Lecture 14a: Introduction to State Based Control

... in which we design controllers using the full state of the system and place the poles wherever we want to.

I. Phase Variables

The phase canonical, or controller, form of

$$G(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0}$$

is

$$\dot{\vec{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \end{pmatrix} \vec{x}$$

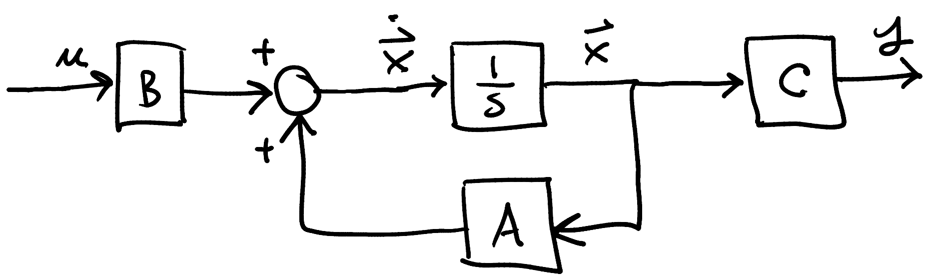
Tuesday, November 22, 2005
7:41 AM

Note that we can read the characteristic polynomial from the bottom line.

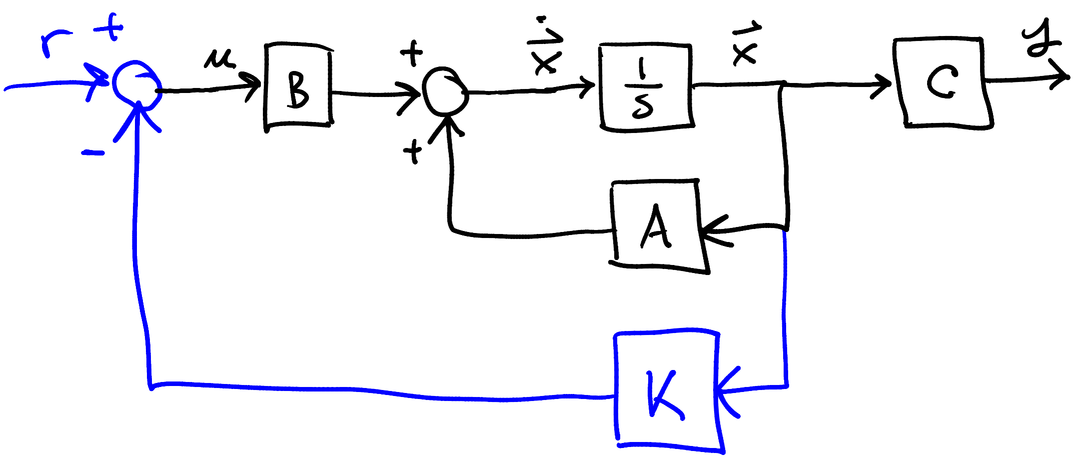
Suppose we set

$$u = -K\vec{x} + r = -(K_1\ K_2\ K_3\ K_4) + r$$

Then we go from:



to:



Tuesday, November 22, 2005
7:50 AM

Substituting $u = -K\vec{x} + r$ into the state equations gives

$$\dot{\vec{x}} = A\vec{x} - B(K\vec{x} + r)$$

$$= (A - BK)\vec{x} + Br$$

and (still using the phase variables from before):

$$A - BK = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} (k_1\ k_2\ k_3\ k_4)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(a_0+k_1) & -(a_1+k_2) & -(a_2+k_3) & -(a_3+k_4) \end{pmatrix}$$

which has characteristic polynomial

$$s^4 + (a_3+k_4)s^3 + (a_2+k_3)s^2 + (a_1+k_2)s + (a_0+k_1)$$

Note that in this polynomial we have a gain on each term, so we can put the poles of the system wherever we want to.

Example: Say $G(s) = \dfrac{1}{s^2+s}$

and we want poles at $-1 \pm j$.

Setting $u = -K\vec{x} + r$ gives the new polynomial

$$s^2 + (a_1 + k_2)s + (a_0 + k_1)$$

$$= s^2 + (k_2+1)s + k_1 = (s+1-j)(s+1+j)$$

$$= s^2 + 2s + 2$$

So we take $k_2 + 1 = 2 \Rightarrow k_2 = 1$

$k_1 = 2$.

II. Arbitrary States

Systems that are not in phase canonical form are amenable to this treatment too, except you have to work to take the determinant of $sI - A$.

Example: Design a controller for the system

$$\dot{\vec{x}} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u$$

that has 10% overshoot and a settling time of 2s. Translating this into 2nd order poles, we want dominant poles at

$$s = -2 \pm 2.7j.$$

The system is 3rd order, however. So we pick a third pole at −5.

The resulting polynomial is

$$(s+5)(s+2-2.7j)(s+2+2.7j)$$

$$= (s+5)(s^2+4s+11.3)$$

$$= s^3+5s^2+4s^2+20s+11.3s+56.5$$

$$= s^3+9s^2+31.3s+56.5.$$

Now we evaluate $|sI-(A-BK)| =$

$$\begin{pmatrix} s & & 0 \\ & s & \\ 0 & & s \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ k_1 & k_2 & k_3 \\ k_1 & k_2 & k_3 \end{pmatrix}$$

$$= \begin{pmatrix} s-1 & -2 & -1 \\ k_1 & s+1+k_2 & k_3 \\ k_1 & k_2 & s-1+k_3 \end{pmatrix}$$

The determinant of $(sI - A + BK)$ is

$$(s-1)\left[(s+1+k_2)(s-1+k_3) - k_2 k_3\right]$$
$$- (-2)\left[k_1(s-1+k_3) - k_1 k_3\right]$$
$$+ (-1)\left[k_1 k_2 - k_1(s+1+k_2)\right]$$

which evaluates to

$$s^3 + (-1+k_2+k_3)s^2 + (-1+3k_1+2k_2)s + (1-k_1+k_2-k_3)$$

Setting the coeficients equal to the desired values gives

$$-1+k_2+k_3 = 9$$
$$-1+3k_1+2k_2 = 31.3$$
$$1-k_1+k_2-k_3 = 56.5$$

which implies

$$k_1 = 48.9, \quad k_2 = 57.2, \quad k_3 = -47.2$$

III. Integral Control

Note, you can also do integral control this way. You add a new state for the integrator.

$$\dot{x}_I = r - y$$
$$= r - C\vec{x}.$$

Together with $\dot{\vec{x}} = Ax + Bu$, $y = Cx$ we have

$$\begin{pmatrix} \dot{\vec{x}} \\ \dot{x}_I \end{pmatrix} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ x_I \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix} \vec{x}.$$

But

$$u = -K\vec{x} + k_I x_I$$

$$= -\begin{pmatrix} K & -k_I \end{pmatrix} \begin{pmatrix} \vec{x} \\ x_I \end{pmatrix}$$

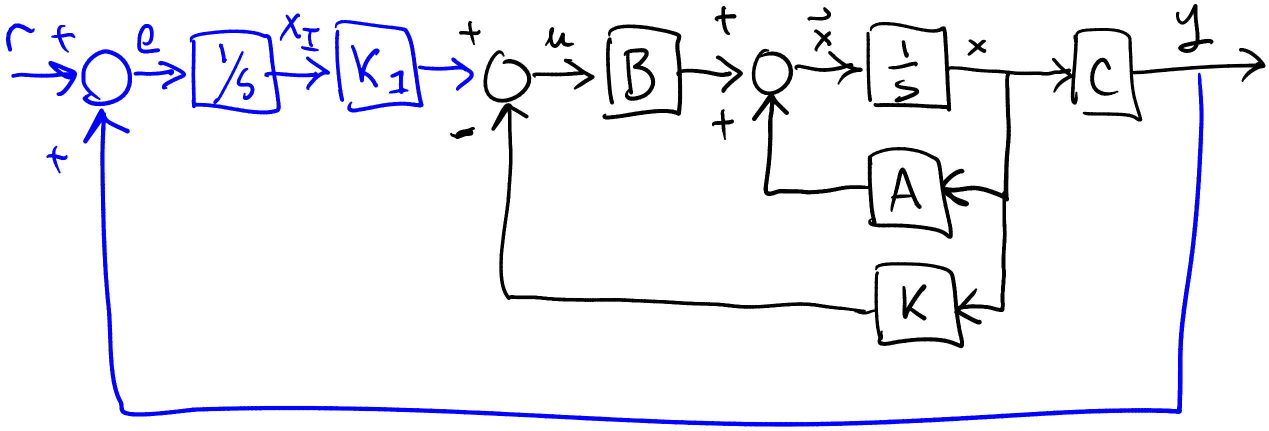
So the state equations become

$$\begin{pmatrix} \dot{\vec{x}} \\ x_I \end{pmatrix} = \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}\begin{pmatrix} \vec{x} \\ x_I \end{pmatrix} - \begin{pmatrix} B \\ 0 \end{pmatrix}\begin{pmatrix} K & -k_I \end{pmatrix}\begin{pmatrix} \vec{x} \\ x_I \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

$$= \begin{pmatrix} A & 0 \\ -C & 0 \end{pmatrix}\begin{pmatrix} \vec{x} \\ x_I \end{pmatrix} - \begin{pmatrix} BK & -Bk_I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \vec{x} \\ x_I \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

$$= \begin{pmatrix} A-BK & Bk_I \\ -C & 0 \end{pmatrix}\begin{pmatrix} \vec{x} \\ x_I \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r.$$

In block diagram form



Example: Say $G(s) = \frac{1}{s}$, the rocket example. Then

$$\dot{x} = 0x + u$$
$$y = x$$

Adding an integrator and controller gives

$$A - BK = 0 - 1 \cdot K_1 = -k_1$$

$$Bk_I = k_I$$

so

$$\begin{pmatrix} \dot{x} \\ \dot{x}_I \end{pmatrix} = \overbrace{\begin{pmatrix} -k_1 & k_I \\ -1 & 0 \end{pmatrix}}^{\tilde{A}} \begin{pmatrix} x \\ x_I \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ x_I \end{pmatrix}.$$

The characteristic equation is

$$|sI - \tilde{A}|$$

$$= \begin{vmatrix} s+K_1 & K_I \\ 1 & s \end{vmatrix} = s(s+K_1) + K_I$$

$$= s^2 + K_1 s + K_I$$

which is the PI controller we found before. Clearly, we can place the poles anywhere we want to.

V. This May not Always Work

Consider the system

$$\dot{\vec{x}} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \vec{x}$$

Note that the input u doesn't affect x_1 in any way. When we apply full-state-feedback, we get

$$A-BK = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_1 \;\; k_2)$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ -k_1 & -k_2 \end{pmatrix}$$

which has characteristic polynomial

$$|sI - A + BK|$$

$$= \begin{vmatrix} s+1 & 0 \\ k_1 & s+k_2 \end{vmatrix} = (s+1)(s+k_2)$$

$$= s^2 + (k_2+1)s + k_2.$$

The gain k_1 is not present. This means that we cannot put the poles anywhere in the s-plane, only on a 1D line parameterized by k_2.