AE640A: Autonomous Navigation (Spring'22)
Assignment #0: Mathematics Basics

Due Date: TBD Maximum Marks: 100

Instructions

- This assignment is based on topics from **Module 0** and is designed to get you familiar with the basic mathematical foundations required in the course and brush up your concepts.
- You may choose to typeset your answers in LaTeX or Markdown etc. We encourage this and acknowledge the efforts put in by you. Hence, if the document is submitted with proper source code is not plagiarised and is properly typeset, a **5% bonus** will be added to whatever score you obtain in that assignment.
- For handwritten answers, use a scanning application on your phone to scan the handwritten answers and convert them to a pdf. Use ilovepdf to merge various pdf files. As the name suggests, we love pdf's, so please submit a single pdf file as **rollnumber.pdf.**

Set 1 - Basics of Probability Theory (20 points)

1. Prove $P(A|B,C) = \frac{P(B|A,C)P(A|C)}{P(B|C)}$. *(5 points)*
2. Prove $P(x_t|z_t, x_{t-1}, u_{t-1}) = \frac{P(z_t|x_t,x_{t-1},u_{t-1})P(x_t|x_{t-1},u_{t-1})}{P(z_t|x_{t-1},u_{t-1})}$. *(5 points)*
3. Prove $\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$. *(5 points)*
4. Find mean and variance of x_{t+1}, given that $x_{t+1} = Fx_t + Gu_t$ where $x_t \sim \mathcal{N}(\hat{x}_t, \Sigma_t)$, $u_t = u + \eta_t$, $\eta_t \sim \mathcal{N}(0, Q_t)$. *(10 points)*

Set 2 - Random Variable Transformation (15 points)

Let X and Y be i.i.d random variables having uniform distribution $U[0,1]$. Find the probability density functions of following random variables: *(15 points)*

1. $Z_1 := log(X)$
2. $Z_2 := exp(X)$
3. $Z_3 := X + Y$

Set 3 - MLE (15 points)

1. Estimate the parameters (mean and variance) of a Gaussian distribution using MLE.

2. Find the optimal classification threshold for two equally probable classes represented by the univariate distributions $\mathbb{N}(23, \sigma^2)$ and $\mathbb{N}(33, \sigma^2)$.

3. For the classes in Q2, how does the optimal classification threshold change if the variance of the first class becomes double of the current value? Try out the same when the variance of the second class becomes double, keeping that of the first class unchanged.

Set 4 - Gradients and Optimization (50 points)

1. (Gradient and Hessian) Compute the gradient and the hessian of the three-variable function $f(x, y, z) = 5xy^2z$.

2. (Taylor Series) Find a Maclaurin Series (which is a Taylor Series whose derivatives are computed at zero) approximation for $f(x) = cos(6x)$ upto the fifth derivative.

3. (Lagrange Multipliers) Find the maximum and minimum values of $f(x, y) = 3x - 6y$ subject to the constraint $4x^2 + 2y^2 = 25$. You should try plotting the function using an online graph calculator like desmos and incude the graph in your pdf. You can check out similar problems here.

4. (Gradient Descent) The Gradient Descent algorithm aims to minimise the objective function by taking steps in the direction of the negative of the gradient. We start at a point x_0 and move a positive distance α in the direction of the negative gradient. We keep doing this until we reach the optimum value. You can answer the following questions in 2-3 sentences:

   - Write the gradient descent equation which expresses the update in the value using the gradient. Use the following notation: x_{n+1} is the updated value, x_n is the current value, α is the constant multiplied with the gradient ∇ of the function $f(x_n)$.
   - We aim to minimise our objective function. In gradient descent, why do we follow the **negative** of the gradient, and not the positive? What is the direction the gradient of the function is pointing us in?
   - The α (the term being multiplied with the gradient ∇) in the gradient descent equation is called the *step size*. Very simply, by looking at the words *step* and *size*, can you explain what it may mean in the context of the blindfolded mountain climber we discussed in the lecture?
   - Do you think gradient descent might be able to tell whether the minimum value it has found is local or global?