Monday, October 17, 2005
8:32 PM

Lecture 6a: From transfer functions to state space

... in which we examine several ways to obtain state space representations from block diagrams or signal flow graphs.

I. Motivating Example

Recall the rocket system

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -g/m + f \end{pmatrix}$$

thruster force

Define $u = -g/m + f$. Then the system has the form

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

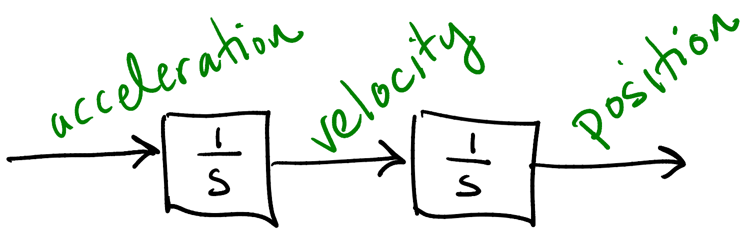
The transfer function is

$$\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B$$

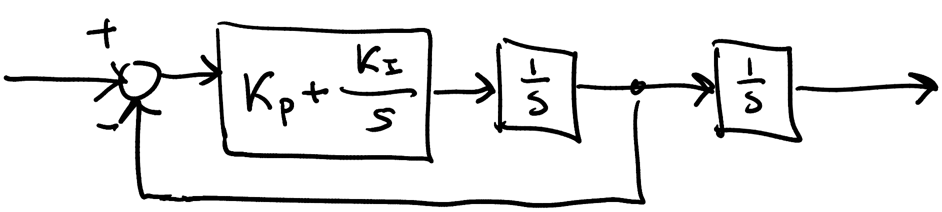
$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{s^2} \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{s^2} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s \\ 1 \end{pmatrix} = \frac{1}{s^2}$$

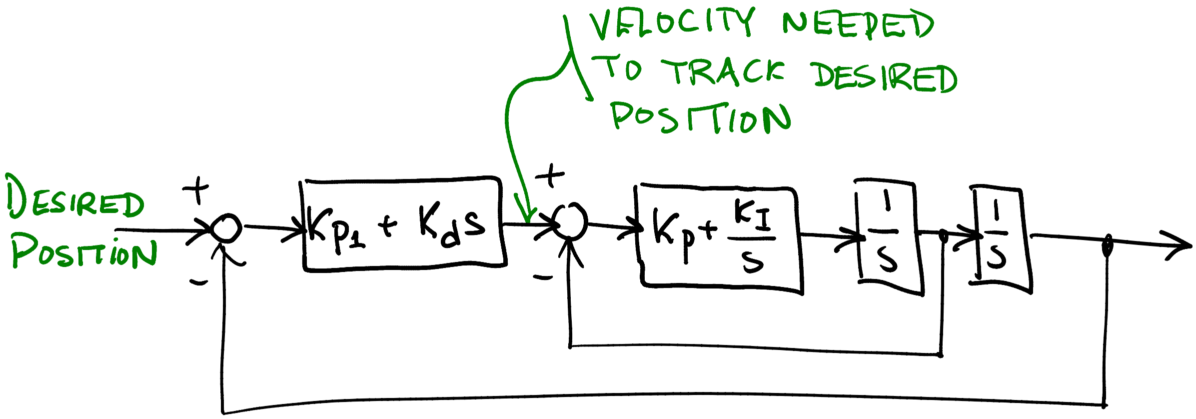
One way to look at this is as a series of blocks



The velocity controller we made looked essentially like:



One can then do position control assumming that velocity is directly actuated:



This is called inner-loop/outer-loop control. The inner loop is sometimes called the <u>slave</u> and the outer loop is called the <u>master</u>.

The inner loop has T.F. $L(s) = \dfrac{G(s)}{1+G(s)}$ where

$$G(s) = \left(K_p + \frac{K_I}{s}\right)\frac{1}{s} = \frac{K_p s + K_I}{s^2}$$

So that $L(s) = (K_p s + K_I)/(s^2 + K_p s + K_I)$.

The whole system's TF is then

$$T(s) = \frac{H(s)}{1 + H(s)}$$

where $H(s) = (K_{P1} + K_D s)\,L(s) \cdot \frac{1}{s}$

$$= \frac{(K_{P1} + K_D s)(K_P s + K_I)}{s^3 + K_P s^2 + K_I s}$$

$$= \frac{K_P K_D s^2 + (K_D K_I + K_{P1} K_P)s + K_{P1} K_I}{s^3 + K_P s^2 + K_I s}$$

So,

$$T(s) = \frac{K_P K_D s^2 + (K_D K_I + K_{P1} K_P)s + K_{P1} K_I}{s^3 + (K_P + K_P K_D)s^2 + (K_I + K_D K_I + K_{P1} K_P)s + K_{P1} K_I}$$

Our goal is to understand this system better by looking at it in state space.

Monday, October 17, 2005
10:03 PM

II. Phase Canonical Form

Goal: Write Down A, B and C matrices from a T.F. So say

$$\frac{Y(s)}{R(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}.$$

Multiply by $Z(s)/Z(s)$ to get

$$Y(s) = (b^3 s^3 + \dots + b_0) Z(s)$$

$$U(s) = (s^4 + \dots + a_0) Z(s)$$

$$\Updownarrow \mathcal{L}$$

$$y = b_3 \dddot{z} + b_2 \ddot{z} + b_1 \dot{z} + b_0 z$$

$$u = \ddddot{z} + a_3 \dddot{z} + a_2 \ddot{z} + a_1 \dot{z} + a_0 z$$

Put $x_1 = z$

$x_2 = \dot{z} = \dot{x}_1$

$x_3 = \ddot{z} = \dot{x}_2$

$x_4 = \dddot{z} = \dot{x}_3$

Then

$$\dot{x}_4 = \ddddot{z} = -a_0 z - a_1 \dot{z} - a_2 \ddot{z} - a_3 \dddot{z} + u$$

$$= -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u$$

And $y = b_3 x_4 + b_2 x_3 + b_1 x_2 + b_0 x_1$

In matrix form this is

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \end{pmatrix} \vec{x}$$

For example, with the inner-loop $L(s) = (K_p s + K_I) / (s^2 + K_p s + K_I)$ we get

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -K_I & -K_p \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} K_I & K_p \end{pmatrix} \vec{x}$$

In this case $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where $x_2 = V$ is the velocity and x_1 is the internal state of the integrator part of the controller.

Note also that in this case the output $y = K_I x_1 + K_p x_2$ is the ouptut of the controller, which is kind-of wierd!

One way to look at phase canonical form is to look at the signal flow graph of $Y(s)/U(s)$. First note:

$$\dot{x}_i \xrightarrow{1/s} x_i$$

represents integration. Now, multiply $Y(s)/R(s)$ by s^{-4} to get

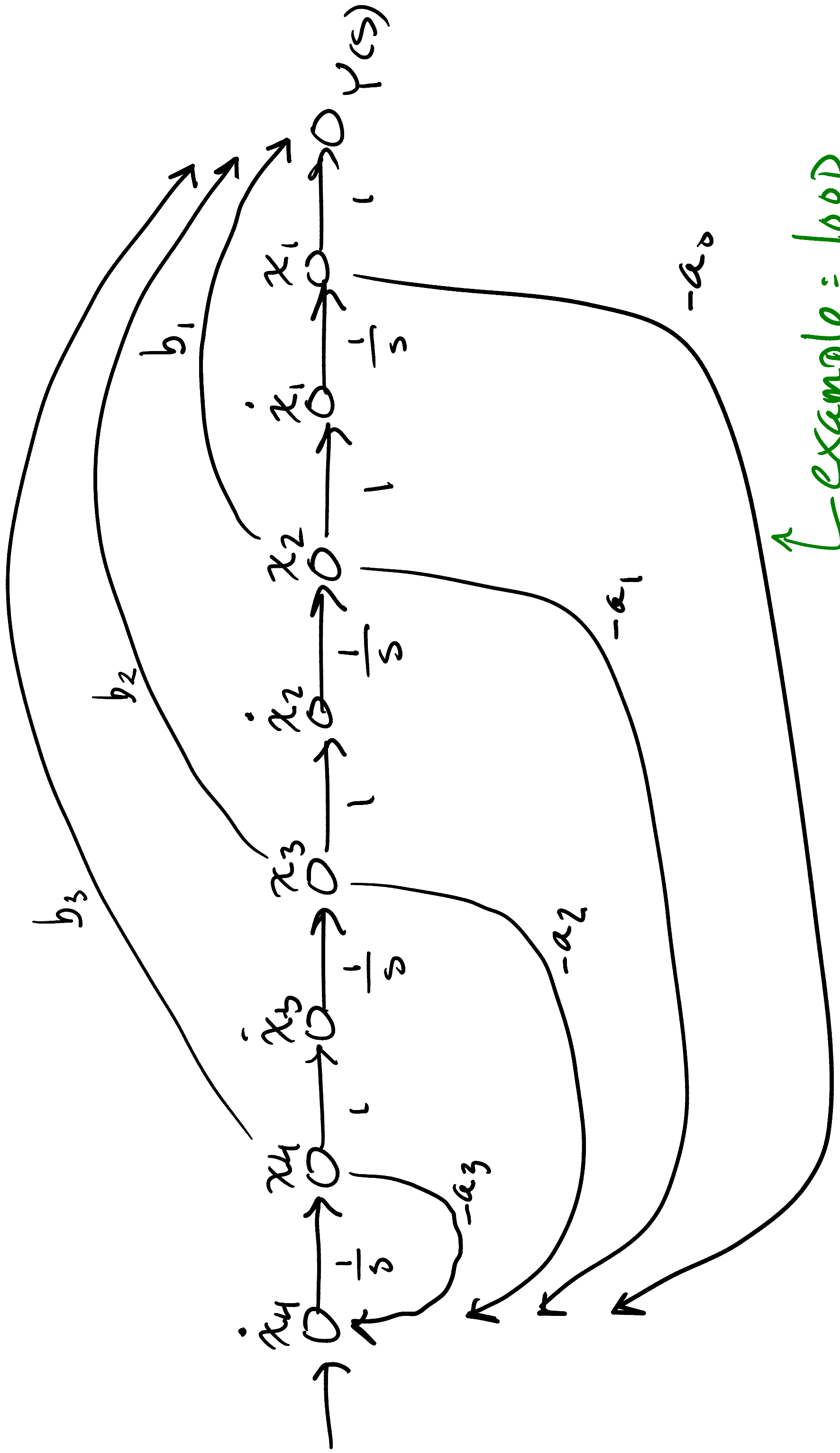
$$\frac{Y(s)}{R(s)} = \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

$$= \frac{\sum T_k \Delta_k}{\Delta}$$

forward paths

loop terms

Using Mason's rule, we suppose the TF came from a signal flow graph:



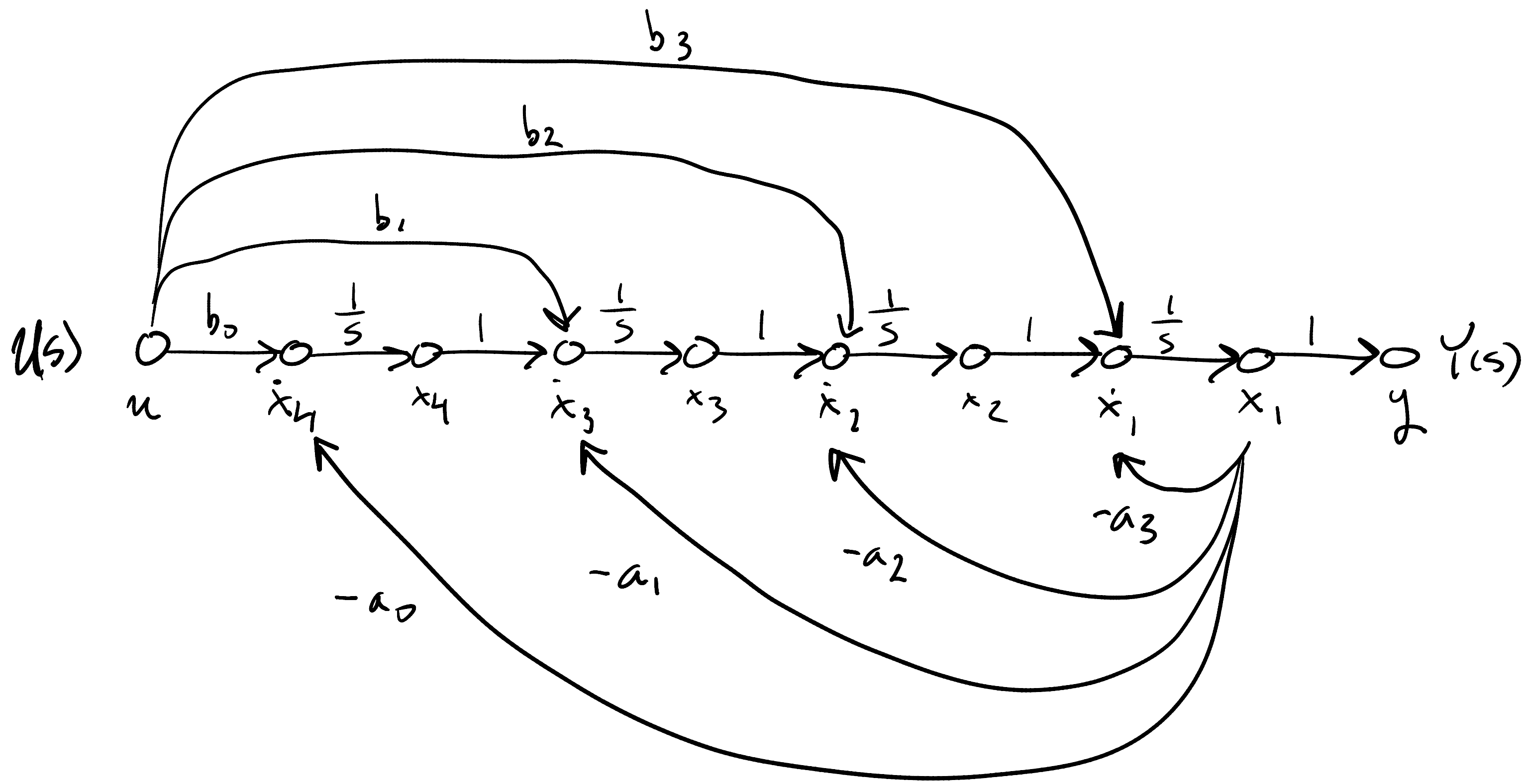
Example: loop gain is $\left(\frac{1}{s}\right)^3(-a_0)$

$= a_0 s^{-3}$

Monday, October 17, 2005
10:42 PM

III. Feedforward Canonical Form

The above analysis suggests another form.



$$\frac{Y(s)}{R(s)} = \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-1} + a_0}$$

In this case, we get

$$\dot{\vec{x}} = \begin{pmatrix} -a_3 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 1 \\ -a_0 & 0 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \vec{x}$$

This is a bit more natural for the output. Now look at $L(s)$:

$$\dot{\vec{x}} = \begin{pmatrix} -K_P & 1 \\ -K_I & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} K_P \\ K_I \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \vec{x}$$

But niether of the states correspond to a physical variable!