Lecture 3a

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Lecture 3a: Input and Output

... in which we analyze the form of dynamical systems with inputs and outputs and focus on linear systems.

I. General Control Systems

The general form is

$$\dot{\vec{x}} = f(\vec{x}, u)$$

$$y = g(\vec{x})$$

where $u \in \mathbb{R}$ is the input and y is the output.

To linearize around $\vec{x}=0,\ u=0$:

$$\dot{\vec{x}} \approx \underbrace{\left.\frac{\partial f}{\partial \vec{x}}\right|_{\substack{\vec{x}=0\\ u=0}}}_{A} \vec{x} + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{\substack{\vec{x}=0\\ u=0}} u}_{B}$$

$$y \approx \underbrace{\left.\frac{\partial g}{\partial \vec{x}}\right|_{\vec{x}=0}}_{C}$$

where $\frac{\partial f}{\partial \vec{x}}$ is as before and

$$\frac{\partial f}{\partial u} = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{pmatrix} \text{ and } \frac{\partial g}{\partial \vec{x}} = \left(\frac{\partial g}{\partial x_1} \cdots \frac{\partial g}{\partial x_n} \right).$$

So the linearized form is

$$\boxed{\begin{aligned} \dot{\vec{x}} &= A\vec{x} + Bu \\ y &= C\vec{x} \end{aligned}}$$

Example: Linearize

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ a\sin x_1 + bu\cos x_1 \end{pmatrix}$$

$$y = x_1 \quad \text{near } \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Solution:

$$A = \left.\begin{pmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \\ \frac{\partial}{\partial x_1}(a\sin x_1 + bu\cos x_1) & \frac{\partial}{\partial x_2}(a\sin x_1 + bu\cos x_1) \end{pmatrix}\right|_{\substack{\breve{x}=0 \\ u=0}}$$

$$= \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$$

[Note that $|\lambda I - A| = \lambda^2 - a$

$\Rightarrow \lambda = \pm\sqrt{a}$

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a saddle point.]

$$B = \left.\begin{pmatrix} \dfrac{\partial x_2}{\partial u} \\ \dfrac{\partial}{\partial u}(a\sin x_1 + bu\cos x_1) \end{pmatrix}\right|_0$$

$$= \left.\begin{pmatrix} 0 \\ b\cos x_1 \end{pmatrix}\right|_{x_1=0} = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \vec{x} \end{aligned}}$$

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II. Matrix Exponentials

Recall that if A is a matrix, then

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

It turns out that the solution to

$\dot{\vec{x}} = A\vec{x}$ is

$$\vec{x}(t) = e^{At}\vec{x}(0).$$

This is because

$$\begin{aligned}\frac{d}{dt}e^{At}\vec{x}(0) &= \frac{d}{dt}\left(I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots\right)\vec{x}(0)\\ &= \left(A + A^2t + \frac{A^3t^2}{2!} + \cdots\right)\vec{x}(0)\\ &= A\left(I + At + \frac{(At)^2}{2!} + \cdots\right)\vec{x}(0)\\ &= Ae^{At}\vec{x}(0) = A\vec{x}(t).\end{aligned}$$

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Example: What is e^{At} when $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$?

Solution:

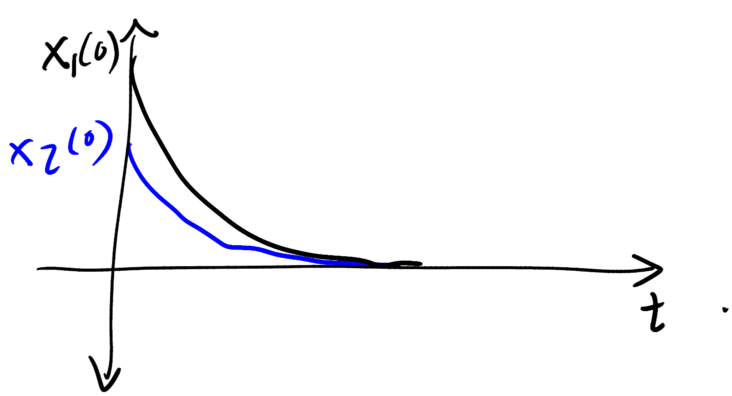
$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} t + \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \frac{t^2}{2!} + \cdots$$

$$= \left(1 - t - \frac{t^2}{2!} + \frac{t^3}{3!} - \cdots\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

This means $\dot{\vec{x}} = A\vec{x}$ has solution

$$x_1(t) = e^{-t} x_1(0)$$
$$x_2(t) = e^{-t} x_2(0)$$



III. Solutions in State Space

Recall that the solution to

$$\dot{\vec{x}} = A\vec{x}$$

is $e^{At}\vec{x}_0$. We now figure out the solution to

$$\dot{\vec{x}} = A\vec{x} + Bu.$$

$$\Leftrightarrow \quad \dot{\vec{x}} - A\vec{x} = Bu$$

multiply by e^{-At}

$$\Leftrightarrow e^{-At}(\dot{\vec{x}} - A\vec{x}) = e^{-At}Bu$$

$$\left(\text{Note: } \frac{d}{dt}\left[e^{-At}\vec{x}\right] = e^{-At}\dot{x} - Ae^{-At}\vec{x} \right.$$
$$\left. = e^{-At}(\dot{\vec{x}} - A\vec{x}) \right)$$

$$\Leftrightarrow \frac{d}{dt}\left[e^{-At}\vec{x}\right] = e^{-At}Bu$$

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Integrate both sides

$$\left(e^{-At}\vec{x}\right)\Big|_0^t = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$\Leftrightarrow$

$$e^{-At}\vec{x}(t) - \vec{x}(0) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$\Leftrightarrow$

$$\vec{x}(t) = \underbrace{e^{At}\vec{x}(0)}_{\text{natural response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{forced response}}$$

Example: Consider the system

$$\dot{\vec{x}} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

and suppose that $u = 1$.

First, computing e^{At} in MATLAB gives

$$\boxed{e^{At} = \begin{pmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{pmatrix}.}$$

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> syms t
> A=[-1 0; 1 -1]
> expm(A*t)
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you can also use MAPLE or Mathematica. Some EE machines may not have "syms."

Next, for the forced response:

$$\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$= \int_0^t \begin{pmatrix} e^{\tau-t} & 0 \\ (t-\tau)e^{\tau-t} & e^{\tau-t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau$$

$$= \int_0^t \begin{pmatrix} 0 \\ e^{\tau-t} \end{pmatrix} d\tau = e^{-t} \int_0^t \begin{pmatrix} 0 \\ e^{\tau} \end{pmatrix} d\tau$$

$$= e^{-t} \begin{pmatrix} 0 \\ e^{t} - 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 1 - e^{-t} \end{pmatrix}}$$

So the solution is

$$\vec{x}(t) = \begin{pmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} 0 \\ 1-e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} x_1(0) \\ te^{-t} x_1(0) + e^{-t} x_2(0) + 1 - e^{-t} \end{pmatrix}$$

Note that as $t \to \infty$,

$\vec{x}(t) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This is what happens when you force a stable system. The natural response goes to 0 and the forced response is all that's left!