities and express precisely the intended meaning.

For example take the English sentence “Father Ted married Jack and Jill”. Does this mean Jack married Jill? In translating we might make the following assignments: **Individual Constants**

- *a*: Father Ted
- *b*: Jack
- *c*: Jill

Predicates:

- $M\alpha\beta\gamma$: α officiated at the marriage of β to γ

Using these assignments the sentence above could be translated as follows:

- $M(a, b, c)$: Father Ted officiated at the marriage of Jack and Jill.
- $\exists x\ \exists y\ (M(a, b, x) \wedge M(a, c, y))$: Father Ted officiated at the marriage of Jack to somebody and Father Ted officiated at the marriage of Jill to somebody.
- $\exists x\ \exists y\ (M(x, a, b) \wedge M(y, a, c))$: Somebody officiated at the marriage of Father Ted to Jack and somebody officiated at the marriage of Father Ted to Jill.

To establish which is the correct translation of “Father Ted married Jack and Jill”, it would be necessary to ask the speaker exactly what was meant.

Philosophical significance

Atomic sentences are of particular interest in philosophical logic and the theory of truth and, it has been argued, there are corresponding **atomic facts**. An Atomic sentence (or possibly the *meaning* of an atomic sentence) is called an **elementary proposition** by Wittgenstein and an **atomic proposition** by Russell:

- *4.2 The sense of a proposition is its agreement and disagreement with possibilities of existence and non-existence of states of affairs. 4.21 The simplest kind of proposition, an elementary proposition, asserts the existence of a state of affairs.*: Wittgenstein, Tractatus Logico-Philosophicus, s:Tractatus Logico-Philosophicus.
- *A proposition (true or false) asserting an atomic fact is called an atomic proposition.*: Russell, Introduction to Tractatus Logico-Philosophicus, s:Tractatus Logico-Philosophicus/Introduction

- see also [2] and [3] especially regarding *elementary proposition* and *atomic proposition* as discussed by Russell and Wittgenstein

Note the distinction between an *elementary/atomic proposition* and *an atomic fact*

No atomic sentence can be deduced from (is not entailed by) any other atomic sentence, no two atomic sentences are incompatible, and no sets of atomic sentences are self-contradictory. Wittgenstein made much of this in his Tractatus Logico-Philosophicus. If there are any atomic sentences then there must be "atomic facts" which correspond to those that are true, and the conjunction of all true atomic sentences would say all that was the case, i.e. "the world" since, according to Wittegenstein, "The world is all that is the case". (TLP:1). Similarly the set of all sets of atomic sentences corresponds to the set of all possible worlds (all that could be the case).

The T-schema, which embodies the theory of truth proposed by Alfred Tarski, defines the truth of arbitrary sentences from the truth of atomic sentences.

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[1] *Philosophy of Logic*, Willard Van Orman Quine

[2] http://plato.stanford.edu/entries/logical-atomism/

[3] http://plato.stanford.edu/entries/wittgenstein-atomism/

Logical connective

In logic, a **logical connective** (also called a **logical operator** or a **truth function**) is a symbol or word used to connect two or more sentences (of either a formal or a natural language) in a grammatically valid way, such that the sense of the compound sentence produced depends only on the original sentences.

The most common logical connectives are **binary connectives** (also called **dyadic connectives**) which join two sentences which can be thought of as the function's operands. Also commonly, negation is considered to be a **unary connective**.

Logical connectives along with quantifiers are the two main types of logical constants used in formal systems such as propositional logic and predicate logic.

In language

Natural language

In the grammar of natural languages two sentences may be joined by a grammatical conjunction to form a *grammatically* compound sentence. Some but not all such grammatical conjunctions are truth functions. For example, consider the following sentences:

A: Jack went up the hill.

B: Jill went up the hill.

C: Jack went up the hill *and* Jill went up the hill.

D: Jack went up the hill *so* Jill went up the hill.

The words *and* and *so* are *grammatical* conjunctions joining the sentences (A) and (B) to form the compound sentences (C) and (D). The *and* in (C) is a *logical* connective, since the truth of (C) is completely determined by (A) and (B): it would make no sense to affirm (A) and (B) but deny (C). However *so* in (D) is not a logical connective, since it would be quite reasonable to affirm (A) and (B) but deny (D): perhaps, after all, Jill went up the hill to fetch a pail of water, not because Jack had gone up the hill at all.

Various English words and word pairs express logical connectives, and some of them are synonymous. Examples (with the name of the relationship in parentheses) are:

- "and" (conjunction)
- "or" (disjunction)
- "either...or" (exclusive disjunction)
- "implies" (implication)
- "if...then" (implication)
- "if and only if" (equivalence)
- "only if" (implication)
- "just in case" (equivalence)
- "but" (conjunction)
- "however" (conjunction)
- "not both" (NAND)
- "neither...nor" (NOR)

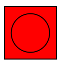
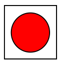
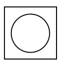
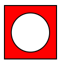
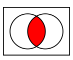
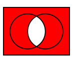
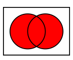
The word "not" (negation) and the phrases "it is false that" (negation) and "it is not the case that" (negation) also express a logical connective – even though they are applied to a single statement, and do not connect two statements.

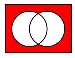
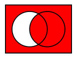
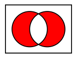
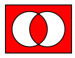
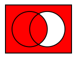
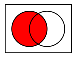
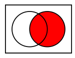
Formal languages

In formal languages, truth functions are represented by unambiguous symbols. These symbols are called "logical connectives", "logical operators", "propositional operators", or, in classical logic, "truth-functional connectives". See well-formed formula for the rules which allow new well-formed formulas to be constructed by joining other well-formed formulas using truth-functional connectives.

Logical connectives can be used to link more than two statements, so one can speak about "n-ary logical connective".

Common logical connectives

Name / Symbol		Truth table					Venndiagram
		P =	0	1			
Truth/Tautology	⊤		1	1			
Proposition P			0	1			
False/Contradiction	⊥		0	0			
Negation	¬		1	0			
Binary connectives		P =	0	0	1	1	
		Q =	0	1	0	1	
Conjunction	∧		0	0	0	1	
Alternative denial	↑		1	1	1	0	
Disjunction	∨		0	1	1	1	

Joint denial	$\downarrow$	1 0 0 0	
Material conditional	$\rightarrow$	1 1 0 1	
Exclusive or	$\nleftrightarrow$	0 1 1 0	
Biconditional	$\leftrightarrow$	1 0 0 1	
Converse implication	$\leftarrow$	1 0 1 1	
Proposition P		0 0 1 1	
Proposition Q		0 1 0 1	

More information

List of common logical connectives

Commonly used logical connectives include:

- Negation (not): $\neg$, N*p*, ~
- Conjunction (and): $\wedge$, K*pq*, & , •
- Disjunction (or): $\vee$, A*pq*
- Material implication (if...then): $\rightarrow$, C*pq*, $\Rightarrow$, $\supset$
- Biconditional (if and only if): $\leftrightarrow$, E*pq*, $\equiv$, $=$

Alternative names for biconditional are "iff", "xnor" and "bi-implication".

For example, the meaning of the statements *it is raining* and *I am indoors* is transformed when the two are combined with logical connectives:

- It is raining **and** I am indoors ($P \wedge Q$)
- **If** it is raining, **then** I am indoors ($P \rightarrow Q$)
- **If** I am indoors, **then** it is raining ($Q \rightarrow P$)
- I am indoors **if and only if** it is raining ($P \leftrightarrow Q$)
- It is **not** raining ($\neg P$)

For statement P = *It is raining* and Q = *I am indoors*.

It is also common to consider the *always true* formula and the *always false* formula to be connective:

- True formula ($\top$, 1, V*pq*, or T)
- False formula ($\bot$, 0, O*pq*, or F)

History of notations

- Negation: the symbol ¬ appeared in Heyting in 1929.[1][2] (compare to Frege's symbol —┬— A in his Begriffsschrift); the symbol ~ appeared in Russell in 1908;[3] an alternative notation is to add an horizontal line on top of the formula, as in $\overline{P}$; another alternative notation is to use a prime symbol as in P'.
- Conjunction: the symbol ∧ appeared in Heyting in 1929[1] (compare to Peano's use of the set-theoretic notation of intersection $\cap$ [4]); & appeared at least in Schönfinkel in 1924;[5] **.** comes from Boole's interpretation of logic as an elementary algebra.
- Disjunction: the symbol ∨ appeared in Russell in 1908 [3] (compare to Peano's use of the set-theoretic notation of union $\cup$); the symbol + is also used, in spite of the ambiguity coming from the fact that the + of ordinary elementary algebra is an exclusive or when interpreted logically in a two-element ring; punctually in the history a + together with a dot in the lower right corner has been used by Peirce,[6]

- Implication: the symbol $\rightarrow$ can be seen in Hilbert in 1917;[7] $\supset$ was used by Russell in 1908[3] (compare to Peano's inverted C notation); $\Rightarrow$ was used in Vax.[8]
- Biconditional: the symbol $\equiv$ was used at least by Russell in 1908;[3] $\leftrightarrow$ was used at least by Tarski in 1940;[9] $\Leftrightarrow$ was used in Vax; other symbols appeared punctually in the history such as $\supset\subset$ in Gentzen,[10] ~ in Schönfinkel[5] or $\subset\supset$ in Chazal.[11]
- True: the symbol 1 comes from Boole's interpretation of logic as an elementary algebra over the two-element Boolean algebra; other notations include $\bigwedge$ to be found in Peano.
- False: the symbol 0 comes also from Boole's interpretation of logic as a ring; other notations include $\bigvee$ to be found in Peano.

Some authors used letters for connectives at some time of the history: **u.** for conjunction (German's "und" for "and") and **o.** for disjunction (German's "oder" for "or") in earlier works by Hilbert (1904); **N*p*** for negation, **K*pq*** for conjunction, **A*pq*** for disjunction, **C*pq*** for implication, **E*pq*** for biconditional in Łukasiewicz (1929).[12]

Redundancy

Such logical connective as converse implication $\leftarrow$ is actually the same as material conditional with swapped arguments, so the symbol for converse implication is redundant. In some logical calculi (notably, in classical logic) certain essentially different compound statements are logically equivalent. Less trivial example of a redundancy is a classical equivalence between $\neg P \vee Q$ and $P \rightarrow Q$. Therefore, a classical-based logical system does not need the conditional operator "$\rightarrow$" if "$\neg$" (not) and "$\vee$" (or) are already in use, or may use the "$\rightarrow$" only as a syntactic sugar for a compound having one negation and one disjunction.

There are sixteen Boolean functions associating the input truth values P and Q with four-digit binary outputs. These correspond to possible choices of binary logical connectives for classical logic. Different implementation of classical logic can choose different functionally complete subsets of connectives.

One approach is to choose a *minimal* set, and define other connectives by some logical form, like in the example with material conditional above. The following are the minimal functionally complete sets of operators in classical logic whose arities do not exceed 2:

One element

$\{\uparrow\}, \{\downarrow\}.$

Two elements

$\{\vee, \neg\}, \{\wedge, \neg\}, \{\rightarrow, \neg\}, \{\leftarrow, \neg\}, \{\rightarrow, \bot\}, \{\leftarrow, \bot\}, \{\rightarrow, \nleftrightarrow\}, \{\leftarrow, \nleftrightarrow\}, \{\rightarrow, \nrightarrow\}, \{\rightarrow, \nleftarrow\}, \{\leftarrow, \nrightarrow\}, \{\leftarrow, \nleftarrow\}, \{\nrightarrow, \neg\}, \{\nleftarrow, \neg\}, \{\nrightarrow, \top\}, \{\nleftarrow, \top\}, \{\nrightarrow, \leftrightarrow\}, \{\nleftarrow, \leftrightarrow\}.$

Three elements

$\{\vee, \leftrightarrow, \bot\}, \{\vee, \leftrightarrow, \nleftrightarrow\}, \{\vee, \nleftrightarrow, \top\}, \{\wedge, \leftrightarrow, \bot\}, \{\wedge, \leftrightarrow, \nleftrightarrow\}, \{\wedge, \nleftrightarrow, \top\}.$

See more details about functional completeness in classical logic at Truth function#Functional completeness.

Another approach is to use on equal rights connectives of a certain convenient and functionally complete, but *not minimal* set. This approach requires more propositional axioms and each equivalence between logical forms must be either an axiom or provable as a theorem.

But intuitionistic logic has the situation more complicated. Of its five connectives $\{\wedge, \vee, \rightarrow, \neg, \bot\}$ only negation $\neg$ has to be reduced to other connectives (see details). Neither of conjunction, disjunction and material conditional has an equivalent form constructed of other four logical connectives.

Properties

Some logical connectives possess properties which may be expressed in the theorems containing the connective. Some of those properties that a logical connective may have are:

- **Associativity**: Within an expression containing two or more of the same associative connectives in a row, the order of the operations does not matter as long as the sequence of the operands is not changed.
- **Commutativity**: The operands of the connective may be swapped preserving logical equivalence to the original expression.
- **Distributivity**: A connective denoted by · distributes over another connective denoted by +, if $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all operands a, b, c.
- **Idempotence**: Whenever the operands of the operation are the same, the compound is logically equivalent to the operand.
- **Absorption**: A pair of connectives $\wedge$, $\vee$ satisfies the absorption law if $a \wedge (a \vee b) = a$ for all operands a, b.
- **Monotonicity**: If $f(a_1, ..., a_n) \le f(b_1, ..., b_n)$ for all $a_1, ..., a_n, b_1, ..., b_n \in \{0,1\}$ such that $a_1 \le b_1, a_2 \le b_2, ..., a_n \le b_n$. E.g., $\vee$, $\wedge$, $\top$, $\bot$.
- **Affinity**: Each variable always makes a difference in the truth-value of the operation or it never makes a difference. E.g., $\neg$, $\leftrightarrow$, $\not\leftrightarrow$, $\top$, $\bot$.
- **Duality**: To read the truth-value assignments for the operation from top to bottom on its truth table is the same as taking the complement of reading the table of the same or another connective from bottom to top. Without resorting to truth tables it may be formulated as $\tilde{g}(\neg a_1, ..., \neg a_n) = \neg g(a_1, ..., a_n)$. E.g., $\neg$.
- **Truth-preserving**: The compound all those argument are tautologies is a tautology itself. E.g., $\vee$, $\wedge$, $\top$, $\rightarrow$, $\leftrightarrow$, $\subset$. (see validity)
- **Falsehood-preserving**: The compound all those argument are contradictions is a contradiction itself. E.g., $\vee$, $\wedge$, $\not\leftrightarrow$, $\bot$, $\not\subset$, $\not\supset$. (see validity)
- **Involutivity** (for unary connectives): $f(f(a)) = a$. E.g. negation in classical logic.

For classical and intuitionistic logic, the "=" symbol means that corresponding implications "…→…" and "…←…" for logical compounds can be both proved as theorems, and the "≤" symbol means that "…→…" for logical compounds is a consequence of corresponding "…→…" connectives for propositional variables. Some of many-valued logics may have incompatible definitions of equivalence and order (entailment).

Both conjunction and disjunction are associative, commutative and idempotent in classical logic, most varieties of many-valued logic and intuitionistic logic. The same is true about distributivity of conjunction over disjunction and disjunction over conjunction, as well as for the absorption law.

In classical logic and some varieties of many-valued logic, conjunction and disjunction are dual, and negation is self-dual, the latter is also self-dual in intuitionistic logic.

Order of precedence

As a way of reducing the number of necessary parentheses, one may introduce precedence rules: ¬ has higher precedence than $\wedge$, $\wedge$ higher than $\vee$, and $\vee$ higher than →. So for example, $P \vee Q \wedge \neg R \rightarrow S$ is short for $(P \vee (Q \wedge (\neg R))) \rightarrow S$.

Here is a table that shows a commonly used precedence of logical operators.

Operator	*Precedence*
¬	1
∧	2
∨	3
→	4
↔	5

The order of precedence determines which connective is the "main connective" when interpreting a non-atomic formula.

Computer science

Truth-functional approach to logical operators is implemented as logic gates in digital circuits. Practically all digital circuits (the major exception is DRAM) are built up from NAND, NOR, NOT, and transmission gates; see more details in Truth function#Computer science. Logical operators over bit vectors (corresponding to finite Boolean algebras) are bitwise operations.

But not any usage of a logical connective in programming has a Boolean semantic. For example, lazy evaluation is sometimes implemented for $P \wedge Q$ and $P \vee Q$, so these connectives are not commutative if some of expressions `P`, `Q` has side effects. Also, a conditional, which in some sense corresponds to the material conditional connective, is essentially non-Boolean because for `if (P) then Q;` the consequent Q is not executed if the antecedent P is false (although a compound as a whole is successful ≈ "true" in such case). This is closer to intuitionist and constructivist views on the material conditional, rather than to classical logic's ones.

Notes

[1] Heyting (1929) *Die formalen Regeln der intuitionistischen Logik*.
[2] Denis Roegel (2002), *Petit panorama des notations logiques du 20e siècle* (http://www.loria.fr/~roegel/cours/symboles-logiques.pdf) (see chart on page 2).
[3] Russell (1908) *Mathematical logic as based on the theory of types* (American Journal of Mathematics 30, p222–262, also in From Frege to Gödel edited by van Heijenoort).
[4] Peano (1889) *Arithmetices principia, nova methodo exposita*.
[5] Schönfinkel (1924) *Über die Bausteine der mathematischen Logik*, translated as *On the building blocks of mathematical logic* in From Frege to Gödel edited by van Heijenoort.
[6] Peirce (1867) *On an improvement in Boole's calculus of logic.*
[7] Hilbert (1917/1918) *Prinzipien der Mathematik* (Bernays' course notes).
[8] Vax (1982) *Lexique logique*, Presses Universitaires de France.
[9] Tarski (1940) *Introduction to logic and to the methodology of deductive sciences.*
[10] Gentzen (1934) *Untersuchungen über das logische Schließen*.
[11] Chazal (1996) : Éléments de logique formelle.
[12] See Roegel

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- Bocheński, Józef Maria (1959), *A Précis of Mathematical Logic*, translated from the French and German editions by Otto Bird, D. Reidel, Dordrecht, South Holland.
- Enderton, Herbert (2001), *A Mathematical Introduction to Logic* (2nd ed.), Boston, MA: Academic Press, ISBN 978-0-12-238452-3
- Gamut, L.T.F (1991), "Chapter 2", *Logic, Language and Meaning*, **1**, University of Chicago Press, pp. 54–64, OCLC 21372380

Further reading

- Lloyd Humberstone (2011). *The Connectives*. MIT Press. ISBN 978-0-262-01654-4.

External links

- Hazewinkel, Michiel, ed. (2001), "Propositional connective" (http://www.encyclopediaofmath.org/index.php?title=p/p075490), *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Lloyd Humberstone (2010), " Sentence Connectives in Formal Logic (http://plato.stanford.edu/entries/connectives-logic/)", Stanford Encyclopedia of Philosophy (An abstract algebraic logic approach to connectives.)
- John MacFarlane (2005), " Logical constants (http://plato.stanford.edu/entries/logical-constants/)", Stanford Encyclopedia of Philosophy.

First-order logic

First-order logic is a formal system used in mathematics, philosophy, linguistics, and computer science. It is also known as **first-order predicate calculus**, the **lower predicate calculus**, **quantification theory**, and predicate logic (a less precise term). First-order logic is distinguished from propositional logic by its use of quantified variables.

A theory about some topic is usually first-order logic together with: a specified domain of discourse over which the quantified variables range, finitely many functions which map from that domain into it, finitely many predicates defined on that domain, and a recursive set of axioms which are believed to hold for those things. Sometimes "theory" is understood in a more formal sense, which is just a set of sentences in first-order logic.

The adjective "first-order" distinguishes first-order logic from higher-order logic in which there are predicates having predicates or functions as arguments, or in which one or both of predicate quantifiers or function quantifiers are permitted.[1] In first-order theories, predicates are often associated with sets. In interpreted higher-order theories, predicates may be interpreted as sets of sets.

There are many deductive systems for first-order logic that are sound (all provable statements are true) and complete (all true statements are provable). Although the logical consequence relation is only semidecidable, much progress has been made in automated theorem proving in first-order logic. First-order logic also satisfies several metalogical theorems that make it amenable to analysis in proof theory, such as the Löwenheim–Skolem theorem and the compactness theorem.

First-order logic is of great importance to the foundations of mathematics, because it is the standard formal logic for axiomatic systems. Many common axiomatic systems, such as first-order Peano arithmetic and axiomatic set theory, including the canonical Zermelo–Fraenkel set theory (ZF), can be formalized as first-order theories. No first-order theory, however, has the strength to describe fully and categorically structures with an infinite domain, such as the natural numbers or the real line. Categorical axiom systems for these structures can be obtained in stronger logics such as second-order logic.

For a history of first-order logic and how it came to be the dominant formal logic, see José Ferreirós 2001.

Introduction

While propositional logic deals with simple declarative propositions, first-order logic additionally covers predicates and quantification.

A predicate resembles a function that returns either True or False. Consider the following sentences: "Socrates is a philosopher", "Plato is a philosopher". In propositional logic these are treated as two unrelated propositions, denoted for example by p and q. In first-order logic, however, the sentences can be expressed in a more parallel manner using the predicate Phil(a), which asserts that the object represented by a is a philosopher. Thus if a represents Socrates then Phil(a) asserts the first proposition, p; if a instead represents Plato then Phil(a) asserts the second proposition, q. A key aspect of first-order logic is visible here: the string "Phil" is a syntactic entity which is given semantic meaning by declaring that Phil(a) holds exactly when a is a philosopher. An assignment of semantic meaning is called an interpretation.

First-order logic allows reasoning about properties that are shared by many objects, through the use of variables. For example, let Phil(a) assert that a is a philosopher and let Schol(a) assert that a is a scholar. Then the formula

$$\text{Phil}(a) \to \text{Schol}(a)$$

asserts that if a is a philosopher then a is a scholar. The symbol $\rightarrow$ is used to denote a conditional (if/then) statement. The hypothesis lies to the left of the arrow and the conclusion to the right. The truth of this formula depends on which object is denoted by a, and on the interpretations of "Phil" and "Schol".

Assertions of the form "for every a, if a is a philosopher then a is a scholar" require both the use of variables and the use of a quantifier. Again, let Phil(a) assert a is a philosopher and let Schol(a) assert that a is a scholar. Then the first-order sentence

$$\forall a(\text{Phil}(a) \to \text{Schol}(a))$$

asserts that no matter what a represents, if a is a philosopher then a is scholar. Here $\forall$, the universal quantifier, expresses the idea that the claim in parentheses holds for *all* choices of a.

To show that the claim "If a is a philosopher then a is a scholar" is false, one would show there is some philosopher who is not a scholar. This counterclaim can be expressed with the existential quantifier $\exists$:

$$\exists a(\text{Phil}(a) \land \neg\text{Schol}(a)) .$$

Here:

- $\neg$ is the negation operator: $\neg\text{Schol}(a)$ is true if and only if $\text{Schol}(a)$ is false, in other words if and only if a is not a scholar.
- $\land$ is the conjunction operator: $\text{Phil}(a) \land \neg\text{Schol}(a)$ asserts that a is a philosopher and also not a scholar.

The predicates Phil(a) and Schol(a) take only one parameter each. First-order logic can also express predicates with more than one parameter. For example, "there is someone who can be fooled every time" can be expressed as:

$$\exists x(\text{Person}(x) \land \forall y(\text{Time}(y) \to \text{Canfool}(x, y))) .$$

Here Person(x) is interpreted to mean x is a person, Time(y) to mean that y is a moment of time, and Canfool(x,y) to mean that (person) x can be fooled at (time) y. For clarity, this statement asserts that there is at least one person who can be fooled at all times, which is stronger than asserting that at all times at least one person exists who can be fooled. This would be expressed as:

$$\forall y(\text{Time}(y) \to \exists x(\text{Person}(x) \land \text{Canfool}(x, y))) .$$

Asserting the latter (that there is always at least one foolable person) does not signify whether this foolable person is always the same for all moments of time.

The **range** of the quantifiers is the set of objects that can be used to satisfy them. (In the informal examples in this section, the range of the quantifiers was left unspecified.) In addition to specifying the meaning of predicate symbols such as Person and Time, an interpretation must specify a nonempty set, known as the domain of discourse or universe, as a range for the quantifiers. Thus a statement of the form $\exists a \mathrm{Phil}(a)$ is said to be true, under a particular interpretation, if there is some object in the domain of discourse of that interpretation that satisfies the predicate that the interpretation uses to assign meaning to the symbol Phil.

Syntax

There are two key parts of first order logic. The syntax determines which collections of symbols are legal expressions in first-order logic, while the semantics determine the meanings behind these expressions.

Alphabet

Unlike natural languages, such as English, the language of first-order logic is completely formal, so that it can be mechanically determined whether a given expression is legal. There are two key types of legal expressions: **terms**, which intuitively represent objects, and **formulas**, which intuitively express predicates that can be true or false. The terms and formulas of first-order logic are strings of **symbols** which together form the **alphabet** of the language. As with all formal languages, the nature of the symbols themselves is outside the scope of formal logic; they are often regarded simply as letters and punctuation symbols.

It is common to divide the symbols of the alphabet into **logical symbols**, which always have the same meaning, and **non-logical symbols**, whose meaning varies by interpretation. For example, the logical symbol $\wedge$ always represents "and"; it is never interpreted as "or". On the other hand, a non-logical predicate symbol such as Phil(*x*) could be interpreted to mean "*x* is a philosopher", "*x* is a man named Philip", or any other unary predicate, depending on the interpretation at hand.

Logical symbols

There are several logical symbols in the alphabet, which vary by author but usually include:

- The quantifier symbols $\forall$ and $\exists$
- The logical connectives: $\wedge$ for conjunction, $\vee$ for disjunction, $\rightarrow$ for implication, $\leftrightarrow$ for biconditional, $\neg$ for negation. Occasionally other logical connective symbols are included. Some authors use $\Rightarrow$, or C*pq*, instead of $\rightarrow$, and $\Leftrightarrow$, or E*pq*, instead of $\leftrightarrow$, especially in contexts where $\rightarrow$ is used for other purposes. Moreover, the horseshoe $\supset$ may replace $\rightarrow$; the triple-bar $\equiv$ may replace $\leftrightarrow$, and a tilde (~), N*p*, or F*pq*, may replace $\neg$; ||, or A*pq* may replace $\vee$; and &, K*pq*, or the middle dot, $\cdot$, may replace $\wedge$, especially if these symbols are not available for technical reasons. (*Note*: the aforementioned symbols C*pq*, E*pq*, N*p*, A*pq*, and K*pq* are used in Polish notation.)
- Parentheses, brackets, and other punctuation symbols. The choice of such symbols varies depending on context.
- An infinite set of **variables**, often denoted by lowercase letters at the end of the alphabet *x*, *y*, *z*, … . Subscripts are often used to distinguish variables: $x_0, x_1, x_2, \ldots$.
- An **equality symbol** (sometimes, **identity symbol**) =; see the section on equality below.

It should be noted that not all of these symbols are required - only one of the quantifiers, negation and conjunction, variables, brackets and equality suffice. There are numerous minor variations that may define additional logical symbols:

- Sometimes the truth constants T, V*pq*, or $\top$, for "true" and F, O*pq*, or $\bot$, for "false" are included. Without any such logical operators of valence 0, these two constants can only be expressed using quantifiers.
- Sometimes additional logical connectives are included, such as the Sheffer stroke, D*pq* (NAND), and exclusive or, J*pq*.

Non-logical symbols

The non-logical symbols represent predicates (relations), functions and constants on the domain of discourse. It used to be standard practice to use a fixed, infinite set of non-logical symbols for all purposes. A more recent practice is to use different non-logical symbols according to the application one has in mind. Therefore it has become necessary to name the set of all non-logical symbols used in a particular application. This choice is made via a **signature**.[2]

The traditional approach is to have only one, infinite, set of non-logical symbols (one signature) for all applications. Consequently, under the traditional approach there is only one language of first-order logic.[3] This approach is still common, especially in philosophically oriented books.

1. For every integer $n \geq 0$ there is a collection of n-**ary**, or n-**place**, **predicate symbols**. Because they represent relations between n elements, they are also called **relation symbols**. For each arity n we have an infinite supply of them:

   $$P^n{}_0, P^n{}_1, P^n{}_2, P^n{}_3, \ldots$$

2. For every integer $n \geq 0$ there are infinitely many n-ary **function symbols**:

   $$f^n{}_0, f^n{}_1, f^n{}_2, f^n{}_3, \ldots$$

In contemporary mathematical logic, the signature varies by application. Typical signatures in mathematics are {1, ×} or just {×} for groups, or {0, 1, +, ×, <} for ordered fields. There are no restrictions on the number of non-logical symbols. The signature can be empty, finite, or infinite, even uncountable. Uncountable signatures occur for example in modern proofs of the Löwenheim-Skolem theorem.

In this approach, every non-logical symbol is of one of the following types.

1. A **predicate symbol** (or **relation symbol**) with some **valence** (or **arity**, number of arguments) greater than or equal to 0. These which are often denoted by uppercase letters P, Q, R,... .
   - Relations of valence 0 can be identified with propositional variables. For example, P, which can stand for any statement.
   - For example, $P(x)$ is a predicate variable of valence 1. One possible interpretation is "x is a man".
   - $Q(x,y)$ is a predicate variable of valence 2. Possible interpretations include "x is greater than y" and "x is the father of y".
2. A **function symbol**, with some valence greater than or equal to 0. These are often denoted by lowercase letters f, g, h,... .
   - Examples: $f(x)$ may be interpreted as for "the father of x". In arithmetic, it may stand for "-x". In set theory, it may stand for "the power set of x". In arithmetic, $g(x,y)$ may stand for "$x+y$". In set theory, it may stand for "the union of x and y".
   - Function symbols of valence 0 are called **constant symbols**, and are often denoted by lowercase letters at the beginning of the alphabet a, b, c,... . The symbol a may stand for Socrates. In arithmetic, it may stand for 0. In set theory, such a constant may stand for the empty set.

The traditional approach can be recovered in the modern approach by simply specifying the "custom" signature to consist of the traditional sequences of non-logical symbols.

Formation rules

The formation rules define the terms and formulas of first order logic. When terms and formulas are represented as strings of symbols, these rules can be used to write a formal grammar for terms and formulas. These rules are generally context-free (each production has a single symbol on the left side), except that the set of symbols may be allowed to be infinite and there may be many start symbols, for example the variables in the case of terms.

Terms

The set of **terms** is inductively defined by the following rules:

1. **Variables.** Any variable is a term.
2. **Functions.** Any expression $f(t_1,...,t_n)$ of n arguments (where each argument t_i is a term and f is a function symbol of valence n) is a term. In particular, symbols denoting individual constants are 0-ary function symbols, and are thus terms.

Only expressions which can be obtained by finitely many applications of rules 1 and 2 are terms. For example, no expression involving a predicate symbol is a term.

Formulas

The set of **formulas** (also called **well-formed formulas**[4] or **wff**s) is inductively defined by the following rules:

1. **Predicate symbols.** If P is an n-ary predicate symbol and t_1, ..., t_n are terms then $P(t_1,...,t_n)$ is a formula.
2. **Equality.** If the equality symbol is considered part of logic, and t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.
3. **Negation.** If φ is a formula, then $\neg\varphi$ is a formula.
4. **Binary connectives.** If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula. Similar rules apply to other binary logical connectives.
5. **Quantifiers.** If φ is a formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are formulas.

Only expressions which can be obtained by finitely many applications of rules 1–5 are formulas. The formulas obtained from the first two rules are said to be **atomic formulas**.

For example,

$$\forall x \forall y (P(f(x)) \rightarrow \neg(P(x) \rightarrow Q(f(y), x, z)))$$

is a formula, if f is a unary function symbol, P a unary predicate symbol, and Q a ternary predicate symbol. On the other hand, $\forall x\, x \rightarrow$ is not a formula, although it is a string of symbols from the alphabet.

The role of the parentheses in the definition is to ensure that any formula can only be obtained in one way by following the inductive definition (in other words, there is a unique parse tree for each formula). This property is known as **unique readability** of formulas. There are many conventions for where parentheses are used in formulas. For example, some authors use colons or full stops instead of parentheses, or change the places in which parentheses are inserted. Each author's particular definition must be accompanied by a proof of unique readability.

This definition of a formula does not support defining an if-then-else function ite(c,a,b) where "c" is a condition expressed as a formula, that would return "a" if c is true, and "b" if it is false. This is because both predicates and functions can only accept terms as parameters, but the first parameter is a formula. Some languages built on first-order logic, such as SMT-LIB 2.0, add this. [5]

Notational conventions

For convenience, conventions have been developed about the precedence of the logical operators, to avoid the need to write parentheses in some cases. These rules are similar to the order of operations in arithmetic. A common convention is:

- $\neg$ is evaluated first
- $\land$ and $\lor$ are evaluated next
- Quantifiers are evaluated next
- $\to$ is evaluated last.

Moreover, extra punctuation not required by the definition may be inserted to make formulas easier to read. Thus the formula

$$(\neg\forall x P(x) \to \exists x \neg P(x))$$

might be written as

$$(\neg[\forall x P(x)]) \to \exists x[\neg P(x)].$$

In some fields, it is common to use infix notation for binary relations and functions, instead of the prefix notation defined above. For example, in arithmetic, one typically writes "2 + 2 = 4" instead of "=(+(2,2),4)". It is common to regard formulas in infix notation as abbreviations for the corresponding formulas in prefix notation.

The definitions above use infix notation for binary connectives such as $\to$. A less common convention is Polish notation, in which one writes $\to$, $\land$, and so on in front of their arguments rather than between them. This convention allows all punctuation symbols to be discarded. Polish notation is compact and elegant, but rarely used in practice because it is hard for humans to read it. In Polish notation, the formula

$$\forall x \forall y (P(f(x)) \to \neg(P(x) \to Q(f(y), x, z)))$$

becomes "∀x∀y→Pfx¬→ PxQfyxz".

Free and bound variables

In a formula, a variable may occur **free** or **bound**. Intuitively, a variable is free in a formula if it is not quantified: in $\forall y\, P(x, y)$, variable *x* is free while *y* is bound. The free and bound variables of a formula are defined inductively as follows.

1. **Atomic formulas.** If φ is an atomic formula then *x* is free in φ if and only if *x* occurs in φ. Moreover, there are no bound variables in any atomic formula.
2. **Negation.** *x* is free in $\neg$ φ if and only if *x* is free in φ. *x* is bound in $\neg$ φ if and only if *x* is bound in φ.
3. **Binary connectives.** *x* is free in (φ $\to$ ψ) if and only if *x* is free in either φ or ψ. *x* is bound in (φ $\to$ ψ) if and only if *x* is bound in either φ or ψ. The same rule applies to any other binary connective in place of $\to$.
4. **Quantifiers.** *x* is free in $\forall$ *y* φ if and only if *x* is free in φ and *x* is a different symbol from *y*. Also, *x* is bound in $\forall$ *y* φ if and only if *x* is *y* or *x* is bound in φ. The same rule holds with $\exists$ in place of $\forall$.

For example, in $\forall$ *x* $\forall$ *y* (*P*(*x*) $\to$ *Q*(*x*,*f*(*x*),*z*)), *x* and *y* are bound variables, *z* is a free variable, and *w* is neither because it does not occur in the formula.

Freeness and boundness can be also specialized to specific occurrences of variables in a formula. For example, in $P(x) \to \forall x\, Q(x)$, the first occurrence of *x* is free while the second is bound. In other words, the *x* in $P(x)$ is free while the x in $\forall x\, Q(x)$ is bound.

A formula in first-order logic with no free variables is called a **first-order sentence**. These are the formulas that will have well-defined truth values under an interpretation. For example, whether a formula such as Phil(*x*) is true must depend on what *x* represents. But the sentence $\exists x\, \mathrm{Phil}(x)$ will be either true or false in a given interpretation.

Examples

Abelian groups

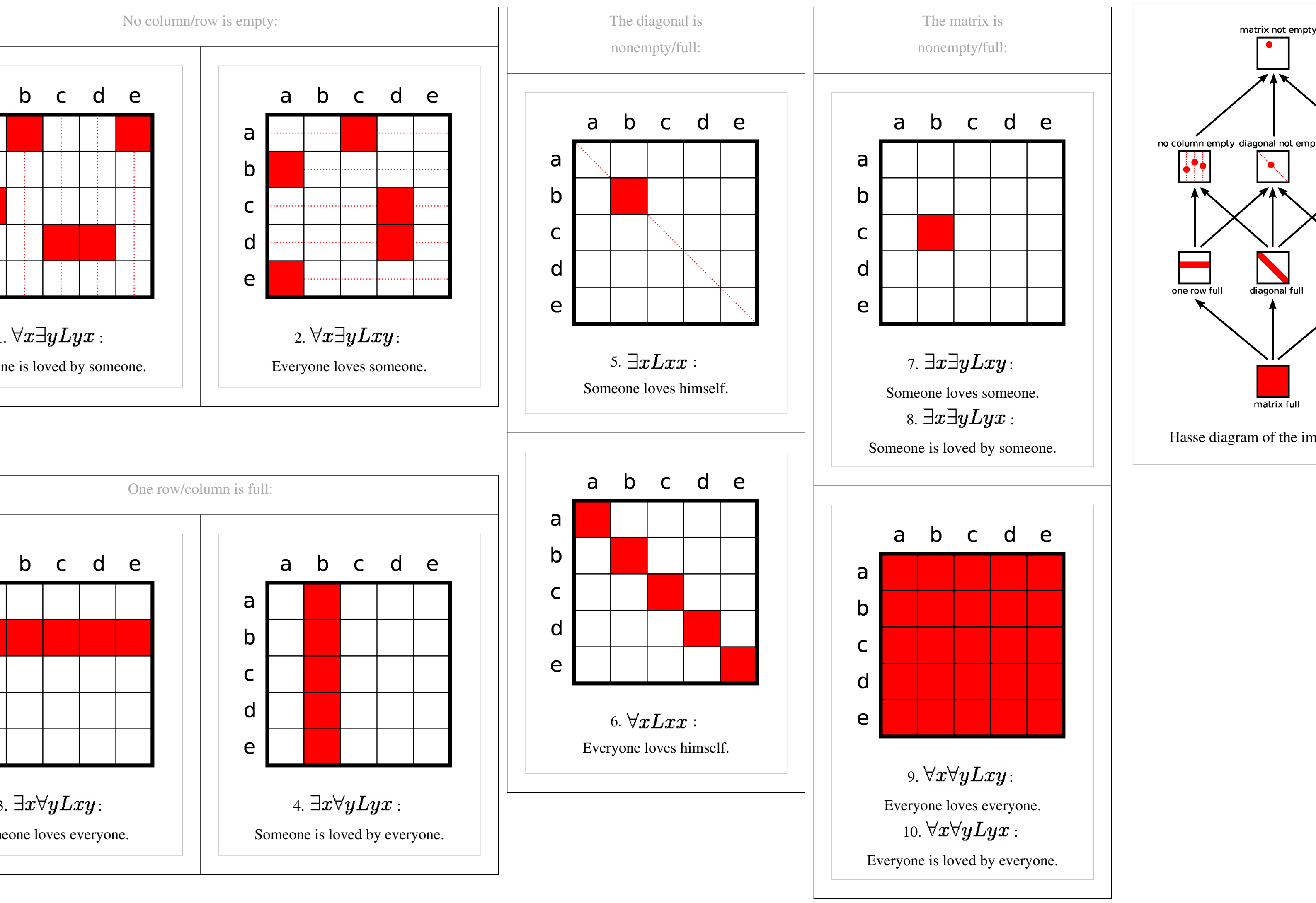
In mathematics the language of ordered abelian groups has one constant symbol 0, one unary function symbol −, one binary function symbol +, and one binary relation symbol ≤. Then:

- The expressions +(*x*, *y*) and +(*x*, +(*y*, −(*z*))) are **terms**. These are usually written as *x* + *y* and *x* + *y* − *z*.
- The expressions +(*x*, *y*) = 0 and ≤(+(*x*, +(*y*, −(*z*))), +(*x*, *y*)) are **atomic formulas**.

  These are usually written as *x* + *y* = 0 and *x* + *y* − *z* ≤ *x* + *y*.
- The expression $(\forall x \forall y \le(+(x,y),z) \to \forall x\, \forall y\, +(x,y)=0)$ is a **formula**, which is usually written as $\forall x \forall y(x+y \le z) \to \forall x \forall y(x+y=0)$.

Loving relation

There are 10 different formulas with 8 different meanings, that use the *loving* relation Lxy ("x loves y.") and the quantifiers ∀ and ∃:



The logical matrices represent the formulas for the case that there are five individuals that can love (vertical axis) and be loved (horizontal axis). Except for the sentences 9/10 they are examples. E.g. the matrix representing sentence 5 stands for "b loves himself."; the matrix representing sentences 7/8 stands for "c loves b."

It's important and instructive to distinguish sentence 1, $\forall x \exists y Lyx$, and 3, $\exists x \forall y Lxy$: In both cases everyone is loved; but in the first case everyone is loved by someone, in the second case everyone is loved by the same person. Some sentences imply each other — e.g. if 3 is true also 1 is true, but not vice versa. (See Hasse diagram)

Semantics

An interpretation of a first-order language assigns a denotation to all non-logical constants in that language. It also determines a domain of discourse that specifies the range of the quantifiers. The result is that each term is assigned an object that it represents, and each sentence is assigned a truth value. In this way, an interpretation provides semantic meaning to the terms and formulas of the language. The study of the interpretations of formal languages is called formal semantics. What follows is a description of the standard or Tarskian semantics for first-order logic. (It is also possible to define game semantics for first-order logic, but aside from requiring the axiom of choice, game semantics agree with Tarskian semantics for first-order logic, so game semantics will not be elaborated herein.)

The domain of discourse *D* is a nonempty set of "objects" of some kind. Intuitively, a first-order formula is a statement about these objects; for example, $\exists x P(x)$ states the existence of an object *x* such that the predicate *P* is true where referred to it. The domain of discourse is the set of considered objects. For example, one can take D to be the set of integer numbers.

The interpretation of a function symbol is a function. For example, if the domain of discourse consists of integers, a function symbol *f* of arity 2 can be interpreted as the function that gives the sum of its arguments. In other words, the symbol *f* is associated with the function *I(f)* which, in this interpretation, is addition.

The interpretation of a constant symbol is a function from the one-element set D^0 to *D*, which can be simply identified with an object in *D*. For example, an interpretation may assign the value $I(c) = 10$ to the constant symbol c .

The interpretation of an *n*-ary predicate symbol is a set of *n*-tuples of elements of the domain of discourse. This means that, given an interpretation, a predicate symbol, and *n* elements of the domain of discourse, one can tell whether the predicate is true of those elements according to the given interpretation. For example, an interpretation *I(P)* of a binary predicate symbol *P* may be the set of pairs of integers such that the first one is less than the second. According to this interpretation, the predicate *P* would be true if its first argument is less than the second.

First-order structures

The most common way of specifying an interpretation (especially in mathematics) is to specify a **structure** (also called a **model**; see below). The structure consists of a nonempty set *D* that forms the domain of discourse and an interpretation *I* of the non-logical terms of the signature. This interpretation is itself a function:

- Each function symbol *f* of arity *n* is assigned a function *I(f)* from D^n to D . In particular, each constant symbol of the signature is assigned an individual in the domain of discourse.
- Each predicate symbol *P* of arity *n* is assigned a relation *I(P)* over D^n or, equivalently, a function from D^n to $\{true, false\}$. Thus each predicate symbol is interpreted by a Boolean-valued function on *D*.

Evaluation of truth values

A formula evaluates to true or false given an interpretation, and a **variable assignment** μ that associates an element of the domain of discourse with each variable. The reason that a variable assignment is required is to give meanings to formulas with free variables, such as $y = x$. The truth value of this formula changes depending on whether *x* and *y* denote the same individual.

First, the variable assignment μ can be extended to all terms of the language, with the result that each term maps to a single element of the domain of discourse. The following rules are used to make this assignment:

1. **Variables.** Each variable *x* evaluates to μ(*x*)
2. **Functions.** Given terms $t_1, \ldots, t_n$ that have been evaluated to elements $d_1, \ldots, d_n$ of the domain of discourse, and a *n*-ary function symbol *f*, the term $f(t_1, \ldots, t_n)$ evaluates to $(I(f))(d_1, \ldots, d_n)$.

Next, each formula is assigned a truth value. The inductive definition used to make this assignment is called the T-schema.

1. **Atomic formulas (1).** A formula $P(t_1, \ldots, t_n)$ is associated the value true or false depending on whether $\langle v_1, \ldots, v_n \rangle \in I(P)$, where $v_1, \ldots, v_n$ are the evaluation of the terms $t_1, \ldots, t_n$ and $I(P)$ is the interpretation of P, which by assumption is a subset of D^n.
2. **Atomic formulas (2).** A formula $t_1 = t_2$ is assigned true if t_1 and t_2 evaluate to the same object of the domain of discourse (see the section on equality below).
3. **Logical connectives.** A formula in the form $\neg\phi$, $\phi \rightarrow \psi$, etc. is evaluated according to the truth table for the connective in question, as in propositional logic.
4. **Existential quantifiers.** A formula $\exists x \phi(x)$ is true according to *M* and μ if there exists an evaluation μ' of the variables that only differs from μ regarding the evaluation of *x* and such that φ is true according to the interpretation *M* and the variable assignment μ'. This formal definition captures the idea that $\exists x \phi(x)$ is true if and only if there is a way to choose a value for *x* such that φ(*x*) is satisfied.
5. **Universal quantifiers.** A formula $\forall x \phi(x)$ is true according to *M* and μ if φ(*x*) is true for every pair composed by the interpretation *M* and some variable assignment μ' that differs from μ only on the value of *x*. This captures the idea that $\forall x \phi(x)$ is true if every possible choice of a value for *x* causes φ(*x*) to be true.

If a formula does not contain free variables, and so is a sentence, then the initial variable assignment does not affect its truth value. In other words, a sentence is true according to *M* and μ if and only if is true according to *M* and every other variable assignment μ'.

There is a second common approach to defining truth values that does not rely on variable assignment functions. Instead, given an interpretation *M*, one first adds to the signature a collection of constant symbols, one for each element of the domain of discourse in *M*; say that for each *d* in the domain the constant symbol c_d is fixed. The interpretation is extended so that each new constant symbol is assigned to its corresponding element of the domain. One now defines truth for quantified formulas syntactically, as follows:

1. **Existential quantifiers (alternate).** A formula $\exists x \phi(x)$ is true according to *M* if there is some *d* in the domain of discourse such that $\phi(c_d)$ holds. Here $\phi(c_d)$ is the result of substituting c_d for every free occurrence of *x* in φ.
2. **Universal quantifiers (alternate).** A formula $\forall x \phi(x)$ is true according to *M* if, for every *d* in the domain of discourse, $\phi(c_d)$ is true according to *M*.

This alternate approach gives exactly the same truth values to all sentences as the approach via variable assignments.

Validity, satisfiability, and logical consequence

If a sentence φ evaluates to True under a given interpretation *M*, one says that *M* **satisfies** φ; this is denoted $M \vDash \phi$. A sentence is **satisfiable** if there is some interpretation under which it is true.

Satisfiability of formulas with free variables is more complicated, because an interpretation on its own does not determine the truth value of such a formula. The most common convention is that a formula with free variables is said to be satisfied by an interpretation if the formula remains true regardless which individuals from the domain of discourse are assigned to its free variables. This has the same effect as saying that a formula is satisfied if and only if its universal closure is satisfied.

A formula is **logically valid** (or simply **valid**) if it is true in every interpretation. These formulas play a role similar to tautologies in propositional logic.

A formula φ is a **logical consequence** of a formula ψ if every interpretation that makes ψ true also makes φ true. In this case one says that φ is logically implied by ψ.

Algebraizations

An alternate approach to the semantics of first-order logic proceeds via abstract algebra. This approach generalizes the Lindenbaum–Tarski algebras of propositional logic. There are three ways of eliminating quantified variables from first-order logic, that do not involve replacing quantifiers with other variable binding term operators:

- Cylindric algebra, by Alfred Tarski and his coworkers;
- Polyadic algebra, by Paul Halmos;
- Predicate functor logic, mainly due to Willard Quine.

These algebras are all lattices that properly extend the two-element Boolean algebra.

Tarski and Givant (1987) showed that the fragment of first-order logic that has no atomic sentence lying in the scope of more than three quantifiers, has the same expressive power as relation algebra. This fragment is of great interest because it suffices for Peano arithmetic and most axiomatic set theory, including the canonical ZFC. They also prove that first-order logic with a primitive ordered pair is equivalent to a relation algebra with two ordered pair projection functions.

First-order theories, models, and elementary classes

A **first-order theory** consists of a set of axioms in a particular first-order signature. The set of axioms is often finite or recursively enumerable, in which case the theory is called **effective**. Some authors require theories to also include all logical consequences of the axioms.

A first-order structure that satisfies all sentences in a given theory is said to be a **model** of the theory. An **elementary class** is the set of all structures satisfying a particular theory. These classes are a main subject of study in model theory.

Many theories have an intended interpretation, a certain model that is kept in mind when studying the theory. For example, the intended interpretation of Peano arithmetic consists of the usual natural numbers with their usual operations. However, the Löwenheim–Skolem theorem shows that most first-order theories will also have other, nonstandard models.

A theory is **consistent** if it is not possible to prove a contradiction from the axioms of the theory. A theory is **complete** if, for every formula in its signature, either that formula or its negation is a logical consequence of the axioms of the theory. Gödel's incompleteness theorem shows that effective first-order theories that include a sufficient portion of the theory of the natural numbers can never be both consistent and complete.

Empty domains

The definition above requires that the domain of discourse of any interpretation must be a nonempty set. There are settings, such as inclusive logic, where empty domains are permitted. Moreover, if a class of algebraic structures includes an empty structure (for example, there is an empty poset), that class can only be an elementary class in first-order logic if empty domains are permitted or the empty structure is removed from the class.

There are several difficulties with empty domains, however:

- Many common rules of inference are only valid when the domain of discourse is required to be nonempty. One example is the rule stating that $\phi \lor \exists x \psi$ implies $\exists x(\phi \lor \psi)$ when *x* is not a free variable in φ. This rule, which is used to put formulas into prenex normal form, is sound in nonempty domains, but unsound if the empty domain is permitted.
- The definition of truth in an interpretation that uses a variable assignment function cannot work with empty domains, because there are no variable assignment functions whose range is empty. (Similarly, one cannot assign interpretations to constant symbols.) This truth definition requires that one must select a variable assignment function (μ above) before truth values for even atomic formulas can be defined. Then the truth value of a sentence is defined to be its truth value under any variable assignment, and it is proved that this truth value does not depend on which assignment is chosen. This technique does not work if there are no assignment functions at all; it must be changed to accommodate empty domains.

Thus, when the empty domain is permitted, it must often be treated as a special case. Most authors, however, simply exclude the empty domain by definition.

Deductive systems

A **deductive system** is used to demonstrate, on a purely syntactic basis, that one formula is a logical consequence of another formula. There are many such systems for first-order logic, including Hilbert-style deductive systems, natural deduction, the sequent calculus, the tableaux method, and resolution. These share the common property that a deduction is a finite syntactic object; the format of this object, and the way it is constructed, vary widely. These finite deductions themselves are often called **derivations** in proof theory. They are also often called proofs, but are completely formalized unlike natural-language mathematical proofs.

A deductive system is **sound** if any formula that can be derived in the system is logically valid. Conversely, a deductive system is **complete** if every logically valid formula is derivable. All of the systems discussed in this article are both sound and complete. They also share the property that it is possible to effectively verify that a purportedly valid deduction is actually a deduction; such deduction systems are called **effective**.

A key property of deductive systems is that they are purely syntactic, so that derivations can be verified without considering any interpretation. Thus a sound argument is correct in every possible interpretation of the language, regardless whether that interpretation is about mathematics, economics, or some other area.

In general, logical consequence in first-order logic is only semidecidable: if a sentence A logically implies a sentence B then this can be discovered (for example, by searching for a proof until one is found, using some effective, sound, complete proof system). However, if A does not logically imply B, this does not mean that A logically implies the negation of B. There is no effective procedure that, given formulas A and B, always correctly decides whether A logically implies B.

Rules of inference

A **rule of inference** states that, given a particular formula (or set of formulas) with a certain property as a hypothesis, another specific formula (or set of formulas) can be derived as a conclusion. The rule is sound (or truth-preserving) if it preserves validity in the sense that whenever any interpretation satisfies the hypothesis, that interpretation also satisfies the conclusion.

For example, one common rule of inference is the **rule of substitution**. If *t* is a term and φ is a formula possibly containing the variable *x*, then φ[*t*/*x*] (often denoted φ[*x*/*t*]) is the result of replacing all free instances of *x* by *t* in φ. The substitution rule states that for any φ and any term *t*, one can conclude φ[*t*/*x*] from φ provided that no free variable of *t* becomes bound during the substitution process. (If some free variable of *t* becomes bound, then to substitute *t* for *x* it is first necessary to change the bound variables of φ to differ from the free variables of *t*.)

To see why the restriction on bound variables is necessary, consider the logically valid formula φ given by $\exists x(x = y)$, in the signature of (0,1,+,×,=) of arithmetic. If *t* is the term "x + 1", the formula φ[*t*/*y*] is $\exists x(x = x + 1)$, which will be false in many interpretations. The problem is that the free variable *x* of *t* became bound during the substitution. The intended replacement can be obtained by renaming the bound variable *x* of φ to something else, say *z*, so that the formula after substitution is $\exists z(z = x + 1)$, which is again logically valid.
The substitution rule demonstrates several common aspects of rules of inference. It is entirely syntactical; one can tell whether it was correctly applied without appeal to any interpretation. It has (syntactically-defined) limitations on when it can be applied, which must be respected to preserve the correctness of derivations. Moreover, as is often the case, these limitations are necessary because of interactions between free and bound variables that occur during syntactic manipulations of the formulas involved in the inference rule.

Hilbert-style systems and natural deduction

A deduction in a Hilbert-style deductive system is a list of formulas, each of which is a **logical axiom**, a hypothesis that has been assumed for the derivation at hand, or follows from previous formulas via a rule of inference. The logical axioms consist of several axiom schemes of logically valid formulas; these encompass a significant amount of propositional logic. The rules of inference enable the manipulation of quantifiers. Typical Hilbert-style systems have a small number of rules of inference, along with several infinite schemes of logical axioms. It is common to have only modus ponens and universal generalization as rules of inference.

Natural deduction systems resemble Hilbert-style systems in that a deduction is a finite list of formulas. However, natural deduction systems have no logical axioms; they compensate by adding additional rules of inference that can be used to manipulate the logical connectives in formulas in the proof.

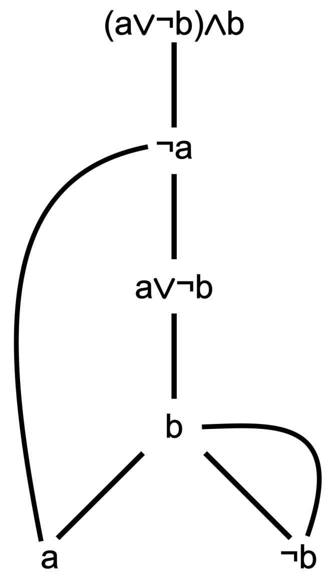
Sequent calculus

The sequent calculus was developed to study the properties of natural deduction systems. Instead of working with one formula at a time, it uses **sequents**, which are expressions of the form

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_k,$$

where A_1, ..., A_n, B_1, ..., B_k are formulas and the turnstile symbol $\vdash$ is used as punctuation to separate the two halves. Intuitively, a sequent expresses the idea that $(A_1 \wedge \cdots \wedge A_n)$ implies $(B_1 \vee \cdots \vee B_k)$.

Tableaux method



A tableaux proof for the propositional formula ((a ∨ ~b) & b) → a.

Unlike the methods just described, the derivations in the tableaux method are not lists of formulas. Instead, a derivation is a tree of formulas. To show that a formula A is provable, the tableaux method attempts to demonstrate that the negation of A is unsatisfiable. The tree of the derivation has $\neg A$ at its root; the tree branches in a way that reflects the structure of the formula. For example, to show that $C \vee D$ is unsatisfiable requires showing that C and D are each unsatisfiable; the corresponds to a branching point in the tree with parent $C \vee D$ and children C and D.

Resolution

The resolution rule is a single rule of inference that, together with unification, is sound and complete for first-order logic. As with the tableaux method, a formula is proved by showing that the negation of the formula is unsatisfiable. Resolution is commonly used in automated theorem proving.

The resolution method works only with formulas that are disjunctions of atomic formulas; arbitrary formulas must first be converted to this form through Skolemization. The resolution rule states that from the hypotheses $A_1 \vee \cdots \vee A_k \vee C$ and $B_1 \vee \cdots \vee B_l \vee \neg C$, the conclusion $A_1 \vee \cdots \vee A_k \vee B_1 \vee \cdots \vee B_l$ can be obtained.

Provable identities

The following sentences can be called "identities" because the main connective in each is the biconditional.

$$\neg\forall x\, P(x) \Leftrightarrow \exists x\, \neg P(x)$$
$$\neg\exists x\, P(x) \Leftrightarrow \forall x\, \neg P(x)$$
$$\forall x\, \forall y\, P(x,y) \Leftrightarrow \forall y\, \forall x\, P(x,y)$$
$$\exists x\, \exists y\, P(x,y) \Leftrightarrow \exists y\, \exists x\, P(x,y)$$
$$\forall x\, P(x) \wedge \forall x\, Q(x) \Leftrightarrow \forall x\, (P(x) \wedge Q(x))$$
$$\exists x\, P(x) \vee \exists x\, Q(x) \Leftrightarrow \exists x\, (P(x) \vee Q(x))$$

$P \wedge \exists x\, Q(x) \Leftrightarrow \exists x\, (P \wedge Q(x))$ (where x must not occur free in P)

$P \vee \forall x\, Q(x) \Leftrightarrow \forall x\, (P \vee Q(x))$ (where x must not occur free in P)

Equality and its axioms

There are several different conventions for using equality (or identity) in first-order logic. The most common convention, known as **first-order logic with equality**, includes the equality symbol as a primitive logical symbol which is always interpreted as the real equality relation between members of the domain of discourse, such that the "two" given members are the same member. This approach also adds certain axioms about equality to the deductive system employed. These equality axioms are:

1. **Reflexivity**. For each variable *x*, *x* = *x*.
2. **Substitution for functions.** For all variables *x* and *y*, and any function symbol *f*,

   x = *y* → *f*(...,*x*,...) = *f*(...,*y*,...).
3. **Substitution for formulas**. For any variables *x* and *y* and any formula φ(*x*), if φ' is obtained by replacing any number of free occurrences of *x* in φ with *y*, such that these remain free occurrences of *y*, then

$x = y \to (\varphi \to \varphi')$.

These are axiom schemes, each of which specifies an infinite set of axioms. The third scheme is known as **Leibniz's law**, "the principle of substitutivity", "the indiscernibility of identicals", or "the replacement property". The second scheme, involving the function symbol f, is (equivalent to) a special case of the third scheme, using the formula

$x = y \to (f(\ldots,x,\ldots) = z \to f(\ldots,y,\ldots) = z)$.

Many other properties of equality are consequences of the axioms above, for example:

1. **Symmetry.** If $x = y$ then $y = x$.
2. **Transitivity.** If $x = y$ and $y = z$ then $x = z$.

First-order logic without equality

An alternate approach considers the equality relation to be a non-logical symbol. This convention is known as **first-order logic without equality**. If an equality relation is included in the signature, the axioms of equality must now be added to the theories under consideration, if desired, instead of being considered rules of logic. The main difference between this method and first-order logic with equality is that an interpretation may now interpret two distinct individuals as "equal" (although, by Leibniz's law, these will satisfy exactly the same formulas under any interpretation). That is, the equality relation may now be interpreted by an arbitrary equivalence relation on the domain of discourse that is congruent with respect to the functions and relations of the interpretation.

When this second convention is followed, the term **normal model** is used to refer to an interpretation where no distinct individuals a and b satisfy $a = b$. In first-order logic with equality, only normal models are considered, and so there is no term for a model other than a normal model. When first-order logic without equality is studied, it is necessary to amend the statements of results such as the Löwenheim–Skolem theorem so that only normal models are considered.

First-order logic without equality is often employed in the context of second-order arithmetic and other higher-order theories of arithmetic, where the equality relation between sets of natural numbers is usually omitted.

Defining equality within a theory

If a theory has a binary formula $A(x,y)$ which satisfies reflexivity and Leibniz's law, the theory is said to have equality, or to be a theory with equality. The theory may not have all instances of the above schemes as axioms, but rather as derivable theorems. For example, in theories with no function symbols and a finite number of relations, it is possible to define equality in terms of the relations, by defining the two terms s and t to be equal if any relation is unchanged by changing s to t in any argument.

Some theories allow other *ad hoc* definitions of equality:

- In the theory of partial orders with one relation symbol $\leq$, one could define $s = t$ to be an abbreviation for $s \leq t \land t \leq s$.
- In set theory with one relation $\in$, one may define $s = t$ to be an abbreviation for $\forall x\, (s \in x \leftrightarrow t \in x) \land \forall x\, (x \in s \leftrightarrow x \in t)$. This definition of equality then automatically satisfies the axioms for equality. In this case, one should replace the usual axiom of extensionality, $\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$, by $\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow \forall z (x \in z \Leftrightarrow y \in z)]$, i.e. if x and y have the same elements, then they belong to the same sets.

Metalogical properties

One motivation for the use of first-order logic, rather than higher-order logic, is that first-order logic has many metalogical properties that stronger logics do not have. These results concern general properties of first-order logic itself, rather than properties of individual theories. They provide fundamental tools for the construction of models of first-order theories.

Completeness and undecidability

Gödel's completeness theorem, proved by Kurt Gödel in 1929, establishes that there are sound, complete, effective deductive systems for first-order logic, and thus the first-order logical consequence relation is captured by finite provability. Naively, the statement that a formula φ logically implies a formula ψ depends on every model of φ; these models will in general be of arbitrarily large cardinality, and so logical consequence cannot be effectively verified by checking every model. However, it is possible to enumerate all finite derivations and search for a derivation of ψ from φ. If ψ is logically implied by φ, such a derivation will eventually be found. Thus first-order logical consequence is semidecidable: it is possible to make an effective enumeration of all pairs of sentences (φ,ψ) such that ψ is a logical consequence of φ.

Unlike propositional logic, first-order logic is undecidable (although semidecidable), provided that the language has at least one predicate of arity at least 2 (other than equality). This means that there is no decision procedure that determines whether arbitrary formulas are logically valid. This result was established independently by Alonzo Church and Alan Turing in 1936 and 1937, respectively, giving a negative answer to the Entscheidungsproblem posed by David Hilbert in 1928. Their proofs demonstrate a connection between the unsolvability of the decision problem for first-order logic and the unsolvability of the halting problem.

There are systems weaker than full first-order logic for which the logical consequence relation is decidable. These include propositional logic and monadic predicate logic, which is first-order logic restricted to unary predicate symbols and no function symbols. The Bernays–Schönfinkel class of first-order formulas is also decidable. Decidable subsets of first-order logic are also studied in the framework of description logics.

The Löwenheim–Skolem theorem

The Löwenheim–Skolem theorem shows that if a first-order theory of cardinality λ has any infinite model then it has models of every infinite cardinality greater than or equal to λ. One of the earliest results in model theory, it implies that it is not possible to characterize countability or uncountability in a first-order language. That is, there is no first-order formula $\varphi(x)$ such that an arbitrary structure M satisfies φ if and only if the domain of discourse of M is countable (or, in the second case, uncountable).

The Löwenheim–Skolem theorem implies that infinite structures cannot be categorically axiomatized in first-order logic. For example, there is no first-order theory whose only model is the real line: any first-order theory with an infinite model also has a model of cardinality larger than the continuum. Since the real line is infinite, any theory satisfied by the real line is also satisfied by some nonstandard models. When the Löwenheim–Skolem theorem is applied to first-order set theories, the nonintuitive consequences are known as Skolem's paradox.

The compactness theorem

The compactness theorem states that a set of first-order sentences has a model if and only if every finite subset of it has a model. This implies that if a formula is a logical consequence of an infinite set of first-order axioms, then it is a logical consequence of some finite number of those axioms. This theorem was proved first by Kurt Gödel as a consequence of the completeness theorem, but many additional proofs have been obtained over time. It is a central tool in model theory, providing a fundamental method for constructing models.

The compactness theorem has a limiting effect on which collections of first-order structures are elementary classes. For example, the compactness theorem implies that any theory that has arbitrarily large finite models has an infinite model. Thus the class of all finite graphs is not an elementary class (the same holds for many other algebraic structures).

There are also more subtle limitations of first-order logic that are implied by the compactness theorem. For example, in computer science, many situations can be modeled as a directed graph of states (nodes) and connections (directed edges). Validating such a system may require showing that no "bad" state can be reached from any "good" state. Thus one seeks to determine if the good and bad states are in different connected components of the graph. However, the compactness theorem can be used to show that connected graphs are not an elementary class in first-order logic, and there is no formula $\varphi(x,y)$ of first-order logic, in the signature of graphs, that expresses the idea that there is a path from x to y. Connectedness can be expressed in second-order logic, however, but not with only existential set quantifiers, as Σ_1^1 also enjoys compactness.

Lindström's theorem

Per Lindström showed that the metalogical properties just discussed actually characterize first-order logic in the sense that no stronger logic can also have those properties (Ebbinghaus and Flum 1994, Chapter XIII). Lindström defined a class of abstract logical systems, and a rigorous definition of the relative strength of a member of this class. He established two theorems for systems of this type:

- A logical system satisfying Lindström's definition that contains first-order logic and satisfies both the Löwenheim–Skolem theorem and the compactness theorem must be equivalent to first-order logic.
- A logical system satisfying Lindström's definition that has a semidecidable logical consequence relation and satisfies the Löwenheim–Skolem theorem must be equivalent to first-order logic.

Limitations

Although first-order logic is sufficient for formalizing much of mathematics, and is commonly used in computer science and other fields, it has certain limitations. These include limitations on its expressiveness and limitations of the fragments of natural languages that it can describe.

For instance, first-order logic is undecidable, meaning a sound, complete and terminating decision algorithm is impossible. This has led to the study of interesting decidable fragments such as C_2, first-order logic with two variables and the counting quantifiers $\exists^{\geq n}$ and $\exists^{\leq n}$ (these quantifiers are, respectively, "there exists at least n" and "there exists at most n") (Horrocks 2010).

Expressiveness

The Löwenheim–Skolem theorem shows that if a first-order theory has any infinite model, then it has infinite models of every cardinality. In particular, no first-order theory with an infinite model can be categorical. Thus there is no first-order theory whose only model has the set of natural numbers as its domain, or whose only model has the set of real numbers as its domain. Many extensions of first-order logic, including infinitary logics and higher-order logics, are more expressive in the sense that they do permit categorical axiomatizations of the natural numbers or real numbers. This expressiveness comes at a metalogical cost, however: by Lindström's theorem, the compactness theorem and the downward Löwenheim–Skolem theorem cannot hold in any logic stronger than first-order.

Formalizing natural languages

First-order logic is able to formalize many simple quantifier constructions in natural language, such as "every person who lives in Perth lives in Australia". But there are many more complicated features of natural language that cannot be expressed in (single-sorted) first-order logic. "Any logical system which is appropriate as an instrument for the analysis of natural language needs a much richer structure than first-order predicate logic" (Gamut 1991, p. 75).

Type	Example	Comment
Quantification over properties	If John is self-satisfied, then there is at least one thing he has in common with Peter	Requires a quantifier over predicates, which cannot be implemented in single-sorted first-order logic: Zj→ ∃X(Xj∧Xp)
Quantification over properties	Santa Claus has all the attributes of a sadist	Requires quantifiers over predicates, which cannot be implemented in single-sorted first-order logic: ∀X(∀x(Sx → Xx)→Xs)
Predicate adverbial	John is walking quickly	Cannot be analysed as Wj ∧ Qj; predicate adverbials are not the same kind of thing as second-order predicates such as colour
Relative adjective	Jumbo is a small elephant	Cannot be analysed as Sj ∧ Ej; predicate adjectives are not the same kind of thing as second-order predicates such as colour
Predicate adverbial modifier	John is walking very quickly	-
Relative adjective modifier	Jumbo is terribly small	An expression such as "terribly", when applied to a relative adjective such as "small", results in a new composite relative adjective "terribly small"
Prepositions	Mary is sitting next to John	The preposition "next to" when applied to "John" results in the predicate adverbial "next to John"

Restrictions, extensions and variations

There are many variations of first-order logic. Some of these are inessential in the sense that they merely change notation without affecting the semantics. Others change the expressive power more significantly, by extending the semantics through additional quantifiers or other new logical symbols. For example, infinitary logics permit formulas of infinite size, and modal logics add symbols for possibility and necessity.

Restricted languages

First-order logic can be studied in languages with fewer logical symbols than were described above.

- Because $\exists x\phi(x)$ can be expressed as $\neg\forall x\neg\phi(x)$, and $\forall x\phi(x)$ can be expressed as $\neg\exists x\neg\phi(x)$, either of the two quantifiers $\exists$ and $\forall$ can be dropped.
- Since $\phi \lor \psi$ can be expressed as $\neg(\neg\phi \land \neg\psi)$ and $\phi \land \psi$ can be expressed as $\neg(\neg\phi \lor \neg\psi)$, either $\lor$ or $\land$ can be dropped. In other words, it is sufficient to have $\neg$ and $\lor$, or $\neg$ and $\land$, as the only logical connectives.
- Similarly, it is sufficient to have only $\neg$ and $\rightarrow$ as logical connectives, or to have only the Sheffer stroke (NAND) or the Peirce arrow (NOR) operator.
- It is possible to entirely avoid function symbols and constant symbols, rewriting them via predicate symbols in an appropriate way. For example, instead of using a constant symbol 0 one may use a predicate $0(x)$ (interpreted as $x = 0$), and replace every predicate such as $P(0, y)$ with $\forall x\ (0(x) \rightarrow P(x, y))$. A function such as $f(x_1, x_2, ..., x_n)$ will similarly be replaced by a predicate $F(x_1, x_2, ..., x_n, y)$ interpreted as $y = f(x_1, x_2, ..., x_n)$. This change requires adding additional axioms to the theory at hand, so that interpretations of the predicate symbols used have the correct semantics.

Restrictions such as these are useful as a technique to reduce the number of inference rules or axiom schemes in deductive systems, which leads to shorter proofs of metalogical results. The cost of the restrictions is that it becomes more difficult to express natural-language statements in the formal system at hand, because the logical connectives used in the natural language statements must be replaced by their (longer) definitions in terms of the restricted collection of logical connectives. Similarly, derivations in the limited systems may be longer than derivations in systems that include additional connectives. There is thus a trade-off between the ease of working within the formal system and the ease of proving results about the formal system.

It is also possible to restrict the arities of function symbols and predicate symbols, in sufficiently expressive theories. One can in principle dispense entirely with functions of arity greater than 2 and predicates of arity greater than 1 in theories that include a pairing function. This is a function of arity 2 that takes pairs of elements of the domain and returns an ordered pair containing them. It is also sufficient to have two predicate symbols of arity 2 that define projection functions from an ordered pair to its components. In either case it is necessary that the natural axioms for a pairing function and its projections are satisfied.

Many-sorted logic

Ordinary first-order interpretations have a single domain of discourse over which all quantifiers range. **Many-sorted first-order logic** allows variables to have different **sorts**, which have different domains. This is also called **typed first-order logic**, and the sorts called **types** (as in data type), but it is not the same as first-order type theory. Many-sorted first-order logic is often used in the study of second-order arithmetic.

When there are only finitely many sorts in a theory, many-sorted first-order logic can be reduced to single-sorted first-order logic. One introduces into the single-sorted theory a unary predicate symbol for each sort in the many-sorted theory, and adds an axiom saying that these unary predicates partition the domain of discourse. For example, if there are two sorts, one adds predicate symbols $P_1(x)$ and $P_2(x)$ and the axiom

$$\forall x(P_1(x) \vee P_2(x)) \wedge \neg\exists x(P_1(x) \wedge P_2(x)).$$

Then the elements satisfying P_1 are thought of as elements of the first sort, and elements satisfying P_2 as elements of the second sort. One can quantify over each sort by using the corresponding predicate symbol to limit the range of quantification. For example, to say there is an element of the first sort satisfying formula φ(x), one writes

$$\exists x(P_1(x) \wedge \phi(x)).$$

Additional quantifiers

Additional quantifiers can be added to first-order logic.

- Sometimes it is useful to say that "*P(x)* holds for exactly one *x*", which can be expressed as $\exists! x\, P(x)$. This notation, called uniqueness quantification, may be taken to abbreviate a formula such as $\exists\, x\, (P(x) \wedge \forall\, y\, (P(y) \rightarrow (x = y)))$.
- **First-order logic with extra quantifiers** has new quantifiers *Qx*,..., with meanings such as "there are many *x* such that ...". Also see branching quantifiers and the plural quantifiers of George Boolos and others.
- **Bounded quantifiers** are often used in the study of set theory or arithmetic.

Infinitary logics

Infinitary logic allows infinitely long sentences. For example, one may allow a conjunction or disjunction of infinitely many formulas, or quantification over infinitely many variables. Infinitely long sentences arise in areas of mathematics including topology and model theory.

Infinitary logic generalizes first-order logic to allow formulas of infinite length. The most common way in which formulas can become infinite is through infinite conjunctions and disjunctions. However, it is also possible to admit generalized signatures in which function and relation symbols are allowed to have infinite arities, or in which quantifiers can bind infinitely many variables. Because an infinite formula cannot be represented by a finite string, it is necessary to choose some other representation of formulas; the usual representation in this context is a tree. Thus formulas are, essentially, identified with their parse trees, rather than with the strings being parsed.

The most commonly studied infinitary logics are denoted $L_{\alpha\beta}$, where α and β are each either cardinal numbers or the symbol ∞. In this notation, ordinary first-order logic is $L_{\omega\omega}$. In the logic $L_{\infty\omega}$, arbitrary conjunctions or disjunctions are allowed when building formulas, and there is an unlimited supply of variables. More generally, the logic that permits conjunctions or disjunctions with less than κ constituents is known as $L_{\kappa\omega}$. For example, $L_{\omega}1\omega$ permits countable conjunctions and disjunctions.

The set of free variables in a formula of $L_{\kappa\omega}$ can have any cardinality strictly less than κ, yet only finitely many of them can be in the scope of any quantifier when a formula appears as a subformula of another.[6] In other infinitary logics, a subformula may be in the scope of infinitely many quantifiers. For example, in $L_{\kappa\infty}$, a single universal or existential quantifier may bind arbitrarily many variables simultaneously. Similarly, the logic $L_{\kappa\lambda}$ permits simultaneous quantification over fewer than λ variables, as well as conjunctions and disjunctions of size less than κ.

Non-classical and modal logics

- **Intuitionistic first-order logic** uses intuitionistic rather than classical propositional calculus; for example, ¬¬φ need not be equivalent to φ.
- First-order **modal logic** allows one to describe other possible worlds as well as this contingently true world which we inhabit. In some versions, the set of possible worlds varies depending on which possible world one inhabits. Modal logic has extra *modal operators* with meanings which can be characterized informally as, for example "it is necessary that φ" (true in all possible worlds) and "it is possible that φ" (true in some possible world). With standard first-order logic we have a single domain and each predicate is assigned one extension. With first-order modal logic we have a *domain function* that assigns each possible world its own domain, so that each predicate gets an extension only relative to these possible worlds. This allows us to model cases where, for example, Alex is a Philosopher, but might have been a Mathematician, and might not have existed at all. In the first possible world $P(a)$ is true, in the second $P(a)$ is false, and in the third possible world there is no a in the domain at all.
- **first-order fuzzy logics** are first-order extensions of propositional fuzzy logics rather than classical propositional calculus.

Higher-order logics

The characteristic feature of first-order logic is that individuals can be quantified, but not predicates. Thus

$$\exists a(\mathrm{Phil}(a))$$

is a legal first-order formula, but

$$\exists \mathrm{Phil}(\mathrm{Phil}(a))$$

is not, in most formalizations of first-order logic. Second-order logic extends first-order logic by adding the latter type of quantification. Other higher-order logics allow quantification over even higher types than second-order logic permits. These higher types include relations between relations, functions from relations to relations between relations, and other higher-type objects. Thus the "first" in first-order logic describes the type of objects that can be quantified.

Unlike first-order logic, for which only one semantics is studied, there are several possible semantics for second-order logic. The most commonly employed semantics for second-order and higher-order logic is known as **full semantics**. The combination of additional quantifiers and the full semantics for these quantifiers makes higher-order logic stronger than first-order logic. In particular, the (semantic) logical consequence relation for second-order and higher-order logic is not semidecidable; there is no effective deduction system for second-order logic that is sound and complete under full semantics.

Second-order logic with full semantics is more expressive than first-order logic. For example, it is possible to create axiom systems in second-order logic that uniquely characterize the natural numbers and the real line. The cost of this expressiveness is that second-order and higher-order logics have fewer attractive metalogical properties than first-order logic. For example, the Löwenheim–Skolem theorem and compactness theorem of first-order logic become false when generalized to higher-order logics with full semantics.

Automated theorem proving and formal methods

Automated theorem proving refers to the development of computer programs that search and find derivations (formal proofs) of mathematical theorems. Finding derivations is a difficult task because the search space can be very large; an exhaustive search of every possible derivation is theoretically possible but computationally infeasible for many systems of interest in mathematics. Thus complicated heuristic functions are developed to attempt to find a derivation in less time than a blind search.

The related area of automated proof verification uses computer programs to check that human-created proofs are correct. Unlike complicated automated theorem provers, verification systems may be small enough that their correctness can be checked both by hand and through automated software verification. This validation of the proof verifier is needed to give confidence that any derivation labeled as "correct" is actually correct.

Some proof verifiers, such as Metamath, insist on having a complete derivation as input. Others, such as Mizar and Isabelle, take a well-formatted proof sketch (which may still be very long and detailed) and fill in the missing pieces by doing simple proof searches or applying known decision procedures: the resulting derivation is then verified by a small, core "kernel". Many such systems are primarily intended for interactive use by human mathematicians: these are known as proof assistants. They may also use formal logics that are stronger than first-order logic, such as type theory. Because a full derivation of any nontrivial result in a first-order deductive system will be extremely long for a human to write,[7] results are often formalized as a series of lemmas, for which derivations can be constructed separately.

Automated theorem provers are also used to implement formal verification in computer science. In this setting, theorem provers are used to verify the correctness of programs and of hardware such as processors with respect to a formal specification. Because such analysis is time-consuming and thus expensive, it is usually reserved for projects in which a malfunction would have grave human or financial consequences.

Notes

[1] Mendelson, Elliott (1964). *Introduction to Mathematical Logic*. Van Nostrand Reinhold. pp. 56.

[2] The word *language* is sometimes used as a synonym for signature, but this can be confusing because "language" can also refer to the set of formulas.

[3] More precisely, there is only one language of each variant of one-sorted first-order logic: with or without equality, with or without functions, with or without propositional variables, ….

[4] Some authors who use the term "well-formed formula" use "formula" to mean any string of symbols from the alphabet. However, most authors in mathematical logic use "formula" to mean "well-formed formula" and have no term for non-well-formed formulas. In every context, it is only the well-formed formulas that are of interest.

[5] The SMT-LIB Standard: Version 2.0, by Clark Barrett, Aaron Stump, and Cesare Tinelli. http://goedel.cs.uiowa.edu/smtlib/

[6] Some authors only admit formulas with finitely many free variables in $L_{\kappa\omega}$, and more generally only formulas with $< \lambda$ free variables in $L_{\kappa\lambda}$.

[7] Avigad *et al.* (2007) discuss the process of formally verifying a proof of the prime number theorem. The formalized proof required approximately 30,000 lines of input to the Isabelle proof verifier.

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- Metamath (http://us.metamath.org/index.html): an ongoing online project to reconstruct mathematics as a huge first-order theory, using first-order logic and the axiomatic set theory ZFC. *Principia Mathematica* modernized.
- Podnieks, Karl; *Introduction to mathematical logic* (http://www.ltn.lv/~podnieks/)
- *Cambridge Mathematics Tripos Notes* (http://john.fremlin.de/schoolwork/logic/index.html) (typeset by John Fremlin). These notes cover part of a past Cambridge Mathematics Tripos course taught to undergraduates students (usually) within their third year. The course is entitled "Logic, Computation and Set Theory" and covers Ordinals and cardinals, Posets and Zorn's Lemma, Propositional logic, Predicate logic, Set theory and Consistency issues related to ZFC and other set theories.

Second-order logic

In logic and mathematics **second-order logic** is an extension of first-order logic, which itself is an extension of propositional logic.[1] Second-order logic is in turn extended by higher-order logic and type theory.

First-order logic uses only variables that range over individuals (elements of the domain of discourse); second-order logic has these variables as well as additional variables that range over sets of individuals. For example, the second-order sentence $\forall P\,\forall x(x \in P \lor x \notin P)$ says that for every set *P* of individuals and every individual *x*, either *x* is in *P* or it is not (this is the principle of bivalence). Second-order logic also includes variables quantifying over functions, and other variables as explained in the section *Syntax* below. Both first-order and second-order logic use the idea of a domain of discourse (often called simply the "domain" or the "universe"). The domain is a set of individual elements which can be quantified over.

Syntax and fragments

The syntax of second-order logic tells which expressions are well formed formulas. In addition to the syntax of first-order logic, second-order logic includes many new **sorts** (sometimes called **types**) of variables. These are:

- A sort of variables that range over sets of individuals. If *S* is a variable of this sort and *t* is a first-order term then the expression $t \in S$ (also written $S(t)$ or St) is an atomic formula. Sets of individuals can also be viewed as unary relations on the domain.
- For each natural number *k* there is a sort of variable that ranges over all *k*-ary relations on the individuals. If *R* is such a *k*-ary relation variable and $t_1,\ldots, t_k$ are first-order terms then the expression $R(t_1,\ldots,t_k)$ is an atomic formula.
- For each natural number *k* there is a sort of variable that ranges over functions that take *k* elements of the domain and return a single element of the domain. If *f* is such a *k*-ary function symbol and $t_1,\ldots,t_k$ are first-order terms then the expression $f(t_1,\ldots,t_k)$ is a first-order term.

For each of the sorts of variable just defined, it is permissible to build up formulas by using universal and/or existential quantifiers. Thus there are many sorts of quantifiers, two for each sort of variable. A **sentence** in second-order logic, as in first-order logic, is a well-formed formula with no free variables (of any sort).

It's possible to forgo the introduction function variables in the definition given above (and some authors do this) because an *n*-ary function variable can be represented by a relation variable of arity *n*+1 and an appropriate formula for the uniqueness of the "result" in the *n*+1 argument of the relation. (Shapiro 2000, p. 63)

In **monadic second-order logic** (MSOL), only variables for subsets of the domain are added. The second-order logic with all the sorts of variables just described is sometimes called **full second-order logic** to distinguish it from the monadic version.

Just as in first-order logic, second-order logic may include non-logical symbols in a particular second-order language. These are restricted, however, in that all terms that they form must be either first-order terms (which can be substituted for a first-order variable) or second-order terms (which can be substituted for a second-order variable of an appropriate sort).

A formula in second-order logic is said to be of first-order (and sometimes denoted Σ^1_0or Π^1_0) if its quantifiers (which may be of either type) range only over variables of first order, although it may have free variables of second order. A Σ^1_1(existential second-order) formula is one additionally having some existential quantifiers over second order variables, i.e. $\exists R_0 \ldots \exists R_m \phi$, where ϕ is a first-order formula. The fragment of second order logic consisting only of existential second-order formulas is called **existential second-order logic** and abbreviated ESO, Σ^1_1or even ∃SO. Π^1_1formulas are defined dually, the their fragment is called universal second-order logic. More expressive fragments are defined recursively for any $k > 0$, Σ^1_{k+1}has the form $\exists R_0 \ldots \exists R_m \phi$, where ϕ is a Π^1_k formula. (See analytical hierarchy for the analogous construction of second-order arithmetic.)

Semantics

The semantics of second-order logic establish the meaning of each sentence. Unlike first-order logic, which has only one standard semantics, there are two different semantics that are commonly used for second-order logic: **standard semantics** and **Henkin semantics**. In each of these semantics, the interpretations of the first-order quantifiers and the logical connectives are the same as in first-order logic. Only the ranges of quantifiers over second-order variables differ in the two types of semantics (Väänänen 2001).

In standard semantics, also called full semantics, the quantifiers range over *all* sets or functions of the appropriate sort. Thus once the domain of the first-order variables is established, the meaning of the remaining quantifiers is fixed. It is these semantics that give second-order logic its expressive power, and they will be assumed for the remainder of this article.

In Henkin semantics, each sort of second-order variable has a particular domain of its own to range over, which may be a proper subset of all sets or functions of that sort. Leon Henkin (1950) defined these semantics and proved that Gödel's completeness theorem and compactness theorem, which hold for first-order logic, carry over to second-order logic with Henkin semantics. This is because Henkin semantics are almost identical to many-sorted first-order semantics, where additional sorts of variables are added to simulate the new variables of second-order logic. Second-order logic with Henkin semantics is not more expressive than first-order logic. Henkin semantics are commonly used in the study of second-order arithmetic.

Väänänen (2001) argued that the choice between Henkin models and full models for second-order logic is analogous to the choice between ZFC and V as a basis for set theory: "As with second-order logic, we cannot really choose whether we axiomatize mathematics using V or ZFC. The result is the same in both cases, as ZFC *is* the best attempt so far to use V as an axiomatization of mathematics."

Expressive power

Second-order logic is more expressive than first-order logic. For example, if the domain is the set of all real numbers, one can assert in first-order logic the existence of an additive inverse of each real number by writing $\forall x \; \exists y \; (x + y = 0)$ but one needs second-order logic to assert the least-upper-bound property for sets of real numbers, which states that every bounded, nonempty set of real numbers has a supremum. If the domain is the set of all real numbers, the following second-order sentence (split over two lines) expresses the least upper bound property:

$$(\forall \mathrm{A}) \, ([(\exists \mathrm{w}) \, (\mathrm{w} \in \mathrm{A}) \wedge (\exists \mathrm{z})(\forall \mathrm{w})(\mathrm{w} \in \mathrm{A} \rightarrow \mathrm{w} \leq \mathrm{z})]$$

$$\rightarrow (\exists \mathrm{x})(\forall \mathrm{y})[(\forall \mathrm{w})(\mathrm{w} \in \mathrm{A} \rightarrow \mathrm{w} \leq \mathrm{y}) \leftrightarrow (\mathrm{x} \leq \mathrm{y})])$$

In second-order logic, it is possible to write formal sentences which say "the domain is finite" or "the domain is of countable cardinality." To say that the domain is finite, use the sentence that says that every surjective function from the domain to itself is injective. To say that the domain has countable cardinality, use the sentence that says that there is a bijection between every two infinite subsets of the domain. It follows from the compactness theorem and the upward Löwenheim–Skolem theorem that it is not possible to characterize finiteness or countability, respectively, in first-order logic.

Certain fragments of second order logic like ESO are also more expressive than first-order logic even though they are strictly less expressive than the full second-order logic. ESO also enjoys translation equivalence with some extensions of first-order logic which allow non-linear ordering of quantifier dependencies, like first-order logic extended with Henkin quantifiers, Hintikka and Sandu's independence-friendly logic, and Väänänen's dependence logic.

Deductive systems

A deductive system for a logic is a set of inference rules and logical axioms that determine which sequences of formulas constitute valid proofs. Several deductive systems can be used for second-order logic, although none can be complete for the standard semantics (see below). Each of these systems is sound, which means any sentence they can be used to prove is logically valid in the appropriate semantics.

The weakest deductive system that can be used consists of a standard deductive system for first-order logic (such as natural deduction) augmented with substitution rules for second-order terms.[2] This deductive system is commonly used in the study of second-order arithmetic.

The deductive systems considered by Shapiro (1991) and Henkin (1950) add to the augmented first-order deductive scheme both comprehension axioms and choice axioms. These axioms are sound for standard second-order semantics. They are sound for Henkin semantics if only Henkin models that satisfy the comprehension and choice axioms are considered.[3]

Non-reducibility to first-order logic

One might attempt to reduce the second-order theory of the real numbers, with full second-order semantics, to the first-order theory in the following way. First expand the domain from the set of all real numbers to a two-sorted domain, with the second sort containing all *sets of* real numbers. Add a new binary predicate to the language: the membership relation. Then sentences that were second-order become first-order, with the formerly second-order quantifiers ranging over the second sort instead. This reduction can be attempted in a one-sorted theory by adding unary predicates that tell whether an element is a number or a set, and taking the domain to be the union of the set of real numbers and the power set of the real numbers.

But notice that the domain was asserted to include ***all*** sets of real numbers. That requirement cannot be reduced to a first-order sentence, as the Löwenheim-Skolem theorem shows. That theorem implies that there is some countably infinite subset of the real numbers, whose members we will call *internal numbers*, and some countably infinite collection of sets of internal numbers, whose members we will call "internal sets", such that the domain consisting of internal numbers and internal sets satisfies exactly the same first-order sentences satisfied as the domain of real numbers and sets of real numbers. In particular, it satisfies a sort of least-upper-bound axiom that says, in effect:

> Every nonempty *internal* set that has an *internal* upper bound has a least *internal* upper bound.

Countability of the set of all internal numbers (in conjunction with the fact that those form a densely ordered set) implies that that set does not satisfy the full least-upper-bound axiom. Countability of the set of all *internal* sets implies that it is not the set of *all* subsets of the set of all *internal* numbers (since Cantor's theorem implies that the set of all subsets of a countably infinite set is an uncountably infinite set). This construction is closely related to Skolem's paradox.

Thus the first-order theory of real numbers and sets of real numbers has many models, some of which are countable. The second-order theory of the real numbers has only one model, however. This follows from the classical theorem that there is only one Archimedean complete ordered field, along with the fact that all the axioms of an Archimedean complete ordered field are expressible in second-order logic. This shows that the second-order theory of the real numbers cannot be reduced to a first-order theory, in the sense that the second-order theory of the real numbers has only one model but the corresponding first-order theory has many models.

There are more extreme examples showing that second-order logic with standard semantics is more expressive than first-order logic. There is a finite second-order theory whose only model is the real numbers if the continuum hypothesis holds and which has no model if the continuum hypothesis does not hold (cf. Shapiro 2000 p. 105). This theory consists of a finite theory characterizing the real numbers as a complete Archimedean ordered field plus an axiom saying that the domain is of the first uncountable cardinality. This example illustrates that the question of whether a sentence in second-order logic is consistent is extremely subtle.

Additional limitations of second order logic are described in the next section.

Metalogical results

It is a corollary of Gödel's incompleteness theorem that there is no deductive system (that is, no notion of *provability*) for second-order formulas that simultaneously satisfies these three desired attributes:[4]

- (Soundness) Every provable second-order sentence is universally valid, i.e., true in all domains under standard semantics.
- (Completeness) Every universally valid second-order formula, under standard semantics, is provable.
- (Effectiveness) There is a proof-checking algorithm that can correctly decide whether a given sequence of symbols is a valid proof or not.

This corollary is sometimes expressed by saying that second-order logic does not admit a complete proof theory. In this respect second-order logic with standard semantics differs from first-order logic; Quine (1970, pp. 90–91) pointed to the lack of a complete proof system as a reason for thinking of second-order logic as not *logic*, properly speaking.

As mentioned above, Henkin proved that the standard deductive system for first-order logic is sound, complete, and effective for second-order logic with Henkin semantics, and the deductive system with comprehension and choice principles is sound, complete, and effective for Henkin semantics using only models that satisfy these principles.

The compactness theorem and the Löwenheim-Skolem theorem do not hold for full models of second-order logic. They do hold however for Henkin models. (Väänänen 2001)

History and disputed value

Predicate logic was primarily introduced to the mathematical community by C. S. Peirce, who coined the term *second-order logic* and whose notation is most similar to the modern form (Putnam 1982). However, today most students of logic are more familiar with the works of Frege, who actually published his work several years prior to Peirce but whose works remained in obscurity until Bertrand Russell and Alfred North Whitehead made them famous. Frege used different variables to distinguish quantification over objects from quantification over properties and sets; but he did not see himself as doing two different kinds of logic. After the discovery of Russell's paradox it was realized that something was wrong with his system. Eventually logicians found that restricting Frege's logic in various ways—to what is now called first-order logic—eliminated this problem: sets and properties cannot be quantified over in first-order-logic alone. The now-standard hierarchy of orders of logics dates from this time.

It was found that set theory could be formulated as an axiomatized system within the apparatus of first-order logic (at the cost of several kinds of completeness, but nothing so bad as Russell's paradox), and this was done (see Zermelo-Fraenkel set theory), as sets are vital for mathematics. Arithmetic, mereology, and a variety of other powerful logical theories could be formulated axiomatically without appeal to any more logical apparatus than first-order quantification, and this, along with Gödel and Skolem's adherence to first-order logic, led to a general decline in work in second (or any higher) order logic.

This rejection was actively advanced by some logicians, most notably W. V. Quine. Quine advanced the view that in predicate-language sentences like *Fx* the "*x*" is to be thought of as a variable or name denoting an object and hence can be quantified over, as in "For all things, it is the case that . . ." but the "*F*" is to be thought of as an *abbreviation* for an incomplete sentence, not the name of an object (not even of an abstract object like a property). For example, it might mean " . . . is a dog." But it makes no sense to think we can quantify over something like this. (Such a position is quite consistent with Frege's own arguments on the concept-object distinction). So to use a predicate as a variable is to have it occupy the place of a name which only individual variables should occupy. This reasoning has been rejected by Boolos.

In recent years second-order logic has made something of a recovery, buoyed by George Boolos' interpretation of second-order quantification as plural quantification over the same domain of objects as first-order quantification (Boolos 1984). Boolos furthermore points to the claimed nonfirstorderizability of sentences such as "Some critics admire only each other" and "Some of Fianchetto's men went into the warehouse unaccompanied by anyone else" which he argues can only be expressed by the full force of second-order quantification. However, generalized quantification and partially ordered, or branching, quantification may suffice to express a certain class of purportedly nonfirstorderizable sentences as well and it does not appeal to second-order quantification.

Relation to computational complexity

The expressive power of various forms of second-order logic on finite structures is intimately tied to computational complexity theory. The field of descriptive complexity studies which computational complexity classes can be characterized by the power of the logic needed to express languages (sets of finite strings) in them. A string $w = w_1 \cdots w_n$ in a finite alphabet A can be represented by a finite structure with domain $D = \{1,...,n\}$, unary predicates P_a for each $a \in A$, satisfied by those indices i such that $w_i = a$, and additional predicates which serve to uniquely identify which index is which (typically, one takes the graph of the successor function on D or the order relation $<$, possibly with other arithmetic predicates). Conversely, the table of any finite structure can be encoded by a finite string.

With this identification, we have the following characterizations of variants of second-order logic over finite structures:

- REG (the set of regular languages) is definable by monadic, second-order formulas (Büchi's theorem, 1960)
- NP is the set of languages definable by existential, second-order formulas (Fagin's theorem, 1974).
- co-NP is the set of languages definable by universal, second-order formulas.
- PH is the set of languages definable by second-order formulas.
- PSPACE is the set of languages definable by second-order formulas with an added transitive closure operator.
- EXPTIME is the set of languages definable by second-order formulas with an added least fixed point operator.

Relationships among these classes directly impact the relative expressiveness of the logics over finite structures; for example, if **PH** = **PSPACE**, then adding a transitive closure operator to second-order logic would not make it any more expressive over finite structures.

Notes

[1] Shapiro (1991) and Hinman (2005) give complete introductions to the subject, with full definitions.

[2] Such a system is used without comment by Hinman (2005).

[3] These are the models originally studied by Henkin (1950).

[4] The proof of this corollary is that a sound, complete, and effective deduction system for standard semantics could be used to produce a recursively enumerable completion of Peano arithmetic, which Gödel's theorem shows cannot exist.

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Many-sorted logic

Many-sorted logic can reflect formally our intention not to handle the universe as a homogeneous collection of objects, but to partition it in a way that is similar to types in typeful programming. Both functional and assertive "parts of speech" in the language of the logic reflect this typeful partitioning of the universe, even on the syntax level: substitution and argument passing can be done only accordingly, respecting the “sorts”.

There are more ways to formalize the intention mentioned above; a *many-sorted logic* is any package of information which fulfills it. In most cases, the following are given:

- a set of sorts, *S*
- an appropriate generalization of the notion of *signature* to be able to handle the additional information that comes with the sorts.

The domain of discourse of any structure of that signature is then fragmented into disjoint subsets, one for every sort.

Algebraization

The algebraization of many-sorted logic is explained in *On the Algebraization of Many-sorted Logics* [1] by Carlos Caleiro and Ricardo Gonçalves. The book generalizes abstract algebraic logic to the many-sorted case, but it can also be used as introductory material.

External links

- "Many-sorted Logic" [2], the first chapter in *Lecture Notes on Decision Procedures* [3] by Calogero G. Zarba [4]

References

[1] http://sqig.math.ist.utl.pt/pub/CaleiroC/06-CG-manysorted.pdf
[2] http://react.cs.uni-sb.de/teaching/decision-procedures-verification-06/ch01.pdf
[3] http://react.cs.uni-sb.de/~zarba/notes.html
[4] http://react.cs.uni-sb.de/~zarba/

Infinitary logic

Those unfamiliar with mathematical logic or the concept of ordinals are advised to consult those articles first.

An **infinitary logic** is a logic that allows infinitely long statements and/or infinitely long proofs. Some infinitary logics may have different properties from those of standard first-order logic. In particular, infinitary logics may fail to be compact or complete. Notions of compactness and completeness that are equivalent in finitary logic sometimes are not so in infinitary logic. So for infinitary logics the notions of strong compactness and strong completeness are defined. This article addresses Hilbert-type infinitary logics, as these have been extensively studied and constitute the most straightforward extensions of finitary logic. These are not, however, the only infinitary logics that have been formulated or studied.

Considering whether a certain infinitary logic named Ω-logic is complete promises to throw light on the continuum hypothesis.

A word on notation and the axiom of choice

As a language with infinitely long formulae is being presented, it is not possible to write expressions down as they should be written. To get around this problem a number of notational conveniences, which, strictly speaking, are not part of the formal language, are used. $\cdots$ is used to point out an expression that is infinitely long. Where it is unclear, the length of the sequence is noted afterwards. Where this notation becomes ambiguous or confusing, suffixes such as $\vee_{\gamma<\delta}A_{\gamma}$ are used to indicate an infinite disjunction over a set of formulae of cardinality δ. The same notation may be applied to quantifiers for example $\forall_{\gamma<\delta}V_{\gamma}:$. This is meant to represent an infinite sequence of quantifiers for each V_{γ} where $\gamma<\delta$.
All usage of suffixes and $\cdots$ are not part of formal infinitary languages. The axiom of choice is assumed (as is often done when discussing infinitary logic) as this is necessary to have sensible distributivity laws.

Definition of Hilbert-type infinitary logics

A first-order infinitary logic $L_{\alpha,\beta}$, α regular, β = 0 or ω ≤ β ≤ α, has the same set of symbols as a finitary logic and may use all the rules for formation of formulae of a finitary logic together with some additional ones:

- Given a set of variables $V=\{V_{\gamma}|\gamma<\delta<\beta\}$ and a formula A_0 then $\forall V_0:\forall V_1\cdots(A_0)$ and $\exists V_0:\exists V_1\cdots(A_0)$ are formulae (In each case the sequence of quantifiers has length δ).
- Given a set of formulae $A=\{A_{\gamma}|\gamma<\delta<\alpha\}$ then $(A_0\vee A_1\vee\cdots)$ and $(A_0\wedge A_1\wedge\cdots)$ are formulae (In each case the sequence has length δ).

The concepts of bound variables apply in the same manner to infinite sentences. Note that the number of brackets in these formulae is always finite. Just as in finitary logic, a formula all of whose variables are bound is referred to as a *sentence*.

A theory T in infinitary logic $L_{\alpha,\beta}$ is a set of statements in the logic. A proof in infinitary logic from a theory T is a sequence of statements of length γ which obeys the following conditions: Each statement is either a logical axiom, an element of T, or is deduced from previous statements using a rule of inference. As before, all rules of inference in finitary logic can be used, together with an additional one:

- Given a set of statements $A=\{A_{\gamma}|\gamma<\delta<\alpha\}$ which have occurred previously in the proof then the statement $\wedge_{\gamma<\delta}A_{\gamma}$ can be inferred.

The logical axiom schemata specific to infinitary logic are presented below. Global schemata variables: δ and γ such that $0<\delta<\alpha$.

- $((\wedge_{\epsilon<\delta}(A_{\delta}\implies A_{\epsilon}))\implies(A_{\delta}\implies\wedge_{\epsilon<\delta}A_{\epsilon}))$
- For each $\gamma<\delta$, $((\wedge_{\epsilon<\delta}A_{\epsilon})\implies A_{\gamma})$

- Chang's distributivity laws (for each γ): $(\bigvee_{\mu<\gamma}(\bigwedge_{\delta<\gamma}A_{\mu,\delta}))$ where $\forall\mu:\forall\delta:\exists\epsilon<\gamma:A_{\mu,\delta}=A_\epsilon$ and $\forall g\in\gamma^\gamma:\exists\epsilon<\gamma:\{A_\epsilon,\neg A_\epsilon\}\subseteq\{A_{\mu,g(\mu)}:\mu<\gamma\}$
- For $\gamma<\alpha$, $((\bigwedge_{\mu<\gamma}(\bigvee_{\delta<\gamma}A_{\mu,\delta}))\implies(\bigvee_{\epsilon<\gamma^\gamma}(\bigwedge_{\mu<\gamma}A_{\mu,\gamma_\epsilon})))$ where γ_ϵ is a well ordering of γ^γ

The last two axiom schemata require the axiom of choice because certain sets must be well orderable. The last axiom schema is strictly speaking unnecessary as Chang's distributivity laws imply it, however it is included as a natural way to allow natural weakenings to the logic.

Completeness, compactness, and strong completeness

A theory is any set of statements. The truth of statements in models are defined by recursion and will agree with the definition for finitary logic where both are defined. Given a theory T a statement is said to be valid for the theory T if it is true in all models of T.

A logic $L_{\alpha,\beta}$ is complete if for every sentence S valid in every model there exists a proof of S. It is strongly complete if for any theory T for every sentence S valid in T there is a proof of S from T. An infinitary logic can be complete without being strongly complete.

A logic is compact if for every theory T of cardinality α if all subsets S of T have models then T has a model. A logic is strongly compact if for every theory T if all subsets S of T, where S has cardinality $<\alpha$, have models then T has a model. If a logic is strongly compact, and complete, then it is strongly complete.

The cardinal $\kappa\neq\omega$ is weakly compact if $L_{\kappa,\kappa}$ is compact and κ is strongly compact if $L_{\kappa,\kappa}$ is strongly compact.

Concepts expressible in infinitary logic

In the language of set theory the following statement expresses foundation:

$$\forall_{\gamma<\omega}V_\gamma:\neg\bigwedge_{\gamma<\omega}V_{\gamma+}\in V_\gamma.$$

Unlike the axiom of foundation, this statement admits no non-standard interpretations. The concept of well foundedness can only be expressed in a logic which allows infinitely many quantifiers in an individual statement. As a consequence many theories, including Peano arithmetic, which cannot be properly axiomatised in finitary logic, can be in a suitable infinitary logic. Other examples include the theories of non-archimedean fields and torsion-free groups. These three theories can be defined without the use of infinite quantification; only infinite junctions are needed.

Complete infinitary logics

Two infinitary logics stand out in their completeness. These are $L_{\omega,\omega}$ and $L_{\omega_1,\omega}$. The former is standard finitary first-order logic and the latter is an infinitary logic that only allows statements of countable size.

$L_{\omega,\omega}$ is also strongly complete, compact and strongly compact.

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Quantification

Quantification has several distinct senses. In mathematics and empirical science, it is the act of counting and measuring that maps human sense observations and experiences into members of some set of numbers. Quantification in this sense is fundamental to the scientific method.

In logic, **quantification** is the binding of a variable ranging over a domain of discourse. The variable thereby becomes bound by an operator called a **quantifier**. Academic discussion of quantification refers more often to this meaning of the term than the preceding one.

In grammar, a **quantifier** is a type of determiner, such as *all* or *many*, that indicates quantity. These items have been argued to correspond to logical quantifiers at the semantic level.

Natural language

All known human languages make use of quantification (Wiese 2004). For example, in English:

- *Every* glass in my recent order was chipped.
- *Some of* the people standing across the river have white armbands.
- *Most of* the people I talked to didn't have a clue who the candidates were.
- *A lot of* people are smart.

The words in italics are quantifiers. There exists no simple way of reformulating any one of these expressions as a conjunction or disjunction of sentences, each a simple predicate of an individual such as *That wine glass was chipped*. These examples also suggest that the construction of quantified expressions in natural language can be syntactically very complicated. Fortunately, for mathematical assertions, the quantification process is syntactically more straightforward.

The study of quantification in natural languages is much more difficult than the corresponding problem for formal languages. This comes in part from the fact that the grammatical structure of natural language sentences may conceal the logical structure. Moreover, mathematical conventions strictly specify the range of validity for formal language quantifiers; for natural language, specifying the range of validity requires dealing with non-trivial semantic problems. For example the sentence *"Someone gets mugged in New York every 10 minutes"* does not identify whether it is the same person getting mugged every 10 minutes.

Montague grammar gives a novel formal semantics of natural languages. Its proponents argue that it provides a much more natural formal rendering of natural language than the traditional treatments of Frege, Russell and Quine.

Logic

In language and logic, quantification is a construct that specifies the quantity of specimens in the domain of discourse that apply to (or satisfy) an open formula. For example, in arithmetic, it allows the expression of the statement that every natural number has a successor. A language element which generates a quantification is called a quantifier. The resulting expression is a quantified expression; and the expression is said to be quantified over the predicate or function expression whose free variable is bound by the quantifier. Quantification is used in both natural languages and formal languages. Examples of quantifiers in English are *for all*, *for some*, *many*, *few*, *a lot*, and *no*. In formal languages, quantification is a formula constructor that produces new formulas from old ones. The semantics of the language specifies how the constructor is interpreted as an extent of validity.

The two fundamental kinds of quantification in predicate logic are universal quantification and existential quantification. The traditional symbol for the universal quantifier "all" is "$\forall$", an inverted letter "A", and for the existential quantifier "exists" is "$\exists$", a rotated letter "E". These quantifiers have been generalized beginning with the work of Mostowski and Lindström.

Mathematics

Consider the following statement:

1·2 = 1 + 1, and 2·2 = 2 + 2, and 3 · 2 = 3 + 3, ..., and $n \cdot 2 = n + n$, etc.

This has the appearance of an *infinite conjunction* of propositions. From the point of view of formal languages this is immediately a problem, since syntax rules are expected to generate finite objects. The example above is fortunate in that there is a procedure to generate all the conjuncts. However, if an assertion were to be made about every irrational number, there would be no way to enumerate all the conjuncts, since irrationals cannot be enumerated. A succinct formulation which avoids these problems uses *universal quantification*:

For any natural number n, $n \cdot 2 = n + n$.

A similar analysis applies to the disjunction,

1 is equal to 5 + 5, or 2 is equal to 5 + 5, or 3 is equal to 5 + 5, ... , or n is equal to 5 + 5, etc.

which can be rephrased using *existential quantification*:

For some natural number n, n is equal to 5 + 5.

It is possible to devise abstract algebras whose models include formal languages with quantification, but progress has been slow and interest in such algebra has been limited. Three approaches have been devised to date:

- Relation algebra, invented by DeMorgan, and developed by Charles Sanders Peirce, Ernst Schröder, Tarski, and Tarski's students. Relation algebra cannot represent any formula with quantifiers nested more than three deep. Surprisingly, the models of relation algebra include the axiomatic set theory ZFC and Peano arithmetic;
- Cylindric algebra, devised by Tarski, Henkin, and others;
- The polyadic algebra of Paul Halmos.

Notation

There are two quantifiers, the universal quantifier and the existential quantifier. The traditional symbol for the universal quantifier is "∀", an inverted letter "A", which stands for "for all" or "all". The corresponding symbol for the existential quantifier is "∃", a rotated letter "E", which stands for "there exists" or "exists".

An example of quantifying an English statement would be as follows. Given the statement, "All of Peters friends either like to dance or like to go to the beach", we can identify key aspects and rewrite using symbols including quantifiers. So, let x be any one particular friend of Peter, X the set of all Peter's friends, P(x) be the predicate (mathematical logic) "x likes to dance", and lastly Q(x) the predicate "x likes to go to the beach". Then we have, : $\forall x \in X, P(x) \lor Q(x)$. Which is read, "for all x that's a member of X, P of x or Q of x."

Some other quantified expressions are constructed as follows,

$$\exists x\, P \qquad \forall x\, P$$

for a formula P. Variant notations include, for set X and set members x:

$$(\exists x)P \qquad (\exists x\,.\,P) \qquad \exists x \cdot P \qquad (\exists x : P) \qquad \exists x(P) \qquad \exists_x\, P \qquad \exists x, P \qquad \exists x{\in}X\, P \qquad \exists\, x{:}X\, P$$

All of these variations also apply to universal quantification. Other variations for the universal quantifier are

$$(x)\, P \qquad \bigwedge_x P$$

Some versions of the notation explicitly mention the range of quantification. The range of quantification must always be specified; for a given mathematical theory, this can be done in several ways:

- Assume a fixed domain of discourse for every quantification, as is done in Zermelo–Fraenkel set theory,
- Fix several domains of discourse in advance and require that each variable have a declared domain, which is the *type* of that variable. This is analogous to the situation in statically typed computer programming languages, where variables have declared types.

- Mention explicitly the range of quantification, perhaps using a symbol for the set of all objects in that domain or the type of the objects in that domain.

One can use any variable as a quantified variable in place of any other, under certain restrictions in which *variable capture* does not occur. Even if the notation uses typed variables, variables of that type may be used.

Informally or in natural language, the "$\forall x$" or "$\exists x$" might appear after or in the middle of $P(x)$. Formally, however, the phrase that introduces the dummy variable is placed in front.

Mathematical formulas mix symbolic expressions for quantifiers, with natural language quantifiers such as

For any natural number x, ...

There exists an x such that ....

For at least one x.

Keywords for uniqueness quantification include:

For exactly one natural number x, ...

There is one and only one x such that ....

Further, x may be replaced by a pronoun. For example,

For any natural number, its product with 2 equals to its sum with itself

Some natural number is prime.

Equivalent Expressions

If X is a domain of x and $P(x)$ is a predicate dependent on x, then the universal proposition is expressed in Boolean algebra terms as

$$\forall x \in X, P(x) \equiv \{x \in X\} \to P(x) \equiv \{x \notin X\} \vee P(x),$$

which equivalently reads "if x is in X, then $P(x)$ is true." If x is not in X, then $P(x)$ is indeterminate. Note that the truth of the expression requires only that x be in X, so it can be *any* x in X, independent of $P(x)$, whereas the *falsity* of the expression, or the truth of

$$\{x \in X\} \wedge \neg P(x),$$

additionally requires that x be such that $P(x)$ evaluates to false; this is the reason behind calling x a "bound variable." This last expression can thus be read as "for some x in X, $P(x)$ is false," or "there exists an x in X such that $P(x)$ is false." So, we now have the equivalent Boolean expression for the existential proposition:

$$\exists x \in X : P(x) \equiv \{x \in X\} \wedge P(x).$$

Thus, together with negation, only one of either the universal or existential quantifier is needed to perform both tasks:

$$\neg(\forall x \in X, P(x)) \equiv \exists x \in X : \neg P(x),$$

which shows that to disprove a "for all x" proposition, one needs no more than to find an x for which the predicate is false. Similarly,

$$\neg(\exists x \in X : P(x)) \equiv \forall x \in X, \neg P(x),$$

to disprove a "there exists an x" proposition, one needs to show that the predicate is false for all x.

Nesting

Consider the following statement:

> For any natural number n, there is a natural number s such that $s = n^2$.

This is clearly true; it just asserts that every natural number has a square. The meaning of the assertion in which the quantifiers are turned around is different:

> There is a natural number s such that for any natural number n, $s = n^2$.

This is clearly false; it asserts that there is a single natural number *s* that is at once the square of *every* natural number. This is because the syntax directs that any newly introduced variable cannot be a function of subsequently introduced variables.

This illustrates that the order of quantifiers is critical to meaning.

A less trivial example is the important concept of uniform continuity from analysis, which differs from the more familiar concept of pointwise continuity only by an exchange in the positions of two quantifiers. To illustrate this, let *f* be a real-valued function on **R**.

- A: Pointwise continuity of f on **R**:

$$\underbrace{\forall x \in \mathbb{R}}\,\underbrace{\forall \epsilon > 0}\,\exists \delta > 0\,\forall h \in \mathbb{R}\,\left(|h| < \delta\ \rightarrow\ |f(x) - f(x+h)| < \epsilon\right)$$

interchanging the universal quantifiers over the braces, this is the same as

- A': Pointwise continuity of *f* on **R**:

$$\forall \epsilon > 0\,\underbrace{\forall x \in \mathbb{R}}\,\underbrace{\exists \delta > 0}\,\forall h \in \mathbb{R}\,\left(|h| < \delta\ \rightarrow\ |f(x) - f(x+h)| < \epsilon\right)$$

Thus, it is implied that the particular value chosen for δ can only be a function of ε and x, the variables that precede it; whereas in

- B: Uniform continuity of *f* on **R**:

$$\forall \epsilon > 0\,\underbrace{\exists \delta > 0}\,\underbrace{\forall x \in \mathbb{R}}\,\forall h \in \mathbb{R}\,\left(|h| < \delta\ \rightarrow\ |f(x) - f(x+h)| < \epsilon\right)$$

by interchanging the existential and universal quantifiers over the braces in A', δ is asserted to be independent of x.

Ambiguity is avoided with the quantifiers in front:

- $\exists$ A: $\forall$ B: C – unambiguous
- there is an A such that $\forall$ B: C – unambiguous
- there is an A such that for all B, C – unambiguous, provided that the separation between B and C is clear
- there is an A such that C for all B – it is often clear that what is meant is

  > there is an A such that (C for all B)

  but it could be interpreted as

  > (there is an A such that C) for all B

- there is an A such that C $\forall$ B — suggests more strongly that the first is meant; this may be reinforced by the layout, for example by putting "C $\forall$ B" on a new line.

The maximum depth of nesting of quantifiers inside a formula is called its Quantifier rank.

Range of quantification

Every quantification involves one specific variable and a *domain of discourse* or *range of quantification* of that variable. The range of quantification specifies the set of values that the variable takes. In the examples above, the range of quantification is the set of natural numbers. Specification of the range of quantification allows us to express the difference between, asserting that a predicate holds for some natural number or for some real number. Expository conventions often reserve some variable names such as "*n*" for natural numbers and "*x*" for real numbers, although relying exclusively on naming conventions cannot work in general since ranges of variables can change in the course of a mathematical argument.

A more natural way to restrict the domain of discourse uses *guarded quantification*. For example, the guarded quantification

For some natural number n, n is even and n is prime

means

For some even number n, n is prime.

In some mathematical theories a single domain of discourse fixed in advance is assumed. For example, in Zermelo–Fraenkel set theory, variables range over all sets. In this case, guarded quantifiers can be used to mimic a smaller range of quantification. Thus in the example above to express

For any natural number n, $n \cdot 2 = n + n$

in Zermelo–Fraenkel set theory, it can be said

For any n, if n belongs to **N**, then $n \cdot 2 = n + n$,

where **N** is the set of all natural numbers.

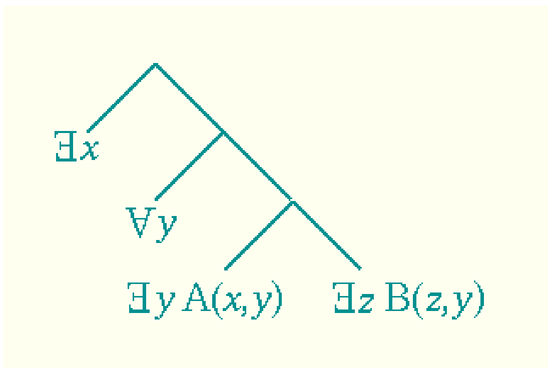
Formal semantics

Mathematical Semantics is the application of mathematics to study the meaning of expressions in a formal language. It has three elements: A mathematical specification of a class of objects via syntax, a mathematical specification of various semantic domains and the relation between the two, which is usually expressed as a function from syntactic objects to semantic ones. This article only addresses the issue of how quantifier elements are interpreted.

Given a model theoretical logical framework, the syntax of a formula can be given by a syntax tree. Quantifiers have scope and a variable x is free if it is not within the scope of a quantification for that variable. Thus in

$$\forall x(\exists y B(x,y)) \vee C(y,x)$$

the occurrence of both x and y in $C(y,x)$ is free.



Syntactic tree illustrating scope and variable capture

An interpretation for first-order predicate calculus assumes as given a domain of individuals X. A formula A whose free variables are $x_1, ..., x_n$ is interpreted as a boolean-valued function $F(v_1, ..., v_n)$ of n arguments, where each argument ranges over the domain X. Boolean-valued means that the function assumes one of the values **T** (interpreted as truth) or **F** (interpreted as falsehood). The interpretation of the formula

$$\forall x_n A(x_1, \ldots, x_n)$$

is the function G of n-1 arguments such that $G(v_1, ...,v_{n-1})$ = **T** if and only if $F(v_1, ..., v_{n-1}, w)$ = **T** for every w in X. If $F(v_1, ..., v_{n-1}, w)$ = **F** for at least one value of w, then $G(v_1, ...,v_{n-1})$ = **F**. Similarly the interpretation of the formula

$$\exists x_n A(x_1, \ldots, x_n)$$

is the function H of n-1 arguments such that $H(v_1, ...,v_{n-1})$ = **T** if and only if $F(v_1, ...,v_{n-1}, w)$ = **T** for at least one w and $H(v_1, ..., v_{n-1})$ = **F** otherwise.

The semantics for uniqueness quantification requires first-order predicate calculus with equality. This means there is given a distinguished two-placed predicate "="; the semantics is also modified accordingly so that "=" is always interpreted as the two-place equality relation on X. The interpretation of

$$\exists! x_n A(x_1, \ldots, x_n)$$

then is the function of n-1 arguments, which is the logical *and* of the interpretations of

$$\exists x_n A(x_1, \ldots, x_n)$$
$$\forall y, z \, \{A(x_1, \ldots, x_{n-1}, y) \wedge A(x_1, \ldots, x_{n-1}, z) \implies y = z\} \, .$$

Paucal, multal and other degree quantifiers

None of the quantifiers previously discussed apply to a quantification such as

There are many integers $n < 100$, such that n is divisible by 2 or 3 or 5.

One possible interpretation mechanism can be obtained as follows: Suppose that in addition to a semantic domain *X*, we have given a probability measure P defined on *X* and cutoff numbers $0 < a \le b \le 1$. If *A* is a formula with free variables $x_1,\ldots,x_n$ whose interpretation is the function *F* of variables $v_1,\ldots,v_n$ then the interpretation of

$$\exists^{\text{many}} x_n A(x_1, \ldots, x_{n-1}, x_n)$$

is the function of $v_1,\ldots,v_{n-1}$ which is **T** if and only if

$$\mathrm{P}\{w : F(v_1, \ldots, v_{n-1}, w) = \mathbf{T}\} \ge b$$

and **F** otherwise. Similarly, the interpretation of

$$\exists^{\text{few}} x_n A(x_1, \ldots, x_{n-1}, x_n)$$

is the function of $v_1,\ldots,v_{n-1}$ which is **F** if and only if

$$0 < \mathrm{P}\{w : F(v_1, \ldots, v_{n-1}, w) = \mathbf{T}\} \le a$$

and **T** otherwise.

Other quantifiers

A few other quantifiers have been proposed over time. In particular, the solution quantifier,[1] noted § (section sign) and read "those". For example:

$$\left[\S n \in \mathbb{N} \quad n^2 \le 4\right] = \{0, 1, 2\}$$

is read "those *n* in **N** such that $n^2 \le 4$ are in {0,1,2}." The same construct is expressible in set-builder notation:

$$\{n \in \mathbf{N} : n^2 \le 4\} = \{0, 1, 2\}$$

History

Term logic treats quantification in a manner that is closer to natural language, and also less suited to formal analysis. Aristotelian logic treated *All*, *Some* and *No* in the 1st century BC, in an account also touching on the alethic modalities.

Gottlob Frege, in his 1879 *Begriffsschrift*, was the first to employ a quantifier to bind a variable ranging over a domain of discourse and appearing in predicates. He would universally quantify a variable (or relation) by writing the variable over a dimple in an otherwise straight line appearing in his diagrammatic formulas. Frege did not devise an explicit notation for existential quantification, instead employing his equivalent of ~∀*x*~, or contraposition. Frege's treatment of quantification went largely unremarked until Bertrand Russell's 1903 *Principles of Mathematics*.

In work that culminated in Peirce (1885), Charles Sanders Peirce and his student Oscar Howard Mitchell independently invented universal and existential quantifiers, and bound variables. Peirce and Mitchell wrote Π_x and Σ_x where we now write ∀*x* and ∃*x*. Peirce's notation can be found in the writings of Ernst Schröder, Leopold Loewenheim, Thoralf Skolem, and Polish logicians into the 1950s. Most notably, it is the notation of Kurt Gödel's landmark 1930 paper on the completeness of first-order logic, and 1931 paper on the incompleteness of Peano arithmetic.

Peirce's approach to quantification also influenced William Ernest Johnson and Giuseppe Peano, who invented yet another notation, namely (*x*) for the universal quantification of *x* and (in 1897) ∃*x* for the existential quantification of *x*. Hence for decades, the canonical notation in philosophy and mathematical logic was (*x*)*P* to express "all individuals in the domain of discourse have the property *P*," and "(∃*x*)*P*" for "there exists at least one individual in the domain of discourse having the property *P*." Peano, who was much better known than Peirce, in effect diffused the latter's thinking throughout Europe. Peano's notation was adopted by the *Principia Mathematica* of Whitehead and Russell, Quine, and Alonzo Church. In 1935, Gentzen introduced the ∀ symbol, by analogy with Peano's ∃ symbol. ∀ did not become canonical until the 1960s.

Around 1895, Peirce began developing his existential graphs, whose variables can be seen as tacitly quantified. Whether the shallowest instance of a variable is even or odd determines whether that variable's quantification is universal or existential. (Shallowness is the contrary of depth, which is determined by the nesting of negations.) Peirce's graphical logic has attracted some attention in recent years by those researching heterogeneous reasoning and diagrammatic inference.

Natural science

Some measure of the undisputed general importance of quantification in the natural sciences can be gleaned from the following comments: "these are mere facts, but they are quantitative facts and the basis of science."[2] It seems to be held as universally true that "the foundation of quantification is measurement."[3] There is little doubt that "quantification provided a basis for the objectivity of science."[4] In ancient times, "musicians and artists...rejected quantification, but merchants, by definition, quantified their affairs, in order to survive, made them visible on parchment and paper."[5] Any reasonable "comparison between Aristotle and Galileo shows clearly that there can be no unique lawfulness discovered without detailed quantification."[6] Even today, "universities use imperfect instruments called 'exams' to indirectly quantify something they call knowledge."[7] This meaning of quantification comes under the heading of pragmatics.

In some instances in the natural sciences a seemingly intangible concept may be quantified by creating a scale—for example, a pain scale in medical research, or a discomfort scale at the intersection of meteorology and human physiology such as the heat index measuring the combined perceived effect of heat and humidity, or the wind chill factor measuring the combined perceived effects of cold and wind.

Social sciences

In the social sciences, quantification is an integral part of economics and psychology. Both disciplines gather data—economics by empirical observation and psychology by experimentation, and both use statistical techniques such as regression analysis to draw conclusions from it.

In some instances a seemingly intangible property may be quantified by asking subjects to rate something on a scale—for example, a happiness scale or a quality of life scale—or by the construction of a scale by the researcher, as with the index of economic freedom. In other cases, an unobservable variable may be quantified by replacing it with a proxy variable with which it is highly correlated—for example, per capita gross domestic product is often used as a proxy for standard of living or quality of life.

Frequently in the use of regression, the presence or absence of a trait is quantified by employing a dummy variable, which takes on the value 1 in the presence of the trait or the value 0 in the absence of the trait.

Quantitative linguistics is an area of linguistics that relies on quantification. For example,[8] indices of grammaticalization of morphemes, such as phonological shortness, dependence on surroundings, and fusion with the verb, have been developed and found to be significantly correlated across languages with stage of evolution of function of the morpheme.

Hard versus soft science

The ease of quantification is one of the features used to distinguish hard and soft sciences from each other. Hard sciences are often considered to be more scientific, rigorous, or accurate. In some social sciences such as sociology, specific accurate data are difficult to obtain, either because laboratory conditions are not present or because the issues involved are conceptual but not directly quantifiable.

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Analytic philosophy

Analytic philosophy (sometimes **analytical philosophy**) is a generic term for a style of philosophy that came to dominate English-speaking countries in the 20th century. In the United States, United Kingdom, Canada, Scandinavia, Australia, and New Zealand, the vast majority of university philosophy departments identify themselves as "analytic" departments.[1]

The term "analytic philosophy" can refer to:

- A broad philosophical tradition[2][3] characterized by an emphasis on clarity and argument (often achieved via modern formal logic and analysis of language) and a respect for the natural sciences.[4][5][6]
- The more specific set of developments of early 20th-century philosophy that were the historical antecedents of the broad sense: e.g., the work of Bertrand Russell, Ludwig Wittgenstein, G. E. Moore, Gottlob Frege, and the logical positivists.

In this latter, narrower sense, analytic philosophy is identified with specific philosophical commitments (many of which are rejected by contemporary analytic philosophers), such as:

- The logical positivist principle that there are not any specifically philosophical truths and that the object of philosophy is the logical clarification of thoughts. This may be contrasted with the traditional foundationalism, which considers philosophy as a special, elite science that investigates the fundamental reasons and principles of everything.[7] As a result, many analytic philosophers have considered their inquiries as continuous with, or subordinate to, those of the natural sciences.[8]
- The principle that the logical clarification of thoughts can only be achieved by analysis of the logical form of philosophical propositions.[9] The logical form of a proposition is a way of representing it (often using the formal grammar and symbolism of a logical system) to display its similarity with all other propositions of the same type. However, analytic philosophers disagree widely about the correct logical form of ordinary language.[10]
- The rejection of sweeping philosophical systems in favour of attention to detail,[11] or ordinary language.[12]

According to a characteristic paragraph by Bertrand Russell:

> "Modern analytical empiricism [...] differs from that of Locke, Berkeley, and Hume by its incorporation of mathematics and its development of a powerful logical technique. It is thus able, in regard to certain problems, to achieve definite answers, which have the quality of science rather than of philosophy. It has the advantage, as compared with the philosophies of the system-builders, of being able to tackle its problems one at a time, instead of having to invent at one stroke a block theory of the whole universe. Its methods, in this respect, resemble those of science. I have no doubt that, in so far as philosophical knowledge is possible, it is by such methods that it must be sought; I have also no doubt that, by these methods, many ancient problems are completely soluble."[13]

Analytic philosophy is often understood in contrast to other philosophical traditions, most notably continental philosophy, and also Indian philosophy, Thomism, and Marxism.[14]

History

Late 19th-century English philosophy was dominated by British idealism, as taught by philosophers such as F.H. Bradley and Thomas Hill Green. It was against this intellectual background that the founders of analytic philosophy, G. E. Moore and Bertrand Russell, articulated the program of early analytic philosophy.

Since its beginning, a basic principle of analytic philosophy has been conceptual clarity,[15] in the name of which Moore and Russell rejected Hegelianism, which they accused of obscurity and idealism.[16][17] Inspired by developments in modern logic, the early Russell claimed that the problems of philosophy can be solved by showing the simple constituents of complex notions.[15]

Russell, during his early career, along with collaborator Alfred North Whitehead, was much influenced by Gottlob Frege, who developed predicate logic, which allowed a much greater range of sentences to be parsed into logical form than was possible by the ancient Aristotlean logic. Frege was also an influential philosopher of mathematics in Germany at the beginning of the 20th century. In contrast to Husserl's 1891 book *Philosophie der Arithmetik*, which attempted to show that the concept of the cardinal number derived from psychical acts of grouping objects and counting them,[18] Frege sought to show that mathematics and logic have their own validity, independent of the judgments or mental states of individual mathematicians and logicians (which were the basis of arithmetic according to the "psychologism" of Husserl's *Philosophie*). Frege further developed his philosophy of logic and mathematics in *The Foundations of Arithmetic* and *The Basic Laws of Arithmetic* where he provided an alternative to psychologistic accounts of the concept of number.

Like Frege, Bertrand Russell and Alfred North Whitehead attempted to show that mathematics is reducible to fundamental logical principles. Their *Principia Mathematica* (1910–1913) encouraged many philosophers to renew their interest with the development of symbolic logic. Additionally, Bertrand Russell adopted Frege's predicate logic as his primary philosophical method, a method he thought could expose the underlying structure of philosophical problems. For example, the English word “is” has three distinct meanings by predicate logic:

- For the sentence 'the cat *is* asleep', the *is* of predication means that "x is P" (denoted as P(x))
- For the sentence 'there *is* a cat', the *is* of existence means that "there is an x" (∃x);
- For the sentence 'three *is* half of six', the *is* of identity means that "x is the same as y" (x=y).

Russell sought to resolve various philosophical issues by applying such definite distinctions, most famously in his analysis of definite descriptions in "On Denoting."[19]

Ideal language analysis

From about 1910 to 1930, analytic philosophers like Russell and Ludwig Wittgenstein emphasized creating an ideal language for philosophical analysis, which would be free from the ambiguities of ordinary language that, in their opinion, often made philosophy invalid. This philosophical trend can be called "ideal-language analysis" or "formalism". During this phase, Russell and Wittgenstein sought to understand language, and hence philosophical problems, by using formal logic to formalize the way in which philosophical statements are made. Ludwig Wittgenstein developed a comprehensive system of logical atomism in his *Tractatus Logico-Philosophicus*. He thereby argued that the world is the totality of actual states of affairs and that these states of affairs can be expressed by the language of first-order predicate logic. So a *picture* of the world can be made by expressing atomic facts as atomic propositions, and linking them using logical operators.

Logical positivism

During the late 1920s, '30s, and '40s, Russell and Wittgenstein's formalism was developed by a group of philosophers in Vienna and Berlin, who were known as the Vienna Circle and Berlin Circle respectively, into a doctrine known as logical positivism (or logical empiricism). Logical positivism used formal logical methods to develop an empiricist account of knowledge.[20] Philosophers such as Rudolf Carnap and Hans Reichenbach, along with other members of the Vienna Circle, claimed that the truths of logic and mathematics were tautologies, and those of science were verifiable empirical claims. These two constituted the entire universe of meaningful judgments; anything else was nonsense. The claims of ethics, aesthetics and theology were, accordingly, pseudo-statements, neither true nor false, simply meaningless nonsense. Karl Popper's insistence upon the role of falsification in the philosophy of science was a reaction to the logical positivists.[21] With the coming to power of Adolf Hitler and National Socialism in Germany and Austria, many members of the Vienna and Berlin Circles fled Germany, most commonly to Britain and America, which helped to reinforce the dominance of logical positivism and analytic philosophy in the Anglophone countries.[22]

Logical positivists typically considered philosophy as having a very limited function. For them, philosophy concerned the clarification of thoughts, rather than having a distinct subject matter of its own. The positivists adopted the verification principle, according to which every meaningful statement is either analytic or is capable of being verified by experience. This caused the logical positivists to reject many traditional problems of philosophy, especially those of metaphysics or ontology, as meaningless.

Ordinary language analysis

After World War II, during the late 1940s and 1950s, analytic philosophy took a turn toward ordinary-language analysis. This movement had two main strands. One followed in the wake of Wittgenstein's later philosophy, which departed dramatically from his early work of the *Tractatus*. The other, known as "Oxford philosophy", involved J. L. Austin. In contrast to earlier analytic philosophers (including the early Wittgenstein) who thought philosophers should avoid the deceptive trappings of natural language by constructing ideal languages, ordinary language philosophers claimed that ordinary language already represented a large number of subtle distinctions that had been unrecognized in the formulation of traditional philosophical theories or problems. While schools such as logical positivism emphasize logical terms, supposed to be universal and separate from contingent factors (such as culture, language, historical conditions), ordinary language philosophy emphasizes the use of language by ordinary people. Some have argued that ordinary language philosophy is of a more sociological grounding, as it essentially emphasizes on the use of language within social contexts. The best-known ordinary language philosophers during the 1950s were Austin and Gilbert Ryle. Some say that this movement marked a return to the common sense philosophy advocated by G.E. Moore.

Ordinary language philosophy often sought to disperse philosophical problems by showing them to be the result of misunderstanding ordinary language. See for example Ryle (who attempted to dispose of "Descartes' myth") and Wittgenstein, among others.

Contemporary analytic philosophy

Although contemporary philosophers who self-identify as "analytic" have widely divergent interests, assumptions, and methods—and have often rejected the fundamental premises that defined the analytic movement before 1960—analytic philosophy, in its contemporary state, is usually taken to be defined by a particular style[4] characterized by precision and thoroughness about a narrow topic, and resistance to "imprecise or cavalier discussions of broad topics."[23]

In the 1950s, logical positivism was influentially challenged by Wittgenstein in the *Philosophical Investigations*, Quine in "Two Dogmas of Empiricism", and Sellars in *Empiricism and the Philosophy of Mind*. Following 1960, Anglophone philosophy began to incorporate a wider range of interests, views, and methods. Still, many philosophers in Britain and America still consider themselves to be "analytic philosophers."[1][4] Largely, they have done so by expanding the notion of "analytic philosophy" from the specific programs that dominated Anglophone philosophy before 1960 to a much more general notion of an "analytic" style, characterized by precision and thoroughness about a narrow topic and opposed to "imprecise or cavalier discussions of broad topics".[23] This interpretation of the history is far from universally accepted, and its opponents would say that it grossly downplays the role of Wittgenstein in the sixties and seventies.

Many philosophers and historians have attempted to define or describe analytic philosophy. Those definitions often include a focus on conceptual analysis: A.P. Martinich draws an analogy between analytic philosophy's interest in conceptual analysis and analytic chemistry, which "aims at determining chemical compositions."[24] Steven D. Hales described analytic philosophy as one of three types of philosophical method practiced in the West: "[i]n roughly reverse order by number of proponents, they are phenomenology, ideological philosophy, and analytic philosophy".[25]

Scott Soames agrees that clarity is important: analytic philosophy, he says, has "an implicit commitment—albeit faltering and imperfect—to the ideals of clarity, rigor and argumentation" and it "aims at truth and knowledge, as opposed to moral or spiritual improvement [...] the goal in analytic philosophy is to discover what is true, not to provide a useful recipe for living one's life". Soames also states that analytic philosophy is characterised by "a more piecemeal approach. There is, I think, a widespread presumption within the tradition that it is often possible to make philosophical progress by intensively investigating a small, circumscribed range of philosophical issues while holding broader, systematic questions in abeyance".[26]

A few of the most important and active fields and subfields in analytic philosophy are summarized in the following sections.

Philosophy of mind and cognitive science

Motivated by the logical positivists' interest in verificationism, behaviorism was the most prominent theory of mind in analytic philosophy for the first half of the twentieth century. Behaviorists tended to hold either that statements about the mind were equivalent to *statements about* behavior and dispositions to behave in particular ways or that mental states were directly equivalent to behavior and dispositions to behave. Behaviorism later became far less popular, in favor of type physicalism or functionalism, theories that identified mental states with brain states. During this period, topics in the philosophy of mind were often in close contact with issues in cognitive science such as modularity or innateness. Finally, analytic philosophy has featured a few philosophers who were dualists, and recently forms of property dualism have had a resurgence, with David Chalmers as the most prominent representative.[27]

John Searle suggests that the obsession with linguistic philosophy of the last century has been superseded by an emphasis on the philosophy of mind,[28] in which functionalism is currently the dominant theory. In recent years, a central focus for research in the philosophy of mind has been consciousness. And while there is a general consensus for the global neuronal workspace model of consciousness,[29] there are many views as to how the specifics work out. The best known theories are Daniel Dennett's heterophenomenology, Fred Dretske and Michael Tye's representationalism, and the higher-order theories of either David M. Rosenthal—who advocates a higher-order thought (HOT) model—or David Armstrong and William Lycan—who advocate a higher-order perception (HOP) model. An alternative higher-order theory, the higher-order global states (HOGS) model, is offered by Robert van Gulick.[30]

Ethics in analytic philosophy

Philosophers working in the analytic tradition have gradually come to distinguish three major branches of moral philosophy.

- Normative ethics whose function is the examination and production of normative ethical judgments
- Meta-ethics whose function is the investigation of moral terms and concepts,
- Applied ethics whose function is the investigation of how existing normative principles should be applied in difficult or borderline cases, often cases created by the appearance of new technologies or new scientific knowledge.

Normative ethics

The first half of the twentieth century was marked by skepticism toward, and neglect of, normative ethics. Related subjects, such as social and political philosophy, aesthetics, and philosophy of history, moved to the fringes of English-language philosophy during this period.

During this time, utilitarianism was the only non-skeptical approach to ethics to remain popular. However, as the influence of logical positivism began to wane mid-century, contemporary analytic philosophers began to have a renewed interest in ethics. G.E.M. Anscombe’s 1958 Modern Moral Philosophy sparked a revival of Aristotle's virtue ethical approach and John Rawls’s 1971 *A Theory of Justice* restored interest in Kantian ethical philosophy. At present, contemporary normative ethics is dominated by three schools: utilitarianism, virtue ethics, and deontology.

Meta-ethics

Twentieth-century meta-ethics has two roots. The first is G. E. Moore's investigation into the nature of ethical terms (e.g. good) in his *Principia Ethica* (1903), which identified the naturalistic fallacy. Along with Hume's famous is/ought distinction, the naturalistic fallacy was a central point of investigation for analytical philosophers.

The second is in logical positivism and its attitude that statements which are unverifiable are meaningless. Although that attitude was adopted originally as a means to promote scientific investigation of the world by rejecting grand metaphysical systems, it had the side effect of making (ethical and aesthetic) value judgments (as well as religious statements and beliefs) meaningless. But since value judgments are obviously of major importance in human life, it became incumbent on logical positivism to develop an explanation of the nature and meaning of value judgements. As a result, analytic philosophers avoided normative ethics, and instead began meta-ethical investigations into the nature of moral terms, statements, and judgments.

The logical positivists held that statements about value—including all ethical and aesthetic judgments—are non-cognitive; that is, they make no statements that can be objectively verified or falsified. Instead, the logical positivists adopted an emotivist position, which held that value judgments expressed the attitude of the speaker. Saying, "Killing is wrong", they thought, was equivalent to saying, "Boo to murder", or saying the word "murder" with a particular tone of disapproval.

While non-cognitivism was generally accepted by analytic philosophers, emotivism had many deficiencies, and evolved into more sophisticated non-cognitivist positions such as the expressivism of Charles Stevenson, and the universal prescriptivism of R. M. Hare, which had its foundations in J. L. Austin's philosophy of speech acts.

These positions were not without their critics. Phillipa Foot contributed several essays attacking all these positions. J. O. Urmson's article "On Grading" called the is/ought distinction into question.

As non-cognitivism, the is/ought distinction, and the naturalistic fallacy began to be called into question, analytic philosophers began to show a renewed interest in the traditional questions of moral philosophy. Perhaps most influential in this area was Elizabeth Anscombe, whose landmark monograph "Intention" was called by Donald Davidson "the most important treatment of action since Aristotle", and is widely regarded as a masterpiece of moral psychology. A favorite student and close friend of Ludwig Wittgenstein, her 1958 article "Modern Moral Philosophy" introduced the term "consequentialism" into the philosophical lexicon, declared the "is-ought" impasse to be a dead end, and led to a revival in virtue ethics.

Applied ethics

A significant feature of analytic philosophy since approximately 1970 has been the emergence of applied ethics—an interest in the application of moral principles to specific practical issues.

Areas of special interest for applied ethics include environmental issues, animal rights issues, and the many challenges created by advancing medical science.[31][32][33]

Analytic philosophy of religion

As with the study of ethics, early analytic philosophy tended to avoid the study of philosophy of religion, largely dismissing (as per the logical positivists view) the subject as part of metaphysics and therefore meaningless.[34] The collapse of logical positivism renewed interest in philosophy of religion, prompting philosophers like William Alston, John Mackie, Alvin Plantinga, Robert Merrihew Adams, Richard Swinburne, and Antony Flew not only to introduce new problems, but to re-open classical topics such as the nature of miracles, theistic arguments, the problem of evil, (see existence of God) the rationality of belief in God, concepts of the nature of God, and many more.[35]

Plantinga, Mackie and Flew debated the logical validity of the *free will defense* as a way to solve the problem of evil.[36] Alston, grappling with the consequences of analytic philosophy of language, worked on the nature of religious language. Adams worked on the relationship of faith and morality.[37] Analytic epistemology and metaphysics has formed the basis for a number of philosophically-sophisticated theistic arguments, like those of the reformed epistemologists like Plantinga.

Analytic philosophy of religion has also been preoccupied with Ludwig Wittgenstein, as well as his interpretation of Søren Kierkegaard's philosophy of religion.[38] Using first-hand remarks (which was later published in *Philosophical Investigations*, *Culture and Value*, and other works), philosophers such as Peter Winch and Norman Malcolm developed what has come to be known as *contemplative philosophy*, a Wittgensteinian school of thought rooted in the "Swansea tradition," and which includes Wittgensteinians such as Rush Rhees, Peter Winch and D. Z. Phillips, among others. The name "contemplative philosophy" was first coined by D. Z. Phillips in *Philosophy's Cool Place*, which rests on an interpretation of a passage from Wittgenstein's "Culture and Value."[39] This interpretation was first labeled, "Wittgensteinian Fideism," by Kai Nielsen but those who consider themselves Wittgensteinians in the Swansea tradition have relentlessly and repeatedly rejected this construal as caricature of Wittgenstein's considered position; this is especially true of D. Z. Phillips.[40] Responding to this interpretation, Kai Nielsen and D.Z. Phillips became two of the most prominent philosophers on Wittgenstein's philosophy of religion.[41]

Political philosophy

Liberalism

Current analytic political philosophy owes much to John Rawls, who in a series of papers from the 1950s onward (most notably "Two Concepts of Rules" and "Justice as Fairness") and his 1971 book *A Theory of Justice*, produced a sophisticated and closely argued defence of a liberalism in politics. This was followed in short order by Rawls's colleague Robert Nozick's book *Anarchy, State, and Utopia*, a defence of free-market libertarianism. Isaiah Berlin has had a notable influence on both analytic political philosophy and Liberalism with his lecture the Two Concepts of Liberty.

Recent decades have also seen the rise of several critiques of liberalism, including the feminist critiques of Catharine MacKinnon and Andrea Dworkin, the communitarian critiques of Michael Sandel and Alasdair MacIntyre (though it should be noted both shy away from the term), and the multiculturalist critiques of Amy Gutmann and Charles Taylor. Although not an analytic philosopher, Jürgen Habermas is another important—if controversial—figure in contemporary analytic political philosophy, whose social theory is a blend of social science, Marxism, neo-Kantianism, and American pragmatism.

Consequentialist libertarianism also derives from the analytic tradition.

Analytical Marxism

Another development in the area of political philosophy has been the emergence of a school known as Analytical Marxism. Members of this school seek to apply the techniques of analytic philosophy, along with tools of modern social science such as rational choice theory to the elucidation of the theories of Karl Marx and his successors. The best-known member of this school is Oxford University philosopher G.A. Cohen, whose 1978 work, *Karl Marx's Theory of History: A Defence* is generally taken as representing the genesis of this school. In that book, Cohen applied the tools of logical and linguistic analysis to the elucidation and defense of Marx's materialist conception of history. Other prominent Analytical Marxists include the economist John Roemer, the social scientist Jon Elster, and the sociologist Erik Olin Wright. The work of these later philosophers have furthered Cohen's work by bringing to bear modern social science methods, such as rational choice theory, to supplement Cohen's use of analytic philosophical techniques in the interpretation of Marxian theory.

Cohen himself would later engage directly with Rawlsian political philosophy to advance a socialist theory of justice that stands in contrast to both traditional Marxism and the theories advanced by Rawls and Nozick. In particular, he points to Marx's principle of from each according to his ability, to each according to his need.

Communitarianism

Communitarians such as Alasdair MacIntyre, Charles Taylor, Michael Walzer and Michael Sandel advance a critique of Liberalism that uses analytic techniques to isolate the key assumptions of Liberal individualists, such as Rawls, and then challenges these assumptions. In particular, Communitarians challenge the Liberal assumption that the individual can be viewed as fully autonomous from the community in which he lives and is brought up. Instead, they push for a conception of the individual that emphasizes the role that the community plays in shaping his or her values, thought processes and opinions.

Analytic metaphysics

One striking break with early analytic philosophy was the revival of metaphysical theorizing in the second half of the twentieth century. Philosophers such as David Kellogg Lewis and David Armstrong developed elaborate theories on a range of topics such as universals, causation, possibility and necessity, and abstract objects.

Among the developments that led to the revival of metaphysical theorizing were Quine's attack on the analytic-synthetic distinction, which was generally taken to undermine Carnap's distinction between existence questions internal to a framework and those external to it.[42]

Metaphysics remains a fertile area for research, having recovered from the attacks of A.J. Ayer and the logical positivists. And though many were inherited from previous decades, the debate remains fierce. The philosophy of fiction, the problem of empty names, and the debate over existence's status as a property have all risen out of relative obscurity to become central concerns, while perennial issues such as free will, possible worlds, and the philosophy of time have had new life breathed into them.[43][44]

Science has also played an increasingly significant role in metaphysics. The theory of special relativity has had a profound effect on the philosophy of time, and quantum physics is routinely discussed in the free will debate.[45] The weight given to scientific evidence is largely due to widespread commitments among philosophers to scientific realism and naturalism.

Philosophy of language

Philosophy of language is another area that has slowed down over the course of the last four decades, as evidenced by the fact that few major figures in contemporary philosophy treat it as a primary research area. Indeed, while the debate remains fierce, it is still strongly under the influence of those figures from the first half of the century: Gottlob Frege, Bertrand Russell, Ludwig Wittgenstein, J.L. Austin, Alfred Tarski, and W.V.O. Quine.

In *Naming and Necessity*, Kripke influentially argued that flaws in common theories of proper names are indicative of larger misunderstandings of the metaphysics of necessity and possibility. By wedding the tools of modal logic to a causal theory of reference, Kripke was widely regarded as reviving theories of essence and identity as respectable topics of philosophical discussion.

Philosophy of science

Reacting against the earlier philosopher of science Sir Karl Popper, who had suggested the falsifiability criterion on which to judge the demarcation between science and non-science, discussions in philosophy of science in the last forty years were dominated by social constructivist and cognitive relativist theories of science. Thomas Samuel Kuhn is one of the major philosophers of science representative of the former theory, while Paul Feyerabend is representative of the latter theory. Philosophy of biology has also undergone considerable growth, particularly due to the considerable debate in recent years over evolution. Here again, Daniel Dennett and his 1995 book *Darwin's Dangerous Idea* stand at the foreground of this debate.

Epistemology

Owing largely to Gettier's 1963 paper "Is Justified True Belief Knowledge?", epistemology saw a resurgence in analytic philosophy over the last 50 years. A large portion of current epistemological research aims to resolve the problems that Gettier's examples presented to the traditional justified true belief model of knowledge. Other areas of contemporary research include basic knowledge, the nature of evidence, the value of knowledge, epistemic luck, virtue epistemology, the role of intuitions in justification, and treating knowledge as a primitive concept.

Aesthetics

In the wake of attacks on the traditional aesthetic notions of beauty and sublimity from post-modern thinkers, analytic philosophers were slow in taking on analyses of art and aesthetic judgment. Susanne Langer[46] and Nelson Goodman[47] addressed these problems in an analytic style in the 1950s and 60s. Rigorous efforts to pursue analyses of traditional aesthetic concepts were undertaken by Guy Sircello in the 1970s and 80s, resulting in new analytic theories of love,[48] sublimity,[49] and beauty.[50]

Notes

[1] "Without exception, the best philosophy departments in the United States are dominated by analytic philosophy, and among the leading philosophers in the United States, all but a tiny handful would be classified as analytic philosophers. Practitioners of types of philosophizing that are not in the analytic tradition—such as phenomenology, classical pragmatism, existentialism, or Marxism—feel it necessary to define their position in relation to analytic philosophy." John Searle (2003) *Contemporary Philosophy in the United States* in N. Bunnin and E.P. Tsui-James (eds.), *The Blackwell Companion to Philosophy*, 2nd ed., (Blackwell, 2003), p. 1.

[2] See, e.g., Avrum Stroll, *Twentieth-Century Analytic Philosophy* (Columbia University Press, 2000), p. 5: "[I]t is difficult to give a precise definition of 'analytic philosophy' since it is not so much a specific doctrine as a loose concatenation of approaches to problems." Also, see ibid., p. 7: "I think Sluga is right in saying 'it may be hopeless to try to determine the essence of analytic philosophy.' Nearly every proposed definition has been challenged by some scholar. [...] [W]e are dealing with a family resemblance concept."

[3] See Hans-Johann Glock, *What Is Analytic Philosophy* (Cambridge University Press, 2008), p. 205: "The answer to the title question, then, is that analytic philosophy is a tradition held together *both* by ties of mutual influence *and* by family resemblances."

[4] Brian Leiter (2006) webpage *"Analytic" and "Continental" Philosophy* (http://www.philosophicalgourmet.com/analytic.asp). Quote on the definition: "'Analytic' philosophy today names a style of doing philosophy, not a philosophical program or a set of substantive views. Analytic philosophers, crudely speaking, aim for argumentative clarity and precision; draw freely on the tools of logic; and often identify, professionally and intellectually, more closely with the sciences and mathematics, than with the humanities."

[5] H. Glock, "Was Wittgenstein an Analytic Philosopher?", *Metaphilosophy*, 35:4 (2004), pp. 419–444.

[6] Colin McGinn, *The Making of a Philosopher: My Journey through Twentieth-Century Philosophy* (HarperCollins, 2002), p. xi.: "analytical philosophy [is] too narrow a label, since [it] is not generally a matter of taking a word or concept and analyzing it (whatever exactly that might be). [...] This tradition emphasizes clarity, rigor, argument, theory, truth. It is not a tradition that aims primarily for inspiration or consolation or ideology. Nor is it particularly concerned with 'philosophy of life,' though parts of it are. This kind of philosophy is more like science than religion, more like mathematics than poetry – though it is neither science nor mathematics."

[7] See Aristotle Metaphysics (Book II 993a), Kenny (1973) p. 230.

[8] This is an attitude that begins with John Locke, who described his work as that of an "underlabourer" to the achievements of natural scientists such as Newton. During the twentieth century, the most influential advocate of the continuity of philosophy with science was Willard Van Orman Quine: see, e.g., his papers "Two Dogmas of Empiricism" and "Epistemology Naturalized".

[9] A.P. Martinich, "Introduction," in Martinich & D. Sosa (eds.), *A Companion to Analytic Philosophy* (Blackwell, 2001), p. 1: "To use a general name for the kind of analytic philosophy practiced during the first half of the twentieth century, [...] 'conceptual analysis' aims at breaking down complex concepts into their simpler components."

[10] Wittgenstein, op. cit., 4.111

[11] Scott Soames, *Philosophical Analysis in the Twentieth Century* Vol. 1 (Princeton UP, 2003), p. xv: "There is, I think, a widespread presumption within the tradition that it is often possible to make philosophical progress by intensively investigating a small, circumscribed range of philosophical issues while holding broader, systematic questions in abeyance. What distinguishes twentieth-century analytical philosophy from at least some philosophy in other traditions, or at other times, is not a categorical rejection of philosophical systems, but rather the acceptance of a wealth of smaller, more thorough and more rigorous, investigations that need not be tied to any overarching philosophical view." See also, e.g., "Philosophical Analysis" (catalogued under "Analysis, Philosophical") in *Encyclopedia of Philosophy* , Vol. 1 (Macmillan, 1967), *esp.* sections on "Bertrand Russell" at p. 97*ff*, "G.E. Moore" at p. 100*ff*, and "Logical Positivism" at p. 102*ff.*

[12] See, e.g., the works of G.E. Moore and J.L. Austin.

[13] *A History of Western Philosophy* (Simon & Schuster, 1945), p. 834.

[14] A.C. Grayling (ed.), *Philosophy 2: Further through the Subject* (Oxford University Press, 1998), p. 2: "Analytic philosophy is mainly associated with the contemporary English-speaking world, but it is by no means the only important philosophical tradition. In this volume two other immensely rich and important such traditions are introduced: Indian philosophy, and philosophical thought in Europe from the time of Hegel." L.J. Cohen, *The Dialogue of Reason: An Analysis of Analytical Philosophy* (Oxford University Press, 1986), p. 5: "So, despite a few overlaps, analytical philosophy is not difficult to distinguish broadly [...] from other modern movements, like phenomenology, say, or existentialism, or from the large amount of philosophizing that has also gone on in the present century within frameworks deriving from other influential thinkers like Aquinas, Hegel, or Marx." H.-J. Glock, *What Is Analytic Philosophy?* (Cambridge University Press, 2008), p. 86: "Most non-analytic philosophers of the twentieth century do not belong to continental philosophy."

[15] Mautner, Thomas (editor) (2005) *The Penguin Dictionary of Philosophy*, entry for 'Analytic philosophy*, pp.22–3*

[16] See for example Moore's *A Defence of Common Sense* and Russell's critique of the Doctrine of internal relations,

[17] "Analytic philosophy opposed right from its beginning English neo-Hegelianism of Bradley's sort and similar ones. It did not only criticize the latter's denial of the existence of an external world (anyway an unjust criticism), but also the bombastic, obscure style of Hegel's writings." Jonkers, Peter (2003). "Perspectives on Twentieth Century Philosophy:A Reply to Tom Rockmore" (http://www.arsdisputandi.org/publish/articles/000129/article.pdf). *Ars Disputandi* **3**. ISSN 1566-5399. .

[18] Willard, Dallas. "Husserl on a Logic that Failed". *Philosophical Review* **89** (1): 52–53.

[19] Russell, Bertrand (1905). "On Denoting" (http://www.fh-augsburg.de/~harsch/anglica/Chronology/20thC/Russell/rus_deno.html). *Mind* **14**: 473–93. .

[20] Carnap, R. (1928). The Logical Structure of the World. ?.

[21] Popper, Karl R. (2002). The Logic of Scientific Discovery. Routledge. ISBN 0-415-27844-9.

[22] Important amongst these were Ludwig Wittgenstein and Rudolf Carnap. Karl Popper might also be included, since despite his rejection of the term his method is similar to the analytic tradition.
[23] Analytic Philosophy [Internet Encyclopedia of Philosophy] (http://www.iep.utm.edu/a/analytic.htm)
[24] A.P. Martinich, ed. (2001). *A companion to analytic philosophy*. Malden, Mass.: Blackwell. pp. 1–5. ISBN 0-631-21415-1.
[25] Hales, Steven D. (2002). *Analytic philosophy : classic readings*. Belmont, CA: Wadsworth/Thomson Learning. pp. 1–10. ISBN 0-534-51277-1.
[26] Soames, Scott (2003). *The dawn of analysis* (2nd print., 1st paperb. print. ed.). Princeton, NJ: Princeton Univ. Press. pp. xiii-xvii. ISBN 0-691-11573-7.
[27] Dualism (http://plato.stanford.edu/entries/dualism) entry in the *Stanford Encyclopedia of Philosophy*
[28] Postrel and Feser, February 2000, *Reality Principles: An Interview with John R. Searle* at http://www.reason.com/news/show/27599.html
[29] Dennett, Daniel C. (2001) "Are We Explaining Consciousness Yet?" *Cognition* 79 (1–2):221-37.
[30] For summaries and some criticism of the different higher-order theories, see Van Gulick, Robert (2006) "Mirror Mirror—Is That All?" In Kriegel & Williford (eds.), *Self-Representational Approaches to Consciousness*. Cambridge, MA: MIT Press. The final draft is also available here (http://web.syr.edu/~rnvangul/mirror-mirror.final.pdf). For Van Gulick's own view, see Van Gulick, Robert. "Higher-Order Global States HOGS: An Alternative Higher-Order Model of Consciousness." In Gennaro, R.J., (ed.) *Higher-Order Theories of Consciousness: An Anthology.* Amsterdam and Philadelphia: John Benjamins.
[31] Brennan, Andrew and Yeuk-Sze Lo (2002). "Environmental Ethics" §2 (http://plato.stanford.edu/entries/ethics-environmental/#2), in *The Stanford Encyclopedia of Philosophy*.
[32] Gruen, Lori (2003). " The Moral Status of Animals (http://plato.stanford.edu/entries/moral-animal/)," in *The Stanford Encyclopedia of Philosophy*.
[33] See Hursthouse, Rosalind (2003). "Virtue Ethics" §3 (http://plato.stanford.edu/entries/ethics-virtue/#3), in *The Stanford Encyclopedia of Philosophy* and Donchin, Anne (2004). " Feminist Bioethics (http://plato.stanford.edu/entries/feminist-bioethics/)" in *The Stanford Encyclopedia of Philosophy*.
[34] (a notable exception is the series of Michael B. Forest's 1934–36 *Mind* articles involving the Christian doctrine of creation and the rise of modern science).
[35] Peterson, Michael et al. (2003). *Reason and Religious Belief*
[36] Mackie, John L. (1982). *The Miracle of Theism: Arguments For and Against the Existence of God*
[37] Adams, Robert M. (1987). *The Virtue of Faith And Other Essays in Philosophical Theology*
[38] Creegan, Charles. (1989). *Wittgenstein and Kierkegaard: Religion, Individuality and Philosophical Method*
[39] Phillips, D. Z. (1999). *Philosophy's Cool Place.* Cornell University Press. The quote is from Wittgenstein's *Culture and Value* (2e): "My ideal is a certain coolness. A temple providing a setting for the passions without meddling with them.
[40] Fideism (http://plato.stanford.edu/entries/fideism) entry in the *Stanford Encyclopedia of Philosophy*
[41] Nielsen, Kai and D.Z. Phillips. (2005). *Wittgensteinian Fideism?*
[42] S. Yablo and A. Gallois, *Does Ontology Rest on a Mistake?*, Proceedings of the Aristotelian Society, Supplementary Volumes, Vol. 72, (1998), pp. 229-261+263-283 first part (http://www.mit.edu/~yablo/om.pdf)
[43] Everett, Anthony and Thomas Hofweber (eds.) (2000), *Empty Names, Fiction and the Puzzles of Non-Existence.*
[44] Van Inwagen, Peter, and Dean Zimmerman (eds.) (1998), *Metaphysics: The Big Questions.*
[45] Ibid.
[46] Susanne Langer, *Feeling and Form: A Theory of Art* (1953)
[47] Nelson Goodman, *Languages of Art: An Approach to a Theory of Symbols*. Indianapolis: Bobbs-Merrill, 1968. 2nd ed. Indianapolis: Hackett, 1976. Based on his 1960-61 John Locke lectures.
[48] Guy Sircello, *Love and Beauty.* Princeton, NJ: Princeton University Press, 1989.
[49] Guy Sircello "How Is a Theory of the Sublime Possible?" *The Journal of Aesthetics and Art Criticism*, Vol. 51, No. 4 (Autumn, 1993), pp. 541-550
[50] Guy Sircello, *A New Theory of Beauty.* Princeton Essays on the Arts, 1. Princeton, NJ: Princeton University Press, 1975.

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- Wittgenstein, *Tractatus Logico-Philosophicus*

Further reading

- The London Philosophy Study Guide (http://www.ucl.ac.uk/philosophy/LPSG/) offers many suggestions on what to read, depending on the student's familiarity with the subject: Frege, Russell, and Wittgenstein (http://www.ucl.ac.uk/philosophy/LPSG/FRW.htm)
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External links

- Analytic philosophy (http://www.iep.utm.edu/analytic) entry in the *Internet Encyclopedia of Philosophy*
- Analytic philosophy (http://plato.stanford.edu/entries/analysis/s6) entry in the *Stanford Encyclopedia of Philosophy*
- Analytic philosophy (http://www.dmoz.org/Society/Philosophy/Analytic_Philosophy//) at the Open Directory Project

Principles of Mathematical Logic

Principles of Mathematical Logic is the 1950 American translation of the 1938 second edition of David Hilbert's and Wilhelm Ackermann's classic text *Grundzüge der theoretischen Logik*, on elementary mathematical logic. The 1928 first edition thereof is considered the first elementary text clearly grounded in the formalism now known as first-order logic (FOL). Hilbert and Ackermann also formalized FOL in a way that subsequently achieved canonical status. FOL is now a core formalism of mathematical logic, and is presupposed by contemporary treatments of Peano arithmetic and nearly all treatments of axiomatic set theory.

The 1928 edition included a clear statement of the Entscheidungsproblem (decision problem) for FOL, and also asked whether that logic was complete (i.e., whether all semantic truths of FOL were theorems derivable from the FOL axioms and rules). The first problem was answered in the negative by Alonzo Church in 1936. The second was answered affirmatively by Kurt Gödel in 1929.

The text also touched on set theory and relational algebra as ways of going beyond FOL. Contemporary notation for logic owes more to this text than it does to the notation of *Principia Mathematica*, long popular in the English speaking world.

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Modal logic

Modal logic is a type of formal logic primarily developed in the 1960s that extends classical propositional and predicate logic to include operators expressing modality. Modals—words that express modalities—qualify a statement. For example, the statement "John is happy" might be qualified by saying that John is usually happy, in which case the term "usually" is functioning as a modal. The traditional alethic modalities, or modalities of truth, include possibility ("Possibly, p", "It is possible that p"), necessity ("Necessarily, p", "It is necessary that p"), and impossibility ("It is impossible that p").[1] Other modalities that have been formalized in modal logic include temporal modalities, or modalities of time (notably, "It was the case that p", "It has always been that p", "It will be that p", "It will always be that p"),[2][3] deontic modalities (notably, "It is obligatory that p", and "It is permissible that p"), epistemic modalities, or modalities of knowledge ("It is known that p")[4] and doxastic modalities, or modalities of belief ("It is believed that p").[5]

A formal modal logic represents modalities using modal operators. For example, "It might rain today" and "It is possible that rain will fall today" both contain the notion of possibility. In a modal logic this is represented as an operator, Possibly, attached to the sentence "It will rain today".

The basic unary (1-place) modal operators are usually written $\Box$ for Necessarily and $\Diamond$ for Possibly. In a classical modal logic, each can be expressed by the other with negation:

$$\Diamond P \leftrightarrow \neg\Box\neg P;$$
$$\Box P \leftrightarrow \neg\Diamond\neg P.$$

Thus it is *possible* that it will rain today if and only if it is *not necessary* that it will *not* rain today; and it is *necessary* that it will rain today if and only if it is *not possible* that it will *not* rain today. Alternative symbols used for the modal operators are "L" for Necessarily and "M" for Possibly.[6]

Development of modal logic

Although Aristotle's logic is in large parts concerned with the theory of non-modalized categorical syllogisms, he also developed a modal syllogistic.[7] Moreover, there are passages in his work, such as the famous sea-battle argument in *De Interpretatione* § 9, that are now seen as anticipations of modal logic and its connection with potentiality and time. So are the modal systems developed by Diodorus Cronus, Philo of Megara and the Stoic Chrysippus, which each contained precursors of modal axioms T and D as well as inter-defined notions of necessity and possibility.[8] Modal logic as a self-aware subject owes much to the writings of the Scholastics, in particular William of Ockham and John Duns Scotus, who reasoned informally in a modal manner, mainly to analyze statements about essence and accident.

C. I. Lewis founded modern modal logic in his 1910 Harvard thesis and in a series of scholarly articles beginning in 1912. This work culminated in his 1932 book *Symbolic Logic* (with C. H. Langford), which introduced the five systems *S1* through *S5*.

Ruth C. Barcan (later Ruth Barcan Marcus) developed the first axiomatic systems of quantified modal logic — first and second order extensions of Lewis's "S2", "S4", and "S5".

The contemporary era in modal semantics began in 1959, when Saul Kripke (then only a 19-year-old Harvard University undergraduate) introduced the now-standard Kripke semantics for modal logics. These are commonly referred to as "possible worlds" semantics. Kripke and A. N. Prior had previously corresponded at some length. Kripke semantics is basically simple, but proofs are eased using semantic-tableaux or analytic tableaux, as explained by E. W. Beth.

A. N. Prior created modern temporal logic, closely related to modal logic, in 1957 by adding modal operators [F] and [P] meaning "eventually" and "previously". Vaughan Pratt introduced dynamic logic in 1976. In 1977, Amir Pnueli proposed using temporal logic to formalise the behaviour of continually operating concurrent programs. Flavors of temporal logic include propositional dynamic logic (PDL), propositional linear temporal logic (PLTL), linear temporal logic (LTL), computational tree logic (CTL), Hennessy–Milner logic, and *T*.

The mathematical structure of modal logic, namely Boolean algebras augmented with unary operations (often called modal algebras), began to emerge with J. C. C. McKinsey's 1941 proof that *S2* and *S4* are decidable, and reached full flower in the work of Alfred Tarski and his student Bjarni Jonsson (Jonsson and Tarski 1951–52). This work revealed that *S4* and *S5* are models of interior algebra, a proper extension of Boolean algebra originally designed to capture the properties of the interior and closure operators of topology. Texts on modal logic typically do little more than mention its connections with the study of Boolean algebras and topology. For a thorough survey of the history of formal modal logic and of the associated mathematics, see [[Robert Goldblatt [9]] (2006).]

Formalizations

Semantics

The semantics for modal logic are usually given like so:[10] First we define a *frame*, which consists of a non-empty set, *G*, whose members are generally called possible worlds, and a binary relation, *R*, that holds (or not) between the possible worlds of *G*. This binary relation is called the *accessibility relation*. For example, *w R v* means that the world *v* is accessible from world *w*. That is to say, the state of affairs known as *v* is a live possibility for *w*. This gives a pair, <*G*, *R*>.

Next, the *frame* is extended to a *model* by specifying the truth-values of all propositions at each of the worlds in *G*. We do so by defining a relation ⊨ between possible worlds and propositional letters. If there is a world *w* such that *w* ⊨ P, then P is true at *w*. A model is thus an ordered triple, <*G*, *R*, ⊨>.

Then we recursively define the truth of a formula in a model:

- $w \vDash \neg P$ if and only if $w \nvDash P$
- $w \vDash (P \wedge Q)$ if and only if $w \vDash P$ and $w \vDash Q$
- $w \vDash \Box P$ if and only if for every element v of G, if $w R v$ then $v \vDash P$
- $w \vDash \Diamond P$ if and only if for some element v of G, it holds that $w R v$ and $v \vDash P$

According to these semantics, a truth is *necessary* with respect to a possible world *w* if it is true at every world that is accessible to *w*, and *possible* if it is true at some world that is accessible to *w*. Possibility thereby depends upon the accessibility relation *R*, which allows us to express the relative nature of possibility. For example, we might say that given our laws of physics it is not possible for humans to travel faster than the speed of light, but that given other circumstances it could have been possible to do so. Using the accessibility relation we can translate this scenario as follows: At all of the worlds accessible to our own world, it is not the case that humans can travel faster than the speed of light, but at one of these accessible worlds there is *another* world accessible from *those* worlds but not accessible from our own at which humans can travel faster than the speed of light.

It should also be noted that the definition of $\Box$ makes vacuously true certain sentences, since when it speaks of "every world that is accessible to w" it takes for granted the usual mathematical interpretation of the word "every" (see vacuous truth). Hence, if a world w doesn't have any accessible worlds, any sentence beginning with $\Box$ is true.

The different systems of modal logic are distinguished by the properties of their corresponding accessibility relations. There are several systems that have been espoused (often called *frame conditions*). An accessibility relation is:

- **reflexive** iff $w\ R\ w$, for every w in G
- **symmetric** iff $w\ R\ v$ implies $v\ R\ w$, for all w and v in G
- **transitive** iff $w\ R\ v$ and $v\ R\ q$ together imply $w\ R\ q$, for all w, v, q in G.
- **serial** iff, for each w in G there is some v in G such that $w\ R\ v$.
- **euclidean** iff, for every u,v and w, $w\ R\ u$ and $w\ R\ v$ implies $u\ R\ v$ (note that it also implies: $v\ R\ u$)

The logics that stem from these frame conditions are:

- **K** := no conditions
- **D** := serial
- **T** := reflexive
- **S4** := reflexive and transitive
- **S5** := reflexive, symmetric, transitive and Euclidean

S5 models are reflexive transitive and euclidean. The accessibility relation R is an equivalence relation. The relation *R* is reflexive, symmetric and transitive. It is interesting to note how the euclidean property along with reflexivity yields symmetry and transitivity. We can prove that these frames produce the same set of valid sentences as do any frames where all worlds can see all other worlds of *W*(*i.e.*, where *R* is a "total" relation). This gives the corresponding *modal graph* which is total complete (*i.e.*, no more edges (relations) can be added).

For example, in S4:

$w \models \Diamond P$ if and only if for some element v of G, it holds that $v \models P$ and $w\ R\ v$.

However, in S5, we can just say that

$w \models \Diamond P$ if and only if for some element v of G, it holds that $v \models P$.

We can drop the accessibility clause from the latter stipulation because it is trivially true of all S5 frames that $w\ R\ v$.

All of these logical systems can also be defined axiomatically, as is shown in the next section. For example, in S5, the axioms $P \to \Box\Diamond P$, $\Box P \to \Box\Box P$, and $\Box P \to P$ (corresponding to *symmetry*, *transitivity* and *reflexivity*, respectively) hold, whereas at least one of these axioms does not hold in each of the other, weaker logics.

Axiomatic systems

The first formalizations of modal logic were axiomatic. Numerous variations with very different properties have been proposed since C. I. Lewis began working in the area in 1910. Hughes and Cresswell (1996), for example, describe 42 normal and 25 non-normal modal logics. Zeman (1973) describes some systems Hughes and Cresswell omit.

Modern treatments of modal logic begin by augmenting the propositional calculus with two unary operations, one denoting "necessity" and the other "possibility". The notation of C. I. Lewis, much employed since, denotes "necessarily *p*" by a prefixed "box" ($\Box p$) whose scope is established by parentheses. Likewise, a prefixed "diamond" ($\Diamond p$) denotes "possibly *p*". Regardless of notation, each of these operators is definable in terms of the other:

- $\Box p$ (necessarily *p*) is equivalent to $\neg\Diamond\neg p$ ("not possible that not-*p*")
- $\Diamond p$ (possibly *p*) is equivalent to $\neg\Box\neg p$ ("not necessarily not-*p*")

Hence $\Box$ and $\Diamond$ form a dual pair of operators.

In many modal logics, the necessity and possibility operators satisfy the following analogs of de Morgan's laws from Boolean algebra:

"It is **not necessary that** *X*" is logically equivalent to "It is **possible that not** *X*".

"It is **not possible that** *X*" is logically equivalent to "It is **necessary that not** *X*".

Precisely what axioms and rules must be added to the propositional calculus to create a usable system of modal logic is a matter of philosophical opinion, often driven by the theorems one wishes to prove; or, in computer science, it is a matter of what sort of computational or deductive system one wishes to model. Many modal logics, known collectively as normal modal logics, include the following rule and axiom:

- **N**, **Necessitation Rule**: If *p* is a theorem (of any system invoking **N**), then $\Box p$ is likewise a theorem.
- **K**, **Distribution Axiom**: $\Box(p \to q) \to (\Box p \to \Box q)$.

The weakest normal modal logic, named *K* in honor of Saul Kripke, is simply the propositional calculus augmented by $\Box$, the rule **N**, and the axiom **K**. *K* is weak in that it fails to determine whether a proposition can be necessary but only contingently necessary. That is, it is not a theorem of *K* that if $\Box p$ is true then $\Box\Box p$ is true, i.e., that necessary truths are "necessarily necessary". If such perplexities are deemed forced and artificial, this defect of *K* is not a great one. In any case, different answers to such questions yield different systems of modal logic.

Adding axioms to *K* gives rise to other well-known modal systems. One cannot prove in *K* that if "*p* is necessary" then *p* is true. The axiom **T** remedies this defect:

- **T**, **Reflexivity Axiom**: $\Box p \to p$ (If *p* is necessary, then *p* is the case.) **T** holds in most but not all modal logics. Zeman (1973) describes a few exceptions, such as $S1^0$.

Other well-known elementary axioms are:

- **4**: $\Box p \to \Box\Box p$
- **B**: $p \to \Box\Diamond p$
- **D**: $\Box p \to \Diamond p$
- **5**: $\Diamond p \to \Box\Diamond p$

These yield the systems (axioms in bold, systems in italics):

- *K* := **K** + **N**
- *T* := *K* + **T**
- *S4* := *T* + **4**
- *S5* := *S4* + **5**
- *D* := *K* + **D**.

K through *S5* form a nested hierarchy of systems, making up the core of normal modal logic. But specific rules or sets of rules may be appropriate for specific systems. For example, in deontic logic, $\Box p \to \Diamond p$ (If it ought to be that *p*, then it is permitted that *p*) seems appropriate, but we should probably not include that $p \to \Box\Diamond p$. In fact, to do so is to commit the naturalistic fallacy (i.e. to state that what is natural is also good, by saying that if *p* is the case, *p* ought to be permitted).

The commonly employed system *S5* simply makes all modal truths necessary. For example, if *p* is possible, then it is "necessary" that *p* is possible. Also, if *p* is necessary, then it is necessary that *p* is necessary. Other systems of modal logic have been formulated, in part because *S5* does not describe every kind of modality of interest.

Alethic logic

Modalities of necessity and possibility are called *alethic* modalities. They are also sometimes called *special* modalities, from the Latin *species*. Modal logic was first developed to deal with these concepts, and only afterward was extended to others. For this reason, or perhaps for their familiarity and simplicity, necessity and possibility are often casually treated as *the* subject matter of modal logic. Moreover it is easier to make sense of relativizing necessity, e.g. to legal, physical, nomological, epistemic, and so on, than it is to make sense of relativizing other notions.

In classical modal logic, a proposition is said to be

- **possible** if and only if it is *not necessarily false* (regardless of whether it is actually true or actually false);
- **necessary** if and only if it is *not possibly false*; and
- **contingent** if and only if it is *not necessarily false* and *not necessarily true* (i.e. possible but not necessarily true).

In classical modal logic, therefore, either the notion of possibility or necessity may be taken to be basic, where these other notions are defined in terms of it in the manner of De Morgan duality. Intuitionistic modal logic treats possibility and necessity as not perfectly symmetric.

For those with difficulty with the concept of something being possible but not true, the meaning of these terms may be made more comprehensible by thinking of multiple "possible worlds" (in the sense of Leibniz) or "alternate universes"; something "necessary" is true in all possible worlds, something "possible" is true in at least one possible world. These "possible world semantics" are formalized with Kripke semantics.

Physical possibility

Something is physically, or nomically, possible if it is permitted by the laws of physics. For example, current theory is thought to allow for there to be an atom with an atomic number of 126,[11] even if there are no such atoms in existence. In contrast, while it is logically possible to accelerate beyond the speed of light,[12] modern science stipulates that it is not physically possible for material particles or information.[13]

Metaphysical possibility

Philosophers ponder the properties that objects have independently of those dictated by scientific laws. For example, it might be metaphysically necessary, as some have thought, that all thinking beings have bodies and can experience the passage of time. Saul Kripke has argued that every person necessarily has the parents they do have: anyone with different parents would not be the same person.[14]

Metaphysical possibility is generally thought to be more restricting than bare logical possibility (i.e., fewer things are metaphysically possible than are logically possible). Its exact relation to physical possibility is a matter of some dispute. Philosophers also disagree over whether metaphysical truths are necessary merely "by definition", or whether they reflect some underlying deep facts about the world, or something else entirely.

Confusion with epistemic modalities

Alethic modalities and epistemic modalities (see below) are often expressed in English using the same words. "It is possible that bigfoot exists" can mean either "Bigfoot *could* exist, whether or not bigfoot does in fact exist" (alethic), or more likely, "For all I know, bigfoot exists" (epistemic).

It has been questioned whether these modalities should be considered distinct from each other. The criticism states that there is no real difference between "the truth in the world" (alethic) and "the truth in an individual's mind" (epistemic).[15] An investigation has not found a single language in which alethic and epistemic modalities are formally distinguished, as by the means of a grammatical mood.[16]

Epistemic logic

Epistemic modalities (from the Greek *episteme*, knowledge), deal with the *certainty* of sentences. The $\Box$ operator is translated as "x knows that…", and the $\Diamond$ operator is translated as "For all x knows, it may be true that…" In ordinary speech both metaphysical and epistemic modalities are often expressed in similar words; the following contrasts may help:

A person, Jones, might reasonably say *both*: (1) "No, it is *not* possible that Bigfoot exists; I am quite certain of that"; *and*, (2) "Sure, Bigfoot possibly *could* exist". What Jones means by (1) is that given all the available information, there is no question remaining as to whether Bigfoot exists. This is an epistemic claim. By (2) he makes the *metaphysical* claim that it is *possible for* Bigfoot to exist, *even though he does not* (which is not equivalent to "it is *possible that* Bigfoot exists – for all I know", which contradicts (1)).

From the other direction, Jones might say, (3) "It is *possible* that Goldbach's conjecture is true; but also *possible* that it is false", and *also* (4) "if it *is* true, then it is necessarily true, and not possibly false". Here Jones means that it is *epistemically possible* that it is true or false, for all he knows (Goldbach's conjecture has not been proven either true or false), but if there *is* a proof (heretofore undiscovered), then it would show that it is not *logically* possible for Goldbach's conjecture to be false—there could be no set of numbers that violated it. Logical possibility is a form of *alethic* possibility; (4) makes a claim about whether it is possible (i.e., logically speaking) that a mathematical truth to have been false, but (3) only makes a claim about whether it is possible, for all Jones knows, (i.e., speaking of certitude) that the mathematical claim is specifically either true or false, and so again Jones does not contradict himself. It is worthwhile to observe that Jones is not necessarily correct: It is possible (epistemically) that Goldbach's conjecture is both true and unprovable.[17]

Epistemic possibilities also bear on the actual world in a way that metaphysical possibilities do not. Metaphysical possibilities bear on ways the world *might have been,* but epistemic possibilities bear on the way the world *may be* (for all we know). Suppose, for example, that I want to know whether or not to take an umbrella before I leave. If you tell me "it is *possible that* it is raining outside" – in the sense of epistemic possibility – then that would weigh on whether or not I take the umbrella. But if you just tell me that "it is *possible for* it to rain outside" – in the sense of *metaphysical possibility* – then I am no better off for this bit of modal enlightenment.

Some features of epistemic modal logic are in debate. For example, if *x* knows that *p*, does *x* know that it knows that *p*? That is to say, should $\Box P \to \Box\Box P$ be an axiom in these systems? While the answer to this question is unclear, there is at least one axiom that is generally included in epistemic modal logic, because it is minimally true of all normal modal logics (see the section on axiomatic systems):

- **K**, *Distribution Axiom*: $\Box(p \to q) \to (\Box p \to \Box q)$.

But this is disconcerting, because with **K**, we can prove that we know all the logical consequences of our beliefs: If *q* is a logical consequence of *p*, then $\Box(p \to q)$. And if so, then we can deduce that $(\Box p \to \Box q)$ using **K**. When we translate this into epistemic terms, this says that if *q* is a logical consequence of *p*, then we know that it is, and if we know *p*, we know *q*. That is to say, we know all the logical consequences of our beliefs. This must be true for all possible Kripkean modal interpretations of epistemic cases where $\Box$ is translated as "knows that". But then, for example, if *x* knows that prime numbers are divisible only by themselves and the number one, then *x* knows that 8683317618811886495518194401279999999 is prime (since this number is only divisible by itself and the number one). That is to say, under the modal interpretation of knowledge, anyone who knows the definition of a prime number knows that this number is prime. This shows that epistemic modal logics that are based on normal modal systems provide an idealized account of knowledge, and explain objective, rather than subjective knowledge (if anything).

Temporal logic

Temporal logic is an approach to the semantics of expressions with tense, that is, expressions with qualifications of when. Some expressions, such as '2 + 2 = 4', are true at all times, while tensed expressions such as 'John is happy' are only true sometimes.

In temporal logic, tense constructions are treated in terms of modalities, where a standard method for formalizing talk of time is to use *two* pairs of operators, one for the past and one for the future (P will just mean 'it is presently the case that P'). For example:

F*P* : It will sometimes be the case that *P*

G*P* : It will always be the case that *P*

P*P* : It was sometime the case that *P*

H*P* : It has always been the case that *P*

There are then at least three modal logics that we can develop. For example, we can stipulate that,

$\Diamond P$ = *P* is the case at some time *t*

$\Box P$ = *P* is the case at every time *t*

Or we can trade these operators to deal only with the future (or past). For example,

$\Diamond_1 P$ = **F***P*

$\Box_1 P$ = **G***P*

or,

$\Diamond_2 P$ = *P* and/or **F***P*

$\Box_2 P$ = *P* and **G***P*

The operators **F** and **G** may seem initially foreign, but they create normal modal systems. Note that **F***P* is the same as ¬**G**¬*P*. We can combine the above operators to form complex statements. For example, **P***P* → $\Box$**P***P* says (effectively), *Everything that is past and true is necessary*.

It seems reasonable to say that possibly it will rain tomorrow, and possibly it won't; on the other hand, since we can't change the past, if it is true that it rained yesterday, it probably isn't true that it may not have rained yesterday. It seems the past is "fixed", or necessary, in a way the future is not. This is sometimes referred to as accidental necessity. But if the past is "fixed", and everything that is in the future will eventually be in the past, then it seems plausible to say that future events are necessary too.

Similarly, the problem of future contingents considers the semantics of assertions about the future: is either of the propositions 'There will be a sea battle tomorrow', or 'There will not be a sea battle tomorrow' now true? Considering this thesis led Aristotle to reject the principle of bivalence for assertions concerning the future.

Additional binary operators are also relevant to temporal logics, *q.v.* Linear Temporal Logic.

Versions of temporal logic can be used in computer science to model computer operations and prove theorems about them. In one version, $\Diamond P$ means "at a future time in the computation it is possible that the computer state will be such that P is true"; $\Box P$ means "at all future times in the computation P will be true". In another version, $\Diamond P$ means "at the immediate next state of the computation, P might be true"; $\Box P$ means "at the immediate next state of the computation, P will be true". These differ in the choice of Accessibility relation. (P always means "P is true at the current computer state".) These two examples involve nondeterministic or not-fully-understood computations; there are many other modal logics specialized to different types of program analysis. Each one naturally leads to slightly different axioms.

A variation, closely related to Temporal or Chronological or Tense logics, are Modal logics based upon "topology", "place", or "spatial position".[18][19] One might also take note that in the Russian language, verbs have an aspect, based commonly upon time, but position also.

Deontic logic

Likewise talk of morality, or of obligation and norms generally, seems to have a modal structure. The difference between "You must do this" and "You may do this" looks a lot like the difference between "This is necessary" and "This is possible". Such logics are called *deontic*, from the Greek for "duty".

Deontic logics commonly lack the axiom **T** semantically corresponding to the reflexivity of the accessibility relation in Kripke semantics: in symbols, $\Box\phi \to \phi$. Interpreting $\Box$ as "it is obligatory that", **T** informally says that every obligation is true. For example, if it is obligatory not to kill others (i.e. killing is morally forbidden), then **T** implies that people actually do not kill others. The consequent is obviously false.

Instead, using Kripke semantics, we say that though our own world does not realize all obligations, the worlds accessible to it do (i.e., **T** holds at these worlds). These worlds are called idealized worlds. *P* is obligatory with respect to our own world if at all idealized worlds accessible to our world, *P* holds. Though this was one of the first interpretations of the formal semantics, it has recently come under criticism.[20]

One other principle that is often (at least traditionally) accepted as a deontic principle is *D*, $\Box\phi \to \Diamond\phi$, which corresponds to the seriality (or extendability or unboundedness) of the accessibility relation. It is an embodiment of the Kantian idea that "ought implies can". (Clearly the "can" can be interpreted in various senses, e.g. in a moral or alethic sense.)

Intuitive problems with deontic logic

When we try and formalize ethics with standard modal logic, we run into some problems. Suppose that we have a proposition *K*: you have stolen some money, and another, *Q*: you have stolen a small amount of money. Now suppose we want to express the thought that "if you have stolen some money, it ought to be a small amount of money". There are two likely candidates,

(1) $(K \to \Box Q)$

(2) $\Box(K \to Q)$

But (1) and *K* together entail $\Box Q$, which says that it ought to be the case that you have stolen a small amount of money. This surely isn't right, because you ought not to have stolen anything at all. And (2) doesn't work either: If the right representation of "if you have stolen some money it ought to be a small amount" is (2), then the right representation of (3) "if you have stolen some money then it ought to be a large amount" is $\Box(K \to (K \land \neg Q))$. Now suppose (as seems reasonable) that you ought not to steal anthing, or $\Box\neg K$. But then we can deduce $\Box(K \to (K \land \neg Q))$ via $\Box(\neg K) \to \Box(K \to K \land \neg K)$ and $\Box(K \land \neg K \to (K \land \neg Q))$ (the contrapositive of $Q \to K$); so sentence (3) follows from our hypothesis (of course the same logic shows sentence (2)). But that can't be right, and is not right when we use natural language. Telling someone they should not steal certainly does not imply that they should steal large amounts of money if they do engage in theft.[21]

Doxastic logic

Doxastic logic concerns the logic of belief (of some set of agents). The term doxastic is derived from the ancient Greek *doxa* which means "belief". Typically, a doxastic logic uses $\Box$, often written "B", to mean "It is believed that", or when relativized to a particular agent s, "It is believed by s that".

Other modal logics

Significantly, modal logics can be developed to accommodate most of these idioms; it is the fact of their common logical structure (the use of "intensional" sentential operators) that make them all varieties of the same thing.

The ontology of possibility

In the most common interpretation of modal logic, one considers "logically possible worlds". If a statement is true in all possible worlds, then it is a necessary truth. If a statement happens to be true in our world, but is not true in all possible worlds, then it is a contingent truth. A statement that is true in some possible world (not necessarily our own) is called a possible truth.

Under this "possible worlds idiom," to maintain that Bigfoot's existence is possible but not actual, one says, "There is some possible world in which Bigfoot exists; but in the actual world, Bigfoot does not exist". However, it is unclear what this claim commits us to. Are we really alleging the existence of possible worlds, every bit as real as our actual world, just not actual? Saul Kripke believes that 'possible world' is something of a misnomer – that the term 'possible world' is just a useful way of visualizing the concept of possibility.[22] For him, the sentences "you could have rolled a 4 instead of a 6" and "there is a possible world where you rolled a 4, but you rolled a 6 in the actual world" are not significantly different statements, and neither commit us to the existence of a possible world.[23] David Lewis, on the other hand, made himself notorious by biting the bullet, asserting that all merely possible worlds are as real as our own, and that what distinguishes our world as *actual* is simply that it is indeed our world – *this* world.[24] That position is a major tenet of "modal realism". Some philosophers decline to endorse any version of modal realism, considering it ontologically extravagant, and prefer to seek various ways to paraphrase away these ontological commitments. Robert Adams holds that 'possible worlds' are better thought of as 'world-stories', or consistent sets of propositions. Thus, it is possible that you rolled a 4 if such a state of affairs can be described coherently.[25]

Computer scientists will generally pick a highly specific interpretation of the modal operators specialized to the particular sort of computation being analysed. In place of "all worlds", you may have "all possible next states of the computer", or "all possible future states of the computer".

Applications

- Modality has also been treated from the viewpoint of "counter-factuals" in literature (see Victorian Studies).[26][27][28]
- Modality modifies propositions and modalities provide closure (i.e.: propositions with modalities are still propositions). Thus, as propositions constitute a part of language, they may be understood as subject to linguistic analysis such as that of Noam Chomsky. Modalities might then be viewed as being context-free, context-sensitive, or even fully phrase-structured (Chomsky type-0) languages. This broadens the view of modalities which are usually viewed as context-free. A discussion of this may be found under the Philosophy of language.
- Aristotle classified and discussed rhetoric as being based upon the enthymeme, thus closely related to logic. However, it is clear that if logic is extended by modal logics, multi-valued logics, etc., then rhetoric must also be extended by modern developments.

Further applications

Modal logics have begun to be used in areas of the humanities such as literature, poetry, art and history.[29][30][31]

Controversies

Modal logic has been rejected by many philosophers. (Historically, philosophers starting with Aristotle seem to have had priority of interest in modal logic over mathematicians.)

Nicholas Rescher has argued that Bertrand Russell rejected Model Logic, and that this rejection led to the theory of modal logic languishing for decades.[32] However, Jan Dejnozka has argued against this view, stating that a modal system which Dejnozka calls *MDL* is described in Russell's works, although Russell did believe the concept of modality to "come from confusing propositions with propositional functions," as he wrote in *The Analysis of Matter*.[33]

Arthur Norman Prior warned his protégé Ruth Barcan to prepare well in the debates concerning Quantified Modal Logic with Willard Van Orman Quine, due to the biases against Modal Logic.[34]

Notes

[1] "Formal Logic", by A. N. Prior, Oxford Univ. Press, 1962, p. 185

[2] "Temporal Logic", by Rescher and Urquhart, Springer-Verlag, 1971, p. 52

[3] "Past, Present and Future", by A. N. Prior, Oxford Univ. Press, 1967

[4] "Knowledge and Belief", by Jaakko Hinntikka, Cornell Univ. Press, 1962

[5] "Topics in Philosophical Logic", by N. Rescher, Humanities Press, 1968, p. 41

[6] So in the standard work *A New Introduction to Modal Logic*, by G. E. Hughes and M. J. Cresswell, Routledge, 1996, *passim*.

[7] (http://plato.stanford.edu/entries/logic-ancient/#ModLog) Stanford Encyclopedia of Philosophy: Susanne Bobzien, *Ancient Logic*

[8] (http://plato.stanford.edu/entries/logic-ancient/#DioCroPhiLog) Stanford Encyclopedia of Philosophy: Susanne Bobzien, *Ancient Logic*

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External links

- Stanford Encyclopedia of Philosophy:
  - " Modal logic (http://plato.stanford.edu/entries/logic-modal)" – by James Garson.
  - " Provability Logic (http://plato.stanford.edu/entries/logic-provability/)" – by Rineke Verbrugge.
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- List of Logic Systems (http://www.cc.utah.edu/~nahaj/logic/structures/systems/index.html) List of many modal logics with sources, by John Halleck.
- Advances in Modal Logic. (http://aiml.net/) Biannual international conference and book series in modal logic.
- S4prover (http://teachinglogic.imag.fr/TableauxS4) A tableaux prover for S4 logic

Alethic modality

Alethic modality is a linguistic modality which indicates logical necessity, possibility or impossibility.[1]

Alethic modality is often associated with epistemic modality in research. However, it has been questioned whether this modality should be considered distinct from epistemic modality which denotes the speaker's evaluation or judgment of the truth. The criticism states that there is no real difference between "the truth in the world" (alethic) and "the truth in an individual's mind" (epistemic).[2] An investigation has not found a single language in which alethic and epistemic modalities are formally distinguished, as by the means of a grammatical mood.[3] In such a language, "A circle can't be square", "can't be" would be expressed by an alethic mood, whereas for "He can't be that wealthy", "can't be" would be expressed by an epistemic mood. As we can see, this is not a distinction drawn in English grammar.

"You can't give these plants too much water." is a well-known play on the distinction between this so-called alethic modality and (perhaps hortatory or injunctive) modality. The dilemma is fairly easily resolved when listening through paralinguistic cues and particularly suprasegmental cues (intonation). So while there may not be a morphologically based alethic mood, this does not seem to preclude the usefulness of distinguishing between these two types of modes. Alethic modality might then concern what are considered to be apodictic statements.

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Necessity

In U.S. criminal law, **necessity** may be either a possible justification or an exculpation for breaking the law. Defendants seeking to rely on this defense argue that they should not be held liable for their actions as a crime because their conduct was *necessary* to prevent some greater harm and when that conduct is not excused under some other more specific provision of law such as self defense. Except for a few statutory exemptions and in some medical cases [1] there is no corresponding defense in English law.[2]

For example, a drunk driver might contend that he drove his car to get away from a kidnap (cf. *North by Northwest*). Most common law and civil law jurisdictions recognize this defense, but only under limited circumstances. Generally, the defendant must affirmatively show (i.e., introduce some evidence) that (a) the harm he sought to avoid outweighs the danger of the prohibited conduct he is charged with; (b) he had no reasonable alternative; (c) he ceased to engage in the prohibited conduct as soon as the danger passed; and (d) he did not himself create the danger he sought to avoid. Thus, with the "drunk driver" example cited above, the necessity defense will not be recognized if the defendant drove further than was reasonably necessary to get away from the kidnapper, or if some other reasonable alternative was available to him. However case law suggests necessity is narrowed to medical cases.

The political necessity defense saw its demise in the case of *United States v. Schoon*.[3] In that case, thirty people, including appellants, gained admittance to the IRS office in Tucson, where they chanted "keep America's tax dollars out of El Salvador," splashed simulated blood on the counters, walls, and carpeting, and generally obstructed the office's operation. The court ruled that the elements of necessity did not exist in this case.[4]

General discussion

As a matter of political expediency, states usually allow some classes of person to be excused from liability when they are engaged in socially useful functions but intentionally cause injury, loss or damage. For example, the fire services and other civil defence organizations have a general duty to keep the community safe from harm. If a fire or flood is threatening to spread out of control, it may be reasonably necessary to destroy other property to form a fire break, or to trespass on land to throw up mounds of earth to prevent the water from spreading. These examples have the common feature of individuals intentionally breaking the law because they believe it to be urgently necessary to protect others from harm, but some states distinguish between a response to a crisis arising from an entirely natural cause (an inanimate force of nature), e.g. a fire from a lightning strike or rain from a storm, and a response to an entirely human crisis. Thus, parents who lack the financial means to feed their children cannot use necessity as a defense if they steal food. The existence of welfare benefits and strategies other than self-help defeat the claim of an urgent necessity that cannot be avoided in any way other than by breaking the law. Further, some states apply a test of proportionality. So the defense would only be allowed where the degree of harm actually caused was a reasonably proportionate response to the degree of harm threatened. This is a legal form of cost–benefit analysis.

Specific jurisdictions

- England
- Canada
- International

International Law

Customary International Law

Under International law, a customary international obligation or an obligation granted under a Bilateral Investment Treaty may be suspended under the Doctrine of Necessity. It is "an exception from illegality and in certain cases even as an exception from responsibility." *See Continental Casualty Company v Argentine Republic, ICSID Case No ARB/03/09.* In order to invoke the Doctrine of Necessity: (1) Invoking State must not have contributed to the state of necessity, (2) Actions taken were only way to safeguard an essential interest from grave and impending danger. *Id. at page 72, paragraph 165.*

United States

In specific states

- Alabama
- Alaska
- Arizona
- Arkansas
- California
- Colorado
- Connecticut
- Delaware
- Florida
- Georgia
- Hawaii
- Idaho
- Illinois
- Indiana
- Iowa
- Kansas
- Kentucky
- Louisiana
- Maine
- Maryland
- Massachusetts
- Michigan
- Minnesota
- Mississippi
- Missouri
- Montana
- Nebraska
- Nevada
- New Hampshire
- New Jersey
- New Mexico
- New York
- North Carolina
- North Dakota
- Ohio
- Oklahoma
- Oregon
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- Rhode Island
- South Carolina
- South Dakota
- Tennessee
- Texas
- Utah
- Vermont
- Virginia
- Washington
- West Virginia
- Wisconsin
- Wyoming

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Logical possibility

A **logically possible** proposition is one that can be asserted without implying a logical contradiction. This is to say that a proposition is logically possible if there is some coherent way for the world to be, under which the proposition would be true. Thus, "the sky is blue" (and all other actually true propositions) is logically possible: there exists some logically coherent way for the world to be such that it is true, viz., the way that the world actually is. But this "way for the world to be" need not be the way the world actually is; it need only be logically coherent. So, for example, the false proposition the sky is green is also logically possible, so long as we are able (as we indeed seem to be) to conceive of some logically coherent world in which the sky is green. Philosophers generally consider logical possibility to be the broadest sort of subjunctive possibility in modal logic.

Logical possibility should be distinguished from other sorts of subjunctive possibilities. For example, it may be logically possible for the universe's physical laws to be different from what they actually are. If it is, then many things that we would normally consider to be demonstrably impossible can be logically possible: for example, that travel might be possible at speeds faster-than-light or that escape from black holes is not impossible. Many philosophers, then, have held that these scenarios are logically possible but nomologically impossible (impossible under the actual laws of nature).

These propositions are also to be contrasted with logically **im**possible propositions, i.e., propositions which could not possibly be true under any circumstances in any universe because they are formal contradictions. While it is logically possible for the sky to be green, it is not logically possible for a square to be circular in shape.[1] Some combinations of physical laws are also known to result in contradictions. For instance, *if* a given universe's physical laws are invariant through time, *then* the law of conservation of energy holds in that universe. This is a consequence of Noether's theorem, which can be proven mathematically. Thus, a universe whose physical laws do not vary with time and which does not exhibit conservation of energy is not logically possible.

With this understanding of logical possibility in mind, other logical modalities may be defined in terms of it: a proposition is logically necessary if it is not logically possible for it to be false, logically impossible if it is not logically possible for it to be true, and logically contingent if it is logically possible for it to be true and also logically possible for it to be false.

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External links

- Do Modal Claims Imply the Existence of Possible Worlds? (http://home.sprynet.com/~owl1/lewis.htm), paper criticizing David Lewis' theory of possibility.

De Morgan's laws

In propositional logic and boolean algebra, **De Morgan's laws**[1][2][3] are a pair of transformation rules that are both valid rules of inference. The rules allow the expression of conjunctions and disjunctions purely in terms of each other via negation.

The rules can be expressed in English as:

The negation of a conjunction is the disjunction of the negations.
The negation of a disjunction is the conjunction of the negations.

The rules can be expressed in formal language with two propositions *P* and *Q* as:

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$$
$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$$

where:

- ¬ is the negation operator (NOT)
- $\wedge$ is the conjunction operator (AND)
- $\vee$ is the disjunction operator (OR)
- ⇔ is a metalogical symbol meaning "can be replaced in a logical proof with"

Applications of the rules include simplification of logical expressions in computer programs and digital circuit designs. De Morgan's laws are an example of a more general concept of mathematical duality.

Formal notation

The *negation of conjunction* rule may be written in sequent notation:

$$\neg(P \wedge Q) \vdash (\neg P \vee \neg Q)$$

The *negation of disjunction* rule may be written as:

$$\neg(P \vee Q) \vdash (\neg P \wedge \neg Q)$$

In rule form: *negation of conjunction*

$$\frac{\neg(P \wedge Q)}{\therefore \neg P \vee \neg Q}$$

and *negation of disjunction*

$$\frac{\neg(P \vee Q)}{\therefore \neg P \wedge \neg Q}$$

and expressed as a truth-functional tautology or theorem of propositional logic:

$$\neg(P \wedge Q) \to (\neg P \vee \neg Q)$$
$$\neg(P \vee Q) \to (\neg P \wedge \neg Q)$$

where P, and Q are propositions expressed in some formal system.

Substitution form

De Morgan's laws are normally shown in the compact form above, with negation of the output on the left and negation of the inputs on the right. A clearer form for substitution can be stated as:

$$(P \wedge Q) \equiv \neg(\neg P \vee \neg Q)$$
$$(P \vee Q) \equiv \neg(\neg P \wedge \neg Q)$$

This emphasizes the need to invert both the inputs and the output, as well as change the operator, when doing a substitution.

Set theory and Boolean algebra

In set theory and Boolean algebra, it is often stated as "Union and intersection interchange under complementation",[4] which can be formally expressed as:

- $\overline{A \cup B} \equiv \overline{A} \cap \overline{B}$
- $\overline{A \cap B} \equiv \overline{A} \cup \overline{B}$

where:

- A is the negation of A, the overline being written above the terms to be negated
- $\cap$ is the intersection operator (AND)
- $\cup$ is the union operator (OR)

The generalized form is:

$$\overline{\bigcap_{i \in I} A_i} \equiv \bigcup_{i \in I} \overline{A_i}$$
$$\overline{\bigcup_{i \in I} A_i} \equiv \bigcap_{i \in I} \overline{A_i}$$

where I is some, possibly uncountable, indexing set.

In set notation, De Morgan's law can be remembered using the mnemonic "break the line, change the sign".[5]

Engineering

In electrical and computer engineering, De Morgan's law is commonly written as:

$$\overline{A \cdot B} \equiv \overline{A} + \overline{B}$$
$$\overline{A + B} \equiv \overline{A} \cdot \overline{B}$$

where:

- $\cdot$ is a logical AND
- $+$ is a logical OR
- the overbar is the logical NOT of what is underneath the overbar.

History

The law is named after Augustus De Morgan (1806–1871)[6] who introduced a formal version of the laws to classical propositional logic. De Morgan's formulation was influenced by algebraization of logic undertaken by George Boole, which later cemented De Morgan's claim to the find. Although a similar observation was made by Aristotle and was known to Greek and Medieval logicians[7] (in the 14th century, William of Ockham wrote down the words that would result by reading the laws out),[8] De Morgan is given credit for stating the laws formally and incorporating them in to the language of logic. De Morgan's Laws can be proved easily, and may even seem trivial.[9] Nonetheless, these laws are helpful in making valid inferences in proofs and deductive arguments.

Informal proof

De Morgan's theorem may be applied to the negation of a disjunction or the negation of a conjunction in all or part of a formula.

Negation of a disjunction

In the case of its application to a disjunction, consider the following claim: "it is false that either of A or B is true", which is written as:

$$\neg(A \lor B)$$

In that it has been established that *neither* A nor B is true, then it must follow that both A is not true and B is not true, which may be written directly as:

$$(\neg A) \wedge (\neg B)$$

If either A or B *were* true, then the disjunction of A and B would be true, making its negation false. Presented in English, this follows the logic that "Since two things are both false, it is also false that either of them is true."

Working in the opposite direction, the second expression asserts that A is false and B is false (or equivalently that "not A" and "not B" are true). Knowing this, a disjunction of A and B must be false also. The negation of said disjunction must thus be true, and the result is identical to the first claim.

Negation of a conjunction

The application of De Morgan's theorem to a conjunction is very similar to its application to a disjunction both in form and rationale. Consider the following claim: "it is false that A and B are both true", which is written as:

$$\neg(A \wedge B)$$

In order for this claim to be true, either or both of A or B must be false, for if they both were true, then the conjunction of A and B would be true, making its negation false. Thus, one (at least) or more of A and B must be false (or equivalently, one or more of "not A" and "not B" must be true). This may be written directly as:

$$(\neg A) \lor (\neg B)$$

Presented in English, this follows the logic that "Since it is false that two things are both true, at least one of them must be false."

Working in the opposite direction again, the second expression asserts that at least one of "not A" and "not B" must be true, or equivalently that at least one of A and B must be false. Since at least one of them must be false, then their conjunction would likewise be false. Negating said conjunction thus results in a true expression, and this expression is identical to the first claim.

Formal proof

The laws may be proven directly using truth tables; "1" represents true, "0" represents false.

First we prove: $\neg(p \lor q) \Leftrightarrow (\neg p) \wedge (\neg q)$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Since the values in the 4th and last columns are the same for all rows (which cover all possible truth value assignments to the variables), we can conclude that the two expressions are logically equivalent.

Now we prove $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$ by the same method:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

Extensions

In extensions of classical propositional logic, the duality still holds (that is, to any logical operator we can always find its dual), since in the presence of the identities governing negation, one may always introduce an operator that is the De Morgan dual of another. This leads to an important property of logics based on classical logic, namely the existence of negation normal forms: any formula is equivalent to another formula where negations only occur applied to the non-logical atoms of the formula. The existence of negation normal forms drives many applications, for example in digital circuit design, where it is used to manipulate the types of logic gates, and in formal logic, where it is a prerequisite for finding the conjunctive normal form and disjunctive normal form of a formula. Computer programmers use them to simplify or properly negate complicated logical conditions. They are also often useful in computations in elementary probability theory.

Let us define the dual of any propositional operator P(p, q, ...) depending on elementary propositions p, q, ... to be the operator P^d defined by

$$\mathrm{P}^d(p, q, ...) = \neg P(\neg p, \neg q, \dots).$$

This idea can be generalised to quantifiers, so for example the universal quantifier and existential quantifier are duals:

$$\forall x\, P(x) \equiv \neg \exists x\, \neg P(x),$$
$$\exists x\, P(x) \equiv \neg \forall x\, \neg P(x).$$

To relate these quantifier dualities to the De Morgan laws, set up a model with some small number of elements in its domain D, such as

$$D = \{a, b, c\}.$$

Then

$$\forall x\, P(x) \equiv P(a) \wedge P(b) \wedge P(c)$$

and

$$\exists x\, P(x) \equiv P(a) \vee P(b) \vee P(c).$$

But, using De Morgan's laws,

$$P(a) \wedge P(b) \wedge P(c) \equiv \neg(\neg P(a) \vee \neg P(b) \vee \neg P(c))$$

and

$$P(a) \vee P(b) \vee P(c) \equiv \neg(\neg P(a) \wedge \neg P(b) \wedge \neg P(c)),$$

verifying the quantifier dualities in the model.

Then, the quantifier dualities can be extended further to modal logic, relating the box ("necessarily") and diamond ("possibly") operators:

$$\Box p \equiv \neg \Diamond \neg p,$$
$$\Diamond p \equiv \neg \Box \neg p.$$

In its application to the alethic modalities of possibility and necessity, Aristotle observed this case, and in the case of normal modal logic, the relationship of these modal operators to the quantification can be understood by setting up models using Kripke semantics.

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[3] Moore and Parker
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[5] 2000 Solved Problems in Digital Electronics (http://books.google.com/books?id=NdAjEDP5mDsC&pg=PA81&lpg=PA81&dq=break+the+line+change+the+sign&source=web&ots=BtUl4oQOja&sig=H1Wz9e6Uv_bNeSbTvN6lr3s47PQ#PPA81,M1) By S. P. Bali
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[7] Bocheński's *History of Formal Logic*
[8] William of Ockham, Summa Logicae, part II, sections 32 & 33.
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- de Morgan's laws (http://planetmath.org/encyclopedia/DeMorgansLaws.html) at PlanetMath

Temporal logic

In logic, the term **temporal logic** is used to describe any system of rules and symbolism for representing, and reasoning about, propositions qualified in terms of time. In a temporal logic we can then express statements like "I am *always* hungry", "I will *eventually* be hungry", or "I will be hungry *until* I eat something". Temporal logic is sometimes also used to refer to **tense logic**, a particular modal logic-based system of temporal logic introduced by Arthur Prior in the late 1950s, and important results obtained were by Hans Kamp. Subsequently it has been developed further by computer scientists, notably Amir Pnueli, and logicians.

Temporal logic has found an important application in formal verification, where it is used to state requirements of hardware or software systems. For instance, one may wish to say that *whenever* a request is made, access to a resource is *eventually* granted, but it is *never* granted to two requestors simultaneously. Such a statement can conveniently be expressed in a temporal logic.

Motivation

Consider the statement: "I am hungry." Though its meaning is constant in time, the truth value of the statement can vary in time. Sometimes the statement is true, and sometimes the statement is false, but the statement is never true and false simultaneously. In a temporal logic, statements can have a truth value which can vary in time. Contrast this with an atemporal logic, which can only discuss statements whose truth value is constant in time. This treatment of truth values over time differentiates temporal logic from computational verb logic.

Temporal logic always has the ability to reason about a time line. So-called linear time logics are restricted to this type of reasoning. Branching logics, however, can reason about multiple time lines. This presupposes an environment that may act unpredictably. To continue the example, in a branching logic we may state that "there is a possibility that I will stay hungry forever." We may also state that "there is a possibility that eventually I am no longer hungry." If we do not know whether or not I will ever get fed, these statements are both true some times.

History

Although Aristotle's logic is almost entirely concerned with the theory of the categorical syllogism, there are passages in his work that are now seen as anticipations of temporal logic, and may imply an early, partially developed form of first-order temporal modal binary logic. Aristotle was particularly concerned with the problem of future contingents, where he could not accept that the principle of bivalence applies to statements about future events, i.e. that we can presently decide if a statement about a future event is true or false, such as "there will be a sea battle tomorrow".[1]

There was little development for millennia, Charles Sanders Peirce noted in the 19th century:[2]

> “Time has usually been considered by logicians to be what is called 'extralogical' matter. I have never shared this opinion. But I have thought that logic had not yet reached the state of development at which the introduction of temporal modifications of its forms would not result in great confusion; and I am much of that way of thinking yet.”

Arthur Prior was concerned with the philosophical matters of free will and predestination. According to his wife, he first considered formalizing temporal logic in 1953. He gave lectures on the topic at the University of Oxford in 1955-6, and in 1957 published a book, *Time and Modality*, in which he introduced a propositional modal logic with two temporal connectives (modal operators), F and P, corresponding to "sometime in the future" and "sometime in the past". In this early work, Prior considered time to be linear. In 1958 however, he received a letter from Saul Kripke, who pointed out that this assumption is perhaps unwarranted. In a development that foreshadowed a similar one in computer science, Prior took this under advisement, and developed two theories of branching time, which he called "Ockhamist" and "Peircean".[2] Between 1958 and 1965 Prior also corresponded with Charles Leonard

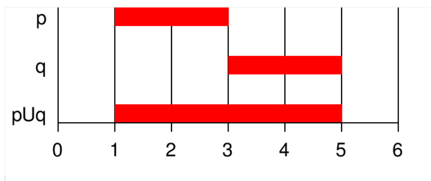
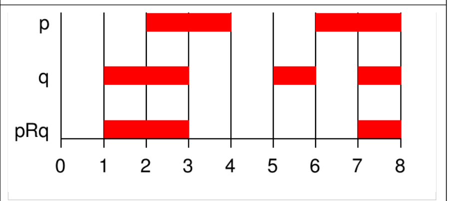
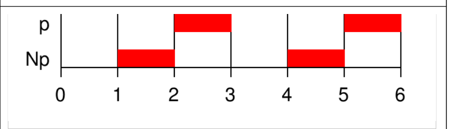
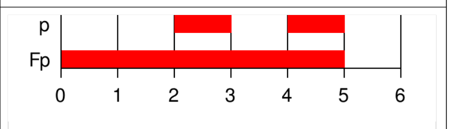
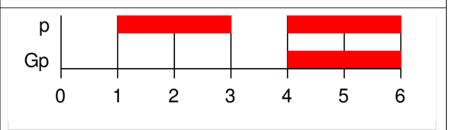
Hamblin, and a number of early developments in the field can be traced to this correspondence, for example Hamblin implications. Prior published his most mature work on the topic, the book *Past, Present, and Future* in 1967. He died two years later.[3]

The binary temporal operators *Since* and *Until* were introduced by Hans Kamp in his 1968 Ph. D. thesis,[4] which also contains an important result relating temporal logic to first order logic—a result now known as Kamp's theorem.[5][2][6]

Two early contenders in formal verifications were Linear Temporal Logic (a linear time logic by Amir Pnueli) and Computation Tree Logic, a branching time logic by E.M. Clarke and E.A. Emerson. The fact that the second logic is more efficient than the first does not reflect on branching and linear logics in general, as has sometimes been argued. Rather, Emerson and Lei show that any linear logic can be extended to a branching logic that can be decided with the same complexity.

Temporal operators

Temporal logic has two kinds of operators: logical operators and modal operators [7]. Logical operators are usual truth-functional operators ($\neg, \lor, \land, \rightarrow$). The modal operators used in Linear Temporal Logic and Computation Tree Logic are defined as follows.

Textual	Symbolic	Definition	Explanation	Diagram
		Binary operators		
ϕ **U** ψ	$\phi\,\mathcal{U}\,\psi$	$(B\,\mathcal{U}\,C)(\phi) = (\exists i : C(\phi_i) \land (\forall j < i : B(\phi_j)))$	**U**ntil: ψ holds at the current or a future position, and ϕ has to hold until that position. At that position ϕ does not have to hold any more.	
ϕ **R** ψ	$\phi\,\mathcal{R}\,\psi$	$(B\,\mathcal{R}\,C)(\phi) = (\forall i : C(\phi_i) \lor (\exists j < i : B(\phi_j)))$	**R**elease: ϕ releases ψ if ψ is true until the first position in which ϕ is true (or forever if such a position does not exist).	
		Unary operators		
N ϕ	$\bigcirc\phi$	$\mathcal{N}B(\phi_i) = B(\phi_{i+1})$	**N**ext: ϕ has to hold at the next state. (**X** is used synonymously.)	
F ϕ	$\Diamond\phi$	$\mathcal{F}B(\phi) = (true\,\mathcal{U}\,B)(\phi)$	**F**uture: ϕ eventually has to hold (somewhere on the subsequent path).	
G ϕ	$\Box\phi$	$\mathcal{G}B(\phi) = \neg\mathcal{F}\neg B(\phi)$	**G**lobally: ϕ has to hold on the entire subsequent path.	
A ϕ	$\forall\phi$	$(\mathcal{A}B)(\psi) = (\forall\phi : \phi_0 = \psi \rightarrow B(\phi))$	**A**ll: ϕ has to hold on all paths starting from the current state.	
E ϕ	$\exists\phi$	$(\mathcal{E}B)(\psi) = (\exists\phi : \phi_0 = \psi \land B(\phi))$	**E**xists: there exists at least one path starting from the current state where ϕ holds.	

Alternate symbols:

- operator **R** is sometimes denoted by **V**
- The operator **W** is the *weak until* operator: fWg is equivalent to $fUg \vee Gf$

Unary operators are well-formed formulas whenever B(ϕ) is well-formed. Binary operators are well-formed formulas whenever B(ϕ) and C(ϕ) are well-formed.

In some logics, some operators cannot be expressed. For example, **N** operator cannot be expressed in Temporal Logic of Actions.

Temporal logics

Temporal logics include

- Interval temporal logic (ITL)
- μ calculus. which includes as a subset
  - Hennessy-Milner logic (HML)
  - CTL*, which includes as a subset
    - Computational tree logic (CTL)
    - Linear temporal logic (LTL)
    - Metric Interval Temporal Logic (MITL) [ref: Maler,Nickovic,2004, monitoring temporal properties of continuous signals]
    - Signal Temporal Logic (STL) [ref: Maler,Nickovic,2004, monitoring temporal properties of continuous signals]

A variation, closely related to Temporal or Chronological or Tense logics, are Modal logics based upon "topology", "place", or "spatial position".[8][9] One might also take note that in the Russian language, verbs have an aspect, based commonly upon time, but position also.

Notes

[1] Vardi 2008, p. 153
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Further reading

- Peter Øhrstrøm; Per F. V. Hasle (1995). *Temporal logic: from ancient ideas to artificial intelligence*. Springer. ISBN 978-0-7923-3586-3.

External links

- Stanford Encyclopedia of Philosophy: " Temporal Logic (http://plato.stanford.edu/entries/logic-temporal/)" -- by Anthony Galton.
- Temporal Logic (http://staff.science.uva.nl/~yde/papers/TempLog.pdf) by Yde Venema, formal description of syntax and semantics, questions of axiomatization. Treating also Kamp's dyadic temporal operators (since, until)
- Notes on games in temporal logic (http://www.doc.ic.ac.uk/~imh/papers/sa.ps.gz) by Ian Hodkinson, including a formal description of first-order temporal logic
- CADP - provides generic model checkers for various temporal logic (http://www.inrialpes.fr/vasy/cadp)
- PAT (http://www.comp.nus.edu.sg/~pat/) is a powerful free model checker, LTL checker, simulator and refinement checker for CSP and its extensions (with shared variable, arrays, wide range of fairness).

Kripke semantics

Kripke semantics (also known as **relational semantics** or **frame semantics**, and often confused with possible world semantics) is a formal semantics for non-classical logic systems created in the late 1950s and early 1960s by Saul Kripke. It was first made for modal logics, and later adapted to intuitionistic logic and other non-classical systems. The discovery of Kripke semantics was a breakthrough in the theory of non-classical logics, because the model theory of such logics was nonexistent before Kripke.

Semantics of modal logic

The language of propositional modal logic consists of a countably infinite set of propositional variables, a set of truth-functional connectives (in this article $\rightarrow$ and $\neg$), and the modal operator $\Box$ ("necessarily"). The modal operator $\Diamond$ ("possibly") is the dual of $\Box$ and may be defined in terms of it like so: $\Diamond A := \neg\Box\neg A$ ("possibly A" is defined as equivalent to "not necessarily not A").

Basic definitions

A **Kripke frame** or **modal frame** is a pair $\langle W, R \rangle$, where *W* is a non-empty set, and *R* is a binary relation on *W*. Elements of *W* are called *nodes* or *worlds*, and *R* is known as the accessibility relation.

A **Kripke model** is a triple $\langle W, R, \Vdash \rangle$, where $\langle W, R \rangle$ is a Kripke frame, and $\Vdash$ is a relation between nodes of *W* and modal formulas, such that:

- $w \Vdash \neg A$ if and only if $w \nVdash A$,
- $w \Vdash A \rightarrow B$ if and only if $w \nVdash A$ or $w \Vdash B$,
- $w \Vdash \Box A$ if and only if $u \Vdash A$ for all u such that $w \; R \; u$.

We read $w \Vdash A$ as "*w* satisfies *A*", "*A* is satisfied in *w*", or "*w* forces *A*". The relation $\Vdash$ is called the *satisfaction relation*, *evaluation*, or *forcing relation*. The satisfaction relation is uniquely determined by its value on propositional variables.

A formula *A* is **valid** in:

- a model $\langle W, R, \Vdash \rangle$, if $w \Vdash A$ for all *w* ∈ *W*,
- a frame $\langle W, R \rangle$, if it is valid in $\langle W, R, \Vdash \rangle$ for all possible choices of $\Vdash$,
- a class *C* of frames or models, if it is valid in every member of *C*.

We define Thm(*C*) to be the set of all formulas that are valid in *C*. Conversely, if *X* is a set of formulas, let Mod(*X*) be the class of all frames which validate every formula from *X*.

A modal logic (i.e., a set of formulas) *L* is **sound** with respect to a class of frames *C*, if *L* ⊆ Thm(*C*). *L* is **complete** wrt *C* if *L* ⊇ Thm(*C*).

Correspondence and completeness

Semantics is useful for investigating a logic (i.e. a derivation system) only if the semantic consequence relation reflects its syntactical counterpart, the *syntactic consequence* relation (*derivability*). It is vital to know which modal logics are sound and complete with respect to a class of Kripke frames, and to determine also which class that is.

For any class *C* of Kripke frames, Thm(*C*) is a normal modal logic (in particular, theorems of the minimal normal modal logic, *K*, are valid in every Kripke model). However, the converse does not hold in general. There are Kripke incomplete normal modal logics, which is not a problem, because most of the modal systems studied are complete of classes of frames described by simple conditions.

A normal modal logic *L* **corresponds** to a class of frames *C*, if *C* = Mod(*L*). In other words, *C* is the largest class of frames such that *L* is sound wrt *C*. It follows that *L* is Kripke complete if and only if it is complete of its corresponding class.

Consider the schema **T** : $\Box A \to A$. **T** is valid in any reflexive frame $\langle W, R\rangle$: if $w \Vdash \Box A$, then $w \Vdash A$ since *w R w*. On the other hand, a frame which validates **T** has to be reflexive: fix $w \in W$, and define satisfaction of a propositional variable *p* as follows: $u \Vdash p$ if and only if *w R u*. Then $w \Vdash \Box p$, thus $w \Vdash p$ by **T**, which means *w R w* using the definition of $\Vdash$. **T** corresponds to the class of reflexive Kripke frames.

It is often much easier to characterize the corresponding class of *L* than to prove its completeness, thus correspondence serves as a guide to completeness proofs. Correspondence is also used to show *incompleteness* of modal logics: suppose $L_1 \subseteq L_2$ are normal modal logics that correspond to the same class of frames, but L_1 does not prove all theorems of L_2. Then L_1 is Kripke incomplete. For example, the schema $\Box(A \equiv \Box A) \to \Box A$ generates an incomplete logic, as it corresponds to the same class of frames as **GL** (viz. transitive and converse well-founded frames), but does not prove the **GL**-tautology $\Box A \to \Box\Box A$.

The table below is a list of common modal axioms together with their corresponding classes. The naming of the axioms often varies.

Common modal axiom schemata

Name	Axiom	Frame condition
K	$\Box(A \to B) \to (\Box A \to \Box B)$	N/A
T	$\Box A \to A$	reflexive: $w\,R\,w$
4	$\Box A \to \Box\Box A$	transitive: $w\,R\,v \wedge v\,R\,u \Rightarrow w\,R\,u$
	$\Box\Box A \to \Box A$	dense: $w\,R\,u \Rightarrow \exists v\,(w\,R\,v \wedge v\,R\,u)$
D	$\Box A \to \Diamond A$	serial: $\forall w\,\exists v\,(w\,R\,v)$
B	$A \to \Box\Diamond A$	symmetric : $w\,R\,v \Rightarrow v\,R\,w$
5	$\Diamond A \to \Box\Diamond A$	Euclidean: $w\,R\,u \wedge w\,R\,v \Rightarrow u\,R\,v$
GL	$\Box(\Box A \to A) \to \Box A$	*R* transitive, R^{-1} well-founded
Grz	$\Box(\Box(A \to \Box A) \to A) \to A$	*R* reflexive and transitive, R^{-1}–*Id* well-founded
H	$\Box(\Box A \to B) \vee \Box(\Box B \to A)$	$w\,R\,u \wedge w\,R\,v \Rightarrow u\,R\,v \vee v\,R\,u$
M	$\Box\Diamond A \to \Diamond\Box A$	(a complicated second-order property)
G	$\Diamond\Box A \to \Box\Diamond A$	$w\,R\,u \wedge w\,R\,v \Rightarrow \exists x\,(u\,R\,x \wedge v\,R\,x)$
	$A \to \Box A$	$w\,R\,v \Rightarrow w = v$
	$\Diamond A \to \Box A$	partial function: $w\,R\,u \wedge w\,R\,v \Rightarrow u = v$
	$\Diamond A \leftrightarrow \Box A$	function: $\forall w\,\exists! u\,w\,R\,u$

Here is a list of several common modal systems. Frame conditions for some of them were simplified: the logics are *complete* with respect to the frame classes given in the table, but they may *correspond* to a larger class of frames.

Common normal modal logics

name	axioms	frame condition
K	—	all frames
T	T	reflexive
K4	4	transitive
S4	T, 4	preorder
S5	T, 5 or D, B, 4	equivalence relation
S4.3	T, 4, H	total preorder
S4.1	T, 4, M	preorder, $\forall w\,\exists u\,(w\,R\,u \land \forall v\,(u\,R\,v \Rightarrow u = v))$
S4.2	T, 4, G	directed preorder
GL	GL or 4, GL	finite strict partial order
Grz, S4Grz	Grz or T, 4, Grz	finite partial order
D	D	serial
D45	D, 4, 5	transitive, serial, and Euclidean

Canonical models

For any normal modal logic *L*, a Kripke model (called the **canonical model**) can be constructed, which validates precisely the theorems of *L*, by an adaptation of the standard technique of using maximal consistent sets as models. Canonical Kripke models play a role similar to the Lindenbaum–Tarski algebra construction in algebraic semantics.

A set of formulas is *L-consistent* if no contradiction can be derived from them using the axioms of *L*, and Modus Ponens. A *maximal L-consistent set* (an *L-MCS* for short) is an *L*-consistent set which has no proper *L*-consistent superset.

The **canonical model** of *L* is a Kripke model $\langle W, R, \Vdash \rangle$, where *W* is the set of all *L-MCS*, and the relations *R* and $\Vdash$ are as follows:

$X\ R\ Y$ if and only if for every formula A, if $\Box A \in X$ then $A \in Y$,

$X \Vdash A$ if and only if $A \in X$.

The canonical model is a model of *L*, as every *L-MCS* contains all theorems of *L*. By Zorn's lemma, each *L*-consistent set is contained in an *L-MCS*, in particular every formula unprovable in *L* has a counterexample in the canonical model.

The main application of canonical models are completeness proofs. Properties of the canonical model of **K** immediately imply completeness of **K** with respect to the class of all Kripke frames. This argument does *not* work for arbitrary *L*, because there is no guarantee that the underlying *frame* of the canonical model satisfies the frame conditions of *L*.

We say that a formula or a set *X* of formulas is **canonical** with respect to a property *P* of Kripke frames, if

- *X* is valid in every frame which satisfies *P*,
- for any normal modal logic *L* which contains *X*, the underlying frame of the canonical model of *L* satisfies *P*.

A union of canonical sets of formulas is itself canonical. It follows from the preceding discussion that any logic axiomatized by a canonical set of formulas is Kripke complete, and compact.

The axioms T, 4, D, B, 5, H, G (and thus any combination of them) are canonical. GL and Grz are not canonical, because they are not compact. The axiom M by itself is not canonical (Goldblatt, 1991), but the combined logic **S4.1** (in fact, even **K4.1**) is canonical.

In general, it is undecidable whether a given axiom is canonical. We know a nice sufficient condition: H. Sahlqvist identified a broad class of formulas (now called Sahlqvist formulas) such that

- a Sahlqvist formula is canonical,
- the class of frames corresponding to a Sahlqvist formula is first-order definable,
- there is an algorithm which computes the corresponding frame condition to a given Sahlqvist formula.

This is a powerful criterion: for example, all axioms listed above as canonical are (equivalent to) Sahlqvist formulas.

Finite model property

A logic has the **finite model property** (FMP) if it is complete with respect to a class of finite frames. An application of this notion is the decidability question: it follows from Post's theorem that a recursively axiomatized modal logic *L* which has FMP is decidable, provided it is decidable whether a given finite frame is a model of *L*. In particular, every finitely axiomatizable logic with FMP is decidable.

There are various methods for establishing FMP for a given logic. Refinements and extensions of the canonical model construction often work, using tools such as filtration or unravelling. As another possibility, completeness proofs based on cut-free sequent calculi usually produce finite models directly.

Most of the modal systems used in practice (including all listed above) have FMP.

In some cases, we can use FMP to prove Kripke completeness of a logic: every normal modal logic is complete with respect to a class of modal algebras, and a *finite* modal algebra can be transformed into a Kripke frame. As an example, Robert Bull proved using this method that every normal extension of **S4.3** has FMP, and is Kripke complete.

Multimodal logics

Kripke semantics has a straightforward generalization to logics with more than one modality. A Kripke frame for a language with $\{\Box_i \mid i \in I\}$ as the set of its necessity operators consists of a non-empty set *W* equipped with binary relations R_i for each $i \in I$. The definition of a satisfaction relation is modified as follows:

$w \Vdash \Box_i A$ if and only if $\forall u\,(w\ R_i\ u \Rightarrow u \Vdash A)$.

A simplified semantics, discovered by Tim Carlson, is often used for polymodal provability logics. A **Carlson model** is a structure $\langle W, R, \{D_i\}_{i\in I}, \Vdash\rangle$ with a single accessibility relation *R*, and subsets $D_i \subseteq W$ for each modality. Satisfaction is defined as

$w \Vdash \Box_i A$ if and only if $\forall u \in D_i\,(w\ R\ u \Rightarrow u \Vdash A)$.

Carlson models are easier to visualize and to work with than usual polymodal Kripke models; there are, however, Kripke complete polymodal logics which are Carlson incomplete.

Semantics of intuitionistic logic

Kripke semantics for the intuitionistic logic follows the same principles as the semantics of modal logic, but it uses a different definition of satisfaction.

An **intuitionistic Kripke model** is a triple $\langle W, \le, \Vdash\rangle$, where $\langle W, \le\rangle$ is a preordered Kripke frame, and $\Vdash$ satisfies the following conditions:

- if *p* is a propositional variable, $w \le u$, and $w \Vdash p$, then $u \Vdash p$ (*persistency* condition),
- $w \Vdash A \wedge B$ if and only if $w \Vdash A$ and $w \Vdash B$,
- $w \Vdash A \vee B$ if and only if $w \Vdash A$ or $w \Vdash B$,
- $w \Vdash A \to B$ if and only if for all $u \ge w$, $u \Vdash A$ implies $u \Vdash B$,
- not $w \Vdash \bot$.

The negation of A, $\neg A$, could be defined as an abbreviation for $A \to \bot$. If for all u such that $w \le u$, not $u \Vdash A$, then $w \Vdash A \to \bot$ is vacuously true, so $w \Vdash \neg A$.

Intuitionistic logic is sound and complete with respect to its Kripke semantics, and it has FMP.

Intuitionistic first-order logic

Let L be a first-order language. A Kripke model of L is a triple $\langle W, \le, \{M_w\}_{w\in W}\rangle$, where $\langle W, \le\rangle$ is an intuitionistic Kripke frame, M_w is a (classical) L-structure for each node $w \in W$, and the following compatibility conditions hold whenever $u \le v$:

- the domain of M_u is included in the domain of M_v,
- realizations of function symbols in M_u and M_v agree on elements of M_u,
- for each n-ary predicate P and elements $a_1,\dots,a_n \in M_u$: if $P(a_1,\dots,a_n)$ holds in M_u, then it holds in M_v.

Given an evaluation e of variables by elements of M_w, we define the satisfaction relation $w \Vdash A[e]$:

- $w \Vdash P(t_1,\dots,t_n)[e]$ if and only if $P(t_1[e],\dots,t_n[e])$ holds in M_w,
- $w \Vdash (A \wedge B)[e]$ if and only if $w \Vdash A[e]$ and $w \Vdash B[e]$,
- $w \Vdash (A \vee B)[e]$ if and only if $w \Vdash A[e]$ or $w \Vdash B[e]$,
- $w \Vdash (A \to B)[e]$ if and only if for all $u \ge w$, $u \Vdash A[e]$ implies $u \Vdash B[e]$,
- not $w \Vdash \bot[e]$,
- $w \Vdash (\exists x\, A)[e]$ if and only if there exists an $a \in M_w$ such that $w \Vdash A[e(x \to a)]$,
- $w \Vdash (\forall x\, A)[e]$ if and only if for every $u \ge w$ and every $a \in M_u$, $u \Vdash A[e(x \to a)]$.

Here $e(x\to a)$ is the evaluation which gives x the value a, and otherwise agrees with e.

See a slightly different formalization in.[1]

Kripke–Joyal semantics

As part of the independent development of sheaf theory, it was realised around 1965 that Kripke semantics was intimately related to the treatment of existential quantification in topos theory.[2] That is, the 'local' aspect of existence for sections of a sheaf was a kind of logic of the 'possible'. Though this development was the work of a number of people, the name **Kripke–Joyal semantics** is often used in this connection.

Model constructions

As in the classical model theory, there are methods for constructing a new Kripke model from other models.

The natural homomorphisms in Kripke semantics are called **p-morphisms** (which is short for *pseudo-epimorphism*, but the latter term is rarely used). A p-morphism of Kripke frames $\langle W, R\rangle$ and $\langle W', R'\rangle$ is a mapping $f\colon W \to W'$ such that

- f preserves the accessibility relation, i.e., $u\,R\,v$ implies $f(u)\,R'\,f(v)$,
- whenever $f(u)\,R'\,v'$, there is a $v \in W$ such that $u\,R\,v$ and $f(v) = v'$.

A p-morphism of Kripke models $\langle W, R, \Vdash\rangle$ and $\langle W', R', \Vdash'\rangle$ is a p-morphism of their underlying frames $f\colon W \to W'$, which satisfies

$w \Vdash p$ if and only if $f(w) \Vdash' p$, for any propositional variable p.

P-morphisms are a special kind of bisimulations. In general, a **bisimulation** between frames $\langle W, R\rangle$ and $\langle W', R'\rangle$ is a relation $B \subseteq W \times W'$, which satisfies the following “zig-zag” property:

- if $u\,B\,u'$ and $u\,R\,v$, there exists $v' \in W'$ such that $v\,B\,v'$ and $u'\,R'\,v'$,
- if $u\,B\,u'$ and $u'\,R'\,v'$, there exists $v \in W$ such that $v\,B\,v'$ and $u\,R\,v$.

A bisimulation of models is additionally required to preserve forcing of atomic formulas:

if $w\ B\ w'$, then $w \Vdash p$ if and only if $w' \Vdash' p$, for any propositional variable p.

The key property which follows from this definition is that bisimulations (hence also p-morphisms) of models preserve the satisfaction of *all* formulas, not only propositional variables.

We can transform a Kripke model into a tree using **unravelling**. Given a model $\langle W, R, \Vdash \rangle$ and a fixed node $w_0 \in W$, we define a model $\langle W', R', \Vdash' \rangle$, where W' is the set of all finite sequences $s = \langle w_0, w_1, \ldots, w_n \rangle$ such that $w_i\ R\ w_{i+1}$ for all $i < n$, and $s \Vdash p$ if and only if $w_n \Vdash p$ for a propositional variable p. The definition of the accessibility relation R' varies; in the simplest case we put

$$\langle w_0, w_1, \ldots, w_n \rangle\ R'\ \langle w_0, w_1, \ldots, w_n, w_{n+1} \rangle,$$

but many applications need the reflexive and/or transitive closure of this relation, or similar modifications.

Filtration is a usefull construction which uses to prove FMP for many logics. Let X be a set of formulas closed under taking subformulas. An X-filtration of a model $\langle W, R, \Vdash \rangle$ is a mapping f from W to a model $\langle W', R', \Vdash' \rangle$ such that

- f is a surjection,
- f preserves the accessibility relation, and (in both directions) satisfaction of variables $p \in X$,
- if $f(u)\ R'\ f(v)$ and $u \Vdash \Box A$, where $\Box A \in X$, then $v \Vdash A$.

It follows that f preserves satisfaction of all formulas from X. In typical applications, we take f as the projection onto the quotient of W over the relation

$u \sim_X v$ if and only if for all $A \in X$, $u \Vdash A$ if and only if $v \Vdash A$.

As in the case of unravelling, the definition of the accessibility relation on the quotient varies.

General frame semantics

The main defect of Kripke semantics is the existence of Kripke incomplete logics, and logics which are complete but not compact. It can be remedied by equipping Kripke frames with extra structure which restricts the set of possible valuations, using ideas from algebraic semantics. This gives rise to the general frame semantics.

Computer science applications

Blackburn et al. (2001) point out that because a relational structure is simply a set together with a collection of relations on that set, it is unsurprising that relational structures are to be found just about everywhere. As an example from theoretical computer science, they give labeled transition systems, which model program execution. Blackburn et al. thus claim because of this connection that modal languages are ideally suited in providing "internal, local perspective on relational structures." (p. xii)

History and terminology

Kripke semantics does not originate with Kripke, but instead the idea of giving semantics in the style given above, that is based on valuations made that are relative to nodes, predates Kripke by a long margin:

- Rudolf Carnap seems to have been the first to have the idea that one can give a **possible world semantics** for the modalities of necessity and possibility by means of giving the valuation function a parameter that ranges over Leibnizian possible worlds. Bayart develops this idea further, but neither gave recursive definitions of satisfaction in the style introduced by Tarski;
- J.C.C. McKinsey and Alfred Tarski developed an approach to modeling modal logics that is still influential in modern research, namely the algebraic approach, in which Boolean algebras with operators are used as models. Bjarni Jónsson and Tarski established the representability of Boolean algebras with operators in terms of frames. If the two ideas had been put together, the result would have been precisely frame models, which is to say Kripke models, years before Kripke. But no one (not even Tarski) saw the connection at the time.

- Arthur Prior, building on unpublished work of C. A. Meredith, developed a translation of sentential modal logic into classical predicate logic that, if he had combined it with the usual model theory for the latter, would have produced a model theory equivalent to Kripke models for the former. But his approach was resolutely syntactic and anti-model-theoretic.
- Stig Kanger gave a rather more complex approach to the interpretation of modal logic, but one that contains many of the key ideas of Kripke's approach. He first noted the relationship between conditions on accessibility relations and Lewis-style axioms for modal logic. Kanger failed, however, to give a completeness proof for his system;
- Jaakko Hintikka gave a semantics in his papers introducing epistemic logic that is a simple variation of Kripke's semantics, equivalent to the characterisation of valuations by means of maximal consistent sets. He doesn't give inference rules for epistemic logic, and so cannot give a completeness proof;
- Richard Montague had many of the key ideas contained in Kripke's work, but he did not regard them as significant, because he had no completeness proof, and so did not publish until after Kripke's papers had created a sensation in the logic community;
- Evert Willem Beth presented a semantics of intuitionistic logic based on trees, which closely resembles Kripke semantics, except for using a more cumbersome definition of satisfaction.

Though the essential ideas of **Kripke semantics** were very much in the air by the time Kripke first published, Saul Kripke's work on modal logic is rightly regarded as ground-breaking. Most importantly, it was Kripke who proved the completeness theorems for modal logic, and Kripke who identified the weakest normal modal logic.

Despite the seminal contribution of Kripke's work, many modal logicians deprecate the term **Kripke semantics** as disrespectful of the important contributions these other pioneers made. The other most widely used term **possible world semantics** is deprecated as inappropriate when applied to modalities other than possibility and necessity, such as in epistemic or deontic logic. Instead they prefer the terms **relational semantics** or **frame semantics**. The use of "semantics" for "model theory" has been objected to as well, on the grounds that it invites confusion with linguistic semantics: whether the apparatus of "possible worlds" that appears in models has anything to do with the linguistic meaning of modal constructions in natural language is a contentious issue.

Notes

[1] Intuitionistic Logic (http://www.seop.leeds.ac.uk/archives/spr2004/entries/logic-intuitionistic/). Written by Joan Moschovakis (http://www.math.ucla.edu/~joan/). Published in Stanford Encyclopedia of Philosophy.

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External links

- The Stanford Encyclopedia of Philosophy: " Modal Logic (http://plato.stanford.edu/archives/win2001/entries/logic-modal)" – by James Garson.
- Intuitionistic Logic (http://www.seop.leeds.ac.uk/archives/spr2004/entries/logic-intuitionistic/). Written by Joan Moschovakis (http://www.math.ucla.edu/~joan/). Published in Stanford Encyclopedia of Philosophy.
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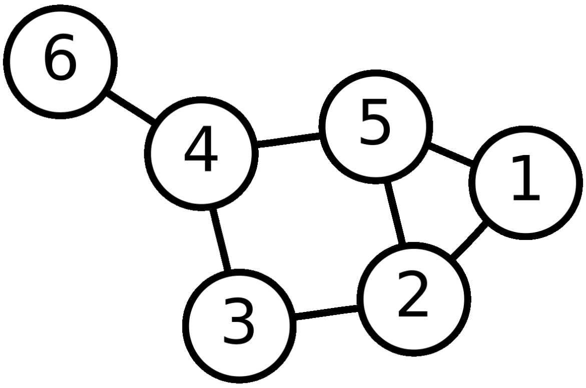
- Burgess, John P., " Kripke Models. (http://www.princeton.edu/~jburgess/Kripke1.doc)"

Graph theory

In mathematics and computer science, **graph theory** is the study of *graphs*, which are mathematical structures used to model pairwise relations between objects from a certain collection. A "graph" in this context is a collection of "vertices" or "nodes" and a collection of *edges* that connect pairs of vertices. A graph may be *undirected*, meaning that there is no distinction between the two vertices associated with each edge, or its edges may be *directed* from one vertex to another; see graph (mathematics) for more detailed definitions and for other variations in the types of graph that are commonly considered. Graphs are one of the prime objects of study in discrete mathematics.



A drawing of a graph

The graphs studied in graph theory should not be confused with the graphs of functions or other kinds of graphs.

Refer to the glossary of graph theory for basic definitions in graph theory.

Applications

Graphs are among the most ubiquitous models of both natural and human-made structures. They can be used to model many types of relations and process dynamics in physical, biological[1] and social systems. Many problems of practical interest can be represented by graphs.

In computer science, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc. One practical example: The link structure of a website could be represented by a directed graph. The vertices are the web pages available at the website and a directed edge from page *A* to page *B* exists if and only if *A* contains a link to *B*. A similar approach can be taken to problems in travel, biology, computer chip design, and many other fields. The development of algorithms to handle graphs is therefore of major interest in computer science. There, the transformation of graphs is often formalized and represented by graph rewrite systems. They are either directly used or properties of the rewrite systems (e.g. confluence) are studied. Complementary to graph transformation systems focussing on rule-based in-memory manipulation of graphs are graph databases geared towards transaction-safe, persistent storing and querying of graph-structured data.

Graph-theoretic methods, in various forms, have proven particularly useful in linguistics, since natural language often lends itself well to discrete structure. Traditionally, syntax and compositional semantics follow tree-based structures, whose expressive power lies in the Principle of Compositionality, modeled in a hierarchical graph. More contemporary approaches such as Head-driven phrase structure grammar (HPSG) model syntactic constructions via the unification of typed feature structures, which are directed acyclic graphs. Within lexical semantics, especially as applied to computers, modeling word meaning is easier when a given word is understood in terms of related words; semantic networks are therefore important in computational linguistics. Still other methods in phonology (e.g. Optimality Theory, which uses lattice graphs) and morphology (e.g. finite-state morphology, using finite-state transducers) are common in the analysis of language as a graph. Indeed, the usefulness of this area of mathematics to linguistics has borne organizations such as TextGraphs [2], as well as various 'Net' projects, such as WordNet, VerbNet, and others.

Graph theory is also used to study molecules in chemistry and physics. In condensed matter physics, the three dimensional structure of complicated simulated atomic structures can be studied quantitatively by gathering statistics on graph-theoretic properties related to the topology of the atoms. For example, Franzblau's shortest-path (SP) rings. In chemistry a graph makes a natural model for a molecule, where vertices represent atoms and edges bonds. This approach is especially used in computer processing of molecular structures, ranging from chemical editors to database searching. In statistical physics, graphs can represent local connections between interacting parts of a system, as well as the dynamics of a physical process on such systems.

Graph theory is also widely used in sociology as a way, for example, to measure actors' prestige or to explore diffusion mechanisms, notably through the use of social network analysis software.

Likewise, graph theory is useful in biology and conservation efforts where a vertex can represent regions where certain species exist (or habitats) and the edges represent migration paths, or movement between the regions. This information is important when looking at breeding patterns or tracking the spread of disease, parasites or how changes to the movement can affect other species.

In mathematics, graphs are useful in geometry and certain parts of topology, e.g. Knot Theory. Algebraic graph theory has close links with group theory.

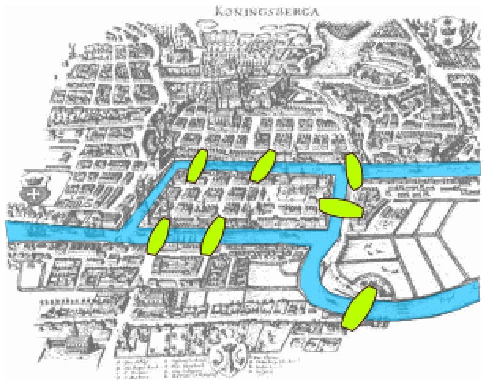
A graph structure can be extended by assigning a weight to each edge of the graph. Graphs with weights, or weighted graphs, are used to represent structures in which pairwise connections have some numerical values. For example if a graph represents a road network, the weights could represent the length of each road.

A digraph with weighted edges in the context of graph theory is called a network. Network analysis have many practical applications, for example, to model and analyze traffic networks. Applications of network analysis split broadly into three categories:

1. First, analysis to determine structural properties of a network, such as the distribution of vertex degrees and the diameter of the graph. A vast number of graph measures exist, and the production of useful ones for various domains remains an active area of research.
2. Second, analysis to find a measurable quantity within the network, for example, for a transportation network, the level of vehicular flow within any portion of it.
3. Third, analysis of dynamical properties of networks.

History

The paper written by Leonhard Euler on the *Seven Bridges of Königsberg* and published in 1736 is regarded as the first paper in the history of graph theory.[3] This paper, as well as the one written by Vandermonde on the *knight problem,* carried on with the *analysis situs* initiated by Leibniz. Euler's formula relating the number of edges, vertices, and faces of a convex polyhedron was studied and generalized by Cauchy[4] and L'Huillier,[5] and is at the origin of topology.



The Königsberg Bridge problem

More than one century after Euler's paper on the bridges of Königsberg and while Listing introduced topology, Cayley was led by the study of particular analytical forms arising from differential calculus to study a particular class of graphs, the *trees*. This study had many implications in theoretical chemistry. The involved techniques mainly concerned the enumeration of graphs having particular properties. Enumerative graph theory then rose from the results of Cayley and the fundamental results published by Pólya between 1935 and 1937 and the generalization of these by De Bruijn in 1959. Cayley linked his results on trees with the contemporary studies of chemical composition.[6] The fusion of the ideas coming from mathematics with those coming from chemistry is at the origin of a part of the standard terminology of graph theory.

In particular, the term "graph" was introduced by Sylvester in a paper published in 1878 in *Nature*, where he draws an analogy between "quantic invariants" and "co-variants" of algebra and molecular diagrams:[7]

> "[...] Every invariant and co-variant thus becomes expressible by a *graph* precisely identical with a Kekuléan diagram or chemicograph. [...] I give a rule for the geometrical multiplication of graphs, *i.e.* for constructing a *graph* to the product of in- or co-variants whose separate graphs are given. [...]" (italics as in the original).

The first textbook on graph theory was written by Dénes Kőnig, and published in 1936.[8] A later textbook by Frank Harary, published in 1969, was enormously popular, and enabled mathematicians, chemists, electrical engineers and social scientists to talk to each other. Harary donated all of the royalties to fund the Pólya Prize.[9]

One of the most famous and productive problems of graph theory is the four color problem: "Is it true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors?" This problem was first posed by Francis Guthrie in 1852 and its first written record is in a letter of De Morgan addressed to Hamilton the same year. Many incorrect proofs have been proposed, including those by Cayley, Kempe, and others. The study and the generalization of this problem by Tait, Heawood, Ramsey and Hadwiger led to the study of the colorings of the graphs embedded on surfaces with arbitrary genus. Tait's reformulation generated a new class of problems, the *factorization problems*, particularly studied by Petersen and Kőnig. The works of Ramsey on colorations and more specially the results obtained by Turán in 1941 was at the origin of another branch of graph theory, *extremal graph theory*.

The four color problem remained unsolved for more than a century. In 1969 Heinrich Heesch published a method for solving the problem using computers.[10] A computer-aided proof produced in 1976 by Kenneth Appel and Wolfgang Haken makes fundamental use of the notion of "discharging" developed by Heesch.[11][12] The proof involved checking the properties of 1,936 configurations by computer, and was not fully accepted at the time due to its complexity. A simpler proof considering only 633 configurations was given twenty years later by Robertson, Seymour, Sanders and Thomas.[13]

The autonomous development of topology from 1860 and 1930 fertilized graph theory back through the works of Jordan, Kuratowski and Whitney. Another important factor of common development of graph theory and topology came from the use of the techniques of modern algebra. The first example of such a use comes from the work of the physicist Gustav Kirchhoff, who published in 1845 his Kirchhoff's circuit laws for calculating the voltage and current in electric circuits.

The introduction of probabilistic methods in graph theory, especially in the study of Erdős and Rényi of the asymptotic probability of graph connectivity, gave rise to yet another branch, known as *random graph theory*, which has been a fruitful source of graph-theoretic results.

Drawing graphs

Graphs are represented graphically by drawing a dot or circle for every vertex, and drawing an arc between two vertices if they are connected by an edge. If the graph is directed, the direction is indicated by drawing an arrow.

A graph drawing should not be confused with the graph itself (the abstract, non-visual structure) as there are several ways to structure the graph drawing. All that matters is which vertices are connected to which others by how many edges and not the exact layout. In practice it is often difficult to decide if two drawings represent the same graph. Depending on the problem domain some layouts may be better suited and easier to understand than others.

The pioneering work of W. T. Tutte was very influential in the subject of graph drawing. Among other achievements, he introduced the use of linear algebraic methods to obtain graph drawings.

Graph drawing also can be said to encompass problems that deal with the crossing number and its various generalizations. The crossing number of a graph is the minimum number of intersections between edges that a drawing of the graph in the plane must contain. For a planar graph, the crossing number is zero by definition.

Drawings on surfaces other than the plane are also studied.

Graph-theoretic data structures

There are different ways to store graphs in a computer system. The data structure used depends on both the graph structure and the algorithm used for manipulating the graph. Theoretically one can distinguish between list and matrix structures but in concrete applications the best structure is often a combination of both. List structures are often preferred for sparse graphs as they have smaller memory requirements. Matrix structures on the other hand provide faster access for some applications but can consume huge amounts of memory.

List structures

Incidence list

The edges are represented by an array containing pairs (tuples if directed) of vertices (that the edge connects) and possibly weight and other data. Vertices connected by an edge are said to be *adjacent*.

Adjacency list

Much like the incidence list, each vertex has a list of which vertices it is adjacent to. This causes redundancy in an undirected graph: for example, if vertices A and B are adjacent, A's adjacency list contains B, while B's list contains A. Adjacency queries are faster, at the cost of extra storage space.

Matrix structures

Incidence matrix

The graph is represented by a matrix of size |*V* | (number of vertices) by |*E*| (number of edges) where the entry [vertex, edge] contains the edge's endpoint data (simplest case: 1 - incident, 0 - not incident).

Adjacency matrix

This is an *n* by *n* matrix *A*, where *n* is the number of vertices in the graph. If there is an edge from a vertex *x* to a vertex *y*, then the element $a_{x,y}$ is 1 (or in general the number of *xy* edges), otherwise it is 0. In computing, this matrix makes it easy to find subgraphs, and to reverse a directed graph.

Laplacian matrix or "Kirchhoff matrix" or "Admittance matrix"

This is defined as *D* − *A*, where *D* is the diagonal degree matrix. It explicitly contains both adjacency information and degree information. (However, there are other, similar matrices that are also called "Laplacian matrices" of a graph.)

Distance matrix

A symmetric *n* by *n* matrix *D*, where *n* is the number of vertices in the graph. The element $d_{x,y}$ is the length of a shortest path between *x* and *y*; if there is no such path $d_{x,y}$ = infinity. It can be derived from powers of *A*
$d_{x,y} = \min\{n \mid A^n[x, y] \neq 0\}$.

Problems in graph theory

Enumeration

There is a large literature on graphical enumeration: the problem of counting graphs meeting specified conditions. Some of this work is found in Harary and Palmer (1973).

Subgraphs, induced subgraphs, and minors

A common problem, called the subgraph isomorphism problem, is finding a fixed graph as a subgraph in a given graph. One reason to be interested in such a question is that many graph properties are *hereditary* for subgraphs, which means that a graph has the property if and only if all subgraphs have it too. Unfortunately, finding maximal subgraphs of a certain kind is often an NP-complete problem.

- Finding the largest complete graph is called the clique problem (NP-complete).

A similar problem is finding induced subgraphs in a given graph. Again, some important graph properties are hereditary with respect to induced subgraphs, which means that a graph has a property if and only if all induced subgraphs also have it. Finding maximal induced subgraphs of a certain kind is also often NP-complete. For example,

- Finding the largest edgeless induced subgraph, or independent set, called the independent set problem (NP-complete).

Still another such problem, the *minor containment problem*, is to find a fixed graph as a minor of a given graph. A minor or **subcontraction** of a graph is any graph obtained by taking a subgraph and contracting some (or no) edges. Many graph properties are hereditary for minors, which means that a graph has a property if and only if all minors have it too. A famous example:

- A graph is planar if it contains as a minor neither the complete bipartite graph $K_{3,3}$(See the Three-cottage problem) nor the complete graph K_5.

Another class of problems has to do with the extent to which various species and generalizations of graphs are determined by their *point-deleted subgraphs*, for example:

- The reconstruction conjecture.

Graph coloring

Many problems have to do with various ways of coloring graphs, for example:

- The four-color theorem
- The strong perfect graph theorem
- The Erdős–Faber–Lovász conjecture (unsolved)
- The total coloring conjecture (unsolved)
- The list coloring conjecture (unsolved)
- The Hadwiger conjecture (graph theory) (unsolved).

Subsumption and unification

Constraint modeling theories concern families of directed graphs related by a partial order. In these applications, graphs are ordered by specificity, meaning that more constrained graphs—which are more specific and thus contain a greater amount of information—are subsumed by those that are more general. Operations between graphs include evaluating the direction of a subsumption relationship between two graphs, if any, and computing graph unification. The unification of two argument graphs is defined as the most general graph (or the computation thereof) that is consistent with (i.e. contains all of the information in) the inputs, if such a graph exists; efficient unification algorithms are known.

For contraint frameworks which are strictly compositional, graph unification is the sufficient satisfiability and combination function. Well-known applications include automatic theorem proving and modeling the elaboration of linguistic structure.

Route problems

- Hamiltonian path and cycle problems
- Minimum spanning tree
- Route inspection problem (also called the "Chinese Postman Problem")
- Seven Bridges of Königsberg
- Shortest path problem
- Steiner tree
- Three-cottage problem
- Traveling salesman problem (NP-hard)

Network flow

There are numerous problems arising especially from applications that have to do with various notions of flows in networks, for example:

- Max flow min cut theorem

Visibility graph problems

- Museum guard problem

Covering problems

Covering problems are specific instances of subgraph-finding problems, and they tend to be closely related to the clique problem or the independent set problem.

- Set cover problem

- Vertex cover problem

Graph classes

Many problems involve characterizing the members of various classes of graphs. Some examples of such questions are below:

- Enumerating the members of a class
- Characterizing a class in terms of forbidden substructures
- Ascertaining relationships among classes (e.g., does one property of graphs imply another)
- Finding efficient algorithms to decide membership in a class
- Finding representations for members of a class.

Notes

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- Digraphs: Theory Algorithms and Applications (http://www.cs.rhul.ac.uk/books/dbook/) 2007 by Jorgen Bang-Jensen and Gregory Gutin
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- Graph theory tutorial (http://www.utm.edu/departments/math/graph/)
- A searchable database of small connected graphs (http://www.gfredericks.com/main/sandbox/graphs)
- Image gallery: graphs (http://web.archive.org/web/20060206155001/http://www.nd.edu/~networks/gallery.htm)
- Concise, annotated list of graph theory resources for researchers (http://www.babelgraph.org/links.html)
- rocs (http://www.kde.org/applications/education/rocs/) — a graph theory IDE

Computational linguistics

Computational linguistics is an interdisciplinary field dealing with the statistical or rule-based modeling of natural language from a computational perspective.

Traditionally, computational linguistics was usually performed by computer scientists who had specialized in the application of computers to the processing of a natural language. Computational linguists often work as members of interdisciplinary teams, including linguists (specifically trained in linguistics), language experts (persons with some level of ability in the languages relevant to a given project), and computer scientists. In general, computational linguistics draws upon the involvement of linguists, computer scientists, experts in artificial intelligence, mathematicians, logicians, philosophers, cognitive scientists, cognitive psychologists, psycholinguists, anthropologists and neuroscientists, among others.

Computational linguistics has theoretical and applied components, where theoretical computational linguistics takes up issues in theoretical linguistics and cognitive science, and applied computational linguistics focuses on the practical outcome of modelling human language use.[1]

Origins

Computational linguistics as a field predates artificial intelligence, a field under which it is often grouped. Computational linguistics originated with efforts in the United States in the 1950s to use computers to automatically translate texts from foreign languages, particularly Russian scientific journals, into English.[2] Since computers can make arithmetic calculations much faster and more accurately than humans, it was thought to be only a short matter of time before the technical details could be taken care of that would allow them the same remarkable capacity to process language.[3]

When machine translation (also known as mechanical translation) failed to yield accurate translations right away, automated processing of human languages was recognized as far more complex than had originally been assumed.

Computational linguistics was born as the name of the new field of study devoted to developing algorithms and software for intelligently processing language data. When artificial intelligence came into existence in the 1960s, the field of computational linguistics became that sub-division of artificial intelligence dealing with human-level comprehension and production of natural languages.

In order to translate one language into another, it was observed that one had to understand the grammar of both languages, including both morphology (the grammar of word forms) and syntax (the grammar of sentence structure). In order to understand syntax, one had to also understand the semantics and the lexicon (or 'vocabulary'), and even to understand something of the pragmatics of language use. Thus, what started as an effort to translate between languages evolved into an entire discipline devoted to understanding how to represent and process natural languages using computers.[4]

Nowadays research within the scope of computational linguistics is done at computational linguistics departments,[5] computational linguistics laboratories,[6] computer science departments,[7] and linguistics departments.[8][9]

Approaches

Just as computational linguistics can be performed by experts in a variety of fields, and through a plethora of departments, so too can the research fields broach a diverse range of topics. The following sections discuss some of the literature available across the entire field broken into four main area of discourse: developmental linguistics, structural linguistics, linguistic production, and linguistic comprehension.

Developmental Approaches

Language is a skill which develops throughout the life of an individual. This developmental process has been examined using a number of techniques, and a computational approach is one of them. Human language development does provide some constraints which make it feasible to apply a computational method to understanding it. For instance, during language acquisition, human children are largely only exposed to positive evidence.[10] This means that during the linguistic development of an individual, only evidence for what is a correct form is provided, and not evidence for what is not correct. This is insufficient information for a simple hypothesis testing procedure for information as complex as language,[11] and so provides certain boundaries for a computational approach to modeling language development and acquisition in an individual.

Attempts have been made to model the developmental process of language acquisition in children from a computational angle. Work in this realm has also been proposed as a method to explain the evolution of language through history. Using models, it has been shown that languages can be learned most efficiently with a combination of simple input at first presented incrementally and the child develops better memory and longer attention span.[12] This was simultaneously posed as a reason for the long developmental period of human children.[13] Both conclusions were drawn because of the strength of the neural network which the project created.

The ability of infants to develop language has also been modeled using robots[14] in order to test linguistic theories. Enabled to learn as children might, a model was created based on an affordance model in which mappings between actions, perceptions, and effects were created and linked to spoken words. Crucially, these robots were able to acquire functioning word-to-meaning mappings without needing grammatical structure, vastly simplifying the learning process and shedding light on information which furthers the current understanding of linguistic development. It is important to note that this information could only have been empirically tested using a computational approach.

As our understanding of the linguistic development of an individual within a lifetime is continually improved using neural networks and learning robotic systems, it is also important to keep in mind that languages themselves change and develop through time. Computational approaches to understanding this phenomenon have unearthed very interesting information. Using the Price Equation and Pólya urn dynamics, researchers have created a system which not only predicts future linguistic evolution, but also gives insight into the evolutionary history of modern day languages.[15] This modeling effort achieved through computational linguistics what would have otherwise been impossible.

It is clear that the understanding of linguistic development in humans as well as throughout evolutionary time has been fantastically improved because of advances in computational linguistics. The ability to model and modify systems at will affords science an ethical method of testing hypotheses that would otherwise be intractable.

Structural Approaches

In order to create better computational models of language, an understanding of language's structure is crucial. To this end, the English language has been meticulously studied using computational approaches to better understand how the language works on a structural level. One of the most important pieces of being able to study linguistic structure is the availability of large linguistic corpora. This grants computational linguists the raw data necessary to run their models and gain a better understanding of the underlying structures present in the vast amount of data which is contained in any single language. One of the most cited English linguistic corpora is the Penn Treebank.[16] Containing over 4.5 million words of American English, this corpus has been annotated for part-of-speech information. This type of annotated corpus allows other researchers to apply hypotheses and measures that would otherwise be impossible to perform.

Theoretical approaches to the structure of languages have also been submitted. These works allow computational linguistics to have a framework within which to work out hypotheses that will further the understanding of the language in a myriad of ways. One of the original theoretical theses on internalization of grammar and structure of language proposed two types of models.[17] In these models, rules or patterns learned increase in strength with the frequency of their encounter.[18] The work also created a question for computational linguists to answer: how does an infant learn a specific and non-normal grammar (Chomsky Normal Form) without learning an overgeneralized version and getting stuck[19]? Theoretical efforts like these set the direction for research to go early in the lifetime of a field of study, and are crucial to the growth of the field.

Structural information about languages allows for the discovery and implementation of similarity recognition between pairs of text utterances.[20] For instance, it has recently been proven that based on the structural information present in patterns of human discourse, conceptual recurrence plots can be used to model and visualize trends in data and create reliable measures of similarity between natural textual utterances.[21] This technique is a strong tool for further probing the structure of human discourse. Without the computational approach to this question, the vastly complex information present in discourse data would have remained inaccessible to scientists.

Information regarding the structural data of a language is not simply available for English, but can also be found in other languages, such as Japanese.[22] Using computational methods, Japanese sentence corpora were analyzed and a pattern of log-normality was found in relation to sentence length.[23] Though the exact cause of this lognormality remains as of yet unknown, it is precisely this sort of intriguing information which computational linguistics is designed to uncover. This information could lead to further important discoveries regarding the underlying structure of Japanese, and could have any number of effects on the understanding of Japanese as a language. Computational linguistics allows for very exciting additions to the scientific knowledge base to happen quickly and with very little room for doubt.

Without a computational approach to the structure of linguistic data, much of the information that is available now would still be hidden under the vastness of data within any single language. Computational linguistics allows scientists to parse huge amounts of data reliably and efficiently, creating the possibility for discoveries unlike any seen in most other approaches.

Production Approaches

The production of language is equally as complex in the information it provides and the necessary skills which a fluent producer must have. That is to say, comprehension is only half the battle of communication. The other half is how a system produces language, and computational linguistics has made some very interesting discoveries in this area.

In a now famous paper published in 1950 Alan Turing proposed the possibility that machines might one day have the ability to "think". As a thought experiment for what might define the concept of thought in machines, he proposed an "imitation test" in which a human subject has two text-only conversations, one with a fellow human and another with a machine attempting to respond like a human. Turing proposes that if the subject cannot tell the difference between the human and the machine, it may be concluded that the machine is capable of thought.[24] Today this test is known as the Turing test and it remains an influential idea in the area of artificial intelligence.

One of the earliest and best known examples of a computer program designed to converse naturally with humans is the ELIZA program developed by Joseph Weizenbaum at MIT in 1966. The program emulated a Rogerian psychotherapist when responding to written statements and questions posed by a user. It appeared capable of understanding what was said to it and responding intelligently, but in truth it simply followed a pattern matching routine that relied on only understanding a few keywords in each sentence. Its responses were generated by recombining the unknown parts of the sentence around properly translated versions of the known words. For example in the phrase "It seems that you hate me" ELIZA understands "you" and "me" which matches the general pattern "you [some words] me", allowing ELIZA to update the words "you" and "me" to "I" and "you" and replying "What makes you think I hate you?". In this example ELIZA has no understanding of the word "hate", but it is not required for a logical response in the context of this type of psychotherapy.[25]

Some projects are still trying to solve the problem which first started computational linguistics off as its own field in the first place. However, the methods have become more refined and clever, and consequently the results generated by computational linguists have become more enlightening. In an effort to improve computer translation, several models have been compared, including hidden Markov models, smoothing techniques, and the specific refinements of those to apply them to verb translation.[26] The model which was found to produce the most natural translations of German and French words was a refined alignment model with a first-order dependence and a fertility model[16]. They also provide efficient training algorithms for the models presented, which can give other scientists the ability to improve further on their results. This type of work is specific to computational linguistics, and has applications which could vastly improve understanding of how language is produced and comprehended by computers.

Work has also been done in making computers produce language in a more naturalistic manner. Using linguistic input from humans, algorithms have been constructed which are able to modify a system's style of production based on a factor such as linguistic input from a human, or more abstract factors like politeness or any of the five main dimensions of personality.[27] This work takes a computational approach via parameter estimation models to categorize the vast array of linguistic styles we see across individuals and simplify it for a computer to work in the same way[11], making human-computer interaction much more natural.

Comprehension Approaches

Much of the focus of modern computational linguistics is on comprehension. WIth the proliferation of the internet and the abundance of easily accessible written human language, the ability to create a program capable of understanding human language would have many broad and exciting possibilities, including improved search engines, automated customer service, and online education.

Early work in comprehension included applying Bayesian statistics to the task of optical character recognition, as illustrated by Bledsoe and Browing in 1959 in which a large dictionary of possible letters were generated by "learning" from example letters and then the probability that any one of those learned examples matched the new input was combined to make a final decision.[28] Other attempts at applying Bayesian statistics to language analysis included the work of Mosteller and Wallace (1963) in which an analysis of the words used in the Federalist papers was used to attempt to determine their authorship (concluding that Madison most likely authored the majority of the papers).[29]

In 1979 Terry Winograd developed an early natural language processing engine capable of interpreting naturally written commands within a simple rule governed environment. The primary language parsing program in this project was called SHRDLU, which was capable of carrying out a somewhat natural conversation with the user giving it commands, but only within the scope of the toy environment designed for the task. This environment consisted of different shaped and colored blocks, and SHRDLU was capable of interpreting commands such as "Find a block which is taller than the one you are holding and put it into the box." and asking questions such as "I don't understand which pyramid you mean." in response to the user's input.[30] While impressive, this kind of natural language processing has proven much more difficult outside the limited scope of the toy environment. Similarly a project developed by NASA called LUNAR was designed to provide answers to naturally written questions about the geological analysis of lunar rocks returned by the Apollo missions.[31] These kinds of problems are referred to as question answering.

Initial attempts at understanding spoken language were based on work done in the 1960s and 70s in signal modeling where an unknown signal is analyzed to look for patterns and to make predictions based on its history. An initial and somewhat successful approach to applying this kind of signal modeling to language was achieved with the use of hidden Marcov models as detailed by Rabiner in 1989.[32] This approach attempts to determine probabilities for the arbitrary number of models that could be being used in generating speech as well as modeling the probabilities for various words generated from each of these possible models. Similar approaches were employed in early speech recognition attempts starting in the late 70s at IBM using word/part-of-speech pair probabilities.[33]

More recently these kinds of statistical approaches have been applied to more difficult tasks such as topic identification using Bayesian parameter estimation to infer topic probabilities in text documents.[34]

Subfields

Computational linguistics can be divided into major areas depending upon the medium of the language being processed, whether spoken or textual; and upon the task being performed, whether analyzing language (recognition) or synthesizing language (generation).

Speech recognition and speech synthesis deal with how spoken language can be understood or created using computers. Parsing and generation are sub-divisions of computational linguistics dealing respectively with taking language apart and putting it together. Machine translation remains the sub-division of computational linguistics dealing with having computers translate between languages.

Some of the areas of research that are studied by computational linguistics include:

- Computational complexity of natural language, largely modeled on automata theory, with the application of context-sensitive grammar and linearly bounded Turing machines.

- Computational semantics comprises defining suitable logics for linguistic meaning representation, automatically constructing them and reasoning with them
- Computer-aided corpus linguistics
- Design of parsers or chunkers for natural languages
- Design of taggers like POS-taggers (part-of-speech taggers)
- Machine translation as one of the earliest and most difficult applications of computational linguistics draws on many subfields.
- Simulation and study of language evolution in historical linguistics/glottochronology.

The Association for Computational Linguistics defines computational linguistics as:

> ...the scientific study of language from a computational perspective. Computational linguists are interested in providing computational models of various kinds of linguistic phenomena.[35]

Counter-views

The status of Computational linguistics may be questioned from four perspectives: from the standpoints of philosophy of science, natural science (mismatch between human Cognitive domain and machine algorithms), Social Science Problem (Plurilingual condition and the advent of the simulated hyper-real cf. Baudrillard), Algocentricism (the discourse, which is only controlled or appropriated by the algorithm) in contrast with post-formal subjective and substantive task of Linguistics.

These problems are summarized as follows:

Philosophy of science problem: There is nothing called 'pen-paper-card linguistics', when these tools were used to taxonomize corpus. If computer manipulates linguistic data through the 'pen-paper-card' method, is it justified to label it as a separate epistemological discipline?

Natural science problem: This problem deals with the matching condition between 'human cognitive domain' and 'machine algorithm' (identity and difference between computer and human being) on the basis of Russel's paradox and Goedel's theorem Problem raised by Roger Penrose (1990,1994) and Searle's Chinese room puzzle. Computer's halting problem. Fuzziness of so- called natural language (The paper deals with some Bangla usages of numerals, where the status of number one is not always equal to one. The value of this fuzzy one is determined by the context, speakers socio-economic status etc.) Post-formalists deny the analytical procedures proposed by structralists. According to them, speaking subject perceives language as a whole (*gestalt*). Fragmenting language-object by deploying grammatical rules implies understanding symbolic order by means of another (meta-)symbolic order. A binary machine appropriates language-object according its own algorithmic program. It leads to a metonymic transformation of speaking subject as subjects' non-algorithmic capability is ignored.[36][37][38]

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External links

- Association for Computational Linguistics (ACL) (http://www.aclweb.org/)
  - ACL Anthology of research papers (http://www.aclweb.org/anthology)
  - ACL Wiki for Computational Linguistics (http://aclweb.org/aclwiki/)
- CICLing annual conferences on Computational Linguistics (http://www.CICLing.org/)
- Computational Linguistics – Applications workshop (http://CLA.imcsit.org)
- Free online introductory book on Computational Linguistics (http://web.archive.org/web/20080125103030/http://www.gelbukh.com/clbook/) (Internet Archive copy)
- Language Technology World (http://www.lt-world.org/)
- Resources for Text, Speech and Language Processing (http://www.cs.technion.ac.il/~gabr/resources/resources.html)

Dynamic logic (modal logic)

Dynamic logic is an extension of modal logic originally intended for reasoning about computer programs and later applied to more general complex behaviors arising in linguistics, philosophy, AI, and other fields.

Language

Modal logic is characterized by the modal operators $\Box p$ (box p) asserting that p is necessarily the case, and $\Diamond p$ (diamond p) asserting that p is possibly the case. Dynamic logic extends this by associating to every action a the modal operators $[a]$ and $\langle a\rangle$, thereby making it a multimodal logic. The meaning of $[a]p$ is that after performing action a it is necessarily the case that p holds, that is, a must bring about p. The meaning of $\langle a\rangle p$ is that after performing a it is possible that p holds, that is, a might bring about p. These operators are related by $[a]p \equiv \neg\langle a\rangle\neg p$ and $\langle a\rangle p \equiv \neg[a]\neg p$, analogously to the relationship between the universal ($\forall$) and existential ($\exists$) quantifiers.

Dynamic logic permits compound actions built up from smaller actions. While the basic control operators of any programming language could be used for this purpose, Kleene's regular expression operators are a good match to modal logic. Given actions a and b , the compound action $a \cup b$, *choice*, also written $a + b$ or $a|b$, is performed by performing one of a or b . The compound action $a;b$, *sequence*, is performed by performing first a and then b . The compound action $a*$, *iteration*, is performed by performing a zero or more times, sequentially. The constant action 0 or **BLOCK** does nothing and does not terminate, whereas the constant action 1 or **SKIP** or **NOP**, definable as $0*$, does nothing but does terminate.

Axioms

These operators can be axiomatized in dynamic logic as follows, taking as already given a suitable axiomatization of modal logic including such axioms for modal operators as the above-mentioned axiom $[a]p \equiv \neg\langle a\rangle\neg p$ and the two inference rules *modus ponens* ($\vdash p$ and $\vdash p \rightarrow q$ implies $\vdash q$) and *necessitation* ($\vdash p$ implies $\vdash [a]p$).

A1. $[0]p$

A2. $[1]p \equiv p$

A3. $[a \cup b]p \equiv [a]p \wedge [b]p$

A4. $[a;b]p \equiv [a][b]p$

A5. $[a*]p \equiv p \wedge [a][a*]p$

A6. $p \wedge [a*](p \rightarrow [a]p) \rightarrow [a*]p$

Axiom A1 makes the empty promise that when **BLOCK** terminates, p will hold, even if p is the proposition **false**. (Thus **BLOCK** abstracts the essence of the action of hell freezing over.)
A2 says that **[NOP]** acts as the identity function on propositions, that is, it transforms p into itself.
A3 says that if doing one of a or b must bring about p, then a must bring about p and likewise for b, and conversely.
A4 says that if doing a and then b must bring about p, then a must bring about a situation in which b must bring about p.
A5 is the evident result of applying A2, A3 and A4 to the equation $a* = 1 \cup a; a*$ of Kleene algebra.
A6 asserts that if p holds now, and no matter how often we perform a it remains the case that the truth of p after that performance entails its truth after one more performance of a, then p must remain true no matter how often we perform a. A6 is recognizable as mathematical induction with the action **n := n+1** of incrementing **n** generalized to arbitrary actions a.

Derivations

The modal logic axiom $[a]p \equiv \neg\langle a\rangle\neg p$ permits the derivation of the following six theorems corresponding to the above:
T1. $\neg\langle 0\rangle p$
T2. $\langle 1\rangle p \equiv p$
T3. $\langle a \cup b\rangle p \equiv \langle a\rangle p \vee \langle b\rangle p$
T4. $\langle a; b\rangle p \equiv \langle a\rangle\langle b\rangle p$
T5. $\langle a*\rangle p \equiv p \vee \langle a\rangle\langle a*\rangle p$
T6. $\langle a*\rangle p \rightarrow p \vee \langle a*\rangle(\neg p \wedge \langle a\rangle p)$
T1 asserts the impossibility of bringing anything about by performing **BLOCK**.
T2 notes again that **NOP** changes nothing, bearing in mind that **NOP** is both deterministic and terminating whence $[1]$ and $\langle 1\rangle$ have the same force.
T3 says that if the choice of a or b could bring about p, then either a or b alone could bring about p.
T4 is just like A4.
T5 is explained as for A5.
T6 asserts that if it is possible to bring about p by performing a sufficiently often, then either p is true now or it is possible to perform a repeatedly to bring about a situation where p is (still) false but one more performance of a could bring about p.
Box and diamond are entirely symmetric with regard to which one takes as primitive. An alternative axiomatization would have been to take the theorems T1-T6 as axioms, from which we could then have derived A1-A6 as theorems.

The difference between implication and inference is the same in dynamic logic as in any other logic: whereas the implication $p \rightarrow q$ asserts that if p is true then so is q, the inference $p \vdash q$ asserts that if p is valid then so is q. However the dynamic nature of dynamic logic moves this distinction out of the realm of abstract axiomatics into the common-sense experience of situations in flux. The inference rule $p \vdash [a]p$, for example, is sound because its premise asserts that p holds at all times, whence no matter where a might take us, p will be true there. The implication $p \rightarrow [a]p$ is not valid, however, because the truth of p at the present moment is no guarantee of its truth after performing a. For example, $p \rightarrow [a]p$ will be true in any situation where p is false, or in any situation where $[a]p$ is true, but the assertion $(x = 1) \rightarrow [x := x + 1](x = 1)$ is false in any situation where x has value 1, and therefore is not valid.

Derived rules of inference

As for modal logic, the inference rules *modus ponens* and *necessitation* suffice also for dynamic logic as the only primitive rules it needs, as noted above. However, as usual in logic, many more rules can be derived from these with the help of the axioms. An example instance of such a derived rule in dynamic logic is that if kicking a broken TV once can't possibly fix it, then repeatedly kicking it can't possibly fix it either. Writing k for the action of kicking the TV, and b for the proposition that the TV is broken, dynamic logic expresses this inference as $b \to [k]b \vdash b \to [k*]b$, having as premise $b \to [k]b$ and as conclusion $b \to [k*]b$. The meaning of $[k]b$ is that it is guaranteed that after kicking the TV, it is broken. Hence the premise $b \to [k]b$ means that if the TV is broken, then after kicking it once it will still be broken. $k*$ denotes the action of kicking the TV zero or more times. Hence the conclusion $b \to [k*]b$ means that if the TV is broken, then after kicking it zero or more times it will still be broken. For if not, then after the second-to-last kick the TV would be in a state where kicking it once more would fix it, which the premise claims can never happen under any circumstances.

The inference $b \to [k]b \vdash b \to [k*]b$ is sound. However the implication $(b \to [k]b) \to (b \to [k*]b)$ is not valid because we can easily find situations in which $b \to [k]b$ holds but $b \to [k*]b$ does not. In any such counterexample situation, b must hold but $[k*]b$ must be false, while $[k]b$ however must be true. But this could happen in any situation where the TV is broken but can be revived with two kicks. The implication fails (is not valid) because it only requires that $b \to [k]b$ hold now, whereas the inference succeeds (is sound) because it requires that $b \to [k]b$ hold in all situations, not just the present one.

An example of a valid implication is the proposition $(x \geq 3) \to [x := x+1](x \geq 4)$. This says that if x is greater or equal to 3, then after incrementing x , x must be greater or equal to 4. In the case of deterministic actions a that are guaranteed to terminate, such as $x := x+1$, *must* and *might* have the same force, that is, $[a]$ and $\langle a \rangle$ have the same meaning. Hence the above proposition is equivalent to $(x \geq 3) \to \langle x := x+1 \rangle (x \geq 4)$ asserting that if x is greater or equal to 3 then after performing $x := x+1$, x might be greater or equal to 4.

Assignment

The general form of an assignment statement is $x := e$ where x is a variable and e is an expression built from constants and variables with whatever operations are provided by the language, such as addition and multiplication. The Hoare axiom for assignment is not given as a single axiom but rather as an axiom schema.

A7. $[x := e]\Phi(x) \equiv \Phi(e)$

This is a schema in the sense that $\Phi(x)$ can be instantiated with any formula Φ containing zero or more instances of a variable x . The meaning of $\Phi(e)$ is Φ with those occurrences of x that occur free in Φ, i.e. not bound by some quantifier as in $\forall x$, replaced by e. For example we may instantiate A7 with $[x := e](x = y^2) \equiv e = y^2$, or with $[x := e](b = c + x) \equiv b = c + e$. Such an axiom schema allows infinitely many axioms having a common form to be written as a finite expression connoting that form.

The instance $[x := x+1](x \geq 4) \equiv (x+1) \geq 4$ of A7 allows us to calculate mechanically that the example $[x := x+1]x \geq 4$ encountered a few paragraphs ago is equivalent to $(x+1) \geq 4$, which in turn is equivalent to $x \geq 3$ by elementary algebra.

An example illustrating assignment in combination with $*$ is the proposition $\langle (x := x+1)* \rangle x = 7$. This asserts that it is possible, by incrementing x sufficiently often, to make x equal to 7. This of course is not always true, e.g. if x is 8 to begin with, or 6.5, whence this proposition is not a theorem of dynamic logic. If x is of type integer however, then this proposition is true if and only if x is at most 7 to begin with, that is, it is just a roundabout way of saying $x \leq 7$.

Mathematical induction can be obtained as the instance of A6 in which the proposition p is instantiated as $\Phi(n)$, the action a as $n := n+1$, and n as 0. The first two of these three instantiations are straightforward, converting A6 to $(\Phi(n) \wedge [(n := n+1)*](\Phi(n) \to [n := n+1]\Phi(n))) \to [(n := n+1)*]\Phi(n)$.

However, the ostensibly simple substitution of 0 for n is not so simple as it brings out the so-called *referential opacity* of modal logic in the case when a modality can interfere with a substitution.

When we substituted $\Phi(n)$ for p, we were thinking of the proposition symbol p as a rigid designator with respect to the modality $[n := n+1]$, meaning that it is the same proposition after incrementing n as before, even though incrementing n may impact its truth. Likewise, the action a is still the same action after incrementing n, even though incrementing n will result in its executing in a different environment. However, n itself is not a rigid designator with respect to the modality $[n := n+1]$; if it denotes 3 before incrementing n, it denotes 4 after. So we can't just substitute 0 for n everywhere in A6.

One way of dealing with the opacity of modalities is to eliminate them. To this end, expand $[(n := n+1)*]\Phi(n)$ as the infinite conjunction $[(n := n+1)^0]\Phi(n) \wedge [(n := n+1)^1]\Phi(n) \wedge [(n := n+1)^2]\Phi(n) \wedge \ldots$, that is, the conjunction over all i of $[(n := n+1)^i]\Phi(n)$. Now apply A4 to turn $[(n := n+1)^i]\Phi(n)$ into $[n := n+1][n := n+1]\ldots\Phi(n)$, having i modalities. Then apply Hoare's axiom i times to this to produce $\Phi(n+i)$, then simplify this infinite conjunction to $\forall i\Phi(n+i)$. This whole reduction should be applied to both instances of $[(n := n+1)*]$ in A6, yielding $(\Phi(n) \wedge \forall i(\Phi(n+i) \to [n := n+1]\Phi(n+i))) \to \forall i\Phi(n+i)$. The remaining modality can now be eliminated with one more use of Hoare's axiom to give $(\Phi(n) \wedge \forall i(\Phi(n+i) \to \Phi(n+i+1))) \to \forall i\Phi(n+i)$.

With the opaque modalities now out of the way, we can safely substitute 0 for n in the usual manner of first-order logic to obtain Peano's celebrated axiom $(\Phi(0) \wedge \forall i(\Phi(i) \to \Phi(i+1))) \to \forall i\Phi(i)$, namely mathematical induction.

One subtlety we glossed over here is that $\forall i$ should be understood as ranging over the natural numbers, where i is the superscript in the expansion of $a*$ as the union of a^i over all natural numbers i. The importance of keeping this typing information straight becomes apparent if n had been of type *integer*, or even *real*, for any of which A6 is perfectly valid as an axiom. As a case in point, if n is a real variable and $\Phi(n)$ is the predicate n *is a natural number*, then axiom A6 after the first two substitutions, that is, $(\Phi(n) \wedge \forall i(\Phi(n+i) \to \Phi(n+i+1))) \to \forall i\Phi(n+i)$, is just as valid, that is, true in every state regardless of the value of n in that state, as when n is of type *natural number*. If in a given state n is a natural number, then the antecedent of the main implication of A6 holds, but then $n+i$ is also a natural number so the consequent also holds. If n is not a natural number, then the antecedent is false and so A6 remains true regardless of the truth of the consequent. We could strengthen A6 to an equivalence $p \wedge [a*](p \to [a]p) \equiv [a*]p$ without impacting any of this, the other direction being provable from A5, from which we see that if the antecedent of A6 does happen to be false somewhere, then the consequent *must* be false.

Test

Dynamic logic associates to every proposition p an action $p?$ called a test. When p holds, the test $p?$ acts as a **NOP**, changing nothing while allowing the action to move on. When p is false, $p?$ acts as **BLOCK**. Tests can be axiomatized as follows.

A8. $[p?]q \equiv p \to q$

The corresponding theorem for $\langle p?\rangle$ is:

T8. $\langle p?\rangle q \equiv p \wedge q$

The construct **if p then a else b** is realized in dynamic logic as $(p?;a) \cup (\neg p?;b)$. This action expresses a guarded choice: if p holds then $p?;a$ is equivalent to a, whereas $\neg p?;b$ is equivalent to BLOCK, and $a \cup 0$ is equivalent to a. Hence when p is true the performer of the action can only take the left branch, and when p is false the right.

The construct **while p do a** is realized as $(p?;a)*;\neg p?$. This performs $p?;a$ zero or more times and then performs $\neg p?$. As long as p remains true, the $\neg p?$ at the end blocks the performer from terminating the iteration prematurely, but as soon as it becomes false, further iterations of the body p are blocked and the performer then has no choice but to exit via the test $\neg p?$.

Quantification as random assignment

The random-assignment statement $x := ?$ denotes the nondeterministic action of setting x to an arbitrary value. $[x := ?]p$ then says that p holds no matter what you set x to, while $\langle x := ?\rangle p$ says that it is possible to set x to a value that makes p true. $[x := ?]$ thus has the same meaning as the universal quantifier $\forall x$, while $\langle x := ?\rangle$ similarly corresponds to the existential quantifier $\exists x$. That is, first-order logic can be understood as the dynamic logic of programs of the form $x := ?$.

Possible-world semantics

Modal logic is most commonly interpreted in terms of possible world semantics or Kripke structures. This semantics carries over naturally to dynamic logic by interpreting worlds as states of a computer in the application to program verification, or states of our environment in applications to linguistics, AI, etc. One role for possible world semantics is to formalize the intuitive notions of truth and validity, which in turn permit the notions of soundness and completeness to be defined for axiom systems. An inference rule is sound when validity of its premises implies validity of its conclusion. An axiom system is sound when all its axioms are valid and its inference rules are sound. An axiom system is complete when every valid formula is derivable as a theorem of that system. These concepts apply to all systems of logic including dynamic logic.

Propositional dynamic logic (PDL)

Ordinary or first-order logic has two types of terms, respectively assertions and data. As can be seen from the examples above, dynamic logic adds a third type of term denoting actions. The dynamic logic assertion $[x := x+1](x \geq 4)$ contains all three types: x, $x+1$, and 4 are data, $x := x+1$ is an action, and $x \geq 4$ and $[x := x+1](x \geq 4)$ are assertions. Propositional logic is derived from first-order logic by omitting data terms and reasons only about abstract propositions, which may be simple propositional variables or atoms or compound propositions built with such logical connectives as *and*, *or*, and *not*.

Propositional dynamic logic, or PDL, was derived from dynamic logic in 1977 by Michael J. Fischer and Richard Ladner. PDL blends the ideas behind propositional logic and dynamic logic by adding actions while omitting data; hence the terms of PDL are actions and propositions. The TV example above is expressed in PDL whereas the next example involving $x := x+1$ is in first-order DL. PDL is to (first-order) dynamic logic as propositional logic is to first-order logic.

Fischer and Ladner showed in their 1977 paper that PDL satisfiability was of computational complexity at most nondeterministic exponential time, and at least deterministic exponential time in the worst case. This gap was closed in 1978 by Vaughan Pratt who showed that PDL was decidable in deterministic exponential time. In 1977, Krister Segerberg proposed a complete axiomatization of PDL, namely any complete axiomatization of modal logic K together with axioms A1-A6 as given above. Completeness proofs for Segerberg's axioms were found by Gabbay (unpublished note), Parikh (1978), Pratt (1979), and Kozen and Parikh (1981).

History

Dynamic logic was developed by Vaughan Pratt in 1974 in notes for a class on program verification as an approach to assigning meaning to Hoare logic by expressing the Hoare formula $p\{a\}q$ as $p \to [a]q$. The approach was later published in 1976 as a logical system in its own right. The system parallels A. Salwicki's system of Algorithmic Logic and Edsger Dijkstra's notion of weakest-precondition predicate transformer $wp(a,p)$, with $[a]p$ corresponding to Dijkstra's $wlp(a,p)$, weakest liberal precondition. Those logics however made no connection with either modal logic, Kripke semantics, regular expressions, or the calculus of binary relations; dynamic logic therefore can be viewed as a refinement of algorithmic logic and Predicate Transformers that connects them up to the axiomatics and Kripke semantics of modal logic as well as to the calculi of binary relations and regular expressions.

The Concurrency Challenge

Hoare logic, algorithmic logic, weakest preconditions, and dynamic logic are all well suited to discourse and reasoning about sequential behavior. Extending these logics to concurrent behavior however has proved problematic. There are various approaches but all of them lack the elegance of the sequential case. In contrast Amir Pnueli's 1977 system of temporal logic, another variant of modal logic sharing many common features with dynamic logic, differs from all of the above-mentioned logics by being what Pnueli has characterized as an "endogenous" logic, the others being "exogenous" logics. By this Pnueli meant that temporal logic assertions are interpreted within a universal behavioral framework in which a single global situation changes with the passage of time, whereas the assertions of the other logics are made externally to the multiple actions about which they speak. The advantage of the endogenous approach is that it makes no fundamental assumptions about what causes what as the environment changes with time. Instead a temporal logic formula can talk about two unrelated parts of a system, which because they are unrelated tacitly evolve in parallel. In effect ordinary logical conjunction of temporal assertions is the concurrent composition operator of temporal logic. The simplicity of this approach to concurrency has resulted in temporal logic being the modal logic of choice for reasoning about concurrent systems with its aspects of synchronization, interference, independence, deadlock, livelock, fairness, etc.

These concerns of concurrency would appear to be less central to linguistics, philosophy, and artificial intelligence, the areas in which dynamic logic is most often encountered nowadays.

For a comprehensive treatment of dynamic logic see the book by David Harel et al. cited below.

References

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- David Harel, "Dynamic Logic", In D. Gabbay and F. Guenthner, editors, Handbook of Philosophical Logic, volume II: Extensions of Classical Logic, chapter 10, pages 497-604. Reidel, Dordrecht, 1984.

External links

- Semantical Considerations on Floyd-Hoare Logic [1] (original paper on dynamic logic)

References

[1] http://boole.stanford.edu/pub/semcon.pdf

Rhetoric

Painting depicting a lecture in a knight academy, painted by Pieter Isaacsz or Reinhold Timm for Rosenborg Castle as part of a series of seven paintings depicting the seven independent arts. This painting illustrates rhetorics

Rhetoric is the art of discourse, an art that aims to improve the facility of speakers or writers who attempt to inform, persuade, or motivate particular audiences in specific situations.[1] As a subject of formal study and a productive civic practice, rhetoric has played a central role in the Western tradition.[2] Its best known definition comes from Aristotle, who considers it a counterpart of both logic and politics, and calls it "the faculty of observing in any given case the available means of persuasion."[3] Rhetorics typically provide heuristics for understanding, discovering, and developing arguments for particular situations, such as Aristotle's three persuasive audience appeals, logos, pathos, and ethos. The five canons of rhetoric, which trace the traditional tasks in designing a persuasive speech, were first codified in classical Rome, invention, arrangement, style, memory, and delivery. Along with grammar and logic (or dialectic – see Martianus Capella), rhetoric is one of the three ancient arts of discourse.

From ancient Greece to the late 19th century, it was a central part of Western education, filling the need to train public speakers and writers to move audiences to action with arguments.[4] The word is derived from the Greek ῥητορικός (*rhētorikós*), "oratorical",[5] from ῥήτωρ (*rhḗtōr*), "public speaker",[6] related to ῥῆμα (*rhêma*), "that which is said or spoken, word, saying",[7] and ultimately derived from the verb λέγω (*loqui*), "to speak, say".[8]

Uses of rhetoric

The scope of rhetoric

Scholars have debated the scope of rhetoric since ancient times. Although some have limited rhetoric to the specific realm of political discourse, many modern scholars liberate it to encompass every aspect of culture. Contemporary studies of rhetoric address a more diverse range of domains than was the case in ancient times. While classical rhetoric trained speakers to be effective persuaders in public forums and institutions such as courtrooms and assemblies, contemporary rhetoric investigates human discourse writ large. Rhetoricians have studied the discourses of a wide variety of domains, including the natural and social sciences, fine art, religion, journalism, digital media, fiction, history, cartography, and architecture, along with the more traditional domains of politics and the law.[9] Many contemporary approaches treat rhetoric as human communication that includes purposeful and strategic manipulation of symbols. Public relations, lobbying, law, marketing, professional and technical writing, and advertising are modern professions that employ rhetorical practitioners.

Because the ancient Greeks highly valued public political participation, rhetoric emerged as a crucial tool to influence politics. Consequently, rhetoric remains associated with its political origins. However, even the original instructors of Western speech—the Sophists—disputed this limited view of rhetoric. According to the Sophists, such as Gorgias, a successful rhetorician could speak convincingly on any topic, regardless of his experience in that field. This method suggested rhetoric could be a means of communicating any expertise, not just politics. In his *Encomium to Helen*, Gorgias even applied rhetoric to fiction by seeking for his own pleasure to prove the blamelessness of the mythical Helen of Troy in starting the Trojan War.[10]

Looking to another key rhetorical theorist, Plato defined the scope of rhetoric according to his negative opinions of the art. He criticized the Sophists for using rhetoric as a means of deceit instead of discovering truth. In "Gorgias," one of his Socratic Dialogues, Plato defines rhetoric as the persuasion of ignorant masses within the courts and assemblies.[11] Rhetoric, in Plato's opinion, is merely a form of flattery and functions similarly to cookery, which masks the undesirability of unhealthy food by making it taste good. Thus, Plato considered any speech of lengthy prose aimed at flattery as within the scope of rhetoric.

Aristotle both redeemed rhetoric from his teacher and narrowed its focus by defining three genres of rhetoric—deliberative, forensic or judicial, and epideictic.[12] Yet, even as he provided order to existing rhetorical theories, Aristotle extended the definition of rhetoric, calling it the ability to identify the appropriate means of persuasion in a given situation, thereby making rhetoric applicable to all fields, not just politics. When one considers that rhetoric included torture (in the sense that the practice of torture is a form of persuasion or coercion), it is clear that rhetoric cannot be viewed only in academic terms. However, the enthymeme based upon logic (especially, based upon the syllogism) was viewed as the basis of rhetoric. However, since the time of Aristotle, logic has also changed, for example, Modal logic has undergone a major development which also modifies rhetoric.[13] Yet, Aristotle also outlined generic constraints that focused the rhetorical art squarely within the domain of public political practice. He restricted rhetoric to the domain of the contingent or probable: those matters that admit multiple legitimate opinions or arguments.

The contemporary neo-Aristotelian and neo-Sophistic positions on rhetoric mirror the division between the Sophists and Aristotle. Neo-Aristotelians generally study rhetoric as political discourse, while the neo-Sophistic view contends that rhetoric cannot be so limited. Rhetorical scholar Michael Leff characterizes the conflict between these positions as viewing rhetoric as a "thing contained" versus a "container." The neo-Aristotelian view threatens the study of rhetoric by restraining it to such a limited field, ignoring many critical applications of rhetorical theory, criticism, and practice. Simultaneously, the neo-Sophists threaten to expand rhetoric beyond a point of coherent theoretical value.

Over the past century, people studying rhetoric have tended to enlarge its object domain beyond speech texts. Kenneth Burke asserted humans use rhetoric to resolve conflicts by identifying shared characteristics and interests in symbols. By nature, humans engage in identification, either to identify themselves or another individual with a group. This definition of rhetoric as identification broadened the scope from strategic and overt political persuasion to the more implicit tactics of identification found in an immense range of sources.[14]

Among the many scholars who have since pursued Burke's line of thought, James Boyd White sees rhetoric as a broader domain of social experience in his notion of constitutive rhetoric. Influenced by theories of social construction, White argues that culture is "reconstituted" through language. Just as language influences people, people influence language. Language is socially constructed, and depends on the meanings people attach to it. Because language is not rigid and changes depending on the situation, the very usage of language is rhetorical. An author, White would say, is always trying to construct a new world and persuading his or her readers to share that world within the text.[15]

Individuals engage in the rhetorical process anytime they speak or produce meaning. Even in the field of science, the practices of which were once viewed as being merely the objective testing and reporting of knowledge, scientists must persuade their audience to accept their findings by sufficiently demonstrating that their study or experiment was conducted reliably and resulted in sufficient evidence to support their conclusions.

The vast scope of rhetoric is difficult to define; however, political discourse remains, in many ways, the paradigmatic example for studying and theorizing specific techniques and conceptions of persuasion, considered by many a synonym for "rhetoric."[16]

Rhetoric as a civic art

Throughout European History, rhetoric has concerned itself with persuasion in public and political settings such as assemblies and courts. Because of its associations with democratic institutions, rhetoric is commonly said to flourish in open and democratic societies with rights of free speech, free assembly, and political enfranchisement for some portion of the population. Those who classify rhetoric as a civic art believe that rhetoric has the power to shape communities, form the character of citizens and greatly impact civic life.

Rhetoric was viewed as a civic art by several of the ancient philosophers. Aristotle and Isocrates were two of the first to see rhetoric in this light. In his work, *Antidosis*, Isocrates states, "we have come together and founded cities and made laws and invented arts; and, generally speaking, there is not institution devised by man which the power of speech has not helped us to establish". With this statement he argues that rhetoric is a fundamental part of civic life in every society and that it has been necessary in the foundation of all aspects of society. He further argues in his piece *Against the Sophists* that rhetoric, although it cannot be taught to just anyone, is capable of shaping the character of man. He writes, "I do think that the study of political discourse can help more than any other thing to stimulate and form such qualities of character". Aristotle, writing several years after Isocrates, supported many of his arguments and continued to make arguments for rhetoric as a civic art.

In the words of Aristotle, in his essay *Rhetoric*, rhetoric is "the faculty of observing in any given case the available means of persuasion". According to Aristotle, this art of persuasion could be used in public settings in three different ways. He writes in Book I, Chapter III, "A member of the assembly decides about future events, a juryman about past events: while those who merely decide on the orator's skill are observers. From this it follows that there are three divisions of oratory- (1) political, (2) forensic, and (3) the ceremonial oratory of display". Eugene Garver, in his critique of "Aristotle's Rhetoric", confirms that Aristotle viewed rhetoric as a civic art. Garver writes, "Rhetoric articulates a civic art of rhetoric, combining the almost incompatible properties of techne and appropriateness to citizens".[17] Each of Aristotle's divisions plays a role in civic life and can be used in a different way to impact cities.

Because rhetoric is a public art capable of shaping opinion, some of the ancients including Plato found fault in it. They claimed that while it could be used to improve civic life, it could be used equally easily to deceive or manipulate with negative effects on the city. The masses were incapable of analyzing or deciding anything on their own and would therefore be swayed by the most persuasive speeches. Thus, civic life could be controlled by the one who could deliver the best speech. Plato's explores the problematic moral status of rhetoric twice: in *Gorgias*, a dialogue named for the famed Sophist, and in *The Phaedrus*, a dialogue best known for its commentary on love.

More trusting in the power of rhetoric to support a republic, the Roman orator Cicero argued that art required something more than eloquence. A good orator needed also to be a good man, a person enlightened on a variety of civic topics. He describes the proper training of the orator in his major text on rhetoric, *De Oratore*, modeled on Plato's dialogues.

Modern day works continue to support the claims of the ancients that rhetoric is an art capable of influencing civic life. In his work *Political Style*, Robert Hariman claims, "Furthermore, questions of freedom, equality, and justice often are raised and addressed through performances ranging from debates to demonstrations without loss of moral content".[18] James Boyd White argues further that rhetoric is capable not only of addressing issues of political interest but that it can influence culture as a whole. In his book, *When Words Lose Their Meaning*, he argues that words of persuasion and identification define community and civic life. He states that words produce "the methods by which culture is maintained, criticized, and transformed".[19] Both White and Hariman agree that words and rhetoric have the power to shape culture and civic life.

In modern times, rhetoric has consistently remained relevant as a civic art. In speeches, as well as in non-verbal forms, rhetoric continues to be used as a tool to influence communities from local to national levels.

Rhetoric as a course of study

Rhetoric as a course of study has evolved significantly since its ancient beginnings. Through the ages, the study and teaching of rhetoric has adapted to the particular exigencies of the time and venue.[20] The study of rhetoric has conformed to a multitude of different applications, ranging from architecture to literature.[21] Although the curriculum has transformed in a number of ways, it has generally emphasized the study of principles and rules of composition as a means for moving audiences. Generally speaking, the study of rhetoric trains students to speak and/or write effectively, as well as critically understand and analyze discourse.

Rhetoric began as a civic art in Ancient Greece where students were trained to develop tactics of oratorical persuasion, especially in legal disputes. Rhetoric originated in a school of pre-Socratic philosophers known as the Sophists circa 600 BC. Demosthenes and Lysias emerged as major orators during this period, and Isocrates and Gorgias as prominent teachers. Rhetorical education focused on five particular canons: inventio (invention), dispositio (arrangement), elocutio (style), memoria (memory), and actio (delivery). Modern teachings continue to reference these rhetorical leaders and their work in discussions of classical rhetoric and persuasion.

Rhetoric was later taught in universities during the Middle Ages as one of the three original liberal arts or trivium (along with logic and grammar).[22] During the medieval period, political rhetoric declined as republican oratory died out and the emperors of Rome garnered increasing authority. With the rise of European monarchs in following centuries, rhetoric shifted into the courtly and religious applications. Augustine exerted strong influence on Christian rhetoric in the Middle Ages, advocating the use of rhetoric to lead audiences to truth and understanding, especially in the church. The study of liberal arts, he believed, contributed to rhetorical study: "In the case of a keen and ardent nature, fine words will come more readily through reading and hearing the eloquent than by pursuing the rules of rhetoric."[23] Poetry and letter writing, for instance, became a central component of rhetorical study during the Middle Ages.[24] After the fall of the Republic in Rome, poetry became a tool for rhetorical training since there were fewer opportunities for political speech.[25] Letter writing was the primary form through which business was conducted both in state and church, so it became an important aspect of rhetorical education.[26]

Rhetorical education became more restrained as style and substance separated in 16th-century France with Peter Ramus, and attention turned to the scientific method. That is, influential scholars like Ramus argued that the processes of invention and arrangement should be elevated to the domain of philosophy, while rhetorical instruction should be chiefly concerned with the use of figures and other forms of the ornamentation of language. Scholars such as Francis Bacon developed the study of "scientific rhetoric."[27] This concentration rejected the elaborate style characteristic of the classical oration. This plain language carried over to John Locke's teaching, which emphasized concrete knowledge and steered away from ornamentation in speech, further alienating rhetorical instruction, which was identified wholly with this ornamentation, from the pursuit of knowledge.

In the 18th century, rhetoric assumed a more social role, initiating the creation of new education systems. "Elocution schools" arose (predominantly in England) in which females analyzed classic literature, most notably the works of William Shakespeare, and discussed pronunciation tactics.[28]

The study of rhetoric underwent a revival with the rise of democratic institutions during the late 18th and early 19th centuries. Scotland's author and theorist Hugh Blair served as a key leader of this movement during the late 18th century. In his most famous work "Lectures on Rhetoric and Belles Lettres", he advocates rhetorical study for common citizens as a resource for social success. Many American colleges and secondary schools used Blair's text throughout the 19th century to train students of rhetoric.[29]

Political rhetoric also underwent renewal in the wake of the US and French revolutions. The rhetorical studies of ancient Greece and Rome were resurrected in the studies of the era as speakers and teachers looked to Cicero and others to inspire defense of the new republic. Leading rhetorical theorists included John Quincy Adams of Harvard who advocated the democratic advancement of rhetorical art. Harvard's founding of the Boylston Professorship of Rhetoric and Oratory sparked the growth of rhetorical study in colleges across the United States.[26] Harvard's rhetoric program drew inspiration from literary sources to guide organization and style.

Debate clubs and lyceums also developed as forums in which common citizens could hear speakers and sharpen debate skills. The American lyceum in particular was seen as both an educational and social institution, featuring group discussions and guest lecturers.[30] These programs cultivated democratic values and promoted active participation in political analysis.

Throughout the 20th century, rhetoric developed as a concentrated field of study with the establishment of rhetorical courses in high schools and universities. Courses such as public speaking and speech analysis apply fundamental Greek theories (such as the modes of persuasion: ethos, pathos, and logos) as well as trace rhetorical development throughout the course of history. Rhetoric has earned a more esteemed reputation as a field of study with the emergence of Communication Studies departments in university programs and in conjunction with the linguistic turn. Rhetorical study has broadened in scope, and is especially utilized by the fields of marketing, politics, and literature.

Rhetoric, as an area of study, is concerned with how humans use symbols, especially language, to reach agreement that permits coordinated effort of some sort.[31] Harvard University, the first university in the United States, based on the European model, taught a basic curriculum, including rhetoric. Rhetoric, in this sense, how to properly give speeches, played an important role in their training. Rhetoric was soon taught in departments of English as well.[32]

Epistemology

The relationship between rhetoric and knowledge is one of the oldest and most interesting problems. The contemporary stereotype of rhetoric as "empty speech" or "empty words" reflects a radical division of rhetoric from knowledge, a division that has had influential adherents within the rhetorical tradition, most notably Plato and Peter Ramus. It is a division that has been strongly associated with Enlightenment thinking about language, which attempted to make language a neutral, transparent medium. A philosophical argument has ensued for centuries about whether or not rhetoric and truth have any correlation to one another. In ancient Greece, the sophists generally believed that humans were incapable of determining truth but used logos to determine what was best (or worst) for the community. Sophists like Protagoras put great emphasis on speech as a means that could help in making these decisions for the community.

However, Plato was critical of the sophists' views because he believed that rhetoric was simply too dangerous, being based in skill and common opinion (doxa). Plato set out to instead discover episteme, or "truth," through the dialectical method. Since Plato's argument has shaped western philosophy, rhetoric has mainly been regarded as an evil that has no epistemic status.

Over the 20th century, with the influence of social constructionism and pragmatism, this tradition began to change. Robert L. Scott states that rhetoric is, in fact, epistemic.[33] His argument is based on the belief that truth is not a central, objective set of facts but that truth is based on the situation at hand. Scott goes as far as stating that if a man believes in an ultimate truth and argues it, he is only fooling himself by convincing himself of one argument among many possible options. Ultimately, truth is relative to situated experiences, and rhetoric is necessary to give meaning to individual circumstances. Researchers in the rhetoric of science, have shown how the two are difficult to separate, and how discourse helps to create knowledge. This perspective is often called "epistemic rhetoric", where communication among interlocutors is fundamental to the creation of knowledge in communities.

Truth has also been theorized as a mutual agreement amongst the community. Academics like Thomas Farrell discuss the importance of social consensus as knowledge.[34] Furthermore, Brummett points out, "A worldview in which truth is agreement must have rhetoric at its heart, for agreement is gained in no other way."[35] So, if one agrees with the statement that truth is mutual agreement, truth must be relative and necessarily arise in persuasion. Emphasizing this close relationship between discourse and knowledge, contemporary rhetoricians have been associated with a number of philosophical and social scientific theories that see language and discourse as central to, rather than in conflict with, knowledge-making (see critical theory, post-structuralism, hermeneutics, dramatism, reflexivity).

History

Rhetoric has its origins in the earliest civilization, Mesopotamia.[36] Some of the earliest examples of rhetoric can be found in the Akkadian writings of the princess and priestess Enheduanna (ca. 2285-2250 BC),[37] while later examples can be found in the Neo-Assyrian Empire during the time of Sennacherib (704–681 BC).[38] In ancient Egypt, rhetoric has existed since at least the Middle Kingdom period (ca. 2080-1640 BC). The Egyptians held eloquent speaking in high esteem, and it was a skill that had a very high value in their society. The "Egyptian rules of rhetoric" also clearly specified that "knowing when not to speak is essential, and very respected, rhetorical knowledge." Their "approach to rhetoric" was thus a "balance between eloquence and wise silence." Their rules of speech also strongly emphasized "adherence to social behaviors that support a conservative status quo" and they held that "skilled speech should support, not question, society."[39] In ancient China, rhetoric dates back to the Chinese philosopher, Confucius (551-479 BC), and continued with later followers. The tradition of Confucianism emphasized the use of eloquence in speaking.[40] The use of rhetoric can also be found in the ancient Biblical tradition.[41]

In ancient Greece, the earliest mention of oratorical skill occurs in Homer's *Iliad*, where heroes like Achilles, Hektor, and Odysseus were honored for their ability to advise and exhort their peers and followers (the *Laos* or army) in wise and appropriate action. With the rise of the democratic *polis*, speaking skill was adapted to the needs of the public and political life of cities in ancient Greece, much of which revolved around the use of oratory as the medium through which political and judicial decisions were made, and through which philosophical ideas were developed and disseminated. For modern students today, it can be difficult to remember that the wide use and availability of written texts is a phenomenon that was just coming into vogue in Classical Greece. In Classical times, many of the great thinkers and political leaders performed their works before an audience, usually in the context of a competition or contest for fame, political influence, and cultural capital; in fact, many of them are known only through the texts that their students, followers, or detractors wrote down. As has already been noted, *rhetor* was the Greek term for *orator:* A *rhetor* was a citizen who regularly addressed juries and political assemblies and who was thus understood to have gained some knowledge about public speaking in the process, though in general facility with language was often referred to as *logôn techne*, "skill with arguments" or "verbal artistry."[42]

Rhetoric thus evolved as an important art, one that provided the orator with the forms, means, and strategies for persuading an audience of the correctness of the orator's arguments. Today the term *rhetoric* can be used at times to refer only to the form of argumentation, often with the pejorative connotation that rhetoric is a means of obscuring the truth. Classical philosophers believed quite the contrary: the skilled use of rhetoric was essential to the discovery of truths, because it provided the means of ordering and clarifying arguments.

The Sophists

In Europe, organized thought about public speaking began in ancient Greece.[43] Possibly, the first study about the power of language may be attributed to the philosopher Empedocles (d. ca. 444 BC), whose theories on human knowledge would provide a basis for many future rhetoricians. The first written manual is attributed to Corax and his pupil Tisias. Their work, as well as that of many of the early rhetoricians, grew out of the courts of law; Tisias, for example, is believed to have written judicial speeches that others delivered in the courts. Teaching in oratory was popularized in the 5th century BC by itinerant teachers known as sophists, the best known of whom were Protagoras (c.481-420 BC), Gorgias (c.483-376 BC), and Isocrates (436-338 BC). The Sophists were a disparate group who travelled from city to city, teaching in public places to attract students and offer them an education. Their central focus was on logos or what we might broadly refer to as discourse, its functions and powers. They defined parts of speech, analyzed poetry, parsed close synonyms, invented argumentation strategies, and debated the nature of reality. They claimed to make their students "better," or, in other words, to teach virtue. They thus claimed that human "excellence" was not an accident of fate or a prerogative of noble birth, but an art or "*techne*" that could be taught and learned. They were thus among the first humanists. Several sophists also questioned received wisdom about the gods and the Greek culture, which they believed was taken for granted by Greeks of their time, making them among the first agnostics. For example, they argued that cultural practices were a function of convention or *nomos* rather than blood or birth or *phusis*. They argued even further that morality or immorality of any action could not be judged outside of the cultural context within which it occurred. The well-known phrase, "Man is the measure of all things" arises from this belief. One of their most famous, and infamous, doctrines has to do with probability and counter arguments. They taught that every argument could be countered with an opposing argument, that an argument's effectiveness derived from how "likely" it appeared to the audience (its probability of seeming true), and that any probability argument could be countered with an inverted probability argument. Thus, if it seemed likely that a strong, poor man were guilty of robbing a rich, weak man, the strong poor man could argue, on the contrary, that this very likelihood (that he would be a suspect) makes it unlikely that he committed the crime, since he would most likely be apprehended for the crime. They also taught and were known for their ability to make the weaker (or worse) argument the stronger (or better). Aristophanes famously parodies the clever inversions that sophists were known for in his play *The Clouds*.

The word "sophistry" developed strong negative connotations in ancient Greece that continue today, but in ancient Greece sophists were nevertheless popular and well-paid professionals, widely respected for their abilities but also widely criticized for their excesses.

Isocrates

Isocrates (436-338 BC), like the sophists, taught public speaking as a means of human improvement, but he worked to distinguish himself from the Sophists, whom he saw as claiming far more than they could deliver. He suggested that while an art of virtue or excellence did exist, it was only one piece, and the least, in a process of self-improvement that relied much more heavily on native talent and desire, constant practice, and the imitation of good models. Isocrates believed that practice in speaking publicly about noble themes and important questions would function to improve the character of both speaker and audience while also offering the best service to a city. In fact, Isocrates was an outspoken champion of rhetoric as a mode of civic engagement.[44] He thus wrote his speeches as "models" for his students to imitate in the same way that poets might imitate Homer or Hesiod, seeking to inspire in them a desire to attain fame through civic leadership. His was the first permanent school in Athens and it is likely that Plato's Academy and Aristotle's Lyceum were founded in part as a response to Isocrates. Though he left no handbooks, his speeches (*"Antidosis"* and *"Against the Sophists"* are most relevant to students of rhetoric) became models of oratory (he was one of the canonical "Ten Attic Orators") and keys to his entire educational program. He had a marked influence on Cicero and Quintilian, and through them, on the entire educational system of the west.

Plato

Plato (427-347 BC) famously outlined the differences between true and false rhetoric in a number of dialogues; particularly the *Gorgias* and *Phaedrus* wherein Plato disputes the sophistic notion that the art of persuasion (the sophists' art which he calls "rhetoric"), can exist independent of the art of dialectic. Plato claims that since sophists appeal only to what seems probable, they are not advancing their students and audiences, but simply flattering them with what they want to hear. While Plato's condemnation of rhetoric is clear in the *Gorgias*, in the *Phaedrus* he suggests the possibility of a true art wherein rhetoric is based upon the knowledge produced by dialectic, and relies on a dialectically informed rhetoric to appeal to the main character: Phaedrus, to take up philosophy. Thus Plato's rhetoric is actually dialectic (or philosophy) "turned" toward those who are not yet philosophers and are thus unready to pursue dialectic directly. Plato's animosity against rhetoric, and against the sophists, derives not only from their inflated claims to teach virtue and their reliance on appearances, but from the fact that his teacher, Socrates, was sentenced to death after sophists' efforts.

Aristotle

A marble bust of Aristotle

Aristotle (384-322 BC) was a student of Plato who famously set forth an extended treatise on rhetoric that still repays careful study today. In the first sentence of The Art of Rhetoric, Aristotle says that "rhetoric is the counterpart [literally, the antistrophe] of dialectic." As the "antistrophe" of a Greek ode responds to and is patterned after the structure of the "strophe" (they form two sections of the whole and are sung by two parts of the chorus), so the art of rhetoric follows and is structurally patterned after the art of dialectic because both are arts of discourse production. Thus, while dialectical methods are necessary to find truth in theoretical matters, rhetorical methods are required in practical matters such as adjudicating somebody's guilt or innocence when charged in a court of law, or adjudicating a prudent course of action to be taken in a deliberative assembly. The core features dialectic include the absence of determined subject matter, its elaboration on earlier empirical practice, the explication of its aims, the type of utility and the definition of the proper function. For Plato and Aristotle, dialectic involves persuasion, so when Aristotle says that rhetoric is the antistrophe of dialectic, he means that rhetoric as he uses the term has a domain or scope of application that is parallel to but different from the domain or scope of application of dialectic. In *Nietzsche Humanist* (1998: 129), Claude Pavur explains that "[t]he Greek prefix 'anti' does not merely designate opposition, but it can also mean 'in place of.'" When Aristotle characterizes rhetoric as the antistrophe of dialectic, he no doubt means that rhetoric is used in place of dialectic when we are discussing civic issues in a court of law or in a legislative assembly. The domain of rhetoric is civic affairs and practical decision making in civic affairs, not theoretical considerations of operational definitions of terms and clarification of thought – these, for him, are in the domain of dialectic.

Aristotle's treatise on rhetoric is an attempt to systematically describe civic rhetoric as a human art or skill (techne). It is more of an objective theory than it is an interpretive theory with a rhetorical tradition. Aristotle’s “art” of rhetoric emphasizes on persuasion to be the purpose of rhetoric. His definition of rhetoric as "the faculty of observing in any given case the available means of persuasion," essentially a mode of discovery, seems to limit the art to the inventional process, and Aristotle heavily emphasizes the logical aspect of this process. In his world, rhetoric is the art of discovering all available means of persuasion. A speaker supports the probability of a message by logical, ethical, and emotional proofs. Some form of logos, ethos, and pathos is present in every possible public presentation that exists. But the treatise in fact also discusses not only elements of style and (briefly) delivery, but also emotional appeals (pathos) and characterological appeals (ethos). He thus identifies three steps or "offices" of rhetoric—invention, arrangement, and style—and three different types of rhetorical proof:

- ethos: Aristotle’s theory of character and how the character and credibility of a speaker can influence an audience to consider him/her to be believable.
  - This could be any position in which the speaker—whether an acknowledged expert on the subject, or an acquaintance of a person who experienced the matter in question—knows about the topic.
  - For instance, when a magazine claims that *An MIT professor predicts that the robotic era is coming in 2050,* the use of big-name "MIT" (a world-renowned American university for the advanced research in math, science, and technology) establishes the "strong" credibility.
  - There are three qualities that contribute to a credible ethos and they include perceived intelligence, virtuous character, and goodwill.
  - Audience is more likely to be persuaded by a credible source because they are more reliable.

- pathos: the use of emotional appeals to alter the audience's judgment.
  - This can be done through metaphor, amplification, storytelling, or presenting the topic in a way that evokes strong emotions in the audience.
  - Aristotle used pathos as a corrective measure to help the speaker create appeals to emotion to motivate decision making.
  - George Kennedy claims that pathos was an early discussion of human psychology.
  - Strong emotions are likely to persuade when there is a connection with the audience.
- logos: the use of reasoning, either inductive or deductive, to construct an argument.
  - Logos appeals include appeals to statistics, math, logic, and *objectivity.* For instance, when advertisements claim that their product is *37% more effective than the competition,* they are making a logical appeal.
  - Inductive reasoning uses examples (historical, mythical, or hypothetical) to draw conclusions.
  - Deductive reasoning, or "enthymematic" reasoning, uses generally accepted propositions to derive specific conclusions. The term *logic* evolved from *logos*. Aristotle emphasized enthymematic reasoning as central to the process of rhetorical invention, though later rhetorical theorists placed much less emphasis on it. An “enthymeme” would follow today’s form of a syllogism; however it would exclude either the major or minor premise. An enthymeme is persuasive because the audience is providing the missing premise. Because the audience is able to provide the missing premise, they are more likely to be persuaded by the message.

Aristotle also identifies three different types or genres of civic rhetoric: *forensic* (also known as judicial, was concerned with determining *truth* or *falsity* of events that took place in the *past*, issues of guilt. An example of forensic rhetoric would be in a courtroom), *deliberative* (also known as political, was concerned with determining whether or not particular actions *should* or should not be taken in the *future.* Making laws would be an example of deliberative rhetoric), and *epideictic* (also known as ceremonial, was concerned with praise and blame, values, right and wrong, demonstrating beauty and skill in the *present.* Examples of epideictic rhetoric would include a eulogy or a wedding toast).

The Five Canons of Rhetoric serve as a guide to creating persuasive messages and arguments:

- Invention - the process of developing arguments
- Style - determining how to present the arguments
- Arrangement - organizing the arguments for extreme effect
- Delivery - the gestures, pronunciation, tone and pace used when presenting the persuasive arguments
- Memory - the process of learning and memorizing the speech and persuasive messages (This was the last canon of rhetoric that was added much later to the original four canons.)

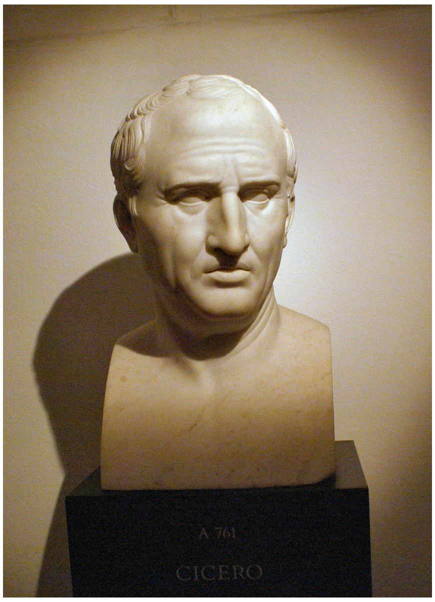
In the rhetoric field, there is an intellectual debate about Aristotle's definition of rhetoric. Some believe that Aristotle defines rhetoric in *On Rhetoric* as the art of persuasion, while others think he defines it as the art of judgment. Rhetoric as the art of judgment would mean the rhetor discerns the available means of persuasion with a choice. Aristotle also says rhetoric is concerned with judgment because the audience judges the rhetor's ethos.

One of the most famous of Aristotelian doctrines was the idea of topics (also referred to as common topics or commonplaces). Though the term had a wide range of application (as a memory technique or compositional exercise, for example) it most often referred to the "seats of argument"—the list of categories of thought or modes of reasoning—that a speaker could use in order to generate arguments or proofs. The topics were thus a heuristic or inventional tool designed to help speakers categorize and thus better retain and apply frequently used types of argument. For example, since we often see effects as "like" their causes, one way to invent an argument (about a future effect) is by discussing the cause (which it will be "like"). This and other rhetorical topics derive from Aristotle's belief that there are certain predictable ways in which humans (particularly non-specialists) draw conclusions from premises. Based upon and adapted from his dialectical Topics, the rhetorical topics became a central feature of later rhetorical theorizing, most famously in Cicero's work of that name.

It has been questioned whether or not it is ethical to alter a message to make it appear to be more persuasive. Aristotle stands between the two extremes. For example, there is lying and being brutally honest, but Aristotle would be more in the middle by using truthful statements. This is called the Golden Mean. The golden mean is a level of moderation that is quite middle ground. Golden mean seems to be the most effective way to persuade others. It is known that Aristotle spoke of ethics in terms of character as opposed to conduct. Rhetoric is about creating convincing arguments. Aristotle has argued that rhetoric depends on the nature of the argument and that audience’s opinion of the speaker. As a result, in order to determine whether or not the persuasive message is ethical, the speaker must understand how the audience characterizes the speaker’s good character.

Another critique (or weakness) that questions the aspects of Aristotle's rhetoric theory is that he does not focus much on the emotional (pathos) proof, leaving his work a jumbled, unorganized mess.

Cicero



Bust of Marcus Tullius Cicero

For the Romans, oration became an important part of public life. Cicero (106-43 BC) was chief among Roman rhetoricians and remains the best known ancient orator and the only orator who both spoke in public and produced treatises on the subject. *Rhetorica ad Herennium*, formerly attributed to Cicero but now considered to be of unknown authorship, is one of the most significant works on rhetoric and is still widely used as a reference today. It is an extensive reference on the use of rhetoric, and in the Middle Ages and Renaissance, it achieved wide publication as an advanced school text on rhetoric.

Cicero is considered one of the most significant rhetoricians of all time. His works include the early and very influential De Inventione (On Invention, often read alongside the *Ad Herennium* as the two basic texts of rhetorical theory throughout the Middle Ages and into the Renaissance), De Oratore (a fuller statement of rhetorical principles in dialogue form), Topics (a rhetorical treatment of common topics, highly influential through the Renaissance), Brutus (Cicero) (a discussion of famous orators) and Orator (a defense of Cicero's style). Cicero also left a large body of speeches and letters which would establish the outlines of Latin eloquence and style for generations to come. It was the rediscovery of Cicero's speeches (such as the defense of Archias) and letters (to Atticus) by Italians like Petrarch that, in part, ignited the cultural innovations that we know as the Renaissance. He championed the learning of Greek (and Greek rhetoric), contributed to Roman ethics, linguistics, philosophy, and politics, and emphasized the importance of all forms of appeal (emotion, humor, stylistic range, irony and digression in addition to pure reasoning) in oratory. But perhaps his most significant contribution to subsequent rhetoric, and education in general, was his argument that orators learn not only about the specifics of their case (the *hypothesis*) but also about the general questions from which they were derived (the *theses*). Thus, in giving a speech in defense of a poet whose Roman citizenship had been questioned, the orator should examine not only the specifics of that poet's civic status, he should also examine the role and value of poetry and of literature more generally in Roman culture and political life. The orator, said Cicero, needed to be knowledgeable about all areas of human life and culture, including law, politics, history, literature, ethics, warfare, medicine, even arithmetic and geometry. Cicero gave rise to the idea that the "ideal orator" be well-versed in all branches of learning: an idea that was rendered as "liberal humanism," and that lives on today in liberal arts or general education requirements in colleges and universities around the world.

Quintilian

Quintilian (35-100 AD) began his career as a pleader in the courts of law; his reputation grew so great that Vespasian created a chair of rhetoric for him in Rome. The culmination of his life's work was the *Institutio oratoria* (*Institutes of Oratory,* or alternatively, *The Orator's Education*), a lengthy treatise on the training of the orator in which he discusses the training of the "perfect" orator from birth to old age and, in the process, reviews the doctrines and opinions of many influential rhetoricians who preceded him.

In the Institutes, Quintilian organizes rhetorical study through the stages of education that an aspiring orator would undergo, beginning with the selection of a nurse. Aspects of elementary education (training in reading and writing, grammar, and literary criticism) are followed by preliminary rhetorical exercises in composition (the progymnasmata) that include maxims and fables, narratives and comparisons, and finally full legal or political speeches. The delivery of speeches within the context of education or for entertainment purposes became widespread and popular under the term "declamation." Rhetorical training proper was categorized under five canons that would persist for centuries in academic circles:

- *Inventio* (invention) is the process that leads to the development and refinement of an argument.
- Once arguments are developed, *dispositio* (disposition, or arrangement) is used to determine how it should be organized for greatest effect, usually beginning with the *exordium*.
- Once the speech content is known and the structure is determined, the next steps involve *elocutio* (style) and *pronuntiatio* (presentation).
- *Memoria* (memory) comes to play as the speaker recalls each of these elements during the speech.
- *Actio* (delivery) is the final step as the speech is presented in a gracious and pleasing way to the audience - the Grand Style.

This work was available only in fragments in medieval times, but the discovery of a complete copy at the Abbey of St. Gall in 1416 led to its emergence as one of the most influential works on rhetoric during the Renaissance.

Quintilian's work describes not just the art of rhetoric, but the formation of the perfect orator as a politically active, virtuous, publicly minded citizen. His emphasis was on the ethical application of rhetorical training, in part a reaction against the growing tendency in Roman schools toward standardization of themes and techniques. At the same time that rhetoric was becoming divorced from political decision making, rhetoric rose as a culturally vibrant and important mode of entertainment and cultural criticism in a movement known as the "second sophistic," a development which gave rise to the charge (made by Quintilian and others) that teachers were emphasizing style over substance in rhetoric.

Medieval to Enlightenment

After the breakup of the western Roman Empire, the study of rhetoric continued to be central to the study of the verbal arts; but the study of the verbal arts went into decline for several centuries, followed eventually by a gradual rise in formal education, culminating in the rise of medieval universities. But rhetoric transmuted during this period into the arts of letter writing (*ars dictaminis*) and sermon writing (*ars praedicandi*). As part of the *trivium*, rhetoric was secondary to the study of logic, and its study was highly scholastic: students were given repetitive exercises in the creation of discourses on historical subjects (*suasoriae*) or on classic legal questions (*controversiae*).

Although he is not commonly regarded as a rhetorician, St. Augustine (354-430) was trained in rhetoric and was at one time a professor of Latin rhetoric in Milan. After his conversion to Christianity, he became interested in using these "pagan" arts for spreading his religion. This new use of rhetoric is explored in the Fourth Book of his *De Doctrina Christiana*, which laid the foundation of what would become homiletics, the rhetoric of the sermon. Augustine begins the book by asking why "the power of eloquence, which is so efficacious in pleading either for the erroneous cause or the right", should not be used for righteous purposes (IV.3).

One early concern of the medieval Christian church was its attitude to classical rhetoric itself. Jerome (d. 420) complained, "What has Horace to do with the Psalms, Virgil with the Gospels, Cicero with the Apostles?" Augustine is also remembered for arguing for the preservation of pagan works and fostering a church tradition which led to conservation of numerous pre-Christian rhetorical writings.

Rhetoric would not regain its classical heights until the renaissance, but new writings did advance rhetorical thought. Boethius (480?-524), in his brief *Overview of the Structure of Rhetoric*, continues Aristotle's taxonomy by placing rhetoric in subordination to philosophical argument or dialectic.[45] The introduction of Arab scholarship from European relations with the Muslim empire (in particular Al-Andalus) renewed interest in Aristotle and Classical thought in general, leading to what some historians call the 12th century renaissance. A number of medieval grammars and studies of poetry and rhetoric appeared.

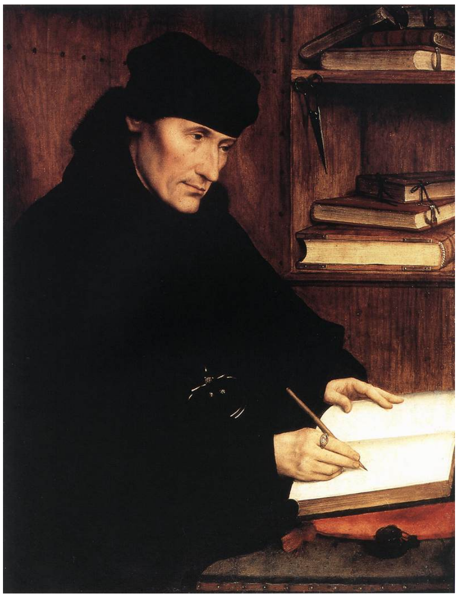
Late medieval rhetorical writings include those of St. Thomas Aquinas (1225?-1274), Matthew of Vendome (*Ars Versificatoria*, 1175?), and Geoffrey of Vinsauf (*Poetria Nova*, 1200–1216). Pre-modern female rhetoricians, outside of Socrates' friend Aspasia, are rare; but medieval rhetoric produced by women either in religious orders, such as Julian of Norwich (d. 1415), or the very well-connected Christine de Pizan (1364?-1430?), did occur if not always recorded in writing.

In his 1943 Cambridge University doctoral dissertation in English, Canadian Marshall McLuhan (1911–1980) surveys the verbal arts from approximately the time of Cicero down to the time of Thomas Nashe (1567-1600?).[46] His dissertation is still noteworthy for undertaking to study the history of the verbal arts together as the trivium, even though the developments that he surveys have been studied in greater detail since he undertook his study. As noted below, McLuhan became one of the most widely publicized thinkers in the 20th century, so it is important to note his scholarly roots in the study of the history of rhetoric and dialectic.

Another interesting record of medieval rhetorical thought can be seen in the many animal debate poems popular in England and the continent during the Middle Ages, such as The Owl and the Nightingale (13th century) and Geoffrey Chaucer's Parliament of Fowls (1382?).

Sixteenth century

Walter J. Ong's encyclopedia article "Humanism" in the 1967 *New Catholic Encyclopedia* provides a well-informed survey of Renaissance humanism, which defined itself broadly as disfavoring medieval scholastic logic and dialectic and as favoring instead the study of classical Latin style and grammar and philology and rhetoric. (Reprinted in Ong's *Faith and Contexts* (Scholars Press, 1999; 4: 69-91.))



Portrait of Erasmus of Rotterdam

One influential figure in the rebirth of interest in classical rhetoric was Erasmus (c.1466-1536). His 1512 work, *De Duplici Copia Verborum et Rerum* (also known as *Copia: Foundations of the Abundant Style*), was widely published (it went through more than 150 editions throughout Europe) and became one of the basic school texts on the subject. Its treatment of rhetoric is less comprehensive than the classic works of antiquity, but provides a traditional treatment of *res-verba* (matter and form): its first book treats the subject of elocutio, showing the student how to use schemes and tropes; the second book covers inventio. Much of the emphasis is on abundance of variation (*copia* means "plenty" or "abundance", as in copious or cornucopia), so both books focus on ways to introduce the maximum amount of variety into discourse. For instance, in one section of the *De Copia*, Erasmus presents two hundred variations of the sentence *"Semper, dum vivam, tui meminero."* Another of his works, the extremely popular The Praise of Folly, also had considerable influence on the teaching of rhetoric in the later 16th century. Its orations in favour of qualities such as madness spawned a type of exercise popular in Elizabethan grammar schools, later called adoxography, which required pupils to compose passages in praise of useless things.

Juan Luis Vives (1492–1540) also helped shape the study of rhetoric in England. A Spaniard, he was appointed in 1523 to the Lectureship of Rhetoric at Oxford by Cardinal Wolsey, and was entrusted by Henry VIII to be one of the tutors of Mary. Vives fell into disfavor when Henry VIII divorced Catherine of Aragon and left England in 1528. His best-known work was a book on education, *De Disciplinis*, published in 1531, and his writings on rhetoric included *Rhetoricae, sive De Ratione Dicendi, Libri Tres* (1533), *De Consultatione* (1533), and a rhetoric on letter writing, *De Conscribendis Epistolas* (1536).

It is likely that many well-known English writers would have been exposed to the works of Erasmus and Vives (as well as those of the Classical rhetoricians) in their schooling, which was conducted in Latin (not English) and often included some study of Greek and placed considerable emphasis on rhetoric. See, for example, T.W. Baldwin's *William Shakspere's Small Latine and Lesse Greeke*, 2 vols. (University of Illinois Press, 1944).

The mid-16th century saw the rise of vernacular rhetorics — those written in English rather than in the Classical languages; adoption of works in English was slow, however, due to the strong orientation toward Latin and Greek. Leonard Cox's *The Art or Crafte of Rhetoryke* (c. 1524-1530; second edition published in 1532) is considered to be the earliest text on rhetorics in English; it was, for the most part, a translation of the work of Philipp Melanchthon.[47] A successful early text was Thomas Wilson's *The Arte of Rhetorique* (1553), which presents a traditional treatment of rhetoric. For instance, Wilson presents the five canons of rhetoric (Invention, Disposition, Elocutio, Memoria, and Utterance or Actio). Other notable works included Angel Day's *The English Secretorie* (1586, 1592), George Puttenham's *The Arte of English Poesie* (1589), and Richard Rainholde's *Foundacion of Rhetorike* (1563).

During this same period, a movement began that would change the organization of the school curriculum in Protestant and especially Puritan circles and lead to rhetoric losing its central place. A French scholar, Pierre de la Ramée, in Latin Petrus Ramus (1515–1572), dissatisfied with what he saw as the overly broad and redundant organization of the trivium, proposed a new curriculum. In his scheme of things, the five components of rhetoric no longer lived under the common heading of rhetoric. Instead, invention and disposition were determined to fall exclusively under the heading of dialectic, while style, delivery, and memory were all that remained for rhetoric. See Walter J. Ong, *Ramus, Method, and the Decay of Dialogue: From the Art of Discourse to the Art of Reason* (Harvard University Press, 1958; reissued by the University of Chicago Press, 2004, with a new foreword by Adrian Johns). Ramus, rightly accused of sodomy and erroneously of atheism, was martyred during the French Wars of Religion. His teachings, seen as inimical to Catholicism, were short-lived in France but found a fertile ground in the Netherlands, Germany and England.[48]

One of Ramus' French followers, Audomarus Talaeus (Omer Talon) published his rhetoric, *Institutiones Oratoriae*, in 1544. This work provided a simple presentation of rhetoric that emphasized the treatment of style, and became so popular that it was mentioned in John Brinsley's (1612) *Ludus literarius; or The Grammar Schoole* as being the "most used in the best schooles." Many other Ramist rhetorics followed in the next half-century, and by the 17th century, their approach became the primary method of teaching rhetoric in Protestant and especially Puritan circles. See Walter J. Ong, *Ramus and Talon Inventory* (Harvard University Press, 1958); Joseph S. Freedman, *Philosophy and the Art Europe, 1500-1700: Teaching and Texts at Schools and Universities* (Ashgate, 1999). John Milton (1608–1674) wrote a textbook in logic or dialectic in Latin based on Ramus' work, which has now been translated into English by Walter J. Ong and Charles J. Ermatinger in *The Complete Prose Works of John Milton* (Yale University Press, 1982; 8: 206-407), with a lengthy introduction by Ong (144-205). The introduction is reprinted in Ong's *Faith and Contexts* (Scholars Press, 1999; 4: 111-41).

Ramism could not exert any influence on the established Catholic schools and universities, which remained loyal to Scholasticism, or on the new Catholic schools and universities founded by members of the religious orders known as the Society of Jesus or the Oratorians, as can be seen in the Jesuit curriculum (in use right up to the 19th century, across the Christian world) known as the Ratio Studiorum (that Claude Pavur, S.J., has recently translated into English, with the Latin text in the parallel column on each page (St. Louis: Institute of Jesuit Sources, 2005)). If the influence of Cicero and Quintilian permeates the Ratio Studiorum, it is through the lenses of devotion and the militancy of the Counter-Reformation. The *Ratio* was indeed imbued with a sense of the divine, of the incarnate logos, that is of rhetoric as an eloquent and humane means to reach further devotion and further action in the Christian city, which was absent from Ramist formalism. The Ratio is, in rhetoric, the answer to St Ignatius Loyola's practice, in devotion, of "spiritual exercises." This complex oratorical-prayer system is absent from Ramism.

Seventeenth century New England

In New England and at Harvard College (founded 1636), Ramus and his followers dominated, as Perry Miller shows in *The New England Mind: The Seventeenth Century* (Harvard University Press, 1939). However, in England, several writers influenced the course of rhetoric during the 17th century, many of them carrying forward the dichotomy that had been set forth by Ramus and his followers during the preceding decades. Of greater importance is that this century saw the development of a modern, vernacular style that looked to English, rather than to Greek, Latin, or French models.

Francis Bacon (1561–1626), although not a rhetorician, contributed to the field in his writings. One of the concerns of the age was to find a suitable style for the discussion of scientific topics, which needed above all a clear exposition of facts and arguments, rather than the ornate style favored at the time. Bacon in his *The Advancement of Learning* criticized those who are preoccupied with style rather than "the weight of matter, worth of subject, soundness of argument, life of invention, or depth of judgment." On matters of style, he proposed that the style conform to the subject matter and to the audience, that simple words be employed whenever possible, and that the style should be agreeable.[49]

Thomas Hobbes (1588–1679) also wrote on rhetoric. Along with a shortened translation of Aristotle's *Rhetoric*, Hobbes also produced a number of other works on the subject. Sharply contrarian on many subjects, Hobbes, like Bacon, also promoted a simpler and more natural style that used figures of speech sparingly.

Perhaps the most influential development in English style came out of the work of the Royal Society (founded in 1660), which in 1664 set up a committee to improve the English language. Among the committee's members were John Evelyn (1620–1706), Thomas Sprat (1635–1713), and John Dryden (1631–1700). Sprat regarded "fine speaking" as a disease, and thought that a proper style should "reject all amplifications, digressions, and swellings of style" and instead "return back to a primitive purity and shortness" (*History of the Royal Society*, 1667).

While the work of this committee never went beyond planning, John Dryden is often credited with creating and exemplifying a new and modern English style. His central tenet was that the style should be proper "to the occasion, the subject, and the persons." As such, he advocated the use of English words whenever possible instead of foreign ones, as well as vernacular, rather than Latinate, syntax. His own prose (and his poetry) became exemplars of this new style.

Eighteenth century

Arguably one of the most influential schools of rhetoric during this time was Scottish Belletristic rhetoric, exemplified by such professors of rhetoric as Hugh Blair whose Lectures on Rhetoric and Belles Lettres saw international success in various editions and translations.

Modern rhetoric

At the turn of the 20th century, there was a revival of rhetorical study manifested in the establishment of departments of rhetoric and speech at academic institutions, as well as the formation of national and international professional organizations.[50] Theorists generally agree that a significant reason for the revival of the study of rhetoric was the renewed importance of language and persuasion in the increasingly mediated environment of the 20th century (see Linguistic turn) and through the 21st century, with the media focus on the wide variations and analyses of political rhetoric and its consequences. The rise of advertising and of mass media such as photography, telegraphy, radio, and film brought rhetoric more prominently into people's lives. More recently the term rhetoric has been applied to media forms other than verbal language, e.g. Visual rhetoric.

Notable modern theorists

- **Chaim Perelman** was a philosopher of law, who studied, taught, and lived most of his life in Brussels. He was among the most important argumentation theorists of the 20th century. His chief work is the *Traité de l'argumentation - la nouvelle rhétorique* (1958), with Lucie Olbrechts-Tyteca, which was translated into English as *The New Rhetoric: A Treatise on Argumentation*, by John Wilkinson and Purcell Weaver (1969). Perelman and Olbrechts-Tyteca move rhetoric from the periphery to the center of argumentation theory. Among their most influential concepts are "dissociation," "the universal audience," "quasi-logical argument," and "presence."
- **Kenneth Burke** was a rhetorical theorist, philosopher, and poet. Many of his works are central to modern rhetorical theory: *A Rhetoric of Motives* (1950), *A Grammar of Motives* (1945), *Language as Symbolic Action* (1966), and *Counterstatement* (1931). Among his influential concepts are "identification," "consubstantiality," and the "dramatistic pentad." He described rhetoric as "the use of language as a symbolic means of inducing cooperation in beings that by nature respond to symbols."[51] In relation to Aristotle's theory, Aristotle was more interested in constructing rhetoric, while Burke was interested in "debunking" it.
- **Edwin Black** was a rhetorical critic best known for his book *Rhetorical Criticism: A Study in Method*[52] (1965) in which he criticized the dominant "neo-Aristotelian" tradition in American rhetorical criticism as having little in common with Aristotle "besides some recurrent topics of discussion and a vaguely derivative view of rhetorical

discourse." Furthermore, he contended, because rhetorical scholars had been focusing primarily on Aristotelian logical forms they often overlooked important, alternative types of discourse. He also published several highly influential essays including: "Secrecy and Disclosure as Rhetorical Forms.",[53] "The Second Persona,"[54] and "A Note on Theory and Practice in Rhetorical Criticism."[55]

- **Marshall McLuhan** was a media theorist whose discoveries are important to the study of rhetoric. McLuhan's famous dictum "the medium is the message" highlights the significance of the medium itself. No other scholar of the history and theory of rhetoric was as widely publicized in the 20th century as McLuhan.[56]
- **I.A. Richards** was a literary critic and rhetorician. His *The Philosophy of Rhetoric* is an important text in modern rhetorical theory. In this work, he defined rhetoric as "a study of misunderstandings and its remedies,"[57] and introduced the influential concepts *tenor* and *vehicle* to describe the components of a metaphor—the main idea and the concept to which it is compared.[58]
- **The Groupe µ**. This interdisciplinary team has contributed to the renovation of the elocutio in the context of poetics and modern linguistics, significantly with *Rhétorique générale* (1970; translated into English as *A General Rhetoric*, by Paul B. Burrell et Edgar M. Slotkin, Johns Hopkins University Press, 1981) and *Rhétorique de la poésie* (1977).
- **Stephen Toulmin** was a philosopher whose models of argumentation have had great influence on modern rhetorical theory. His *Uses of Argument* is an important text in modern rhetorical theory and argumentation theory.[59]

Methods of analysis

There does not exist an analytic method that is widely recognized as "the" rhetorical method, partly because many in rhetorical study see rhetoric as merely produced by reality (see dissent from that view below). It is important to note that the object of rhetorical analysis is typically discourse, and therefore the principles of "rhetorical analysis" would be difficult to distinguish from those of "discourse analysis." However, rhetorical analytic methods can also be applied to almost anything, including objects—a car, a castle, a computer, a comportment.

Generally speaking, rhetorical analysis makes use of rhetorical concepts (ethos, logos, kairos, mediation, etc.) to describe the social or epistemological functions of the object of study. When the object of study happens to be some type of discourse (a speech, a poem, a joke, a newspaper article), the aim of rhetorical analysis is not simply to describe the claims and arguments advanced within the disourse, but (more important) to identify the specific semiotic strategies employed by the speaker to accomplish specific persuasive goals. Therefore, after a rhetorical analyst discovers a use of language that is particularly important in achieving persuasion, she typically moves onto the question of "How does it work?" That is, what effects does this particular use of rhetoric have on an audience, and how does that effect provide more clues as to the speaker's (or writer's) objectives?

There are some scholars who do partial rhetorical analysis and defer judgments about rhetorical success. In other words, some analysts attempt to avoid the question of "Was this use of rhetoric successful [in accomplishing the aims of the speaker]?" To others, however, that is the preeminent point: is the rhetoric strategically effective and what did the rhetoric accomplish? This question allows a shift in focus from the speaker's objectives to the effects and functions of the rhetoric itself.

Rhetorical criticism

Modern rhetorical criticism explores the relationship between text and context; that is, how an instance of rhetoric relates to circumstances. In his Rhetorical Criticism: A Study in Method, scholar Edwin Black states, "It is the task of criticism not to measure… discourses dogmatically against some parochial standard of rationality but, allowing for the immeasurable wide range of human experience, to see them as they really are."[60] While the language "as they really are" is debatable, rhetorical critics explain texts and speeches by investigating their rhetorical situation, typically placing them in a framework of speaker/audience exchange.[61]

Following the neo-Aristotelian approaches to criticism, scholars began to derive methods from other disciplines, such as history, philosophy, and the social sciences.[62] The importance of critics' personal judgment decreased in explicit coverage while the analytical dimension of criticism began to gain momentum. Throughout the 1960s and 1970s, methodological pluralism replaced the singular neo-Aristotelian method. Methodological rhetorical criticism is typically done by deduction, where a broad method is used to examine a specific case of rhetoric.[63] These types include:

- **Ideological criticism** – critics engage rhetoric as it suggests the beliefs, values, assumptions, and interpretations held by the rhetor or the larger culture. Ideological criticism also treats ideology as an artifact of discourse, one that is embedded in key terms (called "ideographs") as well as material resources and discursive embodiment.
- **Cluster criticism** – a method developed by Kenneth Burke that seeks to help the critic understand the rhetor's worldview. This means identifying terms that are 'clustered' around key symbols in the rhetorical artifact and the patterns in which they appear.
- **Generic criticism** – a method that assumes certain situations call for similar needs and expectations within the audience, therefore calling for certain types of rhetoric. It studies rhetoric in different times and locations, looking at similarities in the rhetorical situation and the rhetoric that responds to them. Examples include eulogies, inaugural addresses, and declarations of war.
- **Narrative criticism** – narratives help to organize experiences in order to endow meaning to historical events and transformations. Narrative criticism focuses on the story itself and how the construction of the narrative directs the interpretation of the situation.

By the mid-1980s, however, the study of rhetorical criticism began to move away from precise methodology towards conceptual issues. Conceptually driven criticism operates more through abduction, according to scholar James Jasinski, who argues that this emerging type of criticism can be thought of as a back-and-forth between the text and the concepts, which are being explored at the same time. The concepts remain "works in progress," and understanding those terms develops through the analysis of a text.[64]

Criticism is considered rhetorical when it focuses on the way some types of discourse react to situational exigencies – problems or demands – and constraints. This means that modern rhetorical criticism is based in how the rhetorical case or object persuades, defines, or constructs the audience. In modern terms, what can be considered rhetoric includes, but it is not limited to, speeches, scientific discourse, pamphlets, literary work, works of art, and pictures. Contemporary rhetorical criticism has maintained aspects of early neo-Aristotelian thinking through close reading, which attempts to explore the organization and stylistic structure of a rhetorical object.[65] Using close textual analysis means rhetorical critics use the tools of classical rhetoric and literary analysis to evaluate the style and strategy used to communicate the argument.

Rhetorical criticism serves several purposes or functions. First, rhetorical criticism hopes to help form or improve public taste. It helps educate audiences and develops them into better judges of rhetorical situations by reinforcing ideas of value, morality, and suitability. Rhetorical criticism can thus contribute to the audience's understanding of themselves and society.

French rhetoric

Rhetoric was part of the curriculum in Jesuit and, to a lesser extent, Oratorian colleges until the French Revolution. For Jesuits, right from the foundation of the Society in France, rhetoric was an integral part of the training of young men toward taking up leadership positions in the Church and in State institutions, as Marc Fumaroli has shown it in his foundational *Âge de l'éloquence* (1980). The Oratorians, by contrast, reserved it a lesser place, in part due to the stress they placed on modern language acquisition and a more sensualist philosophy (like Bernard Lamy's *La Rhétorique ou l'Art de parler* (1675) which is an excellent example of their approach). Nonetheless, in the 18th Century, rhetoric was the structure and crown of secondary education, with works such as Rollin's *Treatise of Studies* achieving a wide and enduring fame across the Continent.[66] Later, with Nicolas Boileau and François de Malherbe, rhetoric is the instrument of the clarity of the comment and speech ; the literature that ensues from it is named "Sublime". The main representative remains Rivarol.

The French Revolution, however, turned this around. Philosophers such as Condorcet, who drafted the French revolutionary chart for a people's education under the rule of reason, dismissed rhetoric as an instrument of oppression in the hands of clerics in particular. The Revolution went as far as to suppress the Bar, arguing that forensic rhetoric did disservice to a rational system of justice, by allowing fallacies and emotions to come into play. Nonetheless, as later historians of the 19th century were keen to explain, the Revolution was a high moment of eloquence and rhetorical prowess, although set against a background of rejecting rhetoric.

Under the First Empire and its wide-ranging educational reforms, imposed on or imitated across the Continent, rhetoric regained little ground. In fact, instructions to the newly founded Polytechnic School, tasked with training the scientific and technical elites, made it clear that written reporting was to supersede oral reporting. Rhetoric reentered secondary curriculum in fits and starts, but never regained the prominence it had enjoyed under the *ancien régime*, although the penultimate year of secondary education was known as the Class of Rhetoric. When manuals were redrafted in the mid-century, in particular after the 1848 Revolution to formulate a national curriculum, care was taken to distance their approach to rhetoric from that of the Church, which was seen as an agent of conservatism and reactionary politics.

By the end of the 1870s, a major change had taken place: philosophy of the rationalist or eclectic kind, generally Kantian, had taken over rhetoric as the true end stage of secondary education (the so-called Class of Philosophy bridged secondary and university education). Rhetoric was then relegated to the study of literary figures of speech, a discipline later on taught as Stylistics within the French literature curriculum. More decisively, in 1890, a new standard written exercise superseded the rhetorical exercises of speech writing, letter writing and narration. The new genre, called dissertation, had been invented in 1866, for the purpose of rational argument in the philosophy class. Typically, in a dissertation, a question is asked, such as: "Is history a sign of humanity's freedom?" The structure of a dissertation consists in an introduction that elucidates the basic definitions involved in the question as set, followed by an argument or thesis, a counter-argument or antithesis, and a resolving argument or synthesis that is not a compromise between the former but the production of a new argument, ending with a conclusion that does not sum up the points but opens onto a new problem. Hegelianism influenced the dissertation design. It remains today the standard of writing in French humanities.

By the beginning of the 20th century, rhetoric was fast losing the remains of its former importance, and eventually was taken out of the school curriculum altogether at the time of the Separation of State and Churches (1905). Part of the argument was that rhetoric remained the last element of irrationality, driven by religious arguments, in what was perceived as inimical to Republican education. The move, initiated in 1789, found its resolution in 1902 when rhetoric was expunged from all curricula. At the same time, Aristotelian rhetoric, owing to a revival of Thomistic philosophy initiated by Rome, regained ground in what was left of Catholic education in France, in particular at the prestigious Faculty of Theology of Paris, now a private entity. Yet, rhetoric vanished substantially from the French scene, educational or intellectual, for some 60 years..

In the early 1960s a change began to take place, as the word rhetoric and the body of knowledge it covers began to be used again, in a modest and almost secret manner. The new linguistic turn, through the rise of semiotics as well as of structural linguistics, brought to the fore a new interest in figures of speech as signs, the metaphor in particular (in the works of Roman Jakobson, Groupe μ, Michel Charles, Gérard Genette) while famed Structuralist Roland Barthes, a classicist by training, perceived how some basic elements of rhetoric could be of use in the study of narratives, fashion and ideology. Knowledge of rhetoric was so dim in the early 1970s that his short memoir on rhetoric was seen as highly innovative. Basic as it was, it did help rhetoric regain some currency in avant-garde circles. Psychoanalyst Jacques Lacan, his contemporary, makes references to rhetoric, in particular to the Pre-Socratics. Philosopher Jacques Derrida wrote on Voice.

At the same time, more profound work was taking place that eventually gave rise to the French school of rhetoric as it exists today.[67]

This rhetorical revival took place on two fronts.[68] First, in 17th-century French studies, the mainstay of French literary education, awareness grew that rhetoric was necessary to push the limits of knowledge further, and also to provide an antidote to Structuralism and its denial of historicism in culture. This was the pioneering work of Marc Fumaroli who, building on the work of classicist and Neo-Latinist Alain Michel and French scholars such as Roger Zuber, published his famed *Age de l'Eloquence* (1980), was one of the founders of the International Society for the History of Rhetoric and was eventually elevated to a chair in rhetoric at the prestigious College de France. He is the editor in chief of a monumental *History of Rhetoric in Modern Europe*.[69] His disciples form the second generation,[70] with rhetoricians such as Françoise Waquet and Delphine Denis, both of the Sorbonne, or Philippe-Joseph Salazar (fr:Philippe-Joseph Salazar on the French Wikipedia), until recently at Derrida's College international de philosophie, laureate of the Harry Oppenheimer prize and whose recent book on *Hyperpolitique* has attracted the French media's attention on a "re-appropriation of the means of production of persuasion".[71]

Second, in the area of Classical studies, in the wake of Alain Michel, Latin scholars fostered a renewal in Cicero studies. They broke away from a pure literary reading of his orations, in an attempt to embed Cicero in European ethics. Meanwhile, among Greek scholars, the literary historian and philologist Jacques Bompaire, the philologist and philosopher E. Dupréel, and later the literature historian Jacqueline de Romilly pioneered new studies in the Sophists and the Second Sophistic. The second generation of Classicists, often trained in philosophy as well (following Heidegger and Derrida, mainly), built on their work, with authors such as Marcel Detienne (now at Johns Hopkins), Nicole Loraux, Medievalist and logician Alain De Libera (Geneva),[72] Ciceronian scholar Carlos Lévy (Sorbonne, Paris) and Barbara Cassin (Collége international de philosophie, Paris).[73] Sociologist of science Bruno Latour and economist Romain Laufer may also be considered part of, or close to this group. Also French philosophers specialized in Arabic commentaries on Aristotle's *Rhetoric*.[74]

Links between the two strands – literary and philosophical – of the French school of rhetoric are strong and collaborative, and bear witness to the revival of rhetoric in France.[75] A recent issue of *Philosophy & Rhetoric* presents current writing in the field.[76]

Notes

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[2] See, e..g., Thomas Conley, Rhetoric in the European Tradition (University of Chicago, 1991).

[3] "...rhetoric is a combination of the science of logic and of the ethical branch of politics..." Aristotle. *Rhetoric.* (trans. W. Rhys Roberts). I:4:1359. (http://www.public.iastate.edu/~honeyl/Rhetoric/rhet1-4.html); Aristotle, Rhetoric 1.2.1, (http://rhetoric.eserver.org/aristotle/rhet1-2.html)

[4] The definition of rhetoric is a controversial subject within the field and has given rise to philological battles over its meaning in Ancient Greece. See, for instance, Johnstone, Henry W. Jr. (1995). "On Schiappa versus Poulakos." *Rhetoric Review.* 14:2. (Spring), 438-440.

[5] Perseus.Tufts.edu (http://www.perseus.tufts.edu/cgi-bin/ptext?doc=Perseus:text:1999.04.0057:entry=#92374), Rhetorikos, Henry George Liddell, Robert Scott, *A Greek-English Lexicon*, at Perseus

[6] Perseus.Tufts.edu (http://www.perseus.tufts.edu/cgi-bin/ptext?doc=Perseus:text:1999.04.0057:entry=#92385), Rhetor, Henry George Liddell, Robert Scott, *A Greek-English Lexicon*, at Perseus

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[9] John S. Nelson, Allan Megill, and Donald N. McCloskey *The Rhetoric of Human Sciences: Language and Argument in Scholarship and Public Affairs* (http://hhs.sagepub.com/cgi/content/citation/3/2/305?ck=nck), London: University of Wisconsin Press, 1987. ; "In the last ten years, many scholars have investigated exactly how rhetoric works within a particular field." Theodora Polito, *Educational Theory as Theory of Culture: A Vichian perspective on the educational theories of John Dewey and Kieran Egan* (http://www.ingentaconnect.com/content/bpl/epat/2005/00000037/00000004/art00003) Educational Philosophy and Theory, Vol. 37, No. 4, 2005; Deirdre N. McCloskey (1985) *The Rhetoric of Economics* (http://books.google.com/books?id=WtcKAAAACAAJ); JSTOR.org (http://links.jstor.org/

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[11] Plato, "Gorgias," The Classical Library (http://www.classicallibrary.org/plato/dialogues/15_gorgias.htm)

[12] Rapp, Christof. Aristotle's Rhetoric - The Agenda of the Rhetoric (http://plato.stanford.edu/entries/aristotle-rhetoric/#agenda), *Stanford Encyclopedia of Philosophy*

[13] George A. Kennedy, Aristotle, On Rhetoric: A Theory of Civic Discourse (New York: Oxford University Press, 1991).

[14] Kenneth Burke, A Rhetoric of Motives, (Berkeley: University of California Press, 1969).

[15] James Boyd White, When Words Lose Their Meaning (Chicago: The University of Chicago Press, 1984).

[16] Michael Leff, "The Habitation of Rhetoric" in Contemporary Rhetorical Theory: A Reader, ed. John Louis Lucaites, et al. (New York: Guilford Press, 1993).

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[22] cf. Conley, T.M. (1990) Rhetoric in the European Tradition. (University of Chicago Press.; Kennedy, G.A., 1994). A New History of Classical Rhetoric. Princeton University Press.

[23] "Rhetoric." Augnet. N.p., 2010. Web. 12 April 2010. Augnet.org (http://www.augnet.org/default.asp?ipageid=412)

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Dialectic

Dialectic (also *dialectics* and *the dialectical method*) is a method of argument for resolving disagreement that has been central to Indian and European philosophy since antiquity. The word *dialectic* originated in ancient Greece, and was made popular by Plato in the Socratic dialogues. The dialectical method is dialogue between two or more people holding different points of view about a subject, who wish to establish the truth of the matter by dialogue, with reasoned arguments.[1] Dialectics is different from debate, wherein the debaters are committed to their points of view, and mean to win the debate, either by persuading the opponent, proving their argument correct, or proving the opponent's argument incorrect – thus, either a judge or a jury must decide who wins the debate. Dialectics is also different from rhetoric, wherein the speaker uses logos, pathos, or ethos to persuade listeners to take their side of the argument.

The Sophists taught *aretē* (Greek: ἀρετή, *quality*, *excellence*) as the highest value, and the determinant of one's actions in life. The Sophists taught artistic quality in oratory (motivation via speech) as a manner of demonstrating one's *aretē*. Oratory was taught as an art form, used to please and to influence other people via excellent speech; nonetheless, the Sophists taught the pupil to seek *aretē* in all endeavours, not solely in oratory.

Socrates favoured *truth* as the highest value, proposing that it could be discovered through reason and logic in discussion: ergo, *dialectic*. Socrates valued rationality (appealing to logic, not emotion) as the proper means for persuasion, the discovery of truth, and the determinant for one's actions. To Socrates, *truth*, not *aretē*, was the greater good, and each person should, above all else, seek truth to guide one's life. Therefore, Socrates opposed the Sophists and their teaching of rhetoric as art and as emotional oratory requiring neither logic nor proof.[2] Different forms of dialectical reasoning have emerged throughout history from the Indosphere (Greater India) and the West (Europe). These forms include the Socratic method, Hindu, Buddhist, Medieval, Hegelian dialectics, Marxist, Talmudic, and Neo-orthodoxy.

Principles

The purpose of the dialectic method of reasoning is resolution of disagreement through rational discussion, and, ultimately, the search for truth.[3][4] One way to proceed—the Socratic method—is to show that a given hypothesis (with other admissions) leads to a contradiction; thus, forcing the withdrawal of the hypothesis as a candidate for truth (see reductio ad absurdum). Another dialectical resolution of disagreement is by denying a presupposition of the contending thesis and antithesis; thereby, proceeding to *sublation* (transcendence) to *synthesis*, a third thesis.

It is also possible that the rejection of the participants' presuppositions is resisted, which then might generate a second-order controversy.[5]

Fichtean Dialectics (Hegelian Dialectics) is based upon four concepts:

1. Everything is transient and finite, existing in the medium of time.
2. Everything is composed of contradictions (opposing forces).
3. Gradual changes lead to crises, turning points when one force overcomes its opponent force (quantitative change leads to qualitative change).
4. Change is helical (spiral), not circular (negation of the negation).[6]

The concept of *dialectic* existed in the philosophy of Heraclitus of Ephesus, who proposed that everything is in constant change, as a result of inner strife and opposition.[7][8][9] Hence, the history of the dialectical method is the history of philosophy.[10]

Western dialectical forms

Classical philosophy

In classical philosophy, dialectic (Greek: διαλεκτική) is a form of reasoning based upon dialogue of arguments and counter-arguments, advocating *propositions* (theses) and *counter-propositions* (antitheses). The outcome of such a dialectic might be the refutation of a relevant proposition, or of a synthesis, or a combination of the opposing assertions, or a qualitative improvement of the dialogue.[11][12]

Moreover, the term "dialectic" owes much of its prestige to its role in the philosophies of Socrates and Plato, in the Greek Classical period (4–5 c. BCE). Aristotle said that it was the pre-Socratic philosopher Zeno of Elea who invented dialectic, of which the dialogues of Plato are the examples of the Socratic dialectical method.[13]

Socratic dialogue

In Plato's dialogues and other Socratic dialogues, Socrates attempts to examine someone's beliefs, at times even first principles or premises by which we all reason and argue. Socrates typically argues by cross-examining his interlocutor's claims and premises in order to draw out a contradiction or inconsistency among them. According to Plato, the rational detection of error amounts to finding the proof of the antithesis.[14] However, important as this objective is, the principal aim of Socratic activity seems to be to improve the soul of his interlocutors, by freeing them from unrecognized errors.

For example, in the *Euthyphro*, Socrates asks Euthyphro to provide a definition of piety. Euthyphro replies that the pious is that which is loved by the gods. But, Socrates also has Euthyphro agreeing that the gods are quarrelsome and their quarrels, like human quarrels, concern objects of love or hatred. Therefore, Socrates reasons, at least one thing exists that certain gods love but other gods hate. Again, Euthyphro agrees. Socrates concludes that if Euthyphro's definition of piety is acceptable, then there must exist at least one thing that is both pious and impious (as it is both loved and hated by the gods)—which Euthyphro admits is absurd. Thus, Euthyphro is brought to a realization by this dialectical method that his definition of piety is not sufficiently meaningful.

There is another interpretation of the dialectic, as a method of intuition suggested in The Republic.[15] Simon Blackburn writes that the dialectic in this sense is used to understand "the total process of enlightenment, whereby the philosopher is educated so as to achieve knowledge of the supreme good, the Form of the Good".[16]

Medieval philosophy

Dialectics (also called logic) was one of the three liberal arts taught in medieval universities as part of the trivium. The trivium also included rhetoric and grammar.[17][18][19][20]

Based mainly on Aristotle, the first medieval philosopher to work on dialectics was Boethius.[21] After him, many scholastic philosophers also made use of dialectics in their works, such as Abelard,[22] William of Sherwood,[23] Garlandus Compotista,[24] Walter Burley, Roger Swyneshed and William of Ockham.[25]

This dialectic was formed as follows:

1. The Question to be determined
2. The principal objections to the question
3. An argument in favor of the Question, traditionally a single argument ("On the contrary..")
4. The determination of the Question after weighing the evidence. ("I answer that...")
5. The replies to each objection

Modern philosophy

The concept of dialectics was given new life by Georg Wilhelm Friedrich Hegel (following Fichte), whose dialectically dynamic model of nature and of history made it, as it were, a fundamental aspect of the nature of reality (instead of regarding the contradictions into which dialectics leads as a sign of the sterility of the dialectical method, as Immanuel Kant tended to do in his *Critique of Pure Reason*).[26][27] In the mid-19th century, the concept of "dialectic" was appropriated by Karl Marx (see, for example, *Das Kapital*, published in 1867) and Friedrich Engels and retooled in a non-idealist manner, becoming a crucial notion in their philosophy of dialectical materialism. Thus this concept has played a prominent role on the world stage and in world history. In contemporary polemics, "dialectics" may also refer to an understanding of how we can or should perceive the world (epistemology); an assertion that the nature of the world outside one's perception is interconnected, contradictory, and dynamic (ontology); or it can refer to a method of presentation of ideas and conclusions (discourse). According to Hegel, "dialectic" is the method by which human history unfolds; that is to say, history progresses as a dialectical process.

Hegelian dialectic

Hegelian dialectic, usually presented in a threefold manner, was stated by Heinrich Moritz Chalybäus as comprising three dialectical stages of development: a thesis, giving rise to its reaction, an antithesis, which contradicts or negates the thesis, and the tension between the two being resolved by means of a synthesis. Although this model is often named after Hegel, he himself never used that specific formulation. Hegel ascribed that terminology to Kant.[28] Carrying on Kant's work, Fichte greatly elaborated on the synthesis model, and popularized it.

On the other hand, Hegel did use a three-valued logical model that is very similar to the antithesis model, but Hegel's most usual terms were: Abstract-Negative-Concrete. Hegel used this writing model as a backbone to accompany his points in many of his works.

The formula, thesis-antithesis-synthesis, does not explain why the thesis requires an Antithesis. However, the formula, abstract-negative-concrete, suggests a flaw, or perhaps an incomplete-ness, in any initial thesis—it is too abstract and lacks the negative of trial, error and experience. For Hegel, the concrete, the synthesis, the absolute, must always pass through the phase of the negative, in the journey to completion, that is, mediation. This is the actual essence of what is popularly called Hegelian Dialectics.

To describe the activity of overcoming the negative, Hegel also often used the term *Aufhebung*, variously translated into English as "sublation" or "overcoming," to conceive of the working of the dialectic. Roughly, the term indicates preserving the useful portion of an idea, thing, society, etc., while moving beyond its limitations. (Jacques Derrida's preferred French translation of the term was *relever*).[29]

In the *Logic*, for instance, Hegel describes a dialectic of existence: first, existence must be posited as pure Being (*Sein*); but pure Being, upon examination, is found to be indistinguishable from Nothing (*Nichts*). When it is realized that what is coming into being is, at the same time, also returning to nothing (in life, for example, one's living is also a dying), both Being and Nothing are united as Becoming.[30]

As in the Socratic dialectic, Hegel claimed to proceed by making implicit contradictions explicit: each stage of the process is the product of contradictions inherent or implicit in the preceding stage. For Hegel, the whole of history is one tremendous dialectic, major stages of which chart a progression from self-alienation as slavery to self-unification and realization as the rational, constitutional state of free and equal citizens. The Hegelian dialectic cannot be mechanically applied for any chosen thesis. Critics argue that the selection of any antithesis, other than the logical negation of the thesis, is subjective. Then, if the logical negation is used as the antithesis, there is no rigorous way to derive a synthesis. In practice, when an antithesis is selected to suit the user's subjective purpose, the resulting "contradictions" are rhetorical, not logical, and the resulting synthesis is not rigorously defensible against a multitude of other possible syntheses. The problem with the Fichtean "Thesis-Antithesis-Synthesis" model is that it implies that contradictions or negations come from outside of things. Hegel's point is that they are inherent in and internal to things. This conception of dialectics derives ultimately from Heraclitus.

Hegel has outlined that the purpose of dialectics is "to study things in their own being and movement and thus to demonstrate the finitude of the partial categories of understanding"[31]

One important dialectical principle for Hegel is the transition from quantity to quality, which he terms the Measure. The measure is the qualitative quantum, the quantum is the existence of quantity.[32]

> "The identity between quantity and quality, which is found in Measure, is at first only implicit, and not yet explicitly realised. In other words, these two categories, which unite in Measure, each claim an independent authority. On the one hand, the quantitative features of existence may be altered, without affecting its quality. On the other hand, this increase and diminution, immaterial though it be, has its limit, by exceeding which the quality suffers change. [...] But if the quantity present in measure exceeds a certain limit, the quality corresponding to it is also put in abeyance. This however is not a negation of quality altogether, but only of this definite quality, the place of which is at once occupied by another. This process of measure, which appears alternately as a mere change in quantity, and then as a sudden revulsion of quantity into quality, may be envisaged under the figure of a nodal (knotted) line".[33]

As an example, Hegel mentions the states of aggregation of water: "Thus the temperature of water is, in the first place, a point of no consequence in respect of its liquidity: still with the increase or diminution of the temperature of the liquid water, there comes a point where this state of cohesion suffers a qualitative change, and the water is converted into steam or ice".[34] As other examples Hegel mentions the reaching of a point where a single additional grain makes a heap of wheat; or where the bald-tail is produced, if we continue plucking out single hairs.

Another important principle for Hegel is the negation of the negation, which he also terms *Aufhebung* (sublation): Something is only what it is in its relation to another, but by the negation of the negation this something incorporates the other into itself. The dialectical movement involves two moments that negate each other, something and its other. As a result of the negation of the negation, "something becomes its other; this other is itself something; therefore it likewise becomes an other, and so on ad infinitum".[35] Something in its passage into other only joins with itself, it is self-related.[36] In becoming there are two moments:[37] coming-to-be and ceasing-to-be: by sublation, i.e., negation of the negation, being passes over into nothing, it ceases to be, but something new shows up, is coming to be. What is sublated (*aufgehoben*) on the one hand ceases to be and is put to an end, but on the other hand it is preserved and maintained.[38] In dialectics, a totality transforms itself; it is self-related, then self-forgetful, relieving the original tension.

Marxist dialectics

Karl Marx and Friedrich Engels proposed that G.F. Hegel had rendered philosophy too abstractly ideal:

> The mystification which dialectic suffers in Hegel's hands, by no means prevents him from being the first to present its general form of working in a comprehensive and conscious manner. With him it is standing on its head. It must be turned right side up again, if you would discover the rational kernel within the mystical shell.[39]

In contradiction to Hegelian idealism, Karl Marx presented Dialectical materialism (Marxist dialectics):

> My dialectic method is not only different from the Hegelian, but is its direct opposite. To Hegel, the life-process of the human brain, i.e. the process of thinking, which, under the name of 'the Idea', he even transforms into an independent subject, is the demiurgos of the real world, and the real world is only the external, phenomenal form of 'the Idea'. With me, on the contrary, the ideal is nothing else than the material world reflected by the human mind, and translated into forms of thought. (*Capital*, Afterword, Second German Ed., Moscow, 1970, vol. 1, p. 29).

In Marxism, the dialectical method of historical study became intertwined with historical materialism, the school of thought exemplified by the works of Marx, Engels, and Vladimir Lenin. In the USSR, under Joseph Stalin, Marxist dialectics became "diamat" (short for dialectical materialism), a theory emphasizing the primacy of the material way of life, social "praxis," over all forms of social consciousness and the secondary, dependent character of the "ideal."

The term "dialectical materialism" was coined by the 19th-century social theorist Joseph Dietzgen who used the theory to explain the nature of socialism and social development. The original populariser of Marxism in Russia, Georgi Plekhanov used the terms "dialectical materialism" and "historical materialism" interchangeably. For Lenin, the primary feature of Marx's "dialectical materialism" (Lenin's term) was its application of materialist philosophy to history and social sciences. Lenin's main input in the philosophy of dialectical materialism was his theory of reflection, which presented human consciousness as a dynamic reflection of the objective material world that fully shapes its contents and structure. Later, Stalin's works on the subject established a rigid and formalistic division of Marxist-Leninist theory in the dialectical materialism and historical materialism parts. While the first was supposed to be the key method and theory of the philosophy of nature, the second was the Soviet version of the philosophy of history.

A dialectical method was fundamental to Marxist politics, e.g., the works of Karl Korsch, Georg Lukács and certain members of the Frankfurt School. Soviet academics, notably Evald Ilyenkov and Zaid Orudzhev, continued pursuing unorthodox philosophic study of Marxist dialectics; likewise in the West, notably the philosopher Bertell Ollman at New York University.

Friedrich Engels proposed that Nature is dialectical, thus, in Anti-Dühring he said that the negation of negation is:

> A very simple process, which is taking place everywhere and every day, which any child can understand as soon as it is stripped of the veil of mystery in which it was enveloped by the old idealist philosophy.[40]

In *Dialectics of Nature*, Engels said:

> Probably the same gentlemen who up to now have decried the transformation of quantity into quality as mysticism and incomprehensible transcendentalism will now declare that it is indeed something quite self-evident, trivial, and commonplace, which they have long employed, and so they have been taught nothing new. But to have formulated for the first time in its universally valid form a general law of development of Nature, society, and thought, will always remain an act of historic importance.[41]

Marxist dialectics is exemplified in *Das Kapital* (Capital), which outlines two central theories: (i) surplus value and (ii) the materialist conception of history; Marx explains dialectical materialism:

> In its rational form, it is a scandal and abomination to bourgeoisdom and its doctrinaire professors, because it includes in its comprehension an affirmative recognition of the existing state of things, at the same time, also, the recognition of the negation of that state, of its inevitable breaking up; because it regards every historically developed social form as in fluid movement, and therefore takes into account its transient nature not less than its momentary existence; because it lets nothing impose upon it, and is in its essence critical and revolutionary.[42]

Class struggle is the central contradiction to be resolved by Marxist dialectics, because of its central role in the social and political lives of a society. Nonetheless, Marx and Marxists developed the concept of class struggle to comprehend the dialectical contradictions between mental and manual labor, and between town and country. Hence, philosophic contradiction is central to the development of dialectics — the progress from quantity to quality, the acceleration of gradual social change; the negation of the initial development of the *status quo*; the negation of that negation; and the high-level recurrence of features of the original *status quo*. In the USSR, Progress Publishers issued anthologies of dialectical materialism by Lenin, wherein he also quotes Marx and Engels:

> As the most comprehensive and profound doctrine of development, and the richest in content, Hegelian dialectics was considered by Marx and Engels the greatest achievement of classical German philosophy.... “The great basic thought”, Engels writes, “that the world is not to be comprehended as a complex of ready-made things, but as a complex of processes, in which the things, apparently stable no less than their mind images in our heads, the concepts, go through an uninterrupted change of coming into being and passing away... this great fundamental thought has, especially since the time of Hegel, so thoroughly permeated ordinary consciousness that, in its generality, it is now scarcely ever contradicted.

> But, to acknowledge this fundamental thought in words, and to apply it in reality in detail to each domain of investigation, are two different things.... For dialectical philosophy nothing is final, absolute, sacred. It reveals the transitory character of everything and in everything; nothing can endure before it, except the uninterrupted process of becoming and of passing away, of endless ascendancy from the lower to the higher. And dialectical philosophy, itself, is nothing more than the mere reflection of this process in the thinking brain.” Thus, according to Marx, dialectics is “the science of the general laws of motion both of the external world and of human thought”.[43]

Lenin describes his dialectical understanding of the concept of *development*:

> A development that repeats, as it were, stages that have already been passed, but repeats them in a different way, on a higher basis (“the negation of the negation”), a development, so to speak, that proceeds in spirals, not in a straight line; a development by leaps, catastrophes, and revolutions; “breaks in continuity”; the transformation of quantity into quality; inner impulses towards development, imparted by the contradiction and conflict of the various forces and tendencies acting on a given body, or within a given phenomenon, or within a given society; the interdependence and the closest and indissoluble connection between all aspects of any phenomenon (history constantly revealing ever new aspects), a connection that provides a uniform, and universal process of motion, one that follows definite laws — these are some of the features of dialectics as a doctrine of development that is richer than the conventional one.[43]

In practice, marxist dialectics was frequently used as a tool of eristics and propaganda. In 1857 Marx explained that in a letter to Engels, commenting on his predictions published in New York Times:

> It is possible that I could disgrace myself. But there's always a bit of Dialectic to help out. I have naturally expressed my statements so that I am also right if the opposite thing happens.[44]

Indian forms of dialectic

Indian continental debate: an intra- and inter-Dharmic dialectic

Anacker (2005: p. 20), in the introduction to his translation of seven works by Vasubandhu (fl. 4th c.), a famed dialectician of the Gupta Empire, contextualizes the prestige of dialectic and cut-throat debate in classical India and makes references to the possibly apocryphal story of the banishment of Moheyan post-debate with Kamalaśīla (fl. 713–763):

> Philosophical debating was in classical India often a spectator-sport, much as contests of poetry-improvisation were in Germany in its High Middle Ages, and as they still are in the Telegu country today. The king himself was often the judge at these debates, and loss to an opponent could have serious consequences. To take an atrociously extreme example, when the Tamil Śaivite Ñānasambandar Nāyanār defeated the Jain ācāryas in Madurai before the Pāṇḍya King Māravarman Avaniśūlāmani (620-645) this debate is said to have resulted in the impalement of 8000 Jains, an event still celebrated in the Mīnāksi Temple of Madurai today. Usually, the results were not so drastic; they could mean formal recognition by the defeated side of the superiority of the winning party, forced conversions, or, as in the case of the *Council of Lhasa*, which was conducted by Indians, banishment of the losers.[45]

Brahmin/Vedic/Hindu dialectic

While Western philosophy traces dialectics to ancient Greek thought of Socrates and Plato, the idea of tension between two opposing forces leading to synthesis is much older and present in Hindu Philosophy.[46] Indian philosophy, for the most part subsumed within the Indian religions, has an ancient tradition of dialectic polemics. The two complements, "purusha" (the active cause) and the "prakriti" (the passive nature) brings everything into existence. They follow the "rta", the Dharma (Universal Law of Nature).

Jain dialectic

Anekantavada and Syadvada are the sophisticated dialectic traditions developed by the Jains to arrive at truth. As per Jainism, the truth or the reality is perceived differently from different points of view, and that no single point of view is the complete truth.[47][48] Jain doctrine of Anekantavada states that an object has infinite modes of existence and qualities and, as such, they cannot be completely perceived in all its aspects and manifestations, due to the inherent limitations of being human. Only the Kevalis—the omniscient beings—can comprehend the object in all its aspects and manifestations, and that all others are capable of knowing only a part of it. Consequently, no one view can claim to represent the absolute truth. According to Jains, the ultimate principle should always be logical and no principle can be devoid of logic or reason.[49] Thus one finds in the Jain texts, deliberative exhortations on any subject in all its facts, may they be constructive or obstructive, inferential or analytical, enlightening or destructive.[50]

Syādvāda is a theory of conditioned predication that provides an expression to anekānta by recommending that epithet *Syād* be attached to every expression.[51] Syādvāda is not only an extension of Anekānta ontology, but a separate system of logic capable of standing on its own force. The Sanskrit etymological root of the term Syād is "perhaps" or "maybe", but in context of syādvāda, it means "in some ways" or "from a perspective." As reality is complex, no single proposition can express the nature of reality fully. Thus the term "syāt" should be prefixed before each proposition giving it a conditional point of view and thus removing any dogmatism in the statement.[52] Since it ensures that each statement is expressed from seven different conditional and relative view points or propositions, it is known as theory of conditioned predication. These seven propositions also known as saptabhangi are:[53]

1. ***syād-asti***: "in some ways it is"
2. ***syād-nāsti***: "in some ways it is not"
3. ***syād-asti-nāsti***: "in some ways it is and it is not"
4. ***syād-asti-avaktavyaḥ***: "in some ways it is and it is indescribable"
5. ***syād-nāsti-avaktavyaḥ***: "in some ways it is not and it is indescribable"
6. ***syād-asti-nāsti-avaktavyaḥ***: "in some ways it is, it is not and it is indescribable"
7. ***syād-avaktavyaḥ***: "in some ways it is indescribable"

Buddhist dialectic

Buddhism has developed sophisticated, and sometimes highly institutionalized traditions of dialectics during its long history. Nalanda University, and later the Gelugpa Buddhism of Tibet, are examples. The historical development and clarification of Buddhist doctrine and polemics, through dialectics and formal debate, is well documented. Buddhist doctrine was rigorously critiqued (though not ultimately refuted) in the 2nd century by Nagarjuna, whose uncompromisingly logical approach to the realisation of truth, became the basis for the development of a vital stream of Buddhist thought. This dialectical approach of Buddhism, to the elucidation and articulation of an account of the Cosmos as the truth it really is, became known as the Perfection of Wisdom and was later developed by other notable thinkers, such as Dignaga and Dharmakirti (between 500 and 700). The dialectical method of truth-seeking is evident throughout the traditions of Madhyamaka, Yogacara, and Tantric Buddhism. Trisong Detsen, and later Je Tsongkhapa, championed the value of dialectic and of formalised training in debate in Tibet.

Dialectical theology

Neo-orthodoxy, in Europe also known as theology of crisis and dialectical theology,[54][55] is an approach to theology in Protestantism that was developed in the aftermath of the First World War (1914–1918). It is characterized as a reaction against doctrines of 19th-century liberal theology and a more positive reevaluation of the teachings of the Reformation, much of which had been in decline (especially in western Europe) since the late 18th century.[56] It is primarily associated with two Swiss professors and pastors, Karl Barth[57] (1886–1968) and Emil Brunner (1899–1966),[54][55] even though Barth himself expressed his unease in the use of the term.[58]

Dialectical method and dualism

Another way to understand dialectics is to view it as a method of thinking to overcome formal dualism and monistic reductionism.[59] For example, formal dualism regards the opposites as mutually exclusive entities, whilst monism finds each to be an epiphenomenon of the other. Dialectical thinking rejects both views. The dialectical method requires focus on both at the same time. It looks for a transcendence of the opposites entailing a leap of the imagination to a higher level, which (1) provides justification for rejecting both alternatives as false and/or (2) helps elucidate a real but previously veiled integral relationship between apparent opposites that have been kept apart and regarded as distinct. For example, the superposition principle of quantum physics can be explained using the dialectical method of thinking—likewise the example below from dialectical biology. Such examples showing the relationship of the dialectic method of thinking to the scientific method to a large part negates the criticism of Popper (see text below) that the two are mutually exclusive. The dialectic method also examines false alternatives presented by formal dualism (materialism vs idealism; rationalism vs empiricism; mind vs body, etc.) and looks for ways to transcend the opposites and form synthesis. In the dialectical method, both have something in common, and understanding of the parts requires understanding their relationship with the whole system. The dialectical method thus views the whole of reality as an evolving process.

Dialectical biology

In *The Dialectical Biologist* (Harvard U.P. 1985 ISBN 0-674-20281-3), Richard Levins and Richard Lewontin sketch a dialectical approach to biology. They see "dialectics" more as a set of questions to ask about biological research, a weapon against dogmatism, than as a set of pre-determined answers. They focus on the (dialectical) relationship between the "whole" (or totality) and the "parts." "Part makes whole, and whole makes part" (p. 272). That is, a biological system of some kind consists of a collection of heterogeneous parts. All of these contribute to the character of the whole, as in reductionist thinking. On the other hand, the whole has an existence independent of the parts and feeds back to affect and determine the nature of the parts. This back-and-forth (dialectic) of causation implies a dynamic process. For example, Darwinian evolution points to the competition of a variety of species, each with heterogeneous members, within a given environment. This leads to changing species and even to new species arising. A dialectical biologist fully accepts this picture then looks for ways in which the competing creatures (which serve as the internal conflicts in the environment) lead to changes. The changes manifest in the creatures themselves, through the creatures embracing biological adaptations that provide them with advantages, and in the environment itself, as when the action of microbes encourages the erosion of rocks. Further, each species is part of the "environment" of all the others.

Criticisms

Many philosophers have offered critiques of dialectic, and it can even be said that hostility or receptivity to dialectics is one of the things that divides twentieth-century Anglo-American philosophy from the so-called "continental" tradition, a divide that only a few contemporary philosophers (among them, G.H. von Wright, Paul Ricoeur, Hans-Georg Gadamer, Richard Rorty, Charles Taylor) have ventured to bridge.

It is generally thought dialectics has become central to "Continental" philosophy, while it plays no part in "Anglo-American" philosophy. In other words, on the continent of Europe, dialectics has entered intellectual culture as what might be called a legitimate part of thought and philosophy, whereas in America and Britain, the dialectic plays no discernible part in the intellectual culture, which instead tends toward positivism. A prime example of the European tradition is Jean-Paul Sartre's *Critique of Dialectical Reason*, which is very different from the works of Popper, whose philosophy was for a time highly influential in the UK where he resided (see below). Sartre states:

> "Existentialism, like Marxism, addresses itself to experience in order to discover there concrete syntheses. It can conceive of these syntheses only within a moving, dialectical totalisation, which is nothing else but history

> or—from the strictly cultural point of view adopted here—'philosophy-becoming-the world'."[60]

Karl Popper has attacked the dialectic repeatedly. In 1937 he wrote and delivered a paper entitled "What Is Dialectic?" in which he attacked the dialectical method for its willingness "to put up with contradictions".[61] Popper concluded the essay with these words: "The whole development of dialectic should be a warning against the dangers inherent in philosophical system-building. It should remind us that philosophy should not be made a basis for any sort of scientific system and that philosophers should be much more modest in their claims. One task which they can fulfill quite usefully is the study of the critical methods of science" (Ibid., p. 335).

In chapter 12 of volume 2 of *The Open Society and Its Enemies* (1944; 5th rev. ed., 1966) Popper unleashed a famous attack on Hegelian dialectics, in which he held that Hegel's thought (unjustly, in the view of some philosophers, such as Walter Kaufmann,[62]) was to some degree responsible for facilitating the rise of fascism in Europe by encouraging and justifying irrationalism. In section 17 of his 1961 "addenda" to *The Open Society*, entitled "Facts, Standards and Truth: A Further Criticism of Relativism," Popper refused to moderate his criticism of the Hegelian dialectic, arguing that it "played a major role in the downfall of the liberal movement in Germany,... by contributing to historicism and to an identification of might and right, encouraged totalitarian modes of thought. . . . [and] undermined and eventually lowered the traditional standards of intellectual responsibility and honesty".[63]

Formalism

In the past few decades, European and American logicians have attempted to provide mathematical foundations for dialectical logic or argument. There had been pre-formal treatises on argument and dialectic, from authors such as Stephen Toulmin (*The Uses of Argument*), Nicholas Rescher (*Dialectics*), and van Eemeren and Grootendorst (Pragma-dialectics). One can include the communities of informal logic and paraconsistent logic. However, building on theories of defeasible reasoning (see John L. Pollock), systems have been built that define well-formedness of arguments, rules governing the process of introducing arguments based on fixed assumptions, and rules for shifting burden. Many of these logics appear in the special area of artificial intelligence and law, though the computer scientists' interest in formalizing dialectic originates in a desire to build decision support and computer-supported collaborative work systems.[64]

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[5] The musicologist Rose Rosengard Subotnick gives this example: "A question posed in a Fred Friendly Seminar entitled *Hard Drugs, Hard Choices: The Crisis Beyond Our Borders* (http://www.fredfriendly.org/index.cfm?fuseaction=tv.detail&pid=40) (aired on WNET on February 26, 1990), illustrates that others, too, seem to find this dynamic enlightening: 'Are our lives so barren because we use drugs? Or do we use drugs because our lives are so barren?' The question is dialectical to the extent that it enables one to grasp the two opposed priorities as simultaneously valid."

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External links

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- David Walls, Dialectical Social Science (http://www.sonoma.edu/users/w/wallsd/dialectical.shtml)
- Dialectics for Kids (http://www.dialectics4kids.com)

Critical thinking

Critical thinking is a type of reasonable, reflective thinking that is aimed at deciding what to believe or what to do.[1] It is a way of deciding whether a claim is always true, sometimes true, partly true, or false. Critical thinking can be traced in Western thought to the Socratic method of Ancient Greece and in the East, to the Buddhist kalama sutta and Abhidharma. Critical thinking is an important component of most professions. It is a part of the formal education process and is increasingly significant as students progress through university to graduate education, although there is debate among educators about its precise meaning and scope.[2]

Definitions

Different sources define critical thinking variously as:

- "reasonable reflective thinking focused on deciding what to believe or do"[3]
- "the intellectually disciplined process of actively and skillfully conceptualizing, applying, analyzing, synthesizing, and/or evaluating information gathered from, or generated by, observation, experience, reflection, reasoning, or communication, as a guide to belief and action"[4]
- "purposeful, self-regulatory judgment which results in interpretation, analysis, evaluation, and inference, as well as explanation of the evidential, conceptual, methodological, criteriological, or contextual considerations upon which that judgment is based"[5]
- " includes a commitment to using reason in the formulation of our beliefs"[6]

Within the philosophical frame of critical social theory, critical thinking is commonly understood to involve commitment to the social and political practice of participatory democracy, willingness to imagine or remain open to considering alternative perspectives, willingness to integrate new or revised perspectives into our ways of thinking and acting, and willingness to foster criticality in others.[7]

History and etymology

The critical thinking philosophical frame traces its roots in analytic philosophy and pragmatist constructivism which dates back over 2,500 years, as in the Buddha's Teachings: mainly in the kalama sutta and the Abhidharma; as well as the Greek Socratic tradition in which probing questions were used to determine whether claims to knowledge based on authority could be rationally justified with clarity and logical consistency. The one sense of the term **critical** means *crucial*; a second sense derives from κριτικός **kritikos**, which means *discerning judgment*.[8] The movement represented a pragmatic response to expectations and demands for the kind of thinking required of the modern workforce.[9] The critical-theory philosophical frame traces its roots to the Frankfurt School of Critical Social Theory that attempted to amend Marxist theory for applicability in 20th-century Germany. Critical thinking within this philosophical frame was introduced by Max Horkheimer in his book *Traditional and Critical Theory* (1937).

Meaning

Critical thinking clarifies goals, examines assumptions, discerns hidden values, evaluates evidence, accomplishes actions, and assesses conclusions.

"Critical" as used in the expression "critical thinking" connotes involving skillful judgment as to truth, merit, etc. "Critical" in this context does not mean "disapproval" or "negative." There are many positive and useful uses of critical thinking, for example formulating a workable solution to a complex personal problem, deliberating as a group about what course of action to take, or analyzing the assumptions and the quality of the methods used in scientifically arriving at a reasonable level of confidence about a given hypothesis.

To add further clarification on what is meant by thinking critically, Richard Paul (1995) articulated critical thinking as either weak or strong.

The weak-sense critical thinker is a highly skilled but selfishly motivated pseudo-intellectual who works to advance one's personal agenda without seriously considering the ethical consequences and implications. Conceived as such, the weak-sense critical thinker is often highly skilled but uses those skills selectively so as to pursue unjust and selfish ends (Paul, 1995).

Conversely, the strong-sense critical thinker skillfully enters into the logic of problems and issues to see the problem for what it is without egocentric and/or socio-centric bias. Thus conceived, the strong-sense mind seeks to actively, systematically, reflectively, and fair-mindedly construct insight with sensitivity to expose and address the many obstacles that compromise high quality thought and learning. Using strong critical thinking we might evaluate an argument, for example, as worthy of acceptance because it is valid and based on true premises. Upon reflection, a speaker may be evaluated as a credible source of knowledge on a given topic.

Critical thinking can occur whenever one judges, decides, or solves a problem; in general, whenever one must figure out what to believe or what to do, and do so in a reasonable and reflective way. Reading, writing, speaking, and listening can all be done critically or uncritically. Critical thinking is crucial to becoming a close reader and a substantive writer. Expressed in most general terms, critical thinking is "a way of taking up the problems of life."[10]

Skills

The list of core critical thinking skills includes observation, interpretation, analysis, inference, evaluation, explanation, and meta-cognition. There is a reasonable level of consensus among experts that an individual or group engaged in strong critical thinking gives due consideration to establish:

- Evidence through observation
- Context
- Relevant criteria for making the judgment well
- Applicable methods or techniques for forming the judgment
- Applicable theoretical constructs for understanding the problem and the question at hand

In addition to possessing strong critical-thinking skills, one must be disposed to engage problems and decisions using those skills. Critical thinking employs not only logic but broad intellectual criteria such as clarity, credibility, accuracy, precision, relevance, depth, breadth, significance, and fairness.[11]

Procedure

Critical thinking calls for the ability to:

- Recognize problems, to find workable means for meeting those problems
- Understand the importance of prioritization and order of precedence in problem solving
- Gather and marshal pertinent (relevant) information
- Recognize unstated assumptions and values
- Comprehend and use language with accuracy, clarity, and discernment
- Interpret data, to appraise evidence and evaluate arguments
- Recognize the existence (or non-existence) of logical relationships between propositions
- Draw warranted conclusions and generalizations
- Put to test the conclusions and generalizations at which one arrives
- Reconstruct one's patterns of beliefs on the basis of wider experience
- Render accurate judgments about specific things and qualities in everyday life

In sum:

"A persistent effort to examine any belief or supposed form of knowledge in the light of the evidence that supports it and the further conclusions to which it tends."[12]

Example thinker

Irrespective of the sphere of thought, "a well-cultivated critical thinker":[13]

- raises important questions and problems, formulating them clearly and precisely
- gathers and assesses relevant information, using abstract ideas to interpret it effectively
- comes to well-reasoned conclusions and solutions, testing them against relevant criteria and standards
- thinks open-mindedly within alternative systems of thought, recognizing and assessing, as need be, their assumptions, implications, and practical consequences
- communicates effectively with others in figuring out solutions to complex problems, without being unduly influenced by others' thinking on the topic.

Principles and dispositions

Willingness to criticize oneself

Critical thinking is about being both willing and able to evaluate one's thinking. Thinking might be criticized because one does not have all the relevant information – indeed, important information may remain undiscovered, or the information may not even be knowable – or because one makes unjustified inferences, uses inappropriate concepts, or fails to notice important implications. One's thinking may be unclear, inaccurate, imprecise, irrelevant, narrow, shallow, illogical, or trivial, due to ignorance or misapplication of the appropriate learned skills of thinking.

On the other hand, one's thinking might be criticized as being the result of a sub-optimal disposition. The dispositional dimension of critical thinking is characterological. Its focus is in learning and developing the *habitual* intention to be truth-seeking, open-minded, systematic, analytical, inquisitive, confident in reasoning, and prudent in making judgments. Those who are ambivalent on one or more of these aspects of the disposition toward critical thinking or who have an opposite disposition (intellectually arrogant, biased, intolerant, emotional, disorganized, lazy, heedless of consequences, indifferent toward new information, mistrustful of reasoning, or imprudent) are more likely to encounter problems in using their critical-thinking skills. Failure to recognize the importance of correct dispositions can lead to various forms of self-deception and closed-mindedness, both individually and collectively.[14]

Reflective thought

In reflective problem solving and thoughtful decision making using critical thinking, one considers evidence (like investigating evidence), the context of judgment, the relevant criteria for making the judgment well, the applicable methods or techniques for forming the judgment, and the applicable theoretical constructs for understanding the problem and the question at hand.

The deliberation characteristic of strong critical thinking associates critical thinking with the reflective aspect of human reasoning. Those who would seek to improve our individual and collective capacity to engage problems using strong critical thinking skills are, therefore, recommending that we bring greater reflection and deliberation to decision making.

Critical thinking is based on self-corrective concepts and principles, not on hard and fast, or step-by-step, procedures.[15]

Competence

Critical thinking employs not only logic (either formal or, much more often, informal) but also broad intellectual criteria such as clarity, credibility, accuracy, precision, relevance, depth, breadth, significance and fairness.

Habits or traits of mind

The habits of mind that characterize a person strongly disposed toward critical thinking include a desire to follow reason and evidence wherever they may lead, a systematic approach to problem solving, inquisitiveness, even-handedness, and confidence in reasoning.[16]

When individuals possess intellectual skills alone, without the intellectual traits of mind, *weak sense critical thinking* results. Fair-minded or *strong sense critical thinking* requires intellectual humility, empathy, integrity, perseverance, courage, autonomy, confidence in reason, and other intellectual traits. Thus, critical thinking without essential intellectual traits often results in clever, but manipulative and often unethical or subjective thought.

Importance

Critical thinking is an important element of all professional fields and academic disciplines (by referencing their respective sets of permissible questions, evidence sources, criteria, etc.). Within the framework of scientific skepticism, the process of critical thinking involves the careful acquisition and interpretation of information and use of it to reach a well-justified conclusion. The concepts and principles of critical thinking can be applied to any context or case but only by reflecting upon the nature of that application. Critical thinking forms, therefore, a system of related, and overlapping, modes of thought such as anthropological thinking, sociological thinking, historical thinking, political thinking, psychological thinking, philosophical thinking, mathematical thinking, chemical thinking, biological thinking, ecological thinking, legal thinking, ethical thinking, musical thinking, thinking like a painter, sculptor, engineer, business person, etc. In other words, though critical thinking principles are universal, their application to disciplines requires a process of reflective contextualization.

Critical thinking is considered important in the academic fields because it enables one to analyze, evaluate, explain, and restructure their thinking, thereby decreasing the risk of adopting, acting on, or thinking with, a false belief. However, even with knowledge of the methods of logical inquiry and reasoning, mistakes can happen due to a thinker's inability to apply the methods or because of character traits such as egocentrism. Critical thinking includes identification of prejudice, bias, propaganda, self-deception, distortion, misinformation, etc. Given research in cognitive psychology, some educators believe that schools should focus on teaching their students critical thinking skills and cultivation of intellectual traits.

Socratic method is defined as "a prolonged series of questions and answers which refutes a moral assertion by leading an opponent to draw a conclusion that contradicts his own viewpoint."[17] Critical thinking skills through Socratic method taught in schools help create leaders. Instructors that promote critical thinking skills can benefit the students by increasing their confidence and creating a repeatable thought process to question and confidently approach a solution. Students also accomplish follower-ship skills that can be used to probe the leader's foundations. Critical thinking skills through Socratic method serve to produce professionals that are self-governing. However, Socratic method for critical thinking skills can become confusing if an instructor or leader uses the method too rigidly, the student may not know what the instructor or leader wants from him. An instructor or leader may disillusion the students if he uses particular style of questioning. Instructors must reveal their reasoning behind the questions in order to guide the students in the right direction. "Socratic method can serve twenty-first-century leaders to instruct students, mentor protégés, motivate followers, advise other leaders, and influence peers."[17]

Critical thinking skills can help nurses apply the process of examination. Nurses through critical thinking skills can question, evaluate, and reconstruct the nursing care process by challenging the established theory and practice. Critical thinking skills can helps nurse problem solve, reflect, and make a conclusive decision about the current situation they face. Critical thinking creates "new possibilities for the development of the nursing knowledge."[18] Due to the sociocultural, environmental, and political issues that are affecting healthcare delivery, it would be helpful to embody new techniques in nursing. Nurses can acquire critical thinking skills through the Socratic method of dialogue and reflection. Critical thinking also is considered important for human rights education for toleration. The Declaration of Principles on Tolerance adopted by UNESCO in 1995 affirms that "education for tolerance could aim at countering factors that lead to fear and exculsion of others, and could help young people to develop capacities for independent judgement, *critical thinking* and ethical reasoning."[19]

There is currently a growing recognition that the Western emphasis on critical thinking has a broader and deeper impact than relates simply to cognitive skills. Le Cornu (2009) argues a case which links critical thinking to a heightened individualism which she considers is not so prevalent in the East, and suggests that education at all levels should train people in three principal types of thinking and reflection: receptive, appreciative and critical.

Research

Edward Glaser proposed that the ability to think critically involves three elements:[12]

1. An attitude of being disposed to consider in a thoughtful way the problems and subjects that come within the range of one's experiences
2. Knowledge of the methods of logical inquiry and reasoning
3. Some skill in applying those methods.

Educational programs aimed at developing critical thinking in children and adult learners, individually or in group problem solving and decision making contexts, continue to address these same three central elements.

Contemporary cognitive psychology regards human reasoning as a complex process that is both reactive and reflective.[20]

The relationship between critical thinking skills and critical thinking dispositions is an empirical question. Some people have both in abundance, some have skills but not the disposition to use them, some are disposed but lack strong skills, and some have neither. Two measures of critical thinking dispositions are the California Critical Thinking Disposition Inventory[21] and the California Measure of Mental Motivation.[22]

In schooling

John Dewey is just one of many educational leaders who recognized that a curriculum aimed at building thinking skills would be a benefit not only to the individual learner, but to the community and to the entire democracy.

The key to seeing the significance of critical thinking in academics is in understanding the significance of critical thinking in learning. There are two meanings to the learning of this content. The first occurs when learners (for the first time) construct in their minds the basic ideas, principles, and theories that are inherent in content. This is a process of internalization. The second occurs when learners effectively use those ideas, principles, and theories as they become relevant in learners' lives. This is a process of application. Good teachers cultivate critical thinking (intellectually engaged thinking) at every stage of learning, including initial learning. This process of intellectual engagement is at the heart of the Oxford, Durham, Cambridge and London School of Economics tutorials. The tutor questions the students, often in a Socratic manner (see Socratic questioning). The key is that the teacher who fosters critical thinking fosters reflectiveness in students by asking questions that stimulate thinking essential to the construction of knowledge.

As emphasized above, each discipline adapts its use of critical thinking concepts and principles (principles like in school). The core concepts are always there, but they are embedded in subject-specific content. For students to learn content, intellectual engagement is crucial. All students must do their own thinking, their own construction of knowledge. Good teachers recognize this and therefore focus on the questions, readings, activities that stimulate the mind to take ownership of key concepts and principles underlying the subject.

In the UK school system, *Critical Thinking* is offered as a subject that 16- to 18-year-olds can take as an A-Level. Under the OCR exam board, students can sit two exam papers for the AS: "Credibility of Evidence" and "Assessing and Developing Argument". The full Advanced GCE is now available: in addition to the two AS units, candidates sit the two papers "Resolution of Dilemmas" and "Critical Reasoning". The A-level tests candidates on their ability to think critically about, and analyze, arguments on their deductive or inductive validity, as well as producing their own arguments. It also tests their ability to analyze certain related topics such as credibility and ethical decision-making. However, due to its comparative lack of subject content, many universities do not accept it as a main A-level for admissions.[23] Nevertheless, the AS is often useful in developing reasoning skills, and the full Advanced GCE is useful for degree courses in politics, philosophy, history or theology, providing the skills required for critical analysis that are useful, for example, in biblical study.

There used to also be an Advanced Extension Award offered in Critical Thinking in the UK, open to any A-level student regardless of whether they have the Critical Thinking A-level. Cambridge International Examinations have an A-level in Thinking Skills.[24]

From 2008, Assessment and Qualifications Alliance has also been offering an A-level Critical Thinking specification;[25]

OCR exam board have also modified theirs for 2008. Many examinations for university entrance set by universities, on top of A-level examinations, also include a critical thinking component, such as the LNAT, the UKCAT, the BioMedical Admissions Test and the Thinking Skills Assessment.

Research in efficiency of critical thinking instruction

A meta-analysis of the literature on teaching effectiveness in higher education has been undertaken.[26] The study noted concerns from higher education, politicians and business people that higher education was failing to meet society's requirements for well-educated citizens. The study concluded that although faculty may aspire to develop students' thinking skills, in practice they tend to aim at facts and concepts in the disciplines, at the lowest cognitive levels, rather than development of intellect or values.

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- Critical thinking (https://inpho.cogs.indiana.edu/taxonomy/2410) at the Indiana Philosophy Ontology Project
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- What "Critical" means in "Critical Thinking" (http://www.citigraphics.net/jenner/djenner/archive/CritiqueAndCriticalThinking.pdf) by Donald Jenner
- Critical Thinking Means Business (http://www.talentlens.com/en/downloads/whitepapers/Pearson_TalentLens_Critical_Thinking_Means_Business.pdf) – A guide to developing critical thinking ability by Pearson

Logicism

Logicism is one of the schools of thought in the philosophy of mathematics, putting forth the theory that mathematics is an extension of logic and therefore some or all mathematics is reducible to logic.[1] Bertrand Russell and Alfred North Whitehead championed this theory fathered by Richard Dedekind and Gottlob Frege. Dedekind's path to logicism had a turning point when he was able to reduce the theory of real numbers to the rational number system by means of set theory. This and related ideas convinced him that arithmetic, algebra and analysis were reducible to the natural numbers plus a "logic" of sets; furthermore by 1872 he had concluded that the naturals themselves were reducible to sets and mappings. It is likely that other logicists, most importantly Frege, were also guided by the new theories of the real numbers published in the year 1872. This started a period of expansion of logicism, with Dedekind and Frege as its main exponents, which however was brought to a deep crisis with the discovery of the classical paradoxes of set theory (Cantor 1896, Zermelo and Russell 1900-1901). Frege gave up on the project after Russell recognized and communicated his paradox exposing an inconsistency in naive set theory. On the other hand, Russell wrote The Principles of Mathematics in 1903 using the paradox and developments of Giuseppe Peano's school of geometry. Since he treated the subject of primitive notions in geometry and set theory, this text is a watershed in the development of logicism. Evidence of the assertion of logicism was collected by Russell and Whitehead in their *Principia Mathematica*.[2]

Today, the bulk of modern mathematics is believed to be reducible to a logical foundation using the axioms of Zermelo-Fraenkel set theory (or one of its extensions, such as ZFC), which has no known inconsistencies (although it remains possible that inconsistencies in it may still be discovered). Thus to some extent Dedekind's project was proved viable, but in the process the theory of sets and mappings came to be regarded as transcending pure logic.

Kurt Gödel's incompleteness theorem is sometimes alleged to undermine logicism because it shows that no particular axiomatization of mathematics can decide all statements. However, the basic spirit of logicism remains valid, as that theorem is proved with logic just like other theorems.

Logicism was key in the development of analytic philosophy in the twentieth century.

Origin of the name "logicism"

Grattan-Guiness states that the French word 'Logistique' was "introduced by Couturat and others at the 1904 International of Congress of Philosophy', and was used by Russell and others from then on, in versions appropriate for various languages" (G-G 2000:4502).

Apparently the first (and only) usage by Russell appeared in his 1919: "Russell referred several time [sic] to Frege, introducing him as one 'who first succeeded in "logicising" mathematics' (p. 7). Apart from the mis-representation (which Russell partly rectified by explaining his own view of the role of arithmetic in mathematics), the passage is notable for the word which he put in quotation marks, but their presence suggests nervousness, and he never used the word again, so that 'logicism' did not emerge until the later 1920s" (G-G 2002:434).[3]

About same time as Carnap (1929), but apparently independently, Fraenkel (1928) used the word: "Without comment he used the name 'logicism' to characterise the Whitehead/Russell position (in the title of the section on p. 244, explanation on p. 263)" (G-G 2002:269). Carnap used a slightly different word 'Logistik'; Behmann complained about its use in Carnap's manuscript so Carnap proposed the word "Logizisumus', but he finally stuck to his word-choice 'Logistik' (G-G 2002:501). Ultimately "the spread was mainly due to Carnap, from 1930 onwards." (G-G 2000:502).

Intent, or goal, of Logicism

Symbolic logic: The overt intent of Logicism is to reduce all of philosophy to symbolic logic (Russell), and/or to reduce all of mathematics to symbolic logic (Frege, Dedekind, Peano, Russell). As contrasted with algebraic logic (Boolean logic) that employs arithmetic concepts, symbolic logic begins with a very reduced set of marks (non-arithmetic symbols), a (very-)few "logical" axioms that embody the three "laws of thought," and a couple construction rules that dictate how the marks are to be assembled and manipulated—substitution and *modus ponens* (inference of the true from the true). Logicism also adopts from Frege's groundwork the reduction of natural language statements from "subject|predicate" into either propositional "atoms" or the "argument|function" of "generalization"—the notions "all," "some," "class" (collection, aggregate) and "relation."

As perhaps its core tenet, logicism forbids any "intuition" of number to sneak in either as an axiom or by accident. The goal is to derive all of mathematics, starting with the counting numbers and then the irrational numbers, from the "laws of thought" alone, without any tacit (hidden) assumptions of "before" and "after" or "less" and "more" or to the point: "successor" and "predecessor." Gödel 1944 summarized Russell's logicistic "constructions," when compared to "constructions" in the foundational systems of Intuitionism and Formalism ("the Hilbert School") as follows: "Both of these schools base their constructions on a mathematical intuition whose avoidance is exactly one of the principal aims of Russell's constructivism" (Gödel 1944 in *Collected Works* 1990:119).

History: Gödel 1944 summarized the historical background from Leibniz's in *Characteristica universalis*, through Frege and Peano to Russell: "Frege was chiefly interested in the analysis of thought and used his calculus in the first place for deriving arithmetic from pure logic", whereas Peano "was more interested in its applications within mathematics". But "It was only [Russell's] *Principia Mathematica* that full use was made of the new method for actually deriving large parts of mathematics from a very few logical concepts and axioms. In addition, the young science was enriched by a new instrument, the abstract theory of relations" (p. 120-121).

Kleene 1952 states it this way: "Leibniz (1666) first conceived of logic as a science containing the ideas and principles underlying all other sciences. Dedekind (1888) and Frege (1884, 1893, 1903) were engaged in defining mathematical notions in terms of logical ones, and Peano (1889, 1894-1908) in expressing mathematical theorems in a logical symbolism" (p. 43); in the previous paragraph he includes Russell and Whitehead as exemplars of the "logicistic school," the other two "foundational" schools being the intuitionistic and the "formalistic or axiomatic school" (p. 43).

Dedekind 1887 describes his intent in the 1887 Preface to the First Edition of his *The Nature and Meaning of Numbers*. He believed that in the "foundations of the simplest science; viz., that part of logic which deals with the theory of numbers" had not been properly argued -- "nothing capable of proof ought to be accepted without proof":

> In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number-concept entirely independent of the notions of intuitions of space and time, that I consider it an immediate result from the laws of thought . . . numbers are free creations of the human mind . . . [and] only through the purely logical process of building up the science of numbers . . . are we prepared accurately to investigate our notions of space and time by bringing them into relation with this number-domain created in our mind" (Dedekind 1887 Dover republication 1963 :31).

Peano 1889 states his intent in his Preface to his 1889 *Principles of Arithmetic*:

> Questions that pertain to the foundations of mathematics, although treated by many in recent times, still lack a satisfactory solution. The difficulty has its main source in the ambiguity of language. ¶ That is why it is of the utmost importance to examine attentively the very words we use. My goal has been to undertake this examination" (Peano 1889 in van Heijenoort 1967:85).

Frege 1879 describes his intent in the Preface to his 1879 *Begriffsschrift*: He started with a consideration of arithmetic: did it derive from "logic" or from "facts of experience"?

"I first had to ascertain how far one could proceed in arithmetic by means of inferences alone, with the sole support of those laws of thought that transcend all particulars. My initial step was to attempt to reduce the concept of ordering in a sequence to that of *logical* consequence, so as to proceed from there to the concept of number. To prevent anything intuitive from penetrating here unnoticed I had to bend every effort to keep the chain of inferences free of gaps . . . I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present ideography. Its first purpose, therefore, is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed" (Frege 1879 in van Heijenoort 1967:5).

Russell 1903 describes his intent in the Preface to his 1903 *Principles of Mathematics*:

"THE present work has two main objects. One of these, the *proof* that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles" (Preface 1903:vi).

"A few words as to the origin of the present work may serve to show the importance of the questions discussed. About six years ago, I began an investigation into the philosophy of Dynamics. . . . [From two questions -- acceleration and absolute motion in a "relational theory of space"] I was led to a re-examination of the principles of Geometry, thence to the philosophy of continuity and infinity, and then, with a view to discovering the meaning of the word *any*, to Symbolic Logic" (Preface 1903:vi-vii).

Epistemology behind logicism

TBD: [Dedekind's and Frege's epistemology needs expansion]

Dedekind and Frege: The epistemology of Dedekind and Frege is not as well-defined as that of the philosopher Russell, but both seem accepting as *a priori* the customary "laws of thought" concerning simple propositional statements (usually of belief); these laws would be sufficient in themselves if augmented with theory of classes and relations (e.g. x R y) between individuals x and y linked by the generalization R.

Dedekind's "free formations of the human mind" rebels against the strictures of Kronecker: Dedekind's argument begins with "1. In what follows I understand by *thing* every object of our thought"; we humans use symbols to discuss these "things" of our minds; "A thing is completely determined by all that can be affirmed or thougt concerning it" (p. 44). In a subsequent paragraph Dedekind is discusses what a "system *S* is: it is an aggregate, a manifold, a totality of associated elements (things) *a*, *b*, *c*; he asserts that "such a system *S* . . . *as an object of our thought is likewise a thing* (1); it is completely determined when with respect to every thing it is determined whether it is an element of *S* or not.*" (p. 45, italics added). The * indicates a footnote where he states that:

"Kronecker not long ago (*Crelle's Journal*, Vol. 99, pp. 334-336) has endeavored to impose certain limitations upon the free formation of concepts in mathematics which I do not believe to be justified" (p. 45).

Indeed he awaits Kronecker's "publishing his reasons for the necessity or merely the expediency of these limitations" (p. 45).

Leopold Kronecker, famous for his assertion that "God made the integers, all else is the work of man"[4] had his foes, among them the formidable Hilbert. Hilbert called Kronecker a " *dogmatist*, to the extent that he accepts the integer with its essential properties as a dogma and does not look back"[5] and equate his extreme constructivist stance with that of Brouwer's Intuitionism, accusing both of "subjectivism": "It is part of the task of science to liberate us from arbitrariness, sentiment and habit and to protect us from the subjectivism that already made itself felt in Kronecker's views and, it seems to me, finds its culmination in intuitionism".[6] Hilbert then baldly states that "mathematics is a presuppositionless science. To found it I do not need God, as does Kronecker . . ."(p. 479).

[TBD: There is more discussion to be found in Grattan-Guinness re Kronecker, Cantor, the Crelle journal edited by Kronecker et. al., philosophies of Cantor and Kronecker.]

Russell the realist: Russell's Realism served him as an antidote to British Idealism,[7] with portions borrowed from European Rationalism and British empiricism.[8] To begin with, "Russell was a realist about two key issues: universals and material objects" (Russell 1912:xi). For Russell, tables are real things that exist independent of Russell the observer. Rationalism would contribute the notion of *a priori* knowledge,[9] while empiricism would contribute the role of experiential knowledge (induction from experience).[10] Russell would credit Kant with the idea of "a priori" knowledge, but he offers an objection to Kant he deems "fatal": "The facts [of the world] must always conform to logic and arithmetic. To say that logic and arithmetic are contributed by us does not account for this" (1912:87); Russell concludes that the *a priori* knowledge that we possess is "about things, and not merely about thoughts" (1912:89). And in this Russell's epistemology seems different from that of Dedekind's belief that "numbers are free creations of the human mind" (Dedekind 1887:31)[11]

But his epistemology about the innate (he prefers the word *a priori* when applied to logical principles, cf 1912:74) is intricate. He would strongly, unambiguously express support for the Platonic "universals" (cf 1912:91-118) and he would conclude that truth and falsity are "out there"; minds create *beliefs* and what makes a belief true is a fact, "and this fact does not (except in exceptional cases) involve the mind of the person who has the belief" (1912:130).

Where did Russell derive these epistemic notions? He tells us in the Preface to his 1903 *Principles of Mathematics*. Note that he asserts that the belief: "Emily is a rabbit" is non-existent, and yet the truth of this non-existent proposition is independent of any knowing mind; if Emily really is a rabbit, the fact of this truth exists whether or not Russell or any other mind is alive or dead, and the relation of Emily to rabbit-hood is "ultimate" :

> "On fundamental questions of philosophy, my position, in all its chief features, is derived from Mr G. E. Moore. I have accepted from him the non-existential nature of propositions (except such as happen to assert existence) and their independence of any knowing mind; also the pluralism which regards the world, both that of existents and that of entities, as composed of an infinite number of mutually independent entities, with relations which are ultimate, and not reducible to adjectives of their terms or of the whole which these compose. . . . The doctrines just mentioned are, in my opinion, quite indispensable to any even tolerably satisfactory philosophy of mathematics, as I hope the following pages will show. . . . Formally, my premisses are simply assumed; but the fact that they allow mathematics to be true, which most current philosophies do not, is surely a powerful argument in their favour." (Preface 1903:viii)

Russell and the paradox: In 1902 Russell discovered of a "vicious circle" (the so-called Russell's paradox) in Frege's *Begriffsschrift* and he was determined not to repeat it in his 1903 *Principles of Mathematics*. In two Appendices that he tacked on at the last minute he devotes 28 pages to a detailed analysis of, first Frege's theory contrasted against his own, and secondly a fix for the paradox. Unfortunately he was not optimistic about the outcome:

> "In the case of classes, I must confess, I have failed to perceive any concept fulfilling the conditions requisite for the notion of class. And the contradiction discussed in Chapter x. proves that something is amiss, but what this is I have hitherto failed to discover. (Preface to Russell 1903:vi)"

"Fictionalism" and Russell's no-class theory: Gödel in his 1944 would disagree with the young Russell of 1903 ("[my premisses] allow mathematics to be true") but would probably agree with Russell's statement quoted above ("something is amiss"); Russell's theory had failed to arrive at a satisfactory foundation of mathematics: the result was "essentially negative; i.e. the classes and concepts introduced this way do not have all the properties required for the use of mathematics" (Gödel 1944:132).

How did Russell arrive in this situation? Gödel observes that Russell is a surprising "realist" with a twist: he cites Russell's 1919:169 "Logic is concerned with the real world just as truly as zoology" (Gödel 1944:120). But he observes that "when he started on a concrete problem, the objects to be analyzed (e.g. the classes or propositions) soon for the most part turned into "logical fictions" . . . [meaning] only that we have no direct perception of them."

(Gödel 1944:120)

In an observation pertainent to Russell's brand of logicism, Perry remarks that Russell went through three phases of realism -- extreme, moderate and constructive (Perry 1997:xxv). In 1903 he was in his extreme phase; by 1905 he would be in his moderate phase. In a few years he would "dispense with physical or material objects as basic bits of the furniture of the world. He would attempt to construct them out of sense-data" in his next book *Our knowledge of the External World*[1914]" (Perry 1997:xxvi).

These constructions in what Gödel 1944 would call "nominalistic constructivism . . . which might better be called fictionalism" derived from Russell's "more radical idea, the "no-class theory" (p. 125):

> "according to which classes or concepts *never* exist as real objects, and sentences containing these terms are meaningful only as they can be interpreted as . . . a manner of speaking about other things" (p. 125).

See more in the Criticism sections, below.

The Logistic construction of the natural numbers

The attempt to construct the natural numbers is summarized succinctly by Bernays 1930-1931.[12] But rather than use Bernays' précis, which is incomplete in the details, the construction is best given as a simple finite example together with the details to be found in Russell 1919.

In *general* the logicism of Dedekind-Frege is similar to that of Russell, but with significant (and critical) differences in the particulars (see Criticisms, below). Overall, though, the logicistic construction-process [Dedekind-Frege-Russell] is far different than that of contemporary set theory. Whereas in set theory the notion of "number" begins from an *axiom*—the axiom of pairing that leads to the definition of "ordered pair"—no *overt* number-axiom exists in logicism. Rather, logicism begins its *construction* of the numbers from "primitive propositions" that include "class", "propositional function", and *in particular*, "relations" of "similarity" ("equinumerosity": placing the elements of collections in one-to-one correspondence) and "ordering" (using "the successor of" relation to order the collections of the equinumerous classes)".[13] The logicistic derivation equates the cardinal numbers *constructed* this way to the natural numbers, and these numbers end up all of the same "type"—as equivalence classes of classes—whereas in set theory each number is of a higher class than its predecessor (thus each successor contains its predecessor as a subset). Kleene observes that:

> "The viewpoint here is very different from that [of Kronecker's supposition that 'God made the integers' plus Peano's axioms of number and mathematical induction,] where we supposed an intuitive conception of the number sequence and elicited from it the principle that, whenever a particular property P of natural numbers is given such that (1) and (2), then any given natural number must have the property P" (Kleene 1952:44).

The importance to logicism of the construction of the natural numbers derives from Russell's contention that "That all traditional pure mathematics can be derived from the natural numbers is a fairly recent discovery, though it had long been suspected" (1919:4). The derivation of the *real* numbers (rationals, irrationals) derives from the theory of Dedekind cuts on the continuous "number line". While an example of how this is done is useful, it relies first on the derivation of the natural numbers. So, if philosophic problems appear in the logistic attempt to derive the natural numbers, these problems will be sufficient to stop the program until these are fixed (see Criticisms, below).

Preliminaries

For Dedekind, Frege and Russell, collections (classes) are aggregates of "things" specified by proper names, that come about as the result of propositions (utterances about something that asserts a fact about that thing or things). Russell tore the general notion down in the following manner. He begins with "terms" in sentences that he decomposes as follows:

Terms: For Russell, "terms" are either "things" or "concepts": "Whatever may be an object of thought, or may occur in any true or false proposition, or can be counted as one, I call a *term*. This, then, is the widest word in the philosophical vocabulary. I shall use as synonymous with it the words, unit, individual, and entity. The first two emphasize the fact that every term is one, while the third is derived from the fact that every term has being, i.e. is in some sense. A man, a moment, a number, a class, a relation, a chimaera, or anything else that can be mentioned, is sure to be a term; and to deny that such and such a thing is a term must always be false" (Russell 1903:43)

Things are indicated by proper names; concepts are indicated by adjectives or verbs: "Among terms, it is possible to distinguish two kinds, which I shall call respectively *things* and *concepts*; the former are the terms indicated by proper names, the latter those indicated by all other words . . . Among concepts, again, two kinds at least must be distinguished, namely those indicated by adjectives and those indicated by verbs" (1903:44).

Concept-adjectives are "predicates"; concept-verbs are "relations": "The former kind will often be called predicates or class-concepts; the latter are always or almost always relations." (1903:44)

The notion of a "variable" subject appearing in a proposition: "I shall speak of the *terms* of a proposition as those terms, however numerous, which occur in a proposition and may be regarded as subjects about which the proposition is. It is a characteristic of the terms of a proposition that anyone of them may be replaced by any other entity without our ceasing to have a proposition. Thus we shall say that "Socrates is human" is a proposition having only one term; of the remaining component of the proposition, one is the verb, the other is a predicate.. . . Predicates, then, are concepts, other than verbs, which occur in propositions having only one term or subject." (1903:45)

In other words, a "term" can be place-holder that indicates (denotes) one or more things that can be put into the placeholder. (1903:45).

Truth and falsehood: Suppose Russell were to point to an object and utter: "This object in front of me named "Emily" is a woman." This is a proposition, an assertion of Russell's belief to be tested against the "facts" of the outer world: "Minds do not *create* truth or falsehood. They create beliefs . . . what makes a belief true is a *fact*, and this fact does not (except in exceptional cases) in any way involve the mind of the person who has the belief" (1912:130). If by investigation of the utterance and correspondence with "fact", Russell discovers that Emily is a rabbit, then his utterance is considered "false"; if Emily is a female human (a female "featherless biped" as Russell likes to call humans), then his utterance is considered "true".

If Russell were to utter a *generalization* about all Emilys then these object/s (entity/ies) must be examined, one after another in order to verify the truth of the generalization. Thus if Russell were to assert "All Emilys are women", then the "All" is a tipoff that the utterance is about all entities "Emily" in correspondence with a concept labeled "woman" and a methodical examination of all creatures with human names would have to commence.

Classes (aggregates, complexes): "The class, as opposed to the class-concept, is the sum or conjunction of all the terms which have the given predicate" (1903 p. 55). Classes can be specified by extension (listing their members) or by intension, i.e. by a "propositional function" such as "x is a u" or "x is v". But "if we take extension pure, our class is defined by enumeration of its terms, and this method will not allow us to deal, as Symbolic Logic does, with infinite classes. Thus our classes must in general be regarded as objects denoted by concepts, and to this extent the point of view of intension is essential." (1909 p. 66)

Propositional functions: "The characteristic of a class concept, as distinguished from terms in general, is that "x is a u" is a propositional function when, and only when, u is a class-concept." (1903:56)

Extensional versus intensional definition of a class: "71. Class may be defined either extensionally or intensionally. That is to say, we may define the kind of object which is a class, or the kind of concept which denotes a class: this is the precise meaning of the opposition of extension and intension in this connection. But although the general notion can be defined in this two-fold manner, particular classes, except when they happen to be finite, can only be defined intensionally, i.e. as the objects denoted by such and such concepts. . . logically; the extensional definition appears to be equally applicable to infinite classes, but practically, if we were to attempt it, Death would cut short our laudable endeavour before it had attained its goal.(1903:69)

The definition of the natural numbers

The natural numbers derive from *ALL* propositions (i.e. completely unrestricted) in this and all other possible worlds, that can be uttered about *ANY* collection of entities whatsoever. Russell makes this clear in the second (italicized) sentence:

> "In the first place, numbers themselves form an infinite collection, and cannot therefore be defined by enumeration. *In the second place, the collections having a given number of terms themselves presumably form an infinite collection: it is to be presumed, for example, that there are an infinite collection of trios in the world*, for if this were not the case the total number of things in the world would be finite, which, though possible, seems unlikely. In the third place, we wish to define "number" in such a way that infinite numbers may be possible; thus we must be able to speak of the number of terms in an infinite collection, and such a collection must be defined by intension, i.e. by a property common to all its members and peculiar to them." (1919:13)

To begin, devise a finite example. Suppose there are 12 families on a street. Some have children, some do not. To discuss the names of the children in these households requires 12 propositions asserting " *childname* is the name of a child in family Fn" applied this collection of households on the particular street of families with names F1, F2, . . . F12. Each of the 12 propositions regards whether or not the "argument" *childname* applies to a child in a particular household. The children's names (*childname*) can be thought of as the x in a propositional function f(x), where the function is "name of a child in the family with name Fn".[14]

To keep things simple all 26 letters of the alphabet are used up in this example, each letter representing the name of a particular child (in real life there could be repeats). Notice that, in the Russellian view these collections are not sets, but rather "aggregates" or "collections" or "classes"—listings of names that satisfy the predicates F1, F2, ... As noted in Step 1, For Russell, these "classes" are "symbolic fictions" that exist only as their aggregate members, i.e. as the *extensions of their propositional functions*, and not as unit-things in themselves.

Step 1: Assemble *ALL* the classes: Whereas the following example is finite over the very-finite propositional function " *childnames* of the children in family Fn'" on the very-finite street of a finite number of (12) families, Russell intended the following to extend to *ALL* propositional functions extending over an infinity of this and all other possible worlds; this would allow him to create *ALL* the numbers (to infinity).

Kleene observes that already Russell has set himself up with an impredicative definition that he will have to resolve, or otherwise he will be confronted with his Russell paradox. "Here instead we presuppose the totality of all properties of cardinal numbers, as existing in logic, prior to the definition of the natural number sequence" (Kleene 1952:44). The problem will appear, even in the finite example presented here, when Russell confronts the unit class (cf Russell 1903:517).

The matter of debate comes down to this: what exactly *is* a "class"? For Dedekind and Frege, a class is a distinct entity all its own, a "unity" that can be identified with all those entities *x* that satisfy the propositional function F( ). (This symbolism appears in Russell, attributing it to Frege: "The essence of a function is what is left when the *x* is taken away, i.e in the above instance, $2(\ )^3 + (\ )$. The argument *x* does not belong to the function, but the two together make a whole (ib. p. 6 [i.e. Frege's 1891 *Function und Begriff*]" (Russell 1903:505).) For example, a particular "unity" could be given a name; suppose a family Fα has the children with the names Annie, Barbie and Charles:

[a, b, c]$_{F\alpha}$

This Dedekind-Frege construction could be symbolized by a bracketing process similar to, but to be distinguished from, the symbolism of contemporary set theory { a, b, c }, i.e. [ ] with the elements that satisfy the proposition separated by commas (an index to label each collection-as-a-unity will not be used, but could be):

[a, b, c], [d], [ ], [e, f, g], [h, i], [j, k], [l, m, n, o, p], [ ], [q, r], [s], [t, u], [v, w, x, y, z]

This notion of collection-or or class-as-object, when used without restriction, results in Russell's paradox; see more below about impredecative definitions. Russell's solution was to define the notion of a class to be only those elements that satisfy the proposition, his argument being that, indeed, the arguments *x* do not belong to the propositional function aka "class" created by the function. The class itself is not to be regarded as a unitary object in its own right, it exists only as a kind of useful fiction: "We have avoided the decision as to whether a class of things has in any sense an existence as one object. A decision of this question in either way is indifferent to our logic" (First edition of *Principia Mathematica* 1927:24).

Russell does not waver from this opinion in his 1919; observe the words "symbolic fictions":

> "When we have decided that classes cannot be things of the same sort as their members, that they cannot be just heaps or aggregates, and also that they cannot be identified with propositional functions, it becomes very difficult to see what they can be, if they are to be more than *symbolic fictions*. And if we can find any way of dealing with them as *symbolic fictions*, we increase the logical security of our position, since we avoid the need of assuming that there are classes without being compelled to make the opposite assumption that there are no classes. We merely abstain from both assumptions. . . . But when we refuse to assert that there are classes, we must not be supposed to be asserting dogmatically that there are none. We are merely agnostic as regards them . . .." (1919:184)

And by the second edition of *PM* (1927) Russell would insist that "functions occur only through their values, . . . all functions of functions are extensional, . . . [and] consequently there is no reason to distinguish between functions and classes . . . Thus classes, as distinct from functions, loose even that shadowy being which they retain in *20" (p. xxxix). In other words, classes as a separate notion have vanished altogether.

Given Russell's insistence that classes are not singular objects-in-themselves, but only collected aggregates, the only correct way to symbolize the above listing is to eliminate the brackets. But this is visually confusing, especially with regards to the null class, so a dashed vertical line at each end of the collection will be used to symbolize the collection-as-aggregate:

¦a, b, c¦, ¦d¦, ¦ ¦, ¦e, f, g¦, ¦h, i¦, ¦j, k¦, ¦l, m, n, o, p¦, ¦ ¦, ¦q, r¦, ¦s¦, ¦t, u¦, ¦v, w, x, y, z¦

Step 2: Collect "similar" classes into bundles (equivalence classes): These above collections can be put into a "binary relation" (comparing for) similarity by "equinumerosity", symbolized here by , i.e. one-one correspondence of the elements,[15] and thereby create Russellian classes of classes or what Russell called "bundles". "We can suppose all couples in one bundle, all trios in another, and so on. In this way we obtain various bundles of collections, each bundle consisting of all the collections that have a certain number of terms. Each bundle is a class whose members are collections, i.e. classes; thus each is a class of classes" (Russell 1919:14).

Take for example ¦h,i¦. Its terms h, i cannot be put into one-one correspondence with the terms of ¦a,b,c¦, ¦d¦, ¦ ¦, ¦e,f,g¦, etc. But it can be put in correspondence with itself and with ¦j,k¦, ¦q,r¦, and ¦t,u¦. These similar collections can be assembled into a "bundle" (equivalence class) as shown below.

¦ ¦h,i¦, ¦j,k¦, ¦q,r¦, ¦t,u¦ ¦

The bundles (equivalence classes) are shown below.

¦ ¦a, b, c¦, ¦e, f, g¦ ¦

¦ ¦d¦, ¦s¦ ¦

¦ ¦ ¦, ¦ ¦ ¦

┊ ┊h, i┊, ┊j, k┊, ┊q, r┊, ┊t, u┊ ┊

┊ ┊ l, m, n, o, p┊, ┊v, w, x, y, z┊ ┊

Step 3: Define the null-class: Notice that the third class-of-classes, ┊ ┊┊, ┊┊ ┊ , is special because its classes contain no elements, i.e. no elements satisfy the predicates that created this particular class/collection. Example: the predicates are:

"For all *childnames*: "*childname* is the name of a child in family F_ρ".

"For all *childnames*: "*childname* is the name of a child in family F_σ".

Thes particular predicates cannot be satisfied because families F_ρ and F_σ are childless. There are no terms (names) that satisfy these particular predicates. Remarkably, the class of things, signified by the fictitious ┊┊, that satisfy each of these this classes is not only empty, *it does not exist at all* (more or less, for Russell the agnostic-about-class-existence); for Dedekind-Frege it does exist.

This peculiar non-existent entity ┊┊ is nicknamed the "null class" or the "empty class". This is not the same as the class of all null classes ┊ ┊┊ ┊: the class of all null classes is destined to become "0"; see below. Russell symbolized the null/empty class ┊┊ with Λ. So what exactly is the Russellian null class? In *PM* Russell says that "A class is said to *exist* when it has at least one member . . . the class which has no members is called the "null class" . . . "α is the null-class" is equivalent to "α does not exist". One is left uneasy: *Does the null class itself "exist"?* This problem bedeviled Russell throughout his writing of 1903.[16] After he discovered the paradox in Frege's *Begriffsschrift* he added Appendix A to his 1903 where through the analysis of the nature of the null and unit classes, he discovered the need for a "doctrine of types"; see more about the unit class, the problem of impredicative definitions and Russell's "vicious circle principle" below.[17]

Step 4: Assign a "numeral" to each bundle: For purposes of abbreviation and identification, to each bundle assign a unique symbol (aka a "numeral"). These symbols are arbitrary. (The symbol ≡ means "is an abbreviation for" or "is a definition of"):

┊ ┊a, b, c┊, ┊e, f, g┊ ┊ ≡ ▯

┊ ┊d┊, ┊s┊ ┊ ≡ ■

┊ ┊┊ ┊ ≡ ♣

┊ ┊h, i┊, ┊j, k┊ ┊, ┊q, r┊, ┊t, u┊ ┊ ≡ ▯

┊ ┊ l, m, n, o, p┊, ┊v, w, x, y, z┊ ┊ ≡ ♦

Step 5: Define "0": In order to "order" the bundles into the familiar number-line a starting point traditionally called "zero", is required. Russell picked the empty or *null* class of classes to fill this role. This null class-of-classes ┊ ┊┊ ┊┊ ┊ has been labeled "0" ≡ ♣[18]

Step 6: Define the notion of "successor": Russell defined a new characteristic "hereditary", a property of certain classes with the ability to "inherit" a characteristic from another class (or class-of-classes) i.e. "A property is said to be "hereditary" in the natural-number series if, whenever it belongs to a number *n*, it also belongs to *n*+1, the successor of *n*." (1903:21). He asserts that "the natural numbers are the *posterity* -- the "children", the inheritors of the "successor"—of 0 with respect to the relation "the immediate predecessor of (which is the converse of "successor") (1919:23).

Note Russell has used a few words here without definition, in particular "number series", "number n", and "successor". He will define these in due course. *Observe in particular that Russell does not use the unit class-of-classes "1" to construct the successor* (in our example ┊ ┊d┊, ┊s┊ ┊ ≡ ■) . The reason is that, in Russell's detailed analysis,[19] if a unit class ■ becomes a entity in its own right, then it too can be an element in its own proposition; this causes the proposition to become "impredicative" and result in a "vicious circle". Rather, he states (confusingly): "We saw in Chapter II that a cardinal [natural] number is to be defined as a class of classes, and in Chapter III that the number 1 is to be defined as the class of all unit classes, of all that have just one member, as we should say but for the vicious circle. Of course, when the number 1 is defined as the class of all unit classes, *unit classes* must be defined so as not to assume that we know what is meant by *one* (1919:181).

For his definition of successor, Russell will use for his "unit" a single entity or "term" as follows:

> "It remains to define "successor." Given any number *n* let *α* be a class which has *n* members, and let *x* be a term which is not a member of *α*. Then the class consisting of *α* with *x* added on will have *+1* members. Thus we have the following definition:
>
> *the successor of the number of terms in the class α is the number of terms in the class consisting of α together with x where x is not any term belonging to the class*." (1919:23)

Russell's definition requires a new "term" (name, thing) which is "added into" the collections inside the bundles. To keep the example abstract this will be abbreviated by the name "Smiley" ≡ ☺ (on the assumption that no one has ever actually named their child "Smiley").

Step 7: Construct the successor of the null class: For example into the null class Λ stick the smiley face. From the previous, it is not obvious how to do this. The predicate:

"For all *childnames*: "*childname* is the name of a child in family F_{α}".

has to be modified to creating a predicate that contains a term that is always true:

"For all *childnames*: "*childname* is the name of a child in family F_{α} *AND* *Smiley*";

In the case of the family with no children, "Smiley" is the only "term" that satisfies the predicate. Russell fretted over the use of the word *AND* here, as in "Barbie AND Smiley", and called this kind of AND (symbolized below with *&*) a "numerical conjunction"[20]:

¦ ¦¦ ¦ *&* ☺ → ¦¦☺¦¦

By the relation of similarity ▯, this new class can be put into the equivalence class (the unit class) defined by ■:

¦¦☺¦¦ ▯ ¦d¦,¦s¦ → ¦¦☺¦,¦d¦,¦s¦¦≡ ■, i.e.

0 *&* ☺ → ■,

Step 8: For every equivalence class, create its successor: Note that the smiley-face symbol must be inserted into every collection/class in a particular equivalence-class bundle, then by the relation of similarity ≈ each newly-generated class-of-classes must be put into the equivalence class that defines n+1:

▯ *&* ☺ ≡¦¦h,i¦, ¦j,k¦, ¦q,r¦, ¦t,u¦¦ *&* ☺ → ¦¦h, i, ☺¦, ¦j, k, ☺¦, ¦q, r, ☺¦, ¦t, u, ☺¦, ¦a, b, c¦, ¦e, f, g¦¦ ≡ ▯, i.e.

▯ *&* ☺ → ▯

And in a similar manner, by use of the abbreviations set up above, for each numeral its successor is created:

0

0 *&* ☺ = ■

■ *&* ☺ = ▯

▯ *&* ☺ = ▯

▯ *&* ☺ = ? [no symbol]

? *&* ☺ = ♦

♦ *&* ☺ = etc, etc

Step 9: Order the numbers: The process of creating a successor requires the relation " . . . is the successor of . . .", call it "S", between the various "numerals", for example ■ S 0, ▯ S ■, and so forth. "We must now consider the *serial* character of the natural numbers in the order 0, 1, 2, 3, . . . We ordinarily think of the numbers as in this order, and it is an essential part of the work of analysing our data to seek a definition of "order" or "series " in logical terms. . . . The order lies, not in the *class* of terms, but in a relation among the members of the class, in respect of which some appear as earlier and some as later." (1919:31)

Russell applies to the notion of "ordering relation" three criteria: First, he defines the notion of "asymmetry" i.e. given the relation such as S (" . . . is the successor of . . . ") between two terms x, and y: x S y ≠ y S x. Second, he defines the notion of transitivity for three numerals x, y and z: if x S y and y S z then x S z. Third, he defines the notion of "connected": "Given any two terms of the class which is to be ordered, there must be one which precedes and the other which follows. . . . A relation is connected when, given any two different terms of its field [both domain and converse domain of a relation e.g. husbands versus wives in the relation of married] the relation holds between the first and the second or between the second and the first (not excluding the possibility that both may happen, though both cannot happen if the relation is asymmetrical).(1919:32)

He concludes: ". . . [natural] number *m* is said to be less than another number *n* when n possesses every hereditary property possessed by the successor of *m*. It is easy to see, and not difficult to prove, that the relation "less than," so defined, is asymmetrical, transitive, and connected, and has the [natural] numbers for its field [i.e. both domain and converse domain are the numbers]." (1919:35)

Criticism

The problem of presuming the "extralogical" notion of "iteration": Kleene points out that, "the logicistic thesis can be questioned finally on the ground that logic already presupposes mathematical ideas in its formulation. In the Intuitionistic view, an essential mathematical kernel is contained in the idea of iteration" (Kleene 1952:46)

Bernays 1930-1931 observes that this notion "two things" already presupposes something, even without the claim of existence of two things, and also without reference to a predicate, which applies to the two things; it means, simply, "a thing and one more thing. . . . With respect to this simple definition, the Number concept turns out to be an elementary *structural concept* . . . the claim of the logicists that mathematics is purely logical knowledge turns out to be blurred and misleading upon closer observation of theoretical logic. . . . [one can extend the definition of "logical"] however, through this definition what is epistemologically essential is concealed, and what is peculiar to mathematics is overlooked" (in Mancosu 1998:243).

Hilbert 1931:266-7, like Bernays, detects "something extra-logical" in mathematics: "Besides experience and thought, there is yet a third source of knowledge. Even if today we can no longer agree with Kant in the details, nevertheless the most general and fundamental idea of the Kantian epistemology retains its significance: to ascertain the intuitive *a priori* mode of thought, and thereby to investigate the condition of the possibility of all knowledge. In my opinion this is essentially what happens in my investigations of the principles of mathematics. The *a priori* is here nothing more and nothing less than a fundamental mode of thought, which I also call the finite mode of thought: something is already given to us in advance in our faculty of representation: certain *extra-logical concrete objects* that exist intuitively as an immediate experience before all thought. If logical inference is to be certain, then these objects must be completely surveyable in all their parts, and their presentation, their differences, their succeeding one another or their being arrayed next to one another is immediately and intuitively given to us, along with the objects, as something that neither can be reduced to anything else, nor needs such a reduction." (Hilbert 1931 in Mancosu 1998: 266, 267).

In brief: the notion of "sequence" or "successor" is an *a priori* notion that lies outside symbolic logic.

Hilbert dismissed logicism as a "false path": "Some tried to define the numbers purely logically; others simply took the usual number-theoretic modes of inference to be self-evident. On both paths they encountered obstacles that proved to be insuperable." (Hilbert 1931 in Mancoso 1998:267) .

Mancosu states that Brouwer concluded that: "the classical laws or principles of logic are part of [the] perceived regularity [in the symbolic representation]; they are derived from the post factum record of mathematical constructions . . . Theoretical logic . . . [is] an empirical science and an application of mathematics" (Brouwer quoted by Mancosu 1998:9).

Gödel 1944: With respect to the *technical* aspects of Russellian logicism as it appears in *Principia Mathematic* (either edition), Gödel is flat-out disappointed:

> "It is to be regretted that this first comprehensive and thorough-going presentation of a mathematical logic and the derivation of mathematics from it [is?] so greatly lacking in formal precision in the foundations (contained in *1 - *21 of *Principia*) that it presents in this respect a considerable step backwards as compared with Frege. What is missing, above all, is a precise statement of the syntax of the formalism"(cf footnote 1 in Gödel 1944 *Collected Works* 1990:120).

In particular he pointed out that "The matter is especially doubtful for the rule of substitution and of replacing defined symbols by their *definiens*" (Russell 1944:120)

With respect the *philosophy* that formed these foundations, Gödel would home in on Russell's "no-class theory", or what Gödel would call his "nominalistic kind of contructivism, such as that embodied in Russell's "no class theory" . . . which might better be called fictionalism" (cf footnote 1 in Gödel 1944:119). See more in "Gödel's criticism and suggestions" below.

Grattan-Guinness: [TBD] A complicated theory of relations continued to strangle Russell's explanatory 1919 *Introduction to Mathematical Philosophy* and his 1927 second edition of *Principia*. Set theory, meanwhile had moved on with its reduction of relation to the ordered pair of sets. Grattan-Guiness observes that in the second edition of *Principia* Russell ignored this reduction that had been achieved by his own student Norbert Wiener (1914). Perhaps because of "residual annoyance, Russell did not react at all".[21] By 1914 Hausdorff would provide another, equivalent definition, and Kuratowski in 1921 would provide the one in use today.[22]

The unit class, impredicativity and the vicious circle principle

A benign impredicative definition: Suppose the local librarian wants to catalog (index) her collection into a single book (call it I for "index"). Her index must list ALL the books and their locations in the library. As it turns out, there are only three books, and these have titles Ά, β, and Γ. To form her index-book I, she goes out and buys a book of 200 blank pages and labels it "I". Now she has four books: I, Ά, β, and Γ. Her task is not difficult. When completed, the contents of her index I is 4 pages, each with a unique title and unique location (each entry abbreviated as Title.Location_T):

> I ← { I.L_I, Ά.$L_{Ά}$, β.L_β, Γ.L_Γ }.

This sort of definition of I was deemed by Poincaré to be "impredicative". He opined that only predicative definitions can be allowed in mathematics:

> "a definition is 'predicative' and logically admissible only if it *excludes* all objects that are dependent upon the notion defined, that is, that can in any way be determined by it".[23]

By Poincaré's definition, the librarian's index book is "impredicative" because the definition of I is dependent upon the definition of the totality I, Ά, β, and Γ. As noted below, some commentators insist that impredicativity in commonsense versions is harmless, but as the examples show below there are versions which are not harmless. In the teeth of these, Russell would enunciate a strict prohibition—his "vicious circle principle":

> "No totality can contain members definable only in terms of this totality, or members involving or presupposing this totality" (vicious circle principle)" (Gödel 1944 appearing in *Collected Works Vol. II* 1990:125).[24]

A pernicious impredicativity: α = NOT-α: To create a pernicious paradox, apply input α to the simple function box F(x) with output ω = 1 - α. This is the algebraic-logic equivalent of the symbolic-logical ω = NOT-α for truth values 1 and 0 rather than "true" and "false". In either case, when input α = 0, output ω = 1; when input α = 1, output ω = 0.

To make the function "impredicative", wrap around output ω to input α, i.e. identify (equate) the input with (to) the output (at either the output or input, it does not matter):

$\alpha = 1\text{-}\alpha$

Algebraically the equation is satisfied only when $\alpha = 0.5$. But *logically*, when only "truth values" 0 and 1 are permitted, then the equality *cannot* be satisfied. To see what is happening, employ an illustrative crutch: assume (i) the starting value of $\alpha = \alpha_0$ and (ii) observe the input-output propagation in discrete time-instants that proceed left to right in sequence across the page:

$\alpha_0 \rightarrow F(x) \rightarrow 1\text{-}\alpha_0 \rightarrow F(x) \rightarrow (1\ \text{-}(1\text{-}\alpha_o)) \rightarrow F(x) \rightarrow (1\text{-}(1\text{-}(1\text{-}\alpha_o))) \rightarrow F(x) \rightarrow$ *ad nauseam*

Start with $\alpha_0 = 0$:

$\alpha_0 = \mathbf{0} \rightarrow F(x) \rightarrow \mathbf{1} \rightarrow F(x) \rightarrow \mathbf{0} \rightarrow F(x) \rightarrow \mathbf{1} \rightarrow F(x) \rightarrow$ *ad nauseam*

Observe that output ω oscillates between **0** and **1**. If the "discrete time-instant" crutch (ii) is dropped, the function-box's output (and input) is both 1 and 0 *simultaneously*.

Fatal impredicativity in the definition of the unit class: The problem that bedeviled the logicists (and set theorists too, but with a different resolution) derives from the α = NOT-α paradox[25] Russell discovered in Frege's 1879 *Begriffsschrift*[26] that Frege had allowed a function to derive its input "functional" (value of its variable) not only from an object (thing, term), but from the function's own output as well.[27]

As described above, Both Frege's and Russell's construction of natural numbers begins with the formation of equinumerous classes-of-classes (bundles), then with an assignment of a unique "numeral" to each bundle, and then placing the bundles into an order via a relation S that is asymmetric: $x\ S\ y \neq y\ S\ x$. But Frege, unlike Russell, allowed the class of unit classes (in the example above [[d], [s]]) to be identified as a unit itself:

[[d], [s]] ≡ ■ ≡ 1

But, since the class ■ or 1 is a single object (unit) in its own right, it too must be included in the class-of-unit-classes as an additional class [■]. And this inclusion results in an "infinite regress" (as Godel called it) of increasing "type" and increasing content:

[[d], [s], [■]] ≡ ■

[[d], [s], [[d], [s], [■]]]] ≡ ■

[[d], [s], [[d], [s], [[[d], [s], [[d], [s], [■]]]]]]] ≡ ■, ad nauseam

Russell would make this problem go away by declaring a class to be a "fiction" (more or less). By this he meant that the class would designate only the elements that satisfied the propositional function (e.g. d and s) and nothing else. As a "fiction" a class cannot be considered to be a thing: an entity, a "term", a singularity, a "unit". It is an *assemblage* e.g. d,s but it is not (in Russell's view) worthy of thing-hood:

> "The class as many . . . is unobjectionable, but is many and not one. We may, if we choose, represent this by a single symbol: thus $x\ \varepsilon\ u$ will mean " x is one of the u 's." This must not be taken as a relation of two terms, x and u, because u as the numerical conjunction is not a single term . . . Thus a class of classes will be many many's; its constituents will each be only many, and cannot therefore in any sense, one might suppose, be single constituents.[etc]" (1903:516).

This supposes that "at the bottom" every single solitary "term" can be listed (specified by a "predicative" predicate) for any class, for any class of classes, for class of classes of classes, etc, but it introduces a new problem—a hierarchy of "types" of classes.

A solution to impredicativity: a hierarchy of types

Classes as non-objects, as useful fictions: Gödel 1944:131 observes that "Russell adduces two reasons against the extensional view of classes, namely the existence of (1) the null class, which cannot very well be a collection, and (2) the unit classes, which would have to be identical with their single elements." He suggests that Russell should have regarded these as fictitious, but not derive the further conclusion that *all* classes (such as the class-of-classes that define the numbers 2, 3, etc) are fictions.

But Russell did not do this. After a detailed analysis in Appendix A: *The Logical and Arithmetical Doctrines of Frege* in his 1903, Russell concludes:

> "The logical doctrine which is thus forced upon us is this: The subject of a proposition may be not a single term, but essentially many terms; this is the case with all propositions asserting numbers other than 0 and 1" (1903:516).

In the following notice the wording "the class as many"—a class is an aggregate of those terms (things) that satisfy the propositional function, but a class is not a thing-in-itself:

> "Thus the final conclusion is, that the correct theory of classes is even more extensional than that of Chapter VI; that the class as many is the only object always defined by a propositional function, and that this is adequate for formal purposes" (1903:518).

It is as if Russell-as-rancher were to round up all his critters (sheep, cows and horses) into three fictitious corrals (one for the sheep, one for the cows, and one for the horses) that are located in his fictitious ranch. What actually exists are the sheep, the cows and the horses (the extensions), but not the fictitious "concepts" corrals and ranch.

Ramified theory of types: function-orders and argument-types, predicative functions: When Russell proclaimed *all* classes are useful fictions he solved the problem of the "unit" class, but the *overall* problem did not go away; rather, it arrived in a new form: "It will now be necessary to distinguish (1) terms, (2) classes, (3) classes of classes, and so on *ad infinitum*; we shall have to hold that no member of one set is a member of any other set, and that $x \varepsilon u$ requires that x should be of a set of a degree lower by one than the set to which u belongs. Thus $x \varepsilon x$ will become a meaningless proposition; and in this way the contradiction is avoided" (1903:517).

This is Russell's "doctrine of types". To guarantee that impredicative expressions such as $x \varepsilon x$ can be treated in his logic, Russell proposed, as a kind of working hypothesis, that all such impredicative definitions have predicative definitions. This supposition requires the notions of function-"orders" and argument-"types". First, functions (and their classes-as-extensions, i.e. "matrices") are to be classified by their "order", where functions of individuals are of order 1, functions of functions (classes of classes) are of order 2, and so forth. Next, he defines the "type" of a function's arguments (the function's "inputs") to be their "range of significance", i.e. what are those inputs α (individuals? classes? classes-of-classes? etc.) that, when plugged into f(x), yield a meaningful output ω. Note that this means that a "type" can be of mixed order, as the following example shows:

> "Joe DiMaggio and the Yankees won the 1947 World Series".

This sentence can be decomposed into two clauses: "x won the 1947 World Series" + "y won the 1947 World Series". The first sentence takes for x an individual "Joe DiMaggio" as its input, the other takes for y an aggregate "Yankees" as its input. Thus the composite-sentence has a (mixed) type of 2, mixed as to order (1 and 2).

By "predicative", Russell meant that the function must be of an order higher than the "type" of its variable(s). Thus a function (of order 2) that creates a class of classes can only entertain arguments for its variable(s) that are classes (type 1) and individuals (type 0), as these are lower types. Type 3 can only entertain types 2, 1 or 0, and so forth. But these types can be mixed (for example, for this sentence to be (sort of) true: " z won the 1947 World Series " could accept the individual (type 0) "Joe DiMaggio" and/or the names of his other teammates, *and* it could accept the class (type 1) of individual players "The Yankees".

The axiom of reducibility: The *axiom of reducibility* is the hypothesis that *any* function of *any* order can be reduced to (or replaced by) an equivalent *predicative* function of the appropriate order.[28] A careful reading of the first edition indicates that an n^{th} order predicative function need not be expressed "all the way down" as a huge "matrix" or aggregate of individual atomic propositions. "For in practice only the *relative* types of variables are relevant; thus the lowest type occurring in a given context may be called that of individuals" (p. 161). But the axiom of reducibility proposes that *in theory* a reduction "all the way down" is possible.

Russell 1927 abandons the axiom of reducibility, and the edifice collapses: By the 2nd edition of *PM* of 1927, though, Russell had given up on the axiom of reducibility and concluded he would indeed force any order of function "all the way down" to its elementary propositions, linked together with logical operators:

> "All propositions, of whatever order, are derived from a matrix composed of elementary propositions combined by means of the stroke" (*PM* 1927 Appendix A, p. 385),

(The "stroke" is Sheffer's inconvenient logical NAND that Russell adopted for the 2nd edition—a single logical function that replaces logical OR and logical NOT).

The net result, though, was a collapse of his theory. Russell arrived at this disheartening conclusion: that "the theory of ordinals and cardinals survives . . . but irrationals, and real numbers generally, can no longer be adequately dealt with. . . .Perhaps some further axiom, less objectionable than the axiom of reducibility, might give these results, but we have not succeeded in finding such an axiom." (*PM* 1927:xiv).

Gödel 1944 agrees that Russell's logicist project was stymied; he seems to disagree that even the integers survived:

> "[In the second edition] The axiom of reducibility is dropped, and it is stated explicitly that all primitive predicates belong to the lowest type and that the only purpose of variables (and evidently also of constants) of higher orders and types is to make it possible to assert more complicated truth-functions of atomic propositions" (Gödel 1944 in *Collected Works*:134).

Gödel asserts, however, that this procedure seems to presuppose arithmetic in some form or other (p. 134). He deduces that "one obtains integers of different orders" (p. 134-135); the proof in Russell 1927 *PM* Appendix B that "the integers of any order higher than 5 are the same as those of order 5" is "not conclusive" and "the question whether (or to what extent) the theory of integers can be obtained on the basis of the ramified hierarchy [classes plus types] must be considered as unsolved at the present time". Gödel concluded that it wouldn't matter anyway because propositional functions of order *n* (any *n*) must be described by finite combinations of symbols (cf all quotes and content derived from page 135).

Gödel's criticism and suggestions

Gödel in his 1944 bores down to the exact place where Russell's logicism fails and offers a few suggestions to rectify the problems. He submits the "vicious circle principle" to reexamination, tearing it apart into three phrases "definable only in terms of", "involving" and "presupposing". It is the first clause that "makes impredicative definitions impossible and thereby destroys the derivation of mathematics from logic, effected by Dedekind and Frege, and a good deal of mathematics itself". Since, he argues, mathematics is doing quite well, thank you, with its various inherent impredicativities (e.g. "real numbers defined by reference to all real numbers"), he concludes that what he has offered is "a proof that the vicious circle principle is false [rather] than that classical mathematics is false" (all quotes Gödel 1944:127).

Russell's no-class theory is the root of the problem: Gödel believes that impredicativity is not "absurd", as it appears throughout mathematics. Where Russell's problem derives from is the "constructivistic (or nominalistic[29]) standpoint toward the objects of logic and mathematics, in particular toward propositions, classes, and notions . . . a notion being a symbol . . . so that a separate object denoted by the symbol appears as a mere fiction" (p. 128).

Indeed, this "no class" theory of Russell, Gödel concludes:

> "is of great interest as one of the few examples, carried out in detail, of the tendency to eliminate assumptions about the existence of objects outside the "data" and to replace them by constructions on the basis of these data[33]. [[33] The "data" are to understand in a relative sense here; i.e. in our case as logic without the assumption of the existence of classes and concepts]. The result has been in this case essentially negative; i.e. the classes and concepts introduced in this way do not have all the properties required from their use in mathematics. . . . All this is only a verification of the view defended above that logic and mathematics (just as physics) are built up on axioms with a real content which cannot be explained away" (p. 132)

He concludes his essay with the following suggestions and observations:

> "One should take a more conservative course, such as would consist in trying to make the meaning of terms "class" and "concept" clearer, and to set up a consisent theory of classes and concepts as objectively existing entities. This is the course which the actual development of mathematical logic has been taking and which Russell himself has been forced to enter upon in the more constructive parts of his work. Major among the attempts in this direction . . . are the simple theory of types . . . and axiomatic set theory, both of which have been successful at least to this extent, that they permit the derivation of modern mathematics and at the same time avoid all known paradoxes . . . ¶ It seems reasonable to suspect that it is this incomplete understanding of the foundations which is responsible for the fact that mathematical logic has up to now remained so far behind the high expectations of Peano and others . . .."(p. 140)

Neo-logicism

Neo-logicism describes a range of views claiming to be the successor of the original logicist program.[30] More narrowly, it is defined as attempts to resurrect Frege's programme through the use of Hume's Principle.[31] This kind of neo-logicism is often referred to as neo-Fregeanism. Two of the major proponents of neo-logicism are Crispin Wright and Bob Hale.[32]

Notes

[1] Logicism (http://www.philosophyprofessor.com/philosophies/logicism.php)

[2] Principia Mathematica (http://plato.stanford.edu/entries/principia-mathematica) entry in the *Stanford Encyclopedia of Philosophy*.

[3] The exact quote from Russell 1919 is the following: "It is time now to turn to the considerations which make it necessary to advance beyond the standpoint of Peano, who represents the last perfection of the "arithetisation" of mathematics, to that of Frege, who first succeeded in "logicising" mathematics, i.e. in reducing to logic the arithmetical notions which his predecessors had shown to be sufficient for mathematics." (Russell 1919/2005:17).

[4] For example, see God made the integers. von Neuman 1925 would cite Kronecker as follows: "The denumerable infinite . . . is nothing more the general notion of the positive integer, on which mathematics rests and of which even Kronecker and Brouwer admit that it was "created by God"" (von Neumann 1925 *An axiomatization of set theory* in van Heijenoort 1967:413.

[5] Hilbert 1904 *On the foundations of logic and arithmetic* in van Heijenoort 1967:130.

[6] Page s474, 475 in Hilbert 1927 *The Foundations of Mathematics* in van Heijenoort 1967:475.

[7] Perry in his 1997 Introduction to Russell 1912:ix)

[8] Cf Russell 1912:74.

[9] "It must be admitted . . . that logical principles are known to us, and cannot be themselves proved by experience, since all proof presupposes them. In this, therefore . . . the rationalists were in the right" (Russell 1912:74).

[10] "Nothing can be known to *exist* except by the help of experience" (Russell 1912:74).

[11] He drives the point home (pages 67-68) where he defines four conditions that determine what we call "the numbers" (cf (71).Definition, page 67: the successor set N' is a part of the collection N, there is a starting-point "1_o" [base number of the number-series *N*], this "1" is not contained in any successor, for any *n* in the collection there exists a transformation φ(*n*) to a *unique* (distinguishable) *n*(cf (26). Definition)). He observes that by establishing these conditions "we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relation to one another . . . by the order-setting transformation φ. . . . With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind." (p. 68)

[12] cf *The Philosophy of Mathematics and Hilbert's Proof Theory* 1930:1931 in Mancosu p. 242.

[13] In his 1903 and in *PM* Russell refers to such assumptions (there are others) as "primitive propositions" ("pp" as opposed to "axioms" (there are some of those, too). But the reader is never certain whether these pp are axioms/axiom-schemas or construction-devices (like substitution or *modus ponens*), or what, exactly. Gödel 1944:120 comments on this absence of formal syntax and the absence of a clearly-specified substitution process.

[14] To be precise both *childname* = variable *x* and family name *Fn* are variables. *Childname* 's domain is "all childnames in this and every other possible world", and family name *Fn* has a domain over the 12 families on the street.

[15] "If the predicates are partitioned into classes with respect to equinumerosity in such a way that all predicates of a class are equinumerous to one another and predicates of different classes are not equinumerous, then each such class represents the *Number*, which applies to the predicates that belong to it" (Bernays 1930-1 in Mancosu 1998:240.

[16] section 487ff (pages 513ff in the Appendix A).

[17] Cf sections 487ff (pages 513ff in the Appendix A).

[18] Whether or not the null class is reducible to is unclear; the conclusion is not important for this example.

[19] 1909 Appendix A

[20] 1903:133ff, Section 130: "Numerical Conjunction" and plurality".

[21] Russell deemed Wiener "the infant phenomenon . . . more infant than phenomenon*; see* Russell's confrontation with Weiner *in Grattan-Guiness 2000:419ff.*

[22] See van Heijenoort's commentary and Norbert Wiener's 1914 *A simplification of the logic of relations* in van Heijenoort 1967:224ff.

[23] Zermelo 1908 in van Heijenoort 1967:190. See the discussion of this very quotation in Mancosu 1998:68.

[24] This same definition appears also in Kleene 1952:42.

[25] An excellent source for details is Fairouz Kamareddine, Twan Laan and Rob Nderpelt, 2004, *A Modern Perspective on Type Theory, From its Origins Until Today*, Kluwer Academic Publishers, Dordrecht, The Netherlands, ISBN. They give a demonsration of how to create the paradox (pages 1–2), as follows: Define an aggregate/class/set y this way: $\exists y \forall x[\ x\ \varepsilon\ y \leftrightarrow \Phi(x)]$. (This says: There exists a class y such that for *ANY* input x, x is an element of set y if and only if x satisfies the given function Φ.) Note that (i) input x is unrestricted as to the "type" of thing that it can be (it can be a thing, or a class), and (ii) function Φ is unrestricted as well. Pick the following tricky function $\Phi(x) = \neg(x\ \varepsilon\ x)$. (This says: Φ(x) is satsified when x is NOT an element of x)). Because y (a class) is also "unrestricted" we can plug "y" in as input: $\exists y[\ y\ \varepsilon\ y \leftrightarrow \neg(y\ \varepsilon\ y)]$. This says that "there exists a class y that is an element of itself only if it is NOT and element of itself. That is the paradox.

[26] Russell's letter to Frege announcing the "disovery", and Frege's letter back to Russell in sad response, together with commentary, can be found in van Heijenoort 1967:124-128. Zermelo in his 1908 claimed priority to the discovery; cf footnote 9 on page 191 in van Heijenoort.

[27] van Heijenoort 1967:3 and pages 124-128

[28] "The axiom of reducibility is the assumption that, given any function $\varphi\hat{z}$, there is a formally equivalent, *predicative* function, i.e. there is a predicative function which is true when φz is true and false when φz is false. In symbols, the axiom is: $\vdash :(\exists\psi) : \varphi z. \equiv_z .\psi!z.$" (*PM* 1913/1962 edition:56, the original uses x with a circumflex). Here $\varphi\hat{z}$ indicates the function with variable $\hat{z}$, i.e. φ(x) where x is argument "z"; φz indicates the value of the function given argument "z"; $\equiv_z$ indicates "equivalence for all z"; ψ!z indicates a predicative function, i.e. one with no variables except individuals.

[29] Perry observes that Plato and Russell are "enthusiastic" about "universals", then in the next sentence writes: " 'Nominalists' think that all that particulars really have in common are the words we apply to them"(Perry in his 1997 Introduction to Russell 1912:xi). Perry adds that while your sweatshirt and mine are different objects generalized by the word "sweatshirt", you have a relation to yours and I have a relation to mine. And Russell "treated relations on par with other universals" (p. xii). But Gödel is saying that Russell's "no-class" theory denies the numbers the status of "universals".

[30] What is Neologicism? (http://mally.stanford.edu/Papers/neologicism2.pdf)

[31] PHIL 30067: Logicism and Neo-Logicism (http://seis.bris.ac.uk/~plxol/Courses/PHIL30067/Syllabus.htm)

[32] http://www.st-andrews.ac.uk/~mr30/papers/EbertRossbergPurpose.pdf

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External links

- Logicism (http://www.rbjones.com/rbjpub/philos/maths/faq001.htm)

Gödel's incompleteness theorems

Gödel's incompleteness theorems are two theorems of mathematical logic that establish inherent limitations of all but the most trivial axiomatic systems capable of doing arithmetic. The theorems, proven by Kurt Gödel in 1931, are important both in mathematical logic and in the philosophy of mathematics. The two results are widely, but not universally, interpreted as showing that Hilbert's program to find a complete and consistent set of axioms for all mathematics is impossible, giving a negative answer to Hilbert's second problem.

The first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an "effective procedure" (e.g., a computer program, but it could be any sort of algorithm) is capable of proving all truths about the relations of the natural numbers (arithmetic). For any such system, there will always be statements about the natural numbers that are true, but that are unprovable within the system. The second incompleteness theorem, an extension of the first, shows that such a system cannot demonstrate its own consistency.

Background

Because statements of a formal theory are written in symbolic form, it is possible to mechanically verify that a formal proof from a finite set of axioms is valid. This task, known as automatic proof verification, is closely related to automated theorem proving. The difference is that instead of constructing a new proof, the proof verifier simply checks that a provided formal proof (or, in instructions that can be followed to create a formal proof) is correct. This process is not merely hypothetical; systems such as Isabelle or Coq are used today to formalize proofs and then check their validity.

Many theories of interest include an infinite set of axioms, however. To verify a formal proof when the set of axioms is infinite, it must be possible to determine whether a statement that is claimed to be an axiom is actually an axiom. This issue arises in first order theories of arithmetic, such as Peano arithmetic, because the principle of mathematical induction is expressed as an infinite set of axioms (an axiom schema).

A formal theory is said to be *effectively generated* if its set of axioms is a recursively enumerable set. This means that there is a computer program that, in principle, could enumerate all the axioms of the theory without listing any statements that are not axioms. This is equivalent to the existence of a program that enumerates all the theorems of the theory without enumerating any statements that are not theorems. Examples of effectively generated theories with infinite sets of axioms include Peano arithmetic and Zermelo–Fraenkel set theory.

In choosing a set of axioms, one goal is to be able to prove as many correct results as possible, without proving any incorrect results. A set of axioms is complete if, for any statement in the axioms' language, either that statement or its negation is provable from the axioms. A set of axioms is (simply) consistent if there is no statement such that both the statement and its negation are provable from the axioms. In the standard system of first-order logic, an inconsistent set of axioms will prove every statement in its language (this is sometimes called the principle of explosion), and is thus automatically complete. A set of axioms that is both complete and consistent, however, proves a maximal set of non-contradictory theorems. Gödel's incompleteness theorems show that in certain cases it is not possible to obtain an effectively generated, complete, consistent theory.

First incompleteness theorem

Gödel's first incompleteness theorem first appeared as "Theorem VI" in Gödel's 1931 paper *On Formally Undecidable Propositions in Principia Mathematica and Related Systems I.*

The formal theorem is written in highly technical language. The broadly accepted natural language statement of the theorem is:

> Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true,[1] but not provable in the theory (Kleene 1967, p. 250).

The true but unprovable statement referred to by the theorem is often referred to as "the Gödel sentence" for the theory. The proof constructs a specific Gödel sentence for each effectively generated theory, but there are infinitely many statements in the language of the theory that share the property of being true but unprovable. For example, the conjunction of the Gödel sentence and any logically valid sentence will have this property.

For each consistent formal theory *T* having the required small amount of number theory, the corresponding Gödel sentence *G* asserts: "*G* cannot be proved within the theory *T*". This interpretation of *G* leads to the following informal analysis. If *G* were provable under the axioms and rules of inference of *T*, then *T* would have a theorem, *G*, which effectively contradicts itself, and thus the theory *T* would be inconsistent. This means that if the theory *T* is consistent then *G* cannot be proved within it, and so the theory *T* is incomplete. Moreover, the claim *G* makes about its own unprovability is correct. In this sense *G* is not only unprovable but true, and provability-within-the-theory-*T* is not the same as truth. This informal analysis can be formalized to make a rigorous proof of the incompleteness theorem, as described in the section "Proof sketch for the first theorem" below. The formal proof reveals exactly the hypotheses required for the theory *T* in order for the self-contradictory nature of *G* to lead to a genuine contradiction.

Each effectively generated theory has its own Gödel statement. It is possible to define a larger theory *T'* that contains the whole of *T*, plus *G* as an additional axiom. This will not result in a complete theory, because Gödel's theorem will also apply to *T'*, and thus *T'* cannot be complete. In this case, *G* is indeed a theorem in *T'*, because it is an axiom. Since *G* states only that it is not provable in *T*, no contradiction is presented by its provability in *T'*. However, because the incompleteness theorem applies to *T'*: there will be a new Gödel statement *G'* for *T'*, showing that *T'* is also incomplete. *G'* will differ from *G* in that *G'* will refer to *T'*, rather than *T*.

To prove the first incompleteness theorem, Gödel represented statements by numbers. Then the theory at hand, which is assumed to prove certain facts about numbers, also proves facts about its own statements, provided that it is effectively generated. Questions about the provability of statements are represented as questions about the properties of numbers, which would be decidable by the theory if it were complete. In these terms, the Gödel sentence states that no natural number exists with a certain, strange property. A number with this property would encode a proof of the inconsistency of the theory. If there were such a number then the theory would be inconsistent, contrary to the consistency hypothesis. So, under the assumption that the theory is consistent, there is no such number.

Meaning of the first incompleteness theorem

Gödel's first incompleteness theorem shows that any consistent effective formal system that includes enough of the theory of the natural numbers is incomplete: there are true statements expressible in its language that are unprovable. Thus no formal system (satisfying the hypotheses of the theorem) that aims to characterize the natural numbers can actually do so, as there will be true number-theoretical statements which that system cannot prove. This fact is sometimes thought to have severe consequences for the program of logicism proposed by Gottlob Frege and Bertrand Russell, which aimed to define the natural numbers in terms of logic (Hellman 1981, p. 451–468). Bob Hale and Crispin Wright argue that it is not a problem for logicism because the incompleteness theorems apply equally to second order logic as they do to arithmetic. They argue that only those who believe that the natural numbers are to be defined in terms of first order logic have this problem.

The existence of an incomplete formal system is, in itself, not particularly surprising. A system may be incomplete simply because not all the necessary axioms have been discovered. For example, Euclidean geometry without the parallel postulate is incomplete; it is not possible to prove or disprove the parallel postulate from the remaining axioms.

Gödel's theorem shows that, in theories that include a small portion of number theory, a complete and consistent finite list of axioms can *never* be created, nor even an infinite list that can be enumerated by a computer program. Each time a new statement is added as an axiom, there are other true statements that still cannot be proved, even with the new axiom. If an axiom is ever added that makes the system complete, it does so at the cost of making the system inconsistent.

There *are* complete and consistent lists of axioms for arithmetic that *cannot* be enumerated by a computer program. For example, one might take all true statements about the natural numbers to be axioms (and no false statements), which gives the theory known as "true arithmetic". The difficulty is that there is no mechanical way to decide, given a statement about the natural numbers, whether it is an axiom of this theory, and thus there is no effective way to verify a formal proof in this theory.

Many logicians believe that Gödel's incompleteness theorems struck a fatal blow to David Hilbert's second problem, which asked for a finitary consistency proof for mathematics. The second incompleteness theorem, in particular, is often viewed as making the problem impossible. Not all mathematicians agree with this analysis, however, and the status of Hilbert's second problem is not yet decided (see "Modern viewpoints on the status of the problem").

Relation to the liar paradox

The liar paradox is the sentence "This sentence is false." An analysis of the liar sentence shows that it cannot be true (for then, as it asserts, it is false), nor can it be false (for then, it is true). A Gödel sentence *G* for a theory *T* makes a similar assertion to the liar sentence, but with truth replaced by provability: *G* says "*G* is not provable in the theory *T*." The analysis of the truth and provability of *G* is a formalized version of the analysis of the truth of the liar sentence.

It is not possible to replace "not provable" with "false" in a Gödel sentence because the predicate "Q is the Gödel number of a false formula" cannot be represented as a formula of arithmetic. This result, known as Tarski's undefinability theorem, was discovered independently by Gödel (when he was working on the proof of the incompleteness theorem) and by Alfred Tarski.

Extensions of Gödel's original result

Gödel demonstrated the incompleteness of the theory of *Principia Mathematica*, a particular theory of arithmetic, but a parallel demonstration could be given for any effective theory of a certain expressiveness. Gödel commented on this fact in the introduction to his paper, but restricted the proof to one system for concreteness. In modern statements of the theorem, it is common to state the effectiveness and expressiveness conditions as hypotheses for the incompleteness theorem, so that it is not limited to any particular formal theory. The terminology used to state these conditions was not yet developed in 1931 when Gödel published his results.

Gödel's original statement and proof of the incompleteness theorem requires the assumption that the theory is not just consistent but *ω-consistent*. A theory is **ω-consistent** if it is not ω-inconsistent, and is ω-inconsistent if there is a predicate *P* such that for every specific natural number *n* the theory proves ~*P*(*n*), and yet the theory also proves that there exists a natural number *n* such that *P*(*n*). That is, the theory says that a number with property *P* exists while denying that it has any specific value. The ω-consistency of a theory implies its consistency, but consistency does not imply ω-consistency. J. Barkley Rosser (1936) strengthened the incompleteness theorem by finding a variation of the proof (Rosser's trick) that only requires the theory to be consistent, rather than ω-consistent. This is mostly of technical interest, since all true formal theories of arithmetic (theories whose axioms are all true statements about natural numbers) are ω-consistent, and thus Gödel's theorem as originally stated applies to them. The stronger version of the incompleteness theorem that only assumes consistency, rather than ω-consistency, is now commonly known as Gödel's incompleteness theorem and as the Gödel–Rosser theorem.

Second incompleteness theorem

Gödel's second incompleteness theorem first appeared as "Theorem XI" in Gödel's 1931 paper *On Formally Undecidable Propositions in Principia Mathematica and Related Systems I.*

The formal theorem is written in highly technical language. The broadly accepted natural language statement of the theorem is:

> For any formal effectively generated theory *T* including basic arithmetical truths and also certain truths about formal provability, if *T* includes a statement of its own consistency then *T* is inconsistent.

This strengthens the first incompleteness theorem, because the statement constructed in the first incompleteness theorem does not directly express the consistency of the theory. The proof of the second incompleteness theorem is obtained by formalizing the proof of the first incompleteness theorem within the theory itself.

A technical subtlety in the second incompleteness theorem is how to express the consistency of *T* as a formula in the language of *T*. There are many ways to do this, and not all of them lead to the same result. In particular, different formalizations of the claim that *T* is consistent may be inequivalent in *T*, and some may even be provable. For example, first-order Peano arithmetic (PA) can prove that the largest consistent subset of PA is consistent. But since PA is consistent, the largest consistent subset of PA is just PA, so in this sense PA "proves that it is consistent". What PA does not prove is that the largest consistent subset of PA is, in fact, the whole of PA. (The term "largest consistent subset of PA" is technically ambiguous, but what is meant here is the largest consistent initial segment of the axioms of PA ordered according to specific criteria; i.e., by "Gödel numbers", the numbers encoding the axioms as per the scheme used by Gödel mentioned above).

For Peano arithmetic, or any familiar explicitly axiomatized theory *T*, it is possible to canonically define a formula Con(*T*) expressing the consistency of *T*; this formula expresses the property that "there does not exist a natural number coding a sequence of formulas, such that each formula is either of the axioms of *T*, a logical axiom, or an immediate consequence of preceding formulas according to the rules of inference of first-order logic, and such that the last formula is a contradiction".

The formalization of Con(*T*) depends on two factors: formalizing the notion of a sentence being derivable from a set of sentences and formalizing the notion of being an axiom of *T*. Formalizing derivability can be done in canonical fashion: given an arithmetical formula A(x) defining a set of axioms, one can canonically form a predicate $\mathrm{Prov}_A(P)$ which expresses that P is provable from the set of axioms defined by A(x).

In addition, the standard proof of the second incompleteness theorem assumes that $\mathrm{Prov}_A(P)$ satisfies that Hilbert–Bernays provability conditions. Letting #(P) represent the Gödel number of a formula P, the derivability conditions say:

1. If T proves P, then T proves $\mathrm{Prov}_A(\#(P))$.
2. T proves 1.; that is, T proves that if T proves P, then T proves $\mathrm{Prov}_A(\#(P))$. In other words, T proves that $\mathrm{Prov}_A(\#(P))$ implies $\mathrm{Prov}_A(\#(\mathrm{Prov}_A(\#(P))))$.
3. T proves that if T proves that $(P \to Q)$ and T proves P then T proves Q. In other words, T proves that $\mathrm{Prov}_A(\#(P \to Q))$ and $\mathrm{Prov}_A(\#(P))$ imply $\mathrm{Prov}_A(\#(Q))$.

Implications for consistency proofs

Gödel's second incompleteness theorem also implies that a theory T_1 satisfying the technical conditions outlined above cannot prove the consistency of any theory T_2 which proves the consistency of T_1. This is because such a theory T_1 can prove that if T_2 proves the consistency of T_1, then T_1 is in fact consistent. For the claim that T_1 is consistent has form "for all numbers n, n has the decidable property of not being a code for a proof of contradiction in T_1". If T_1 were in fact inconsistent, then T_2 would prove for some n that n is the code of a contradiction in T_1. But if T_2 also proved that T_1 is consistent (that is, that there is no such n), then it would itself be inconsistent. This reasoning can be formalized in T_1 to show that if T_2 is consistent, then T_1 is consistent. Since, by second incompleteness theorem, T_1 does not prove its consistency, it cannot prove the consistency of T_2 either.

This corollary of the second incompleteness theorem shows that there is no hope of proving, for example, the consistency of Peano arithmetic using any finitistic means that can be formalized in a theory the consistency of which is provable in Peano arithmetic. For example, the theory of primitive recursive arithmetic (PRA), which is widely accepted as an accurate formalization of finitistic mathematics, is provably consistent in PA. Thus PRA cannot prove the consistency of PA. This fact is generally seen to imply that Hilbert's program, which aimed to justify the use of "ideal" (infinitistic) mathematical principles in the proofs of "real" (finitistic) mathematical statements by giving a finitistic proof that the ideal principles are consistent, cannot be carried out.

The corollary also indicates the epistemological relevance of the second incompleteness theorem. It would actually provide no interesting information if a theory T proved its consistency. This is because inconsistent theories prove everything, including their consistency. Thus a consistency proof of T in T would give us no clue as to whether T really is consistent; no doubts about the consistency of T would be resolved by such a consistency proof. The interest in consistency proofs lies in the possibility of proving the consistency of a theory T in some theory T' which is in some sense less doubtful than T itself, for example weaker than T. For many naturally occurring theories T and T', such as T = Zermelo–Fraenkel set theory and T' = primitive recursive arithmetic, the consistency of T' is provable in T, and thus T' can't prove the consistency of T by the above corollary of the second incompleteness theorem.

The second incompleteness theorem does not rule out consistency proofs altogether, only consistency proofs that could be formalized in the theory that is proved consistent. For example, Gerhard Gentzen proved the consistency of Peano arithmetic (PA) in a different theory which includes an axiom asserting that the ordinal called ε_0 is wellfounded; see Gentzen's consistency proof. Gentzen's theorem spurred the development of ordinal analysis in proof theory.

Examples of undecidable statements

There are two distinct senses of the word "undecidable" in mathematics and computer science. The first of these is the proof-theoretic sense used in relation to Gödel's theorems, that of a statement being neither provable nor refutable in a specified deductive system. The second sense, which will not be discussed here, is used in relation to computability theory and applies not to statements but to decision problems, which are countably infinite sets of questions each requiring a yes or no answer. Such a problem is said to be undecidable if there is no computable function that correctly answers every question in the problem set (see undecidable problem).

Because of the two meanings of the word undecidable, the term independent is sometimes used instead of undecidable for the "neither provable nor refutable" sense. The usage of "independent" is also ambiguous, however. Some use it to mean just "not provable", leaving open whether an independent statement might be refuted.

Undecidability of a statement in a particular deductive system does not, in and of itself, address the question of whether the truth value of the statement is well-defined, or whether it can be determined by other means. Undecidability only implies that the particular deductive system being considered does not prove the truth or falsity of the statement. Whether there exist so-called "absolutely undecidable" statements, whose truth value can never be known or is ill-specified, is a controversial point in the philosophy of mathematics.

The combined work of Gödel and Paul Cohen has given two concrete examples of undecidable statements (in the first sense of the term): The continuum hypothesis can neither be proved nor refuted in ZFC (the standard axiomatization of set theory), and the axiom of choice can neither be proved nor refuted in ZF (which is all the ZFC axioms *except* the axiom of choice). These results do not require the incompleteness theorem. Gödel proved in 1940 that neither of these statements could be disproved in ZF or ZFC set theory. In the 1960s, Cohen proved that neither is provable from ZF, and the continuum hypothesis cannot be proven from ZFC.

In 1973, the Whitehead problem in group theory was shown to be undecidable, in the first sense of the term, in standard set theory.

Gregory Chaitin produced undecidable statements in algorithmic information theory and proved another incompleteness theorem in that setting. Chaitin's incompleteness theorem states that for any theory that can represent enough arithmetic, there is an upper bound c such that no specific number can be proven in that theory to have Kolmogorov complexity greater than c. While Gödel's theorem is related to the liar paradox, Chaitin's result is related to Berry's paradox.

Undecidable statements provable in larger systems

These are natural mathematical equivalents of the Godel "true but undecidable" sentence. They can be proved in a larger system which is generally accepted as a valid form of reasoning, but are undecidable in a more limited system such as Peano Arithmetic.

In 1977, Paris and Harrington proved that the Paris-Harrington principle, a version of the Ramsey theorem, is undecidable in the first-order axiomatization of arithmetic called Peano arithmetic, but can be proven in the larger system of second-order arithmetic. Kirby and Paris later showed Goodstein's theorem, a statement about sequences of natural numbers somewhat simpler than the Paris-Harrington principle, to be undecidable in Peano arithmetic.

Kruskal's tree theorem, which has applications in computer science, is also undecidable from Peano arithmetic but provable in set theory. In fact Kruskal's tree theorem (or its finite form) is undecidable in a much stronger system codifying the principles acceptable based on a philosophy of mathematics called predicativism. The related but more general graph minor theorem (2003) has consequences for computational complexity theory.

Limitations of Gödel's theorems

The conclusions of Gödel's theorems are only proven for the formal theories that satisfy the necessary hypotheses. Not all axiom systems satisfy these hypotheses, even when these systems have models that include the natural numbers as a subset. For example, there are first-order axiomatizations of Euclidean geometry, of real closed fields, and of arithmetic in which multiplication is not *provably* total; none of these meet the hypotheses of Gödel's theorems. The key fact is that these axiomatizations are not expressive enough to define the set of natural numbers or develop basic properties of the natural numbers. Regarding the third example, Dan E. Willard (Willard 2001) has studied many weak systems of arithmetic which do not satisfy the hypotheses of the second incompleteness theorem, and which are consistent and capable of proving their own consistency (see self-verifying theories).

Gödel's theorems only apply to effectively generated (that is, recursively enumerable) theories. If all true statements about natural numbers are taken as axioms for a theory, then this theory is a consistent, complete extension of Peano arithmetic (called true arithmetic) for which none of Gödel's theorems apply in a meaningful way, because this theory is not recursively enumerable.

The second incompleteness theorem only shows that the consistency of certain theories cannot be proved from the axioms of those theories themselves. It does not show that the consistency cannot be proved from other (consistent) axioms. For example, the consistency of the Peano arithmetic can be proved in Zermelo–Fraenkel set theory (ZFC), or in theories of arithmetic augmented with transfinite induction, as in Gentzen's consistency proof.

Relationship with computability

The incompleteness theorem is closely related to several results about undecidable sets in recursion theory.

Stephen Cole Kleene (1943) presented a proof of Gödel's incompleteness theorem using basic results of computability theory. One such result shows that the halting problem is undecidable: there is no computer program that can correctly determine, given a program *P* as input, whether *P* eventually halts when run with a particular given input. Kleene showed that the existence of a complete effective theory of arithmetic with certain consistency properties would force the halting problem to be decidable, a contradiction. This method of proof has also been presented by Shoenfield (1967, p. 132); Charlesworth (1980); and Hopcroft and Ullman (1979).

Franzén (2005, p. 73) explains how Matiyasevich's solution to Hilbert's 10th problem can be used to obtain a proof to Gödel's first incompleteness theorem. Matiyasevich proved that there is no algorithm that, given a multivariate polynomial $p(x_1, x_2,...,x_k)$ with integer coefficients, determines whether there is an integer solution to the equation $p = 0$. Because polynomials with integer coefficients, and integers themselves, are directly expressible in the language of arithmetic, if a multivariate integer polynomial equation $p = 0$ does have a solution in the integers then any sufficiently strong theory of arithmetic *T* will prove this. Moreover, if the theory *T* is ω-consistent, then it will never prove that a particular polynomial equation has a solution when in fact there is no solution in the integers. Thus, if *T* were complete and ω-consistent, it would be possible to determine algorithmically whether a polynomial equation has a solution by merely enumerating proofs of *T* until either "*p* has a solution" or "*p* has no solution" is found, in contradiction to Matiyasevich's theorem. Moreover, for each consistent effectively generated theory *T*, it is possible to effectively generate a multivariate polynomial *p* over the integers such that the equation $p = 0$ has no solutions over the integers, but the lack of solutions cannot be proved in *T* (Davis 2006:416, Jones 1980).

Smorynski (1977, p. 842) shows how the existence of recursively inseparable sets can be used to prove the first incompleteness theorem. This proof is often extended to show that systems such as Peano arithmetic are essentially undecidable (see Kleene 1967, p. 274).

Chaitin's incompleteness theorem gives a different method of producing independent sentences, based on Kolmogorov complexity. Like the proof presented by Kleene that was mentioned above, Chaitin's theorem only applies to theories with the additional property that all their axioms are true in the standard model of the natural numbers. Gödel's incompleteness theorem is distinguished by its applicability to consistent theories that nonetheless include statements that are false in the standard model; these theories are known as ω-inconsistent.

Proof sketch for the first theorem

The proof by contradiction has three essential parts. To begin, choose a formal system that meets the proposed criteria:

1. Statements in the system can be represented by natural numbers (known as Gödel numbers). The significance of this is that properties of statements—such as their truth and falsehood—will be equivalent to determining whether their Gödel numbers have certain properties, and that properties of the statements can therefore be demonstrated by examining their Gödel numbers. This part culminates in the construction of a formula expressing the idea that *"statement S is provable in the system"* (which can be applied to any statement "S" in the system).
2. In the formal system it is possible to construct a number whose matching statement, when interpreted, is self-referential and essentially says that it (i.e. the statement itself) is unprovable. This is done using a technique called "diagonalization" (so-called because of its origins as Cantor's diagonal argument).
3. Within the formal system this statement permits a demonstration that it is neither provable nor disprovable in the system, and therefore the system cannot in fact be ω-consistent. Hence the original assumption that the proposed system met the criteria is false.

Arithmetization of syntax

The main problem in fleshing out the proof described above is that it seems at first that to construct a statement *p* that is equivalent to "*p* cannot be proved", *p* would somehow have to contain a reference to *p*, which could easily give rise to an infinite regress. Gödel's ingenious technique is to show that statements can be matched with numbers (often called the arithmetization of syntax) in such a way that *"proving a statement"* can be replaced with *"testing whether a number has a given property"*. This allows a self-referential formula to be constructed in a way that avoids any infinite regress of definitions. The same technique was later used by Alan Turing in his work on the Entscheidungsproblem.

In simple terms, a method can be devised so that every formula or statement that can be formulated in the system gets a unique number, called its **Gödel number**, in such a way that it is possible to mechanically convert back and forth between formulas and Gödel numbers. The numbers involved might be very long indeed (in terms of number of digits), but this is not a barrier; all that matters is that such numbers can be constructed. A simple example is the way in which English is stored as a sequence of numbers in computers using ASCII or Unicode:

- The word **`HELLO`** is represented by 72-69-76-76-79 using decimal ASCII, ie the number 7269767679.
- The logical statement **`x=y => y=x`** is represented by 120-061-121-032-061-062-032-121-061-120 using octal ASCII, ie the number 120061121032061062032121061120.

In principle, proving a statement true or false can be shown to be equivalent to proving that the number matching the statement does or doesn't have a given property. Because the formal system is strong enough to support reasoning about *numbers in general*, it can support reasoning about *numbers which represent formulae and statements* as well. Crucially, because the system can support reasoning about *properties of numbers*, the results are equivalent to reasoning about *provability of their equivalent statements*.

Construction of a statement about "provability"

Having shown that in principle the system can indirectly make statements about provability, by analyzing properties of those numbers representing statements it is now possible to show how to create a statement that actually does this.

A formula $F(x)$ that contains exactly one free variable x is called a *statement form* or *class-sign*. As soon as x is replaced by a specific number, the statement form turns into a *bona fide* statement, and it is then either provable in the system, or not. For certain formulas one can show that for every natural number n, F(n) is true if and only if it can be proven (the precise requirement in the original proof is weaker, but for the proof sketch this will suffice). In particular, this is true for every specific arithmetic operation between a finite number of natural numbers, such as "2×3=6".

Statement forms themselves are not statements and therefore cannot be proved or disproved. But every statement form $F(x)$ can be assigned a Gödel number denoted by $\mathbf{G}(F)$. The choice of the free variable used in the form $F(x)$ is not relevant to the assignment of the Gödel number $\mathbf{G}(F)$.

Now comes the trick: The notion of provability itself can also be encoded by Gödel numbers, in the following way. Since a proof is a list of statements which obey certain rules, the Gödel number of a proof can be defined. Now, for every statement p, one may ask whether a number x is the Gödel number of its proof. The relation between the Gödel number of p and x, the potential Gödel number of its proof, is an arithmetical relation between two numbers. Therefore there is a statement form Bew(y) that uses this arithmetical relation to state that a Gödel number of a proof of y exists:

> Bew(y) = ∃ x (y is the Gödel number of a formula and x is the Gödel number of a proof of the formula encoded by y).

The name **Bew** is short for *beweisbar*, the German word for "provable"; this name was originally used by Gödel to denote the provability formula just described. Note that "Bew(y)" is merely an abbreviation that represents a particular, very long, formula in the original language of T; the string "Bew" itself is not claimed to be part of this language.

An important feature of the formula Bew(y) is that if a statement p is provable in the system then Bew($\mathbf{G}(p)$) is also provable. This is because any proof of p would have a corresponding Gödel number, the existence of which causes Bew($\mathbf{G}(p)$) to be satisfied.

Diagonalization

The next step in the proof is to obtain a statement that says it is unprovable. Although Gödel constructed this statement directly, the existence of at least one such statement follows from the diagonal lemma, which says that for any sufficiently strong formal system and any statement form F there is a statement p such that the system proves

$$p \leftrightarrow F(\mathbf{G}(p)).$$

By letting F be the negation of Bew(x), we obtain the theorem

$$p \leftrightarrow \sim Bew(\mathbf{G}(p))$$

and the p defined by this roughly states that its own Gödel number is the Gödel number of an unprovable formula.

The statement p is not literally equal to ~Bew($\mathbf{G}(p)$); rather, p states that if a certain calculation is performed, the resulting Gödel number will be that of an unprovable statement. But when this calculation is performed, the resulting Gödel number turns out to be the Gödel number of p itself. This is similar to the following sentence in English:

> ", when preceded by itself in quotes, is unprovable.", when preceded by itself in quotes, is unprovable.

This sentence does not directly refer to itself, but when the stated transformation is made the original sentence is obtained as a result, and thus this sentence asserts its own unprovability. The proof of the diagonal lemma employs a similar method.

Proof of independence

Now assume that the formal system is ω-consistent. Let p be the statement obtained in the previous section.

If p were provable, then Bew(**G**(p)) would be provable, as argued above. But p asserts the negation of Bew(**G**(p)). Thus the system would be inconsistent, proving both a statement and its negation. This contradiction shows that p cannot be provable.

If the negation of p were provable, then Bew(**G**(p)) would be provable (because p was constructed to be equivalent to the negation of Bew(**G**(p))). However, for each specific number x, x cannot be the Gödel number of the proof of p, because p is not provable (from the previous paragraph). Thus on one hand the system supports construction of a number with a certain property (that it is the Gödel number of the proof of p), but on the other hand, for every specific number x, it can be proved that the number does *not* have this property. This is impossible in an ω-consistent system. Thus the negation of p is not provable.

Thus the statement p is undecidable: it can neither be proved nor disproved within the chosen system. So the chosen system is either inconsistent or incomplete. This logic can be applied to any formal system meeting the criteria. The conclusion is that **all** formal systems meeting the criteria are either inconsistent or incomplete. It should be noted that p is not provable (and thus true) in every consistent system. The assumption of ω-consistency is only required for the negation of p to be not provable. So:

- In an ω-consistent formal system, neither p nor its negation can be proved, and so p is undecidable.
- In a consistent formal system either the same situation occurs, or the negation of p can be proved; In the later case, a statement ("not p") is false but provable.

Note that if one tries to fix this by "adding the missing axioms" to avoid the undecidability of the system, then one has to add either p or "not p" as axioms. But this then creates a new formal system$_2$ (old system + p), to which exactly the same process can be applied, creating a new statement form $\mathbf{Bew_2}(x)$ for this new system. When the diagonal lemma is applied to this new form $\mathbf{Bew_2}$, a new statement p_2 is obtained; this statement will be different from the previous one, and this new statement will be undecidable in the new system if it is ω-consistent, thus showing that system$_2$ is equally inconsistent. So adding extra axioms cannot fix the problem.

Proof via Berry's paradox

George Boolos (1989) sketches an alternative proof of the first incompleteness theorem that uses Berry's paradox rather than the liar paradox to construct a true but unprovable formula. A similar proof method was independently discovered by Saul Kripke (Boolos 1998, p. 383). Boolos's proof proceeds by constructing, for any computably enumerable set S of true sentences of arithmetic, another sentence which is true but not contained in S. This gives the first incompleteness theorem as a corollary. According to Boolos, this proof is interesting because it provides a "different sort of reason" for the incompleteness of effective, consistent theories of arithmetic (Boolos 1998, p. 388).

Formalized proofs

Formalized proofs of versions of the incompleteness theorem have been developed by Natarajan Shankar in 1986 using Nqthm (Shankar 1994) and by Russell O'Connor in 2003 using Coq (O'Connor 2005).

Proof sketch for the second theorem

The main difficulty in proving the second incompleteness theorem is to show that various facts about provability used in the proof of the first incompleteness theorem can be formalized within the system using a formal predicate for provability. Once this is done, the second incompleteness theorem follows by formalizing the entire proof of the first incompleteness theorem within the system itself.

Let p stand for the undecidable sentence constructed above, and assume that the consistency of the system can be proven from within the system itself. The demonstration above shows that if the system is consistent, then p is not provable. The proof of this implication can be formalized within the system, and therefore the statement "p is not provable", or "not $P(p)$" can be proven in the system.

But this last statement is equivalent to p itself (and this equivalence can be proven in the system), so p can be proven in the system. This contradiction shows that the system must be inconsistent.

Discussion and implications

The incompleteness results affect the philosophy of mathematics, particularly versions of formalism, which use a single system formal logic to define their principles. One can paraphrase the first theorem as saying the following:

> An all-encompassing axiomatic system can never be found that is able to prove *all* mathematical truths, but no falsehoods.

On the other hand, from a strict formalist perspective this paraphrase would be considered meaningless because it presupposes that mathematical "truth" and "falsehood" are well-defined in an absolute sense, rather than relative to each formal system.

The following rephrasing of the second theorem is even more unsettling to the foundations of mathematics:

> If an axiomatic system can be proven to be consistent from within itself, then it is inconsistent.

Therefore, to establish the consistency of a system S, one needs to use some other system T, but a proof in T is not completely convincing unless T's consistency has already been established without using S.

Theories such as Peano arithmetic, for which any computably enumerable consistent extension is incomplete, are called essentially undecidable or **essentially incomplete**.

Minds and machines

Authors including J. R. Lucas have debated what, if anything, Gödel's incompleteness theorems imply about human intelligence. Much of the debate centers on whether the human mind is equivalent to a Turing machine, or by the Church–Turing thesis, any finite machine at all. If it is, and if the machine is consistent, then Gödel's incompleteness theorems would apply to it.

Hilary Putnam (1960) suggested that while Gödel's theorems cannot be applied to humans, since they make mistakes and are therefore inconsistent, it may be applied to the human faculty of science or mathematics in general. Assuming that it is consistent, either its consistency cannot be proved or it cannot be represented by a Turing machine.

Avi Wigderson (2010) has proposed that the concept of mathematical "knowability" should be based on computational complexity rather than logical decidability. He writes that "when *knowability* is interpreted by modern standards, namely via computational complexity, the Gödel phenomena are very much with us."

Paraconsistent logic

Although Gödel's theorems are usually studied in the context of classical logic, they also have a role in the study of paraconsistent logic and of inherently contradictory statements (*dialetheia*). Graham Priest (1984, 2006) argues that replacing the notion of formal proof in Gödel's theorem with the usual notion of informal proof can be used to show that naive mathematics is inconsistent, and uses this as evidence for dialetheism. The cause of this inconsistency is the inclusion of a truth predicate for a theory within the language of the theory (Priest 2006:47). Stewart Shapiro (2002) gives a more mixed appraisal of the applications of Gödel's theorems to dialetheism. Carl Hewitt (2008) has proposed that (inconsistent) paraconsistent logics that prove their own Gödel sentences may have applications in software engineering.

Appeals to the incompleteness theorems in other fields

Appeals and analogies are sometimes made to the incompleteness theorems in support of arguments that go beyond mathematics and logic. Several authors have commented negatively on such extensions and interpretations, including Torkel Franzén (2005); Alan Sokal and Jean Bricmont (1999); and Ophelia Benson and Jeremy Stangroom (2006). Bricmont and Stangroom (2006, p. 10), for example, quote from Rebecca Goldstein's comments on the disparity between Gödel's avowed Platonism and the anti-realist uses to which his ideas are sometimes put. Sokal and Bricmont (1999, p. 187) criticize Régis Debray's invocation of the theorem in the context of sociology; Debray has defended this use as metaphorical (ibid.).

The role of self-reference

Torkel Franzén (2005, p. 46) observes:

> Gödel's proof of the first incompleteness theorem and Rosser's strengthened version have given many the impression that the theorem can only be proved by constructing self-referential statements [...] or even that only strange self-referential statements are known to be undecidable in elementary arithmetic. To counteract such impressions, we need only introduce a different kind of proof of the first incompleteness theorem.

He then proposes the proofs based on computability, or on information theory, as described earlier in this article, as examples of proofs that should "counteract such impressions".

History

After Gödel published his proof of the completeness theorem as his doctoral thesis in 1929, he turned to a second problem for his habilitation. His original goal was to obtain a positive solution to Hilbert's second problem (Dawson 1997, p. 63). At the time, theories of the natural numbers and real numbers similar to second-order arithmetic were known as "analysis", while theories of the natural numbers alone were known as "arithmetic".

Gödel was not the only person working on the consistency problem. Ackermann had published a flawed consistency proof for analysis in 1925, in which he attempted to use the method of ε-substitution originally developed by Hilbert. Later that year, von Neumann was able to correct the proof for a theory of arithmetic without any axioms of induction. By 1928, Ackermann had communicated a modified proof to Bernays; this modified proof led Hilbert to announce his belief in 1929 that the consistency of arithmetic had been demonstrated and that a consistency proof of analysis would likely soon follow. After the publication of the incompleteness theorems showed that Ackermann's modified proof must be erroneous, von Neumann produced a concrete example showing that its main technique was unsound (Zach 2006, p. 418, Zach 2003, p. 33).

In the course of his research, Gödel discovered that although a sentence which asserts its own falsehood leads to paradox, a sentence that asserts its own non-provability does not. In particular, Gödel was aware of the result now called Tarski's indefinability theorem, although he never published it. Gödel announced his first incompleteness theorem to Carnap, Feigel and Waismann on August 26, 1930; all four would attend a key conference in Königsberg the following week.

Announcement

The 1930 Königsberg conference was a joint meeting of three academic societies, with many of the key logicians of the time in attendance. Carnap, Heyting, and von Neumann delivered one-hour addresses on the mathematical philosophies of logicism, intuitionism, and formalism, respectively (Dawson 1996, p. 69). The conference also included Hilbert's retirement address, as he was leaving his position at the University of Göttingen. Hilbert used the speech to argue his belief that all mathematical problems can be solved. He ended his address by saying,

> For the mathematician there is no *Ignorabimus*, and, in my opinion, not at all for natural science either. ... The true reason why [no one] has succeeded in finding an unsolvable problem is, in my opinion, that there is no unsolvable problem. In contrast to the foolish *Ignoramibus*, our credo avers: We must know. We shall know!

This speech quickly became known as a summary of Hilbert's beliefs on mathematics (its final six words, "*Wir müssen wissen. Wir werden wissen!*", were used as Hilbert's epitaph in 1943). Although Gödel was likely in attendance for Hilbert's address, the two never met face to face (Dawson 1996, p. 72).

Gödel announced his first incompleteness theorem at a roundtable discussion session on the third day of the conference. The announcement drew little attention apart from that of von Neumann, who pulled Gödel aside for conversation. Later that year, working independently with knowledge of the first incompleteness theorem, von Neumann obtained a proof of the second incompleteness theorem, which he announced to Gödel in a letter dated November 20, 1930 (Dawson 1996, p. 70). Gödel had independently obtained the second incompleteness theorem and included it in his submitted manuscript, which was received by *Monatshefte für Mathematik* on November 17, 1930.

Gödel's paper was published in the *Monatshefte* in 1931 under the title *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I* (On Formally Undecidable Propositions in Principia Mathematica and Related Systems I). As the title implies, Gödel originally planned to publish a second part of the paper; it was never written.

Generalization and acceptance

Gödel gave a series of lectures on his theorems at Princeton in 1933–1934 to an audience that included Church, Kleene, and Rosser. By this time, Gödel had grasped that the key property his theorems required is that the theory must be effective (at the time, the term "general recursive" was used). Rosser proved in 1936 that the hypothesis of ω-consistency, which was an integral part of Gödel's original proof, could be replaced by simple consistency, if the Gödel sentence was changed in an appropriate way. These developments left the incompleteness theorems in essentially their modern form.

Gentzen published his consistency proof for first-order arithmetic in 1936. Hilbert accepted this proof as "finitary" although (as Gödel's theorem had already shown) it cannot be formalized within the system of arithmetic that is being proved consistent.

The impact of the incompleteness theorems on Hilbert's program was quickly realized. Bernays included a full proof of the incompleteness theorems in the second volume of *Grundlagen der Mathematik* (1939), along with additional results of Ackermann on the ε-substitution method and Gentzen's consistency proof of arithmetic. This was the first full published proof of the second incompleteness theorem.

Criticisms

Finsler

Paul Finsler (1926) used a version of Richard's paradox to construct an expression that was false but unprovable in a particular, informal framework he had developed. Gödel was unaware of this paper when he proved the incompleteness theorems (Collected Works Vol. IV., p. 9). Finsler wrote Gödel in 1931 to inform him about this paper, which Finsler felt had priority for an incompleteness theorem. Finsler's methods did not rely on formalized provability, and had only a superficial resemblance to Gödel's work (van Heijenoort 1967:328). Gödel read the paper but found it deeply flawed, and his response to Finsler laid out concerns about the lack of formalization (Dawson:89). Finsler continued to argue for his philosophy of mathematics, which eschewed formalization, for the remainder of his career.

Zermelo

In September 1931, Ernst Zermelo wrote Gödel to announce what he described as an "essential gap" in Gödel's argument (Dawson:76). In October, Gödel replied with a 10-page letter (Dawson:76, Grattan-Guinness:512-513). But Zermelo did not relent and published his criticisms in print with "a rather scathing paragraph on his young competitor" (Grattan-Guinness:513). Gödel decided that to pursue the matter further was pointless, and Carnap agreed (Dawson:77). Much of Zermelo's subsequent work was related to logics stronger than first-order logic, with which he hoped to show both the consistency and categoricity of mathematical theories.

Wittgenstein

Ludwig Wittgenstein wrote several passages about the incompleteness theorems that were published posthumously in his 1953 *Remarks on the Foundations of Mathematics*. Gödel was a member of the Vienna Circle during the period in which Wittgenstein's early ideal language philosophy and Tractatus Logico-Philosophicus dominated the circle's thinking. Writings in Gödel's Nachlass express the belief that Wittgenstein deliberately misread his ideas.

Multiple commentators have read Wittgenstein as misunderstanding Gödel (Rodych 2003), although Juliet Floyd and Hilary Putnam (2000), as well as Graham Priest (2004) have provided textual readings arguing that most commentary misunderstands Wittgenstein. On their release, Bernays, Dummett, and Kreisel wrote separate reviews on Wittgenstein's remarks, all of which were extremely negative (Berto 2009:208). The unanimity of this criticism caused Wittgenstein's remarks on the incompleteness theorems to have little impact on the logic community. In 1972, Gödel, stated: "Has Wittgenstein lost his mind? Does he mean it seriously?" (Wang 1996:197) And wrote to Karl Menger that Wittgenstein's comments demonstrate a willful misunderstanding of the incompleteness theorems writing:

> "It is clear from the passages you cite that Wittgenstein did "not" understand [the first incompleteness theorem] (or pretended not to understand it). He interpreted it as a kind of logical paradox, while in fact is just the opposite, namely a mathematical theorem within an absolutely uncontroversial part of mathematics (finitary number theory or combinatorics)." (Wang 1996:197)

Since the publication of Wittgenstein's *Nachlass* in 2000, a series of papers in philosophy have sought to evaluate whether the original criticism of Wittgenstein's remarks was justified. Floyd and Putnam (2000) argue that Wittgenstein had a more complete understanding of the incompleteness theorem than was previously assumed. They are particularly concerned with the interpretation of a Gödel sentence for an ω-inconsistent theory as actually saying "I am not provable", since the theory has no models in which the provability predicate corresponds to actual provability. Rodych (2003) argues that their interpretation of Wittgenstein is not historically justified, while Bays (2004) argues against Floyd and Putnam's philosophical analysis of the provability predicate. Berto (2009) explores the relationship between Wittgenstein's writing and theories of paraconsistent logic.

Notes

[1] The word "true" is used disquotationally here: the Gödel sentence is true in this sense because it "asserts its own unprovability and it is indeed unprovable" (Smoryński 1977 p. 825; also see Franzén 2005 pp. 28–33). It is also possible to read "G_T is true" in the formal sense that primitive recursive arithmetic proves the implication Con(T)→G_T, where Con(T) is a canonical sentence asserting the consistency of T (Smoryński 1977 p. 840, Kikuchi and Tanaka 1994 p. 403)

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Articles by Gödel

- 1931, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I. Monatshefte für Mathematik und Physik 38*: 173-98.

- 1931, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I.* and *On formally undecidable propositions of Principia Mathematica and related systems I* in Solomon Feferman, ed., 1986. *Kurt Gödel Collected works, Vol. I.* Oxford University Press: 144-195. The original German with a facing English translation, preceded by a very illuminating introductory note by Kleene.
  - Hirzel, Martin, 2000, *On formally undecidable propositions of Principia Mathematica and related systems I.* (http://www.research.ibm.com/people/h/hirzel/papers/canon00-goedel.pdf). A modern translation by Hirzel.
- 1951, *Some basic theorems on the foundations of mathematics and their implications* in Solomon Feferman, ed., 1995. *Kurt Gödel Collected works, Vol. III.* Oxford University Press: 304-23.

Translations, during his lifetime, of Gödel's paper into English

None of the following agree in all translated words and in typography. The typography is a serious matter, because Gödel expressly wished to emphasize "those metamathematical notions that had been defined in their usual sense before . . ."(van Heijenoort 1967:595). Three translations exist. Of the first John Dawson states that: "The Meltzer translation was seriously deficient and received a devastating review in the *Journal of Symbolic Logic*; "Gödel also complained about Braithwaite's commentary (Dawson 1997:216). "Fortunately, the Meltzer translation was soon supplanted by a better one prepared by Elliott Mendelson for Martin Davis's anthology *The Undecidable* . . . he found the translation "not quite so good" as he had expected . . . [but because of time constraints he] agreed to its publication" (ibid). (In a footnote Dawson states that "he would regret his compliance, for the published volume was marred throughout by sloppy typography and numerous misprints" (ibid)). Dawson states that "The translation that Gödel favored was that by Jean van Heijenoort"(ibid). For the serious student another version exists as a set of lecture notes recorded by Stephen Kleene and J. B. Rosser "during lectures given by Gödel at to the Institute for Advanced Study during the spring of 1934" (cf commentary by Davis 1965:39 and beginning on p. 41); this version is titled "On Undecidable Propositions of Formal Mathematical Systems". In their order of publication:

- B. Meltzer (translation) and R. B. Braithwaite (Introduction), 1962. *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, Dover Publications, New York (Dover edition 1992), ISBN 0-486-66980-7 (pbk.) This contains a useful translation of Gödel's German abbreviations on pp. 33–34. As noted above, typography, translation and commentary is suspect. Unfortunately, this translation was reprinted with all its suspect content by
  - Stephen Hawking editor, 2005. *God Created the Integers: The Mathematical Breakthroughs That Changed History*, Running Press, Philadelphia, ISBN 0-7624-1922-9. Gödel's paper appears starting on p. 1097, with Hawking's commentary starting on p. 1089.
- Martin Davis editor, 1965. *The Undecidable: Basic Papers on Undecidable Propositions, Unsolvable problems and Computable Functions*, Raven Press, New York, no ISBN. Gödel's paper begins on page 5, preceded by one page of commentary.
- Jean van Heijenoort editor, 1967, 3rd edition 1967. *From Frege to Gödel: A Source Book in Mathematical Logic, 1979-1931*, Harvard University Press, Cambridge Mass., ISBN 0-674-32449-8 (pbk). van Heijenoort did the translation. He states that "Professor Gödel approved the translation, which in many places was accommodated to his wishes."(p. 595). Gödel's paper begins on p. 595; van Heijenoort's commentary begins on p. 592.
- Martin Davis editor, 1965, ibid. "On Undecidable Propositions of Formal Mathematical Systems." A copy with Gödel's corrections of errata and Gödel's added notes begins on page 41, preceded by two pages of Davis's commentary. Until Davis included this in his volume this lecture existed only as mimeographed notes.

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External links

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- Stanford Encyclopedia of Philosophy: " Kurt Gödel (http://plato.stanford.edu/entries/goedel/)" — by Juliette Kennedy.
- MacTutor biographies:
  - Kurt Gödel. (http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Godel.html)
  - Gerhard Gentzen. (http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Gentzen.html)
  - What is Mathematics:Gödel's Theorem and Around (http://podnieks.id.lv/gt.html) by *Karlis Podnieks*. An online free book.
- World's shortest explanation of Gödel's theorem (http://blog.plover.com/math/Gdl-Smullyan.html) using a printing machine as an example.
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Hilbert's program

In mathematics, **Hilbert's program**, formulated by German mathematician David Hilbert, was a proposed solution to the foundational crisis of mathematics, when early attempts to clarify the foundations of mathematics were found to suffer from paradoxes and inconsistencies. As a solution, Hilbert proposed to ground all existing theories to a finite, complete set of axioms, and provide a proof that these axioms were consistent. Hilbert proposed that the consistency of more complicated systems, such as real analysis, could be proven in terms of simpler systems. Ultimately, the consistency of all of mathematics could be reduced to basic arithmetic.

However, some argue that Gödel's incompleteness theorems showed in 1931 that Hilbert's program was unattainable. In his first theorem, Gödel showed that any consistent system with a computable set of axioms which is capable of expressing arithmetic can never be complete: it is possible to construct a statement that can be shown to be true, but that cannot be derived from the formal rules of the system. In his second theorem, he showed that such a system could not prove its own consistency, so it certainly cannot be used to prove the consistency of anything stronger. This refuted Hilbert's assumption that a finitistic system could be used to prove the consistency of a stronger theory.

Statement of Hilbert's program

The main goal of Hilbert's program was to provide secure foundations for all mathematics. In particular this should include:

- A formalization of all mathematics; in other words all mathematical statements should be written in a precise formal language, and manipulated according to well defined rules.
- Completeness: a proof that all true mathematical statements can be proved in the formalism.
- Consistency: a proof that no contradiction can be obtained in the formalism of mathematics. This consistency proof should preferably use only "finitistic" reasoning about finite mathematical objects.
- Conservation: a proof that any result about "real objects" obtained using reasoning about "ideal objects" (such as uncountable sets) can be proved without using ideal objects.
- Decidability: there should be an algorithm for deciding the truth or falsity of any mathematical statement.

Gödel's incompleteness theorems

Kurt Gödel showed that most of the goals of Hilbert's program were impossible to achieve, at least if interpreted in the most obvious way. His second incompleteness theorem stated that any consistent theory powerful enough to encode addition and multiplication of integers cannot prove its own consistency. This wipes out most of Hilbert's program as follows:

- It is not possible to formalize **all** of mathematics, as any attempt at such a formalism will omit some true mathematical statements.
- An easy consequence of Gödel's incompleteness theorem is that there is no complete consistent extension of even Peano arithmetic with a recursively enumerable set of axioms, so in particular most interesting mathematical theories are not complete.
- A theory such as Peano arithmetic cannot even prove its own consistency, so a restricted "finitistic" subset of it certainly cannot prove the consistency of more powerful theories such as set theory.
- There is no algorithm to decide the truth (or provability) of statements in any consistent extension of Peano arithmetic. (Strictly speaking this result only appeared a few years after Gödel's theorem, because at the time the notion of an algorithm had not been precisely defined.)

Hilbert's program after Gödel

Many current lines of research in mathematical logic, proof theory and reverse mathematics can be viewed as natural continuations of Hilbert's original program. Much of it can be salvaged by changing its goals slightly (Zach 2005), and with the following modifications some of it was successfully completed:

- Although it is not possible to formalize **all** mathematics, it is possible to formalize essentially all the mathematics that anyone uses. In particular Zermelo–Fraenkel set theory, combined with first-order logic, gives a satisfactory and generally accepted formalism for essentially all current mathematics.
- Although it is not possible to prove completeness for systems at least as powerful as Peano arithmetic (at least if they have a computable set of axioms), it is possible to prove forms of completeness for many interesting systems. The first big success was by Gödel himself (before he proved the incompleteness theorems) who proved the completeness theorem for first-order logic, showing that any logical consequence of a series of axioms is provable. An example of a non-trivial theory for which completeness has been proved is the theory of algebraically closed fields of given characteristic.
- The question of whether there are finitary consistency proofs of strong theories is difficult to answer, mainly because there is no generally accepted definition of a "finitary proof". Most mathematicians in proof theory seem to regard finitary mathematics as being contained in Peano arithmetic, and in this case it is not possible to give finitary proofs of reasonably strong theories. On the other hand Gödel himself suggested the possibility of giving finitary consistency proofs using finitary methods that cannot be formalized in Peano arithmetic, so he seems to have had a more liberal view of what finitary methods might be allowed. A few years later, Gentzen gave a consistency proof for Peano arithmetic. The only part of this proof that was not clearly finitary was a certain transfinite induction up to the ordinal ε_0. If this transfinite induction is accepted as a finitary method, then one can assert that there is a finitary proof of the consistency of Peano arithmetic. More powerful subsets of second order arithmetic have been given consistency proofs by Gaisi Takeuti and others, and one can again debate about exactly how finitary or constructive these proofs are. (The theories that have been proved consistent by these methods are quite strong, and include most "ordinary" mathematics.)
- Although there is no algorithm for deciding the truth of statements in Peano arithmetic, there are many interesting and non-trivial theories for which such algorithms have been found. For example, Tarski found an algorithm that can decide the truth of any statement in analytic geometry (more precisely, he proved that the theory of real closed fields is decidable). Given the Cantor–Dedekind axiom, this algorithm can be regarded as an algorithm to decide the truth of any statement in Euclidean geometry. This is substantial as few people would consider Euclidean geometry a trivial theory.

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External links

- Entry on Hilbert's program [3] at the Stanford Encyclopedia of Philosophy.

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Russell's paradox

In the foundations of mathematics, **Russell's paradox** (also known as **Russell's antinomy**), discovered by Bertrand Russell in 1901, showed that the naive set theory created by Georg Cantor leads to a contradiction. The same paradox had been discovered a year before by Ernst Zermelo but he did not publish the idea, which remained known only to Hilbert, Husserl and other members of the University of Göttingen.

According to naive set theory, any definable collection is a set. Let *R* be the set of all sets that are not members of themselves. If *R* qualifies as a member of itself, it would contradict its own definition as *a set containing all sets that are not members of themselves*. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox. Symbolically:

$$\text{let } R = \{x \mid x \notin x\}, \text{ then } R \in R \iff R \notin R$$

In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and the Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory (ZF).[1]

Informal presentation

Let us call a set "abnormal" if it is a member of itself, and "normal" otherwise. For example, take the set of all geometrical squares. That set is not itself a square, and therefore is not a member of the set of all squares. So it is "normal". On the other hand, if we take the complementary set that contains all non-squares, that set is itself not a square and so should be one of its own members. It is "abnormal".

Now we consider the set of all normal sets, *R*. Determining whether *R* is normal or abnormal is impossible: If *R* were a normal set, it would be contained in the set of normal sets (itself), and therefore be abnormal; and if *R* were abnormal, it would not be contained in the set of all normal sets (itself), and therefore be normal. This leads to the conclusion that *R* is neither normal nor abnormal: Russell's paradox.

Formal presentation

Define Naive Set Theory (NST) as the theory of predicate logic with a binary predicate $\in$ and the following axiom schema of unrestricted comprehension:

$$\exists y \forall x (x \in y \iff P(x))$$

for any formula *P* with only the variable *x* free. Substitute $x \notin x$ for $P(x)$. Then by existential instantiation (reusing the symbol *y*) and universal instantiation we have

$$y \in y \iff y \notin y$$

a contradiction. Therefore NST is inconsistent.

Set-theoretic responses

In 1908, Ernst Zermelo proposed an axiomatization of set theory that avoided the paradoxes of naive set theory by replacing arbitrary set comprehension with weaker existence axioms, such as his axiom of separation (*Aussonderung*). Modifications to this axiomatic theory proposed in the 1920s by Abraham Fraenkel, Thoralf Skolem, and by Zermelo himself resulted in the axiomatic set theory called ZFC. This theory became widely accepted once Zermelo's axiom of choice ceased to be controversial, and ZFC has remained the canonical axiomatic set theory down to the present day.

ZFC does not assume that, for every property, there is a set of all things satisfying that property. Rather, it asserts that given any set *X*, any subset of *X* definable using first-order logic exists. The object *R* discussed above cannot be constructed in this fashion, and is therefore not a ZFC set. In some extensions of ZFC, objects like *R* are called proper classes. ZFC is silent about types, although some argue that Zermelo's axioms tacitly presuppose a background type theory.

In ZFC, given a set *A*, it is possible to define a set *B* that consists of exactly the sets in *A* that are not members of themselves. *B* cannot be in *A* by the same reasoning in Russell's Paradox. This variation of Russell's paradox shows that no set contains everything.

Through the work of Zermelo and others, especially John von Neumann, the structure of what some see as the "natural" objects described by ZFC eventually became clear; they are the elements of the von Neumann universe, *V*, built up from the empty set by transfinitely iterating the power set operation. It is thus now possible again to reason about sets in a non-axiomatic fashion without running afoul of Russell's paradox, namely by reasoning about the elements of *V*. Whether it is *appropriate* to think of sets in this way is a point of contention among the rival points of view on the philosophy of mathematics.

Other resolutions to Russell's paradox, more in the spirit of type theory, include the axiomatic set theories New Foundations and Scott-Potter set theory.

History

Russell discovered the paradox in May or June 1901.[2] By his own admission in his 1919 *Introduction to Mathematical Philosophy*, he "attempted to discover some flaw in Cantor's proof that there is no greatest cardinal".[3] In a 1902 letter,[4] he announced the discovery to Gottlob Frege of the paradox in Frege's 1879 *Begriffsschrift* and framed the problem in terms of both logic and set theory, and in particular in terms of Frege's definition of function; in the following, p. 17 refers to a page in the original *Begriffsschrift*, and page 23 refers to the same page in van Heijenoort 1967:

> There is just one point where I have encountered a difficulty. You state (p. 17 [p. 23 above]) that a function too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let **w** be the predicate: to be a predicate that cannot be predicated of itself. Can **w** be predicated of itself? From each answer its opposite follows. Therefore we must conclude that **w** is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality.[5]

Russell would go to cover it at length in his 1903 *The Principles of Mathematics* where he repeats his first encounter with the paradox:[6]

> Before taking leave of fundamental questions, it is necessary to examine more in detail the singular contradiction, already mentioned, with regard to predicates not predicable of themselves. ... I may mention that I was led to it in the endeavour to reconcile Cantor's proof...."

Russell wrote to Frege about the paradox just as Frege was preparing the second volume of his *Grundgesetze der Arithmetik*.[7] Frege did not waste time responding to Russell, his letter dated 22 June 1902 appears, with van

Heijenoort's commentary in Heijenoort 1967:126–127. Frege then wrote an appendix admitting to the paradox,[8] and proposed a solution that Russell would endorse in his *Principles of Mathematics*,[9] but was later considered by some unsatisfactory.[10] For his part, Russell had his work at the printers and he added an appendix on the doctrine of types.[11]

Ernst Zermelo in his (1908) *A new proof of the possibility of a well-ordering* (published at the same time he published "the first axiomatic set theory")[12] laid claim to prior discovery of the antinomy in Cantor's naive set theory. He states: "And yet, even the elementary form that Russell[9] gave to the set-theoretic antinomies could have persuaded them [J. König, Jourdain, F. Bernstein] that the solution of these difficulties is not to be sought in the surrender of well-ordering but only in a suitable restriction of the notion of set".[13] Footnote 9 is where he stakes his claim:

> [9]*1903*, pp. 366–368. I had, however, discovered this antinomy myself, independently of Russell, and had communicated it prior to 1903 to Professor Hilbert among others.[14]

A written account of Zermelo's actual argument was discovered in the *Nachlass* of Edmund Husserl.[15]

It is also known that unpublished discussions of set theoretical paradoxes took place in the mathematical community at the turn of the century. van Heijenoort in his commentary before Russell's 1902 *Letter to Frege* states that Zermelo "had discovered the paradox independently of Russell and communicated it to Hilbert, among others, prior to its publication by Russell".[16]

In 1923, Ludwig Wittgenstein proposed to "dispose" of Russell's paradox as follows:

> The reason why a function cannot be its own argument is that the sign for a function already contains the prototype of its argument, and it cannot contain itself. For let us suppose that the function F(fx) could be its own argument: in that case there would be a proposition 'F(F(fx))', in which the outer function F and the inner function F must have different meanings, since the inner one has the form O(f(x)) and the outer one has the form Y(O(fx)). Only the letter 'F' is common to the two functions, but the letter by itself signifies nothing. This immediately becomes clear if instead of 'F(Fu)' we write '(do) : F(Ou) . Ou = Fu'. That disposes of Russell's paradox. (*Tractatus Logico-Philosophicus*, 3.333)

Russell and Alfred North Whitehead wrote their three-volume *Principia Mathematica* (*PM*) hoping to achieve what Frege had been unable to do. They sought to banish the paradoxes of naive set theory by employing a theory of types they devised for this purpose. While they succeeded in grounding arithmetic in a fashion, it is not at all evident that they did so by purely logical means. While *PM* avoided the known paradoxes and allows the derivation of a great deal of mathematics, its system gave rise to new problems.

In any event, Kurt Gödel in 1930–31 proved that while the logic of much of *PM*, now known as first-order logic, is complete, Peano arithmetic is necessarily incomplete if it is consistent. This is very widely – though not universally – regarded as having shown the logicist program of Frege to be impossible to complete.

Applied versions

There are some versions of this paradox that are closer to real-life situations and may be easier to understand for non-logicians. For example, the Barber paradox supposes a barber who shaves all men who do not shave themselves and only men who do not shave themselves. When one thinks about whether the barber should shave himself or not, the paradox begins to emerge.

As another example, consider five lists of encyclopedia entries within the same encyclopedia:

List of articles about people:	List of articles starting with the letter L:	List of articles about places:	List of articles about Japan:	List of all lists that do not contain themselves:
• Ptolemy VII of Egypt • Hermann Hesse • Don Nix • Don Knotts • Nikola Tesla • Sherlock Holmes • Emperor Kōnin	• L • L!VE TV • L&H ... • List of articles starting with the letter K • List of articles starting with the letter L • List of articles starting with the letter M ...	• Leivonmäki • Katase River • Enoshima	• Emperor Showa • Katase River • Enoshima	• List of articles about Japan • List of articles about places • List of articles about people ... • List of articles starting with the letter K • List of articles starting with the letter M ... • **List of all lists that do not contain themselves?**

If the "List of all lists that do not contain themselves" contains itself, then it does not belong to itself and should be removed. However, if it does not list itself, then it should be added to itself.

While appealing, these layman's versions of the paradox share a drawback: an easy refutation of the Barber paradox seems to be that such a barber does not exist, or at least does not shave (a variant of which is that the barber is a woman). The whole point of Russell's paradox is that the answer "such a set does not exist" means the definition of the notion of set within a given theory is unsatisfactory. Note the difference between the statements "such a set does not exist" and "it is an empty set". It is like the difference between saying, "There is no bucket", and saying, "The bucket is empty".

A notable exception to the above may be the Grelling–Nelson paradox, in which words and meaning are the elements of the scenario rather than people and hair-cutting. Though it is easy to refute the Barber's paradox by saying that such a barber does not (and *cannot*) exist, it is impossible to say something similar about a meaningfully defined word.

One way that the paradox has been dramatised is as follows:

> Suppose that every public library has to compile a catalog of all its books. Since the catalog is itself one of the library's books, some librarians include it in the catalog for completeness; while others leave it out as it being one of the library's books is self-evident.
>
> Now imagine that all these catalogs are sent to the national library. Some of them include themselves in their listings, others do not. The national librarian compiles two master catalogs – one of all the catalogs that list themselves, and one of all those that don't.
>
> The question is: should these catalogs list themselves? The 'Catalog of all catalogs that list themselves' is no problem. If the librarian doesn't include it in its own listing, it is still a true catalog of those catalogs that do include themselves. If he *does* include it, it remains a true catalog of those that list themselves.
>
> However, just as the librarian cannot go wrong with the first master catalog, he is doomed to fail with the second. When it comes to the 'Catalog of all catalogs that don't list themselves', the librarian cannot include it in its own listing, because then it would include itself. But in that case, it should belong to the *other* catalog, that of catalogs that do include themselves. However, if the librarian leaves it out, the catalog is incomplete. Either way, it can never be a true catalog of catalogs that do not list themselves.

Applications and related topics

Russell-like paradoxes

As illustrated above for the Barber paradox, Russell's paradox is not hard to extend. Take:

- A transitive verb <V>, that can be applied to its substantive form.

Form the sentence:

> The <V>er that <V>s all (and only those) who don't <V> themselves,

Sometimes the "all" is replaced by "all <V>ers".

An example would be "paint":

> The *paint*er that *paint*s all (and only those) that don't *paint* themselves.

or "elect"

> The *elect*or (representative), that *elect*s all that don't *elect* themselves.

Paradoxes that fall in this scheme include:

- The barber with "shave".
- The original Russell's paradox with "contain": The container (Set) that contains all (containers) that don't contain themselves.
- The Grelling–Nelson paradox with "describer": The describer (word) that describes all words, that don't describe themselves.
- Richard's paradox with "denote": The denoter (number) that denotes all denoters (numbers) that don't denote themselves. (In this paradox, all descriptions of numbers get an assigned number. The term "that denotes all denoters (numbers) that don't denote themselves" is here called *Richardian*.)

Related paradoxes

- The liar paradox and Epimenides paradox, whose origins are ancient
- The Kleene–Rosser paradox, showing that the original lambda calculus is inconsistent, by means of a self-negating statement
- Curry's paradox (named after Haskell Curry), which does not require negation
- The smallest uninteresting integer paradox

Notes

[1] Set theory paradoxes (http://www.suitcaseofdreams.net/Set_theory_Paradox.htm)

[2] Godehard Link (2004), *One hundred years of Russell's paradox* (http://books.google.com/?id=Xg6QpedPpcsC&pg=PA350), p. 350, ISBN 978-3-11-017438-0,

[3] Russell 1920:136

[4] Gottlob Frege, Michael Beaney (1997), *The Frege reader* (http://books.google.com/?id=4ktC0UrG4V8C&pg=PA253), p. 253, ISBN 978-0-631-19445-3, . Also van Heijenoort 1967:124–125

[5] Remarkably, this letter was unpublished until van Heijenoort 1967 – it appears with van Heijenoort's commentary at van Heijenoort 1967:124–125.

[6] Russell 1903:101

[7] cf van Heijenoort's commentary before Frege's *Letter to Russell* in van Heijenoort 1967:126.

[8] van Heijenoort's commentary, cf van Heijenoort 1967:126 ; Frege starts his analysis by this exceptionally honest comment : "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr Bertrand Russell, just when the printing of this volume was nearing its completion" (Appendix of *Grundgesetze der Arithmetik, vol. II*, in *The Frege Reader*, p.279, translation by Michael Beaney

[9] cf van Heijenoort's commentary, cf van Heijenoort 1967:126. The added text reads as follows: " *Note*. The second volume of Gg., which appeared too late to be noticed in the Appendix, contains an interesting discussion of the contradiction (pp. 253–265), suggesting that the solution is to be found by denying that two propositional functions that determine equal classes must be equivalent. As it seems very likely

that this is the true solution, the reader is strongly recommended to examine Frege's argument on the point" (Russell 1903:522); The abbreviation Gg. stands for Frege's *Grundgezetze der Arithmetik*. Begriffsschriftlich abgeleitet. Vol. I. Jena, 1893. Vol. II. 1903.

[10] Livio states that "While Frege did make some desperate attempts to remedy his axiom system, he was unsuccessful. The conclusion appeared to be disastrous...." Livio 2009:188. But van Heijenoort in his commentary before Frege's (1902) *Letter to Russell* describes Frege's proposed "way out" in some detail – the matter has to do with the " 'transformation of the generalization of an equality into an equality of courses-of-values. For Frege a function is something incomplete, 'unsaturated' "; this seems to contradict the contemporary notion of a "function in extension"; see Frege's wording at page 128: "Incidentally, it seems to me that the expession 'a predicate is predicated of itself' is not exact. ...Therefore I would prefer to say that 'a concept is predicated of its own extension' [etc]". But he waffles at the end of his suggestion that a function-as-concept-in-extension can be written as predicated of its function. van Heijenoort cites Quine: "For a late and thorough study of Frege's "way out", see *Quine 1955*": "On Frege's way out", *Mind 64*, 145–159; reprinted in *Quine 1955b*: *Appendix. Completeness of quantification theory. Loewenheim's theorem*, enclosed as a pamphlet with part of the third printing (1955) of *Quine 1950* and incorporated in the revised edition (1959), 253—260" (cf REFERENCES in van Heijenoort 1967:649)

[11] Russell mentions this fact to Frege, cf van Heijenoort's commentary before Frege's (1902) *Letter to Russell* in van Heijenoort 1967:126

[12] van Heijenoort's commentary before Zermelo (1908a) *Investigations in the foundations of set theory I in van Heijenoort 1967:199*

[13] van Heijenoort 1967:190–191. In the section before this he objects strenuously to the notion of impredicativity as defined by Poincaré (and soon to be taken by Russell, too, in his 1908 *Mathematical logic as based on the theory of types* cf van Heijenoort 1967:150–182).

[14] Ernst Zermelo (1908) *A new proof of the possibility of a well-ordering* in van Heijenoort 1967:183–198. Livio 2009:191 reports that Zermelo "discovered Russell's paradox independently as early as 1900"; Livio in turn cites Ewald 1996 and van Heijenoort 1967 (cf Livio 2009:268).

[15] B. Rang and W. Thomas, "Zermelo's discovery of the 'Russell Paradox'", *Historia Mathematica*, v. 8 n. 1, 1981, pp. 15–22. doi:10.1016/0315-0860(81)90002-1

[16] van Heijenoort 1967:124

References

- Potter, Michael (15 January 2004), *Set Theory and its Philosophy*, Clarendon Press (Oxford University Press), ISBN 978-0-19-926973-0
- van Heijenoort, Jean (1967, third printing 1976), *From Frege to Gödel: A Source Book in Mathematical Logic, 1979-1931*, Cambridge, Massachusetts: Harvard University Press, ISBN 0-674-32449-8
- Livio, Mario (6 January 2009), *Is God a Mathematician?*, New York: Simon & Schuster, ISBN 978-0-7432-9405-8

External links

- Russell's Paradox (http://www.cut-the-knot.org/selfreference/russell.shtml) at Cut-the-Knot
- Stanford Encyclopedia of Philosophy: " Russell's Paradox (http://plato.stanford.edu/entries/russell-paradox/)" – by A. D. Irvine.

Gödel's completeness theorem

Gödel's completeness theorem is a fundamental theorem in mathematical logic that establishes a correspondence between semantic truth and syntactic provability in first-order logic. It makes a close link between model theory that deals with what is true in different models, and proof theory that studies what can be formally proved in particular formal systems.

It was first proved by Kurt Gödel in 1929. It was then simplified in 1947, when Leon Henkin observed in his Ph.D. thesis that the hard part of the proof can be presented as the Model Existence Theorem (published in 1949). Henkin's proof was simplified by Gisbert Hasenjaeger in 1953.

Statement of the theorem

Preliminaries

There are numerous deductive systems for first-order logic, including systems of natural deduction and Hilbert-style systems. Common to all deductive systems is the notion of a **formal deduction**. This is a sequence (or, in some cases, a finite tree) of formulas with a specially-designated **conclusion**. The definition of a deduction is such that it is finite and that it is possible to verify algorithmically (by a computer, for example, or by hand) that a given collection of formulas is indeed a deduction.

A first-order formula is called **logically valid** if it is true in every structure for the language of the formula. To formally state, and then prove, the completeness theorem, it is necessary to also define a deductive system. A deductive system is called **complete** if every logically valid formula is the conclusion of some formal deduction, and the completeness theorem for a particular deductive system is the theorem that it is complete in this sense. Thus, in a sense, there is a different completeness theorem for each deductive system. A converse to completeness is **soundness,** the fact that only logically valid formulas are provable in the deductive system.

If some specific deductive system of first-order logic is sound and complete, then is it "perfect" (a formula is provable iff it is a semantic consequence of the axioms), thus equivalent to any other deductive system with the same quality (any proof in one system can be converted into the other).

Gödel's original formulation

The completeness theorem says that if a formula is logically valid then there is a finite deduction (a formal proof) of the formula.

Gödel's completeness theorem says that a deductive system of first-order predicate calculus is "complete" in the sense that no additional inference rules are required to prove all the logically valid formulas. A converse to completeness is **soundness,** the fact that only logically valid formulas are provable in the deductive system. Together with soundness (whose verification is easy), this theorem implies that a formula is logically valid if and only if it is the conclusion of a formal deduction.

Model Existence Theorem

The simplest version of this theorem that suffices in practice for most needs, and has connections with the Löwenheim-Skolem theorem, says:

Every consistent, countable first-order theory has a finite or countable model

A more general version can be expressed as :

Every consistent first-order theory with a well-orderable language, has a model.

This theorem by Henkin is the most directly obtained version of the completeness theorem in its simplest proof.

More general form

It says that for any first-order theory *T* with a well-orderable language, and any sentence *S* in the language of the theory, there is a formal proof of *S* in *T* if and only if *S* is satisfied by every model of *T* (*S* is a semantic consequence of *T*).

This more general theorem is used implicitly, for example, when a sentence is shown to be provable from the axioms of group theory by considering an arbitrary group and showing that the sentence is satisfied by that group. It is deduced from the model existence theorem as follows: if there is no formal proof of a formula then adding its negation to the axioms gives a consisten theory, which has thus a model, so that the formula is not a semantic consequence of the initial theory.

Gödel's original formulation is deduced by taking the particular case of a theory without any axiom.

As a theorem of arithmetic

(Here, Peano Arithmetic (PA) is understood as the first-order theory of arithmetic with symbols of addition and multiplication, and schema of recursion.)

The Model Existence Theorem and its proof can somehow be formalized in the framework of PA. Precisely, we can systematically define a model of any consistent effective first-order theory *T* in PA by interpreting each symbol of *T* by an arithmetical formula whose free variables are the arguments of the symbol. However the definition expressed by this formula is not recursive.

Consequences

An important consequence of the completeness theorem is that it is possible to recursively enumerate the semantic consequences of any effective first-order theory, by enumerating all the possible formal deductions from the axioms of the theory, and use this to produce an enumeration of their conclusions.

This comes in contrast with the direct meaning of the notion of semantic consequence, that quantifies over all structures in a particular language, which is clearly not a recursive definition.

Also, it makes the concept of "provability" and thus of "theorem", a clear concept that only depends on the chosen system of axioms of the theory, and not on the choice of a proof system.

Relationship to the incompleteness theorem

Gödel's incompleteness theorem, another celebrated result, shows that there are inherent limitations in what can be achieved with formal proofs in mathematics. The name for the incompleteness theorem refers to another meaning of *complete* (see model theory - Using the compactness and completeness theorems).

It shows that in any consistent effective theory T containing PA, the formula C_T expressing the consistency of T cannot be proven within T.

Applying the completeness theorem to this result, gives the existence of a model of T where the formula C_T is false. Such a model (precisely, the set of "natural numbers" it contains) is necessarily non-standard, as it contains the code number of a proof of a contradiction of T. But T is consistent when viewed from the outside. Thus this code number of a proof of contradition of T must be a non-standard number.

In fact, the model of *any* theory containing PA obtained by the systematic construction of the arithmetical model existence theorem, is *always* non-standard with a non-equivalent provability predicate and a non-equivalent way to interpret its own construction, so that this construction is non-recursive (as recursive definitions would be unambiguous).

Also, there is no recursive non-standard model of PA.

Relationship to the compactness theorem

The completeness theorem and the compactness theorem are two cornerstones of first-order logic. While neither of these theorems can be proven in a completely effective manner, each one can be effectively obtained from the other.

The compactness theorem says that if a formula φ is a logical consequence of a (possibly infinite) set of formulas Γ then it is a logical consequence of a finite subset of Γ. This is an immediate consequence of the completeness theorem, because only a finite number of axioms from Γ can be mentioned in a formal deduction of φ, and the soundness of the deduction system then implies φ is a logical consequence of this finite set. This proof of the compactness theorem is originally due to Gödel.

Conversely, for many deductive systems, it is possible to prove the completeness theorem as an effective consequence of the compactness theorem.

The ineffectiveness of the completeness theorem can be measured along the lines of reverse mathematics. When considered over a countable language, the completeness and compactness theorems are equivalent to each other and equivalent to a weak form of choice known as weak König's lemma, with the equivalence provable in RCA_0 (a second-order variant of Peano arithmetic restricted to induction over Σ^0_1 formulas). Weak König's lemma is provable in ZF, the system of Zermelo–Fraenkel set theory without axiom of choice, and thus the completeness and compactness theorems for countable languages are provable in ZF. However the situation is different when the language is of arbitrary large cardinality since then, though the completeness and compactness theorems remain provably equivalent to each other in ZF, they are also provably equivalent to a weak form of the axiom of choice known as the ultrafilter lemma. In particular, no theory extending ZF can prove either the completeness or compactness theorems over arbitrary (possibly uncountable) languages without also proving the ultrafilter lemma on a set of same cardinality, knowing that on countable sets, the ultrafilter lemma becomes equivalent to weak König's lemma.

Completeness in other logics

The completeness theorem is a central property of first-order logic that does not hold for all logics. Second-order logic, for example, does not have a completeness theorem for its standard semantics (but does have the completeness property for Henkin semantics), and the same is true of all higher-order logics. It is possible to produce sound deductive systems for higher-order logics, but no such system can be complete. The set of logically-valid formulas in second-order logic is not enumerable.

Lindström's theorem states that first-order logic is the strongest (subject to certain constraints) logic satisfying both compactness and completeness.

A completeness theorem can be proved for modal logic or intuitionistic logic with respect to Kripke semantics.

Proofs

Gödel's original proof of the theorem proceeded by reducing the problem to a special case for formulas in a certain syntactic form, and then handling this form with an *ad hoc* argument.

In modern logic texts, Gödel's completeness theorem is usually proved with Henkin's proof, rather than with Gödel's original proof. Henkin's proof directly constructs a term model for any consistent first-order theory. James Margetson (2004) developed a computerized formal proof using the Isabelle theorem prover. [1] Other proofs are also known.

Further reading

- Gödel, K (1929). *Über die Vollständigkeit des Logikkalküls*. Doctoral dissertation. University Of Vienna.. The first proof of the completeness theorem.
- Gödel, K (1930). "Die Vollständigkeit der Axiome des logischen Funktionenkalküls" (in German). *Monatshefte für Mathematik* **37** (1): 349–360. doi:10.1007/BF01696781. JFM 56.0046.04. The same material as the dissertation, except with briefer proofs, more succinct explanations, and omitting the lengthy introduction.

External links

- Stanford Encyclopedia of Philosophy: "Kurt Gödel [2]" -- by Juliette Kennedy.
- MacTutor biography: Kurt Gödel. [3]
- Detlovs, Vilnis, and Podnieks, Karlis, "Introduction to mathematical logic. [4]"
- Simplified proofs of the completeness and incompleteness theorems [5]

References

[1] http://afp.sourceforge.net/entries/Completeness-paper.pdf
[2] http://plato.stanford.edu/entries/goedel/
[3] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Godel.html
[4] http://www.ltn.lv/~podnieks/
[5] http://settheory.net/incompleteness

Proof calculus

In mathematical logic, a **proof calculus** corresponds to a family of formal systems that use a common style of formal inference for its inference rules. The specific inference rules of a member of such a family characterize the theory of a logic.

Usually a given proof calculus encompasses more than a single particular formal system, since many proof calculi are under-determining and can be used for radically different logics. For example, a paradigmatic case is the sequent calculus, which can be used to express the consequence relations of both intuitionistic logic and relevance logic. Thus, loosely speaking, a proof calculus is a template or design pattern, characterized by a certain style of formal inference, that may be specialized to produce specific formal systems, namely by specifying the actual inference rules for such a system. There is no consensus among logicians on how best to define the term.

Examples of proof calculi

The most widely known proof calculi are those classical calculi that are still in widespread use:

- The class of Hilbert systems, of which the most famous example is the 1928 Hilbert-Ackermann system of first-order logic;
- Gerhard Gentzen's calculus of natural deduction, which is the first formalism of structural proof theory, and which is the cornerstone of the formulae-as-types correspondence relating logic to functional programming;
- Gentzen's sequent calculus, which is the most studied formalism of structural proof theory.

Many other proof calculi were, or might have been, seminal, but are not widely used today.

- Aristotle's syllogistic calculus, presented in the *Organon*, readily admits formalisation. There is still some modern interest in syllogistic, carried out under the aegis of term logic.
- Gottlob Frege's two-dimensional notation of the *Begriffsschrift* is usually regarded as introducing the modern concept of quantifier to logic.
- C.S. Peirce's existential graph might easily have been seminal, had history worked out differently.

Modern research in logic teems with rival proof calculi:

- Several systems have been proposed which replace the usual textual syntax with some graphical syntax.
- Recently, many logicians interested in structural proof theory have proposed calculi with deep inference, for instance display logic, hypersequents, the calculus of structures, and bunched implication.

Church–Turing thesis

In computability theory, the **Church–Turing thesis** (also known as the **Turing-Church thesis**[1], the **Church–Turing conjecture**, **Church's thesis**, **Church's conjecture**, and **Turing's thesis**) is a combined hypothesis ("thesis") about the nature of functions whose values are effectively calculable; or, in more modern terms, functions whose values are algorithmically computable. In simple terms, the Church–Turing thesis states that a function is algorithmically computable if and only if it is computable by a Turing machine.

Several attempts were made in the first half of the 20th Century to formalize the notion of computability:

- American mathematician Alonzo Church created a method for defining functions called the λ-calculus,
- British mathematician Alan Turing created a theoretical model for a machine, now called a universal Turing machine, that could carry out calculations from inputs,
- Church, along with mathematician Stephen Kleene and logician J.B. Rosser created a formal definition of a class of functions whose values could be calculated by recursion.

All three computational processes (recursion, the λ-calculus, and the Turing machine) were shown to be equivalent—all three approaches define the same class of functions.[2][3] This has led mathematicians and computer scientists to believe that the concept of computability is accurately characterized by these three equivalent processes. Informally the Church–Turing thesis states that if some method (algorithm) exists to carry out a calculation, then the same calculation can also be carried out by a Turing machine (as well as by a recursively definable function, and by a λ-function).

The Church–Turing thesis is a statement that characterizes the nature of computation and cannot be formally proven. Even though the three processes mentioned above proved to be equivalent, the fundamental premise behind the thesis — the notion of what it means for a function to be effectively calculable — is "a somewhat vague intuitive one". Thus, the "thesis" remains a hypothesis.[4]

Despite the fact that it has not been formally proven, the Church–Turing thesis now has near-universal acceptance.

Formal statement

Rosser 1939 addresses the notion of "effective computability" as follows: "Clearly the existence of CC and RC (Church's and Rosser's proofs) presupposes a precise definition of "effective". "Effective method" is here used in the rather special sense of a method each step of which is precisely predetermined and which is certain to produce the answer in a finite number of steps".[5] Thus the adverb-adjective "effective" is used in a sense of "1a: producing a decided, decisive, or desired effect", and "capable of producing a result".[6]

In the following, the words "effectively calculable" will mean "produced by any intuitively 'effective' means whatsoever" and "effectively computable" will mean "produced by a Turing-machine or equivalent mechanical device". Turing's "definitions" given in a footnote in his 1939 Ph.D. thesis *Systems of Logic Based on Ordinals*, supervised by Church, are virtually the same:

> "† We shall use the expression 'computable function' to mean a function calculable by a machine, and let 'effectively calculable' refer to the intuitive idea without particular identification with any one of these definitions."[7]

The thesis can be stated as follows:

> *Every effectively calculable function is a computable function.*[8]

Turing stated it this way:

> "It was stated ... that 'a function is effectively calculable if its values can be found by some purely mechanical process.' We may take this literally, understanding that by a purely mechanical process one which could be carried out by a machine. The development ... leads to ... an identification of computability† with effective

calculability." († is the footnote above, ibid.)

History

One of the important problems for logicians in the 1930s was David Hilbert's Entscheidungsproblem, which asked if there was a mechanical procedure for separating mathematical truths from mathematical falsehoods. This quest required that the notion of "algorithm" or "effective calculability" be pinned down, at least well enough for the quest to begin.[9] But from the very outset Alonzo Church's attempts began with a debate that continues to this day.[10] Was the notion of "effective calculability" to be (i) an "axiom or axioms" in an axiomatic system, or (ii) merely a *definition* that "identified" two or more propositions, or (iii) an *empirical hypothesis* to be verified by observation of natural events, or (iv) or just *a proposal* for the sake of argument (i.e. a "thesis").

Circa 1930–1952

In the course of studying the problem, Church and his student Stephen Kleene introduced the notion of λ-definable functions, and they were able to prove that several large classes of functions frequently encountered in number theory were λ-definable.[11] The debate began when Church proposed to Kurt Gödel that one should define the "effectively computable" functions as the λ-definable functions. Gödel, however, was not convinced and called the proposal "thoroughly unsatisfactory".[12] Rather in correspondence with Church (ca 1934–5), Gödel proposed *axiomatizing* the notion of "effective calculability"; indeed, in a 1935 letter to Kleene, Church reported that:

> "His [Gödel's] only idea at the time was that it might be possible, in terms of effective calculability as an undefined notion, to state a set of axioms which would embody the generally accepted properties of this notion, and to do something on that basis".[13]

But Gödel offered no further guidance. Eventually, he would suggest his (primitive) recursion, modified by Herbrand's suggestion, that Gödel had detailed in his 1934 lectures in Princeton NJ (Kleene and another student J. B. Rosser transcribed the notes.). But "he did not think that the two ideas could be satisfactorily identified "except heuristically".[14]

Next, it was necessary to identify and prove the equivalence of two notions of effective calculability. Equipped with the λ-calculus and "general" recursion, Stephen Kleene with help of Church and J. B. Rosser produced proofs (1933, 1935) to show that the two calculi are equivalent. Church subsequently modified his methods to include use of Herbrand–Gödel recursion and then proved (1936) that the Entscheidungsproblem is unsolvable: There is no generalized "effective calculation" (method, algorithm) that can determine whether or not a formula in either the recursive- or λ-calculus is "valid" (more precisely: no method to show that a well formed formula has a "normal form").[15]

Many years later in a letter to Davis (ca 1965), Gödel would confess that "he was, at the time of these [1934] lectures, not at all convinced that his concept of recursion comprised all possible recursions".[16] By 1963–4 Gödel would disavow Herbrand–Gödel recursion and the λ-calculus in favor of the Turing machine as the definition of "algorithm" or "mechanical procedure" or "formal system".[17]

An hypothesis leading to a natural law?: In late 1936 Alan Turing's paper (also proving that the Entscheidungsproblem is unsolvable) was delivered orally, but had not yet appeared in print.[18] On the other hand, Emil Post's 1936 paper had appeared and was certified independent of Turing's work.[19] Post strongly disagreed with Church's "identification" of effective computability with the λ-calculus and recursion, stating:

> "Actually the work already done by Church and others carries this identification considerably beyond the working hypothesis stage. But to mask this identification under a definition . . . blinds us to the need of its continual verification."[20]

Rather, he regarded the notion of "effective calculability" as merely a "working hypothesis" that might lead by inductive reasoning to a "natural law" rather than by "a definition or an axiom".[21] This idea was "sharply" criticized by Church.[22]

Thus Post in his 1936[13] was also discounting Kurt Gödel's suggestion to Church in 1934–5 that the thesis might be expressed as an axiom or set of axioms.[13]

Turing adds another definition, Rosser equates all three: Within just a short time, Turing's 1936–37 paper "On Computable Numbers, with an Application to the Entscheidungsproblem"[18] appeared. In it he stated another notion of "effective computability" with the introduction of his a-machines (now known as the Turing machine abstract computational model). And in a proof-sketch added as an "Appendix" to his 1936–37 paper, Turing showed that the classes of functions defined by λ-calculus and Turing machines coincided.[23]

In a few years (1939) Turing would propose, like Church and Kleene before him, that *his* formal definition of mechanical computing agent was the correct one.[24] Thus, by 1939, both Church (1934) and Turing (1939), neither having knowledge of the other's efforts, had individually proposed that their "formal systems" should be *definitions* of "effective calculability";[25] neither framed their statements as *theses*.

Rosser (1939) formally identified the three notions-as-definitions:

> "All three *definitions* are equivalent, so it does not matter which one is used."[26]

Kleene proposes *Church's Thesis*: This left the overt expression of a "thesis" to Kleene. In his 1943 paper *Recursive Predicates and Quantifiers* Kleene proposed his "THESIS I":

> "This heuristic fact [general recursive functions are effectively calculable]...led Church to state the following thesis(22). The same thesis is implicit in Turing's description of computing machines(23).
>
> > "THESIS I. *Every effectively calculable function (effectively decidable predicate) is general*[27] *recursive* [Kleene's italics]
>
> "Since a precise mathematical definition of the term effectively calculable (effectively decidable) has been wanting, we can take this thesis ... as a definition of it..."[28]
>
> > "(22) references Church 1936
> >
> > "(23) references Turing 1936–7

Kleene goes on to note that:

> "...the thesis has the character of an hypothesis—a point emphasized by Post and by Church(24). If we consider the thesis and its converse as definition, then the hypothesis is an hypothesis about the application of the mathematical theory developed from the definition. For the acceptance of the hypothesis, there are, as we have suggested, quite compelling grounds."[28]
>
> > > "(24) references Post 1936 of Post and Church's *Formal definitions in the theory of ordinal numbers*, *Fund. Math*. vol 28 (1936) pp.11–21 (see ref. #2, Davis 1965:286).

Kleene's Church–Turing Thesis: A few years later (1952) Kleene would overtly name, defend, and express the two "theses" and then "identify" them (show equivalence) by use of his Theorem XXX:

> "Heuristic evidence and other considerations led Church 1936 to propose the following thesis.
>
> > Thesis I. *Every effectively calculable function (effectively decidable predicate) is general recursive*.[29]
>
> Theorem XXX: "The following classes of partial functions are coextensive, i.e. have the same members: (a) the partial recursive functions, (b) the computable functions. . . ".[30]
>
> Turing's thesis: "Turing's thesis that every function which would naturally be regarded as computable is computable under his definition, i.e. by one of his machines, is equivalent to Church's thesis by Theorem XXX."[31]

Later developments

An attempt to understand the notion of "effective computability" better led Robin Gandy (Turing's student and friend) in 1980 to analyze *machine* computation (as opposed to human-computation acted out by a Turing machine). Gandy's curiosity about, and analysis of, "cellular automata", "Conway's game of life", "parallelism" and "crystalline automata" led him to propose four "principles (or constraints) ... which it is argued, any machine must satisfy."[32] His most-important fourth, "the principle of causality" is based on the "finite velocity of propagation of effects and signals; contemporary physics rejects the possibility of instantaneous action at a distance."[33] From these principles and some additional constraints—(1a) a lower bound on the linear dimensions of any of the parts, (1b) an upper bound on speed of propagation (the velocity of light), (2) discrete progress of the machine, and (3) deterministic behavior—he produces a theorem that "What can be calculated by a device satisfying principles I–IV is computable.[34] ".

In the late 1990s Wilfried Sieg analyzed Turing's and Gandy's notions of "effective calculability" with the intent of "sharpening the informal notion, formulating its general features axiomatically, and investigating the axiomatic framework".[35] In his 1997 and 2002 Sieg presents a series of constraints on the behavior of a *computor*—"a human computing agent who proceeds mechanically"; these constraints reduce to:

- "(B.1) (Boundedness) *There is a fixed bound on the number of symbolic configurations a computor can immediately recognize.*
- "(B.2) (Boundedness) *There is a fixed bound on the number of internal states a computor can be in.*
- "(L.1) (Locality) *A computor can change only elements of an observed symbolic configuration.*
- "(L.2) (Locality) *A computor can shift attention from one symbolic configuration to another one, but the new observed configurations must be within a bounded distance of the immediately previously observed configuration.*
- "(D) (Determinacy) *The immediately recognizable (sub-)configuration determines uniquely the next computation step (and id [instantaneous description])*"; stated another way: "*A computor's internal state together with the observed configuration fixes uniquely the next computation step and the next internal state."*[36]

The matter remains in active discussion within the academic community.[37]

Success of the thesis

Other formalisms (besides recursion, the λ-calculus, and the Turing machine) have been proposed for describing effective calculability/computability. Stephen Kleene (1952) adds to the list the functions "*reckonable* in the system S_1" of Kurt Gödel 1936, and Emil Post's (1943, 1946) "*canonical* [also called *normal*] *systems*".[38] In the 1950s Hao Wang and Martin Davis greatly simplified the one-tape Turing-machine model (see Post–Turing machine). Marvin Minsky expanded the model to two or more tapes and greatly simplified the tapes into "up-down counters", which Melzak and Lambek further evolved into what is now known as the counter machine model. In the late 1960s and early 1970s researchers expanded the counter machine model into the register machine, a close cousin to the modern notion of the computer. Other models include combinatory logic and Markov algorithms. Gurevich adds the pointer machine model of Kolmogorov and Uspensky (1953, 1958): "...they just wanted to ... convince themselves that there is no way to extend the notion of computable function."[39]

All these contributions involve proofs that the models are computationally equivalent to the Turing machine; such models are said to be Turing complete. Because all these different attempts at formalizing the concept of "effective calculability/computability" have yielded equivalent results, it is now generally assumed that the Church–Turing thesis is correct. In fact, Gödel (1936) proposed something stronger than this; he observed that there was something "absolute" about the concept of "reckonable in S_1":

> "It may also be shown that a function which is computable ['reckonable'] in one of the systems S_i, or even in a system of transfinite type, is already computable [reckonable] in S_1. Thus the concept 'computable' ['reckonable'] is in a certain definite sense 'absolute', while practically all other familiar metamathematical

concepts (e.g. provable, definable, etc.) depend quite essentially on the system to which they are defined"[40]

Informal usage in proofs

Proofs in computability theory often invoke[41] the Church–Turing thesis in an informal way to establish the computability of functions while avoiding the (often very long) details which would be involved in a rigorous, formal proof. To establish that a function is computable by Turing machine, it is usually considered sufficient to give an informal English description of how the function can be effectively computed, and then conclude "By the Church–Turing thesis" that the function is Turing computable (equivalently partial recursive).

Dirk van Dalen (in Gabbay 2001:284[42]) gives the following example for the sake of illustrating this informal use of the Church–Turing thesis:

> EXAMPLE: Each infinite RE set contains an infinite recursive set.
>
> Proof: Let A be infinite RE. We list the elements of A effectively, n_0, n_1, n_2, n_3, ...
>
> From this list we extract an increasing sublist: put $m_0=n_0$, after finitely many steps we find an n_k such that $n_k > m_0$, put $m_1=n_k$. We repeat this procedure to find $m_2 > m_1$, etc. this yields an effective listing of the subset B={m_0,m_1,m_2,...} of A, with the property $m_i < m_{i+1}$.
>
> *Claim*. B is decidable. For, in order to test k in B we must check if $k=m_i$ for some i. Since the sequence of m_i's is increasing we have to produce at most k+1 elements of the list and compare them with k. If none of them is equal to k, then k not in B. Since this test is effective, B is decidable and, **by Church's thesis**, recursive.

(Emphasis added). In order to make the above example completely rigorous, one would have to carefully construct a Turing Machine, or λ-function, or carefully invoke recursion axioms, or at best, cleverly invoke various theorems of computability theory. But because the computability theorist believes that Turing computability correctly captures what can be computed effectively, and because an effective procedure is spelled out in English for deciding the set B, the computability theorist accepts this as proof that the set is indeed recursive.

As a rule of thumb, the Church–Turing thesis should only be invoked to simplify proofs in cases where the writer would be capable of, and expects the readers also to be capable of, easily (but not necessarily without tedium) producing a rigorous proof if one were demanded.

Variations

The success of the Church–Turing thesis prompted variations of the thesis to be proposed. For example, the **Physical Church–Turing thesis** (PCTT) states:

> "According to Physical CTT, all physically computable functions are Turing-computable"[43]

The Church–Turing thesis says nothing about the efficiency with which one model of computation can simulate another. It has been proved for instance that a (multi-tape) universal Turing machine only suffers a logarithmic slowdown factor in simulating any Turing machine.[44] No such result has been proved in general for an arbitrary but *reasonable* model of computation. A variation of the Church–Turing thesis that addresses this issue is the **Feasibility Thesis**[45] or **(Classical) Complexity-Theoretic Church–Turing Thesis** (SCTT), which is not due to Church or Turing, but rather was realized gradually in the development of complexity theory. It states:[46]

> "*A probabilistic Turing machine can efficiently simulate any realistic model of computation.*"

The word 'efficiently' here means up to polynomial-time reductions. This thesis was originally called *Computational Complexity-Theoretic Church–Turing Thesis* by Ethan Bernstein and Umesh Vazirani (1997). The Complexity-Theoretic Church–Turing Thesis, then, posits that all 'reasonable' models of computation yield the same class of problems that can be computed in polynomial time. Assuming the conjecture that probabilistic polynomial time (BPP) equals deterministic polynomial time (P), the word 'probabilistic' is optional in the Complexity-Theoretic Church–Turing Thesis. A similar thesis, called the *Invariant Thesis*, was introduced by Cees F. Slot and Peter van

Emde Boas. It states: *"Reasonable" machines can simulate each other within a polynomially bounded overhead in time and a constant-factor overhead in space*.[47] The thesis originally appeared in a paper at STOC'84, which was the first paper to show that polynomial-time overhead and constant-space overhead could be *simultaneously* achieved for a simulation of a Random Access Machine on a Turing machine.[48]

If BQP is shown to be a strict superset of BPP, it would invalidate the Complexity-Theoretic Church–Turing Thesis. In other words, there would be efficient quantum algorithms that perform tasks that do not have efficient probabilistic algorithms. This would not however invalidate the original Church–Turing thesis, since a quantum computer can always be simulated by a Turing machine, but it would invalidate the classical Complexity-Theoretic Church–Turing thesis for efficiency reasons. Consequently, the **Quantum Complexity-Theoretic Church–Turing thesis** states:[46]

> *"A quantum Turing machine can efficiently simulate any realistic model of computation."*

Eugene Eberbach and Peter Wegner[49] claim that the Church–Turing thesis is sometimes interpreted too broadly, stating "the broader assertion that algorithms precisely capture what can be computed is invalid". They claim that forms of computation not captured by the thesis are relevant today, terms which they call super-Turing computation.

Philosophical implications

Philosophers have interpreted the Church–Turing thesis as having implications for the philosophy of mind; however, many of the philosophical interpretations of the Thesis involve basic misunderstandings of the thesis statement.[50] B. Jack Copeland states that it's an open empirical question whether there are actual deterministic physical processes that, in the long run, elude simulation by a Turing machine; furthermore, he states that it is an open empirical question whether any such processes are involved in the working of the human brain.[51] There are also some important open questions which cover the relationship between the Church–Turing thesis and physics, and the possibility of hypercomputation. When applied to physics, the thesis has several possible meanings:

1. The universe is equivalent to a Turing machine; thus, computing non-recursive functions is physically impossible. This has been termed the Strong Church–Turing thesis and is a foundation of digital physics.
2. The universe is not equivalent to a Turing machine (i.e., the laws of physics are not Turing-computable), but incomputable physical events are not "harnessable" for the construction of a hypercomputer. For example, a universe in which physics involves real numbers, as opposed to computable reals, might fall into this category. The assumption that incomputable physical events are not "harnessable" has been challenged, however,[52] by a proposed computational process that uses quantum randomness together with a computational machine to hide the computational steps of a Universal Turing Machine with Turing-incomputable firing patterns.
3. The universe is a hypercomputer, and it is possible to build physical devices to harness this property and calculate non-recursive functions. For example, it is an open question whether all quantum mechanical events are Turing-computable, although it is known that rigorous models such as quantum Turing machines are equivalent to deterministic Turing machines. (They are not necessarily efficiently equivalent; see above.) John Lucas and Roger Penrose[53] have suggested that the human mind might be the result of some kind of quantum-mechanically enhanced, "non-algorithmic" computation, although there is no scientific evidence for this proposal.

There are many other technical possibilities which fall outside or between these three categories, but these serve to illustrate the range of the concept.

Non-computable functions

One can formally define functions that are not computable. A well known example of such a function is the Busy Beaver function. This function takes an input *n* and returns the largest number of symbols that a Turing machine with *n* states can print before halting, when run with no input. Finding an upper bound on the busy beaver function is equivalent to solving the halting problem, a problem known to be unsolvable by Turing machines. Since the busy beaver function cannot be computed by Turing machines, the Church–Turing thesis states that this function cannot be effectively computed by any method. For more information see the article busy beaver.

Several computational models allow for the computation of (Church-Turing) non-computable functions. These are known as hypercomputers. Mark Burgin argues that super-recursive algorithms such as inductive Turing machines disprove the Church–Turing thesis. His argument relies on a definition of algorithm broader than the ordinary one, so that non-computable functions obtained from some inductive Turing machines are called computable. This interpretation of the Church–Turing thesis differs from the interpretation commonly accepted in computability theory, discussed above. The argument that super-recursive algorithms are indeed algorithms in the sense of the Church–Turing thesis has not found broad acceptance within the computability research community.

Footnotes

[1] Rabin, Michael O. (June 2012). *Turing, Church, Gödel, Computability, Complexity and Randomization: A Personal View* (http://videolectures.net/turing100_rabin_turing_church_goedel/). .
[2] Church 1934:90 footnote in Davis 1952
[3] Turing 1936–7 in Davis 1952:149
[4] Kleene 1952:317
[5] Rosser 1939 in Davis 1965:225
[6] Merriam Webster's Ninth New Collegiate Dictionary
[7] A. M. Turing (1939), *Systems of Logic Based on Ordinals* (https://webspace.princeton.edu/users/jedwards/Turing Centennial 2012/Mudd Archive files/12285_AC100_Turing_1938.pdf) (Ph.D. thesis). Princeton University. p. 8.
[8] Gandy (Gandy 1980 in Barwise 1980:123) states it this way: *What is effectively calculable is computable.* He calls this "Church's Thesis", a peculiar choice of moniker.
[9] Davis's commentary before Church 1936 *An Unsolvable Problem of Elementary Number Theory* in Davis 1965:88. Church uses the words "effective calculability" on page 100ff.
[10] In his review of *Church's Thesis after 70 Years* edited by Adam Olszewski et al. 2006, Peter Smith's criticism of a paper by Muraswski and Wolenski suggests 4 "lines" re the status of the Church–Turing Thesis: (1) empirical hypothesis (2) axiom or theorem, (3) definition, (4) explication. But Smith opines that (4) is indistinguishable from (3), cf Smith (July 11, 2007) *Church's Thesis after 70 Years* at http://www.phil.cam.ac.uk/teaching_staff/Smith/godelbook/other/CTT.pdf
[11] cf footnote 3 in Church 1936 *An Unsolvable Problem of Elementary Number Theory* in Davis 1965:89
[12] Dawson 1997:99
[13] Sieg 1997:160
[14] Sieg 1997:160 quoting from the 1935 letter written by Church to Kleene, cf Footnote 3 in Gödel 1934 in Davis 1965:44
[15] cf Church 1936 in Davis 1965:105ff
[16] Davis's commentary before Gödel 1934 in Davis 1965:40
[17] For a detailed discussion of Gödel's adoption of Turing's machines as models of computation, see Shagrir date TBD at http://edelstein.huji.ac.il/staff/shagrir/papers/Goedel_on_Turing_on_Computability.pdf
[18] Turing 1937
[19] cf. Editor's footnote to Post 1936 *Finite Combinatory Process. Formulation I.* at Davis 1965:289.
[20] Post 1936 in Davis 1965:291 footnote 8
[21] Post 1936 in Davis 1952:291
[22] Sieg 1997:171 and 176–7
[23] Turing 1936–7 in Davis 1965:263ff
[24] Turing 1939 in Davis:160
[25] cf. Church 1934 in Davis 1965:100, also Turing 1939 in Davis 1965:160
[26] italics added, Rosser 1939 in Davis 1965:226
[27] An archaic usage of Kleene et al. to distinguish Gödel's (1931) "rekursiv" (a few years later named primitive recursion by Rózsa Péter (cf Gandy 1994 in Herken 1994–5:68)) from Herbrand–Gödel's recursion of 1934 i.e. primitive recursion equipped with the additional mu operator; nowadays mu-recursion is called, simply, "recursion".
[28] Kleene 1943 in Davis 1965:274

[29] Kleene 1952:300
[30] Kleene 1952:376
[31] Kleene 1952:376)
[32] Gandy 1980 in Barwise 1980:123ff)
[33] Gandy 1980 in Barwise 1980:135
[34] Gandy 1980 in Barwise:126
[35] (Sieg 1998–9 in Sieg–Somner–Talcott 2002:390ff; also Sieg 1997:154ff)
[36] In a footnote Sieg breaks Post's 1936 (B) into (B.1) and (B.2) and (L) into (L.1) and (L.2) and describes (D) differently. With respect to his proposed Gandy machine he later adds LC.1, LC.2, GA.1 and GA.2. These are complicated; see Sieg 1998–9 in Sieg–Somner–Talcott 2002:390ff.
[37] A collection of papers can be found at *Church's Thesis after 70 Years* edited by Adam Olszewski et al. 2006. Also a review of this collection by Peter Smith (July 11, 2007) *Church's Thesis after 70 Years* at http://www.phil.cam.ac.uk/teaching_staff/Smith/godelbook/other/CTT.pdf
[38] Kleene 1952:320
[39] Gurevich 1988:2
[40] translation of Gödel (1936) by Davis in *The Undecidable* p. 83, differing in the use of the word 'reckonable' in the translation in Kleene (1952) p. 321
[41] Horsten in Olszewski 2006:256
[42] Gabbay 2001:284
[43] Piccinini 2007:101 "Computationalism, the Church–Turing Thesis, and the Church–Turing Fallacy" (http://www.umsl.edu/~piccininig/Computationalism_Church-Turing_Thesis_Church-Turing_Fallacy.pdf). doi:10.1007/s11229-005-0194-z. in Synthese (2007) 154:97–120.
[44] Arora, Sanjeev; Barak, Boaz, "Complexity Theory: A Modern Approach" (http://www.cs.princeton.edu/theory/complexity/), Cambridge University Press, 2009, ISBN 978-0-521-42426-4, section 1.4, "Machines as strings and the universal Turing machine" and 1.7, "Proof of theorem 1.9"
[45] http://www.claymath.org/millennium/P_vs_NP/Official_Problem_Description.pdf
[46] Phillip Kaye, Raymond Laflamme, Michele Mosca, *An introduction to quantum computing*, Oxford University Press, 2007, ISBN 0-19-857049-X, pp. 5–6
[47] Peter van Emde Boas's, *Machine Models and Simulations*, in *Handbook of Theoretical Computer Science A*, Elsevier, 1990, p. 5
[48] C. Slot, P. van Emde Boas, *On tape versus core: an application of space efficient perfect hash functions to the invariance of space*, STOC, December 1984
[49] Eberbach and Wegner, 2003
[50] In particular, see the numerous examples (of errors, of misappropriation of the thesis) at the entry in the Stanford Encyclopedia of Philosophy. For a good place to encounter original papers see David J. Chalmers, ed. 2002, *Philosophy of Mind: Classical and Contemporary Readings*, Oxford University Press, New York.
[51] B. Jack Copeland, *Computation* in Luciano Floridi (ed.), *The Blackwell guide to the philosophy of computing and information*, Wiley-Blackwell, 2004, ISBN 0-631-22919-1, p. 15
[52] Michael Fiske, "Turing Incomputable Computation" in Turing-100 proceedings, The Alan Turing Centenary. http://www.easychair.org/publications/?page=1303694832.
[53] cf his subchapter "The Church–Turing Thesis" (p. 47–49) in his chapter "Algorithms and Turing machines" in his 1990 (2nd edition) *Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics*, Oxford University Press, Oxford UK. Also his a final chapter titled "Where lies the physics of mind?" where, in a subsection he describes "The non-algorithmic nature of mathematical insight" (p. 416–8).

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External links

- The Church–Turing Thesis (http://plato.stanford.edu/entries/church-turing) entry by B. Jack Copeland in the *Stanford Encyclopedia of Philosophy*.
- *Computation in Physical Systems* (http://plato.stanford.edu/entries/computation-physicalsystems/) A comprehensive philosophical treatment of relevant issues.

Complexity class

In computational complexity theory, a **complexity class** is a set of problems of related resource-based complexity. A typical complexity class has a definition of the form:

> the set of problems that can be solved by an abstract machine M using O(f(n)) of resource R, where n is the size of the input.

For example, the class **NP** is the set of decision problems whose solutions can be determined by a non-deterministic Turing machine in polynomial time, while the class **PSPACE** is the set of decision problems that can be solved by a deterministic Turing machine in polynomial space.

The simpler complexity classes are defined by the following factors:

- The type of computational problem: The most commonly used problems are decision problems. However, complexity classes can be defined based on function problems (an example is **FP**), counting problems (e.g. **#P**), optimization problems, promise problems, etc.
- The model of computation: The most common model of computation is the deterministic Turing machine, but many complexity classes are based on nondeterministic Turing machines, boolean circuits, quantum Turing machines, monotone circuits, etc.
- The resource (or resources) that are being bounded and the bounds: These two properties are usually stated together, such as "polynomial time", "logarithmic space", "constant depth", etc.

Many complexity classes can be characterized in terms of the mathematical logic needed to express them; see descriptive complexity.

Bounding the computation time above by some concrete function $f(n)$ often yields complexity classes that depend on the chosen machine model. For instance, the language $\{xx \mid x$ is any binary string$\}$ can be solved in linear time on a multi-tape Turing machine, but necessarily requires quadratic time in the model of single-tape Turing machines. If we allow polynomial variations in running time, Cobham-Edmonds thesis states that "the time complexities in any two reasonable and general models of computation are polynomially related" (Goldreich 2008, Chapter 1.2). This forms the basis for the complexity class P, which is the set of decision problems solvable by a deterministic Turing machine within polynomial time. The corresponding set of function problems is FP.

The Blum axioms can be used to define complexity classes without referring to a concrete computational model.

Important complexity classes

Many important complexity classes can be defined by bounding the time or space used by the algorithm. Some important complexity classes of decision problems defined in this manner are the following:

Complexity class	Model of computation	Resource constraint
DTIME($f(n)$)	Deterministic Turing machine	Time $f(n)$
P	Deterministic Turing machine	Time poly(n)
EXPTIME	Deterministic Turing machine	Time $2^{\text{poly}(n)}$
NTIME($f(n)$)	Non-deterministic Turing machine	Time $f(n)$
NP	Non-deterministic Turing machine	Time poly(n)
NEXPTIME	Non-deterministic Turing machine	Time $2^{\text{poly}(n)}$
DSPACE($f(n)$)	Deterministic Turing machine	Space $f(n)$
L	Deterministic Turing machine	Space O(log n)

PSPACE	Deterministic Turing machine	Space poly(n)
EXPSPACE	Deterministic Turing machine	Space $2^{\text{poly}(n)}$
NSPACE(f(n))	Non-deterministic Turing machine	Space f(n)
NL	Non-deterministic Turing machine	Space O(log n)
NPSPACE	Non-deterministic Turing machine	Space poly(n)
NEXPSPACE	Non-deterministic Turing machine	Space $2^{\text{poly}(n)}$

It turns out that PSPACE = NPSPACE and EXPSPACE = NEXPSPACE by Savitch's theorem.

Other important complexity classes include BPP, ZPP and RP, which are defined using probabilistic Turing machines; AC and NC, which are defined using boolean circuits and BQP and QMA, which are defined using quantum Turing machines. #P is an important complexity class of counting problems (not decision problems). Classes like IP and AM are defined using Interactive proof systems. ALL is the class of all decision problems.

Reduction

Many complexity classes are defined using the concept of a reduction. A reduction is a transformation of one problem into another problem. It captures the informal notion of a problem being at least as difficult as another problem. For instance, if a problem *X* can be solved using an algorithm for *Y*, *X* is no more difficult than *Y*, and we say that *X* *reduces* to *Y*. There are many different types of reductions, based on the method of reduction, such as Cook reductions, Karp reductions and Levin reductions, and the bound on the complexity of reductions, such as polynomial-time reductions or log-space reductions.

The most commonly used reduction is a polynomial-time reduction. This means that the reduction process takes polynomial time. For example, the problem of squaring an integer can be reduced to the problem of multiplying two integers. This means an algorithm for multiplying two integers can be used to square an integer. Indeed, this can be done by giving the same input to both inputs of the multiplication algorithm. Thus we see that squaring is not more difficult than multiplication, since squaring can be reduced to multiplication.

This motivates the concept of a problem being hard for a complexity class. A problem *X* is *hard* for a class of problems *C* if every problem in *C* can be reduced to *X*. Thus no problem in *C* is harder than *X*, since an algorithm for *X* allows us to solve any problem in *C*. Of course, the notion of hard problems depends on the type of reduction being used. For complexity classes larger than P, polynomial-time reductions are commonly used. In particular, the set of problems that are hard for NP is the set of NP-hard problems.

If a problem *X* is in *C* and hard for *C*, then *X* is said to be *complete* for *C*. This means that *X* is the hardest problem in *C* (Since there could be many problems which are equally hard, one might say that *X* is one of the hardest problems in *C*). Thus the class of NP-complete problems contains the most difficult problems in NP, in the sense that they are the ones most likely not to be in P. Because the problem P = NP is not solved, being able to reduce another problem, Π_1, to a known NP-complete problem, Π_2, would indicate that there is no known polynomial-time solution for Π_1. This is because a polynomial-time solution to Π_1 would yield a polynomial-time solution to Π_2. Similarly, because all NP problems can be reduced to the set, finding an NP-complete problem that can be solved in polynomial time would mean that P = NP.

Closure properties of classes

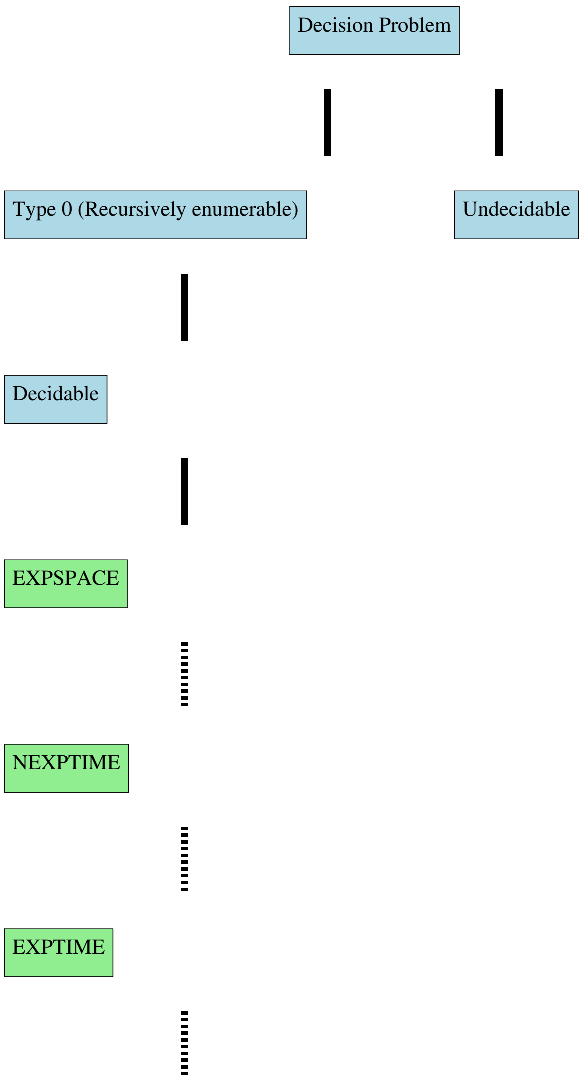
Complexity classes have a variety of closure properties; for example, decision classes may be closed under negation, disjunction, conjunction, or even under all Boolean operations. Moreover, they might also be closed under a variety of quantification schemes. **P**, for instance, is closed under all Boolean operations, and under quantification over polynomially sized domains. However, it is most likely not closed under quantification over exponential sized domains.

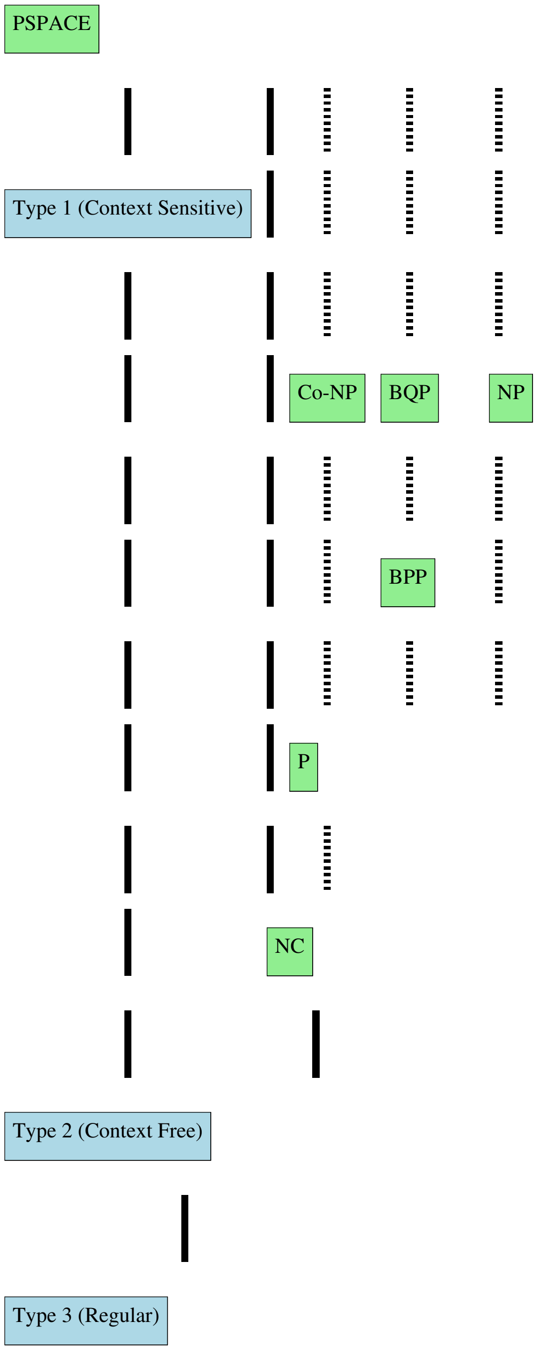
Each class **X** which is not closed under negation has a complement class **Co-Y**, which consists of the complements of the languages contained in **X**. Similarly one can define the Boolean closure of a class, and so on; this is however less commonly done.

One possible route to separating two complexity classes is to find some closure property possessed by one and not by the other.

Relationships between complexity classes

The following table shows some of the classes of problems (or languages, or grammars) that are considered in complexity theory. If class **X** is a strict subset of **Y**, then **X** is shown below **Y**, with a dark line connecting them. If **X** is a subset, but it is unknown whether they are equal sets, then the line is lighter and is dotted. Technically, the breakdown into decidable and undecidable pertains more to the study of computability theory but is useful for putting the complexity classes in perspective.





Hierarchy theorems

For the complexity classes defined in this way, it is desirable to prove that relaxing the requirements on (say) computation time indeed defines a bigger set of problems. In particular, although DTIME(n) is contained in DTIME(n^2), it would be interesting to know if the inclusion is strict. For time and space requirements, the answer to such questions is given by the time and space hierarchy theorems respectively. They are called hierarchy theorems because they induce a proper hierarchy on the classes defined by constraining the respective resources. Thus there are pairs of complexity classes such that one is properly included in the other. Having deduced such proper set inclusions, we can proceed to make quantitative statements about how much more additional time or space is needed in order to increase the number of problems that can be solved.

More precisely, the time hierarchy theorem states that

$$\mathrm{DTIME}\big(f(n)\big) \subsetneq \mathrm{DTIME}\big(f(n)\cdot\log^2(f(n))\big).$$

The space hierarchy theorem states that

$$\mathrm{DSPACE}\big(f(n)\big) \subsetneq \mathrm{DSPACE}\big(f(n)\cdot\log(f(n))\big).$$

The time and space hierarchy theorems form the basis for most separation results of complexity classes. For instance, the time hierarchy theorem tells us that P is strictly contained in EXPTIME, and the space hierarchy theorem tells us that L is strictly contained in PSPACE.

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Further reading

- The Complexity Zoo (http://qwiki.stanford.edu/index.php/Complexity_Zoo_1.0): A huge list of complexity classes, as reference for experts.
- Diagram (http://www.cs.umass.edu/~immerman/complexity_theory.html) by Neil Immerman showing the hierarchy of complexity classes and how they fit together.
- Michael Garey, and David S. Johnson: *Computers and Intractability: A Guide to the Theory of NP-Completeness.* New York: W. H. Freeman & Co., 1979. The standard reference on NP-Complete problems - an important category of problems whose solutions appear to require an impractically long time to compute.

Turing degree

In computer science and mathematical logic the **Turing degree** or **degree of unsolvability** of a set of natural numbers measures the level of algorithmic unsolvability of the set. The concept of Turing degree is fundamental in computability theory, where sets of natural numbers are often regarded as decision problems; the Turing degree of a set tells how difficult it is to solve the decision problem associated with the set.

Two sets are **Turing equivalent** if they have the same level of unsolvability; each Turing degree is a collection of Turing equivalent sets, so that two sets are in different Turing degrees exactly when they are not Turing equivalent. Furthermore, the Turing degrees are partially ordered so that if the Turing degree of a set *X* is less than the Turing degree of a set *Y* then any (noncomputable) procedure that correctly decides whether numbers are in *Y* can be effectively converted to a procedure that correctly decides whether numbers are in *X*. It is in this sense that the Turing degree of a set corresponds to its level of algorithmic unsolvability.

The Turing degrees were introduced by Emil Leon Post (1944), and many fundamental results were established by Stephen Cole Kleene and Post (1954). The Turing degrees have been an area of intense research since then. Many proofs in the area make use of a proof technique known as the **priority method**.

Turing equivalence

For the rest of this article, the word *set* will refer to a set of natural numbers. A set *X* is said to be **Turing reducible** to a set *Y* if there is an oracle Turing machine that decides membership in *X* when given an oracle for membership in *Y*. The notation $X \leq_T Y$ indicates that *X* is Turing reducible to *Y*.

Two sets *X* and *Y* are defined to be **Turing equivalent** if *X* is Turing reducible to *Y* and *Y* is Turing reducible to *X*. The notation $X \equiv_T Y$ indicates that *X* and *Y* are Turing equivalent. The relation $\equiv_T$ can be seen to be an equivalence relation, which means that for all sets *X*, *Y*, and *Z*:

- $X \equiv_T X$
- $X \equiv_T Y$ implies $Y \equiv_T X$
- If $X \equiv_T Y$ and $Y \equiv_T Z$ then $X \equiv_T Z$.

Turing degree

A **Turing degree** is an equivalence class of the relation $\equiv_T$. The notation [*X*] denotes the equivalence class containing a set *X*. The entire collection of Turing degrees is denoted $\mathcal{D}$.

The Turing degrees have a partial order ≤ defined so that [*X*] ≤ [*Y*] if and only if $X \leq_T Y$. There is a unique Turing degree containing all the computable sets, and this degree is less than every other degree. It is denoted **0** (zero) because it is the least element of the poset $\mathcal{D}$. (It is common to use boldface notation for Turing degrees, in order to distinguish them from sets. When no confusion can occur, such as with [*X*], the boldface is not necessary.)

For any sets *X* and *Y*, X **join** Y, written $X \oplus Y$, is defined to be the union of the sets $\{2n : n \in X\}$ and $\{2m+1 : m \in Y\}$. The Turing degree of $X \oplus Y$ is the least upper bound of the degrees of *X* and *Y*. Thus $\mathcal{D}$ is a join-semilattice. The least upper bound of degrees **a** and **b** is denoted **a** ∪ **b**. It is known that $\mathcal{D}$ is not a lattice, as there are pairs of degrees with no greatest lower bound.

For any set *X* the notation *X*′ denotes the set of indices of oracle machines that halt when using *X* as an oracle. The set *X*′ is called the **Turing jump** of *X*. The Turing jump of a degree [*X*] is defined to be the degree [*X*′]; this is a valid definition because $X' \equiv_T Y'$ whenever $X \equiv_T Y$. A key example is **0′**, the degree of the halting problem.

Basic properties of the Turing degrees

- Every Turing degree is countably infinite, that is, it contains exactly $\aleph_0$ sets.
- There are $2^{\aleph_0}$ distinct Turing degrees.
- For each degree **a** the strict inequality **a** < **a′** holds.
- For each degree **a**, the set of degrees below **a** is at most countable. The set of degrees greater than **a** has size $2^{\aleph_0}$.

Structure of the Turing degrees

A great deal of research has been conducted into the structure of the Turing degrees. The following survey lists only some of the many known results. One general conclusion that can be drawn from the research is that the structure of the Turing degrees is extremely complicated.

Order properties

- There are **minimal degrees**. A degree **a** is *minimal* if **a** is nonzero and there is no degree between **0** and **a**. Thus the order relation on the degrees is not a dense order.
- For every nonzero degree **a** there is a degree **b** incomparable with **a**.
- There is a set of $2^{\aleph_0}$ pairwise incomparable Turing degrees.
- There are pairs of degrees with no greatest lower bound. Thus $\mathcal{D}$ is not a lattice.
- Every countable partially ordered set can be embedded in the Turing degrees.
- No infinite, strictly increasing sequence of degrees has a least upper bound.

Properties involving the jump

- For every degree **a** there is a degree strictly between **a** and **a′**. In fact, there is a countable sequence of pairwise incomparable degrees between **a** and **a′**.
- A degree **a** is of the form **b′** if and only if **0′** ≤ **a**.
- For any degree **a** there is a degree **b** such that **a** < **b** and **b′** = **a′**; such a degree **b** is called *low* relative to **a**.
- There is an infinite sequence $\mathbf{a}_i$ of degrees such that $\mathbf{a}'_{i+1} \leq \mathbf{a}_i$ for each *i*.

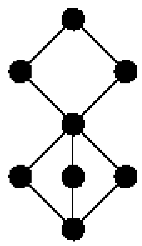
Logical properties

- Simpson (1977) showed that the first-order theory of $\mathcal{D}$ in the language $\langle \leq, = \rangle$ or $\langle \leq, ', = \rangle$ is many-one equivalent to the theory of true second-order arithmetic. This indicates that the structure of $\mathcal{D}$ is extremely complicated.
- Shore and Slaman (1999) showed that the jump operator is definable in the first-order structure of the degrees with the language $\langle \leq, = \rangle$.

Structure of the r.e. Turing degrees

A degree is called r.e. (recursively enumerable) if it contains a recursively enumerable set. Every r.e. degree is less than or equal to **0′** but not every degree less than **0′** is an r.e. degree.

- (G. E. Sacks, 1964) The r.e degrees are dense; between any two r.e. degrees there is a third r.e degree.
- (A. H. Lachlan, 1966a and C. E. M. Yates, 1966) There are two r.e. degrees with no greatest lower bound in the r.e. degrees.
- (A. H. Lachlan, 1966a and C. E. M. Yates, 1966) There is a pair of nonzero r.e. degrees whose greatest lower bound is **0**.
- (S. K. Thomason, 1971) Every finite distributive lattice can be embedded into the r.e. degrees. In fact, the countable atomless Boolean algebra can be embedded in a manner that preserves suprema and infima.
- (A. H. Lachlan and R. I. Soare, 1980) Not all finite lattices can be embedded in the r.e. degrees (via an embedding that preserves suprema and infima). The following particular lattice cannot be embedded in the r.e. degrees:



- (A. H. Lachlan, 1966b) There is no pair of r.e. degrees whose greatest lower bound is **0** and whose least upper bound is **0′**. This result is informally called the *nondiamond theorem*.
- (L. A. Harrington and T. A. Slaman, see Nies, Shore, and Slaman (1998)) The first-order theory of the r.e. degrees in the language $\langle \mathbf{0}, \leq, = \rangle$ is many-one equivalent to the theory of true first order arithmetic.

Post's problem and the priority method

Emil Post studied the r.e. Turing degrees and asked whether there is any r.e. degree strictly between **0** and **0′**. The problem of constructing such a degree (or showing that none exist) became known as **Post's problem**. This problem was solved independently by Friedberg and Muchnik in the 1950s, who showed that these intermediate r.e. degrees do exist. Their proofs each developed the same new method for constructing r.e degrees which came to be known as the **priority method**. The priority method is now the main technique for establishing results about r.e. sets.

The idea of the priority method for constructing an r.e. set X is to list a countable sequence of *requirements* that X must satisfy. For example, to construct an r.e. set X between $\mathbf{0}$ and $\mathbf{0'}$ it is enough to satisfy the requirements A_e and B_e for each natural number e, where A_e requires that the oracle machine with index e does not compute $0'$ from X and B_e requires that the Turing machine with index e (and no oracle) does not compute X. These requirements are put into a *priority ordering*, which is an explicit bijection of the requirements and the natural numbers. The proof proceeds inductively with one stage for each natural number; these stages can be thought of as steps of time during which the set X is enumerated. At each stage, numbers may put into X or forever prevented from entering X in an attempt to *satisfy* requirements (that is, force them to hold once all of X has been enumerated). Sometimes, a number can be enumerated into X to satisfy one requirement but doing this would cause a previously satisfied requirement to become unsatisfied (that is, to be *injured*). The priority order on requirements is used to determine which requirement to satisfy in this case. The informal idea is that if a requirement is injured then it will eventually stop being injured after all higher priority requirements have stopped being injured, although not every priority argument has this property. An argument must be made that the overall set X is r.e. and satisfies all the requirements. Priority arguments can be used to prove many facts about r.e. sets; the requirements used and the manner in which they are satisfied must be carefully chosen to produce the required result.

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Philosophical logic

Philosophical logic is a term introduced by Bertrand Russell to represent his idea that the workings of natural language and thought can only be adequately represented by an artificial language;[1] essentially it was his formalization program for the natural language.[2] Today the term is used with several different meanings.[3]

One modern meaning, espoused mainly by philosophers, is that philosophical logic is the study of the more specifically philosophical aspects of logic in contrast with symbolic logic; for example Sybil Wolfram lists the study of the concepts of argument, meaning, and truth.[4] Colin McGinn includes identity, existence, predication, necessity, and truth as the main topics of his book, which he writes was aimed "to bring philosophy back into philosophical logic".[5] John Woods writes that philosophical logic investigates properties such as truth, meaning and reference in natural languages. As contrasting example he argues that Frege's *Begriffsschrift* is an example of mathematical logic, while Frege's discussion of sense and reference belongs to the philosophical logic realm. Woods also points out that there's substantial overlap between philosophy of language and philosophical logic.[6] Susan Haack argued that there is no distinction between philosophical logic seen this way and philosophy of logic.[7][8] A. C. Grayling disagrees however, writing that when "one does philosophy of logic, one is philosophizing about logic; but when one does philosophical logic one is philosophizing." He concedes however that the distinction is not too sharp.[8] In general there is no agreement whether these two fields coincide or not.[9]

Another meaning assigned to philosophical logic today is that it addresses mainly extensions and alternatives to classical logic, the so-called non-classical logics. In this sense, philosophical logic is a technical subject. Texts such as John P. Burgess' *Philosophical Logic*,[3] the *Blackwell Companion to Philosophical Logic*,[10] or the multi-volume *Handbook of Philosophical Logic*[11](edited by Dov M. Gabbay and Franz Guenthner) address this latter meaning of the term, with classical logic included as a core component however. According to Burgess, philosophical logic in this sense, has its center of gravity in theoretical computer science, because many non-classical logics find applications there.[3] The Springer *Journal of Philosophical Logic* largely addresses this conception of philosophical logic.

Yet another contemporary meaning proposed by Dale Jacquette is that philosophical logic is philosophy in which any recognized methods of logic are used to solve or advance the discussion of philosophical problems.[9]

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[11] http://www.springer.com/series/6024

External links

- Journal of Philosophical Logic, Springer Science+Business Media
- Study Guide to Philosophical Logic and the Philosophy of Logic (http://www.ontology.co/pathways-logic.htm) Annotated selection of books on the subject

Free logic

A **free logic** is a logic with fewer existential presuppositions than classical logic. Free logics may allow for terms that do not denote any object. Free logics may also allow models that have an empty domain. A free logic with the latter property is an **inclusive logic**.

Explanation

In classical logic there are theorems which clearly presuppose that there is something in the domain of discourse. Consider the following classically valid theorems.

1. $\forall xA \to \exists xA$;
2. $\forall xA \to A(r/x)$(where r does not occur free for x in Ax and A(r/x) is the result of substituting r for all free occurrences of x in Ax);
3. $Ar \to \exists xAx$ (where r does not occur free for x in Ax).

A valid scheme in the theory of equality which exhibits the same feature is

4. $\forall x(Fx \to Gx) \land \exists xFx \to \exists x(Fx \land Gx)$.

Informally, if F is '=y', G is 'is Pegasus', and we substitute 'Pegasus' for y, then (4) appears to allow us to infer from 'everything identical with Pegasus is Pegasus' that something is identical with Pegasus. The problem comes from substituting nondesignating constants for variables: in fact, we cannot do this in standard formulations of first-order logic, since there are no nondesignating constants. Classically, ∃x(x=y) is deducible from the open equality axiom y=y by particularization (i.e. (3) above).

In free logic, (1) is replaced with

1b. $\forall xA \land E!t \to \exists xA$, where E! is an existence predicate (in some but not all formulations of free logic, E!t can be defined as ∃y(y=t)).

Similar modifications are made to other theorems with existential import (e.g. the Rule of Particularization becomes (Ar → (E!r → ∃xAx)).

Axiomatizations of free-logic are given in Hintikka (1959),[1] Lambert (1967), Hailperin (1957), and Mendelsohn (1989).

Interpretation

Karel Lambert wrote in 1967[2]:

"In fact, one may regard free logic... literally as a theory about singular existence, in the sense that it lays down certain minimum conditions for that concept." The question which concerned the rest of his paper was then a description of the theory, and to inquire whether it gives a necessary and sufficient condition for existence statements.

Lambert notes the irony in that Willard Van Orman Quine so vigorously defended a form of logic which only accommodates his famous dictum, "To be is to be the value of a variable," when the logic is supplemented with Russellian assumptions of description theory. He criticizes this approach because it puts too much ideology into a logic which is supposed to be philosophically neutral. Rather, he points out, not only does free logic provide for Quine's criterion—it even proves it! This is done by brute force, though, since he takes as axioms $\exists xFx \to (\exists x(E!Fx))$ and $Fy \to (E!y \to \exists xFx)$, which neatly formalizes Quine's dictum. So, Lambert argues, to reject his construction of free logic requires you to reject Quine's philosophy, which requires some argument and also means that whatever logic you develop is always accompanied by the stipulation that you must reject Quine to accept the logic. Likewise, if you reject Quine then you must reject free logic. This amounts to the contribution which free logic makes to ontology.

The point of free logic, though, is to have a formalism which implies no particular ontology, but which merely makes an interpretation of Quine both formally possible and simple. An advantage of this is that formalizing theories of singular existence in free logic brings out their implications for easy analysis. Lambert takes the example of the theory proposed by Wesley C. Salmon and George Nahknikian,[3] which is that to exist is to be self-identical.

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External links

- Free logic (http://plato.stanford.edu/entries/logic-free) entry by John Nolt in the *Stanford Encyclopedia of Philosophy*

Classical logic

Classical logic identifies a class of formal logics that have been most intensively studied and most widely used. The class is sometimes called **standard logic** as well.[1][2] They are characterised by a number of properties:[3]

1. Law of the excluded middle and Double negative elimination;
2. Law of noncontradiction, and the principle of explosion;
3. Monotonicity of entailment and Idempotency of entailment;
4. Commutativity of conjunction;
5. De Morgan duality: every logical operator is dual to another;

While not entailed by the preceding conditions, contemporary discussions of classical logic normally only include propositional and first-order logics.[4][5]

The intended semantics of classical logic is bivalent. With the advent of algebraic logic it became apparent however that classical propositional calculus admits other semantics. In Boolean-valued semantics (for classical propositional logic), the truth values are the elements of an arbitrary Boolean algebra; "true" corresponds to the maximal element of the algebra, and "false" corresponds to the minimal element. Intermediate elements of the algebra correspond to truth values other than "true" and "false". The principle of bivalence holds only when the Boolean algebra is taken to be the two-element algebra, which has no intermediate elements.

Examples of classical logics

- Aristotle's Organon introduces his theory of syllogisms, which is a logic with a restricted form of judgments: assertions take one of four forms, *All Ps are Q*, *Some Ps are Q*, *No Ps are Q*, and *Some Ps are not Q*. These judgments find themselves if two pairs of two dual operators, and each operator is the negation of another, relationships that Aristotle summarised with his square of oppositions. Aristotle explicitly formulated the law of the excluded middle and law of non-contradiction in justifying his system, although these laws cannot be expressed as judgments within the syllogistic framework.
- George Boole's algebraic reformulation of logic, his system of Boolean logic;
- The first-order logic found in Gottlob Frege's Begriffsschrift.

Non-classical logics

- Computability logic is a semantically constructed formal theory of computability, as opposed to classical logic, which is a formal theory of truth; integrates and extends classical, linear and intuitionistic logics.
- Many-valued logic, including fuzzy logic, which rejects the law of the excluded middle and allows as a truth value any real number between 0 and 1.
- Intuitionistic logic rejects the law of the excluded middle, double negative elimination, and the De Morgan's laws;
- Linear logic rejects idempotency of entailment as well;
- Modal logic extends classical logic with non-truth-functional ("modal") operators.
- Paraconsistent logic (e.g., dialetheism and relevance logic) rejects the law of noncontradiction;
- Relevance logic, linear logic, and non-monotonic logic reject monotonicity of entailment;

In *Deviant Logic, Fuzzy Logic: Beyond the Formalism*, Susan Haack divided non-classical logics into deviant, quasi-deviant, and extended logics.[5]

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Further reading

- Graham Priest, *An Introduction to Non-Classical Logic: From If to Is*, 2nd Edition, CUP, 2008, ISBN 978-0-521-67026-5
- Warren Goldfard, "Deductive Logic", 1st edition, 2003, ISBN 0-87220-660-2

Logic in computer science

Logic in computer science describes topics where logic is applied to computer science and artificial intelligence. These include:

1. Investigations into logic that are guided by applications in computer science. For example:
   - Rewriting systems were motivated by solving equations algorithmically;
   - Many developments in type theory were motivated by applications in programming language theory;
   - Abstract interpretation was developed to allow proofs of certain program properties;
   - Logics of knowledge and beliefs (of human and artificial agents);
   - Spatial logics, used for reasoning about interaction between concurrent and distributed processes.
   - Logics for spatial reasoning, e.g. about moving in Euclidean space (which should not be confused with spatial logics used for concurrent systems);
   - some developments in categorical logic;
   - Program logics, such as Hoare logic, Hennessy-Milner logic, and dynamic logic are used to reason about program correctness
   - Process calculi were developed to reason about correctness of concurrent systems.
   - Descriptive complexity theory relates logics to computational complexity
2. Applications of logic in computer science, such as Formal methods:
   - Boolean logic, is used for the design of computer circuits;
   - Specification languages are extended logics for reasoning about whether programs behave correctly, such as the Z specification language. Cf. program logics, below, which can be considered to be minimalist specification logics, and cf. also, automated theorem provers, below; specification languages are often integrated into theorem provers.
   - The notion of institution has been developed as an abstract formalization of the notion of logical system, with the goal of handling the "population explosion" of logics used in formal methods.
   - Predicate logic and logical frameworks are used for proving programs correct, and logics such as temporal logic and #Fundamental concepts in computer science that are naturally expressible in formal logic. For example:
   - Formal semantics of programming languages;
   - Logic programming;
   - Definition of formal languages;

3. Aspects of the theory of computation that cast light on fundamental questions of formal logic. For example: Curry-Howard correspondence and Game semantics;
4. Tools for logicians considered as computer science. For example: Automated theorem proving and Model checking;

The study of basic mathematical logic such as propositional logic and predicate logic (normally in conjunction with set theory) is considered an important theoretical underpinning to any undergraduate computer science course. Higher-order logic is usually considered an advanced topic, but is important in theorem proving tools like HOL.

Books

- *Mathematical Logic for Computer Science* by Mordechai Ben-Ari. Springer-Verlag, 2nd edition, 2003. ISBN 1-85233-319-7.
- *Logic in Computer Science: Modelling and Reasoning about Systems* [1] by Michael Huth, Mark Ryan. Cambridge University Press, 2nd edition, 2004. ISBN 0-521-54310-X.
- *Logic for Mathematics and Computer Science* by Stanley N. Burris. Prentice Hall, 1997. ISBN 0-13-285974-2.

External links

- Article on *Logic and Artificial Intelligence* [2] at the Stanford Encyclopedia of Philosophy.
- IEEE Symposium on Logic in Computer Science [3] (LICS)
- Draft book on Logic in Computer Science by Andrei Voronkov [4]
- Alwen Tiu, Introduction to logic [5] video recording of a lecture at ANU Logic Summer School '09 (aimed mostly at computer scientists)

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[2] http://plato.stanford.edu/entries/logic-ai/
[3] http://www.informatik.hu-berlin.de/lics/
[4] http://www.voronkov.com/lics.cgi
[5] http://videolectures.net/ssll09_tiu_intlo/

Entscheidungsproblem

In mathematics and computer science, the ***Entscheidungsproblem*** (pronounced [ɛntˈʃaɪdʊŋspʁoˌbleːm], German for 'decision problem') is a challenge posed by David Hilbert in 1928. The Entscheidungsproblem asks for an algorithm that takes as input a statement of a first-order logic (possibly with a finite number of axioms beyond the usual axioms of first-order logic) and answers "Yes" or "No" according to whether the statement is *universally valid*, i.e., valid in every structure satisfying the axioms. By the completeness theorem of first-order logic, a statement is universally valid if and only if it can be deduced from the axioms, so the Entscheidungsproblem can also be viewed as asking for an algorithm to decide whether a given statement is provable from the axioms using the rules of logic.

In 1936 and 1937, Alonzo Church and Alan Turing, respectively,[1] published independent papers showing that a general solution to the Entscheidungsproblem is impossible. This result is now known as Church's Theorem or the Church–Turing Theorem (not to be confused with the Church–Turing thesis).

History of the problem

The origin of the Entscheidungsproblem goes back to Gottfried Leibniz, who in the seventeenth century, after having constructed a successful mechanical calculating machine, dreamt of building a machine that could manipulate symbols in order to determine the truth values of mathematical statements.[2] He realized that the first step would have to be a clean formal language, and much of his subsequent work was directed towards that goal. In 1928, David Hilbert and Wilhelm Ackermann posed the question in the form outlined above.

In continuation of his "program" with which he challenged the mathematics community in 1900, at a 1928 international conference David Hilbert asked three questions, the third of which became known as "Hilbert's Entscheidungsproblem".[3] As late as 1930 he believed that there would be no such thing as an unsolvable problem.[4]

Negative answer

Before the question could be answered, the notion of "algorithm" had to be formally defined. This was done by Alonzo Church in 1936 with the concept of "effective calculability" based on his λ calculus and by Alan Turing in the same year with his concept of Turing machines. It was recognized immediately by Turing that these are equivalent models of computation.

The negative answer to the *Entscheidungsproblem* was then given by Alonzo Church in 1935–36 and independently shortly thereafter by Alan Turing in 1936–37. Church proved that there is no computable function which decides for two given λ calculus expressions whether they are equivalent or not. He relied heavily on earlier work by Stephen Kleene. Turing reduced the halting problem for Turing machines to the Entscheidungsproblem. The work of both authors was heavily influenced by Kurt Gödel's earlier work on his incompleteness theorem, especially by the method of assigning numbers (a Gödel numbering) to logical formulas in order to reduce logic to arithmetic.

Turing's argument is as follows. Suppose that we had a general decision algorithm for statements in a first-order language. The question whether a given Turing machine halts or not can be formulated as a first-order statement, which would then be susceptible to the decision algorithm. But Turing had proven earlier that no general algorithm can decide whether a given Turing machine halts.

The Entscheidungsproblem is related to Hilbert's tenth problem, which asks for an algorithm to decide whether Diophantine equations have a solution. The non-existence of such an algorithm, established by Yuri Matiyasevich in 1970, also implies a negative answer to the Entscheidungsproblem.

Some first-order theories are algorithmically decidable; examples of this include Presburger arithmetic, real closed fields and static type systems of (most) programming languages. The general first-order theory of the natural numbers expressed in Peano's axioms cannot be decided with such an algorithm, however.

Notes

[1] Church's paper was presented to the American Mathematical Society on 19 April 1935 and published on 15 April 1936. Turing, who had made substantial progress in writing up his own results, was disappointed to learn of Church's proof upon its publication (see correspondence between Max Newman and Church in Alonzo Church papers (http://diglib.princeton.edu/ead/getEad?eadid=C0948&kw=)). Turing quickly completed his paper and rushed it to publication; it was received by the *Proceedings of the London Mathematical Society* on 28 May 1936, read on 12 November 1936, and published in January 1937 series 2, volume 42 (1936-1937); Turing added corrections in volume 43(1937) pp. 544-546. See Davis 1965:116.

[2] Davis 2000: pp. 3–20

[3] Hodges p. 91

[4] Hodges p. 92, quoting from Hilbert

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Logic programming

Logic programming is one of the 4 main programming paradigms. Its theory of computation is based on first order logic. Programming languages such as Prolog and Datalog implement it.

A form of logical sentences commonly found in logic programming, but not exclusively, is the Horn clause. An example is:

`p(X, Y) if` q(X) and r(Y)

Some logic programming languages accept other logical sentences, such as the "choice" sentence in answer set programming.

Logical sentences can be understood purely declaratively. They can also be understood procedurally as goal-reduction procedures : to solve `p(X, Y)`, first solve `q(X)`, then solve `r(Y)`.

The programmer can use the declarative reading of logic programs to verify their correctness. In addition, the programmer can use the known behaviour of the program executor to develop a procedural understanding of his program. This may be helpful when seeking better execution speed. However, many logically-based program transformation techniques have been developed to transform logic programs automatically and make them efficient.

History

The use of mathematical logic to represent and execute computer programs is also a feature of the lambda calculus, developed by Alonzo Church in the 1930s. However, the first proposal to use the clausal form of logic for representing computer programs was made by Cordell Green (1969). This used an axiomatization of a subset of LISP, together with a representation of an input-output relation, to compute the relation by simulating the execution of the program in LISP. Foster and Elcock's Absys (1969), on the other hand, employed a combination of equations and lambda calculus in an assertional programming language which places no constraints on the order in which operations are performed.

Logic programming in its present form can be traced back to debates in the late 1960s and early 1970s about declarative versus procedural representations of knowledge in Artificial Intelligence. Advocates of declarative representations were notably working at Stanford, associated with John McCarthy, Bertram Raphael and Cordell Green, and in Edinburgh, with John Alan Robinson (an academic visitor from Syracuse University), Pat Hayes, and Robert Kowalski. Advocates of procedural representations were mainly centered at MIT, under the leadership of Marvin Minsky and Seymour Papert.

Although it was based on the proof methods of logic, Planner, developed at MIT, was the first language to emerge within this proceduralist paradigm [Hewitt, 1969]. Planner featured pattern-directed invocation of procedural plans from goals (i.e. goal-reduction or backward chaining) and from assertions (i.e. forward chaining). The most influential implementation of Planner was the subset of Planner, called Micro-Planner, implemented by Gerry Sussman, Eugene Charniak and Terry Winograd. It was used to implement Winograd's natural-language understanding program SHRDLU, which was a landmark at that time. To cope with the very limited memory systems at the time, Planner used a backtracking control structure so that only one possible computation path had to be stored at a time. Planner gave rise to the programming languages QA-4, Popler, Conniver, QLISP, and the concurrent language Ether.

Hayes and Kowalski in Edinburgh tried to reconcile the logic-based declarative approach to knowledge representation with Planner's procedural approach. Hayes (1973) developed an equational language, Golux, in which different procedures could be obtained by altering the behavior of the theorem prover. Kowalski, on the other hand, showed how SL-resolution treats implications as goal-reduction procedures. Kowalski collaborated with Colmerauer in Marseille, who developed these ideas in the design and implementation of the programming language Prolog. Prolog gave rise to the programming languages ALF, Fril, Gödel, Mercury, Oz, Ciao, Visual Prolog, XSB, and λProlog, as well as a variety of concurrent logic programming languages (see Shapiro (1989) for a survey), constraint logic programming languages and datalog.

In 1997, the Association of Logic Programming bestowed to fifteen recognized researchers in logic programming the title *Founders of Logic Programming* to recognize them as pioneers in the field:

- Maurice Bruynooghe (Belgium)
- Jacques Cohen (US)
- Alain Colmerauer (France)
- Keith Clark (UK)
- Veronica Dahl (Canada/Argentina)
- Maarten van Emden (Canada)
- Herve Gallaire (France)
- Robert Kowalski (UK)
- Jack Minker (US)
- Fernando Pereira (US)
- Luis Moniz Pereira (Portugal)
- Ray Reiter (Canada)
- J. Alan Robinson (US)
- Peter Szeredi (Hungary)
- David H. D. Warren (UK)

Prolog

The programming language Prolog was developed in 1972 by Alain Colmerauer. It emerged from a collaboration between Colmerauer in Marseille and Robert Kowalski in Edinburgh. Colmerauer was working on natural language understanding, using logic to represent semantics and using resolution for question-answering. During the summer of 1971, Colmerauer and Kowalski discovered that the clausal form of logic could be used to represent formal grammars and that resolution theorem provers could be used for parsing. They observed that some theorem provers, like hyper-resolution, behave as bottom-up parsers and others, like SL-resolution (1971), behave as top-down parsers.

It was in the following summer of 1972, that Kowalski, again working with Colmerauer, developed the procedural interpretation of implications. This dual declarative/procedural interpretation later became formalised in the Prolog notation

$$H :- B_1, \ldots, B_n.$$

which can be read (and used) both declaratively and procedurally. It also became clear that such clauses could be restricted to definite clauses or Horn clauses, where H, B_1, …, B_n are all atomic predicate logic formulae, and that SL-resolution could be restricted (and generalised) to LUSH or SLD-resolution. Kowalski's procedural interpretation and LUSH were described in a 1973 memo, published in 1974.

Colmerauer, with Philippe Roussel, used this dual interpretation of clauses as the basis of Prolog, which was implemented in the summer and autumn of 1972. The first Prolog program, also written in 1972 and implemented in Marseille, was a French question-answering system. The use of Prolog as a practical programming language was given great momentum by the development of a compiler by David Warren in Edinburgh in 1977. Experiments demonstrated that Edinburgh Prolog could compete with the processing speed of other symbolic programming languages such as Lisp. Edinburgh Prolog became the *de facto* standard and strongly influenced the definition of ISO standard Prolog.

Negation as failure

Micro-Planner had a construct, called "thnot", which when applied to an expression returns the value true if (and only if) the evaluation of the expression fails. An equivalent operator is normally built-in in modern Prolog's implementations and has been called "negation as failure". It is normally written as not(p), where p is an atom whose variables normally have been instantiated by the time not(p) is invoked. A more cryptic (but standard) syntax is \+ p . Negation as failure literals can occur as conditions not(B_i) in the body of program clauses.

The logical status of negation as failure was unresolved until Keith Clark [1978] showed that, under certain natural conditions, it is a correct (and sometimes complete) implementation of classical negation with respect to the completion of the program. Completion amounts roughly to regarding the set of all the program clauses with the same predicate on the left hand side, say

$$\texttt{H :- Body}_1\texttt{.}$$
$$\ldots$$
$$\texttt{H :- Body}_k\texttt{.}$$

as a definition of the predicate

$$\texttt{H iff (Body}_1\texttt{ or } \ldots \texttt{ or Body}_k\texttt{)}$$

where "iff" means "if and only if". Writing the completion also requires explicit use of the equality predicate and the inclusion of a set of appropriate axioms for equality. However, the implementation of negation by failure needs only the if-halves of the definitions without the axioms of equality.

The notion of completion is closely related to McCarthy's circumscription semantics for default reasoning, and to the closed world assumption.

As an alternative to the completion semantics, negation as failure can also be interpreted epistemically, as in the stable model semantics of answer set programming. In this interpretation not(B_i) means literally that B_i is not known or not believed. The epistemic interpretation has the advantage that it can be combined very simply with classical negation, as in "extended logic programming", to formalise such phrases as "the contrary can not be shown", where "contrary" is classical negation and "can not be shown" is the epistemic interpretation of negation as failure.

Problem solving

In the simplified, propositional case in which a logic program and a top-level atomic goal contain no variables, backward reasoning determines an and-or tree, which constitutes the search space for solving the goal. The top-level goal is the root of the tree. Given any node in the tree and any clause whose head matches the node, there exists a set of child nodes corresponding to the sub-goals in the body of the clause. These child nodes are grouped together by an "and". The alternative sets of children corresponding to alternative ways of solving the node are grouped together by an "or".

Any search strategy can be used to search this space. Prolog uses a sequential, last-in-first-out, backtracking strategy, in which only one alternative and one sub-goal is considered at a time. Other search strategies, such as parallel search, intelligent backtracking, or best-first search to find an optimal solution, are also possible.

In the more general case, where sub-goals share variables, other strategies can be used, such as choosing the subgoal that is most highly instantiated or that is sufficiently instantiated so that only one procedure applies. Such strategies are used, for example, in concurrent logic programming.

The fact that there are alternative ways of executing a logic program has been characterised by the equation:

Algorithm = Logic + Control

where "Logic" represents a logic program and "Control" represents different theorem-proving strategies.[1]

Knowledge representation

The fact that Horn clauses can be given a procedural interpretation and, vice versa, that goal-reduction procedures can be understood as Horn clauses + backward reasoning means that logic programs combine declarative and procedural representations of knowledge. The inclusion of negation as failure means that logic programming is a kind of non-monotonic logic.

Despite its simplicity compared with classical logic, this combination of Horn clauses and negation as failure has proved to be surprisingly expressive. For example, it has been shown to correspond, with some further extensions, quite naturally to the semi-formal language of legislation. It is also a natural language for expressing common-sense laws of cause and effect, as in the situation calculus and event calculus.

Abductive logic programming

Abductive Logic Programming is an extension of normal Logic Programming that allows some predicates, declared as abducible predicates, to be incompletely defined. Problem solving is achieved by deriving hypotheses expressed in terms of the abducible predicates as solutions of problems to be solved. These problems can be either observations that need to be explained (as in classical abductive reasoning) or goals to be achieved (as in normal logic programming). It has been used to solve problems in Diagnosis, Planning, Natural Language and Machine Learning. It has also been used to interpret Negation as Failure as a form of abductive reasoning.

Metalogic programming

Because mathematical logic has a long tradition of distinguishing between object language and metalanguage, logic programming also allows metalevel programming. The simplest metalogic program is the so-called "vanilla" meta-interpreter:

```
solve(true).
solve((A,B)):- solve(A),solve(B).
solve(A):- clause(A,B),solve(B).
```

where true represents an empty conjunction, and clause(A,B) means there is an object-level clause of the form A :- B.

Metalogic programming allows object-level and metalevel representations to be combined, as in natural language. It can also be used to implement any logic that is specified by means of inference rules.

Constraint logic programming

Constraint logic programming is an extension of normal Logic Programming that allows some predicates, declared as constraint predicates, to occur as literals in the body of clauses. These literals are not solved by goal-reduction using program clauses, but are added to a store of constraints, which is required to be consistent with some built-in semantics of the constraint predicates.

Problem solving is achieved by reducing the initial problem to a satisfiable set of constraints. Constraint logic programming has been used to solve problems in such fields as civil engineering, mechanical engineering, digital circuit verification, automated timetabling, air traffic control, and finance. It is closely related to abductive logic programming.

Concurrent logic programming

Keith Clark, Steve Gregory, Vijay Saraswat, Udi Shapiro, Kazunori Ueda, etc. developed a family of Prolog-like concurrent message passing systems using unification of shared variables and data structure streams for messages. Efforts were made to base these systems on mathematical logic, and they were used as the basis of the Japanese Fifth Generation Project (ICOT). However, the Prolog-like concurrent systems were based on message passing and consequently were subject to the same indeterminacy as other concurrent message-passing systems, such as Actors (see Indeterminacy in concurrent computation). Consequently, the ICOT languages were not based on logic in the sense that computational steps could not be logically deduced [Hewitt and Agha, 1988].

Concurrent constraint logic programming combines concurrent logic programming and constraint logic programming, using constraints to control concurrency. A clause can contain a guard, which is a set of constraints that may block the applicability of the clause. When the guards of several clauses are satisfied, concurrent constraint logic programming makes a committed choice to the use of only one.

Inductive logic programming

Inductive logic programming is concerned with generalizing positive and negative examples in the context of background knowledge. Generalizations, as well as the examples and background knowledge, are expressed in logic programming syntax. Recent work in this area, combining logic programming, learning and probability, has given rise to the new field of statistical relational learning and probabilistic inductive logic programming.

Higher-order logic programming

Several researchers have extended logic programming with higher-order programming features derived from higher-order logic, such as predicate variables. Such languages include the Prolog extensions HiLog and λProlog.

Linear logic programming

Basing logic programming within linear logic has resulted in the design of logic programming languages that are considerably more expressive than those based on classical logic. Horn clause programs can only represent state change by the change in arguments to predicates. In linear logic programming, one can use the ambient linear logic to support state change. Some early designs of logic programming languages based on linear logic include LO [Andreoli & Pareschi, 1991], Lolli [Hodas & Miller, 1994], ACL [Kobayashi & Yonezawa, 1994], and Forum [Miller, 1996]. Forum provides a goal-directed interpretation of all of linear logic.

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- Evgeny Dantsin, Thomas Eiter, Georg Gottlob, Andrei Voronkov: Complexity and expressive power of logic programming. ACM Comput. Surv. 33(3): 374-425 (2001)
- Ulf Nilsson and Jan Maluszynski, Logic, Programming and Prolog (http://www.ida.liu.se/~ulfni/lpp/)

External links

- *Logic Programming* Virtual Library entry (http://vl.fmnet.info/logic-prog/)
- Bibliographies on Logic Programming (http://liinwww.ira.uka.de/bibliography/LogicProgramming/)
- Association for Logic Programming (ALP) (http://www.cs.kuleuven.be/~dtai/projects/ALP/)
- Theory and Practice of Logic Programming (http://www.cs.kuleuven.be/~dtai/projects/ALP/TPLP/) journal
- Logic programming in C++ with Castor (http://www.mpprogramming.com/Cpp/)
- Logic programming in (http://www.mozart-oz.org/documentation/tutorial/node12.html) Oz
- Prolog Development Center (http://www.pdc.dk/)
- Racklog: Logic Programming in Racket (http://docs.racket-lang.org/racklog/)

ACM Computing Classification System

The **ACM Computing Classification System** is a subject classification system for computer science devised by the Association for Computing Machinery. The system is comparable to the Mathematics Subject Classification in scope, aims and structure, being used by the various ACM journals to organise subjects by area.

History

The system has gone through seven revisions, the first version being published in 1964, and revised versions appearing in 1982, 1983, 1987, 1991, 1998, and the now current version in 2012.

Structure

The ACM Computing Classification System, Version 2012, has a revolutionary change in some areas, for example, in Software that now is called "Software and its engineering" which has three main subjects:

Software organization and properties. This subject addresses the programming language theory and, in a broad sense, what software is. Software notations and tools. This subject classify some practical concerns about software development. Software creation and management. This subject is which we know traditionally as Software Engineering (SE), but SE as category as missed.

The ACM Computing Classification System,version 1998, is hierarchically structured in four levels: three outer levels, coded by capital letters and numbers, and an uncoded fourth level of subject descriptors. Thus, for example, one branch of the hierarchy contains

I. Computing Methodologies, which contains:

I.2 Artificial Intelligence, which contains:

I.2.4 Knowledge representation formalisms and methods, which contains:

Temporal logic.

Each top-level category has two standard subcategories: "general", coded with a "0", and "miscellaneous", coded with a "m". For instance, I.0 denotes the "general" subcategory of Computing Methodologies, while I.m denotes its miscellaneous subcategory. Several subtopics are listed as uncoded subject descriptors in these standard subcategories.

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External links

- ACM Computing Classification System [3] is the homepage of the system, including links to four complete versions of the system, for 1964 [4], 1991 [5], 1998 [6], and the current 2012 version [7].
- The ACM Computing Research Repository [8] uses a classification scheme [9] that is much coarser than the ACM subject classification, and does not cover all areas of CS, but is intended to better cover active areas of research. In addition, papers in this repository are classified according to the ACM subject classification.

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[4] http://www.acm.org/class/1964
[5] http://www.acm.org/class/class91-toc
[6] http://www.acm.org/class/1998
[7] http://www.acm.org/about/class/2012
[8] http://arxiv.org/corr/home
[9] http://arxiv.org/corr/subjectclasses

Principle of bivalence

In logic, the semantic **principle (or law) of bivalence** states that every declarative sentence expressing a proposition (of a theory under inspection) has exactly one truth value, either true or false.[1][2] A logic satisfying this principle is called a **two-valued logic**[3] or **bivalent logic**.[2][4]

In formal logic, the principle of bivalence becomes a property that a semantics may or may not possess. It is not the same as the law of excluded middle, however, and a semantics may satisfy that law without being bivalent.[2]

The principle of bivalence is studied in philosophical logic to address the question of which natural-language statements have a well-defined truth value. Sentences which predict events in the future, and sentences which seem open to interpretation, are particularly difficult for philosophers who hold that the principle of bivalence applies to all declarative natural-language statements.[2] Many-valued logics formalize ideas that a realistic characterization of the notion of consequence requires the admissibility of premises which, owing to vagueness, temporal or quantum indeterminacy, or reference-failure, cannot be considered classically bivalent. Reference failures can also be addressed by free logics.[5]

Relationship with the law of the excluded middle

The principle of bivalence is related to the law of excluded middle though the latter is a syntactic expression of the language of a logic of the form "$P \vee \neg P$". The difference between the principle and the law is important because there are logics which validate the law but which do not validate the principle.[2] For example, the three-valued Logic of Paradox (LP) validates the law of excluded middle, but not the law of non-contradiction, $\neg(P \wedge \neg P)$, and its intended semantics is not bivalent.[6] In classical two-valued logic both the law of excluded middle and the law of non-contradiction hold.[1]

Many modern logic programming systems replace the law of the excluded middle with the concept of negation as failure. The programmer may wish to add the law of the excluded middle by explicitly asserting it as true; however, it is not assumed *a priori*.

Classical logic

The intended semantics of classical logic is bivalent, but this is not true of every semantics for classical logic. In Boolean-valued semantics (for classical propositional logic), the truth values are the elements of an arbitrary Boolean algebra, "true" corresponds to the maximal element of the algebra, and "false" corresponds to the minimal element. Intermediate elements of the algebra correspond to truth values other than "true" and "false". The principle of bivalence holds only when the Boolean algebra is taken to be the two-element algebra, which has no intermediate elements.

Assign Boolean semantics to classical predicate calculus requires that the model be a complete Boolean algebra because the universal quantifier maps to the infimum operation, and the existential quantifier maps to the supremum;[7] this is called a Boolean-valued model. All finite Boolean algebras are complete.

Criticisms

Future contingents

A famous example[2] is the *contingent sea battle* case found in Aristotle's work, *De Interpretatione*, chapter 9:

> Imagine P refers to the statement "There will be a sea battle tomorrow."

The principle of bivalence here asserts:

> Either it is true that there will be a sea battle tomorrow, or it is false that there will be a sea battle tomorrow.

Aristotle hesitated to embrace bivalence for such future contingents; Chrysippus, the Stoic logician, did embrace bivalence for this and all other propositions. The controversy continues to be of central importance in both the philosophy of time and the philosophy of logic.

One of the early motivations for the study of many-valued logics has been precisely this issue. In the early 20th century, the Polish formal logician Jan Łukasiewicz proposed three truth-values: the true, the false and the *as-yet-undetermined*. This approach was later developed by Arend Heyting and L. E. J. Brouwer;[2] see Łukasiewicz logic.

Issues such as this have also been addressed in various temporal logics, where one can assert that "*Eventually*, either there will be a sea battle tomorrow, or there won't be." (Which is true if "tomorrow" eventually occurs.)

Vagueness

Such puzzles as the Sorites paradox and the related continuum fallacy have raised doubt as to the applicability of classical logic and the principle of bivalence to concepts that may be vague in their application. Fuzzy logic and some other multi-valued logics have been proposed as alternatives that handle vague concepts better. Truth (and falsity) in fuzzy logic, for example, comes in varying degrees. Consider the following statement in the circumstance of sorting apples on a moving belt:

> This apple is red.[8]

Upon observation, the apple is an undetermined color between yellow and red, or it is motled both colors. Thus the color falls into neither category " red " nor " yellow ", but these are the only categories available to us as we sort the apples. We might say it is "50% red". This could be rephrased: it is 50% true that the apple is red. Therefore, P is 50% true, and 50% false. Now consider:

> This apple is red and it is not-red.

In other words, P and not-P. This violates the law of noncontradiction and, by extension, bivalence. However, this is only a partial rejection of these laws because P is only partially true. If P were 100% true, not-P would be 100% false, and there is no contradiction because P and not-P no longer holds.

However, the law of the excluded middle is retained, because P and not-P implies P or not-P, since "or" is inclusive. The only two cases where P and not-P is false (when P is 100% true or false) are the same cases considered by two-valued logic, and the same rules apply.

Example of a 3-valued logic applied to vague (undetermined) cases: Kleene 1952[9] (§64, pp. 332–340) offers a 3-valued logic for the cases when algorithms involving partial recursive functions may not return values, but rather end up with circumstances "u" = undecided. He lets "t" = "true", "f" = "false", "u" = "undecided" and redesigns all the propositional connectives. He observes that:

> "We were justified intuitionistically in using the classical 2-valued logic, when we were using the connectives in building primitive and general recursive predicates, since there is a decision procedure for each general recursive predicate; i.e. the law of the excluded middle is proved intuitionistically to apply to general recursive predicates.
>
> "Now if Q(x) is a partial recursive predicate, there is a decision procedure for Q(x) on its range of definition, so the law of the excluded middle or excluded "third" (saying that, Q(x) is either t or f) applies intuitionistically on the range of definition. But there may be no algorithm for deciding, given x, whether Q(x) is defined or not . . .. Hence it is only classically and not intuitionistically that we have a law of the excluded fourth (saying that, for each x, Q(x) is either t, f, or u).
>
> "The third "truth value" u is thus not on par with the other two t and f in our theory. Consideration of its status will show that we are limited to a special kind of truth table".

The following are his "strong tables"[10]:

~Q			**QVR**	**R**	**t**	**f**	**u**	**Q&R**	**R**	**t**	**f**	**u**	**Q→R**	**R**	**t**	**f**	**u**	**Q=R**	**R**	**t**	**f**	**u**
Q	**t**	f	**Q**	**t**	t	t	t	**Q**	**t**	t	f	u	**Q**	**t**	t	f	u	**Q**	**t**	t	f	u
	f	T		**f**	t	f	u		**f**	f	f	f		**f**	t	t	t		**f**	f	T	u
	u	u		**u**	t	u	u		**u**	u	f	u		**u**	t	u	u		**u**	u	u	u

For example, if a determination cannot be made as to whether an apple is red or not-red, then the truth value of the assertion Q: " This apple is red " is " u ". Likewise, the truth value of the assertion R " This apple is not-red " is " u ". Thus the AND of these into the assertion Q AND R, i.e. " This apple is red AND this apple is not-red " will, per the tables, yield " u ". And, the assertion Q OR R, i.e. " This apple is red OR this apple is not-red " will likewise yield " u ".

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[2] Paul Tomassi (1999). *Logic* (http://books.google.com/books?id=TUVQr6InyNYC&pg=PA124). Routledge. p. 124. ISBN 978-0-415-16696-6. .

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[8] Note the use of the (extremely) definite article: " This " as opposed to a more-vague " The ". " The " would have to be accompanied with a pointing-gesture to make it definitive. Ff *Principia Mathematica* (2nd edition), p. 91. Russell & Whitehead observe that this " this " indicates

"something given in sensation" and as such it shall be considered "elementary".

[9] Stephen C. Kleene 1952 *Introduction to Metamathematics*, 6th Reprint 1971, North-Holland Publishing Company, Amsterdam NY, ISBN 0-7294-2130-9.

[10] "Strong tables" is Kleene's choice of words. Note that even though " u " may appear for the value of Q or R, " t " or " f " may, in those occasions, appear as a value in " Q V R ", " Q & R " and " Q → R ". "Weak tables" on the other hand, are "regular", meaning they have " u " appear in all cases when the value " u " is applied to either Q or R or both. Kleene notes that these tables are *not* the same as the original values of the tables of Łukasiewicz 1920. (Kleene gives these differences on page 335). He also concludes that " u " can mean any or all of the following: "undefined", "unknown (or value immaterial)", "value disregarded for the moment", i.e. it is a third category that does not (ultimately) exclude " t " and " f " (page 335).

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External links

- Truth Values (http://plato.stanford.edu/entries/truth-values) entry by Yaroslav Shramko, Heinrich Wansing in the *Stanford Encyclopedia of Philosophy*

Is logic empirical?

"**Is logic empirical?**" is the title of two articles (by Hilary Putnam and Michael Dummett)[1][2] that discuss the idea that the algebraic properties of logic may, or should, be empirically determined; in particular, they deal with the question of whether empirical facts about quantum phenomena may provide grounds for revising classical logic as a consistent logical rendering of reality. The replacement derives from the work of Garrett Birkhoff and John von Neumann on quantum logic. In their work, they showed that the outcomes of quantum measurements can be represented as binary propositions and that these quantum mechanical propositions can be combined in much the same way as propositions in classical logic. However, the algebraic properties of this structure are somewhat different from those of classical propositional logic in that the principle of distributivity fails.

The idea that the principles of logic might be susceptible to revision on empirical grounds has many roots, including the work of W.V. Quine and the foundational studies of Hans Reichenbach.[3]

W.V. Quine

What is the epistemological status of the laws of logic? What sort of arguments are appropriate for criticising purported principles of logic? In his seminal paper "Two Dogmas of Empiricism," the logician and philosopher W.V. Quine argued that all beliefs are in principle subject to revision in the face of empirical data, including the so-called analytic propositions. Thus the laws of logic, being paradigmatic cases of analytic propositions, are not immune to revision.

To justify this claim he cited the so-called *paradoxes of quantum mechanics*. Birkhoff and von Neumann proposed to resolve those paradoxes by abandoning the principle of distributivity, thus substituting their quantum logic for classical logic.

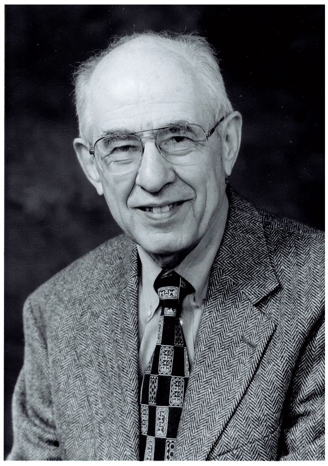
Quine did not at first seriously pursue this argument, providing no sustained argument for the claim in that paper. In *Philosophy of Logic* (the chapter titled "Deviant Logics"), Quine rejects the idea that classical logic should be revised in response to the paradoxes, being concerned with "a serious loss of simplicity", and "the handicap of having to think within a deviant logic". Quine, though, stood by his claim that logic is in principle not immune to revision.

Hans Reichenbach

Reichenbach considered one of the anomalies associated with quantum mechanics, the problem of complementary properties. A pair of properties of a system is said to be *complementary* if each one of them can be assigned a truth value in some experimental setup, but there is no setup which assigns a truth value to both properties. The classic example of complementarity is illustrated by the double-slit experiment in which a photon can be made to exhibit particle-like properties or wave-like properties, depending on the experimental setup used to detect its presence. Another example of complementary properties is that of having a precisely observed position or momentum.

Reichenbach approached the problem within the philosophical program of the logical positivists, wherein the choice of an appropriate language was not a matter of the truth or falsity of a given language – in this case, the language used to describe quantum mechanics – but a matter of "technical advantages of language systems". His solution to the problem was a logic of properties with a three-valued semantics; each property could have one of three possible truth-values: true, false, or indeterminate. The formal properties of such a logical system can be given by a set of fairly simple rules, certainly far simpler than the "projection algebra" that Birkhoff and von Neumann had introduced a few years earlier. However, because of this simplicity, the intended semantics of Reichenbach's three-valued logic is unsuited to provide a foundation for quantum mechanics that can account for observables.

First article: Hilary Putnam


Hilary Putnam

In his paper "Is logic empirical?" Hilary Putnam, whose PhD studies were supervised by Reichenbach, pursued Quine's idea systematically. In the first place, he made an analogy between laws of logic and laws of geometry: at one time Euclid's postulates were believed to be truths about the physical space in which we live, but modern physical theories are based around non-Euclidean geometries, with a different and fundamentally incompatible notion of straight line.

In particular, he claimed that what physicists have learned about quantum mechanics provides a compelling case for abandoning certain familiar principles of classical logic for this reason: realism about the physical world, which Putnam generally maintains, demands that we square up to the anomalies associated with quantum phenomena. Putnam understands realism about physical objects to entail the existence of the properties of momentum and position for quanta. Since the uncertainty principle says that either of them can be determined, but both cannot be determined at the same time, he faces a paradox. He sees the only possible resolution of the paradox as lying in the embrace of quantum logic, which he believes is not inconsistent.

Quantum logic

The formal laws of a physical theory are justified by a process of repeated controlled observations. This from a physicist's point of view is the meaning of the empirical nature of these laws.

The idea of a propositional logic with rules radically different from Boolean logic in itself was not new. Indeed a sort of analogy had been established in the mid-nineteen thirties by Garrett Birkhoff and John von Neumann between a non-classical propositional logic and some aspects of the measurement process in quantum mechanics. Putnam and the physicist David Finkelstein proposed that there was more to this correspondence than a loose analogy: that in fact there was a logical system whose semantics was given by a lattice of projection operators on a Hilbert space. This, actually, was the correct logic for reasoning about the microscopic world.

In this view, classical logic was merely a limiting case of this new logic. If this were the case, then our "preconceived" Boolean logic would have to be rejected by empirical evidence in the same way Euclidean geometry (taken as the correct geometry of physical space) was rejected on the basis of (the facts supporting the theory of) general relativity. This argument is in favour of the view that the rules of logic are empirical.

That logic came to be known as quantum logic. There are, however, few philosophers today who regard this logic as a replacement for classical logic; Putnam himself may no longer hold that view. Quantum logic is still used as a foundational formalism for quantum mechanics: but in a way in which primitive events are not interpreted as atomic sentences but rather in operational terms as possible outcomes of observations. As such, quantum logic provides a unified and consistent mathematical theory of physical observables and quantum measurement.

Second article: Michael Dummett


Michael Dummett

In an article also titled "Is logic empirical?," Michael Dummett argues that Putnam's desire for realism mandates distributivity: the principle of distributivity is essential for the realist's understanding of how propositions are true of the world, in just the same way as he argues the principle of bivalence is. To grasp why: consider why truth tables work for classical logic: firstly, it must be the case that the variable parts of the proposition are either true or false: if they could be other values, or fail to have truth values at all, then the truth table analysis of logical connectives would not exhaust the possible ways these could be applied; for example intutionistic logic respects the classical truth tables, but not the laws of classical logic, because intuitionistic logic allows propositions to be other than true or false. Second, to be able to apply truth tables to describe a connective depends upon distributivity: a truth table is a disjunction of conjunctive possibilities, and the validity of the exercise depends upon the truth of the whole being a consequence of the bivalence of the propositions, which is true only if the principle of distributivity applies.

Hence Putnam cannot embrace realism without embracing classical logic, and hence his argument to endorse quantum logic because of realism about quanta is a hopeless case.

Dummett's argument is all the more interesting because he is not a proponent of classical logic. His argument for the connection between realism and classical logic is part of a wider argument to suggest that, just as the existence of particular class of entities may be a matter of dispute, so a disputation about the objective existence of such entities is also a matter of dispute. Consequently intuitionistic logic is privileged over classical logic, when it comes to disputation concerning phenomena whose objective existence is a matter of controversy.

Thus the question, "Is logic empirical?," for Dummett, leads naturally into the dispute over realism and anti-realism, one of the deepest issues in modern metaphysics.

Notes

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Paradoxes of material implication

The **paradoxes of material implication** are a group of formulas which are truths of classical logic, but which are intuitively problematic. One of these paradoxes is the **paradox of entailment**.

The root of the paradoxes lies in a mismatch between the interpretation of the validity of logical implication in natural language, and its formal interpretation in classical logic, dating back to George Boole's algebraic logic. In classical logic, implication describes conditional if-then statements using a truth-functional interpretation, i.e. "p implies q" is **defined** to be "it is not the case that p is true and q false". Also, "p implies q" is equivalent to "p is false or q is true". For example, "if it is raining, then I will bring an umbrella", is equivalent to "it is not raining, or I will bring an umbrella, or both". This truth-functional interpretation of implication is called material implication or material conditional.

The paradoxes are logical statements which are true but whose truth is intuitively surprising to people who are not familiar with them. If the terms 'p', 'q' and 'r' stand for arbitrary propositions then the main paradoxes are given formally as follows:

1. $(\neg p \wedge p) \to q$, p and its negation imply q. This is the *paradox of entailment*.
2. $p \to (q \to p)$, if p is true then it is implied by every q.
3. $\neg p \to (p \to q)$, if p is false then it implies every q. This is referred to as 'explosion'.
4. $p \to (q \vee \neg q)$, either q or its negation is true, so their disjunction is implied by every p.
5. $(p \to q) \vee (q \to r)$, if p, q and r are three arbitrary propositions, then either p implies q or q implies r. This is because if q is true then p implies it, and if it is false then q implies any other statement. Since r can be p, it follows that given two arbitrary propositions, one must imply the other, even if they are mutually contradictory. For instance, "Nadia is in Barcelona implies Nadia is in Madrid or Nadia is in Madrid implies Nadia is in Barcelona." This truism sounds like nonsense in ordinary discourse.
6. $\neg(p \to q) \to (p \wedge \neg q)$, if p does not imply q then p is true and q is false. NB if p were false then it would imply q, so p is true. If q were also true then p would imply q, hence q is false. This paradox is particularly surprising because it tells us that if one proposition does not imply another then the first is true and the second false.

The paradoxes of material implication arise because of the truth-functional definition of material implication, which is said to be true merely because the antecedent is false or the consequent is true. By this criterion, "If the moon is made of green cheese, then the world is coming to an end," is true merely because the moon isn't made of green cheese. By extension, any contradiction implies anything whatsoever, since a contradiction is never true. (All paraconsistent logics must, by definition, reject (1) as false.) Also, any tautology is implied by anything whatsoever, since a tautology is always true.

To sum up, although it is deceptively similar to what we mean by "logically follows" in ordinary usage, material implication does not capture the meaning of "if... then".

Paradox of entailment

As the most well known of the paradoxes, and most formally simple, the paradox of entailment makes the best introduction.

In natural language, an instance of the paradox of entailment arises:

> *It is raining*

And

> *It is not raining*

Therefore

> *George Washington is made of rakes.*

This arises from the principle of explosion, a law of classical logic stating that inconsistent premises always make an argument valid; that is, inconsistent premises imply any conclusion at all. This seems paradoxical, as it suggests that the above is a valid argument.

Understanding the paradox of entailment

Validity is defined in classical logic as follows:

> *An argument (consisting of premises and a conclusion) is valid if and only if there is no possible situation in which all the premises are true and the conclusion is false.*

For example a valid argument might run:

> *If it is raining, water exists* (1st premise)
>
> *It is raining* (2nd premise)
>
> *Water exists* (Conclusion)

In this example there is no possible situation in which the premises are true while the conclusion is false. Since there is no counterexample, the argument is valid.

But one could construct an argument in which the premises are inconsistent. This would satisfy the test for a valid argument since there would be *no possible situation in which all the premises are true* and therefore *no possible situation in which all the premises are true and the conclusion is false*.

For example an argument with inconsistent premises might run:

> *Matter has mass* (1st premise; true)
>
> *Matter does not have mass* (2nd premise; false)
>
> *All numbers are equal to 12* (Conclusion)

As there is no possible situation where both premises could be true, then there is certainly no possible situation in which the premises could be true while the conclusion was false. So the argument is valid whatever the conclusion is; inconsistent premises imply all conclusions.

Explaining the paradox

The strangeness of the paradox of entailment comes from the fact that the definition of validity in classical logic does not always agree with the use of the term in ordinary language. In everyday use *validity* suggests that the premises are consistent. In classical logic, the additional notion of *soundness* is introduced. A sound argument is a valid argument with all true premises. Hence a valid argument with an inconsistent set of premises can never be sound. A suggested improvement to the notion of logical validity to eliminate this paradox is relevant logic.

Simplification

The classical paradox formulas are closely tied to the formula,

- $(p \wedge q) \rightarrow p$

the principle of Simplification, which can be derived from the paradox formulas rather easily (e.g. from (1) by Importation). In addition, there are serious problems with trying to use material implication as representing the English "if ... then ...". For example, the following are valid inferences:

1. $(p \rightarrow q) \wedge (r \rightarrow s) \ \vdash \ (p \rightarrow s) \vee (r \rightarrow q)$
2. $(p \wedge q) \rightarrow r \ \vdash \ (p \rightarrow r) \vee (q \rightarrow r)$

but mapping these back to English sentences using "if" gives paradoxes. The first might be read "If John is in London then he is in England, and if he is in Paris then he is in France. Therefore, it is either true that if John is in London then he is in France, or that if he is in Paris then he is in England." Either John is in London or John is not in London. If John is in London, then John is in England. Thus the proposition "if John is in Paris, then John is in England" holds because we have prior knowledge that the conclusion is true. If John is not in London, then the proposition "if John is in London, then John is in France" is true because we have prior knowledge that the premise is false.

The second can be read "If both switch A and switch B are closed, then the light is on. Therefore, it is either true that if switch A is closed, the light is on, or if switch B is closed, the light is on." If the two switches are in series, then the premise is true but the conclusion is false. Thus, using classical logic and taking material implication to mean if-then is an unsafe method of reasoning which can give erroneous results.

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Paraconsistent logic

A **paraconsistent logic** is a logical system that attempts to deal with contradictions in a discriminating way. Alternatively, paraconsistent logic is the subfield of logic that is concerned with studying and developing paraconsistent (or "inconsistency-tolerant") systems of logic.

Inconsistency-tolerant logics have been discussed since at least 1910 (and arguably much earlier, for example in the writings of Aristotle); however, the term *paraconsistent* ("beside the consistent") was not coined until 1976, by the Peruvian philosopher Francisco Miró Quesada.[1]

Definition

In classical logic (as well as intuitionistic logic and most other logics), contradictions entail everything. This curious feature, known as the principle of explosion or *ex contradictione sequitur quodlibet* (Latin, "from a contradiction, anything follows")[2] can be expressed formally as

$P \wedge \neg P$	Premise
P	conjunctive elimination
$P \vee A$	weakening
$\neg P$	conjunctive elimination
A	disjunctive syllogism

Which means: if *P* and its negation ¬*P* are both assumed to be true, then *P* is assumed to be true, from which it follows that at least one of the claims *P* and some other (arbitrary) claim *A* is true. However, if we know that either *P* or *A* is true, and also that *P* is not true (that ¬*P* is true) we can conclude that *A*, which could be anything, is true. Thus if a theory contains a single inconsistency, it is trivial—that is, it has every sentence as a theorem. The characteristic or defining feature of a paraconsistent logic is that it rejects the principle of explosion. As a result, paraconsistent logics, unlike classical and other logics, can be used to formalize inconsistent but non-trivial theories.

Paraconsistent logics are propositionally weaker than classical logic

Paraconsistent logics are propositionally *weaker* than classical logic; that is, they deem *fewer* propositional inferences valid. The point is that a paraconsistent logic can never be a propositional extension of classical logic, that is, propositionally validate everything that classical logic does. In some sense, then, paraconsistent logic is more conservative or cautious than classical logic. It is due to such conservativeness that paraconsistent languages can be more *expressive* than their classical counterparts including the hierarchy of metalanguages due to Alfred Tarski et al. According to Solomon Feferman [1984]: *"…natural language abounds with directly or indirectly self-referential yet apparently harmless expressions—all of which are excluded from the Tarskian framework."* This expressive limitation can be overcome in paraconsistent logic.

Motivation

The primary motivation for paraconsistent logic is the conviction that it ought to be possible to reason with inconsistent information in a controlled and discriminating way. The principle of explosion precludes this, and so must be abandoned. In non-paraconsistent logics, there is only one inconsistent theory: the trivial theory that has every sentence as a theorem. Paraconsistent logic makes it possible to distinguish between inconsistent theories and to reason with them.

Research into paraconsistent logic has also led to the establishment of the philosophical school of dialetheism (most notably advocated by Graham Priest), which asserts that true contradictions exist in reality, for example groups of people holding opposing views on various moral issues.[3] Being a dialetheist rationally commits one to some form of paraconsistent logic, on pain of otherwise embracing trivialism, i.e. accepting that all contradictions (and equivalently all statements) are true.[4] However, the study of paraconsistent logics, does not necessarily entail a dialetheist viewpoint. For example, one need not commit to either the existence of true theories or true contradictions, but would rather prefer a weaker standard like empirical adequacy, as proposed by Bas van Fraassen.[5]

The Philosophical Debate on Consistency

In classical logic Aristotle's three laws, namely, the excluded middle (p or $\neg p$), non-contradiction $\neg (p \wedge \neg p)$ and identity (p iff p), are regarded as the same, due to the inter-definition of the connectives. Moreover, traditionally contradictoriness (the presence of contradictions in a theory or in a body of knowledge) and triviality (the fact that such a theory entails all possible consequences) are assumed inseparable, granted that negation is available. These views may be philosophically challenged, precisely on the grounds that they fail to distinguish between contradictoriness and other forms of inconsistency.

On the other hand, it is possible to derive triviality from the 'conflict' between consistency and contradictions, once these notions have been properly distinguished. The very notions of consistency and inconsistency may be furthermore internalized at the object language level.

Tradeoff

Paraconsistency does not come for free: it involves a tradeoff. In particular, abandoning the principle of explosion requires one to abandon at least one of the following three very intuitive principles:[6]

Disjunction introduction	$A \vdash A \vee B$
Disjunctive syllogism	$A \vee B, \neg A \vdash B$
Transitivity or "cut"	$\Gamma \vdash A; A \vdash B \Rightarrow \Gamma \vdash B$

Though each of these principles has been challenged, the most popular approach among logicians is to reject disjunctive syllogism. If one is a dialetheist, it makes perfect sense that disjunctive syllogism should fail. The idea behind this syllogism is that, if $\neg A$, then A is excluded, so the only way A ∨ B could be true would be if B were true. However, if A and $\neg A$ can both be true at the same time, then this reasoning fails.

Another approach is to reject disjunction introduction but keep disjunctive syllogism and transitivity. The disjunction (A ∨ B) is defined as ¬ (¬A ∧ ¬B). In this approach all of the rules of natural deduction hold, except for proof by contradiction and disjunction introduction; moreover, $A \vdash B$ does not mean necessarily that $\vdash A \Rightarrow B$, which is also a difference from natural deduction.[7] Also, the following usual Boolean properties hold: excluded middle and (for conjunction and disjunction) associativity, commutativity, distributivity, De Morgan's laws, and idempotence. Furthermore, by defining the implication (A → B) as ¬ (A ∧ ¬B), there is a Two-Way Deduction Theorem allowing implications to be easily proved. Carl Hewitt favours this approach, claiming that having the usual Boolean properties, Natural Deduction, and Deduction Theorem are huge advantages in software engineering.[7][8]

Yet another approach is to do both simultaneously. In many systems of relevant logic, as well as linear logic, there are two separate disjunctive connectives. One allows disjunction introduction, and one allows disjunctive syllogism. Of course, this has the disadvantages entailed by separate disjunctive connectives including confusion between them and complexity in relating them.

The three principles below, when taken together, also entail explosion, so at least one must be abandoned:

Reductio ad absurdum	$A \to (B \wedge \neg B) \vdash \neg A$
Rule of weakening	$A \vdash B \to A$
Double negation elimination	$\neg\neg A \vdash A$

Both reductio ad absurdum and the rule of weakening have been challenged in this respect, but without much success. Double negation elimination is challenged, but for unrelated reasons. By removing it alone, while upholding the other two one may still be able to prove all negative propositions from a contradiction.

A simple paraconsistent logic

One well-known system of paraconsistent logic is the simple system known as LP ("Logic of Paradox"), first proposed by the Argentinian logician F. G. Asenjo in 1966 and later popularized by Priest and others.[9]

One way of presenting the semantics for LP is to replace the usual functional valuation with a relational one.[10] The binary relation V relates a formula to a truth value: $V(A,1)$ means that A is true, and $V(A,0)$ means that A is false. A formula must be assigned *at least* one truth value, but there is no requirement that it be assigned *at most* one truth value. The semantic clauses for negation and disjunction are given as follows:

- $V(\neg A, 1) \Leftrightarrow V(A, 0)$
- $V(\neg A, 0) \Leftrightarrow V(A, 1)$
- $V(A \vee B, 1) \Leftrightarrow V(A, 1) \ or \ V(B, 1)$
- $V(A \vee B, 0) \Leftrightarrow V(A, 0) \ and \ V(B, 0)$

(The other logical connectives are defined in terms of negation and disjunction as usual.) Or to put the same point less symbolically:

- *not A* is true if and only if *A* is false
- *not A* is false if and only if *A* is true
- *A or B* is true if and only if *A* is true or *B* is true
- *A or B* is false if and only if *A* is false and *B* is false

(Semantic) logical consequence is then defined as truth-preservation:

$\Gamma \vDash A$ if and only if A is true whenever every element of Γ is true.

Now consider a valuation V such that $V(A,1)$ and $V(A,0)$ but it is not the case that $V(B,1)$. It is easy to check that this valuation constitutes a counterexample to both explosion and disjunctive syllogism. However, it is also a counterexample to modus ponens for the material conditional of LP. For this reason, proponents of LP usually advocate expanding the system to include a stronger conditional connective that is not definable in terms of negation and disjunction.[11]

As one can verify, LP preserves most other inference patterns that one would expect to be valid, such as De Morgan's laws and the usual introduction and elimination rules for negation, conjunction, and disjunction. Surprisingly, the logical truths (or tautologies) of LP are precisely those of classical propositional logic.[12] (LP and classical logic differ only in the *inferences* they deem valid.) Relaxing the requirement that every formula be either true or false yields the weaker paraconsistent logic commonly known as FDE ("First-Degree Entailment"). Unlike LP, FDE contains no logical truths.

It must be emphasized that LP is but one of *many* paraconsistent logics that have been proposed.[13] It is presented here merely as an illustration of how a paraconsistent logic can work.

Relation to other logics

One important type of paraconsistent logic is relevance logic. A logic is *relevant* iff it satisfies the following condition:

if $A \to B$ is a theorem, then A and B share a non-logical constant.

It follows that a relevance logic cannot have $(p \wedge \neg p) \to q$ as a theorem, and thus (on reasonable assumptions) cannot validate the inference from $\{p, \neg p\}$ to q.

Paraconsistent logic has significant overlap with many-valued logic; however, not all paraconsistent logics are many-valued (and, of course, not all many-valued logics are paraconsistent). Dialetheic logics, which are also many-valued, are paraconsistent, but the converse does not hold.

Intuitionistic logic allows $A \vee \neg A$ not to be equivalent to true, while paraconsistent logic allows $A \wedge \neg A$ not to be equivalent to false. Thus it seems natural to regard paraconsistent logic as the "dual" of intuitionistic logic. However, intuitionistic logic is a specific logical system whereas paraconsistent logic encompasses a large class of systems. Accordingly, the dual notion to paraconsistency is called paracompleteness, and the "dual" of intuitionistic logic (a specific paracomplete logic) is a specific paraconsistent system called *anti-intuitionistic* or *dual-intuitionistic logic* (sometimes referred to as *Brazilian logic*, for historical reasons).[14] The duality between the two systems is best seen within a sequent calculus framework. While in intuitionistic logic the sequent

$$\vdash A \vee \neg A$$

is not derivable, in dual-intuitionistic logic

$$A \wedge \neg A \vdash$$

is not derivable. Similarly, in intuitionistic logic the sequent

$$\neg\neg A \vdash A$$

is not derivable, while in dual-intuitionistic logic

$$A \vdash \neg\neg A$$

is not derivable. Dual-intuitionistic logic contains a connective # known as *pseudo-difference* which is the dual of intuitionistic implication. Very loosely, $A \# B$ can be read as "A but not B". However, # is not truth-functional as one might expect a 'but not' operator to be; similarly, the intuitionistic implication operator cannot be treated like "$\neg (A \wedge \neg B)$". Dual-intuitionistic logic also features a basic connective $\top$ which is the dual of intuitionistic $\bot$: negation may be defined as $\neg A = (\top \# A)$

A full account of the duality between paraconsistent and intuitionistic logic, including an explanation on why dual-intuitionistic and paraconsistent logics do not coincide, can be found in Brunner and Carnielli (2005).

Applications

Paraconsistent logic has been applied as a means of managing inconsistency in numerous domains, including:[15]

- Semantics. Paraconsistent logic has been proposed as means of providing a simple and intuitive formal account of truth that does not fall prey to paradoxes such as the Liar. However, such systems must also avoid Curry's paradox, which is much more difficult as it does not essentially involve negation.
- Set theory and the foundations of mathematics (see paraconsistent mathematics). Some believe that paraconsistent logic has significant ramifications with respect to the significance of Russell's paradox and Gödel's incompleteness theorems.
- Epistemology and belief revision. Paraconsistent logic has been proposed as a means of reasoning with and revising inconsistent theories and belief systems.
- Knowledge management and artificial intelligence. Some computer scientists have utilized paraconsistent logic as a means of coping gracefully with inconsistent information.[16]

- Deontic logic and metaethics. Paraconsistent logic has been proposed as a means of dealing with ethical and other normative conflicts.
- Software engineering. Paraconsistent logic has been proposed as a means for dealing with the pervasive inconsistencies among the documentation, use cases, and code of large software systems.[7][8]
- Electronics design routinely uses a four valued logic, with "hi-impedance (z)" and "don't care (x)" playing similar roles to "don't know" and "both true and false" respectively, in addition to True and False. This logic was developed independently of Philosophical logics.

Criticism

Some philosophers have argued against dialetheism on the grounds that the counterintuitiveness of giving up any of the three principles above outweighs any counterintuitiveness that the principle of explosion might have.

Others, such as David Lewis, have objected to paraconsistent logic on the ground that it is simply impossible for a statement and its negation to be jointly true.[17] A related objection is that "negation" in paraconsistent logic is not really *negation*; it is merely a subcontrary-forming operator.[18]

Alternatives

Approaches exist that allow for resolution of inconsistent beliefs without violating any of the intuitive logical principles. Most such systems use multi-valued logic with Bayesian inference and the Dempster-Shafer theory, allowing that no non-tautological belief is completely (100%) irrefutable because it must be based upon incomplete, abstracted, interpreted, likely unconfirmed, potentially uninformed, and possibly incorrect knowledge (of course, this very assumption, if non-tautological, entails its own refutability, if by "refutable" we mean "not completely [100%] irrefutable"). These systems effectively give up several logical principles in practice without rejecting them in theory.

Notable figures

Notable figures in the history and/or modern development of paraconsistent logic include:

- Alan Ross Anderson (USA, 1925–1973). One of the founders of relevance logic, a kind of paraconsistent logic.
- F. G. Asenjo (Argentina)
- Diderik Batens (Belgium)
- Nuel Belnap (USA, b. 1930). Worked with Anderson on relevance logic.
- Jean-Yves Béziau (France/Switzerland, b. 1965). Has written extensively on the general structural features and philosophical foundations of paraconsistent logics.
- Ross Brady (Australia)
- Bryson Brown (Canada)
- Walter Carnielli (Brazil). The developer of the *possible-translations semantics*, a new semantics which makes paraconsistent logics applicable and philosophically understood.
- Newton da Costa (Brazil, b. 1929). One of the first to develop formal systems of paraconsistent logic.
- Itala M. L. D'Ottaviano (Brazil)
- J. Michael Dunn (USA). An important figure in relevance logic.
- Stanisław Jaśkowski (Poland). One of the first to develop formal systems of paraconsistent logic.
- R. E. Jennings (Canada)
- David Kellogg Lewis (USA, 1941–2001). Articulate critic of paraconsistent logic.
- Jan Łukasiewicz (Poland, 1878–1956)
- Robert K. Meyer (USA/Australia)
- Chris Mortensen (Australia). Has written extensively on paraconsistent mathematics.

- Lorenzo Peña (Spain, b. 1944). Has developed an original line of paraconsistent logic, gradualistic logic (also known as *transitive logic*, TL), akin to Fuzzy Logic.
- Val Plumwood [formerly Routley] (Australia, b. 1939). Frequent collaborator with Sylvan.
- Graham Priest (Australia). Perhaps the most prominent advocate of paraconsistent logic in the world today.
- Francisco Miró Quesada (Peru). Coined the term *paraconsistent logic*.
- Peter Schotch (Canada)
- B. H. Slater (Australia). Another articulate critic of paraconsistent logic.
- Richard Sylvan [formerly Routley] (New Zealand/Australia, 1935–1996). Important figure in relevance logic and a frequent collaborator with Plumwood and Priest.
- Nicolai A. Vasiliev (Russia, 1880–1940). First to construct logic tolerant to contradiction (1910).

Notes

[1] Priest (2002), p. 288 and §3.3.
[2] Carnielli, W. and Marcos, J. (2001) "Ex contradictione non sequitur quodlibet" (http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.107.70) *Proc. 2nd Conf. on Reasoning and Logic* (Bucharest, July 2000)
[3] Jennifer Fisher (2007). *On the Philosophy of Logic* (http://books.google.com/books?id=k8L_YW-lEEQC&pg=PT142). Cengage Learning. pp. 132–134. ISBN 978-0-495-00888-0. .
[4] Graham Priest; John Woods (2007). "Paraconsistency and Dialetheism" (http://books.google.com/books?id=3TNj1ZkP3qEC&pg=PA131). *The Many Valued and Nonmonotonic Turn in Logic*. Elsevier. p. 131. ISBN 978-0-444-51623-7. .
[5] Otávio Bueno (2010). "Philosophy of Logic" (http://books.google.com/books?id=Vu6cVCLvPt0C&pg=PA55). In Fritz Allhoff. *Philosophies of the Sciences: A Guide*. John Wiley & Sons. p. 55. ISBN 978-1-4051-9995-7. .
[6] See the article on the principle of explosion for more on this.
[7] Hewitt (2008b)
[8] Hewitt (2008a)
[9] Priest (2002), p. 306.
[10] LP is also commonly presented as a many-valued logic with three truth values (*true*, *false*, and *both*).
[11] See, for example, Priest (2002), §5.
[12] See Priest (2002), p. 310.
[13] Surveys of various approaches to paraconsistent logic can be found in Bremer (2005) and Priest (2002), and a large family of paraconsistent logics is developed in detail in Carnielli, Congilio and Marcos (2007).
[14] See Aoyama (2004).
[15] Most of these are discussed in Bremer (2005) and Priest (2002).
[16] See, for example, the articles in Bertossi et al. (2004).
[17] See Lewis (1982).
[18] See Slater (1995), Béziau (2000).

Resources

- Jean-Yves Béziau, Walter Carnielli and Dov Gabbay, eds. (2007). *Handbook of Paraconsistency*. London: King's College. ASIN 1904987737. ISBN 978-1-904987-73-4.
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External links

- Stanford Encyclopedia of Philosophy “Paraconsistent Logic” (http://plato.stanford.edu/entries/logic-paraconsistent/)
- Stanford Encyclopedia of Philosophy “Inconsistent Mathematics” (http://plato.stanford.edu/entries/mathematics-inconsistent/)

Logical truth

Logical truth is one of the most fundamental concepts in logic, and there are different theories on its nature. A logical truth is a statement which is true and remains true under all reinterpretations of its components other than its logical constants. It is a type of analytic statement. All of philosophical logic can be thought of as providing accounts of the nature of logical truth, as well as logical consequence.[1]

Logical truths (including tautologies) are truths which are considered to be **necessarily true**. This is to say that they are considered to be such that they could not be untrue and no situation could arise which would cause us to reject a logical truth. However, it is not universally agreed that there are any statements which are *necessarily* true.

A logical truth was considered by Ludwig Wittgenstein to be a statement which is true in all possible worlds.[2] This is contrasted with facts (which may also be referred to as *contingent claims* or *synthetic claims*) which are true in *this* world, as it has historically unfolded, but which is not true in at least one possible world, as it might have unfolded. The proposition "If p and q, then p" and the proposition "All married people are married" are logical truths because they are true due to their inherent meanings and not because of any facts of the world. Later, with the rise of formal logic a logical truth was considered to be a statement which is true under all possible interpretations.

The existence of logical truths is sometimes put forward as an objection to empiricism because it is impossible to account for our knowledge of logical truths on empiricist grounds.

Logical truths and analytic truths

Logical truths, being analytic statements, do not contain any information about any matters of fact. Other than logical truths, there is also a second class of analytic statements, typified by "No bachelor is married." The characteristic of such a statement is that it can be turned into a logical truth by substituting synonyms for synonyms *salva veritate*. "No bachelor is married." can be turned into "No unmarried man is married." by substituting 'unmarried man' for its synonym 'bachelor.'

In his essay. Two Dogmas of Empiricism, the philosopher W.V.O. Quine called into question the distinction between analytic and synthetic statements. It was this second class of analytic statements that caused him to note that the concept of analyticity itself stands in need of clarification, because it seems to depend on the concept of synonymy, which stands in need of clarification. In his conclusion, Quine rejects that logical truths are necessary truths. Instead he posits that the truth-value of any statement can be changed, including logical truths, given a re-evaluation of the truth-values of every other statement in one's complete theory.

Truth values and tautologies

Considering different interpretations of the same statement leads to the notion of truth value. Simplest approach to truth values means that the statement may be "true" in one case, but "false" in another. In one sense of the term "tautology", it is any type of formula or proposition which turns out to be true under any possible interpretation of its terms (may also be called a valuation or assignment depending upon the context). This is synonymous to logical truth.

However, the term "tautology" is also commonly used to refer to what could more specifically called truth-functional tautologies. Whereas a tautology or logical truth is true solely because of the logical terms it contains in general (e.g. "every", "some", and "is"), a truth-functional tautology is true because of the logical terms it contains which are logical connectives (e.g. "or", "and", and "nor"). Not all logical truths are tautologies of such kind.

Logical truth and logical constants

Logical constants, including logical connectives and quantifiers, can all be reduced conceptually to logical truth. For instance, two statements or more are logically incompatible just in case their conjunction is logically false. One statement logically implies another when it is logically incompatible with the negation of the other. A statement is logically false just in case its negation is logically true, etc. In this way all logical connectives can be expressed in terms of preserving logical truth.

Logical truth and rules of inference

The concept of logical truth is closely connected to the concept of a rule of inference.[3]

Non-classical logics

Non-classical logic is the name given to formal systems which differ in a significant way from standard logical systems such as propositional and predicate logic. There are several ways in which this is done, including by way of extensions, deviations, and variations. The aim of these departures is to make it possible to construct different models of logical consequence and logical truth.[4]

External links

- Logical truth [5] entry in the *Stanford Encyclopedia of Philosophy*
- Logical truth [6] at the Indiana Philosophy Ontology Project
- Logical truth [7] at PhilPapers

References

[1] Quine, Willard Van Orman, *Philosophy of logic*
[2] Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*
[3] Alfred Ayer, *Language, Truth, and Logic*
[4] Theodore Sider, *Logic for philosophy*
[5] http://plato.stanford.edu/entries/logical-truth/
[6] https://inpho.cogs.indiana.edu/taxonomy/2408
[7] http://philpapers.org/browse/logical-semantics-and-logical-truth/

Philosophical skepticism

For a general discussion of skepticism, see Skepticism.

Philosophical skepticism (from Greek σκέψις, *skepsis* meaning "enquiry"; UK spelling **scepticism**) is both a philosophical school of thought and a method that crosses disciplines and cultures. Many skeptics critically examine the meaning systems of their times, and this examination often results in a position of ambiguity or doubt.[1] This skepticism can range from disbelief in contemporary philosophical solutions, to agnosticism, to rejecting the reality of the external world. One kind of scientific skepticism refers to the critical analysis of claims lacking empirical evidence. We are all skeptical of some things, especially since doubt and opposition are not always clearly distinguished. Philosophical skepticism, however, is an old movement with many variations, and contrasts with the view that at least one thing is certain, but if by being certain we mean absolute or unconditional certainty, then it is doubtful if it is rational to claim to be certain about anything. Indeed, for Hellenistic philosophers claiming that at least one thing is certain makes one a dogmatist.

Philosophical skepticism is distinguished from methodological skepticism in that philosophical skepticism is an approach that denies the possibility of certainty in knowledge, whereas methodological skepticism is an approach that subjects all knowledge claims to scrutiny with the goal of sorting out true from false claims.

History

Ancient Western Skepticism

The Western tradition of systematic skepticism goes back at least as far as Pyrrho of Elis (b. circa 360 BC). He was troubled by the disputes that could be found within all philosophical schools of his day. According to a later account of his life, he became overwhelmed by his inability to determine rationally which school was correct. Upon admitting this to himself, he finally achieved the inner peace that he had been seeking.

From a Stoic point of view, Pyrrho found peace by admitting to ignorance and seeming to abandon the criterion by which knowledge is gained. Pyrrho's ignorance was not the ignorance of children or farm animals: it was a *knowledgeable ignorance*, arrived at through the application of logical reasoning and exposition of its inadequacy. The school of thought developed primarily in opposition to what it saw as the dogmatism, or ultimately unfounded assertions of the Stoics; Pyrrhonists made distinctions between "being" and "appearing" and between the identity and the sensing of a phenomenon.

Pyrrho and his school were not actually "skeptics" in the later sense of the word. They had the goal of *αταϱαξια* (*ataraxia* - peace of mind), and pitted one dogmatic philosophy against the next to undermine belief in the whole philosophic enterprise. The idea was to produce in the student a state of aversion towards what the Pyrrhonists considered arbitrary and inconsequential babble. Since no one can observe or otherwise experience causation, external world (its "externality"), ultimate purpose of the universe or life, justice, divinity, soul, etc., they declared no need to believe in such things. The Pyrrhonists pointed out that, despite claims that such notions were necessary, some people "ignorant" of them get by just fine before learning about them. They further noted that science does not require belief and that faith in intelligible realities is different from pragmatic convention for the sake of experiment. For each intuitive notion (e.g. the existence of an external world), the Pyrrhonists cited a contrary opinion to negate it. They added that consensus indicates neither truth nor even probability. For example, the earth is round, and it would remain so even if everyone believed it were flat. Unless, of course, it is flat, and we all simply believe it is round.

The goal of this critique, which Pyrrho's followers realized would ultimately subvert even their own method, was to cultivate a distrust of all grand talk. They expected philosophy to collapse into itself. How far in this direction the Pyrrhonean commitment extended is a matter of debate. The Pyrrhonists confessed a belief in appearances, e.g. in hot and cold, grief and joy. It is impossible to deny, they admitted, that one **seems** to be in pain or **seems** to touch a piece of wood. Their world, thus, was completely phenomenological. An accomplished Pyrrhonist could, ideally, live as well as a dogmatist but with the added benefit of not worrying about truth and falsity, right and wrong, God's will, and so forth.

Later thinkers took up Pyrrho's approach and extended it into modern skepticism. In the process, a split appeared within the movement, never too large or well liked among the literati to begin with. In the Academic skepticism of the *New or Middle Academy*, Arcesilaus (c. 315-241 BCE) and Carneades (c. 213-129 BCE) argued from Stoic premises that the Stoics were actually committed to denying the possibility of knowledge, but seemed to maintain nothing themselves, but Clitomachus, a student of Carneades, interpreted his teacher's philosophy as suggesting an early probabilistic account of knowledge. The Roman politician and philosopher, Cicero, also seems to have been a supporter of the probabilistic position attributed to the Middle Academy, even if the return to a more dogmatic orientation of that school was already beginning to take place.

In the centuries to come, the words *Academician* and *Pyrrhonist* would often be used to mean generally *skeptic*, often ignoring historical changes and distinctions between denial of knowledge and avoidance of belief, between degree of belief and absolute belief, and between possibility and probability.

Sextus Empiricus

Sextus Empiricus (c. CE. 200), the main authority for Pyrrhonian skepticism, worked outside the Academy, which by his time had ceased to be a skeptical or probabilistic school, and argued in a different direction, incorporating aspects of empiricism into the basis for evaluating knowledge, but without the insistence on experience as the absolute standard of it. Sextus' empiricism was limited to the "absolute minimum" already mentioned — that there seem to be appearances. He developed this basic thought of Pyrrho's into lengthy arguments, most of them directed against Stoics and Epicureans, but also the Academic skeptics. The common anti-skeptical argument is that if one knows nothing, one cannot know that one knows nothing, and so may know something after all. It is worth noting that such an argument only succeeds against the complete denial of the possibility of knowledge. Considering dogmatic the claims both to know and not to know, Sextus and his followers claimed neither. Instead, despite the apparent conflict with the goal of ataraxia, they claimed to continue searching for something that might be knowable.

Empiricus, as the most systematic and dogmatic author of the works by Hellenistic sceptics which have survived, noted that there are at least *ten modes* of skepticism. These modes may be broken down into three categories: we may be skeptical of *the subjective perceiver, of the objective world*, and *the relation between perceiver and the world*.[2]

Subjectively, both the powers of the senses and of reasoning may vary across persons. And since knowledge is a product of one and/or the other, and since neither are reliable, knowledge would seem to be in trouble. For instance, a color-blind person sees the world quite differently from everyone else. Moreover, we cannot even give preference on the basis of the power of reason, i.e., by treating the rational animal as a carrier of greater knowledge than the irrational animal. For the irrational animal is still adept at navigating their environment, which presupposes the ability to know about some aspects of the environment.

Secondly, the personality of the individual might also have an impact on what they observe, since (it is argued) preferences are based on sense-impressions, differences in preferences can be attributed to differences in the way that people are affected by the object. (Empiricus:56)

Third, the perceptions of each individual sense seemingly have nothing in common with the other senses: i.e., the color "red" has little to do with the feeling of touching a red object. This is manifest when our senses "disagree" with each other: for example, a mirage presents certain visible features, but is not responsive to any other kind of sense. In that case, our other senses defeat the impressions of sight. But we may also be lacking enough powers of sense to understand the world in its entirety: if we had an extra sense, then we might know of things in a way that the present five senses are unable to advise us of. Given that our senses can be shown to be unreliable by appealing to other senses, and so our senses may be incomplete (relative to some more perfect sense that we lack), then it follows that all of our senses may be unreliable. (Empiricus:58)

Fourth, our circumstances when we do any perceiving may be either natural or unnatural, i.e., we may be either in a state of wakefulness or that of sleep. But it is entirely possible that things in the world really are exactly as they appear to be to those in unnatural states (i.e., if everything were an elaborate dream). (Empiricus:59)

We have reasons for doubt that are based on the *relationship between objective "facts" and subjective experience*. The positions, distances, and places of objects would seem to affect how they are perceived by the person: for instance, the portico may appear tapered when viewed from one end, but symmetrical when viewed at the other; and these features are different. Because they are different features, to believe the object has both properties at the same time is to believe it has two contradictory properties. Since this is absurd, we must suspend judgment about what properties it possesses. (Empiricus:63)

We may also observe that the things we perceive are, in a sense, polluted by experience. Any given perception—say, of a chair—will always be perceived within some context or other (i.e., next to a table, on a mat, etc.) Since this is the case, we can only speak of ideas as they occur in the context of the other things that are paired with it. We can never know of the true nature of the thing, but only how it appears to us in context. (Empiricus: 64)

Along the same lines, the skeptic may insist that all things are relative, by arguing that:

1. Absolute appearances either differ from relative appearances, or they do not.
2. If absolutes do not differ from relatives, then they are themselves relative.
3. But if absolutes do differ from relatives, then they are relative, because all things that differ must differ from something; and to "differ" from something is to be relative to something. (Empiricus:67)

Finally, we have reason to disbelieve that we know anything by looking at problems in understanding *the objects themselves*. Things, when taken individually, may appear to be very different than when they are in mass quantities: for instance, the shavings of a goat's horn are white when taken alone, yet the horn intact is black.

Ancient Eastern Skepticism

Buddhism

Buddhist skepticism (Zen Buddhism) is not concerned with whether a thing exists or not. The Zen masters would answer questions "koans" with seemingly unrelated responses such as hitting the student. This would serve as a means of pulling the student back from the confusion of intellectual pontification, and into a direct experience. Since in Zen, all there is a direct experience, which cannot be explained or clarified beyond the experience itself, this answers the question.

- Buddha is said to have touched the earth at the time of his enlightenment so that it could witness his enlightenment. In this way, Buddhism does not claim that knowledge is unattainable.
- Buddhism places less emphasis on truth and knowledge than western philosophical skepticism. Instead, it emphasizes the goal of *Bodhi*, which, although often translated as *enlightenment*, does not imply truth or knowledge.
- At least in its manifestation of Nagarjuna's texts that form the core of Madhyamaka, the anti-essentialist aspect of Buddhism makes it an anti-philosophy. From that stance, truth exists solely within the contexts that assert them.

Cārvāka philosophy

The Cārvāka (Sanskrit: चार्वाक) school of skepticism, also known as Lokāyata, is a distinct branch of Indian philosophy. The school is named after Cārvāka, author of the Bārhaspatya-sūtras and was founded in approximately 500 BC. Cārvāka is classified as a "heterodox" (nāstika) system, characterized as a materialistic and atheistic school of thought.

Jain Philosophy of Anekantavada and Syadavada

Anekāntavāda also known as the principle of relative pluralism, is one of the basic principles of Jainism. According to this, the truth or the reality is perceived differently from different points of view, and that no single point of view is the complete truth.[3][4] Jain doctrine states that, an object has infinite modes of existence and qualities and, as such, they cannot be completely perceived in all its aspects and manifestations, due to inherent limitations of the humans. Anekāntavāda is literally the doctrine of non-onesidedness or manifoldness; it is often translated as "non-absolutism". Syādvāda is the theory of conditioned predication which provides an expression to anekānta by recommending that epithet “Syād” be attached to every expression.[5] Syādvāda is not only an extension of Anekānta ontology, but a separate system of logic capable of standing on its own force. As reality is complex, no single proposition can express the nature of reality fully. Thus the term “syāt” should be prefixed before each proposition giving it a conditional point of view and thus removing any dogmatism in the statement. .[6] The seven propositions also known as saptabhangi are[7]

1. ***Syād-asti*** – “in some ways it is”,
2. ***syād-nāsti*** - “in some ways it is not”,
3. ***syād-asti-nāsti*** - “in some ways it is and it is not”,
4. ***syād-asti-avaktavyaḥ*** - “in some ways it is and it is indescribable”,
5. ***syād-nāsti-avaktavyaḥ*** - “in some ways it is not and it is indescribable”,
6. ***syād-asti-nāsti-avaktavyaḥ*** - “in some ways it is, it is not and it is indescribable”,
7. ***syād-avaktavyaḥ***- “in some ways it is indescribable”

Each of these seven propositions examines the complex and multifaceted reality from a relative point of view of time, space, substance and mode. To ignore the complexity of the objects is to commit the fallacy of dogmatism. For a rigorous logical and mathematical interpretation see M. K. Jain, Current Science.100, 1663-1672 (2011).

China

In China, the preeminent Daoist work Zhuangzi, attributed to 4th century BC philosopher Zhuangzi during the Hundred Schools of Thought period, is skeptical in nature and provides also two famous skeptical paradoxes, "The Happiness of Fish" and "Zhuangzi dreamed he was a butterfly".

Wang Chong introduced a form of naturalism based on a rational critique of the superstition that was overtaking Confucianism and Daoism in the 1st century CE. His neo-Daoist philosophy was based on a secular, rational practice not unlike the scientific method.

Islam

In Islamic theology and Islamic philosophy, the scholar Al-Ghazali (1058–1111) is considered a pioneer of methodic doubt and skepticism.[8] His 11th century book titled *The Incoherence of the Philosophers* marks a major turn in Islamic epistemology, as Ghazali effectively discovered a methodic form of philosophical skepticism that would not be commonly seen in the West until René Descartes, George Berkeley and David Hume. The encounter with skepticism led Ghazali to embrace a form of theological occasionalism, or the belief that all causal events and interactions are not the product of material conjunctions but rather the immediate and present will of God. While he himself was a critic of the philosophers, Ghazali was a master in the art of philosophy and had immensely studied the field. After such a long education in philosophy, as well as a long process of reflection, he had criticized the philosophical method.

The autobiography Ghazali wrote towards the end of his life, *The Deliverance From Error* (*Al-munqidh min al-ḍalāl*; several English translations[9]) is considered a work of major importance.[10] In it, Ghazali recounts how, once a crisis of epistemological skepticism was resolved by "a light which God Most High cast into my breast...the key to most knowledge,"[11] he studied and mastered the arguments of Kalam, Islamic philosophy and Ismailism. Though appreciating what was valid in the first two of these, at least, he determined that all three approaches were inadequate and found ultimate value only in the mystical experience and spiritual insight (Spiritual intuitive thought – Firasa and Nur) he attained as a result of following Sufi practices. William James, in *Varieties of Religious Experience*, considered the autobiography an important document for "the purely literary student who would like to become acquainted with the inwardness of religions other than the Christian", comparing it to recorded personal religious confessions and autobiographical literature in the Christian tradition.[12]

Scholars have noted the similarities between Descartes' *Discourse on Method* and Ghazali's work[8] and the writer George Henry Lewes went even further by claiming that "had any translation of it [The Revival of Religious Sciences] in the days of Descartes existed, everyone would have cried out against the plagiarism."[13]

Schools of philosophical skepticism

Philosophical skepticism begins with the claim that the skeptic currently does not have knowledge. Some adherents maintain that knowledge is, in theory, possible. It could be argued that Socrates held that view. He appears to have thought that if people continue to ask questions they might eventually come to have knowledge; but that they did not have it yet. Some skeptics have gone further and claimed that true knowledge is impossible, for example the Academic school in Ancient Greece well after the time of Carneades. A third skeptical approach would be neither to accept nor reject the possibility of knowledge.

Skepticism can be either about everything or about particular areas. A 'global' skeptic argues that he does not absolutely know anything to be either true or false. Academic global skepticism has great difficulty in supporting this claim while maintaining philosophical rigor, since it seems to require that nothing can be known — except for the knowledge that nothing can be known, though in its probabilistic form it can use and support the notion of weight of evidence. Thus, some probabilists avoid extreme skepticism by maintaining that they merely are 'reasonably certain' (or 'largely believe') some things are real or true. As for using probabilistic arguments to defend skepticism, in a sense this enlarges or increases scepticism, while the defence of empiricism by Empiricus weakens skepticism and strengthens dogmatism by alleging that sensory appearances are beyond doubt. Much later, Kant would re-define "dogmatism" to make indirect realism about the external world seem objectionable. While many Hellenists, outside of Empiricus, would maintain that everyone who is not sceptical about everything is a dogmatist, this position would seem too extreme for most later philosophers.

Nevertheless, A Pyrrhonian global skeptic labors under no such modern constraint, since he only alleged that he, personally, did not know anything and made no statement about the possibility of knowledge. Nor did Arcesilaus feel bound, since he merely corrected Plato's "I only know that I know nothing" by adding "I don't even know that", thus more fully rejecting dogmatism.

Local skeptics deny that people do or can have knowledge of a particular area. They may be skeptical about the possibility of one form of knowledge without doubting other forms. Different kinds of local skepticism may emerge, depending on the area. A person may doubt the truth value of different types of journalism, for example, depending on the types of media they trust.

In Islamic philosophy, skepticism was established by Al-Ghazali (1058–1111), known in the West as "Algazel", as part of the orthodox Ash'ari school of Islamic theology.

In the West itself, the one Renaissance thinker mostly viewed as the "Father of Modern Skepticism" is Michel de Montaigne, especially in his seminal *Essays*. Francisco Sanches's *That Nothing is Known* (published in 1581 as *Quod* *nihil scitur*) is also one of the crucial texts of **Renaissance skepticism**.[14]

Epistemology and skepticism

Skepticism, as an epistomological argument, poses the question of whether knowledge, in the first place, is possible. Skeptics argue that the belief in something does not necessarily justify an assertion of knowledge of it. In this, skeptics oppose dogmatic foundationalism, which states that there have to be some basic positions that are self-justified or beyond justification, without reference to others. (One example of such functionalism may be found in Spinoza's *Ethics*.) The skeptical response to this can take several approaches. First, claiming that "basic positions" must exist amounts to the logical fallacy of argument from ignorance combined with the slippery slope.

Among other arguments, skeptics used Agrippa's Trilemma, named after Agrippa the Sceptic, to claim no certain belief could be achieved. Foundationalists have used the same *trilemma* as a justification for demanding the validity of basic beliefs.

This skeptical approach is rarely taken to its pyrrhonean extreme by most practitioners. Several modifications have arisen over the years, including the following [15]:

Fictionalism would not claim to have knowledge but will adhere to conclusions on some criterion such as utility, aesthetics, or other personal criteria without claiming that any conclusion is actually "true".

Philosophical fideism (as opposed to religious Fideism) would assert the truth of some propositions, but does so without asserting certainty.

Some forms of pragmatism would accept utility as a provisional guide to truth but not necessarily a universal decision-maker.

Criticism of skepticism

Most philosophies have weaknesses and can be criticized and this is a general principle of progression in philosophy.[16] The philosophy of skepticism asserts that no truth is knowable[17] or only probable.[18] Some say the scientific method also asserts probable findings, because the number of cases tested is always limited and they constitute perceptual observations.[19] To claim that the proposition “no truth is knowable” is knowably true is to refute oneself; as it is contradictory.[20] If we look deeper, we can see that this statement is a self-referencing statement, as proposed by Gödel, which rather than actually disproving itself proves that the logical system itself is not capable of dealing with the statement accurately. Thus extreme skepticism is the only philosophy which embraces Gödel's Incompleteness Theorem (as well as Tarski's Undefinability Theorem which states truth cannot be defined within any logical system)and recognizes the limitations of all logical systems.

Pierre Le Morvan (2011) has distinguished between three broad philosophical approaches to skepticism. The first he calls the "Foil Approach." According to the latter, skepticism is treated as a problem to be solved, or challenge to be met, or threat to be parried; skepticism‘s value on this view, insofar as it is deemed to have one, accrues from its role as a foil contrastively illuminating what is required for knowledge and justified belief. The second he calls the "Bypass Approach" according to which skepticism is bypassed as a central concern of epistemology. Le Morvan advocates a third approach—he dubs it the "Health Approach"—that explores when skepticism is healthy and when it is not, or when it is virtuous and when it is vicious.

Skeptical hypotheses

A skeptical hypothesis is a hypothetical situation which can be used in an argument for skepticism about a particular claim or class of claims. Usually the hypothesis posits the existence of a deceptive power that deceives our senses and undermines the justification of knowledge otherwise accepted as justified. Skeptical hypotheses have received much attention in modern Western philosophy.

The first skeptical hypothesis in modern Western philosophy appears in René Descartes' Meditations on First Philosophy. At the end of the first Meditation Descartes writes: "I will suppose... that some evil demon of the utmost power and cunning has employed all his energies to deceive me."

- The "Brain in a vat" hypothesis is cast in scientific terms. It supposes that one might be a disembodied brain kept alive in a vat, and fed false sensory signals, by a mad scientist.
- The "Dream argument" of Descartes and Zhuangzi supposes reality to be indistinguishable from a dream.
- Descartes' Evil demon is a being "as clever and deceitful as he is powerful, who has directed his entire effort to misleading me."
- The five minute hypothesis (or omphalos hypothesis or Last Thursdayism) suggests that the world was created recently together with records and traces indicating a greater age.
- The Simulated reality hypothesis or Matrix hypothesis suggest that we might be inside a computer simulation or virtual reality.

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[12] William James, *Varieties of Religious Experience*, Harvard University Press, 1985, p. 319 [= 2002 Modern Library Paperback Edition, p. 438].
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[17] Kreeft, Peter & Tacelli, R. K *Handbook of Christian Apologetics*, IVP Academic, Ill. 1994, p. 367)
[18] Popkin, p. 205
[19] Popkin, p. 230
[20] Kreeft p. 373

Further reading

- Breker, Christian: *Einführender Kommentar zu Sextus Empiricus' "Grundriss der pyrrhonischen Skepsis"*, Mainz, 2011: electr. publication, University of Mainz. available online (http://nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:hebis:77-25461) (comment on Sextus Empiricus' "Outlines of Pyrrhonism" in German language)
- Zeller, Eduard; Reichel, Oswald J., *The Stoics, Epicureans and Sceptics* (http://books.google.com/books?id=tfEYAAAAYAAJ&printsec=titlepage), Longmans, Green, and Co., 1892

External links

- Responses to skepticism (http://pantheon.yale.edu/~kd47/responding.htm)
- Excerpts from the "Outlines of Pyrrhonism" by Sextus Empiricus (http://www.philosophy.leeds.ac.uk/GMR/hmp/modules/hdc0405/units/unit05/outlines.html)
- Skepticism (http://plato.stanford.edu/entries/skepticism) entry in the *Stanford Encyclopedia of Philosophy*
- Attempted refutation of external-world scepticism. (http://gehirnimtank.de) On-line version of two recently published books by philosopher Olaf M. Müller (in German)
- Article: Skepticism and Denial (http://www.theness.com/articles.asp?id=23) by Stephen Novella MD, The New England Journal of Skepticism
- Classical Skepticism (http://www.earlham.edu/~peters/writing/skept.htm) by Peter Suber.
- The Matrix as Metaphysics by David Chalmers (http://consc.net/papers/matrix.html)
- Review and summary of *Skepticism and the Veil of Perception* by Mike Huemer (http://ndpr.nd.edu/news/23238-skepticism-and-the-veil-of-perception/)

Logic puzzle

A **logic puzzle** is a puzzle deriving from the mathematics field of deduction.

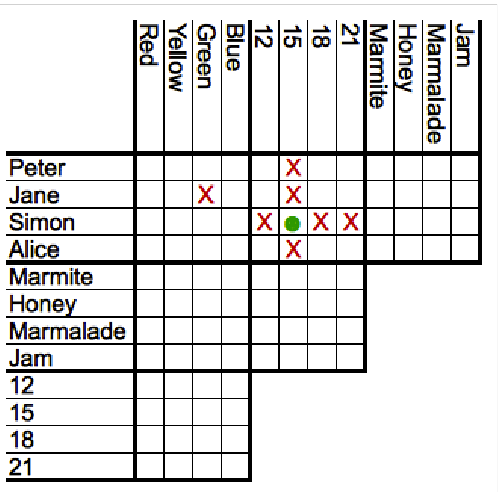
History

The logic puzzle was first produced by Charles Lutwidge Dodgson, who is better known under his pen name Lewis Carroll, the author of *Alice's Adventures in Wonderland*. In his book *The Game of Logic* he introduced a game to solve problems such as confirming the conclusion "Some greyhounds are not fat" from the statements "No fat creatures run well" and "Some greyhounds run well". Puzzles like this, where we are given a list of premises and asked what can be deduced from them, are known as syllogisms. Dodgson goes on to construct much more complex puzzles consisting of up to 8 premises.

In the second half of the 20th century mathematician Raymond M. Smullyan has continued and expanded the branch of logic puzzles with books such as *The Lady or the Tiger?*, *To Mock a Mockingbird* and *Alice in Puzzle-Land*. He popularized the "knights and knaves" puzzles, which involve knights, who always tell the truth, and knaves, who always lie.

There are also logic puzzles that are completely non-verbal in nature. Some popular forms include Sudoku, which involves using deduction to correctly place numbers in a grid; the nonogram, also called "Paint by Numbers", which involves using deduction to correctly fill in a grid with black-and-white squares to produce a picture; and logic mazes, which involve using deduction to figure out the rules of a maze.

Logic grid puzzles



Example logic puzzle grid, with the information that only Simon is 15 and Jane does not like green filled in.

Another form of logic puzzle, popular among puzzle enthusiasts and available in magazines dedicated to the subject, is a format in which the set-up to a scenario is given, as well as the object (for example, determine who brought what dog to a dog show, and what breed each dog was), certain clues are given ("neither Misty nor Rex is the German Shepherd"), and then the reader fills out a matrix with the clues and attempts to deduce the solution. These are often referred to as "logic grid" puzzles. The most famous example may be the so-called Zebra Puzzle, which asks the question *Who Owned the Zebra?*.

Common in logic puzzle magazines are derivatives of the logic grid puzzle called "table puzzles" that are deduced in the same manner as grid puzzles, but lack the grid either because a grid would be too large, or because some other visual aid is provided. For example, a map of a town might be present in lieu of a grid in a puzzle about the location of different shops.

External links

- Puzzles [1] at the Open Directory Project
- Hankies, Snarks, and Triangles [2] – Ivars Peterson's MathTrek

References

[1] http://www.dmoz.org/Games/Puzzles/
[2] http://www.maa.org/mathland/mathland_1_13.html

List of logic symbols

In logic, a set of symbols is commonly used to express logical representation. As logicians are familiar with these symbols, they are not explained each time they are used. So, for students of logic, the following table lists many common symbols together with their name, pronunciation and related field of mathematics. Additionally, the third column contains an informal definition, and the fourth column gives a short example.

Be aware that, outside of logic, different symbols have the same meaning, and the same symbol has, depending on the context, different meanings.

Basic logic symbols

Symbol	Name Should be read as Category	Explanation	Examples	Unicode Value	HTML Entity	LaTeX symbol
⇒ → ⊃	material implication implies; if .. then propositional logic, Heyting algebra	$A \Rightarrow B$ is true just in the case that either A is false or B is true, or both. → may mean the same as ⇒ (the symbol may also indicate the domain and codomain of a function; see table of mathematical symbols). ⊃ may mean the same as ⇒ (the symbol may also mean superset).	$x = 2 \Rightarrow x^2 = 4$ is true, but $x^2 = 4 \Rightarrow x = 2$ is in general false (since x could be −2).	U+21D2 U+2192 U+2283	⇒ → ⊃	⇒ \Rightarrow → \to ⊃ \supset
⇔ ≡ ↔	material equivalence if and only if; iff; means the same as propositional logic	$A \Leftrightarrow B$ is true just in case either both A and B are false, or both A and B are true.	$x + 5 = y + 2 \Leftrightarrow x + 3 = y$	U+21D4 U+2261 U+2194	⇔ ≡ ↔	⇔ \Leftrightarrow ≡ \equiv ↔ \leftrightarrow
¬ ˜ !	negation not propositional logic	The statement $\neg A$ is true if and only if A is false. A slash placed through another operator is the same as "¬" placed in front.	$\neg(\neg A) \Leftrightarrow A$ $x \neq y \Leftrightarrow \neg(x = y)$	U+00AC U+02DC	¬ ˜ ~	¬ \lnot or \neg ∼ \sim
∧ • &	logical conjunction and propositional logic	The statement $A \wedge B$ is true if A and B are both true; else it is false.	$n < 4 \wedge n > 2 \Leftrightarrow n = 3$ when n is a natural number.	U+2227 U+0026	∧ &	∧ \wedge or \land \&[1]
∨ + ∥	logical disjunction or propositional logic	The statement $A \vee B$ is true if A or B (or both) are true; if both are false, the statement is false.	$n \geq 4 \vee n \leq 2 \Leftrightarrow n \neq 3$ when n is a natural number.	U+2228	∨	∨ \lor or \vee

⊕ ⊻	exclusive disjunction xor propositional logic, Boolean algebra	The statement $A \oplus B$ is true when either A or B, but not both, are true. $A ⊻ B$ means the same.	$(\neg A) \oplus A$ is always true, $A \oplus A$ is always false.	**U+2295** **U+22BB**	**⊕**	⊕ **\oplus** ⊻**\veebar**
⊤ T 1	Tautology top, verum propositional logic, Boolean algebra	The statement ⊤ is unconditionally true.	$A \Rightarrow \top$ is always true.	**U+22A4**	**T**	⊤ **\top**
⊥ F 0	Contradiction bottom, falsum propositional logic, Boolean algebra	The statement ⊥ is unconditionally false.	$\bot \Rightarrow A$ is always true.	**U+22A5**	**⊥** **F**	⊥ **\bot**
∀ ()	universal quantification for all; for any; for each first-order logic	$\forall\, x$: $P(x)$ or $(x)\ P(x)$ means $P(x)$ is true for all x.	$\forall\, n \in \mathbf{N}: n^2 \geq n$.	**U+2200**	**&forall**	∀ **\forall**
∃	existential quantification there exists first-order logic	$\exists\, x$: $P(x)$ means there is at least one x such that $P(x)$ is true.	$\exists\, n \in \mathbf{N}$: n is even.	**U+2203**	**∃**	∃ **\exists**
∃!	uniqueness quantification there exists exactly one first-order logic	$\exists!\, x$: $P(x)$ means there is exactly one x such that $P(x)$ is true.	$\exists!\, n \in \mathbf{N}: n + 5 = 2n$.	**U+2203 U+0021**	**∃ !**	∃!**\exists !**
:= ≡ :⇔	definition is defined as everywhere	$x := y$ or $x \equiv y$ means x is defined to be another name for y (but note that ≡ can also mean other things, such as congruence). $P :\Leftrightarrow Q$ means P is defined to be logically equivalent to Q.	$\cosh x := (1/2)(\exp x + \exp(-x))$ A XOR $B :\Leftrightarrow$ $(A \vee B) \wedge \neg(A \wedge B)$	**U+2254** **(U+003A U+003D)** **U+2261** **U+003A U+229C**	**:=** **:** **≡** **⇔**	:=:= ≡ **\equiv** ⇔ **\Leftrightarrow**
( )	precedence grouping everywhere	Perform the operations inside the parentheses first.	$(8/4)/2 = 2/2 = 1$, but $8/(4/2) = 8/2 = 4$.	**U+0028 U+0029**	( )	( ) ( )
⊢	Turnstile provable propositional logic, first-order logic	$x \vdash y$ means y is provable from x (in some specified formal system).	$A \to B \vdash \neg B \to \neg A$	**U+22A2**	**⊢**	⊢ **\vdash**

⊨	double turnstile entails propositional logic, first-order logic	*x* ⊨ *y* means *x* semantically entails *y*	*A* → *B* ⊨ ¬*B* → ¬*A*	**U+22A8**	**⊨**	**⊨\models**

Advanced and rarely used logical symbols

These symbols are sorted by their Unicode value:

- , an outdated way for denoting AND, still in use in electronics; for example "A·B" is the same as "A&B"
- ·: Center dot with a line above it (using HTML style). Outdated way for denoting NAND, for example "A·B" is the same as "A NAND B" or "A|B" or "¬(A & B)". See also Unicode .
- , used as abbreviation for standard numerals. For example, using HTML style "4̅" is a shorthand for the standard numeral "SSSS0".
- Overline, is also a rarely used format for denoting Gödel numbers, for example "AVB" says the Gödel number of "(AVB)"
- Overline is also an outdated way for denoting negation, still in use in electronics; for example "AVB" is the same as "¬(AVB)"
- or : Sheffer stroke, the sign for the NAND operator.
- : strike out existential quantifier same as "¬∃"
- : is a model of
- : is true of
- : negated ⊢, the sign for "does not prove", for example *T* ⊬ *P* says "*P* is not a theorem of *T*"
- : is not true of
- : another NAND operator, can also be rendered as ∧
- : another NOR operator, can also be rendered as V
- : modal operator for "it is possible that", "it is not necessarily not" or rarely "it is not provable not" (in most modal logics it is defined as "¬□¬")
- : usually used for ad-hoc operators
- or : Webb-operator or Peirce arrow, the sign for NOR. Confusingly, "⊥" is also the sign for contradiction or absurdity.
- and : corner quotes, also called "Quine quotes"; the standard symbol used for denoting Gödel number; for example "⌜G⌝" denotes the Gödel number of G. (Typographical note: although the quotes appears as a "pair" in unicode (231C and 231D), they are not symmetrical in some fonts. And in some fonts (for example Arial) they are only symmetrical in certain sizes. Alternatively the quotes can be rendered as ⌈ and ⌉ (U+2308 and U+2309) or by using a negation symbol and a reversed negation symbol ⌐ ¬ in superscript mode.)
- or : modal operator for "it is necessary that" (in modal logic), or "it is provable that" (in provability logic), or "it is obligatory that" (in deontic logic), or "it is believed that" (in doxastic logic).

Note that the following operators are rarely supported by natively installed fonts. If you wish to use these in a web page, you should always embed the necessary fonts so the page viewer can see the web page without having the necessary fonts installed in their computer.

- : modal operator for was never
- : modal operator for will never be
- : modal operator for was always
- : modal operator for will always be

- : sometimes used for "relation", also used for denoting various ad hoc relations (for example, for denoting "witnessing" in the context of Rosser's trick) The fish hook is also used as strict implication by C.I.Lewis p ▯ $q \equiv \Box(p \rightarrow q)$, the corresponding LaTeX macro is \strictif.

See here [2] for an image of glyph. Added to Unicode 3.2.0.

Notes

[1] Although this character is available in LaTeX, the Mediawiki TeX system doesn't support this character.

[2] http://www.fileformat.info/info/unicode/char/297d/index.htm

External links

- Named character entities (http://www.w3.org/TR/WD-html40-970708/sgml/entities.html) in HTML 4.0.

Metalogic

Metalogic is the study of the metatheory of logic. While *logic* is the study of the manner in which logical systems can be used to construct valid and sound arguments, metalogic studies the properties of the logical systems themselves.[1] While logic concerns itself with the truths that may be derived using a logical system, metalogic concerns itself with the truths which may be derived *about* the languages, and systems that are used to express truths.[2]

The basic objects of study in metalogic are formal languages, formal systems, and their interpretations. The study of interpretation of formal systems is the branch of mathematical logic known as model theory, while the study of deductive systems is the branch known as proof theory.

Overview

Formal language

A *formal language* is an organized set of symbols the essential feature of which is that it can be precisely defined in terms of just the shapes and locations of those symbols. Such a language can be defined, then, without any reference to any meanings of any of its expressions; it can exist before any interpretation is assigned to it—that is, before it has any meaning. First order logic is expressed in some formal language. A formal grammar determines which symbols and sets of symbols are formulas in a formal language.

A formal language can be defined formally as a set *A* of strings (finite sequences) on a fixed alphabet α. Some authors, including Carnap, define the language as the ordered pair <α, *A*>.[3] Carnap also requires that each element of α must occur in at least one string in *A*.

Formation rules

Formation rules (also called *formal grammar*) are a precise description of the well-formed formulas of a formal language. It is synonymous with the set of strings over the alphabet of the formal language which constitute well formed formulas. However, it does not describe their semantics (i.e. what they mean).

Formal systems

A *formal system* (also called a *logical calculus*, or a *logical system*) consists of a formal language together with a deductive apparatus (also called a *deductive system*). The deductive apparatus may consist of a set of transformation rules (also called *inference rules*) or a set of axioms, or have both. A formal system is used to derive one expression from one or more other expressions.

A *formal system* can be formally defined as an ordered triple <α, $\mathcal{I}$, $\mathcal{D}$ d>, where $\mathcal{D}$ d is the relation of direct derivability. This relation is understood in a comprehensive sense such that the primitive sentences of the formal system are taken as directly derivable from the empty set of sentences. Direct derivability is a relation between a sentence and a finite, possibly empty set of sentences. Axioms are laid down in such a way that every first place member of $\mathcal{D}$ d is a member of $\mathcal{I}$ and every second place member is a finite subset of $\mathcal{I}$.

It is also possible to define a *formal system* using only the relation $\mathcal{D}$ d. In this way we can omit $\mathcal{I}$, and α in the definitions of *interpreted formal language*, and *interpreted formal system*. However, this method can be more difficult to understand and work with.[3]

Formal proofs

A *formal proof* is a sequence of well-formed formulas of a formal language, the last one of which is a theorem of a formal system. The theorem is a syntactic consequence of all the well formed formulae preceding it in the proof. For a well formed formula to qualify as part of a proof, it must be the result of applying a rule of the deductive apparatus of some formal system to the previous well formed formulae in the proof sequence.

Interpretations

An *interpretation* of a formal system is the assignment of meanings, to the symbols, and truth-values to the sentences of the formal system. The study of interpretations is called Formal semantics. *Giving an interpretation* is synonymous with *constructing a model.*

Important distinctions in metalogic

Metalanguage–Object language

In metalogic, formal languages are sometimes called *object languages*. The language used to make statements about an object language is called a *metalanguage*. This distinction is a key difference between logic and metalogic. While logic deals with *proofs in a formal system*, expressed in some formal language, metalogic deals with *proofs about a formal system* which are expressed in a metalanguage about some object language.

Syntax–semantics

In metalogic, 'syntax' has to do with formal languages or formal systems without regard to any interpretation of them, whereas, 'semantics' has to do with interpretations of formal languages. The term 'syntactic' has a slightly wider scope than 'proof-theoretic', since it may be applied to properties of formal languages without any deductive systems, as well as to formal systems. 'Semantic' is synonymous with 'model-theoretic'.

Use–mention

In metalogic, the words 'use' and 'mention', in both their noun and verb forms, take on a technical sense in order to identify an important distinction.[2] The *use–mention distinction* (sometimes referred to as the *words-as-words distinction*) is the distinction between *using* a word (or phrase) and *mentioning* it. Usually it is indicated that an expression is being mentioned rather than used by enclosing it in quotation marks, printing it in italics, or setting the expression by itself on a line. The enclosing in quotes of an expression gives us the name of an expression, for example:

> 'Metalogic' is the name of this article.
>
> This article is about metalogic.

Type–token

The *type-token distinction* is a distinction in metalogic, that separates an abstract concept from the objects which are particular instances of the concept. For example, the particular bicycle in your garage is a token of the type of thing known as "The bicycle." Whereas, the bicycle in your garage is in a particular place at a particular time, that is not true of "the bicycle" as used in the sentence: "*The bicycle* has become more popular recently." This distinction is used to clarify the meaning of symbols of formal languages.

History

Metalogical questions have been asked since the time of Aristotle. However, it was only with the rise of formal languages in the late 19th and early 20th century that investigations into the foundations of logic began to flourish. In 1904, David Hilbert observed that in investigating the foundations of mathematics that logical notions are presupposed, and therefore a simultaneous account of metalogical and metamathematical principles was required. Today, metalogic and metamathematics are largely synonymous with each other, and both have been substantially subsumed by mathematical logic in academia.

Results in metalogic

Results in metalogic consist of such things as formal proofs demonstrating the consistency, completeness, and decidability of particular formal systems.

Major results in metalogic include:

- Proof of the uncountability of the set of all subsets of the set of natural numbers (Cantor's theorem 1891)
- Löwenheim–Skolem theorem (Leopold Löwenheim 1915 and Thoralf Skolem 1919)
- Proof of the consistency of truth-functional propositional logic (Emil Post 1920)
- Proof of the semantic completeness of truth-functional propositional logic (Paul Bernays 1918),[4] (Emil Post 1920)[2]
- Proof of the syntactic completeness of truth-functional propositional logic (Emil Post 1920)[2]
- Proof of the decidability of truth-functional propositional logic (Emil Post 1920)[2]
- Proof of the consistency of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the semantic completeness of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the decidability of first order monadic predicate logic (Leopold Löwenheim 1915)
- Proof of the consistency of first order predicate logic (David Hilbert and Wilhelm Ackermann 1928)
- Proof of the semantic completeness of first order predicate logic (Gödel's completeness theorem 1930)
- Proof of the undecidability of first order predicate logic (Church's theorem 1936)
- Gödel's first incompleteness theorem 1931
- Gödel's second incompleteness theorem 1931
- Tarski's undefinability theorem (Gödel and Tarski in the 1930s)

References

[1] Harry Gensler, Introduction to Logic, Routledge, 2001, p. 253.
[2] Hunter, Geoffrey, *Metalogic: An Introduction to the Metatheory of Standard First-Order Logic*, University of California Press, 1971
[3] Rudolf Carnap (1958) *Introduction to Symbolic Logic and its Applications*, p. 102.
[4] Hao Wang, Reflections on Kurt Gödel

Outline of logic

The following outline is provided as an overview of and topical guide to logic:

Logic is the formal science of using reason and is considered a branch of both philosophy and mathematics. Logic investigates and classifies the structure of statements and arguments, both through the study of formal systems of inference and through the study of arguments in natural language. The scope of logic can therefore be very large, ranging from core topics such as the study of fallacies and paradoxes, to specialized analyses of reasoning such as probability, correct reasoning, and arguments involving causality. One of the aims of logic is to identify the correct (or valid) and incorrect (or fallacious) inferences. Logicians study the criteria for the evaluation of arguments.

Foundations of logic

- Analytic-synthetic distinction
- Antinomy
- A priori and a posteriori
- Definition
- Description
- Entailment
- Identity (philosophy)
- Inference
- Logical form
- Logical implication
- Logical truth
- Logical consequence
- Name
- Necessity
- Material conditional
- Meaning (linguistic)
- Meaning (non-linguistic)
- Paradox
- Possible world
- Presupposition
- Probability
- Quantification
- Reason
- Reasoning
- Reference
- Semantics
- Strict conditional
- Syntax (logic)
- Truth

- Truth value
- Validity

Philosophical logic

Philosophical logic –

Informal logic and critical thinking

Informal logic – Critical thinking – Argumentation theory –

- Argument –
- Argument map –
- Accuracy and precision –
- Ad hoc hypothesis –
- Ambiguity –
- Analysis –
- Attacking Faulty Reasoning –
- Belief –
- Belief bias –
- Bias –
- Cogency –
- Cognitive bias –
- Confirmation bias –
- Credibility –
- Critical pedagogy –
- Critical reading –
- Decidophobia –
- Decision making –
- Dispositional and occurrent belief –
- Emotional reasoning –
- Evidence –
- Expert –
- Explanation –
- Explanatory power –
- Fact –
- Fallacy –
- Higher-order thinking –
- Inquiry –
- Interpretive discussion –
- Narrative logic –
- Occam's razor –
- Opinion –
- Practical syllogism –
- Precision questioning –
- Propaganda –
- Propaganda techniques –
- Prudence –
- Pseudophilosophy –

- Reasoning –
- Relevance –
- Rhetoric –
- Rigour –
- Socratic questioning –
- Source credibility –
- Source criticism –
- Theory of justification –
- Topical logic –
- Vagueness –
- Weak mindedness –

Deductive reasoning

Theories of deduction

- Anti-psychologism
- Conceptualism
- Constructivism
- Conventionalism
- Counterpart theory
- Deflationary theory of truth
- Dialetheism
- Fictionalism
- Formalism (philosophy)
- Game theory
- Illuminationist philosophy
- Logical atomism
- Logical holism
- Logicism
- Modal fictionalism
- Nominalism
- Object theory
- Polylogism
- Pragmatism
- Preintuitionism
- Proof theory
- Psychologism
- Ramism
- Semantic theory of truth
- Sophism
- Trivialism
- Ultrafinitism

Fallacies

- Fallacy – In logic and rhetoric, this is usually an incorrect argumentation in reasoning resulting in a misconception or presumption. By accident or design, fallacies may exploit emotional triggers in the listener or interlocutor (appeal to emotion), or take advantage of social relationships between people (e.g. argument from authority). Fallacious arguments are often structured using rhetorical patterns that obscure any logical argument. Fallacies can be used to win arguments regardless of the merits. There are dozens of types of fallacies.

Formal logic

- Formal logic – Mathematical logic, symbolic logic and formal logic are largely, if not completely synonymous. The essential feature of this field is the use of formal languages to express the ideas whose logical validity is being studied.

Symbols and strings of symbols

Logical symbols

- Logical variables
  - Propositional variable
  - Predicate variable
  - Literal
  - Metavariable
- Logical constants
  - Logical connective
  - Quantifier
  - Identity
  - Brackets

Logical connectives

Logical connective –

- Converse implication –
- Converse nonimplication –
- Exclusive or –
- Logical NOR –
- Logical biconditional –
- Logical conjunction –
- Logical disjunction –
- Material implication –
- Material nonimplication –
- Negation –
- Sheffer stroke –

Strings of symbols

- Atomic formula
- Open sentence

Types of propositions

- Analytic proposition
- Axiom
- Atomic sentence
- Clause (logic)
- Contingent proposition
- Contradiction
- Logical truth
- Propositional formula
- Rule of inference
- Sentence (mathematical logic)
- Sequent
- Statement (logic)
- Tautology
- Theorem

Rules of inference

- Biconditional elimination
- Biconditional introduction
- Case analysis
- Commutativity of conjunction
- Conjunction introduction
- Constructive dilemma
- Contraposition (traditional logic)
- Conversion (logic)
- De Morgan's laws
- Destructive dilemma
- Disjunction elimination
- Disjunction introduction
- Disjunctive syllogism
- Double negative elimination
- Generalization (logic)
- Hypothetical syllogism
- Law of excluded middle
- Law of identity
- Modus ponendo tollens
- Modus ponens
- Modus tollens
- Obversion
- Principle of contradiction
- Resolution (logic)
- Simplification
- Transposition (logic)

Formal theories

- Formal proof
- List of first-order theories

Expressions in an object language

- Symbol
- Formula
- Formal system
- Theorem
- Formal proof
- Theory

Expressions in a metalanguage

- Metalinguistic variable
- Deductive system
- Metatheorem
- Metatheory
- Interpretation

Propositional and boolean logic

Propositional logic

- Absorption law
- Clause (logic)
- Deductive closure
- Entailment
- Formation rule
- Functional completeness
- Intermediate logic
- Literal (mathematical logic)
- Logical connective
- Logical consequence
- Negation normal form
- Open sentence
- Propositional calculus
- Propositional formula
- Propositional variable
- Rule of inference
- Strict conditional
- Substitution instance
- Truth table
- Zeroth-order logic

Boolean logic

Boolean algebra Boolean logic Boolean algebra (structure)

- Boolean algebras canonically defined
- Introduction to Boolean algebra
- Complete Boolean algebra
- Free Boolean algebra
- Monadic Boolean algebra
- Residuated Boolean algebra
- Two-element Boolean algebra
- Modal algebra
- Derivative algebra (abstract algebra)
- Relation algebra
- Absorption law
- Laws of Form
- De Morgan's laws
- Algebraic normal form
- Canonical form (Boolean algebra)
- Boolean conjunctive query
- Boolean-valued model
- Boolean domain
- Boolean expression
- Boolean ring
- Boolean function
- Boolean-valued function
- Parity function
- Symmetric Boolean function
- Conditioned disjunction
- Field of sets
- Functional completeness
- Implicant
- Logic alphabet
- Logic redundancy
- Logical connective
- Logical matrix
- Minimal negation operator
- Product term
- True quantified Boolean formula
- Truth table

Predicate logic and relations

Predicate logic

- Atomic formula
- Atomic sentence
- Domain of discourse
- Empty domain
- Extension (predicate logic)
- First-order logic
- First-order predicate
- Formation rule
- Free variables and bound variables
- Generalization (logic)
- Monadic predicate calculus
- Predicate (mathematical logic)
- Predicate logic
- Predicate variable
- Quantification
- Second-order predicate
- Sentence (mathematical logic)
- Universal instantiation
- (ε, δ)-definition of limit

Relations

- Finitary relation
- Antisymmetric relation
- Asymmetric relation
- Bijection
- Bijection, injection and surjection
- Binary relation
- Composition of relations
- Concurrent relation
- Congruence relation
- Coreflexive relation
- Covering relation
- Cyclic order
- Dense relation
- Dependence relation
- Dependency relation
- Directed set
- Equivalence relation
- Euclidean relation
- Homogeneous relation
- Idempotence
- Intransitivity
- Inverse relation
- Involutive relation

- Partial equivalence relation
- Partial function
- Partially ordered set
- Preorder
- Prewellordering
- Propositional function
- Quasitransitive relation
- Reflexive relation
- Surjective function
- Symmetric relation
- Ternary relation
- Total relation
- Transitive relation
- Trichotomy (mathematics)
- Well-founded relation

Mathematical logic

Mathematical logic –

Set theory

Set theory –

- Aleph null
- Bijection, injection and surjection
- Binary set
- Cantor's diagonal argument
- Cantor's first uncountability proof
- Cantor's theorem
- Cardinality of the continuum
- Cardinal number
- Codomain
- Complement (set theory)
- Continuum hypothesis
- Countable set
- Decidable set
- Denumerable set
- Disjoint sets
- Disjoint union
- Domain of a function
- Effective enumeration
- Element (mathematics)
- Empty function
- Empty set
- Enumeration
- Extensionality
- Finite set
- Function (mathematics)

- Function composition
- Generalized continuum hypothesis
- Index set
- Infinite set
- Intension
- Intersection (set theory)
- Inverse function
- Löwenheim–Skolem theorem
- Map (mathematics)
- Multiset
- Naïve set theory
- Non-Cantorian set theory
- One to one correspondence
- Ordered pair
- Partition of a set
- Pointed set
- Power set
- Projection (set theory)
- Proper subset
- Proper superset
- Range (mathematics)
- Russell's paradox
- Sequence (mathematics)
- Set (mathematics)
- Set of all sets
- Simple theorems in the algebra of sets
- Singleton (mathematics)
- Skolem paradox
- Subset
- Superset
- Tuple
- Uncountable set
- Union (set theory)
- Zermelo–Fraenkel set theory

Metalogic

Metalogic – The study of the metatheory of logic.

- Completeness
- Syntax (logic)
- Consistency
- Decidability (logic)
- Deductive system
- Interpretation (logic)
- Cantor's theorem
- Church's theorem
- Church's thesis
- Effective method

- Formal system
- Gödel's completeness theorem
- Gödel's first incompleteness theorem
- Gödel's second incompleteness theorem
- Independence (mathematical logic)
- Logical consequence
- Löwenheim-Skolem theorem
- Metalanguage
- Metasyntactic variable
- Metatheorem
- Object language
- Symbol (formal)
- Type-token distinction
- Use–mention distinction
- Well-formed formula

Proof theory

Proof theory – The study of deductive apparatus.

- Axiom
- Deductive system
- Formal proof
- Formal system
- Formal theorem
- Syntactic consequence
- Syntax (logic)
- Transformation rules

Model theory

Model theory – The study of interpretation of formal systems.

- Interpretation (logic)
- Logical validity
- Non-standard model
- Normal model
- Model
- Semantic consequence
- Truth value

Computability theory

Computability theory – branch of mathematical logic that originated in the 1930s with the study of computable functions and Turing degrees. The field has grown to include the study of generalized computability and definability. The basic questions addressed by recursion theory are "What does it mean for a function from the natural numbers to themselves to be computable?" and "How can noncomputable functions be classified into a hierarchy based on their level of noncomputability?". The answers to these questions have led to a rich theory that is still being actively researched.

- Alpha recursion theory
- Arithmetical set

- Church–Turing thesis
- Computability logic
- Computable function
- Computation
- Decision problem
- Effective method
- Entscheidungsproblem
- Enumeration
- Forcing (recursion theory)
- Halting problem
- History of the Church–Turing thesis
- Lambda calculus
- List of undecidable problems
- Post correspondence problem
- Post's theorem
- Primitive recursive function
- Recursion (computer science)
- Recursive language
- Recursive languages and sets
- Recursive set
- Recursively enumerable language
- Recursively enumerable set
- Reduction (recursion theory)
- Turing machine

Classical logic

Classical logic –

- Properties of classical logics:
  - Law of the excluded middle
  - Double negative elimination
  - Law of noncontradiction
  - Principle of explosion
  - Monotonicity of entailment
  - Idempotency of entailment
  - Commutativity of conjunction
  - De Morgan duality – every logical operator is dual to another
- Term logic
- General concepts in classical logic
  - Baralipton
  - Baroco
  - Bivalence
  - Boolean logic
  - Boolean-valued function
  - Categorical proposition
  - Distribution of terms
  - End term

- Enthymeme
- Immediate inference
- Law of contraries
- Logical connective
- Major term
- Middle term
- Minor term
- Organon
- Polysyllogism
- Port-Royal Logic
- Premise
- Prior Analytics
- Relative term
- Sorites paradox
- Square of opposition
- Sum of Logic
- Syllogism
- Tetralemma
- Truth function

Non-classical logic

Non-classical logic – Deviant logic –

- Computability logic –
- Fuzzy logic –
- Linear logic –
- Decision theory –
- Game theory –
- Probability theory –
- Affine logic –
- Bunched logic –
- Description logic –
- Free logic –
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Modal logic

Modal logic –

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- Epistemic logic –
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- Abductive reasoning –

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- Set theory –
- Formal system –
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    - Predicate –
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  - Propositional calculus –
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- Boolean algebra –
  - Boolean logic –
  - Truth value –
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- Categorical logic –
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- Linear logic –
- Metalogic –
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- Temporal logic –
- Sequential logic –
- Provability logic –
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- Relevant logic –
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- Theorem –
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- Axiomatic system –
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- Conditional proof –
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- Degree of truth –
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- Truth condition –
- Truth function –
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- Fallacy –
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- Type theory –
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- Tautology –
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- Logical conditional –
- Logical biconditional –
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- Reductio ad absurdum –
- Rhetoric –
- Self-reference –
- Necessary and sufficient –
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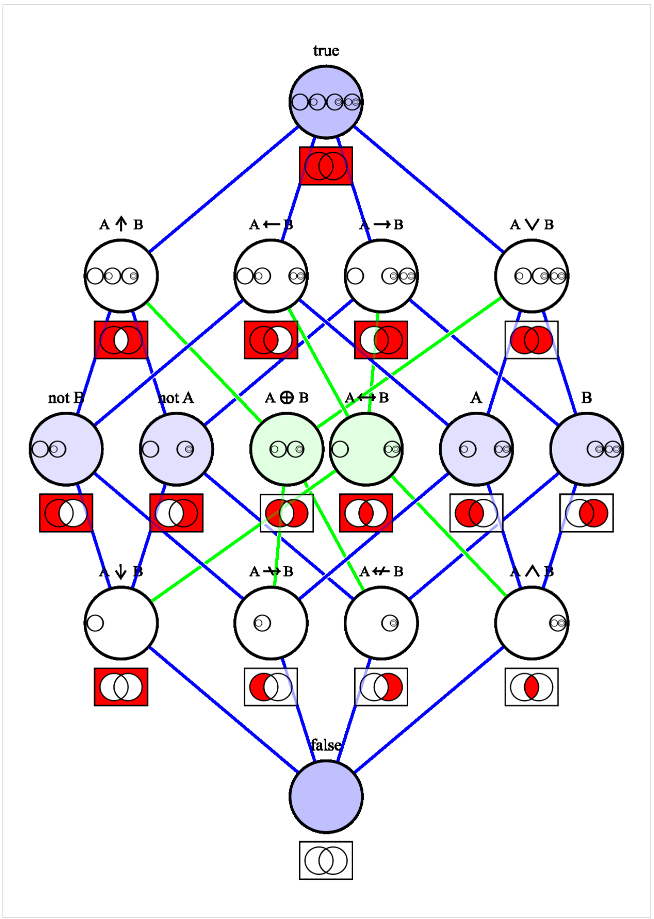
- Nonfirstorderizability –



- Occam's Razor –
- Socratic dialoge –
- Socratic method –
- Argument form –
- Logic programming –
- Unification –

Literature

- A System of Logic
- Association for Symbolic Logic
- Attacking Faulty Reasoning
- Begriffsschrift
- Categories (Aristotle)
- Charles Sanders Peirce bibliography
- De Interpretatione
- Gödel, Escher, Bach
- Introduction to Mathematical Philosophy
- Journal of Logic, Language and Information
- Journal of Philosophical Logic
- Language, Truth, and Logic
- Laws of Form
- Linguistics and Philosophy
- Logic Made Easy
- Metamagical Themas
- Minds, Machines and Gödel
- Novum Organum
- On Formally Undecidable Propositions of Principia Mathematica and Related Systems
- Organon
- Philosophical Investigations
- Philosophy of Arithmetic (book)
- Polish Logic
- Port-Royal Logic
- Posterior Analytics
- Principia Mathematica
- Principles of Mathematical Logic
- Prior Analytics
- Rhetoric (Aristotle)
- Sophistical Refutations
- Sum of Logic
- The Art of Being Right
- The Foundations of Arithmetic
- The Logic of Scientific Discovery
- Topics (Aristotle)
- Tractatus Logico-Philosophicus
- What the Tortoise Said to Achilles
- Where Mathematics Comes From

Lists

- List of Boolean algebra topics
- List of mathematical logic topics
- List of set theory topics
- Index of logic articles
- List of fallacies
- List of logicians
- List of paradoxes
- List of philosophers of language
- List of rules of inference
- For introductory set theory and other supporting material see the outline of discrete mathematics.

External links

- *Taxonomy of Logical Fallacies* [1]
- *An Introduction to Philosophical Logic* [2], by Paul Newall, aimed at beginners
- *forall x: an introduction to formal logic* [3], by P.D. Magnus, covers sentential and quantified logic
- *Translation Tips* [4], by Peter Suber, for translating from English into logical notation
- Math & Logic: The history of formal mathematical, logical, linguistic and methodological ideas. [5] In *The Dictionary of the History of Ideas.*
- *Logic test* [6] Test your logic skills
- *Logic Self-Taught: A Workbook* [7] (originally prepared for on-line logic instruction)

References

[1] http://www.fallacyfiles.org/taxonomy.html
[2] http://www.galilean-library.org/int4.html
[3] http://www.fecundity.com/logic/
[4] http://www.earlham.edu/~peters/courses/log/transtip.htm
[5] http://etext.lib.virginia.edu/DicHist/analytic/anaVII.html
[6] http://www.think-logically.co.uk/lt.htm
[7] http://kpaprzycka.swps.edu.pl/xLogicSelfTaught/LogicSelfTaught.html

List of logic journals

This is a list of academic journals in logic. *See also:* ***List of philosophy journals***.

- *Acta Philosophica Fennica*, Helsinki 1935 ff.
- *Annals of Mathematical Logic*, Vols 1–23, 1970–1982.
- *Annals of Pure and Applied Logic*, 1983 ff. (Successor of the *Annals of Mathematical Logic*).
- *Annals of the Japan Association for the Philosophy of Science*, Tokyo 1956/1957 ff.
- *Analysis*, Oxford 1933/34 ff.
- *Archiv für mathematische Logik und Grundlagenforschung*, vols 1–26, Stuttgart-Berlin-Köln 1950–1987.
- *Archive for Mathematical Logic*, Berlin-Heidelberg 1988 ff. vol. 27 ff. (Successor of the former).
- *Argumentation*. An International Journal on Reasoning, Dordrecht 1987ff.
- *Australasian Journal of Logic*, Melbourne 2003 ff. Electronic Edition.
- *Dialectica*. International Review of Philosophy of Knowledge/Revue Internationale de Philosophie de la Connaissance/Internationale Zeitschrift für Philosophie der Erkenntnis, Neuchâtel-Paris 1947 ff., Vols. 20 ff. Lausanne.
- *Forschungen zur Logistik und zur Grundlegung der exakten Wissenschaften*, Leipzig 1934–1943, Repr. in 1 vol. Hildesheim 1970.
- *History and Philosophy of Logic*, London 1979 ff.
- *Inquiry*. An Interdisciplinary Journal of Philosophy, London 1958 ff.
- *Informal Logic*. Reasoning and Argumentation in Theory and Practice, 1990 ff
- *International Journal of Logic and Computation* 2010 ff.
- *International Logic Review*, Bologna 1970 ff.
- *Journal of Applied Non-Classical Logics*, 1991 ff.
- *Journal of Automated Reasoning*
- *Journal of Logic and Computation*, Oxford 1990 ff.
- *Journal of Logic, Language and Information*, 1992 ff.
- *Journal of Logic Programming*, (Elsevir Publ.) 1984–2000. Continued by **Theory and Practice of Logic*
- *Programming and The Journal of Logic and Algebraic Programming*.
- *Journal of Mathematical Logic*, 2001 ff.
- *Journal of Non-Classical Logic*, 1982–1991.
- *Journal of Multiple-Valued Logic and Soft Computing*, 1994 ff.
- *Journal of Philosophical Logic*, Toronto- Dordrecht 1972 ff.
- *Journal of Symbolic Logic*, Providence, Rhode Island 1936 ff.
- *Linguistics and Philosophy*
- *Logic Journal of the IGPL*, Oxford 1993 ff.
- *Logica Universalis*
- *Logical Analysis and History of Philosophy / Philosophiegeschichte und Logische Analyse*, Bonn 1998 ff.
- *Logique et Analyse*, Nouvelle Serie, Löwen-Paris 1958 ff.
- *Mathematical Logic Quarterly*, Weinheim 1993 ff. (Successor of *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* vols 1 –38, 1955–1992).
- *Methodos*. Rivista trimestriale de Metodologia e di Logica Simbolica, Milano 1949 ff.
- *Miscellanea Logica*
- *Modern Logic*, vols 1–7, 1990–1997.
- *Multiple-Valued Logic*. An International Journal, 1996 ff.
- *Nordic Journal of Philosophical Logic*
- *Notes on Linguistics*

- *Notre Dame Journal of Formal Logic*, Notre Dame, Ind., USA 1960 ff.
- *Philosophia Mathematica*. An International Journal for Philosophy of Modern Mathematics, Hauppauge, N. Y. 1964–1980/1981. 2nd Series 1986–1991 (6 vols). 3rd Series Oxford 1993 ff.
- *Prace z Logiki*, Kracow 1965 ff.
- *Philosophiegeschichte und logische Analyse / Logical Analysis and History of Philosophy*, Paderborn 1998 ff.
- *Proceedings of the Aristotelian Society*, London 1900 ff.
- *Quality and Quantity*. European Journal of Methodology, Bologna 1967 ff.
- *Rassegna Internazionale di Logica / International Logic Review*, Bologna 1970 ff.
- *Rechtstheorie*. Zeitschrift für Logik, Methodenlehre, Kybernetik und Soziologie des Rechts, Berlin- München 1970 ff.
- *Revue d'historie des mathématiques*, Marseille 1995 ff.
- *Scripta Mathematica*. A Quarterly Journal devoted to the Philosophy, History and Expository Treatment of Mathematics, New York 1932 ff.
- *Studia Logica*, Warszawa 1953 ff. Since vol. 54, Studia Logica. An International Journal for Symbolic Logic, Dordrecht 1995 ff. Studia Logica, Prague 1974 and 1975 (only 2 vols). Studies in Logic and the Foundations of Mathematics, Amsterdam 1951 ff.
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- *teorema*, Valencia 1971 ff.
- *The Bulletin of Symbolic Logic*, Champeigne, Ill. 1995 ff. (Electronic Journal of the Cornell University Library).
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- *Quarterly Journal of Mathematics*, Oxford 1930 – vol. 20, 1949. 2nd Series 1950 ff.
- *Theoria (journal)*. Revista de Teoria, Historia y Fundamentos de la Ciencia, San Sebastian 1962 ff.
- *Theory and Decision*. An International Journal for Philosophy and Methodology of the Social Sciences, Dordrecht 1970 ff.
- *Theory and Practice of Logic Programming*, 2001 ff.
- *Zeitschrift für allgemeine Wissenschaftstheorie*, Wiesbaden 1970 ff, Journal for General Philosophy of Science / Zeitschrift für allgemeine Wissenschaftstheorie, Vol 21 ff, Dordrecht 1990 ff
- *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vols. 1 –38, Leipzig 1955–1992. Continued as *Mathematical Logic Quarterly*, vol. 39 ff, 1993 ff..

Truth


Time Saving Truth from Falsehood and Envy, François Lemoyne, 1737

Truth is most often used to mean in accord with fact or reality[1] or fidelity to an original or to a standard or ideal.[1]

The opposite of truth is falsehood, which, correspondingly, can also take on a logical, factual, or ethical meaning. The concept of truth is discussed and debated in several contexts, including philosophy and religion. Many human activities depend upon the concept, which is assumed rather than a subject of discussion, including science, law, and everyday life.

Various theories and views of truth continue to be debated among scholars and philosophers. Language and words are a means by which humans convey information to one another and the method used to recognize a "truth" is termed a criterion of truth. There are differing claims on such questions as what constitutes truth: what things are truthbearers capable of being true or false; how to define and identify truth; the roles that revealed and acquired knowledge play; and whether truth is subjective or objective, relative or absolute.

Many religions consider perfect knowledge of all truth about all things (omniscience) to be an attribute of a divine or supernatural being.

Nomenclature, orthography, and etymology

The English word *truth* is from Old English *tríewþ, tréowþ, trýwþ*, Middle English *trewþe*, cognate to Old High German *triuwida*, Old Norse *tryggð*. Like *troth*, it is a *-th* nominalisation of the adjective *true* (Old English *tréowe*).

The English word *true* is from Old English (West Saxon) *(ge)tríewe, tréowe*, cognate to

Old Saxon *(gi)trûui*, Old High German *(ga)triuwu* (Modern German *treu* "faithful"), Old Norse *tryggr*, Gothic *triggws*,[2] all from a Proto-Germanic **trewwj-* "having good faith". Old Norse *trú*, "faith, word of honour; religious faith, belief"[3] (archaic English *troth* "loyalty, honesty, good faith", compare *Ásatrú*).

Truth, holding a mirror and a serpent (1896). Olin Levi Warner, Library of Congress Thomas Jefferson Building, Washington, D.C.

Thus, 'truth' involves both the quality of "faithfulness, fidelity, loyalty, sincerity, veracity",[4] and that of "agreement with fact or reality", in Anglo-Saxon expressed by *sōþ* (Modern English *sooth*).

All Germanic languages besides English have introduced a terminological distinction between truth "fidelity" and truth "factuality". To express "factuality", North Germanic opted for nouns derived from *sanna* "to assert, affirm", while continental West Germanic (German and Dutch) opted for continuations of *wâra* "faith, trust, pact" (cognate to Slavic *věra* "(religious) faith", but influenced by Latin *verus*). Romance languages use terms following the Latin *veritas*, while the Greek *aletheia*, Russian *pravda* and Serbian *istina* have separate etymological origins.

Major theories of truth

The question of what is a proper basis for deciding how words, symbols, ideas and beliefs may properly be considered true, whether by a single person or an entire society, is dealt with by the five major substantive theories introduced below. Each theory presents perspectives that are widely shared by published scholars.[5][6][7] There also have more recently arisen "deflationary" or "minimalist" theories of truth based on the idea that the application of a term like *true* to a statement does not assert anything significant about it, for instance, anything about its *nature*, but that the label *truth* is a tool of discourse used to express agreement, to emphasize claims, or to form certain types of generalizations.[5][8][9]

Substantive theories

Correspondence theory

Correspondence theories state that true beliefs and true statements correspond to the actual state of affairs.[10] This type of theory posits a relationship between thoughts or statements on one hand, and things or objects on the other. It is a traditional model which goes back at least to some of the classical Greek philosophers such as Socrates, Plato, and Aristotle.[11] This class of theories holds that the truth or the falsity of a representation is determined in principle solely by how it relates to "things", by whether it accurately describes those "things". An example of correspondence theory is the statement by the Thirteenth Century philosopher/theologian Thomas Aquinas: *Veritas est adaequatio rei et intellectus* ("Truth is the equation [or adequation] of things and intellect"), a statement which Aquinas attributed to the Ninth Century neoplatonist Isaac Israeli.[12][13] Aquinas also restated the theory as: "A judgment is said to be true when it conforms to the external reality"[14]

Correspondence theory practically operates on the assumption that truth is a matter of accurately copying what was much later called "objective reality" and then representing it in thoughts, words and other symbols.[15] Many modern theorists have stated that this ideal cannot be achieved independently of some analysis of additional factors.[5][16] For example, language plays a role in that all languages have words that are not easily translatable into another. The German word *Zeitgeist* is one such example: one who speaks or understands the language may "know" what it means, but any translation of the word apparently fails to accurately capture its full meaning (this is a problem with many abstract words, especially those derived in agglutinative languages). Thus, some words add an additional parameter to the construction of an accurate truth predicate. Among the philosophers who grappled with this problem is Alfred Tarski, whose semantic theory is summarized further below in this article.[17]

Proponents of several of the theories below have gone further to assert that there are yet other issues necessary to the analysis, such as interpersonal power struggles, community interactions, personal biases and other factors involved in deciding what is seen as truth.

Coherence theory

Walter Seymour Allward's *Veritas* (Truth) outside Supreme Court of Canada, Ottawa, Ontario Canada

For coherence theories in general, truth requires a proper fit of elements within a whole system. Very often, though, coherence is taken to imply something more than simple logical consistency; often there is a demand that the propositions in a coherent system lend mutual inferential support to each other. So, for example, the completeness and comprehensiveness of the underlying set of concepts is a critical factor in judging the validity and usefulness of a coherent system.[18] A pervasive tenet of coherence theories is the idea that truth is primarily a property of whole systems of propositions, and can be ascribed to individual propositions only according to their coherence with the whole. Among the assortment of perspectives commonly regarded as coherence theory, theorists differ on the question of whether coherence entails many possible true systems of thought or only a single absolute system.

Some variants of coherence theory are claimed to characterize the essential and intrinsic properties of formal systems in logic and mathematics.[19] However, formal reasoners are content to contemplate axiomatically independent and sometimes mutually contradictory systems side by side, for example, the various alternative geometries. On the whole, coherence theories have been criticized as lacking justification in their application to other areas of truth, especially with respect to assertions about the natural world, empirical data in general, assertions about practical matters of psychology and society, especially when used without support from the other major theories of truth.[20]

Coherence theories distinguish the thought of rationalist philosophers, particularly of Spinoza, Leibniz, and G.W.F. Hegel, along with the British philosopher F.H. Bradley.[21] They have found a resurgence also among several proponents of logical positivism, notably Otto Neurath and Carl Hempel.

Constructivist theory

Social constructivism holds that truth is constructed by social processes, is historically and culturally specific, and that it is in part shaped through the power struggles within a community. Constructivism views all of our knowledge as "constructed," because it does not reflect any external "transcendent" realities (as a pure correspondence theory might hold). Rather, perceptions of truth are viewed as contingent on convention, human perception, and social experience. It is believed by constructivists that representations of physical and biological reality, including race, sexuality, and gender are socially constructed.

Giambattista Vico was among the first to claim that history and culture were man-made. Vico's epistemological orientation gathers the most diverse rays and unfolds in one axiom – *verum ipsum factum* – "truth itself is constructed". Hegel and Marx were among the other early proponents of the premise that truth is, or can be, socially constructed. Marx, like many critical theorists who followed, did not reject the existence of objective truth but rather distinguished between true knowledge and knowledge that has been distorted through power or ideology. For Marx scientific and true knowledge is 'in accordance with the dialectical understanding of history' and ideological knowledge 'an epiphenomenal expression of the relation of material forces in a given economic arrangement'.[22]

Consensus theory

Consensus theory holds that truth is whatever is agreed upon, or in some versions, might come to be agreed upon, by some specified group. Such a group might include all human beings, or a subset thereof consisting of more than one person.

Among the current advocates of consensus theory as a useful accounting of the concept of "truth" is the philosopher Jürgen Habermas.[23] Habermas maintains that truth is what would be agreed upon in an ideal speech situation.[24] Among the current strong critics of consensus theory is the philosopher Nicholas Rescher.[25]

Pragmatic theory

The three most influential forms of the *pragmatic theory of truth* were introduced around the turn of the 20th century by Charles Sanders Peirce, William James, and John Dewey. Although there are wide differences in viewpoint among these and other proponents of pragmatic theory, they hold in common that truth is verified and confirmed by the results of putting one's concepts into practice.[26]

Peirce defines truth as follows: "Truth is that concordance of an abstract statement with the ideal limit towards which endless investigation would tend to bring scientific belief, which concordance the abstract statement may possess by virtue of the confession of its inaccuracy and one-sidedness, and this confession is an essential ingredient of truth."[27] This statement emphasizes Peirce's view that ideas of approximation, incompleteness, and partiality, what he describes elsewhere as *fallibilism* and "reference to the future", are essential to a proper conception of truth. Although Peirce uses words like *concordance* and *correspondence* to describe one aspect of the pragmatic sign relation, he is also quite explicit in saying that definitions of truth based on mere correspondence are no more than *nominal* definitions, which he accords a lower status than *real* definitions.

William James's version of pragmatic theory, while complex, is often summarized by his statement that "the 'true' is only the expedient in our way of thinking, just as the 'right' is only the expedient in our way of behaving."[28] By this, James meant that truth is a *quality*, the value of which is confirmed by its effectiveness when applying concepts to practice (thus, "pragmatic").

John Dewey, less broadly than James but more broadly than Peirce, held that inquiry, whether scientific, technical, sociological, philosophical or cultural, is self-corrective over time *if* openly submitted for testing by a community of inquirers in order to clarify, justify, refine and/or refute proposed truths.[29]

Though not widely publicized, a new variation of the pragmatic theory was defined and wielded successfully from the 20th century forward. Defined and named by William Ernest Hocking, this variation is known as "negative pragmatism". Essentially, what works may or may not be true, but what fails cannot be true because the truth always works.[30] Richard Feynman also ascribed to it: "We never are definitely right, we can only be sure we are wrong."[31] This approach incorporates many of the ideas from Peirce, James, and Dewey. For Peirce, the idea of "... endless investigation would tend to bring about scientific belief ..." fits negative pragmatism in that a negative pragmatist would never stop testing. As Feynman noted, an idea or theory "... could never be proved right, because tomorrow's experiment might succeed in proving wrong what you thought was right."[31] Similarly, James and Dewey's ideas also ascribe to repeated testing which is "self-corrective" over time.

Pragmatism and negative pragmatism are also closely aligned with the coherence theory of truth in that any testing should not be isolated but rather incorporate knowledge from all human endeavors and experience. The universe is a whole and integrated system, and testing should recognize and account for its diversity. As Feynman said, "... if it disagrees with experiment, it is wrong."[32]

Minimalist (deflationary) theories

Some philosophers reject the thesis that the concept or term *truth* refers to a real property of sentences or propositions. These philosophers are responding, in part, to the common use of *truth predicates* (e.g., that some particular thing "...is true") which was particularly prevalent in philosophical discourse on truth in the first half of the 20th century. From this point of view, to assert the proposition "'2 + 2 = 4' is true" is logically equivalent to asserting the proposition "2 + 2 = 4", and the phrase "is true" is completely dispensable in this and every other context. These positions are broadly described

- as *deflationary* theories of truth, since they attempt to deflate the presumed importance of the words "true" or *truth*,
- as *disquotational* theories, to draw attention to the disappearance of the quotation marks in cases like the above example, or
- as *minimalist* theories of truth.[5][33]

Whichever term is used, deflationary theories can be said to hold in common that "[t]he predicate 'true' is an expressive convenience, not the name of a property requiring deep analysis."[5] Once we have identified the truth predicate's formal features and utility, deflationists argue, we have said all there is to be said about truth. Among the theoretical concerns of these views is to explain away those special cases where it *does* appear that the concept of truth has peculiar and interesting properties. (See, e.g., Semantic paradoxes, and below.)

In addition to highlighting such formal aspects of the predicate "is true", some deflationists point out that the concept enables us to express things that might otherwise require infinitely long sentences. For example, one cannot express confidence in Michael's accuracy by asserting the endless sentence:

> *Michael says, 'snow is white' and snow is white, or he says 'roses are red' and roses are red or he says ... etc.*

This assertion can also be succinctly expressed by saying: *What Michael says is true*.[34]

Performative theory of truth

Attributed to P. F. Strawson is the performative theory of truth which holds that to say "'Snow is white' is true" is to perform the speech act of signaling one's agreement with the claim that snow is white (much like nodding one's head in agreement). The idea that some statements are more actions than communicative statements is not as odd as it may seem. Consider, for example, that when the bride says "I do" at the appropriate time in a wedding, she is performing the act of taking this man to be her lawful wedded husband. She is not *describing* herself as taking this man, but actually doing so (perhaps the most thorough analysis of such "illocutionary acts" is J. L. Austin, "How to Do Things With Words"[35]).

Strawson holds that a similar analysis is applicable to all speech acts, not just illocutionary ones: "To say a statement is true is not to make a statement about a statement, but rather to perform the act of agreeing with, accepting, or endorsing a statement. When one says 'It's true that it's raining,' one asserts no more than 'It's raining.' The function of [the statement] 'It's true that...' is to agree with, accept, or endorse the statement that 'it's raining.'"[36]

Redundancy and related theories

According to the redundancy theory of truth, asserting that a statement is true is completely equivalent to asserting the statement itself. For example, making the assertion that " 'Snow is white' is true" is equivalent to asserting "Snow is white". Redundancy theorists infer from this premise that truth is a redundant concept; that is, it is merely a word that is traditionally used in conversation or writing, generally for emphasis, but not a word that actually equates to anything in reality. This theory is commonly attributed to Frank P. Ramsey, who held that the use of words like *fact* and *truth* was nothing but a roundabout way of asserting a proposition, and that treating these words as separate problems in isolation from judgment was merely a "linguistic muddle".[5][37][38]

A variant of redundancy theory is the disquotational theory which uses a modified form of Tarski's schema: To say that '"P" is true' is to say that P. A version of this theory was defended by C. J. F. Williams in his book *What is*

Truth?. Yet another version of deflationism is the prosentential theory of truth, first developed by Dorothy Grover, Joseph Camp, and Nuel Belnap as an elaboration of Ramsey's claims. They argue that sentences like "That's true", when said in response to "It's raining", are prosentences, expressions that merely repeat the content of other expressions. In the same way that *it* means the same as *my dog* in the sentence *My dog was hungry, so I fed it*, *That's true* is supposed to mean the same as *It's raining* — if you say the latter and I then say the former. These variations do not necessarily follow Ramsey in asserting that truth is *not* a property, but rather can be understood to say that, for instance, the assertion "P" may well involve a substantial truth, and the theorists in this case are minimalizing only the redundancy or prosentence involved in the statement such as "that's true."[5]

Deflationary principles do not apply to representations that are not analogous to sentences, and also do not apply to many other things that are commonly judged to be true or otherwise. Consider the analogy between the sentence "Snow is white" and the character named Snow White, both of which can be true in some sense. To a minimalist, saying "Snow is white is true" is the same as saying "Snow is white," but to say "Snow White is true" is *not* the same as saying "Snow White."

Pluralist theories

Several of the major theories of truth hold that there is a particular property the having of which makes a belief or proposition true. Pluralist theories of truth assert that there may be more than one property that makes propositions true: ethical propositions might be true by virtue of coherence. Propositions about the physical world might be true by corresponding to the objects and properties they are about.

Some of the pragmatic theories, such as those by Charles Peirce and William James, included aspects of correspondence, coherence and constructivist theories.[27][28] Crispin Wright argued in his 1992 book *Truth and Objectivity* that any predicate which satisfied certain platitudes about truth qualified as a truth predicate. In some discourses, Wright argued, the role of the truth predicate might be played by the notion of superassertibility.[39] Michael Lynch, in a 2009 book *Truth as One and Many*, argued that we should see truth as a functional property capable of being multiply manifested in distinct properties like correspondence or coherence.[40]

Most believed theories

According to a survey of professional philosophers and others on their philosophical views which was carried out in November 2009 (taken by 3226 respondents, including 1803 philosophy faculty members and/or PhDs and 829 philosophy graduate students) 44.9% of respondents accept or lean towards correspondence theories, 20.7% accept or lean towards deflationary theories and 13.8% epistemic theories.[41]

Formal theories

Truth in logic

Logic is concerned with the patterns in reason that can help tell us if a proposition is true or not. However, logic does not deal with truth in the absolute sense, as for instance a metaphysician does. Logicians use formal languages to express the truths which they are concerned with, and as such there is only truth under some interpretation or truth within some logical system.

A logical truth (also called an analytic truth or a necessary truth) is a statement which is true in all possible worlds[42] or under all possible interpretations, as contrasted to a *fact* (also called a *synthetic claim* or a *contingency*) which is only true in this world as it has historically unfolded. A proposition such as "If p and q, then p." is considered to be logical truth because it is true because of the meaning of the symbols and words in it and not because of any facts of any particular world. They are such that they could not be untrue.

Truth in mathematics

There are two main approaches to truth in mathematics. They are the *model theory of truth* and the *proof theory of truth*.

Historically, with the nineteenth century development of Boolean algebra mathematical models of logic began to treat "truth", also represented as "T" or "1", as an arbitrary constant. "Falsity" is also an arbitrary constant, which can be represented as "F" or "0". In propositional logic, these symbols can be manipulated according to a set of axioms and rules of inference, often given in the form of truth tables.

In addition, from at least the time of Hilbert's program at the turn of the twentieth century to the proof of Gödel's incompleteness theorems and the development of the Church-Turing thesis in the early part of that century, true statements in mathematics were generally assumed to be those statements which are provable in a formal axiomatic system.

The works of Kurt Gödel, Alan Turing, and others shook this assumption, with the development of statements that are true but cannot be proven within the system.[43] Two examples of the latter can be found in Hilbert's problems. Work on Hilbert's 10th problem led in the late twentieth century to the construction of specific Diophantine equations for which it is undecidable whether they have a solution,[44] or even if they do, whether they have a finite or infinite number of solutions. More fundamentally, Hilbert's first problem was on the continuum hypothesis.[45] Gödel and Paul Cohen showed that this hypothesis cannot be proved or disproved using the standard axioms of set theory.[46] In the view of some, then, it is equally reasonable to take either the continuum hypothesis or its negation as a new axiom.

Semantic theory of truth

The semantic theory of truth has as its general case for a given language:

> 'P' is true if and only if P

where 'P' is a reference to the sentence (the sentence's name), and P is just the sentence itself.

Logician and philosopher Alfred Tarski developed the theory for formal languages (such as formal logic). Here he restricted it in this way: no language could contain its own truth predicate, that is, the expression *is true* could only apply to sentences in some other language. The latter he called an *object language*, the language being talked about. (It may, in turn, have a truth predicate that can be applied to sentences in still another language.) The reason for his restriction was that languages that contain their own truth predicate will contain paradoxical sentences like the Liar: *This sentence is not true*. See The Liar paradox. As a result Tarski held that the semantic theory could not be applied to any natural language, such as English, because they contain their own truth predicates. Donald Davidson used it as the foundation of his truth-conditional semantics and linked it to radical interpretation in a form of coherentism.

Bertrand Russell is credited with noticing the existence of such paradoxes even in the best symbolic formalizations of mathematics in his day, in particular the paradox that came to be named after him, Russell's paradox. Russell and Whitehead attempted to solve these problems in *Principia Mathematica* by putting statements into a hierarchy of types, wherein a statement cannot refer to itself, but only to statements lower in the hierarchy. This in turn led to new orders of difficulty regarding the precise natures of types and the structures of conceptually possible type systems that have yet to be resolved to this day.

Kripke's theory of truth

Saul Kripke contends that a natural language can in fact contain its own truth predicate without giving rise to contradiction. He showed how to construct one as follows:

- Begin with a subset of sentences of a natural language that contains no occurrences of the expression "is true" (or "is false"). So *The barn is big* is included in the subset, but not " *The barn is big* is true", nor problematic sentences such as "*This sentence* is false".
- Define truth just for the sentences in that subset.
- Then extend the definition of truth to include sentences that predicate truth or falsity of one of the original subset of sentences. So "*The barn is big* is true" is now included, but not either "*This sentence* is false" nor "'*The barn is big* is true' is true".
- Next, define truth for all sentences that predicate truth or falsity of a member of the second set. Imagine this process repeated infinitely, so that truth is defined for *The barn is big*; then for "*The barn is big* is true"; then for "'*The barn is big* is true' is true", and so on.

Notice that truth never gets defined for sentences like *This sentence is false*, since it was not in the original subset and does not predicate truth of any sentence in the original or any subsequent set. In Kripke's terms, these are "ungrounded." Since these sentences are never assigned either truth or falsehood even if the process is carried out infinitely, Kripke's theory implies that some sentences are neither true nor false. This contradicts the Principle of bivalence: every sentence must be either true or false. Since this principle is a key premise in deriving the Liar paradox, the paradox is dissolved.[47]

Notable views

Ancient history

La Vérité "Truth" by Jules Joseph Lefebvre

The ancient Greek origins of the words "true" and "truth" have some consistent definitions throughout great spans of history that were often associated with topics of logic, geometry, mathematics, deduction, induction, and natural philosophy.

Socrates', Plato's and Aristotle's ideas about truth are commonly seen as consistent with correspondence theory. In his *Metaphysics*, Aristotle stated: "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true".[48] The Stanford Encyclopedia of Philosophy proceeds to say of Aristotle:

> *"(...) Aristotle sounds much more like a genuine correspondence theorist in the* Categories *(12b11, 14b14), where he talks of "underlying things" that make statements true and implies that these "things" (pragmata) are logically structured situations or facts (viz., his sitting, his not sitting). Most influential is his claim in* De Interpretatione *(16a3) that thoughts are "likenessess" (homoiosis) of things. Although he nowhere defines truth in terms of a thought's likeness to a thing or fact, it is clear that such a definition would fit well into his overall philosophy of mind. (...)"*[48]

Very similar statements can also be found in Plato (Cratylus 385b2, Sophist 263b).[48]

In Hinduism, Truth is defined as "unchangeable", "that which has no distortion", "that which is beyond distinctions of time, space, and person", "that which pervades the universe in all its constancy". Human body, therefore is not completely true as it changes with time, for example. There are many references, properties and explanations of truth by Hindu sages that explain varied facets of truth, such as "Satyam eva jayate" (Truth alone wins), "Satyam muktaye" (Truth liberates), "Satya' is 'Parahit'artham' va'unmanaso yatha'rthatvam' satyam" (Satya is the benevolent use of words and the mind for the welfare of others or in other words responsibilities is truth too), "When one is firmly established in speaking truth, the fruits of action become subservient to him (patanjali yogasutras, sutra number 2.36), "The face of truth is covered by a golden bowl. *Unveil it, O Pusan (Sun), so that I who have truth as my duty (satyadharma) may see it!*" (Brhadaranyaka V 15 1-4 and the brief IIsa Upanisad 15-18), Truth is superior to silence (Manusmriti), etc. Combined with other words, satya acts as modifier, like "**ultra**" or "**highest**," or more literally "**truest**," connoting **purity and excellence**. For example, satyaloka is the "highest heaven' and Satya Yuga is the "golden age" or best of the four cyclical cosmic ages in Hinduism, and so on.

Medieval age

Avicenna

In early Islamic philosophy, Avicenna (Ibn Sina) defined truth in his Metaphysics of Healing, Book I, Chapter 8, as:

"What corresponds in the mind to what is outside it."Osman Amin (2007), "Influence of Muslim Philosophy on the West", Monthly Renaissance 17 (11).

Avicenna elaborated on his definition of truth in his *Metaphysics* Book Eight, Chapter 6:

"The truth of a thing is the property of the being of each thing which has been established in it."Jan A. Aertsen (1988), Nature and Creature: Thomas Aquinas's Way of Thought, p. 152. BRILL, ISBN 90-04-08451-7.

However, this definition is merely a translation of the Latin translation from the Middle Ages.[51] A modern translation of the original Arabic text states:

"Truth is also said of the veridical belief in the existence [of something]". Avicenna: The Metaphysics of The Healing. Michael E. Marmura. Brigham Young University Press. 2005. p. 284. ISBN 0-934893-77-2.

Aquinas

Reevaluating Avicenna, and also Augustine and Aristotle, Thomas Aquinas stated in his *Disputed Questions on Truth*:

> A natural thing, being placed between two intellects, is called *true* insofar as it conforms to either. It is said to be true with respect to its conformity with the divine intellect insofar as it fulfills the end to which it was ordained by the divine intellect... With respect to its conformity with a human intellect, a thing is said to be true insofar as it is such as to cause a true estimate about itself.[53]

Thus, for Aquinas, the truth of the human intellect (logical truth) is based on the truth in things (ontological truth).[54] Following this, he wrote an elegant re-statement of Aristotle's view in his Summa I.16.1 [55]:

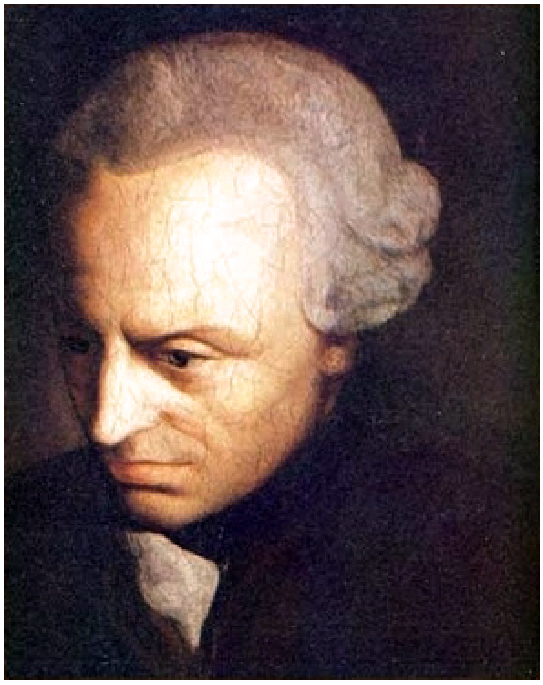
> Veritas est adæquatio intellectus et rei.
> (Truth is the conformity of the intellect to the things.)

Aquinas also said that real things participate in the act of being of the Creator God who is Subsistent Being, Intelligence, and Truth. Thus, these beings possess the light of intelligibility and are knowable. These things (beings; reality) are the foundation of the truth that is found in the human mind, when it acquires knowledge of things, first through the senses, then through the understanding and the judgement done by reason. For Aquinas, human intelligence ("intus", within and "legere", to read) has the capability to reach the essence and existence of things because it has a non-material, spiritual element, although some moral, educational, and other elements might interfere with its capability.

Modern age

Kant

Immanuel Kant endorses a definition of truth along the lines of the correspondence theory of truth.[48] Kant writes in the *Critique of Pure Reason*: "The nominal definition of truth, namely that it is the agreement of cognition with its object, is here granted and presupposed".[56] However, Kant denies that this correspondence definition of truth provides us with a test or criterion to establish which judgements are true. Kant states in his logic lectures:



Immanuel Kant

> :*"(...) Truth, it is said, consists in the agreement of cognition with its object. In consequence of this mere nominal definition, my cognition, to count as true, is supposed to agree with its object. Now I can compare the object with my cognition, however, only* by cognizing it. *Hence my cognition is supposed to confirm itself, which is far short of being sufficient for truth. For since the object is outside me, the cognition in me, all I can ever pass judgement on is whether my cognition of the object agrees with my cognition of the object. The ancients called such a circle in explanation a* diallelon. *And actually the logicians were always reproached with this mistake by the sceptics, who observed that with this definition of truth it is just as when someone makes a statement before a court and in doing so appeals to a witness with whom no one is acquainted, but who wants to establish his credibility by maintaining that the one who called him as witness is an honest man. The accusation was grounded, too. Only the solution of the indicated problem is impossible without qualification and for every man. (...)"*[57]

This passage makes use of his distinction between nominal and real definitions. A nominal definition explains the meaning of a linguistic expression. A real definition describes the essence of certain objects and enable us to determine whether any given item falls within the definition.[58] Kant holds that the definition of truth is merely nominal and, therefore, we cannot employ it to establish which judgements are true. According to Kant, the ancient skeptics criticized the logicians for holding that, by means of a merely nominal definition of truth, they can establish which judgements are true. They were trying to do something that is "impossible without qualification and for every man".[57]

Hegel

Georg Hegel distanced his philosophy from psychology by presenting truth as being an external self–moving object instead of being related to inner, subjective thoughts. Hegel's truth is analogous to the mechanics of a material body in motion under the influence of its own inner force. "Truth is its own self–movement within itself."[59] Teleological truth moves itself in the three–step form of dialectical triplicity toward the final goal of perfect, final, absolute truth. For Hegel, the progression of philosophical truth is a resolution of past oppositions into increasingly more accurate approximations to absolute truth. Chalybäus used the terms "thesis", "antithesis", and "synthesis" to describe Hegel's dialectical triplicity. The "thesis" consists of an incomplete historical movement. To resolve the incompletion, an "antithesis" occurs which opposes the "thesis." In turn, the "synthesis" appears when the "thesis" and "antithesis" become reconciled and a higher level of truth is obtained. This "synthesis" thereby becomes a "thesis," which will again necessitate an "antithesis," requiring a new "synthesis" until a final state is reached as the result of reason's historical movement. History is the Absolute Spirit moving toward a goal. This historical progression will finally conclude itself when the Absolute Spirit understands its own infinite self at the very end of history. Absolute Spirit will then be the complete expression of an infinite God.

Schopenhauer

For Arthur Schopenhauer,[60] a judgment is a combination or separation of two or more concepts. If a judgment is to be an expression of knowledge, it must have a sufficient reason or ground by which the judgment could be called true. *Truth is the reference of a judgment to something different from itself which is its sufficient reason (ground).* Judgments can have material, formal, transcendental, or metalogical truth. A judgment has *material* truth if its concepts are based on intuitive perceptions that are generated from sensations. If a judgment has its reason (ground) in another judgment, its truth is called logical or *formal*. If a judgment, of, for example, pure mathematics or pure science, is based on the forms (space, time, causality) of intuitive, empirical knowledge, then the judgment has *transcendental* truth.

Kierkegaard

When Søren Kierkegaard, as his character *Johannes Climacus*, ends his writings: *My thesis was, subjectivity, heartfelt is the truth*, he does not advocate for subjectivism in its extreme form (the theory that something is true simply because one believes it to be so), but rather that the objective approach to matters of personal truth cannot shed any light upon that which is most essential to a person's life. Objective truths are concerned with the facts of a person's being, while subjective truths are concerned with a person's way of being. Kierkegaard agrees that objective truths for the study of subjects like mathematics, science, and history are relevant and necessary, but argues that objective truths do not shed any light on a person's inner relationship to existence. At best, these truths can only provide a severely narrowed perspective that has little to do with one's actual experience of life.[61]

While objective truths are final and static, subjective truths are continuing and dynamic. The truth of one's existence is a living, inward, and subjective experience that is always in the process of becoming. The values, morals, and spiritual approaches a person adopts, while not denying the existence of objective truths of those beliefs, can only become truly known when they have been inwardly appropriated through subjective experience. Thus, Kierkegaard criticizes all systematic philosophies which attempt to know life or the truth of existence via theories and objective knowledge about reality. As Kierkegaard claims, human truth is something that is continually occurring, and a human being cannot find truth separate from the subjective experience of one's own existing, defined by the values and fundamental essence that consist of one's way of life.[62]

Nietzsche

Friedrich Nietzsche believed the search for truth or 'the will to truth' was a consequence of the *will to power* of philosophers. He thought that truth should be used as long as it promoted life and the will to power, and he thought untruth was better than truth if it had this life enhancement as a consequence. As he wrote in *Beyond Good and Evil*, "*The falseness of a judgment is to us not necessarily an objection to a judgment... The question is to what extent it is life-advancing, life-preserving, species-preserving, perhaps even species-breeding...*" (aphorism 4). He proposed the *will to power* as a truth only because according to him it was the most life affirming and sincere perspective one could have.

Robert Wicks discusses Nietzsche's basic view of truth as follows:

> *"(...) Some scholars regard Nietzsche's 1873 unpublished essay, "On Truth and Lies in a Nonmoral Sense" ("Über Wahrheit und Lüge im außermoralischen Sinn") as a keystone in his thought. In this essay, Nietzsche rejects the idea of universal constants, and claims that what we call "truth" is only "a mobile army of metaphors, metonyms, and anthropomorphisms." His view at this time is that arbitrariness completely prevails within human experience: concepts originate via the very artistic*

> *transference of nerve stimuli into images; "truth" is nothing more than the invention of fixed conventions for merely practical purposes, especially those of repose, security and consistence. (...)"*[63]

Whitehead

Alfred North Whitehead, a British mathematician who became an American philosopher, said: *"There are no whole truths; all truths are half-truths. It is trying to treat them as whole truths that play the devil".*

The logical progression or connection of this line of thought is to conclude that truth can lie, since half-truths are deceptive and may lead to a false conclusion.

Nishida

According to Kitaro Nishida, "knowledge of things in the world begins with the differentiation of unitary consciousness into knower and known and ends with self and things becoming one again. Such unification takes form not only in knowing but in the valuing (of truth) that directs knowing, the willing that directs action, and the feeling or emotive reach that directs sensing."[64]

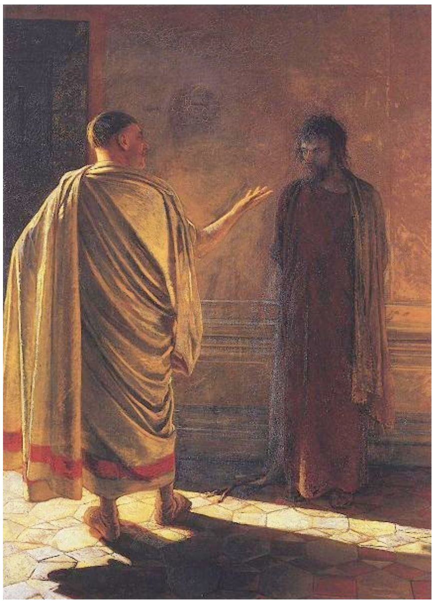
Fromm

Erich Fromm finds that trying to discuss truth as "absolute truth" is sterile and that emphasis ought to be placed on "optimal truth". He considers truth as stemming from the survival imperative of grasping one's environment physically and intellectually, whereby young children instinctively seek truth so as to orient themselves in "a strange and powerful world". The accuracy of their perceived approximation of the truth will therefore have direct consequences on their ability to deal with their environment. Fromm can be understood to define truth as a functional approximation of reality. His vision of optimal truth is described partly in "Man from Himself: An Inquiry into the Psychology of Ethics" (1947), from which excerpts are included below.

> the dichotomy between 'absolute = perfect' and 'relative = imperfect' has been superseded in all fields of scientific thought, where "it is generally recognized that there is no absolute truth but nevertheless that there are objectively valid laws and principles".
>
> In that respect, "a scientifically or rationally valid statement means that the power of reason is applied to all the available data of observation without any of them being suppressed or falsified for the sake of a desired result". The history of science is "a history of inadequate and incomplete statements, and every new insight makes possible the recognition of the inadequacies of previous propositions and offers a springboard for creating a more adequate formulation."
>
> As a result "the history of thought is the history of an ever-increasing approximation to the truth. Scientific knowledge is not absolute but optimal; it contains the optimum of truth attainable in a given historical period." Fromm furthermore notes that "different cultures have emphasized various aspects of the truth" and that increasing interaction between cultures allows for these aspects to reconcile and integrate, increasing further the approximation to the truth.

Foucault

Truth, says Michel Foucault, is problematic when any attempt is made to see truth as an "objective" quality. He prefers not to use the term truth itself but "Regimes of Truth". In his historical investigations he found truth to be something that was itself a part of, or embedded within, a given power structure. Thus Foucault's view shares much in common with the concepts of Nietzsche. Truth for Foucault is also something that shifts through various episteme throughout history.[65]



Quod Est Veritas? Christ and Pilate, by Nikolai Ge

Baudrillard

Jean Baudrillard considered truth to be largely simulated, that is pretending to have something, as opposed to dissimulation, pretending to not have something. He took his cue from iconoclasts who he claims knew that images of God demonstrated that God did not exist.[66] Baudrillard wrote in "Precession of the Simulacra":

> The simulacrum is never that which conceals the truth—it is the truth which conceals that there is none. The simulacrum is true.
>
> —Ecclesiastes[67][68]

Some examples of simulacra that Baudrillard cited were: that prisons simulate the "truth" that society is free; scandals (e.g., Watergate) simulate that corruption is corrected; Disney simulates that the U.S. itself is an adult place. One must remember that though such examples seem extreme, such extremity is an important part of Baudrillard's theory. For a less extreme example, consider how movies usually end with the bad being punished, humiliated, or otherwise failing, thus affirming for viewers the concept that the good end happily and the bad unhappily, a narrative which implies that the status quo and institutionalised power structures are largely legitimate.[66]

In medicine and psychiatry

There is controversy as to the truth value of a proposition made in bad faith self-deception, such as when a hypochondriac has a complaint with no physical symptom.[69]

In religion: omniscience

In a religious context, perfect knowledge of all truth about all things (omniscience) is regarded by some religions, particularly Buddhism and the Abrahamic religions (Christianity, Islam, and Judaism), as an attribute of a divine being.[70] In the Abrahamic view, God can exercise divine judgment, judging the dead on the basis of perfect knowledge of their lives.[71][72]

Notes

[1] Merriam-Webster's Online Dictionary, truth (http://m-w.com/dictionary/truth), 2005

[2] see Holtzmann's law for the *-ww-* : *-gg-* alternation.

[3] *A Concise Dictionary of Old Icelandic* (http://www.northvegr.org/zoega/h442.php), Geir T. Zoëga (1910), Northvegr.org

[4] OED on *true* has "Steadfast in adherence to a commander or friend, to a principle or cause, to one's promises, faith, etc.; firm in allegiance; faithful, loyal, constant, trusty; Honest, honourable, upright, virtuous, trustworthy; free from deceit, sincere, truthful " besides "Conformity with fact; agreement with reality; accuracy, correctness, verity; Consistent with fact; agreeing with the reality; representing the thing as it is; Real, genuine; rightly answering to the description; properly so called; not counterfeit, spurious, or imaginary."

[5] Encyclopedia of Philosophy, Supp., "Truth", auth: Michael Williams, p572-573 (Macmillan, 1996)

[6] Blackburn, Simon, and Simmons, Keith (eds., 1999), *Truth*, Oxford University Press, Oxford, UK. Includes papers by James, Ramsey, Russell, Tarski, and more recent work.

[7] Hale, Bob; Wright, Crispin, eds. (1999). *A Companion to the Philosophy of Language*. pp. 309–330. doi:10.1111/b.9780631213260.1999.00015.x.

[8] Horwich, Paul, *Truth*, (2nd edition, 1988),

[9] Field, Hartry, *Truth and the Absence of Fact* (2001).

[10] Encyclopedia of Philosophy, Vol.2, "Correspondence Theory of Truth", auth: Arthur N. Prior, p223 (Macmillan, 1969) Prior uses Bertrand Russell's wording in defining correspondence theory. According to Prior, Russell was substantially responsible for helping to make correspondence theory widely known under this name.

[11] Encyclopedia of Philosophy, Vol.2, "Correspondence Theory of Truth", auth: Arthur N. Prior, pp. 223-224 (Macmillan, 1969)

[12] Encyclopedia of Philosophy, Vol.2, "Correspondence Theory of Truth", auth: Arthur N. Prior, p224, Macmillan, 1969.

[13] "Correspondence Theory of Truth", in Stanford Encyclopedia of Philosophy (http://plato.stanford.edu/entries/truth-correspondence).

[14] "Correspondence Theory of Truth", in Stanford Encyclopedia of Philosophy (http://plato.stanford.edu/entries/truth-correspondence), (citing De Veritate Q.1, A.1&3; cf. Summa Theologiae Q.16).

[15] *See, e.g.,* Bradley, F.H., "On Truth and Copying", in Blackburn, *et al.* (eds., 1999),*Truth*, 31-45.

[16] Encyclopedia of Philosophy, Vol.2, "Correspondence Theory of Truth", auth: Arthur N. Prior, pp. 223 *ff.* Macmillan, 1969. See especially, section on "Moore's Correspondence Theory", 225-226, "Russell's Correspondence Theory", 226-227, "Remsey and Later Wittgenstein", 228-229, "Tarski's Semantic Theory", 230-231.

[17] Encyclopedia of Philosophy, Vol.2, "Correspondence Theory of Truth", auth: Arthur N. Prior, p. 223 *ff.* Macmillan, 1969). See the section on "Tarski's Semantic Theory", 230-231.

[18] Immanuel Kant, for instance, assembled a controversial but quite coherent system in the early 19th century, whose validity and usefulness continues to be debated even today. Similarly, the systems of Leibniz and Spinoza are characteristic systems that are internally coherent but controversial in terms of their utility and validity.

[19] Encyclopedia of Philosophy, Vol.2, "Coherence Theory of Truth", auth: Alan R. White, p130-131 (Macmillan, 1969)

[20] Encyclopedia of Philosophy, Vol.2, "Coherence Theory of Truth", auth: Alan R. White, p131-133, *see* esp., section on "Epistemological assumptions" (Macmillan, 1969)

[21] Encyclopedia of Philosophy, Vol.2, "Coherence Theory of Truth", auth: Alan R. White, p130

[22] May, Todd, 1993, Between Genealogy and Epistemology: Psychology, politics in the thought of Michel Foucault' with reference to Althusser and Balibar, 1970

[23] *See, e.g.*, Habermas, Jürgen, *Knowledge and Human Interests* (English translation, 1972).

[24] *See, e.g.*, Habermas, Jürgen, *Knowledge and Human Interests* (English translation, 1972), esp. PART III, pp 187 *ff.*

[25] Rescher, Nicholas, *Pluralism: Against the Demand for Consensus* (1995).

[26] Encyclopedia of Philosophy, Vol.5, "Pragmatic Theory of Truth", 427 (Macmillan, 1969).

[27] Peirce, C.S. (1901), "Truth and Falsity and Error" (in part), pp. 716–720 in James Mark Baldwin, ed., *Dictionary of Philosophy and Psychology*, v. 2. Peirce's section is entitled "*Logical*", beginning on p. 718, column 1, and ending on p. 720 with the initials "(C.S.P.)", see Google Books Eprint (http://books.google.com/books?id=Dc8YAAAAIAAJ&pg=PA718). Reprinted, *Collected Papers* v. 5, pp. 565–573.

[28] James, William, *The Meaning of Truth, A Sequel to 'Pragmatism',* (1909).

[29] Encyclopedia of Philosophy, Vol.2, "Dewey, John", auth Richard J. Bernstein, p383 (Macmillan, 1969)

[30] Sahakian, W.S. & Sahakian, M.L., Ideas of the Great Philosophers, NEW YORK: Barnes & Noble, 1966, LCCCN 66-23155

[31] Feynman, The Character of Physical Law, NEW YORK: Random House, 1994, ISBN 0-679-60127-9.

[32] Feynman, *The Character of Physical Law*, p. 150.

[33] Blackburn, Simon, and Simmons, Keith (eds., 1999), *Truth* in the Introductory section of the book.

[34] Kirkham, Theories of Truth, MIT Press, 1992.

[35] J. L. Austin, "How to Do Things With Words". Cambridge: Harvard University Press, 1975

[36] Encyclopedia of Philosophy, Vol.6: *Performative Theory of Truth*, auth: Gertrude Ezorsky, p. 88 (Macmillan, 1969)

[37] Ramsey, F.P. (1927), "Facts and Propositions", Aristotelian Society Supplementary Volume 7, 153–170. Reprinted, pp. 34–51 in F.P. Ramsey, Philosophical Papers, David Hugh Mellor (ed.), Cambridge University Press, Cambridge, UK, 1990

[38] Le Morvan, Pierre. (2004) "Ramsey on Truth and Truth on Ramsey", *The British Journal for the History of Philosophy* 12(4), pp. 705-718.

[39] Truth and Objectivity, Cambridge, MA: Harvard University Press, 1992.

[40] Truth as One and Many (Oxford: Oxford University Press, 2009).
[41] "Philpapers.org" (http://philpapers.org/surveys/results.pl?affil=All+respondents&areas0=0&areas_max=1&grain=medium). Philpapers.org. . Retrieved 2012-05-27.
[42] Ludwig Wittgenstein, Tractatus Logico-Philosophicus
[43] *See, e.g.,* Chaitin, Gregory L., *The Limits of Mathematics* (1997) esp. 89 *ff*.
[44] M. Davis. "Hilbert's Tenth Problem is Unsolvable." *American Mathematical Monthly* 80, pp. 233-269, 1973
[45] Yandell, Benjamin H.. *The Honors Class. Hilbert's Problems and Their Solvers* (2002).
[46] Chaitin, Gregory L., *The Limits of Mathematics* (1997) 1-28, 89 *ff*.
[47] Kripke, Saul. "Outline of a Theory of Truth", Journal of Philosophy, 72 (1975), 690-716
[48] David, Marion (2005). "Correspondence Theory of Truth" (http://plato.stanford.edu/entries/truth-correspondence/#1) in Stanford Encyclopedia of Philosophy
[49] Osman Amin (2007), "Influence of Muslim Philosophy on the West", *Monthly Renaissance* **17** (11).
[50] Jan A. Aertsen (1988), *Nature and Creature: Thomas Aquinas's Way of Thought*, p. 152. BRILL, ISBN 90-04-08451-7.
[51] Simone van Riet (in Latin). *Liber de philosophia prima, sive Scientia divina*. p. 413.
[52] *Avicenna: The Metaphysics of The Healing*. Michael E. Marmura. Brigham Young University Press. 2005. p. 284. ISBN 0-934893-77-2.
[53] *Disputed Questions on Truth*, 1, 2, c, reply to Obj. 1. Trans. Mulligan, McGlynn, Schmidt, *Truth*, vol. I, pp. 10-12.
[54] "Veritas supra ens fundatur" (Truth is founded on being). *Disputed Questions on Truth*, 10, 2, reply to Obj. 3.
[55] http://www.newadvent.org/summa/1016.htm
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[72] The Ancient Egyptians thought that there was a last judgement by the gods similar in many ways to the Abrahamic one, but instead of omniscient knowledge of truth, the life of the dead person was evaluated by Weighing of the Heart, which would record good and bad deeds.

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External links

- An Introduction to Truth (http://www.galilean-library.org/manuscript.php?postid=43788) by Paul Newall, aimed at beginners.
- Stanford Encyclopedia of Philosophy:
  - Truth (http://plato.stanford.edu/entries/truth/)
  - Coherence theory of truth (http://plato.stanford.edu/entries/truth-coherence/)
  - Correspondence theory of truth (http://plato.stanford.edu/entries/truth-correspondence/)
  - Deflationary theory of truth (http://plato.stanford.edu/entries/truth-deflationary/)
  - Identity theory of truth (http://plato.stanford.edu/entries/truth-identity/)
  - Revision theory of truth (http://plato.stanford.edu/entries/truth-revision/)
  - Tarski's definition of truth (http://plato.stanford.edu/entries/tarski-truth/)
  - Axiomatic theories of truth (http://plato.stanford.edu/entries/truth-axiomatic/)
- Heidegger on Truth (Aletheia) as Unconcealment (http://www.ontology.co/heidegger-aletheia.htm)
- History of Truth: The Greek "Aletheia" (http://www.ontology.co/aletheia.htm)
- History of Truth: The Latin "Veritas" (http://www.ontology.co/veritas.htm)

Philosophy

Philosophy is the study of general and fundamental problems, such as those connected with reality, existence, knowledge, values, reason, mind, and language.[1][2] Philosophy is distinguished from other ways of addressing such problems by its critical, generally systematic approach and its reliance on rational argument.[3] The word "philosophy" comes from the Greek φιλοσοφία (*philosophia*), which literally means "love of wisdom".[4][5][6] In more casual speech the "philosophy" of a particular person can refer to the beliefs held by that person.

Areas of inquiry

The main areas of study in philosophy today include metaphysics, epistemology, logic, ethics, and aesthetics.[7][8]

Epistemology

Epistemology is concerned with the nature and scope of knowledge,[9] such as the relationships between truth, belief, and theories of justification.

Skepticism is the position which questions the possibility of completely justifying any truth. The regress argument, a fundamental problem in epistemology, occurs when, in order to completely prove any statement P, its justification itself needs to be supported by another justification. This chain can do three possible options, all of which are unsatisfactory according to the Münchhausen Trilemma. One option is infinitism, where this chain of justification can go on forever. Another option is foundationalism, where the chain of justifications eventually relies on basic beliefs or axioms that are left unproven. The last option, such as in coherentism, is making the chain circular so that a statement is included in its own chain of justification.

Rationalism is the emphasis on reasoning as a source of knowledge. Empiricism is the emphasis on observational evidence via sensory experience over other evidence as the source of knowledge. Rationalism claims that every possible object of knowledge can be deduced from coherent premises without observation. Empiricism claims that at least some knowledge is only a matter of observation. For this, Empiricism often cites the concept of tabula rasa, where individuals are not born with mental content and that knowledge builds from experience or perception. Epistemological solipsism is the idea that the existence of the world outside the mind is an unresolvable question.


René Descartes

Parmenides (fl. 500 BC) argued that it is impossible to doubt that thinking actually occurs. But thinking must have an object, therefore something *beyond* thinking really exists. Parmenides deduced that what really exists must have certain properties—for example, that it cannot come into existence or cease to exist, that it is a coherent whole, that it remains the same eternally (in fact, exists altogether outside time). This is known as the third man argument. Plato (427–347 BC) combined rationalism with a form of realism. The philosopher's work is to consider being, and the essence (ousia) of things. But the characteristic of essences is that they are universal. The nature of a man, a triangle, a tree, applies to all men, all triangles, all trees. Plato argued that these essences are mind-independent "forms", that humans (but particularly philosophers) can come to know by reason, and by ignoring the distractions of sense-perception.

Modern rationalism begins with Descartes. Reflection on the nature of perceptual experience, as well as scientific discoveries in physiology and optics, led Descartes (and also Locke) to the view that we are directly aware of ideas, rather than objects. This view gave rise to three questions:

1. Is an idea a true copy of the real thing that it represents? Sensation is not a direct interaction between bodily objects and our sense, but is a physiological process involving representation (for example, an image on the

retina). Locke thought that a "secondary quality" such as a sensation of green could in no way resemble the arrangement of particles in matter that go to produce this sensation, although he thought that "primary qualities" such as shape, size, number, were really in objects.
2. How can physical objects such as chairs and tables, or even physiological processes in the brain, give rise to mental items such as ideas? This is part of what became known as the mind-body problem.
3. If all the contents of awareness are ideas, how can we know that anything exists apart from ideas?

Descartes tried to address the last problem by reason. He began, echoing Parmenides, with a principle that he thought could not coherently be denied: I *think*, therefore I *am* (often given in his original Latin: *Cogito ergo sum*). From this principle, Descartes went on to construct a complete system of knowledge (which involves proving the existence of God, using, among other means, a version of the ontological argument).[10] His view that reason alone could yield substantial truths about reality strongly influenced those philosophers usually considered modern rationalists (such as Baruch Spinoza, Gottfried Leibniz, and Christian Wolff), while provoking criticism from other philosophers who have retrospectively come to be grouped together as empiricists.

Logic

Logic is the study of the principles of correct reasoning. Arguments use either deductive reasoning or inductive reasoning. Deductive reasoning is when, given certain statements (called premises), other statements (called conclusions) are unavoidably implied. Rules of inferences from premises include the most popular method, modus ponens, where given “A” and “If A then B”, then “B” must be concluded. A common convention for a deductive argument is the syllogism. An argument is termed valid if its conclusion does indeed follow from its premises, whether the premises are true or not, while an argument is sound if its conclusion follows from premises that are true. Propositional logic uses premises that are propositions, which are declarations that are either true or false, while predicate logic uses more complex premises called formulae that contain variables. These can be assigned values or can be quantified as to when they apply with the universal quantifier (always apply) or the existential quantifier (applies at least once). Inductive reasoning makes conclusions or generalizations based on probabilistic reasoning. For example, if “90% of humans are right-handed” and “Joe is human” then “Joe is probably right-handed”. Fields in logic include mathematical logic (formal symbolic logic) and philosophical logic.

Metaphysics

Metaphysics is the study of the most general features of reality, such as existence, time, the relationship between mind and body, objects and their properties, wholes and their parts, events, processes, and causation. Traditional branches of metaphysics include cosmology, the study of the world in its entirety, and ontology, the study of being.

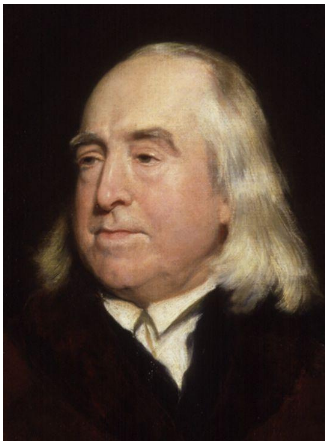
Within metaphysics itself there are a wide range of differing philosophical theories. Idealism, for example, is the belief that reality is mentally constructed or otherwise immaterial while realism holds that reality, or at least some part of it, exists independently of the mind. Subjective idealism describes objects as no more than collections or "bundles" of sense data in the perceiver. The 18th century philosopher George Berkeley contended that existence is fundamentally tied to perception with the phrase *Esse est aut percipi aut percipere* or "To be is to be perceived or to perceive".[11]

In addition to the aforementioned views, however, there is also an ontological dichotomy within metaphysics between the concepts of particulars and universals as well. Particulars are those objects that are said to exist in space and time, as opposed to abstract objects, such as numbers. Universals are properties held by multiple particulars, such as redness or a gender. The type of existence, if any, of universals and abstract objects is an issue of serious debate within metaphysical philosophy. Realism is the philosophical position that universals do in fact exist, while nominalism is the negation, or denial of universals, abstract objects, or both.[12] Conceptualism holds that universals exist, but only within the mind's perception.[13]

The question of whether or not existence is a predicate has been discussed since the Early Modern period. Essence is the set of attributes that make an object what it fundamentally is and without which it loses its identity. Essence is contrasted with accident: a property that the substance has contingently, without which the substance can still retain its identity.

Moral and political philosophy

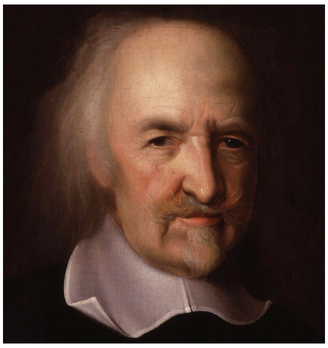
Ethics or "moral philosophy", is concerned primarily with the question of the best way to live, and secondarily, concerning the question of whether this question can be answered. The main branches of ethics are meta-ethics, normative ethics, and applied ethics. Meta-ethics concerns the nature of ethical thought, such as the origins of the words good and bad, and origins of other comparative words of various ethical systems, whether there are absolute ethical truths, and how such truths could be known. Normative ethics are more concerned with the questions of how one ought to act, and what the right course of action is. This is where most ethical theories are generated. Lastly, applied ethics go beyond theory and step into real world ethical practice, such as questions of whether or not abortion is correct. Ethics is also associated with the idea of morality, and the two are often interchangeable.



Jeremy Bentham

One debate that has commanded the attention of ethicists in the modern era has been between consequentialism (actions are to be morally evaluated solely by their *consequences*) and deontology (actions are to be morally evaluated solely by consideration of agents' *duties*, the *rights* of those whom the action concerns, or both). Jeremy Bentham and John Stuart Mill are famous for propagating utilitarianism, which is the idea that the fundamental moral rule is to strive toward the "greatest happiness for the greatest number". However, in promoting this idea they also necessarily promoted the broader doctrine of consequentialism. Adopting a position opposed to consequentialism, Immanuel Kant argued that moral principles were simply products of reason. Kant believed that the incorporation of consequences into moral deliberation was a deep mistake, since it denies the necessity of practical maxims in governing the working of the will. According to Kant, reason requires that we conform our actions to the categorical imperative, which is an absolute duty. An important 20th-century deontologist, W.D. Ross, argued for weaker forms of duties called *prima facie* duties.

More recent works have emphasized the role of character in ethics, a movement known as the *aretaic turn* (that is, the *turn towards virtues*). One strain of this movement followed the work of Bernard Williams. Williams noted that rigid forms of consequentialism and deontology demanded that people behave impartially. This, Williams argued, requires that people abandon their personal projects, and hence their personal integrity, in order to be considered moral. G.E.M. Anscombe, in an influential paper, "Modern Moral Philosophy" (1958), revived virtue ethics as an alternative to what was seen as the entrenched positions of Kantianism and consequentialism. Aretaic perspectives have been inspired in part by research of ancient conceptions of virtue. For example, Aristotle's ethics demands that people follow the *Aristotelian mean*, or balance between two vices; and Confucian ethics argues that virtue consists largely in striving for harmony with other people. Virtue ethics in general has since gained many adherents, and has been defended by such philosophers as Philippa Foot, Alasdair MacIntyre, and Rosalind Hursthouse.



Thomas Hobbes

Political philosophy is the study of government and the relationship of individuals (or families and clans) to communities including the state. It includes questions about justice, law, property, and the rights and obligations of the citizen. Politics and ethics are traditionally inter-linked subjects, as both discuss the question of what is good and how people should live. From ancient times, and well beyond them, the roots of justification for political authority were inescapably tied to outlooks on human nature. In *The Republic*, Plato presented the argument that the ideal society would be run by a council of philosopher-kings, since those best at philosophy are best able to realize the good. Even Plato, however, required philosophers to make their way in the world for many years before beginning their rule at the age of fifty. For Aristotle, humans are political animals (i.e. social animals), and governments are set up to pursue good for the community. Aristotle reasoned that, since the state (*polis*) was the highest form of community, it has the purpose of pursuing the highest good. Aristotle viewed political power as the result of natural inequalities in skill and virtue. Because of these differences, he favored an aristocracy of the able and virtuous. For Aristotle, the person cannot be complete unless he or she lives in a community. His *The Nicomachean Ethics* and *The Politics* are meant to be read in that order. The first book addresses virtues (or "excellences") in the person as a citizen; the second addresses the proper form of government to ensure that citizens will be virtuous, and therefore complete. Both books deal with the essential role of justice in civic life.

Nicolas of Cusa rekindled Platonic thought in the early 15th century. He promoted democracy in Medieval Europe, both in his writings and in his organization of the Council of Florence. Unlike Aristotle and the Hobbesian tradition to follow, Cusa saw human beings as equal and divine (that is, made in God's image), so democracy would be the only just form of government. Cusa's views are credited by some as sparking the Italian Renaissance, which gave rise to the notion of "Nation-States".

Georg Wilhelm Friedrich Hegel

Later, Niccolò Machiavelli rejected the views of Aristotle and Thomas Aquinas as unrealistic. The ideal sovereign is not the embodiment of the moral virtues; rather the sovereign does whatever is successful and necessary, rather than what is morally praiseworthy. Thomas Hobbes also contested many elements of Aristotle's views. For Hobbes, human nature is essentially anti-social: people are essentially egoistic, and this egoism makes life difficult in the natural state of things. Moreover, Hobbes argued, though people may have natural inequalities, these are trivial, since no particular talents or virtues that people may have will make them safe from harm inflicted by others. For these reasons, Hobbes concluded that the state arises from a common agreement to raise the community out of the state of nature. This can only be done by the establishment of a sovereign, in which (or whom) is vested complete control over the community, and is able to inspire awe and terror in its subjects.[14]

Many in the Enlightenment were unsatisfied with existing doctrines in political philosophy, which seemed to marginalize or neglect the possibility of a democratic state. Jean-Jacques Rousseau was among those who attempted to overturn these doctrines: he responded to Hobbes by claiming that a human is by nature a kind of "noble savage", and that society and social contracts corrupt this nature. Another critic was John Locke. In *Second Treatise on Government* he agreed with Hobbes that the nation-state was an efficient tool for raising humanity out of a deplorable state, but he argued that the sovereign might become an abominable institution compared to the relatively benign unmodulated state of nature.[15]

Following the doctrine of the fact-value distinction, due in part to the influence of David Hume and his student Adam Smith, appeals to human nature for political justification were weakened. Nevertheless, many political philosophers, especially moral realists, still make use of some essential human nature as a basis for their arguments.

Marxism is derived from the work of Karl Marx and Friedrich Engels. Their idea that capitalism is based on exploitation of workers and causes alienation of people from their human nature, the historical materialism, their view of social classes, etc., have influenced many fields of study, such as sociology, economics, and politics. Marxism inspired the Marxist school of communism, which brought a huge impact on the history of the 20th century.

Aesthetics

Aesthetics deals with beauty, art, enjoyment, sensory-emotional values, perception, and matters of taste and sentiment.

Specialized branches

- **Philosophy of language** explores the nature, the origins, and the use of language.
- **Philosophy of law** (often called **jurisprudence**) explores the varying theories explaining the nature and the interpretations of the law in society.
- **Philosophy of mind** explores the nature of the mind, and its relationship to the body, and is typified by disputes between dualism and materialism. In recent years there has been increasing similarity between this branch of philosophy and cognitive science.
- **Philosophy of religion**
- **Philosophy of science**

Many academic disciplines have also generated philosophical inquiry. These include history, logic, and mathematics.

Etymology

The introduction of the terms "philosopher" and "philosophy" has been ascribed to the Greek thinker Pythagoras.[16] The ascription is said to be based on a passage in a lost work of Herakleides Pontikos, a disciple of Aristotle. It is considered to be part of the widespread body of legends of Pythagoras of this time. "Philosopher" was understood as a word which contrasted with "sophist" (from *sophoi*). Traveling sophists or "wise men" were important in Classical Greece, often earning money as teachers, whereas philosophers are "lovers of wisdom" and not professionals.

History

Many societies have considered philosophical questions and built philosophical traditions based upon each other's works.

Eastern philosophy is organized by the chronological periods of each region. Historians of western philosophy usually divide the subject into three or more periods, the most important being ancient philosophy, medieval philosophy, and modern philosophy.[17]

Ancient philosophy

Egypt and Babylon

There are authors who date the philosophical maxims of Ptahhotep before the 25th century. For instance, Pulitzer Prize winning historian Will Durant dates these writings as early as 2880 BCE within *The Story of Civilization: Our Oriental History*. Durant claims that Ptahhotep could be considered the very first philosopher in virtue of having the earliest and surviving fragments of moral philosophy (i.e., "The Maxims of Ptah-Hotep").[18][19] Ptahhotep's grandson, Ptahhotep Tshefi, is traditionally credited with being the author of the collection of wise sayings known as *The Maxims of Ptahhotep*,[20] whose opening lines attribute authorship to the vizier Ptahhotep: *Instruction of the Mayor of the city, the Vizier Ptahhotep, under the Majesty of King Isesi*.

The origins of Babylonian philosophy can be traced back to the wisdom of early Mesopotamia, which embodied certain philosophies of life, particularly ethics, in the forms of dialectic, dialogues, epic poetry, folklore, hymns, lyrics, prose, and proverbs. The reasoning and rationality of the Babylonians developed beyond empirical observation.[21] The Babylonian text *Dialog of Pessimism* contains similarities to the agnostic thought of the sophists, the Heraclitean doctrine of contrasts, and the dialogues of Plato, as well as a precursor to the maieutic Socratic method of Socrates and Plato.[22] The Milesian philosopher Thales is also traditionally said to have studied philosophy in Mesopotamia.

Ancient Chinese

Confucius, illustrated in *Myths & Legends of China*, 1922, by E.T.C. Werner.

Philosophy has had a tremendous effect on Chinese civilization, and throughout East Asia. The majority of Chinese philosophy originates in the Spring and Autumn and Warring States era, during a period known as the "Hundred Schools of Thought",[23] which was characterized by significant intellectual and cultural developments.[23] It was during this era that the major philosophies of China, Confucianism, Mohism, Legalism, and Taoism, arose, along with philosophies that later fell into obscurity, like Agriculturalism, Chinese Naturalism, and the Logicians. Of the many philosophical schools of China, only Confucianism and Taoism existed after the Qin Dynasty suppressed any Chinese philosophy that was opposed to Legalism.

Confucianism is humanistic,[24] philosophy that believes that human beings are teachable, improvable and perfectible through personal and communal endeavour especially including self-cultivation and self-creation. Confucianism focuses on the cultivation of virtue and maintenance of ethics, the most basic of which are *ren*, *yi*, and *li*.[25] *Ren* is an obligation of altruism and humaneness for other individuals within a community, *yi* is the upholding of righteousness and the moral disposition to do good, and *li* is a system of norms and propriety that determines how a person should properly act within a community.[25]

Taoism focuses on establishing harmony with the Tao, which is origin of and the totality of everything that exists. The word "Tao" (or "Dao", depending on the romanization scheme) is usually translated as "way", "path" or "principle". Taoist propriety and ethics emphasize the Three Jewels of the Tao: compassion, moderation, and humility, while Taoist thought generally focuses on nature, the relationship between humanity and the cosmos (天人相应); health and longevity; and wu wei, action through inaction. Harmony with the Universe, or the origin of it through the Tao, is the intended result of many Taoist rules and practices.

Ancient Graeco-Roman

Plato (*left*) and Aristotle (*right*): detail from *The School of Athens* by Raffaello Sanzio, 1509

Ancient Graeco-Roman philosophy is a period of Western philosophy, starting in the 6th century [c. 585] BC to the 6th century AD. It is usually divided into three periods: the pre-Socratic period, the period of Plato and Aristotle, and the post-Aristotelian (or Hellenistic) period. A fourth period that is sometimes added includes the Neoplatonic and Christian philosophers of Late Antiquity. The most important of the ancient philosophers (in terms of subsequent influence) are Plato and Aristotle.[26] Plato specifically, is credited as the founder of Western philosophy. The philosopher Alfred North Whitehead said of Plato: "The safest general characterization of the European philosophical tradition is that it consists of a series of footnotes to Plato. I do not mean the systematic scheme of thought which scholars have doubtfully extracted from his writings. I allude to the wealth of general ideas scattered through them."[27]

The main subjects of ancient philosophy are: understanding the fundamental causes and principles of the universe; explaining it in an economical way; the epistemological problem of reconciling the diversity and change of the natural universe, with the possibility of obtaining fixed and certain knowledge about it; questions about things that cannot be perceived by the senses, such as numbers, elements, universals, and gods. Socrates is said to have been the initiator of more focused study upon the human things including the analysis of patterns of reasoning and argument and the nature of the good life and the importance of understanding and knowledge in order to pursue it; the explication of the concept of justice, and its relation to various political systems.[26]

In this period the crucial features of the Western philosophical method were established: a critical approach to received or established views, and the appeal to reason and argumentation. This includes Socrate's dialectic method of inquiry, known as the Socratic method or method of "elenchus", which he largely applied to the examination of key moral concepts such as the Good and Justice. To solve a problem, it would be broken down into a series of questions, the answers to which gradually distill the answer a person would seek. The influence of this approach is most strongly felt today in the use of the scientific method, in which hypothesis is the first stage.

Ancient Indian

The term Indian philosophy (Sanskrit: *Darshanas*), refers to any of several schools of philosophical thought that originated in the Indian subcontinent, including Hindu philosophy, Buddhist philosophy, and Jain philosophy. Having the same or rather intertwined origins, all of these philosophies have a common underlying themes of Dharma and Karma, and similarly attempt to explain the attainment of emancipation. They have been formalized and promulgated chiefly between 1000 BC to a few centuries AD.

India's philosophical tradition dates back to the composition of the Upanisads[28] in the later Vedic period (c. 1000-500 BCE). Subsequent schools (Skt: *Darshanas*) of Indian philosophy were identified as orthodox (Skt: *astika*) or non-orthodox (Skt: *nastika*) depending on whether they regarded the Vedas as an infallible source of knowledge.[29] By some classifications, there are six schools of orthodox Hindu philosophy and three heterodox schools. The orthodox are Nyaya, Vaisesika, Samkhya, Yoga, Purva mimamsa and Vedanta. The Heterodox are Jain, Buddhist and materialist (Cārvāka). Other classifications also include Pashupata, Saiva, Raseśvara and Pāṇini Darśana with the other orthodox schools.[30]

Competition and integration between the various schools was intense during their formative years, especially between 500 BC to 200 AD. Some like the Jain, Buddhist, Shaiva and Vedanta schools survived, while others like Samkhya and Ajivika did not, either being assimilated or going extinct. The Sanskrit term for "philosopher" is *dārśanika*, one who is familiar with the systems of philosophy, or *darśanas*.[31]

In the history of the Indian subcontinent, following the establishment of a Vedic culture, the development of philosophical and religious thought over a period of two millennia gave rise to what came to be called the six schools of *astika*, or orthodox, Indian or Hindu philosophy. These schools have come to be synonymous with the greater religion of Hinduism, which was a development of the early Vedic religion.

Ancient Persian

Persian philosophy can be traced back as far as Old Iranian philosophical traditions and thoughts, with their ancient Indo-Iranian roots. These were considerably influenced by Zarathustra's teachings. Throughout Iranian history and due to remarkable political and social influences such as the Macedonian, the Arab, and the Mongol invasions of Persia, a wide spectrum of schools of thought arose. These espoused a variety of views on philosophical questions, extending from Old Iranian and mainly Zoroastrianism-influenced traditions to schools appearing in the late pre-Islamic era, such as Manicheism and Mazdakism, as well as various post-Islamic schools. Iranian philosophy after Arab invasion of Persia is characterized by different interactions with the old Iranian philosophy, the Greek philosophy and with the development of Islamic philosophy. Illuminationism and the transcendent theosophy are regarded as two of the main philosophical traditions of that era in Persia. Zoroastrianism has been identified as one of the key early events in the development of philosophy.[32]

5th–16th centuries

Europe

Medieval

Medieval philosophy is the philosophy of Western Europe and the Middle East during the Middle Ages, roughly extending from the Christianization of the Roman Empire until the Renaissance.[33] Medieval philosophy is defined partly by the rediscovery and further development of classical Greek and Hellenistic philosophy, and partly by the need to address theological problems and to integrate the then widespread sacred doctrines of Abrahamic religion (Islam, Judaism, and Christianity) with secular learning.

St. Thomas Aquinas

The history of western European medieval philosophy is traditionally divided into two main periods: the period in the Latin West following the Early Middle Ages until the 12th century, when the works of Aristotle and Plato were preserved and cultivated; and the "golden age" of the 12th, 13th and 14th centuries in the Latin West, which witnessed the culmination of the recovery of ancient philosophy, and significant developments in the field of philosophy of religion, logic and metaphysics.

The medieval era was disparagingly treated by the Renaissance humanists, who saw it as a barbaric "middle" period between the classical age of Greek and Roman culture, and the "rebirth" or *renaissance* of classical culture. Yet this period of nearly a thousand years was the longest period of philosophical development in Europe, and possibly the richest. Jorge Gracia has argued that "in intensity, sophistication, and achievement, the philosophical flowering in the thirteenth century could be rightly said to rival the golden age of Greek philosophy in the fourth century B.C."[34]

Some problems discussed throughout this period are the relation of faith to reason, the existence and unity of God, the object of theology and metaphysics, the problems of knowledge, of universals, and of individuation.

Philosophers from the Middle Ages include the Christian philosophers Augustine of Hippo, Boethius, Anselm, Gilbert of Poitiers, Peter Abelard, Roger Bacon, Bonaventure, Thomas Aquinas, Duns Scotus, William of Ockham and Jean Buridan; the Jewish philosophers Maimonides and Gersonides; and the Muslim philosophers Alkindus, Alfarabi, Alhazen, Avicenna, Algazel, Avempace, Abubacer, Ibn Khaldūn, and Averroes. The medieval tradition of Scholasticism continued to flourish as late as the 17th century, in figures such as Francisco Suarez and John of St. Thomas.

Aquinas, father of Thomism, was immensely influential in Catholic Europe, placed a great emphasis on reason and argumentation, and was one of the first to use the new translation of Aristotle's metaphysical and epistemological writing. His work was a significant departure from the Neoplatonic and Augustinian thinking that had dominated much of early Scholasticism.

Renaissance

Giordano Bruno

The Renaissance ("rebirth") was a period of transition between the Middle Ages and modern thought,[35] in which the recovery of classical texts helped shift philosophical interests away from technical studies in logic, metaphysics, and theology towards eclectic inquiries into morality, philology, and mysticism.[36][37] The study of the classics and the humane arts generally, such as history and literature, enjoyed a scholarly interest hitherto unknown in Christendom, a tendency referred to as humanism.[38][39] Displacing the medieval interest in metaphysics and logic, the humanists followed Petrarch in making man and his virtues the focus of philosophy.[40][41]

The study of classical philosophy also developed in two new ways. On the one hand, the study of Aristotle was changed through the influence of Averroism. The disagreements between these Averroist Aristotelians, and more orthodox catholic Aristotelians such as Albertus Magnus and Thomas Aquinas eventually contributed to the development of a "humanist Aristotelianism" developed in the Renaissance, as exemplified in the thought of Pietro Pomponazzi and Giacomo Zabarella. Secondly, as an alternative to Aristotle, the study of Plato and the Neoplatonists became common. This was assisted by the rediscovery of works which had not been well known previously in Western Europe. Notable Renaissance Platonists include Nicholas of Cusa, and later Marsilio Ficino and Giovanni Pico della Mirandola.[41]

The Renaissance also renewed interest in anti-Aristotelian theories of nature considered as an organic, living whole comprehensible independently of theology, as in the work of Nicholas of Cusa, Nicholas Copernicus, Giordano Bruno, Telesius, and Tommaso Campanella.[42] Such movements in natural philosophy dovetailed with a revival of interest in occultism, magic, hermeticism, and astrology, which were thought to yield hidden ways of knowing and mastering nature (e.g., in Marsilio Ficino and Giovanni Pico della Mirandola).[43]

These new movements in philosophy developed contemporaneously with larger religious and political transformations in Europe: the Reformation and the decline of feudalism. Though the theologians of the Protestant Reformation showed little direct interest in philosophy, their destruction of the traditional foundations of theological and intellectual authority harmonized with a revival of fideism and skepticism in thinkers such as Erasmus, Montaigne, and Francisco Sanches.[44][45][46] Meanwhile, the gradual centralization of political power in nation-states was echoed by the emergence of secular political philosophies, as in the works of Niccolò Machiavelli (often described as the first modern political thinker, or a key turning point towards modern political thinking[47]), Thomas More, Erasmus, Justus Lipsius, Jean Bodin, and Hugo Grotius.[48][49]

East Asia

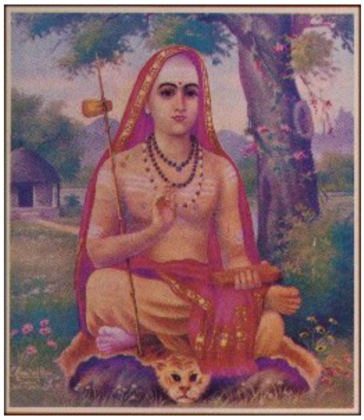
Mid-Imperial Chinese philosophy is primarily defined by the development of Neo-Confucianism. During the Tang Dynasty, Buddhism from Nepal also became a prominent philosophical and religious discipline. (It should be noted that philosophy and religion were clearly distinguished in the West, whilst these concepts were more continuous in the East due to, for example, the philosophical concepts of Buddhism.)

Neo-Confucianism is a philosophical movement that advocated a more rationalist and secular form of Confucianism by rejecting superstitious and mystical elements of Daoism and Buddhism that had influenced Confucianism during and after the Han Dynasty.[50] Although the Neo-Confucianists were critical of Daoism and Buddhism,[51] the two did have an influence on the philosophy, and the Neo-Confucianists borrowed terms and concepts from both. However, unlike the Buddhists and Daoists, who saw metaphysics as a catalyst for spiritual development, religious enlightenment, and immortality, the Neo-Confucianists used metaphysics as a guide for developing a rationalist ethical philosophy.[52]

Neo-Confucianism has its origins in the Tang Dynasty; the Confucianist scholars Han Yu and Li Ao are seen as forbears of the Neo-Confucianists of the Song Dynasty.[51] The Song Dynasty philosopher Zhou Dunyi is seen as the first true "pioneer" of Neo-Confucianism, using Daoist metaphysics as a framework for his ethical philosophy.[52]

Elsewhere in East Asia, Japanese Philosophy began to develop as indigenous Shinto beliefs fused with Buddhism, Confucianism and other schools of Chinese philosophy and Indian philosophy. Similar to Japan, in Korean philosophy the emotional content of Shamanism was integrated into the Neo-Confucianism imported from China.

India


Adi Shankara

The period between 5th and 9th century CE was the most brilliant epoch in the development of Indian philosophy as Hindu and Buddhist philosophies flourished side by side.[53] Of these various schools of thought the non-dualistic Advaita Vedanta emerged as the most influential[54] and most dominant school of philosophy.[55] The major philosophers of this school were Gaudapada, Adi Shankara and Vidyaranya.

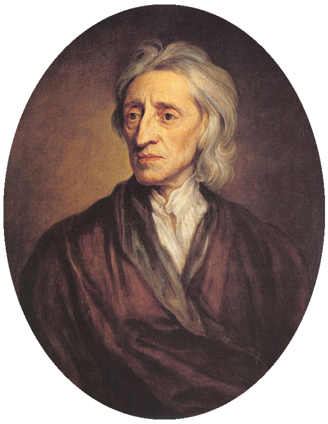
Advaita Vedanta rejects theism and dualism by insisting that “Brahman [ultimate reality] is without parts or attributes…one without a second.” Since, Brahman has no properties, contains no internal diversity and is identical with the whole reality it cannot be understood as God.[56] Brahman though being indescribable is at best described as Satchidananda (merging "Sat" + "Chit" + "Ananda", i.e., Existence, Consciousness and Bliss) by Shankara. Advaita ushered a new era in Indian philosophy and as a result, many new schools of thought arose in the medieval period. Some of them were Visishtadvaita (qualified monism), Dvaita (dualism), Dvaitadvaita (dualism-nondualism), Suddhadvaita (pure non-dualism), Achintya Bheda Abheda and Pratyabhijña (the recognitive school).

Middle East

In early Islamic thought, which refers to philosophy during the "Islamic Golden Age", traditionally dated between the 8th and 12th centuries, two main currents may be distinguished. The first is Kalam, that mainly dealt with Islamic theological questions. These include the Mu'tazili and Ash'ari. The other is Falsafa, that was founded on interpretations of Aristotelianism and Neoplatonism. There were attempts by later philosopher-theologians at harmonizing both trends, notably by Ibn Sina (Avicenna) who founded the school of Avicennism, Ibn Rushd (Averroës) who founded the school of Averroism, and others such as Ibn al-Haytham (Alhacen) and Abū Rayhān al-Bīrūnī.

17th–20th centuries

Early modern philosophy



John Locke

Chronologically, the early modern era of Western philosophy is usually identified with the 17th and 18th centuries, with the 18th century often being referred to as the Enlightenment.[57] Modern philosophy is distinguished from its predecessors by its increasing independence from traditional authorities such as the Church, academia, and Aristotelianism;[58][59] a new focus on the foundations of knowledge and metaphysical system-building;[60][61] and the emergence of modern physics out of natural philosophy.[62] Other central topics of philosophy in this period include the nature of the mind and its relation to the body, the implications of the new natural sciences for traditional theological topics such as free will and God, and the emergence of a secular basis for moral and political philosophy.[63] These trends first distinctively coalesce in Francis Bacon's call for a new, empirical program for expanding knowledge, and soon found massively influential form in the mechanical physics and rationalist metaphysics of Rene Descartes.[64] Thomas Hobbes was the first to apply this methodology systematically to political philosophy and is the originator of modern political philosophy, including the modern theory of a "social contract".[65][66] The academic canon of early modern philosophy generally includes Descartes, Spinoza, Leibniz, Locke, Berkeley, Hume, and Kant,[67][68][69] though influential contributions to philosophy were made by many thinkers in this period, such as Galileo Galilei, Pierre Gassendi, Blaise Pascal, Nicolas Malebranche, Isaac Newton, Christian Wolff, Montesquieu, Pierre Bayle, Thomas Reid, Jean d'Alembert, and Adam Smith. Jean-Jacques Rousseau was a seminal figure in initiating reaction against the Enlightenment. The approximate end of the early modern period is most often identified with Immanuel Kant's systematic attempt to limit metaphysics, justify scientific knowledge, and reconcile both of these with morality and freedom.[70][71][72]

19th-century philosophy

Later modern philosophy is usually considered to begin after the philosophy of Immanuel Kant at the beginning of the 19th century.[73] German philosophy exercised broad influence in this century, owing in part to the dominance of the German university system.[74] German idealists, such as Johann Gottlieb Fichte, Georg Wilhelm Friedrich Hegel, and Friedrich Wilhelm Joseph Schelling, transformed the work of Kant by maintaining that the world is constituted by a rational or mind-like process, and as such is entirely knowable.[75] Arthur Schopenhauer's identification of this world-constituting process as an irrational will to live influenced later 19th- and early 20th-century thinking, such as the work of Friedrich Nietzsche and Sigmund Freud.

After Hegel's death in 1831, 19th-century philosophy largely turned against idealism in favor of varieties of philosophical naturalism, such as the positivism of Auguste Comte, the empiricism of John Stuart Mill, and the materialism of Karl Marx. Logic began a period of its most significant advances since the inception of the discipline, as increasing mathematical precision opened entire fields of inference to formalization in the work of George Boole and Gottlob Frege.[76] Other philosophers who initiated lines of thought that would continue to shape philosophy into the 20th century include

- Gottlob Frege and Henry Sidgwick, whose work in logic and ethics, respectively, provided the tools for early analytic philosophy.
- Charles Sanders Peirce and William James, who founded pragmatism.
- Søren Kierkegaard and Friedrich Nietzsche, who laid the groundwork for existentialism and post-structuralism.

20th-century philosophy

Within the last century, philosophy has increasingly become a professional discipline practiced within universities, like other academic disciplines. Accordingly, it has become less general and more specialized. In the view of one prominent recent historian: "Philosophy has become a highly organized discipline, done by specialists primarily for other specialists. The number of philosophers has exploded, the volume of publication has swelled, and the subfields of serious philosophical investigation have multiplied. Not only is the broad field of philosophy today far too vast to be embraced by one mind, something similar is true even of many highly specialized subfields."[77]

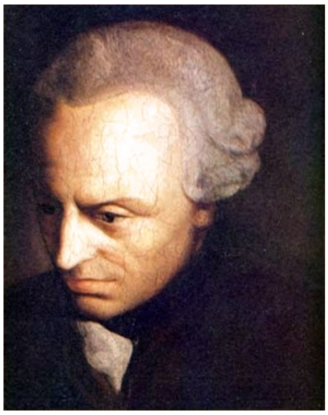
In the English-speaking world, analytic philosophy became the dominant school for much of the 20th century. In the first half of the century, it was a cohesive school, shaped strongly by logical positivism, united by the notion that philosophical problems could and should be solved by attention to logic and language. The pioneering work of Bertrand Russell was a model for the early development of analytic philosophy, moving from a rejection of the idealism dominant in late 19th century British philosophy to an neo-Humean empiricism, strengthened by the conceptual resources of modern mathematical logic.[12][78][79] In the latter half of the 20th century, analytic philosophy diffused into a wide variety of disparate philosophical views, only loosely united by historical lines of influence and a self-identified commitment to clarity and rigor. The post-war transformation of the analytic program led in two broad directions: on one hand, an interest in ordinary language as a way of avoiding or redescribing traditional philosophical problems, and on the other, a more thoroughgoing naturalism that sought to dissolve the puzzles of modern philosophy via the results of the natural sciences (such as cognitive psychology and evolutionary biology). The shift in the work of Ludwig Wittgenstein, from a view congruent with logical positivism to a therapeutic dissolution of traditional philosophy as a linguistic misunderstanding of normal forms of life, was the most influential version of the first direction in analytic philosophy.[80][81] The later work of Russell and the philosophy of W.V.O. Quine are influential exemplars of the naturalist approach dominant in the second half of the 20th century.[82][83][84][85] But the diversity of analytic philosophy from the 1970s onward defies easy generalization: the naturalism of Quine and his epigoni was in some precincts superseded by a "new metaphysics" of possible worlds, as in the influential work of David Lewis.[86][87] Recently, the experimental philosophy movement has sought to reappraise philosophical problems through social science research techniques.

On continental Europe, no single school or temperament enjoyed dominance. The flight of the logical positivists from central Europe during the 1930s and 1940s, however, diminished philosophical interest in natural science, and an emphasis on the humanities, broadly construed, figures prominently in what is usually called "continental philosophy". 20th century movements such as phenomenology, existentialism, modern hermeneutics, critical theory, structuralism, and poststructuralism are included within this loose category. The founder of phenomenology, Edmund Husserl, sought to study consciousness as experienced from a first-person perspective,[88][89] while Martin Heidegger drew on the ideas of Kierkegaard, Nietzsche, and Husserl to propose an unconventional existential approach to ontology.[90][91]

In the Arab-speaking world Arab nationalist philosophy became the dominant school of thought, involving philosophers such as Michel Aflaq, Zaki al-Arsuzi, Salah al-Din al-Bitar of ba'athism and Sati' al-Husri. These people disregarded much of Marx's research, and were mostly concerned with the individual's spirituality which would, in the fight against imperialism and oppression, lead to a united Arab Nation.

Major traditions

German idealism



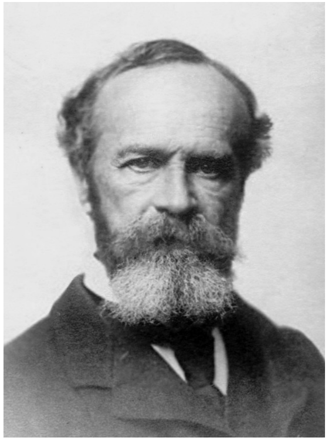
Immanuel Kant

Forms of idealism were prevalent in philosophy from the 18th century to the early 20th century. Transcendental idealism, advocated by Immanuel Kant, is the view that there are limits on what can be understood, since there is much that cannot be brought under the conditions of objective judgment. Kant wrote his *Critique of Pure Reason* (1781–1787) in an attempt to reconcile the conflicting approaches of rationalism and empiricism, and to establish a new groundwork for studying metaphysics. Kant's intention with this work was to look at what we know and then consider what must be true about it, as a logical consequence of the *way* we know it. One major theme was that there are fundamental features of reality that escape our direct knowledge because of the natural limits of the human faculties.[92] Although Kant held that objective knowledge of the world required the mind to impose a conceptual or categorical framework on the stream of pure sensory data—a framework including space and time themselves—he maintained that *things-in-themselves* existed independently of our perceptions and judgments; he was therefore not an idealist in any simple sense. Indeed, Kant's account of *things-in-themselves* is both controversial and highly complex. Continuing his work, Johann Gottlieb Fichte and Friedrich Schelling dispensed with belief in the independent existence of the world, and created a thoroughgoing idealist philosophy.

The most notable work of this German idealism was G. W. F. Hegel's *Phenomenology of Spirit*, of 1807. Hegel admitted his ideas were not new, but that all the previous philosophies had been incomplete. His goal was to correctly finish their job. Hegel asserts that the twin aims of philosophy are to account for the contradictions apparent in human experience (which arise, for instance, out of the supposed contradictions between "being" and "not being"), and also simultaneously to resolve and preserve these contradictions by showing their compatibility at a higher level of examination ("being" and "not being" are resolved with "becoming"). This program of acceptance and reconciliation of contradictions is known as the "Hegelian dialectic". Philosophers influenced by Hegel include Ludwig Andreas Feuerbach, who coined the term projection as pertaining to our inability to recognize anything in the external world without projecting qualities of ourselves upon those things; Karl Marx; Friedrich Engels; and the British idealists, notably T. H. Green, J. M. E. McTaggart and F. H. Bradley.

Few 20th century philosophers have embraced idealism. However, quite a few have embraced Hegelian dialectic. Immanuel Kant's "Copernican Turn" also remains an important philosophical concept today.

Pragmatism



William James

Pragmatism was founded in the spirit of finding a scientific concept of truth that does not depend on personal insight (revelation) or reference to some metaphysical realm. The truth of a statement should be judged by the effect it has on our actions, and truth should be seen as what the whole of scientific enquiry ultimately agrees on.[93] This should probably be seen as a guiding principle more than a definition of what it means for something to be true, though the details of how this principle should be interpreted have been subject to discussion since Charles S. Peirce first conceived it. Peirce's maxim of pragmatism is as follows: "Consider what effects, which might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conceptions of the object."[94] Like postmodern neo-pragmatist Richard Rorty, many are convinced that pragmatism asserts that the truth of beliefs does not consist in their correspondence with reality, but in their usefulness and efficacy.[95]

The late 19th-century American philosophers Charles Sanders Peirce and William James were its co-founders, and it was later developed by John Dewey as instrumentalism. Since the usefulness of any belief at any time might be contingent on circumstance, Peirce and James conceptualised final truth as something only established by the future, final settlement of all opinion.[96] Critics have accused pragmatism of falling victim to a simple fallacy: because something that is true proves useful, that usefulness is the basis for its truth.[97] Thinkers in the pragmatist tradition have included John Dewey, George Santayana, W. V. O. Quine and C. I. Lewis. Pragmatism has more recently been taken in new directions by Richard Rorty, John Lachs, Donald Davidson, Susan Haack, and Hilary Putnam.

Phenomenology

Edmund Husserl's phenomenology was an ambitious attempt to lay the foundations for an account of the structure of conscious experience in general.[98] An important part of Husserl's phenomenological project was to show that all conscious acts are directed at or about objective content, a feature that Husserl called *intentionality*.[99]

In the first part of his two-volume work, the *Logical Investigations* (1901), he launched an extended attack on psychologism. In the second part, he began to develop the technique of *descriptive phenomenology*, with the aim of showing how objective judgments are indeed grounded in conscious experience—not, however, in the first-person experience of particular individuals, but in the properties essential to any experiences of the kind in question.[98]

He also attempted to identify the essential properties of any act of meaning. He developed the method further in *Ideas* (1913) as *transcendental phenomenology*, proposing to ground actual experience, and thus all fields of human knowledge, in the structure of consciousness of an ideal, or transcendental, ego. Later, he attempted to reconcile his transcendental standpoint with an acknowledgement of the intersubjective life-world in which real individual subjects interact. Husserl published only a few works in his lifetime, which treat phenomenology mainly in abstract methodological terms; but he left an enormous quantity of unpublished concrete analyses.

Husserl's work was immediately influential in Germany, with the foundation of phenomenological schools in Munich and Göttingen. Phenomenology later achieved international fame through the work of such philosophers as Martin Heidegger (formerly Husserl's research assistant), Maurice Merleau-Ponty, and Jean-Paul Sartre. Indeed, through the work of Heidegger and Sartre, Husserl's focus on subjective experience influenced aspects of existentialism.

Existentialism

Existentialism is a term applied to the work of a number of late 19th- and 20th-century philosophers who, despite profound doctrinal differences,[100][101] shared the belief that philosophical thinking begins with the human subject—not merely the thinking subject, but the acting, feeling, living human individual.[102] In existentialism, the individual's starting point is characterized by what has been called "the existential attitude", or a sense of disorientation and confusion in the face of an apparently meaningless or absurd world.[103] Many existentialists have also regarded traditional systematic or academic philosophy, in both style and content, as too abstract and remote from concrete human experience.[104][105]


Søren Kierkegaard

Although they did not use the term, the 19th-century philosophers Søren Kierkegaard and Friedrich Nietzsche are widely regarded as the fathers of existentialism. Their influence, however, has extended beyond existentialist thought.[106][107][108]

The main target of Kierkegaard's writings was the idealist philosophical system of Hegel which, he thought, ignored or excluded the inner subjective life of living human beings. Kierkegaard, conversely, held that "truth is subjectivity", arguing that what is most important to an actual human being are questions dealing with an individual's inner relationship to existence. In particular, Kierkegaard, a Christian, believed that the truth of religious faith was a subjective question, and one to be wrestled with passionately.[109][110]

Although Kierkegaard and Nietzsche were among his influences, the extent to which the German philosopher Martin Heidegger should be considered an existentialist is debatable. In *Being and Time* he presented a method of rooting philosophical explanations in human existence (*Dasein*) to be analysed in terms of existential categories (*existentiale*); and this has led many commentators to treat him as an important figure in the existentialist movement. However, in *The Letter on Humanism*, Heidegger explicitly rejected the existentialism of Jean-Paul Sartre.

Sartre became the best-known proponent of existentialism, exploring it not only in theoretical works such as *Being and Nothingness*, but also in plays and novels. Sartre, along with Simone de Beauvoir, represented an avowedly atheistic branch of existentialism, which is now more closely associated with their ideas of nausea, contingency, bad faith, and the absurd than with Kierkegaard's spiritual angst. Nevertheless, the focus on the individual human being, responsible before the universe for the authenticity of his or her existence, is common to all these thinkers.

Structuralism and post-structuralism


Ferdinand de Saussure

Inaugurated by the linguist Ferdinand de Saussure, structuralism sought to clarify systems of signs through analyzing the discourses they both limit and make possible. Saussure conceived of the sign as being delimited by all the other signs in the system, and ideas as being incapable of existence prior to linguistic structure, which articulates thought. This led continental thought away from humanism, and toward what was termed the decentering of man: language is no longer spoken by man to express a true inner self, but language speaks man.

Structuralism sought the province of a hard science, but its positivism soon came under fire by poststructuralism, a wide field of thinkers, some of whom were once themselves structuralists, but later came to criticize it. Structuralists believed they could analyze systems from an external, objective standing, for example, but the poststructuralists argued that this is incorrect, that one cannot transcend structures and thus analysis is itself determined by what it examines, while the distinction between the signifier and signified was treated as crystalline by structuralists, poststructuralists asserted that every attempt to grasp the signified results in more signifiers, so meaning is always in a state of being deferred, making an ultimate interpretation impossible.

Structuralism came to dominate continental philosophy throughout the 1960s and early 1970s, encompassing thinkers as diverse as Claude Lévi-Strauss, Roland Barthes and Jacques Lacan. Post-structuralism came to predominate over the 1970s onwards, including thinkers such as Michel Foucault, Jacques Derrida, Gilles Deleuze and even Roland Barthes; it incorporated a critique of structuralism's limitations.

The analytic tradition

The term *analytic philosophy* roughly designates a group of philosophical methods that stress detailed argumentation, attention to semantics, use of classical logic and non-classical logics and clarity of meaning above all other criteria. Some have held that philosophical problems arise through misuse of language or because of misunderstandings of the logic of our language, while some maintain that there are genuine philosophical problems and that philosophy is continuous with science. Michael Dummett in his *Origins of Analytical Philosophy* makes the case for counting Gottlob Frege's *The Foundations of Arithmetic* as the first analytic work, on the grounds that in that book Frege took the linguistic turn, analyzing philosophical problems through language. Bertrand Russell and G.E. Moore are also often counted as founders of analytic philosophy, beginning with their rejection of British idealism, their defense of realism and the emphasis they laid on the legitimacy of analysis. Russell's classic works *The Principles of Mathematics*,[111] *On Denoting* and *Principia Mathematica* with Alfred North Whitehead, aside from greatly promoting the use of mathematical logic in philosophy, set the ground for much of the research program in the early stages of the analytic tradition, emphasizing such problems as: the reference of proper names, whether 'existence' is a property, the nature of propositions, the analysis of definite descriptions, the discussions on the foundations of mathematics; as well as exploring issues of ontological commitment and even metaphysical problems regarding time, the nature of matter, mind, persistence and change, which Russell tackled often with the aid of mathematical logic. Russell and Moore's philosophy, in the beginning of the 20th century, developed as a critique of Hegel and his British followers in particular, and of grand systems of speculative philosophy in general, though by no means all analytic philosophers reject the philosophy of Hegel (see Charles Taylor) nor speculative philosophy. Some schools in the group include logical positivism, and ordinary language both markedly influenced by Russell and Wittgenstein's development of Logical Atomism the former positively and the latter negatively.

In 1921, Ludwig Wittgenstein, who studied under Russell at Cambridge, published his *Tractatus Logico-Philosophicus*, which gave a rigidly "logical" account of linguistic and philosophical issues. At the time, he understood most of the problems of philosophy as mere puzzles of language, which could be solved by investigating and then minding the logical structure of language. Years later, he reversed a number of the positions he set out in the *Tractatus*, in for example his second major work, *Philosophical Investigations* (1953). *Investigations* was influential in the development of "ordinary language philosophy," which was promoted by Gilbert Ryle, J.L. Austin, and a few others. In the United States, meanwhile, the philosophy of W.V.O. Quine was having a major influence, with such classics as Two Dogmas of Empiricism. In that paper Quine criticizes the distinction between analytic and synthetic statements, arguing that a clear conception of analyticity is unattainable. He argued for holism, the thesis that language, including scientific language, is a set of interconnected sentences, none of which can be verified on its own, rather, the sentences in the language depend on each other for their meaning and truth conditions. A consequence of Quine's approach is that language as a whole has only a thin relation to experience. Some sentences that refer directly to experience might be modified by sense impressions, but as the whole of language is theory-laden, for the whole language to be modified, more than this is required. However, most of the linguistic structure can in principle be revised, even logic, in order to better model the world. Notable students of Quine include Donald Davidson and Daniel Dennett. The former devised a program for giving a semantics to natural language and thereby answer the philosophical conundrum "what is meaning?". A crucial part of the program was the use of Alfred Tarski's semantic theory of truth. Dummett, among others, argued that truth conditions should be dispensed within the theory of meaning, and replaced by assertibility conditions. Some propositions, on this view, are neither true nor false, and thus such a theory of meaning entails a rejection of the law of the excluded middle. This, for Dummett, entails antirealism, as Russell himself pointed out in his *An Inquiry into Meaning and Truth*.

By the 1970s there was a renewed interest in many traditional philosophical problems by the younger generations of analytic philosophers. David Lewis, Saul Kripke, Derek Parfit and others took an interest in traditional metaphysical problems, which they began exploring by the use of logic and philosophy of language. Among those problems some distinguished ones were: free will, essentialism, the nature of personal identity, identity over time, the nature of the mind, the nature of causal laws, space-time, the properties of material beings, modality, etc. In those universities where analytic philosophy has spread, these problems are still being discussed passionately. Analytic philosophers are also interested in the methodology of analytic philosophy itself, with Timothy Williamson, Wykeham Professor of Logic at Oxford, publishing recently a book entitled *The Philosophy of Philosophy*. Some influential figures in contemporary analytic philosophy are: Timothy Williamson, David Lewis, John Searle, Thomas Nagel, Hilary Putnam, Michael Dummett, Peter van Inwagen and Saul Kripke. Analytic philosophy has sometimes been accused of not contributing to the political debate or to traditional questions in aesthetics. However, with the appearance of *A Theory of Justice* by John Rawls and *Anarchy, State and Utopia* by Robert Nozick, analytic political philosophy acquired respectability. Analytic philosophers have also shown depth in their investigations of aesthetics, with Roger Scruton, Nelson Goodman, Arthur Danto and others developing the subject to its current shape.

Applied philosophy

The ideas conceived by a society have profound repercussions on what actions the society performs. The applied study of philosophy yields applications such as those in ethics—applied ethics in particular—and political philosophy. The political and economic philosophies of Confucius, Sun Zi, Chanakya, Ibn Khaldun, Ibn Rushd, Ibn Taimiyyah, Niccolò Machiavelli, Gottfried Wilhelm Leibniz, John Locke, Jean-Jacques Rousseau, Adam Smith, Karl Marx, John Stuart Mill, Mahatma Gandhi, Martin Luther King Jr., and others—all of these have been used to shape and justify governments and their actions.

In the field of philosophy of education, progressive education as championed by John Dewey has had a profound impact on educational practices in the United States in the 20th century. Descendants of this movement include the current efforts in philosophy for children, which are part of philosophy education. Carl von Clausewitz's political philosophy of war has had a profound effect on statecraft, international politics, and military strategy in the 20th century, especially in the years around World War II. Logic has become crucially important in mathematics, linguistics, psychology, computer science, and computer engineering.

Other important applications can be found in epistemology, which aid in understanding the requisites for knowledge, sound evidence, and justified belief (important in law, economics, decision theory, and a number of other disciplines). The philosophy of science discusses the underpinnings of the scientific method and has affected the nature of scientific investigation and argumentation. As such, philosophy has fundamental implications for science as a whole. For example, the strictly empirical approach of Skinner's behaviorism affected for decades the approach of the American psychological establishment. Deep ecology and animal rights examine the moral situation of humans as occupants of a world that has non-human occupants to consider also. Aesthetics can help to interpret discussions of music, literature, the plastic arts, and the whole artistic dimension of life. In general, the various philosophies strive to provide practical activities with a deeper understanding of the theoretical or conceptual underpinnings of their fields.

Often philosophy is seen as an investigation into an area not sufficiently well understood to be its own branch of knowledge. What were once philosophical pursuits have evolved into the modern day fields such as psychology, sociology, linguistics, and economics, for example.

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[2] A.C. Grayling, *Philosophy 1: A Guide through the Subject* (Oxford University Press, 1998), p. 1: "The aim of philosophical inquiry is to gain insight into questions about knowledge, truth, reason, reality, meaning, mind, and value."

[3] Anthony Quinton, in T. Honderich (ed.), *The Oxford Companion to Philosophy* (Oxford University Press, 1995), p. 666: "Philosophy is rationally critical thinking, of a more or less systematic kind about the general nature of the world (metaphysics or theory of existence), the justification of belief (epistemology or theory of knowledge), and the conduct of life (ethics or theory of value). Each of the three elements in this list has a non-philosophical counterpart, from which it is distinguished by its explicitly rational and critical way of proceeding and by its systematic nature. Everyone has some general conception of the nature of the world in which they live and of their place in it. Metaphysics replaces the unargued assumptions embodied in such a conception with a rational and organized body of beliefs about the world as a whole. Everyone has occasion to doubt and question beliefs, their own or those of others, with more or less success and without any theory of what they are doing. Epistemology seeks by argument to make explicit the rules of correct belief formation. Everyone governs their conduct by directing it to desired or valued ends. Ethics, or moral philosophy, in its most inclusive sense, seeks to articulate, in rationally systematic form, the rules or principles involved."

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> "The word 'Nominalism', as used by contemporary philosophers in the Anglo-American tradition, is ambiguous. In one sense, its most traditional sense deriving from the Middle Ages, it implies the rejection of universals. In another, more modern but equally entrenched sense, it implies the rejection of abstract objects"

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[16] See Diogenes Laertius: "Lives of Eminent Philosophers", I, 12 (http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:1999.01.0258:book=8:chapter=1); Cicero: "Tusculanae disputationes", V, 8–9

[17] For example, the multi-author Oxford History of Western Philosophy breaks the subject into eight volumes: ancient, medieval, Renaissance, two volumes covering the period 1600–1750, two volumes covering 1750–1945, and one volume on analytic philosophy since 1945. Anthony Kenny's *New History of Western Philosophy* is divided in four volumes: ancient, medieval, early modern (1500–1830), and later modern (1830 to the present). The more technical Cambridge History of Philosophy divides the topic into nine periods: Greek philosophy to Aristotle, Hellenistic philosophy, later Greek and early medieval, later medieval, Renaissance, three volumes for the 17th–19th centuries, and a final volume on 1870–1945.

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[19] M. Lichtheim, *Ancient Egyptian Literature*, p.61

[20] Grimal, p.79

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[28] p 22, *The Principal Upanisads*, Harper Collins, 1994
[29] *Oxford Dictionary of World Religions*, p. 259
[30] Cowell, E.B.; Gough, A.E. (1882). *Sarva-Darsana Sangraha of Madhava Acharya: Review of Different Systems of Hindu Philosophy*. New Delhi: Indian Books Centre/Sri Satguru Publications. p. xii. ISBN 81-7030-875-5.
[31] Apte, p. 497.
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[34] Gracia, p. 1
[35] Charles Schmitt and Quentin Skinner (eds.), *The Cambridge History of Renaissance Philosophy* (Cambridge University Press, 1988), p. 5, loosely define the period as extending "from the age of Ockham to the revisionary work of Bacon, Descartes and their contemporaries."
[36] Frederick Copleston, *A History of Philosophy, Volume III: From Ockham to Suarez* (The Newman Press, 1953) p. 18: "When one looks at Renaissance philosophy ... one is faced at first sight with a rather bewildering assortment of philosophies."
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[57] Donald Rutherford, *The Cambridge Companion to Early Modern Philosophy* (Cambridge University Press, 2006), p. xiii, defines its subject thus: "what has come to be known as "early modern philosophy"—roughly, philosophy spanning the period between the end of the sixteenth century and the end of the eighteenth century, or, in terms of figures, Montaigne through Kant." Steven Nadler, *A Companion to Early Modern Philosophy* (Blackwell, 2002), p. 1, likewise identifies its subject as "the seventeenth and eighteenth centuries". Anthony Kenny, *The Oxford History of Western Philosophy* (Clarendon: Oxford University Press, 1994), p. 107, introduces "early modern philosophy" as "the writings of the classical philosophers of the seventeenth and eighteenth centuries in Europe".

[58] Steven Nadler, *A Companion to Early Modern Philosophy*, pp. 1–2: "By the seventeenth century [...] it had become more common to find original philosophical minds working outside the strictures of the university—i.e., ecclesiastic—framework. [...] by the end of the eighteenth century, [philosophy] was a secular enterprise."

[59] Anthony Kenny, *A New History of Western Philosophy*, vol. 3 (Oxford University Press, 2006), p. xii: "To someone approaching the early modern period of philosophy from an ancient and medieval background the most striking feature of the age is the absence of Aristotle from the philosophic scene."

[60] Donald Rutherford, *The Cambridge Companion to Early Modern Philosophy* (Cambridge University Press, 2006), p. 1: "epistemology assumes a new significance in the early modern period as philosophers strive to define the conditions and limits of human knowledge."

[61] Kenny, *A New History of Western Philosophy*, vol. 3, p. 211: "The period between Descartes and Hegel was the great age of metaphysical system-building."

[62] Kenny, *A New History of Western Philosophy*, vol. 3, pp. 179–180: "the seventeenth century saw the gradual separation of the old discipline of natural philosophy into the science of physics [...] [b]y the nineteenth century physics was a fully mature empirical science, operating independently of philosophy."

[63] Kenny, *A New History of Western Philosophy*, vol. 3, pp. 212–331.

[64] Nadler, *A Companion to Early Modern Philosophy*, pp. 2–3: "Why should the early modern period in philosophy begin with Descartes and Bacon, for example, rather than with Erasmus and Montaigne? [...] Suffice it to say that at the beginning of the seventeenth century, and especially with Bacon and Descartes, certain questions and concerns come to the fore—a variety of issues that motivated the inquiries and debates that would characterize much philosophical thinking for the next two centuries."

[65] "Hobbes: Moral and Political Philosophy" (http://www.iep.utm.edu/hobmoral/). *Internet Encyclopedia of Philosophy*. .: "Hobbes is the founding father of modern political philosophy. Directly or indirectly, he has set the terms of debate about the fundamentals of political life right into our own times."

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[67] Rutherford, *The Cambridge Companion to Early Modern Philosophy*, p. 1: "Most often this [period] has been associated with the achievements of a handful of great thinkers: the so-called 'rationalists' (Descartes, Spinoza, Leibniz) and 'empiricists' (Locke, Berkeley, Hume), whose inquiries culminate in Kant's 'Critical philosophy.' These canonical figures have been celebrated for the depth and rigor of their treatments of perennial philosophical questions..."

[68] Nadler, *A Companion to Early Modern Philosophy*, p. 2: "The study of early modern philosophy demands that we pay attention to a wide variety of questions and an expansive pantheon of thinkers: the traditional canonical figures (Descartes, Spinoza, Leibniz, Locke, Berkeley, and Hume), to be sure, but also a large 'supporting cast'..."

[69] Bruce Kuklick, "Seven Thinkers and How They Grew: Descartes, Spinoza, Leibniz; Locke, Berkeley, Hume; Kant" in Rorty, Schneewind, and Skinner (eds.), *Philosophy in History* (Cambridge University Press, 1984), p. 125: "Literary, philosophical, and historical studies often rely on a notion of what is *canonical*. In American philosophy scholars go from Jonathan Edwards to John Dewey; in American literature from James Fenimore Cooper to F. Scott Fitzgerald; in political theory from Plato to Hobbes and Locke [...] The texts or authors who fill in the blanks from A to Z in these, and other intellectual traditions, constitute the canon, and there is an accompanying narrative that links text to text or author to author, a 'history of' American literature, economic thought, and so on. The most conventional of such histories are embodied in university courses and the textbooks that accompany them. This essay examines one such course, the History of Modern Philosophy, and the texts that helped to create it. If a philosopher in the United States were asked why the seven people in my title comprise Modern Philosophy, the initial response would be: they were the best, and there are historical and philosophical connections among them."

[70] Rutherford, *The Cambridge Companion to Early Modern Philosophy*, p. 1.

[71] Kenny, *A New History of Western Philosophy*, vol. 3, p. xiii.

[72] Nadler, A Companion to Early Modern Philosophy, p. 3.

[73] Shand, John (ed.) *Central Works of Philosophy, Vol.3 The Nineteenth Century* (McGill-Queens, 2005)

[74] Thomas Baldwin (ed.), *The Cambridge History of Philosophy 1870–1945* (Cambridge University Press, 2003), p. 4: "by the 1870s Germany contained much of the best universities in the world. [...] There were certainly more professors of philosophy in Germany in 1870 than anywhere else in the world, and perhaps more even than everywhere else put together."

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[76] Baldwin (ed.), *The Cambridge History of Philosophy 1870–1945*, p. 119: "within a hundred years of the first stirrings in the early nineteenth century [logic] had undergone the most fundamental transformation and substantial advance in its history."

[77] Scott Soames, *Philosophical Analysis in the Twentieth Century*, vol. 2, p. 463.

[78] Paul Edwards (ed.), *Encyclopedia of Philosophy*, vol. 7 (Macmillan, 1967), p. 239: "Russell has exercised an influence on the course of Anglo-American philosophy in the twentieth century second to that of no other individual."

[79] Thomas Baldwin (ed.), *The Cambridge History of Philosophy 1870–1945* (Cambridge University Press, 2003), p. 376: "[...] the three greatest European philosophers of the twentieth century—Heidegger, Russell, and Wittgenstein."

[80] Avrum Stroll, *Twentieth-Century Analytic Philosophy* (Columbia University Press, 2000), p. 252: "More than any other analytic philosopher, [Wittgenstein] has changed the thinking of a whole generation."

[81] "Wittgenstein, Ludwig" (http://www.iep.utm.edu/wittgens/) in the Internet Encyclopedia of Philosophy: "Ludwig Wittgenstein is one of the most influential philosophers of the twentieth century, and regarded by some as the most important since Immanuel Kant."

[82] Thomas Baldwin, *Contemporary Philosophy* (Oxford University Press, 2001), p. 90: "[Quine] has been, without question, the most influential American philosopher of the second half of the twentieth century."

[83] Peter Hylton, "Quine", in the Stanford Encyclopedia of Philosophy: "Quine's work has been extremely influential and has done much to shape the course of philosophy in the second-half of the twentieth century and into the twenty-first."

[84] Andrew Bailey, *First Philosophy: Knowledge and Reality* (Broadview Press, 2004), p. 274: "Willard Van Orman Quine (1908–2000) was uncontroversially one of the most important philosophers of the twentieth century."

[85] Anthony Kenny, *Philosophy in the Modern World* (Oxford University Press, 2007), p. 64: "After Wittgenstein's death many people regarded W.V.O. Quine (1908–2000) as the doyen of Anglophone philosophy."

[86] Stanford Encyclopedia of Philosophy (http://plato.stanford.edu/entries/david-lewis/): "David Lewis (1941–2001) was one of the most important philosophers of the 20th Century. He made significant contributions to philosophy of language, philosophy of mathematics, philosophy of science, decision theory, epistemology, meta-ethics and aesthetics. In most of these fields he is essential reading; in many of them he is among the most important figures of recent decades. And this list leaves out his two most significant contributions."

[87] John Perry, Michael Bratman, John Martin Fischer (eds.), *Introduction to Philosophy: Classical and Contemporary Readings*, 4th ed. (Oxford University Press, 2006), p. 302: "David Lewis (1941–2001) was one of the most important philosophers of the twentieth century."

[88] "Edmund Husserl" (http://plato.stanford.edu/entries/husserl/), in *The Stanford Encyclopedia of Philosophy*: "Edmund Husserl was the principal founder of phenomenology—and thus one of the most influential philosophers of the 20th century."

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- The Internet Encyclopedia of Philosophy (http://www.iep.utm.edu/)
- Indiana Philosophy Ontology Project (https://inpho.cogs.indiana.edu/)
- PhilPapers (http://philpapers.org) - a comprehensive directory of online philosophical articles and books by academic philosophers
- Philosophy Timeline (http://www.wadsworth.com/philosophy_d/special_features/timeline/timeline.html)
- Map of Western Philosophers (http://maps.google.com/maps/ms?ie=UTF8&hl=en&msa=0&msid=107892646478667659520.0004445545f2b2cffb9ed&ll=47.398349,14.326172&spn=28.597229,79.013672&z=4)
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Article | Category | Glossary | Index | Outline | Portal

Outline of philosophy

The following outline is provided as an overview of and topical guide to philosophy:

Philosophy – study of general and fundamental problems concerning matters such as existence, knowledge, values, reason, mind, and language.[1][2] It is distinguished from other ways of addressing fundamental questions (such as mysticism, myth, or the arts) by its critical, generally systematic approach and its reliance on rational argument.[3] The word "Philosophy" comes from the Greek *philosophia* (φιλοσοφία), which literally means "love of wisdom".[4][5][6]

Core areas of philosophy

The core areas of philosophy are:

- Aesthetics – The study of the nature of beauty, art, and taste, and the creation of personal kinds of truth.
- Epistemology – The study of the nature and scope of knowledge and belief.
- Ethics – The study of the right, the good, and the valuable. Includes study of applied ethics.
- Logic – The study of good reasoning, by examining the validity of arguments and documenting their fallacies.
- Metaphysics – The study of the state of being and the nature of reality.
  - Ontology – The study of being and existence.
- Social philosophy – The study of questions about social behavior.
- Political philosophy – The study of the ideas that become political values.

Major fields of philosophy

Other than the core areas, there are several fields of studied formally within philosophy. They are:

- Philosophy of language
- Philosophy of law
- Philosophy of mind
- Philosophy of religion
- Philosophy of science
- Applied Ethics
- Bioethics
- Environmental ethics
- Mathematical logic
- Philosophical logic
- Meta-ethics
- Applied philosophy
- Meta-philosophy
- Philosophy of artificial intelligence
- Philosophy of biology
- Philosophy of chemistry
- Philosophy of education
- Philosophy of engineering
- Philosophy of history
- Philosophy of mathematics
- Philosophy of music
- Philosophy of perception
- Philosophy of physics

- Philosophy of psychology
- Philosophy of social science
- Philosophy of space and time
- Teleology

History of philosophy

Ancient philosophy

- Axial Age

Western philosophy

- Medieval philosophy
- Renaissance philosophy
- Modern philosophy

Contemporary philosophy

- Analytic philosophy
- Continental philosophy

Philosophical theories

Major traditions in philosophy

Philosophical movements

Ancient

- Confucianism
- Platonic realism
- Aristotelianism
- Pythagoreanism
- Pyrrhonian skepticism
- Epicureanism (hedonism)
- Stoicism
- Cynicism

Medieval

- Neo-Confucianism
- Neoplatonism
- Thomism
- Scotism
- Scholasticism

Modern

- Empiricism
- Existentialism
- German idealism
- Logicism
- Logical Positivism
- Modernism
- Phenomenology
- Pragmatism
- Rationalism
- Utilitarianism

Contemporary

- Deconstructionism (See also deconstructivism).
- Emotivism
- Postmodernism
- Poststructuralism
- Structuralism
- Objectivism

Philosophies by branch

Aesthetics

- Symbolism
- Romanticism
- Historicism
- Classicism
- Modernism
- Postmodernism
- Psychoanalytic theory

Epistemology

- Coherentism
- Constructivist epistemology
- Contextualism
- Determinism
- Empiricism
- Fallibilism
- Foundationalism
- Holism
- Infinitism
- Innatism
- Internalism and externalism
- Naïve realism
- Naturalized epistemology
- Objectivist epistemology
- Phenomenalism

- Positivism
- Reductionism
- Reliabilism
- Representative realism
- Rationalism
- Skepticism
- Theory of Forms
- Transcendental idealism
- Uniformitarianism

Ethics

- Consequentialism
- Deontology
- Virtue ethics
- Ethics of care

Metaphysics

- Anti-realism
- Cartesian dualism
- Free will
- Liberty
- Materialism
- Meaning of life
- Idealism
- Existentialism
- Essentialism
- Libertarianism
- Determinism
- Naturalism
- Monism
- Platonic idealism
- Hindu idealism
- Phenomenalism
- Nihilism
- Realism
- Physicalism
- MOQ
- Relativism
- Scientific realism
- Solipsism
- Subjectivism
- Substance theory
- Type theory

Social and political philosophy

- Anarchism
- Authoritarianism
- Conservatism
- Liberalism
- Libertarianism
- National liberalism
- Socialism
- Utilitarianism
- Conflict theory
- Consensus theory

Philosophy of language

- Causal theory of reference
- Contrast theory of meaning
- Contrastivism
- Conventionalism
- Cratylism
- Deconstruction
- Descriptivist theory of names
- Direct reference theory
- Dramatism
- Expressivism
- Linguistic determinism
- Logical atomism
- Logical positivism
- Mediated reference theory
- Nominalism
- Non-cognitivism
- Phallogocentrism
- Quietism
- Relevance theory
- Semantic externalism
- Semantic holism
- Structuralism
- Supposition theory
- Symbiosism
- Theological noncognitivism
- Theory of descriptions
- Verification theory

Philosophy of law

- Analytical jurisprudence
- Deontological ethics
- Legal moralism
- Legal positivism
- Legal realism
- Libertarian theories of law
- Maternalism
- Natural law
- Paternalism
- Utilitarianism
- Virtue jurisprudence

Philosophy of mind

- Behaviourism
- Biological naturalism
- **Dualism**
- Eliminative materialism
- Emergent materialism
- Epiphenomenalism
- Functionalism
- Identity theory
- Interactionism
- Materialism
- Mind-body problem
- **Monism**
- Naïve realism
- Neutral monism
- Phenomenalism
- Phenomenology *(Existential phenomenology)*
- Physicalism
- Pragmatism
- Property dualism
- Representational theory of mind
- Solipsism
- Substance dualism

Philosophy of religion

- Theories of religion
- Acosmism
- Agnosticism
- Animism
- Antireligion
- Atheism
- Dharmism
- Deism
- Divine command theory

- Dualism
- Esotericism
- Exclusivism
- Existentialism (Christian
- Agnostic
- Atheist)
- Feminist theology
- Fundamentalism
- Gnosticism
- Henotheism
- Humanism (Religious
- Secular
- Christian)
- Inclusivism
- Monism
- Monotheism
- Mysticism
- Naturalism (Metaphysical
- Religious
- Humanistic)
- New Age
- Nondualism
- Nontheism
- Pandeism
- Pantheism
- Perennialism
- Polytheism
- Process theology
- Spiritualism
- Shamanism
- Taoic
- Theism
- Transcendentalism
- *more...*

Metatheory of science

- Confirmation holism
- Coherentism
- Contextualism
- Conventionalism
- Deductive-nomological model
- Determinism
- Empiricism
- Fallibilism
- Foundationalism
- Hypothetico-deductive model
- Infinitism

- Instrumentalism
- Positivism
- Pragmatism
- Rationalism
- Received view of theories
- Reductionism
- Semantic view of theories
- Scientific realism
- Scientism
- Scientific anti-realism
- Skepticism
- Uniformitarianism
- Vitalism

Philosophical literature

- Blackwell Companion to Philosophy
- A History of Western Philosophy, by Bertrand Russell
- A History of Philosophy, by Frederick Copleston

Philosophers

- Timeline of Western philosophers
- Timeline of Eastern philosophers

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[2] A.C. Grayling, *Philosophy 1: A Guide through the Subject* (Oxford University Press, 1998), p. 1: "The aim of philosophical inquiry is to gain insight into questions about knowledge, truth, reason, reality, meaning, mind, and value."

[3] Anthony Quinton, in T. Honderich (ed.), *The Oxford Companion to Philosophy* (Oxford University Press, 1995), p. 666: "Philosophy is rationally critical thinking, of a more or less systematic kind about the general nature of the world (metaphysics or theory of existence), the justification of belief (epistemology or theory of knowledge), and the conduct of life (ethics or theory of value). Each of the three elements in this list has a non-philosophical counterpart, from which it is distinguished by its explicitly rational and critical way of proceeding and by its systematic nature. Everyone has some general conception of the nature of the world in which they live and of their place in it. Metaphysics replaces the unargued assumptions embodied in such a conception with a rational and organized body of beliefs about the world as a whole. Everyone has occasion to doubt and question beliefs, their own or those of others, with more or less success and without any theory of what they are doing. Epistemology seeks by argument to make explicit the rules of correct belief formation. Everyone governs their conduct by directing it to desired or valued ends. Ethics, or moral philosophy, in its most inclusive sense, seeks to articulate, in rationally systematic form, the rules or principles involved."

[4] Philosophia, Henry George Liddell, Robert Scott, *A Greek-English Lexicon*, at Perseus (http://www.perseus.tufts.edu/cgi-bin/ptext?doc=Perseus:text:1999.04.0057:entry=#111487)

[5] Online Etymology Dictionary (http://www.etymonline.com/index.php?search=philosophy&searchmode=none)

[6] The definition of philosophy is: "1.orig., love of, or the search for, wisdom or knowledge 2.theory or logical analysis of the principles underlying conduct, thought, knowledge, and the nature of the universe". *Webster's New World Dictionary* (Second College ed.).

External links

- *Taxonomy of Philosophy* (http://consc.net/taxonomy.html) – topic outline developed by David Chalmers as the category structure for the table of contents of the *PhilPapers* academic directory.
- PhilPapers (http://philpapers.org/) – comprehensive directory of online philosophical articles and books.
- Dictionary of Philosophical Terms and Names (http://www.philosophypages.com/dy/index.htm)
- EpistemeLinks: Philosophy Resources on the Internet (http://www.epistemelinks.org/)
- Guide to Philosophy on the Internet (http://www.earlham.edu/~peters/gpi/index.htm)
- The Internet Encyclopedia of Philosophy (http://www.iep.utm.edu/)
- The Ism Book (http://www.ismbook.com/)
- Introducing Philosophy Series. By Paul Newall (for beginners) (http://www.galilean-library.org/philosophy.html)
- Philosophical positions (http://www.db.dk/jni/lifeboat/Concepts/Position.htm) (philosophy, movement, school, theory, etc.)
- The Problems of Philosophy, by Bertrand Russell (links provided to full text)
- Stanford Encyclopedia of Philosophy (http://plato.stanford.edu/)

Mathematics

Mathematics (from Greek μάθημα *máthēma*, "knowledge, study, learning") is the abstract study of topics encompassing quantity,[2] structure,[3] space,[2] change,[4][5] and other properties;[6] it has no generally accepted definition.[7][8]

Euclid, Greek mathematician, 3rd century BC, as imagined by Raphael in this detail from *The School of Athens*.[1]

Mathematicians seek out patterns[9][10] and formulate new conjectures. Mathematicians resolve the truth or falsity of conjectures by mathematical proof. The research required to solve mathematical problems can take years or even centuries of sustained inquiry. Since the pioneering work of Giuseppe Peano (1858–1932), David Hilbert (1862–1943), and others on axiomatic systems in the late 19th century, it has become customary to view mathematical research as establishing truth by rigorous deduction from appropriately chosen axioms and definitions. When those mathematical structures are good models of real phenomena, then mathematical reasoning can provide insight or predictions about nature.

Through the use of abstraction and logical reasoning, mathematics developed from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Practical mathematics has been a human activity for as far back as written records exist. Rigorous arguments first appeared in Greek mathematics, most notably in Euclid's *Elements*. Mathematics developed at a relatively slow pace until the Renaissance, when mathematical innovations interacting with new scientific discoveries led to a rapid increase in the rate of mathematical discovery that has continued to the present day.[11]

Galileo Galilei (1564–1642) said, "The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word. Without these, one is wandering about in a dark labyrinth."[12] Carl Friedrich Gauss (1777–1855) referred to mathematics as "the Queen of the Sciences."[13] Benjamin Peirce (1809–1880) called mathematics "the science that draws necessary conclusions."[14] David Hilbert said of mathematics: "We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise."[15] Albert Einstein (1879–1955) stated that "as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."[16]

Mathematics is used throughout the world as an essential tool in many fields, including natural science, engineering, medicine, and the social sciences. Applied mathematics, the branch of mathematics concerned with application of mathematical knowledge to other fields, inspires and makes use of new mathematical discoveries, which has led to the development of entirely new mathematical disciplines, such as statistics and game theory. Mathematicians also engage in pure mathematics, or mathematics for its own sake, without having any application in mind. There is no clear line separating pure and applied mathematics, and practical applications for what began as pure mathematics are often discovered.[17]

Etymology

The word *mathematics* comes from the Greek μάθημα (*máthēma*), which, in the ancient Greek language, means "what one learns", "what one gets to know", hence also "study" and "science", and in modern Greek just "lesson". The word *máthēma* is derived from μανθάνω (*manthano*), while the modern Greek equivalent is μαθαίνω (*mathaino*), both of which mean "to learn". In Greece, the word for "mathematics" came to have the narrower and more technical meaning "mathematical study", even in Classical times.[18] Its adjective is μαθηματικός (*mathēmatikós*), meaning "related to learning" or "studious", which likewise further came to mean "mathematical". In particular, μαθηματικὴ τέχνη (*mathēmatikḗ tékhnē*), Latin: *ars mathematica*, meant "the mathematical art".

In Latin, and in English until around 1700, the term *mathematics* more commonly meant "astrology" (or sometimes "astronomy") rather than "mathematics"; the meaning gradually changed to its present one from about 1500 to 1800. This has resulted in several mistranslations: a particularly notorious one is Saint Augustine's warning that Christians should beware of *mathematici* meaning astrologers, which is sometimes mistranslated as a condemnation of mathematicians.

The apparent plural form in English, like the French plural form *les mathématiques* (and the less commonly used singular derivative *la mathématique*), goes back to the Latin neuter plural *mathematica* (Cicero), based on the Greek plural τα μαθηματικά (*ta mathēmatiká*), used by Aristotle (384–322 BC), and meaning roughly "all things mathematical"; although it is plausible that English borrowed only the adjective *mathematic(al)* and formed the noun *mathematics* anew, after the pattern of physics and metaphysics, which were inherited from the Greek.[19] In English, the noun *mathematics* takes singular verb forms. It is often shortened to *maths* or, in English-speaking North America, *math*.[20]

Definitions of mathematics

Aristotle defined mathematics as "the science of quantity", and this definition prevailed until the 18th century.[21] Starting in the 19th century, when the study of mathematics increased in rigor and began to address abstract topics such as group theory and projective geometry, which have no clear-cut relation to quantity and measurement, mathematicians and philosophers began to propose a variety of new definitions.[22] Some of these definitions emphasize the deductive character of much of mathematics, some emphasize its abstractness, some emphasize certain topics within mathematics. Today, no consensus on the definition of mathematics prevails, even among professionals.[7] There is not even consensus on whether mathematics is an art or a science.[8] A great many professional mathematicians take no interest in a definition of mathematics, or consider it undefinable.[7] Some just say, "Mathematics is what mathematicians do."[7]

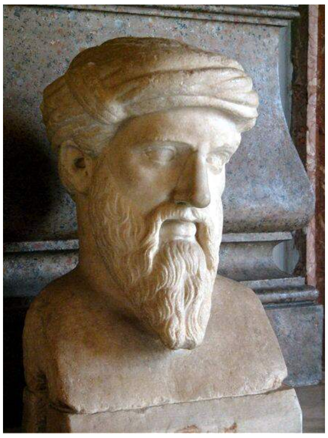
Three leading types of definition of mathematics are called *logicist*, *intuitionist*, and *formalist*, each reflecting a different philosophical school of thought.[23] All have severe problems, none has widespread acceptance, and no reconciliation seems possible.[23]

An early definition of mathematics in terms of logic was Benjamin Peirce's "the science that draws necessary conclusions" (1870).[24] In the *Principia Mathematica*, Bertrand Russell and Alfred North Whitehead advanced the philosophical program known as logicism, and attempted to prove that all mathematical concepts, statements, and principles can be defined and proven entirely in terms of symbolic logic. A logicist definition of mathematics is Russell's "All Mathematics is Symbolic Logic" (1903).[25]

Intuitionist definitions, developing from the philosophy of mathematician L.E.J. Brouwer, identify mathematics with certain mental phenomena. An example of an intuitionist definition is "Mathematics is the mental activity which consists in carrying out constructs one after the other."[23] A peculiarity of intuitionism is that it rejects some mathematical ideas considered valid according to other definitions. In particular, while other philosophies of mathematics allow objects that can be proven to exist even though they cannot be constructed, intuitionism allows only mathematical objects that one can actually construct.

Formalist definitions identify mathematics with its symbols and the rules for operating on them. Haskell Curry defined mathematics simply as "the science of formal systems".[26] A formal system is a set of symbols, or *tokens*, and some *rules* telling how the tokens may be combined into *formulas*. In formal systems, the word *axiom* has a special meaning, different from the ordinary meaning of "a self-evident truth". In formal systems, an axiom is a combination of tokens that is included in a given formal system without needing to be derived using the rules of the system.

History



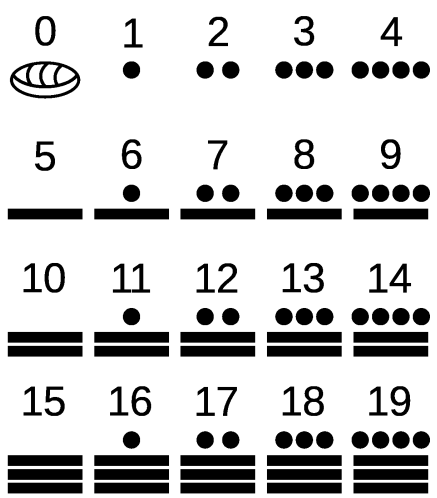
Greek mathematician Pythagoras (c. 570 – c. 495 BC), commonly credited with discovering the Pythagorean theorem.

The evolution of mathematics might be seen as an ever-increasing series of abstractions, or alternatively an expansion of subject matter. The first abstraction, which is shared by many animals,[27] was probably that of numbers: the realization that a collection of two apples and a collection of two oranges (for example) have something in common, namely quantity of their members.

In addition to recognizing how to count *physical* objects, prehistoric peoples also recognized how to count *abstract* quantities, like time – days, seasons, years.[28] Elementary arithmetic (addition, subtraction, multiplication and division) naturally followed.

Since numeracy pre-dated writing, further steps were needed for recording numbers such as tallies or the knotted strings called quipu used by the Inca to store numerical data. Numeral systems have been many and diverse, with the first known written numerals created by Egyptians in Middle Kingdom texts such as the Rhind Mathematical Papyrus.

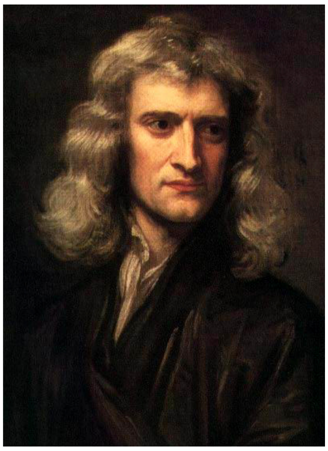
The earliest uses of mathematics were in trading, land measurement, painting and weaving patterns and the recording of time. More complex mathematics did not appear until around 3000 BC, when the Babylonians and Egyptians began using arithmetic, algebra and geometry for taxation and other financial calculations, for building and construction, and for astronomy.[29] The systematic study of mathematics in its own right began with the Ancient Greeks between 600 and 300 BC.[30]



Mayan numerals

Mathematics has since been greatly extended, and there has been a fruitful interaction between mathematics and science, to the benefit of both. Mathematical discoveries continue to be made today. According to Mikhail B. Sevryuk, in the January 2006 issue of the *Bulletin of the American Mathematical Society*, "The number of papers and books included in the *Mathematical Reviews* database since 1940 (the first year of operation of MR) is now more than 1.9 million, and more than 75 thousand items are added to the database each year. The overwhelming majority of works in this ocean contain new mathematical theorems and their proofs."[31]

Inspiration, pure and applied mathematics, and aesthetics


Sir Isaac Newton (1643–1727), an inventor of infinitesimal calculus.

Mathematics arises from many different kinds of problems. At first these were found in commerce, land measurement, architecture and later astronomy; today, all sciences suggest problems studied by mathematicians, and many problems arise within mathematics itself. For example, the physicist Richard Feynman invented the path integral formulation of quantum mechanics using a combination of mathematical reasoning and physical insight, and today's string theory, a still-developing scientific theory which attempts to unify the four fundamental forces of nature, continues to inspire new mathematics.[32] Some mathematics is only relevant in the area that inspired it, and is applied to solve further problems in that area. But often mathematics inspired by one area proves useful in many areas, and joins the general stock of mathematical concepts. A distinction is often made between pure mathematics and applied mathematics. However pure mathematics topics often turn out to have applications, e.g. number theory in cryptography. This remarkable fact that even the "purest" mathematics often turns out to have practical applications is what Eugene Wigner has called "the unreasonable effectiveness of mathematics".[33] As in most areas of study, the explosion of knowledge in the scientific age has led to specialization: there are now hundreds of specialized areas in mathematics and the latest Mathematics Subject Classification runs to 46 pages.[34] Several areas of applied mathematics have merged with related traditions outside of mathematics and become disciplines in their own right, including statistics, operations research, and computer science.

For those who are mathematically inclined, there is often a definite aesthetic aspect to much of mathematics. Many mathematicians talk about the *elegance* of mathematics, its intrinsic aesthetics and inner beauty. Simplicity and generality are valued. There is beauty in a simple and elegant proof, such as Euclid's proof that there are infinitely many prime numbers, and in an elegant numerical method that speeds calculation, such as the fast Fourier transform. G.H. Hardy in *A Mathematician's Apology* expressed the belief that these aesthetic considerations are, in themselves, sufficient to justify the study of pure mathematics. He identified criteria such as significance, unexpectedness, inevitability, and economy as factors that contribute to a mathematical aesthetic.[35] Mathematicians often strive to find proofs that are particularly elegant, proofs from "The Book" of God according to Paul Erdős.[36][37] The popularity of recreational mathematics is another sign of the pleasure many find in solving mathematical questions.

Notation, language, and rigor


Leonhard Euler, who created and popularized much of the mathematical notation used today

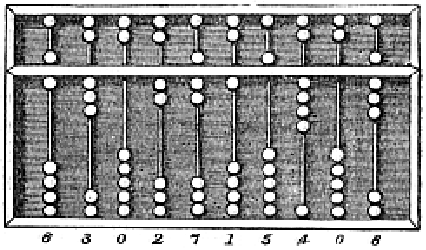
Most of the mathematical notation in use today was not invented until the 16th century.[38] Before that, mathematics was written out in words, a painstaking process that limited mathematical discovery.[39] Euler (1707–1783) was responsible for many of the notations in use today. Modern notation makes mathematics much easier for the professional, but beginners often find it daunting. It is extremely compressed: a few symbols contain a great deal of information. Like musical notation, modern mathematical notation has a strict syntax (which to a limited extent varies from author to author and from discipline to discipline) and encodes information that would be difficult to write in any other way.

Mathematical language can be difficult to understand for beginners. Words such as *or* and *only* have more precise meanings than in everyday speech. Moreover, words such as *open* and *field* have been given specialized mathematical meanings. Technical terms such as *homeomorphism* and *integrable* have precise meanings in mathematics. Additionally, shorthand phrases such as *iff* for "if and only if" belong to mathematical jargon. There is a reason for special notation and technical vocabulary: mathematics requires more precision than everyday speech. Mathematicians refer to this precision of language and logic as "rigor".

Mathematical proof is fundamentally a matter of rigor. Mathematicians want their theorems to follow from axioms by means of systematic reasoning. This is to avoid mistaken "theorems", based on fallible intuitions, of which many instances have occurred in the history of the subject.[40] The level of rigor expected in mathematics has varied over time: the Greeks expected detailed arguments, but at the time of Isaac Newton the methods employed were less rigorous. Problems inherent in the definitions used by Newton would lead to a resurgence of careful analysis and formal proof in the 19th century. Misunderstanding the rigor is a cause for some of the common misconceptions of mathematics. Today, mathematicians continue to argue among themselves about computer-assisted proofs. Since large computations are hard to verify, such proofs may not be sufficiently rigorous.[41]

Axioms in traditional thought were "self-evident truths", but that conception is problematic. At a formal level, an axiom is just a string of symbols, which has an intrinsic meaning only in the context of all derivable formulas of an axiomatic system. It was the goal of Hilbert's program to put all of mathematics on a firm axiomatic basis, but according to Gödel's incompleteness theorem every (sufficiently powerful) axiomatic system has undecidable formulas; and so a final axiomatization of mathematics is impossible. Nonetheless mathematics is often imagined to be (as far as its formal content) nothing but set theory in some axiomatization, in the sense that every mathematical statement or proof could be cast into formulas within set theory.[42]

Fields of mathematics


An abacus, a simple calculating tool used since ancient times.

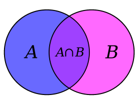
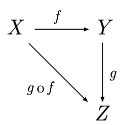
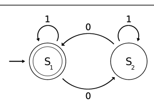
Mathematics can, broadly speaking, be subdivided into the study of quantity, structure, space, and change (i.e. arithmetic, algebra, geometry, and analysis). In addition to these main concerns, there are also subdivisions dedicated to exploring links from the heart of mathematics to other fields: to logic, to set theory (foundations), to the empirical mathematics of the various sciences (applied mathematics), and more recently to the rigorous study of uncertainty.

Foundations and philosophy

In order to clarify the foundations of mathematics, the fields of mathematical logic and set theory were developed. Mathematical logic includes the mathematical study of logic and the applications of formal logic to other areas of mathematics; set theory is the branch of mathematics that studies sets or collections of objects. Category theory, which deals in an abstract way with mathematical structures and relationships between them, is still in development. The phrase "crisis of foundations" describes the search for a rigorous foundation for mathematics that took place from approximately 1900 to 1930.[43] Some disagreement about the foundations of mathematics continues to the present day. The crisis of foundations was stimulated by a number of controversies at the time, including the controversy over Cantor's set theory and the Brouwer–Hilbert controversy.

Mathematical logic is concerned with setting mathematics within a rigorous axiomatic framework, and studying the implications of such a framework. As such, it is home to Gödel's incompleteness theorems which (informally) imply that any effective formal system that contains basic arithmetic, if *sound* (meaning that all theorems that can be proven are true), is necessarily *incomplete* (meaning that there are true theorems which cannot be proved *in that system*). Whatever finite collection of number-theoretical axioms is taken as a foundation, Gödel showed how to construct a formal statement that is a true number-theoretical fact, but which does not follow from those axioms. Therefore no formal system is a complete axiomatization of full number theory. Modern logic is divided into recursion theory, model theory, and proof theory, and is closely linked to theoretical computer science, as well as to category theory.

Theoretical computer science includes computability theory, computational complexity theory, and information theory. Computability theory examines the limitations of various theoretical models of the computer, including the most well-known model – the Turing machine. Complexity theory is the study of tractability by computer; some problems, although theoretically solvable by computer, are so expensive in terms of time or space that solving them is likely to remain practically unfeasible, even with the rapid advancement of computer hardware. A famous problem is the "**P** = **NP**?" problem, one of the Millennium Prize Problems.[44] Finally, information theory is concerned with the amount of data that can be stored on a given medium, and hence deals with concepts such as compression and entropy.

$p \Rightarrow q$			
Mathematical logic	Set theory	Category theory	Theory of computation

Pure mathematics

Quantity

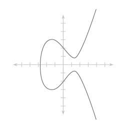
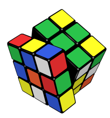
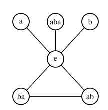
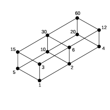
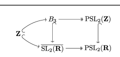
The study of quantity starts with numbers, first the familiar natural numbers and integers ("whole numbers") and arithmetical operations on them, which are characterized in arithmetic. The deeper properties of integers are studied in number theory, from which come such popular results as Fermat's Last Theorem. The twin prime conjecture and Goldbach's conjecture are two unsolved problems in number theory.

As the number system is further developed, the integers are recognized as a subset of the rational numbers ("fractions"). These, in turn, are contained within the real numbers, which are used to represent continuous quantities. Real numbers are generalized to complex numbers. These are the first steps of a hierarchy of numbers that goes on to include quarternions and octonions. Consideration of the natural numbers also leads to the transfinite numbers, which formalize the concept of "infinity". Another area of study is size, which leads to the cardinal numbers and then to another conception of infinity: the aleph numbers, which allow meaningful comparison of the size of infinitely large sets.

$1, 2, 3, \ldots$	$\ldots, -2, -1, 0, 1, 2 \ldots$	$-2, \frac{2}{3}, 1.21$	$-e, \sqrt{2}, 3, \pi$	$2, i, -2+3i, 2e^{i\frac{4\pi}{3}}$
Natural numbers	Integers	Rational numbers	Real numbers	Complex numbers

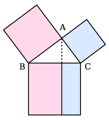
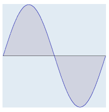
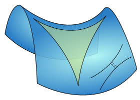
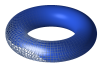
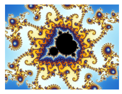
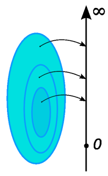
Structure

Many mathematical objects, such as sets of numbers and functions, exhibit internal structure as a consequence of operations or relations that are defined on the set. Mathematics then studies properties of those sets that can be expressed in terms of that structure; for instance number theory studies properties of the set of integers that can be expressed in terms of arithmetic operations. Moreover, it frequently happens that different such structured sets (or structures) exhibit similar properties, which makes it possible, by a further step of abstraction, to state axioms for a class of structures, and then study at once the whole class of structures satisfying these axioms. Thus one can study groups, rings, fields and other abstract systems; together such studies (for structures defined by algebraic operations) constitute the domain of abstract algebra. By its great generality, abstract algebra can often be applied to seemingly unrelated problems; for instance a number of ancient problems concerning compass and straightedge constructions were finally solved using Galois theory, which involves field theory and group theory. Another example of an algebraic theory is linear algebra, which is the general study of vector spaces, whose elements called vectors have both quantity and direction, and can be used to model (relations between) points in space. This is one example of the phenomenon that the originally unrelated areas of geometry and algebra have very strong interactions in modern mathematics. Combinatorics studies ways of enumerating the number of objects that fit a given structure.

$(1,2,3)\ (1,3,2)$ $(2,1,3)\ (2,3,1)$ $(3,1,2)\ (3,2,1)$					
Combinatorics	Number theory	Group theory	Graph theory	Order theory	Algebra

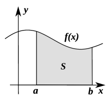
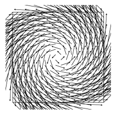
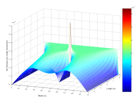
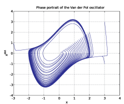
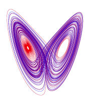
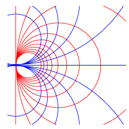
Space

The study of space originates with geometry – in particular, Euclidean geometry. Trigonometry is the branch of mathematics that deals with relationships between the sides and the angles of triangles and with the trigonometric functions; it combines space and numbers, and encompasses the well-known Pythagorean theorem. The modern study of space generalizes these ideas to include higher-dimensional geometry, non-Euclidean geometries (which play a central role in general relativity) and topology. Quantity and space both play a role in analytic geometry, differential geometry, and algebraic geometry. Convex and discrete geometry were developed to solve problems in number theory and functional analysis but now are pursued with an eye on applications in optimization and computer science. Within differential geometry are the concepts of fiber bundles and calculus on manifolds, in particular, vector and tensor calculus. Within algebraic geometry is the description of geometric objects as solution sets of polynomial equations, combining the concepts of quantity and space, and also the study of topological groups, which combine structure and space. Lie groups are used to study space, structure, and change. Topology in all its many ramifications may have been the greatest growth area in 20th century mathematics; it includes point-set topology, set-theoretic topology, algebraic topology and differential topology. In particular, instances of modern day topology are metrizability theory, axiomatic set theory, homotopy theory, and Morse theory. Topology also includes the now solved Poincaré conjecture. Other results in geometry and topology, including the four color theorem and Kepler conjecture, have been proved only with the help of computers.

					
Geometry	Trigonometry	Differential geometry	Topology	Fractal geometry	Measure theory

Change

Understanding and describing change is a common theme in the natural sciences, and calculus was developed as a powerful tool to investigate it. Functions arise here, as a central concept describing a changing quantity. The rigorous study of real numbers and functions of a real variable is known as real analysis, with complex analysis the equivalent field for the complex numbers. Functional analysis focuses attention on (typically infinite-dimensional) spaces of functions. One of many applications of functional analysis is quantum mechanics. Many problems lead naturally to relationships between a quantity and its rate of change, and these are studied as differential equations. Many phenomena in nature can be described by dynamical systems; chaos theory makes precise the ways in which many of these systems exhibit unpredictable yet still deterministic behavior.

					
Calculus	Vector calculus	Differential equations	Dynamical systems	Chaos theory	Complex analysis

Applied mathematics

Applied mathematics concerns itself with mathematical methods that are typically used in science, engineering, business, and industry. Thus, "applied mathematics" is a mathematical science with specialized knowledge. The term *applied mathematics* also describes the professional specialty in which mathematicians work on practical problems; as a profession focused on practical problems, *applied mathematics* focuses on the "formulation, study, and use of mathematical models" in science, engineering, and other areas of mathematical practice.

In the past, practical applications have motivated the development of mathematical theories, which then became the subject of study in pure mathematics, where mathematics is developed primarily for its own sake. Thus, the activity of applied mathematics is vitally connected with research in pure mathematics.

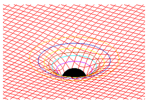
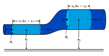
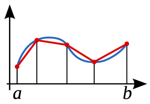
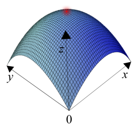
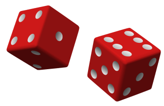
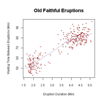
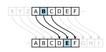
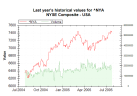
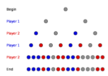
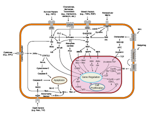
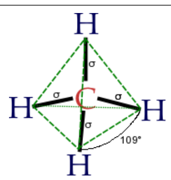
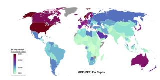
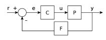
Statistics and other decision sciences

Applied mathematics has significant overlap with the discipline of statistics, whose theory is formulated mathematically, especially with probability theory. Statisticians (working as part of a research project) "create data that makes sense" with random sampling and with randomized experiments;[45] the design of a statistical sample or experiment specifies the analysis of the data (before the data be available). When reconsidering data from experiments and samples or when analyzing data from observational studies, statisticians "make sense of the data" using the art of modelling and the theory of inference – with model selection and estimation; the estimated models and consequential predictions should be tested on new data.[46]

Statistical theory studies decision problems such as minimizing the risk (expected loss) of a statistical action, such as using a procedure in, for example, parameter estimation, hypothesis testing, and selecting the best. In these traditional areas of mathematical statistics, a statistical-decision problem is formulated by minimizing an objective function, like expected loss or cost, under specific constraints: For example, designing a survey often involves minimizing the cost of estimating a population mean with a given level of confidence.[47] Because of its use of optimization, the mathematical theory of statistics shares concerns with other decision sciences, such as operations research, control theory, and mathematical economics.[48]

Computational mathematics

Computational mathematics proposes and studies methods for solving mathematical problems that are typically too large for human numerical capacity. Numerical analysis studies methods for problems in analysis using functional analysis and approximation theory; numerical analysis includes the study of approximation and discretization broadly with special concern for rounding errors. Numerical analysis and, more broadly, scientific computing also study non-analytic topics of mathematical science, especially algorithmic matrix and graph theory. Other areas of computational mathematics include computer algebra and symbolic computation.

						
Mathematical physics	Fluid dynamics	Numerical analysis	Optimization	Probability theory	Statistics	Cryptography
						
Mathematical finance	Game theory	Mathematical biology	Mathematical chemistry	Mathematical economics	Control theory	

Mathematics as profession

Arguably the most prestigious award in mathematics is the Fields Medal,[49][50] established in 1936 and now awarded every four years. The Fields Medal is often considered a mathematical equivalent to the Nobel Prize.

The Wolf Prize in Mathematics, instituted in 1978, recognizes lifetime achievement, and another major international award, the Abel Prize, was introduced in 2003. The Chern Medal was introduced in 2010 to recognize lifetime achievement. These accolades are awarded in recognition of a particular body of work, which may be innovational, or provide a solution to an outstanding problem in an established field.

A famous list of 23 open problems, called "Hilbert's problems", was compiled in 1900 by German mathematician David Hilbert. This list achieved great celebrity among mathematicians, and at least nine of the problems have now been solved. A new list of seven important problems, titled the "Millennium Prize Problems", was published in 2000.

A solution to each of these problems carries a $1 million reward, and only one (the Riemann hypothesis) is duplicated in Hilbert's problems.

Mathematics as science

Carl Friedrich Gauss, known as the "prince of mathematicians".[51]

Gauss referred to mathematics as "the Queen of the Sciences".[13] In the original Latin *Regina Scientiarum*, as well as in German *Königin der Wissenschaften*, the word corresponding to *science* means a "field of knowledge", and this was the original meaning of "science" in English, also. Of course, mathematics is in this sense a field of knowledge. The specialization restricting the meaning of "science" to *natural science* follows the rise of Baconian science, which contrasted "natural science" to scholasticism, the Aristotelean method of inquiring from first principles. Of course, the role of empirical experimentation and observation is negligible in mathematics, compared to natural sciences such as psychology, biology, or physics. Albert Einstein stated that "as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."[16] More recently, Marcus du Sautoy has called mathematics "the Queen of Science ... the main driving force behind scientific discovery".[52]

Many philosophers believe that mathematics is not experimentally falsifiable, and thus not a science according to the definition of Karl Popper.[53] However, in the 1930s Gödel's incompleteness theorems convinced many mathematicians that mathematics cannot be reduced to logic alone, and Karl Popper concluded that "most mathematical theories are, like those of physics and biology, hypothetico-deductive: pure mathematics therefore turns out to be much closer to the natural sciences whose hypotheses are conjectures, than it seemed even recently."[54] Other thinkers, notably Imre Lakatos, have applied a version of falsificationism to mathematics itself.

An alternative view is that certain scientific fields (such as theoretical physics) are mathematics with axioms that are intended to correspond to reality. In fact, the theoretical physicist, J.M. Ziman, proposed that science is *public knowledge* and thus includes mathematics.[55] In any case, mathematics shares much in common with many fields in the physical sciences, notably the exploration of the logical consequences of assumptions. Intuition and experimentation also play a role in the formulation of conjectures in both mathematics and the (other) sciences. Experimental mathematics continues to grow in importance within mathematics, and computation and simulation are playing an increasing role in both the sciences and mathematics, weakening the objection that mathematics does not use the scientific method.

The opinions of mathematicians on this matter are varied. Many mathematicians feel that to call their area a science is to downplay the importance of its aesthetic side, and its history in the traditional seven liberal arts; others feel that to ignore its connection to the sciences is to turn a blind eye to the fact that the interface between mathematics and its applications in science and engineering has driven much development in mathematics. One way this difference of viewpoint plays out is in the philosophical debate as to whether mathematics is *created* (as in art) or *discovered* (as in science). It is common to see universities divided into sections that include a division of *Science and Mathematics*, indicating that the fields are seen as being allied but that they do not coincide. In practice, mathematicians are typically grouped with scientists at the gross level but separated at finer levels. This is one of many issues considered in the philosophy of mathematics.

Notes

[1] No likeness or description of Euclid's physical appearance made during his lifetime survived antiquity. Therefore, Euclid's depiction in works of art depends on the artist's imagination (*see Euclid*).

[2] "mathematics, *n.*" (http://oed.com/view/Entry/114974). *Oxford English Dictionary*. Oxford University Press. 2012. . Retrieved June 16, 2012. "The science of space, number, quantity, and arrangement, whose methods involve logical reasoning and usually the use of symbolic notation, and which includes geometry, arithmetic, algebra, and analysis."

[3] Kneebone, G.T. (1963). *Mathematical Logic and the Foundations of Mathematics: An Introductory Survey*. Dover. pp. 4 (http://books.google.com/books?id=tCXxf4vbXCcC&pg=PA4). ISBN 0486417123. "Mathematics…is simply the study of abstract structures, or formal patterns of connectedness."

[4] LaTorre, Donald R., John W. Kenelly, Iris B. Reed, Laurel R. Carpenter, and Cynthia R Harris (2011). *Calculus Concepts: An Informal Approach to the Mathematics of Change*. Cengage Learning. pp. 2 (http://books.google.com/books?id=1Ebu2Tij4QsC&pg=PA2). ISBN 1439049572. "Calculus is the study of change—how things change, and how quickly they change."

[5] Ramana (2007). *Applied Mathematics*. Tata McGraw–Hill Education. p. 2.10 (http://books.google.com/books?id=XCRC6BeKhIIC&pg=SA2-PA10). ISBN 0070667535. "The mathematical study of change, motion, growth or decay is calculus."

[6] Ziegler, Günter M. (2011). "What Is Mathematics?". *An Invitation to Mathematics: From Competitions to Research*. Springer. pp. 7 (http://books.google.com/books?id=9TATfteVeVYC&pg=PR7). ISBN 3642195326.

[7] Mura, Robert (Dec 1993). "Images of Mathematics Held by University Teachers of Mathematical Sciences". *Educational Studies in Mathematics* **25** (4): 375–385 (http://www.jstor.org/stable/10.2307/3482762).

[8] Tobies, Renate and Helmut Neunzert (2012). *Iris Runge: A Life at the Crossroads of Mathematics, Science, and Industry*. Springer. pp. 9 (http://books.google.com/books?id=EDm0eQqFUQ4C&pg=PA9). ISBN 3034802293. "It is first necessary to ask what is meant by *mathematics* in general. Illustrious scholars have debated this matter until they were blue in the face, and yet no consensus has been reached about whether mathematics is a natural science, a branch of the humanities, or an art form."

[9] Steen, L.A. (April 29, 1988). *The Science of Patterns* Science, 240: 611–616. And summarized at Association for Supervision and Curriculum Development (http://www.ascd.org/publications/curriculum-handbook/409/chapters/The-Future-of-Mathematics-Education.aspx), www.ascd.org.

[10] Devlin, Keith, *Mathematics: The Science of Patterns: The Search for Order in Life, Mind and the Universe* (Scientific American Paperback Library) 1996, ISBN 978-0-7167-5047-5

[11] Eves

[12] Marcus du Sautoy, *A Brief History of Mathematics: 1. Newton and Leibniz* (http://www.bbc.co.uk/programmes/b00sr3fm), BBC Radio 4, 27/09/2010.

[13] Waltershausen

[14] Peirce, p. 97.

[15] Hilbert, D. (1919–20), Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919–1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays (Edited and with an English introduction by David E. Rowe), Basel, Birkhäuser (1992).

[16] Einstein, p. 28. The quote is Einstein's answer to the question: "how can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?" He, too, is concerned with *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*.

[17] Peterson

[18] Both senses can be found in Plato. Liddell and Scott, *s.voce* μαθηματικός

[19] *The Oxford Dictionary of English Etymology*, *Oxford English Dictionary*, *sub* "mathematics", "mathematic", "mathematics"

[20] "maths, *n.*" (http://oed.com/view/Entry/114982) and "math, *n.3*" (http://oed.com/view/Entry/114962). *Oxford English Dictionary,* on-line version (2012).

[21] James Franklin, "Aristotelian Realism" in *Philosophy of Mathematics", ed. A.D. Irvine, p. 104* (http://books.google.com/books?id=mbn35b2ghgkC&pg=PA104#v=onepage&q&f=false)*. Elsevier (2009).*

[22] Cajori, Florian (1893). *A History of Mathematics*. American Mathematical Society (1991 reprint). pp. 285–6 (http://books.google.com/books?id=mGJRjIC9fZgC&pg=PA285). ISBN 0821821024.

[23] Snapper, Ernst (September 1979), "The Three Crises in Mathematics: Logicism, Intuitionism, and Formalism", *Mathematics Magazine* **52** (4): 207–16, doi:10.2307/2689412, JSTOR 2689412.

[24] Peirce, Benjamin (1882). *Linear Associative Algebra* (http://books.google.com/books?id=De0GAAAAYAAJ&pg=PA1#v=onepage&q&f=false). p. 1. .

[25] Bertrand Russell, *The Principles of Mathematics,* p. 5 (http://books.google.com/books?id=kj0a_aV2mxIC&pg=PA5#v=onepage&q&f=false). University Press, Cambridge (1903)

[26] Curry, Haskell (1951). *Outlines of a Formalist Philosophy of Mathematics*. Elsevier. pp. 56 (http://books.google.com/books?id=tZHrBQgp1bkC). ISBN 0444533680.

[27] S. Dehaene; G. Dehaene-Lambertz; L. Cohen (Aug 1998). "Abstract representations of numbers in the animal and human brain". *Trends in Neuroscience* **21** (8): 355–361. doi:10.1016/S0166-2236(98)01263-6. PMID 9720604.

[28] See, for example, Raymond L. Wilder, *Evolution of Mathematical Concepts; an Elementary Study*, *passim*

[29] Kline 1990, Chapter 1.

[30] " *A History of Greek Mathematics: From Thales to Euclid* (http://books.google.com/books?id=drnY3Vjix3kC&pg=PA1&dq&hl=en#v=onepage&q=&f=false)". Thomas Little Heath (1981). ISBN 0-486-24073-8

[31] Sevryuk

[32] Johnson, Gerald W.; Lapidus, Michel L. (2002). *The Feynman Integral and Feynman's Operational Calculus*. Oxford University Press. ISBN 0-8218-2413-9.

[33] Wigner, Eugene (1960). "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" (http://www.dartmouth.edu/~matc/MathDrama/reading/Wigner.html). *Communications on Pure and Applied Mathematics* **13** (1): 1–14. doi:10.1002/cpa.3160130102. .

[34] "Mathematics Subject Classification 2010" (http://www.ams.org/mathscinet/msc/pdfs/classification2010.pdf) (PDF). . Retrieved 2010-11-09.

[35] Hardy, G.H. (1940). *A Mathematician's Apology*. Cambridge University Press. ISBN 0-521-42706-1.

[36] Gold, Bonnie; Simons, Rogers A. (2008). *Proof and Other Dilemmas: Mathematics and Philosophy*. MAA.

[37] Aigner, Martin; Ziegler, Günter M. (2001). *Proofs from* The Book. Springer. ISBN 3-540-40460-0.

[38] Earliest Uses of Various Mathematical Symbols (http://jeff560.tripod.com/mathsym.html) (Contains many further references).

[39] Kline, p. 140, on Diophantus; p. 261, on Vieta.

[40] See *false proof* for simple examples of what can go wrong in a formal proof.

[41] Ivars Peterson, *The Mathematical Tourist*, Freeman, 1988, ISBN 0-7167-1953-3. p. 4 "A few complain that the computer program can't be verified properly", (in reference to the Haken–Apple proof of the Four Color Theorem).

[42] Patrick Suppes, *Axiomatic Set Theory*, Dover, 1972, ISBN 0-486-61630-4. p. 1, "Among the many branches of modern mathematics set theory occupies a unique place: with a few rare exceptions the entities which are studied and analyzed in mathematics may be regarded as certain particular sets or classes of objects."

[43] Luke Howard Hodgkin & Luke Hodgkin, *A History of Mathematics*, Oxford University Press, 2005.

[44] Clay Mathematics Institute (http://www.claymath.org/millennium/P_vs_NP/), P=NP, claymath.org

[45] Rao, C.R. (1997) *Statistics and Truth: Putting Chance to Work*, World Scientific. ISBN 981-02-3111-3

[46] Like other mathematical sciences such as physics and computer science, statistics is an autonomous discipline rather than a branch of applied mathematics. Like research physicists and computer scientists, research statisticians are mathematical scientists. Many statisticians have a degree in mathematics, and some statisticians are also mathematicians.

[47] Rao, C. R. (1981). "Foreword". In Arthanari, T. S.; Dodge, Yadolah. Wiley Series in Probability and Mathematical Statistics. Wiley. pp. vii–viii. ISBN 0-471-08073-X. MR607328.

[48] Whittle (1994, pp. 10–11 and 14–18): Whittle, Peter (1994). "Almost home" (http://www.statslab.cam.ac.uk/History/2history.html#6._1966--72:_The_Churchill_Chair). In Kelly, F. P.. *Probability, statistics and optimisation: A Tribute to Peter Whittle* (previously "A realised path: The Cambridge Statistical Laboratory upto 1993 (revised 2002)" ed.). Chichester: John Wiley. pp. 1–28. ISBN 0-471-94829-2. .

[49] "*The Fields Medal is now indisputably the best known and most influential award in mathematics.*" Monastyrsky

[50] Riehm

[51] Zeidler, Eberhard (2004). *Oxford User's Guide to Mathematics*. Oxford, UK: Oxford University Press. p. 1188. ISBN 0-19-850763-1.

[52] Marcus du Sautoy, *A Brief History of Mathematics: 10. Nicolas Bourbaki* (http://www.bbc.co.uk/programmes/b00stcgv), BBC Radio 4, 01/10/2010.

[53] Shasha, Dennis Elliot; Lazere, Cathy A. (1998). *Out of Their Minds: The Lives and Discoveries of 15 Great Computer Scientists*. Springer. p. 228.

[54] Popper 1995, p. 56

[55] Ziman

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Further reading

- Benson, Donald C., *The Moment of Proof: Mathematical Epiphanies*, Oxford University Press, USA; New Ed edition (December 14, 2000). ISBN 0-19-513919-4.
- Boyer, Carl B., *A History of Mathematics*, Wiley; 2nd edition, revised by Uta C. Merzbach, (March 6, 1991). ISBN 0-471-54397-7.—A concise history of mathematics from the Concept of Number to contemporary Mathematics.
- Davis, Philip J. and Hersh, Reuben, *The Mathematical Experience*. Mariner Books; Reprint edition (January 14, 1999). ISBN 0-395-92968-7.
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External links

- Mathematics (http://www.bbc.co.uk/programmes/p00545hk) on *In Our Time* at the BBC. (listen now (http://www.bbc.co.uk/iplayer/console/p00545hk/In_Our_Time_Mathematics))
- Free Mathematics books (http://freebookcentre.net/SpecialCat/Free-Mathematics-Books-Download.html) Free Mathematics books collection.
- Encyclopaedia of Mathematics online encyclopaedia from Springer (http://www.encyclopediaofmath.org), Graduate-level reference work with over 8,000 entries, illuminating nearly 50,000 notions in mathematics.
- HyperMath site at Georgia State University (http://hyperphysics.phy-astr.gsu.edu/Hbase/hmat.html)
- FreeScience Library (http://www.freescience.info/mathematics.php) The mathematics section of FreeScience library
- Rusin, Dave: *The Mathematical Atlas* (http://www.math-atlas.org/). A guided tour through the various branches of modern mathematics. (Can also be found at NIU.edu (http://www.math.niu.edu/~rusin/known-math/index/index.html).)
- Polyanin, Andrei: *EqWorld: The World of Mathematical Equations* (http://eqworld.ipmnet.ru/). An online resource focusing on algebraic, ordinary differential, partial differential (mathematical physics), integral, and other mathematical equations.
- Cain, George: Online Mathematics Textbooks (http://www.math.gatech.edu/~cain/textbooks/onlinebooks.html) available free online.
- Tricki (http://www.tricki.org/), Wiki-style site that is intended to develop into a large store of useful mathematical problem-solving techniques.
- Mathematical Structures (http://math.chapman.edu/cgi-bin/structures?HomePage), list information about classes of mathematical structures.
- Mathematician Biographies (http://www-history.mcs.st-and.ac.uk/~history/). The MacTutor History of Mathematics archive Extensive history and quotes from all famous mathematicians.
- *Metamath* (http://metamath.org/). A site and a language, that formalize mathematics from its foundations.
- Nrich (http://www.nrich.maths.org/public/index.php), a prize-winning site for students from age five from Cambridge University
- Open Problem Garden (http://garden.irmacs.sfu.ca/), a wiki of open problems in mathematics
- *Planet Math* (http://planetmath.org/). An online mathematics encyclopedia under construction, focusing on modern mathematics. Uses the Attribution-ShareAlike license, allowing article exchange with Wikipedia. Uses TeX markup.
- Some mathematics applets, at MIT (http://www-math.mit.edu/daimp)
- Weisstein, Eric et al.: *MathWorld: World of Mathematics* (http://www.mathworld.com/). An online encyclopedia of mathematics.
- Patrick Jones' Video Tutorials (http://www.youtube.com/user/patrickJMT) on Mathematics
- Citizendium: Theory (mathematics) (http://en.citizendium.org/wiki/Theory_(mathematics)).
- du Sautoy, Marcus, *A Brief History of Mathematics* (http://www.bbc.co.uk/podcasts/series/maths), BBC Radio 4 (2010).
- MathOverflow (http://mathoverflow.net/) A Q&A site for research-level mathematics

List of mathematics articles

This is an alphabetical **index of mathematics articles**.

Outline of mathematics

The following outline is provided as an overview of and topical guide to mathematics:

Mathematics – the search for fundamental truths in pattern, quantity, and change. For more on the relationship between mathematics and science, refer to the article on science.

Nature of mathematics

- Definitions of mathematics – Mathematics has no generally accepted definition. Different schools of thought, particularly in philosophy, have put forth radically different definitions, all of which are controversial.
- Philosophy of mathematics – its aim is to provide an account of the nature and methodology of mathematics and to understand the place of mathematics in people's lives.

Mathematics is

- an academic discipline – branch of knowledge that is taught and researched at the college or university level. Disciplines are defined (in part), and recognized by the academic journals in which research is published, and the learned societies and academic departments or faculties to which their practitioners belong.
- a formal science – branch of knowledge concerned with the properties of formal systems based on definitions and rules of inference. Unlike other sciences, the formal sciences are not concerned with the validity of theories based on observations in the real world.

General reference

Classification systems

- Mathematics in the Dewey Decimal Classification system
- *Mathematics Subject Classification* – alphanumerical classification scheme collaboratively produced by staff of and based on the coverage of the two major mathematical reviewing databases, Mathematical Reviews and Zentralblatt MATH.

Reference databases

- *Mathematical Reviews* – journal and online database published by the American Mathematical Society (AMS) that contains brief synopses (and occasionally evaluations) of many articles in mathematics, statistics and theoretical computer science.
- *Zentralblatt MATH* – service providing reviews and abstracts for articles in pure and applied mathematics, published by Springer Science+Business Media. It is a major international reviewing service which covers the entire field of mathematics. It uses the Mathematics Subject Classification codes for organising their reviews by topic.

Subjects

Quantity

Quantity –

- Arithmetic –
- Natural numbers –
- Integers –
- Rational numbers –
- Real numbers –
- Complex numbers –
- Infinity –

Structure

Structure –

- Abstract algebra –
- Linear algebra –
- Number theory –
- Order theory –
- Function (mathematics) –

Space

Space –

- Geometry –
- Algebraic geometry –
- Trigonometry –
- Differential geometry –
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Influential mathematicians

See Lists of mathematicians

External links

- MAA Reviews – The Basic Library List – Mathematical Association of America [1]
- Naoki's Recommended Books, compiled by Naoki Saito, U. C. Davis [2]
- A List of Recommended Books in Topology, compiled by Allen Hatcher, Cornell U. [3]

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