Lecture 4b

Wednesday, October 05, 2005
10:52 PM

Lecture 4b. Transfer Functions

... in which we begin our desent into the...

!! FREQUENCY DOMAIN !!

I. Transfer Functions for Linear ODEs.

Consider an ODE of the form

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_1\dot{u} + b_0 u$$

where y is the output and u is the input. The transfer function of the ODE is defined by

$$\frac{Y(s)}{U(s)} = \frac{\text{Laplace Transform of the output}}{\text{LaPlace Transform of the Input}}$$

$\boxed{\text{Assumming } y(0) = \dot{y}(0) = u(0) = 0.}$

Recall the LaPlace Transform:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

where $s \in \mathbb{C}$.

The most important consequence of this definition is the following:

Say $F(s)$ is the LaPlace Transform of $f(t)$. Then

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$= -f(0) + sF(s).$$

Also, $f''(t) \xrightarrow{\mathcal{L}} s^2 F(s)$ etc.

Example:

$$2\ddot{y} - \dot{y} + y = \dot{u} - u$$

$$\updownarrow \mathcal{L}$$

$$2s^2 Y(s) - sY(s) + Y(s) = sU(s) - U(s)$$

$$\Leftrightarrow \quad Y(s)(2s^2 - s + 1) = (s-1)U(s)$$

$$T(s) = \frac{Y(s)}{U(s)} = \boxed{\frac{s-1}{2s^2 - s + 1}}$$

- $2s^2 - s + 1$ is the characteristic equation. Its roots are the poles or eigenvalues.
- The roots of $s-1$ are called the zeros of $T(s)$.

Pole zero plots:

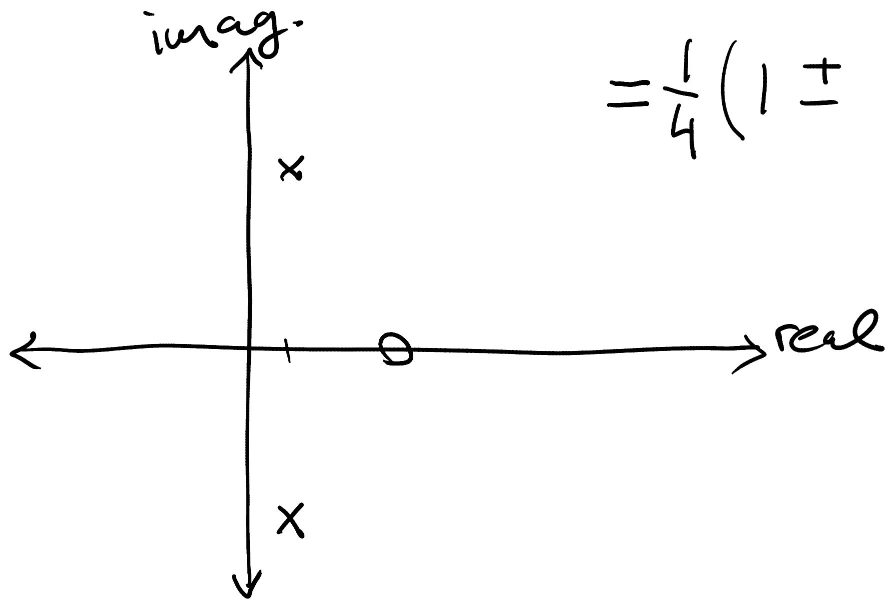
o = zero

x = pole

$2s^2 - s + 1 = 0$

$\Rightarrow s = \dfrac{1 \pm \sqrt{1-8}}{4}$

$= \frac{1}{4}\left(1 \pm \sqrt{7}\, j\right).$



We will get back to these. The point is that a pole-zero plot tells you pretty much everything you need to know about $T(s)$.

II. T.F.s from State Space

Consider the system

$$\dot{\vec{x}} = A\vec{x} + Bu$$

$$y = C\vec{x}.$$

Our goal is to write the T.F. for this system. Taking the Laplace Transform:

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = C\underbrace{X(s)}_{\text{vector of L.P.T. of } x(t).}$$

$$\Leftrightarrow \quad sX(s) - AX(s) = BU(s)$$

$$\Leftrightarrow \quad (sI - A)X(s) = BU(s)$$

$$X(s) = (sI-A)^{-1}BU(s)$$

Wednesday, October 05, 2005
11:16 PM

Since $Y(s) = CX(s)$, we get

$$T(s) = \frac{Y(s)}{U(s)} = \frac{CX(s)}{U(s)}$$

$$= \frac{C(sI-A)^{-1}B\cancel{U(s)}}{\cancel{U(s)}}$$

$$= C(sI-A)^{-1}B$$

The quantity $(sI-A)^{-1}$ is usually denoted $\Phi(s)$.

[Notice the similarity between the eigenvalue formula and $\Phi(s)$.]

We get

$$\boxed{T(s) = C\Phi(s)B}$$

Example:

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \vec{x} + \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix} u$$

$$y = \begin{pmatrix} -\frac{1}{2} & 0 \end{pmatrix} \vec{x}$$

$$\Phi(s) = (sI - A)^{-1} = \begin{pmatrix} s & -1 \\ \frac{1}{2} & s - \frac{1}{2} \end{pmatrix}^{-1}$$

$$= \frac{1}{s(s-\frac{1}{2}) + \frac{1}{2}} \begin{pmatrix} s-\frac{1}{2} & 1 \\ -\frac{1}{2} & s \end{pmatrix}$$

$$= \frac{2}{s^2 - s + 1} \begin{pmatrix} s-\frac{1}{2} & 1 \\ -\frac{1}{2} & s \end{pmatrix}.$$

So

$$T(s) = C\Phi(s)B$$

$$= \frac{\cancel{2}}{2s^2 - s + 1}\begin{pmatrix} -\frac{1}{\cancel{2}} & 0 \end{pmatrix}\begin{pmatrix} s-\frac{1}{2} & 1 \\ -\frac{1}{2} & s \end{pmatrix}\begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2s^2 - s + 1}\begin{pmatrix} \frac{1}{2}-s & -1 \end{pmatrix}\begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2s^2 - s + 1}\begin{pmatrix} s-\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \boxed{\frac{s-1}{2s^2 - s + 1}}$$

Note: This is the same as for

$$2\ddot{y} - \dot{y} + y = \dot{u} - u.$$ $\boxed{\text{Why?}}$

III. Final Value Theorem

If $F(s) \longleftrightarrow f(t)$ then

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$= \int_0^\infty f'(t)\, e^{-st}\, dt$$

Take the limit as $s \to 0$:

$$\int_0^\infty f'(t)\, dt = \lim_{s \to 0} \left(sF(s) - f(0) \right)$$

$$f(\infty) - \cancel{f(0)} = \lim_{s \to 0} sF(s) - \cancel{f(0)}$$

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s).}$$

Note:

1. Can't have more than one pole at 0
2. Must be stable (no poles in RHP).

Example:

$$Y(s) = \frac{s+4}{s(s+1)}$$

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s+4}{s+1} = 4.$$

Check: $$Y(s) = \frac{4}{s} - \frac{3}{s+1}$$

$$\updownarrow$$

$$y(t) = 4 - 3e^{-t} \to 4.$$

This gives a way to check the response of a transfer function to various inputs.

STEP INPUTS:

A <u>unit</u> step is the function

$$r(t) = 1 \xleftrightarrow{\mathcal{L}} R(s) = \frac{1}{s}.$$

$(t \geq 0)$.

So $T(s) = \dfrac{Y(s)}{R(s)} \Rightarrow Y(s) = R(s)T(s)$

$$= \frac{1}{s}T(s).$$

For example, say

$$T(s) = \frac{Y(s)}{R(s)} = \frac{s+2}{(s+1)^2}$$

$$Y(s) = \frac{1}{s} T(s) = \frac{s+2}{s(s+1)^2}$$

$$\lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{s+2}{(s+1)^2}$$

$$= 2.$$