Lecture 15a

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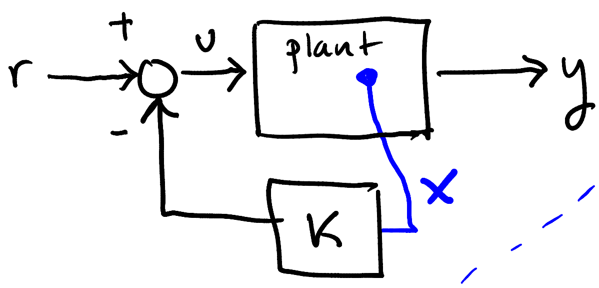
Lecture 15a : Observers

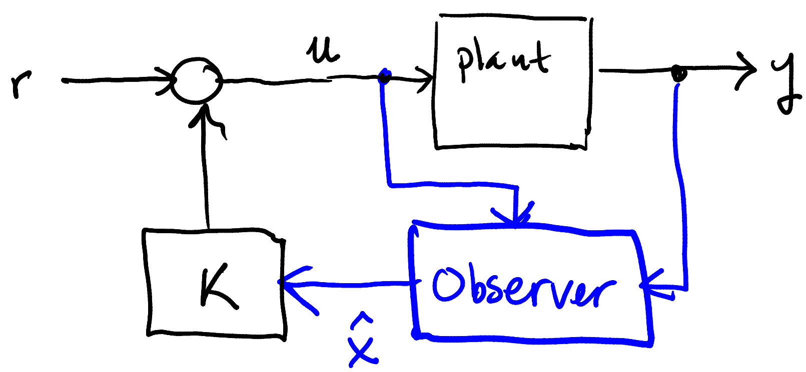
... in which we stop cheating and figure out the full state using only the input and output.

I. The Problem

Full state feedback seems to require information we do not have ...



But in most cases you can estimate the state of the plant based only on $u(t)$ and $y(t)$, using an Observer.



II. The Idea

The observer block simulates the system using u. A very simple observer has the form:

$$\dot{\hat{x}} = A\hat{x} + Bu$$
$$\hat{y} = C\hat{x}$$

A POOR OBSERVER

where $\hat{x}$ is a new vector of the same dimension as $\vec{x}$ called the estimate. The A, B and C matrices come from the plant. In this case

$$\dot{\vec{x}} - \dot{\hat{x}} = A(\vec{x} - \hat{x})$$
$$y - \hat{y} = C(\vec{x} - x).$$

Thus, $\hat{x}$ converges to $\vec{x}$. Unfortunately, it does so at the same rate as $\vec{x}$ converges to $x(\infty)$, which is too slow to be useful.

III. Observer Design

A more useful observer turns out to be

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y-\hat{y})$$
$$\hat{y} = C\hat{x}.$$

L / $n \times 1$ matrix

where L is a matrix of gains (kind of like K in full state feedback). To see how to obtain L, define the error $\vec{e} = \vec{x} - \hat{x}$. Then

$$\dot{\vec{e}} = \dot{\vec{x}} - \dot{\hat{x}} = A\vec{x} - A\hat{x} - L(y-\hat{y})$$
$$= A\vec{e} - LC\vec{x} + LC\hat{x}$$
$$= (A - LC)\vec{e}.$$

Thus, by defining L, we can place the poles of $A-LC$ <u>anywhere</u> to make $\vec{e}$ converge quickly.

This is similar to the problem of finding K so that A-BK has the right poles. But there are differences

OBSERVER DESIGN	FULL STATE FEEDBACK
$A-LC$	$A-BK$
L is $n \times 1$	K is $1 \times n$
easy when A is in feedforward form	easy when A is in phase form
the system may not be <u>observable</u>	the system may not be <u>controllable</u>
want to place poles to be 10 times faster than system	want to place poles for performance

Example: In lec14a we designed a full state feed back controller $K = (10 \quad -2)$ for

$$\dot{\vec{x}} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u.$$

So the poles were at $-5, -6$.

Say that $y = (1 \quad 0)\,\vec{x}$. To define an observer, we evaluate

$$A - LC = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} (1 \quad 0)$$

$$= \begin{pmatrix} -1 - l_1 & 1 \\ -l_2 & -2 \end{pmatrix}$$

And $|sI - (A - LC)|$ is

$$(s + 1 + l_1)(s + 2) + l_2$$

$$= s^2 + (3 + l_1)s + (2l_1 + l_2 + 2).$$

To make $\hat{x}$ converge to $\vec{x}$ 10 times faster than $\vec{x}$ converges, we set the desired char. eqn to

$$(s+50)(s+60) = s^2 + 110s + 3000.$$

which means

$$3 + l_1 = 110 \quad \Rightarrow l_1 = 107$$

$$2l_1 + l_2 + 2 = 3000$$

$$\Leftrightarrow \quad 214 + l_2 = 2998$$

$$\Leftrightarrow \quad l_2 = 2784.$$

IV. Observability

A system is observable if the initial state $\vec{x}(0)$ can be found from watching $u(t)$ and $y(t)$ for a finite intervals of times.

For linear systems, this is equivalent to saying that the poles of $A-LC$ can be placed.

A test for observability is the following fundamental result:

Theorem: A system $\dot{\vec{x}} = A\vec{x} + Bu$, $y = C\vec{x}$ is observable if and only if

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ C^2A \\ \vdots \\ C^{n-1}A \end{pmatrix}$$

has rank n (where n is the dimension of $\vec{x}$).

Example: The initial state of a rocket cannot be observed from position:

A rocket (very simply) has the model:

$$\ddot{y} = u$$

or

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \vec{x} .$$

Here,

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

and $|\mathcal{O}| = 0 \Rightarrow$ unobservable.