Lecture 1b

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Lecture 1b: An Introduction to Dynamical Systems

... in which we introduce continuous dynamical systems and the basic definitions we use to talk about them.

I. Basic Notions

Let $\vec{x}(t) \in \mathbb{R}^n$ be an n-dimensional vector. A continuous, autonomous dynamical system has the form

$$\dot{\vec{x}} = f(\vec{x}),$$

stating how each component changes with time.

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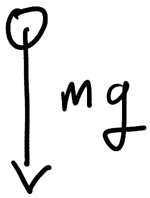
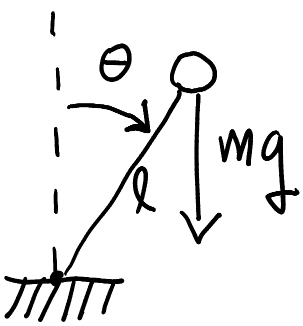
Examples:

(1)



$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -g \end{pmatrix}$$

(2)



θ = position

$\omega = \dot{\theta}$ = angular velocity

friction : $b > 0$

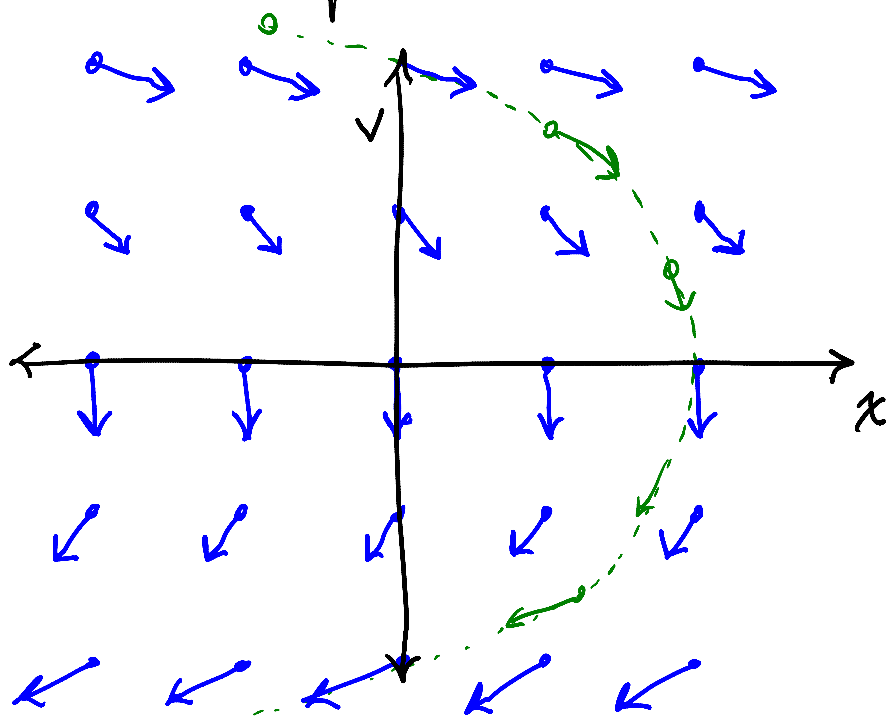
$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \omega \\ \frac{g}{\ell}\sin\theta - b\omega \end{pmatrix}$$

[note, here $\vec{x} = \begin{pmatrix} \theta \\ \omega \end{pmatrix}$]

II. Phase Portraits

A phase portrait is a picture of the vector field defining a dynamical system.

①



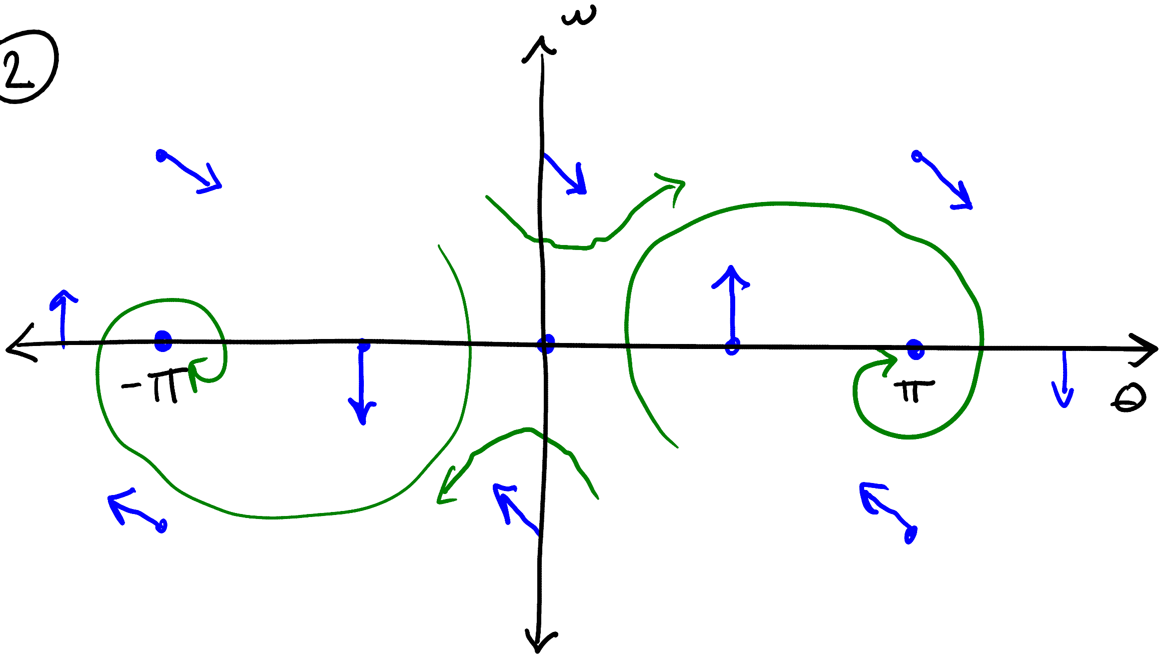
$\begin{pmatrix} x \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -g \end{pmatrix}$

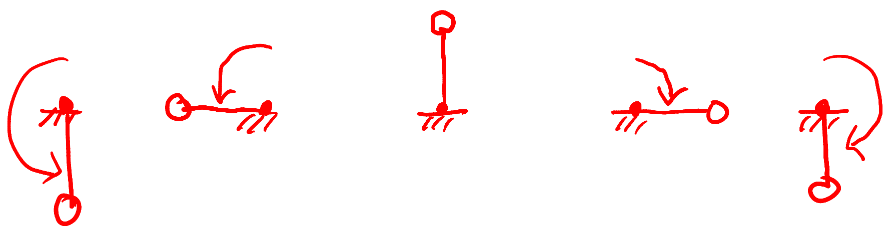
$\begin{pmatrix} x \\ \pm 1 \end{pmatrix} \mapsto \begin{pmatrix} \pm 1 \\ -g \end{pmatrix}$

$\begin{pmatrix} x \\ \pm 2 \end{pmatrix} \mapsto \begin{pmatrix} \pm 2 \\ -g \end{pmatrix}$

We draw a vector from x to $x+f(x)$. or to $x+\varepsilon f(x)$ so the arrows don't overlap.

(2)

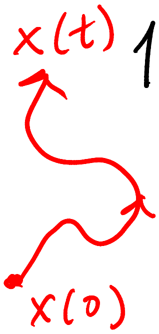




Note: $f\begin{pmatrix} k\pi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

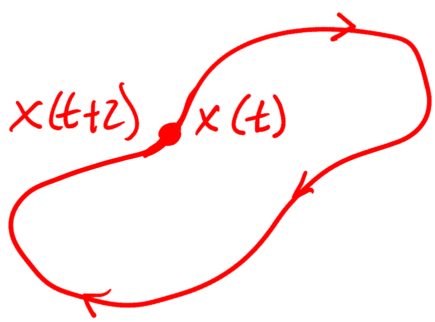
Note: We follow the vectors to draw trajectories (smooth curves) that give a better picture.

Definitions:

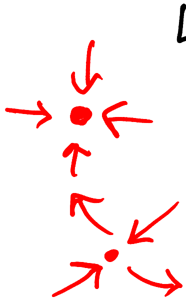


1. A function of time $\vec{x}(t)$ satisfying $\dot{\vec{x}} = f(\vec{x})$ is called a solution or trajectory.

2. If $\vec{x}(t) = \vec{x}(t+\tau)$ for some τ, then $\vec{x}(t)$ is a limit cycle.



3. If $f(\vec{x}) = 0$, then $\vec{x}$ is an equilibrium point.



4. If $\vec{x}$ is an eq. point and all nearby trajectories tend to $\vec{x}$, then $\vec{x}$ is stable. Otherwise, it is unstable.

Examples:

① $$x(t) = x(0) + v(0)t - \frac{1}{2}gt^2$$

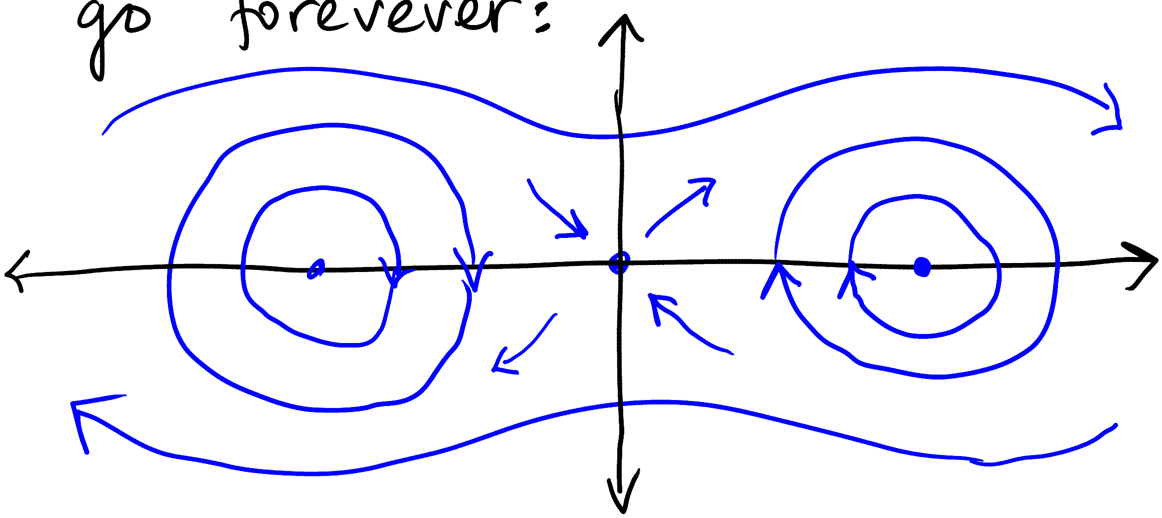
$$v(t) = v(0) - gt$$

defines the general form of a trajectory for the falling-rock system.

The system has no limit cycles and no equilibrium points.

② The points $\begin{pmatrix} k\pi \\ 0 \end{pmatrix}$ are all equilibrium points. From the phase portrait, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ seems unstable and $\begin{pmatrix} \pm\pi \\ 0 \end{pmatrix}$ seem stable.

With $b>0$, there are no limit cycles. However, with $b=0$, the pendulum should go forever:



For example, the pendulum rocking back and forth between $\pi-1$ and $\pi+1$ is a limit cycle.

III. Systems with Control

A dynamical system with control has the form

$$\dot{\vec{x}} = f(\vec{x}, u)$$

control input

observable output

$$y = g(\vec{x})$$

Examples:

① $$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -g + u \end{pmatrix}$$

thruster force?

$$g\begin{pmatrix} x \\ v \end{pmatrix} = v$$

for velocity control, you <u>may</u> only measure v.

Example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 + \underline{u} \\ (x_1 - 1)(x_1 + 1) - x_2 \end{pmatrix}$$

$$y = \underline{x_1}$$

When $u = 0$, we get the natural response.

To find equilibrium points, solve

$$\vec{0} = f(\vec{x}) = \begin{pmatrix} x_2 \\ (x_1 - 1)(x_1 + 1) - x_2 \end{pmatrix}$$

$$\Rightarrow x_2 = 0$$

$$\Rightarrow (x_1 - 1)(x_2 + 1) = 0$$

$$\Rightarrow x_1 = \pm 1$$

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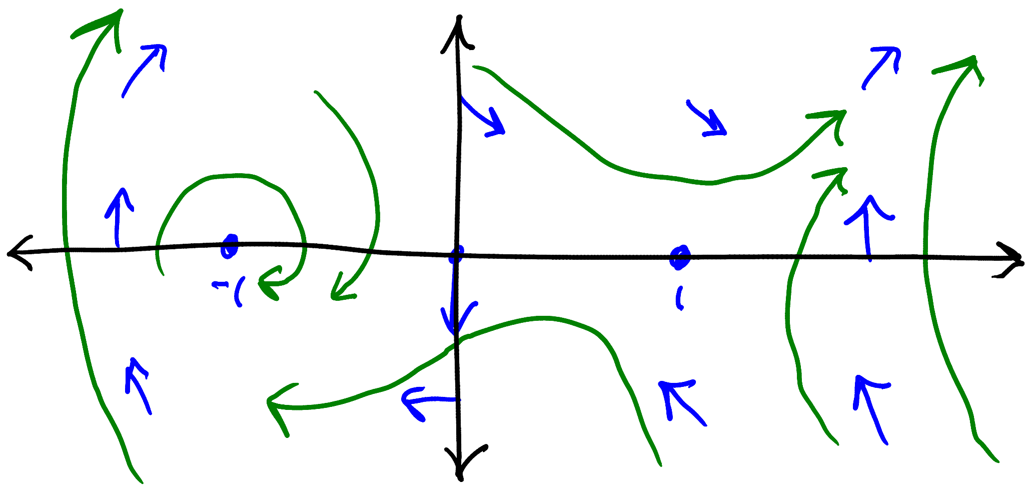
The resulting phase portrait is:



stable unstable

Next we will learn that when f and g are linear, they have the form

$$f(\vec{x}, u) = A\vec{x} + Bu$$

$$g(\vec{x}) = C\vec{x}$$

where A, B and C are matrices.