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Saathi

Assignment - 5

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Q1) We need to prove that the Kalman Filter is a MSME estimator.

⇒ This essentially means that we need to prove that the Kalman Gain minimises the expectation of the squared error. $E[(x-\hat{x})^2]$

⇒ $E[(x-\hat{x})^2] = \text{cov}[x_t - \hat{x}_{t/t}]$

⇒ $\text{cov}[x_t - \hat{x}_{t/t-1} - K_t(x_t - C\hat{x}_{t/t-1})]$

⇒ $\text{cov}[(I-K_tC_t)(x_t - \hat{x}_{t/t-1}) - K_t n_t]$

⇒ $(I-K_tC_t) P_{t/t-1} (I-K_tC_t)^T + K_t R_t K_t^T$

→ Minimising $E[(x-\hat{x})^2]$ by differentiating wrt K_t:

$$\Rightarrow \frac{\partial[E(x-\hat{x})^2]}{\partial K_t} = \frac{\partial\, tr(E(x-\hat{x})^2)}{\partial K_t} = 0$$

$$= tr\left(\frac{\partial}{\partial K_t} E[(x-\hat{x})^2]\right) = 0$$

Upon derivating, we finally get:

⇒ $-2P^T_{t/t-1} C^T_t + 2K_t S_t = 0$

⇒ $K_t S_t = P^T_{t/t-1} C^T_t$

$$\boxed{K_t = P^T_{t/t-1} C_t^T S_t^{-1}}$$

* This is the gain of the Kalman filter. Hence, we can say that Kalman filter is a MSME estimator.

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Q2) Polar to rectangular coordinates :

$x = r\cos\theta$

$y = r\sin\theta$ $\quad$ $\bullet\, r = \hat{r} + r_e$

$f = h(r,\theta)$ $\quad$ $\bullet\, \theta = \hat{\theta} + \theta_e$

$\left(\bar{r} = 1m, \quad \bar{\theta} = \frac{\pi}{2}\right) \rightarrow p(r) = N(\bar{r}, \sigma_r^2)$

$p(\theta) = N(\bar{\theta}, \sigma_\theta^2)$

This function is written as :

$f = h(s)$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = r\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

(i) Mean & covariance of f:

$$\mu = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = E\left[\begin{bmatrix} x \\ y \end{bmatrix}\right] = \begin{bmatrix} E(x) \\ E(y) \end{bmatrix}$$

$$\mu = \begin{bmatrix} E(r\cos\theta) \\ E(r\sin\theta) \end{bmatrix} = \begin{bmatrix} E(r) \times E(\cos\theta) \\ E(r) \times E(\sin\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \bar{r}\, e^{-\frac{\sigma_\theta^2}{2}} \cos\bar{\theta} \\ \bar{r}\, e^{-\frac{\sigma_\theta^2}{2}} \sin\bar{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$

Substitute $\bar{r} = 1, \bar{\theta} = \frac{\pi}{2}$

$$\mu = \begin{bmatrix} 0 \\ e^{-\frac{\sigma_\theta^2}{2}} \end{bmatrix}$$

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Covariance

$$\Rightarrow \Sigma = E[(f-\mu)(f-\mu)^T]$$
$$= E\left[\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}\begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix}\right]$$

$$= E\begin{bmatrix} (x-\mu_x)^2 & (x-\mu_x)(y-\mu_y) \\ (x-\mu_x)(y-\mu_y) & (y-\mu_y)^2 \end{bmatrix}$$

Now, $\Sigma = \begin{bmatrix} E[(x-\mu_x)^2] & E[(x-\mu_x)(y-\mu_y)] \\ E[(x-\mu_x)(y-\mu_y)] & E[(y-\mu_y)^2] \end{bmatrix}$

Now, computing each element individually:

$$\Rightarrow E(x-\mu_x)^2 = E(x^2+\mu_x^2-2x\mu_x)$$
$$= E(x^2) + E[\mu_x^2] - 2\mu_x E[x]$$
$$= E[r^2\cos^2\theta] + \mu_x^2 - 2\mu_x^2$$
$$= E[r^2]\,E[\cos^2\theta] + \mu_x^2 - 2\mu_x^2$$

- $E[r^2] = E[(r+\mu-\mu)^2]$
$$= E[(r-\mu)^2 + \mu^2 + 2(r-\mu)\mu]$$
$$= E[(r-\mu)^2 + \mu^2 - 2\mu^2 + 2r\mu]$$
$$= E[(r-\mu)^2] + E[\mu^2]$$
$$= \sigma_r^2 + \mu_r^2$$
$$= \sigma_r^2 + \bar{r}^2$$

$$\Rightarrow E(x-\mu_x)^2 = (\sigma_r^2+\bar{r}^2)\,E\left(\frac{\cos 2\theta + 1}{2}\right) - \mu_x^2$$
$$=$$
$$= \frac{1}{2}(\sigma_r^2+\bar{r}^2)\left(1+\exp(-2\sigma_\theta^2)\cos 2\theta\right) - \mu_x^2$$

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Substituting values,

$$\bar{r} = 1, \quad \bar{\theta} = \frac{\pi}{2}$$

$$\Rightarrow \frac{1}{2}(\sigma_r^2 + 1)(1 - e^{-2\sigma_\theta^2})$$

$$\Rightarrow E[(x-\mu_x)(y-\mu_y)] = E[xy - x\mu_y - y\mu_x + \mu_x\mu_y]$$

$$\Rightarrow E(xy) - E[x\mu_y] - E[y\mu_x] + E[\mu_x\mu_y]$$

$$\Rightarrow E[r^2] \cdot E[\sin\theta\cos\theta] - \mu_x\mu_y$$

$$\Rightarrow (\bar{r}^2 + \sigma_r^2)\, E\left[\frac{\sin(2\theta)}{2}\right] - \mu_x\mu_y$$

$$\Rightarrow \frac{1}{2}(\sigma_r^2 + \bar{r}^2)\, e^{-2\sigma_\theta^2}\sin(2\theta) - \mu_x\mu_y$$

Substituting values,

$$E[(x-\mu_x)(y-\mu_y)] = 0$$

Now, $E[(y-\mu_y)^2] = E[y^2 + \mu_y^2 - 2y\mu_y]$

$$= E[r^2\sin^2\theta] - \mu_y^2$$

$$= E(r^2)\,E(\sin^2\theta) - \mu_y^2$$

$$= (\sigma_r^2 + \bar{r}^2) \cdot E\left[\frac{1-\cos 2\theta}{2}\right] - \mu_y^2$$

$$= \frac{1}{2}(\sigma_r^2 + \bar{r}^2)(1 - \exp(-2\sigma_\theta^2)\cos 2\theta) - \mu_y^2$$

Substituting values, we get

$$\Rightarrow \frac{1}{2}(\sigma_r^2 + 1)(1 + e^{-2\sigma_\theta^2}) - e^{-\frac{\sigma_\theta^2}{2}}$$

So, $\mu = \begin{bmatrix} 0 \\ e^{-\frac{\sigma_\theta^2}{2}} \end{bmatrix}$

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Finally,

$$\text{covariance} = \begin{bmatrix} \frac{1}{2}(\sigma_r^2+1)(1-e^{-2\sigma_\theta^2}) & 0 \\ 0 & \frac{1}{2}(\sigma_r^2+1)(1+e^{-2\sigma_\theta^2}) - e^{-\frac{\sigma_\theta^2}{2}} \end{bmatrix}$$

(ii) Mean & covariance using first order linearization

Using taylor series:

$$f = h(s_0) + h'(s_0)(s-s_0)$$

$$f = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{r}\cos\bar{\theta} \\ \bar{r}\sin\bar{\theta} \end{bmatrix} + \begin{bmatrix} \cos\bar{\theta} & -\bar{r}\sin\bar{\theta} \\ \sin\bar{\theta} & \bar{r}\cos\bar{\theta} \end{bmatrix} \begin{bmatrix} r-\bar{r} \\ \theta-\bar{\theta} \end{bmatrix}$$

$h'(s) \longrightarrow$ Jacobian of f at s_0.

$$\mu = E\begin{bmatrix} \bar{r}\cos\bar{\theta} \\ \bar{r}\sin\bar{\theta} \end{bmatrix} + E\begin{bmatrix} \cos\bar{\theta} & -\bar{r}\sin\bar{\theta} \\ \sin\bar{\theta} & \bar{r}\cos\bar{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mu = \begin{bmatrix} \bar{r}\cos\bar{\theta} \\ \bar{r}\sin\bar{\theta} \end{bmatrix}$$

Substituting values, $\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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$$\text{Covariance} = E\left[(f - \mu)(f - \mu)^T\right]$$

$$= E\left[h'(s_0) \begin{bmatrix} r - \bar{r} \\ \theta - \bar{\theta} \end{bmatrix} \begin{bmatrix} r - \bar{r} & \theta - \bar{\theta} \end{bmatrix} H^T \right]$$

$$= \begin{bmatrix} \cos\bar{\theta} & -\bar{r}\sin\bar{\theta} \\ \sin\bar{\theta} & \bar{r}\cos\bar{\theta} \end{bmatrix} \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix} \begin{bmatrix} \cos\bar{\theta} & \sin\bar{\theta} \\ -\bar{r}\sin\bar{\theta} & \bar{r}\cos\bar{\theta} \end{bmatrix}$$

Substituting values,

$$\Sigma = \begin{bmatrix} \sigma_\theta^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$