Mathematical Modeling Project

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1 The Simple Population Dynamics Model

$$\begin{aligned} \dot{v} &= v(r - ax) = f_1(v, x), \\ \dot{x} &= -bx + cv = f_2(v, x) \end{aligned} \tag{1}$$

Where v is the viral strain and x is the specific immune response to the strain. r is the rate at which the virus reproduces. a is the rate at which the immune cells destroy the virus. b is the rate at which the immune cells die off. c is rate at which the immune cells reproduce. This is dependent on the number of viruses present, v.

1.1 Getting our eigenvalues

First we take the Jacobian

$$J = \begin{pmatrix} r - ax & -av \\ c & -b \end{pmatrix} \tag{2}$$

Then we find the characteristic equations by evaluating the Jacobian at the two fixed points of our system $(0, 0)$ and (α, β), where $\alpha = \frac{br}{ac}$ and let $\beta = \frac{r}{a}$

$$\begin{aligned} &J|_{(0,0)} - \lambda I = \lambda^2 + \lambda(b - r) - br = 0 \\ &\implies \lambda_{1,2} = r, -b \\ &J|_{(\alpha,\beta)} - \lambda I = \lambda^2 + \lambda\gamma + \delta \\ &\implies \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\delta}}{2} \end{aligned} \tag{3}$$

where

$$\gamma = b - r + a\beta, \text{ and } \delta = ac\alpha + ab\beta - rb$$

1.2 modeling our sytem

We will now model our system of equations with conditions $r = 2.4, a = 2, b = 0.1$, and $c = 1.$. We will also assume that we are starting with no viruses and no immune response.

1.2.1 Determining Stability of Fixed points

The fixed point $(0, 0)$ coresponds to the eigenvalues $\lambda_1 = 2.4, \lambda_2 = -.1$, which implies that $(0, 0)$ is a saddle point. The fixed point $(\alpha, \beta) = (.12, 1.2)$ results in eigenvalues $\lambda_1 = -.05 + 4.873i, \lambda_2 = -.05 - 4873i$, which implies that the point $(.12, 1.2)$ is a spiral sink.

Figure 1: Population size of the viral load and the immune response for a single virus strain with $r = 2.4, a = 2, b = 0.1, c = 1$.