Lecture 4a

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Lecture 4a: Matrix Exponentials & Block Diagrams

I. Computing the Matrix Exponential

(A) The first way to go is to use a trick.

Recall that if A is a matrix with distinct eigenvalues that

$$A = PDP^{-1}$$

where

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and $P = (\vec{v}_1 \cdots \vec{v}_n)$. It turns out that knowing the exponential of D is all we need.

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To see this, note that

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$= I + PDP^{-1} + \frac{(PDP^{-1})^2}{2!} + \frac{(PDP^{-1})^3}{3!} + \cdots .$$

Next, see that

$$(PDP^{-1})^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

etc.

So we get

$$e^A = I + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \cdots$$

$$= Pe^DP^{-1} .$$

Example: What is the matrix exponential of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} ?$$

① Find the eigenvalues

$$|\lambda I - A| = (\lambda-1)(\lambda-4) - 4$$
$$= \lambda^2 - 4\lambda - \lambda + 4 - 4 = \lambda(\lambda-5)$$
so $\lambda = 5, 0$.

② Find the eigenvectors

$$\lambda_1 = 5 \Rightarrow A\vec{v}_1 = 5\vec{v}_1 \Rightarrow u_1 + 2u_2 = 5u_1$$
$$\Rightarrow u_2 = 2u_1$$
$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = 0 \Rightarrow A\vec{v}_2 = \vec{0} \Rightarrow u_1 = -2u_2$$
$$\Rightarrow \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

So $P = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

(3) Find P^{-1}.

Recall $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

So

$$P^{-1} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{1+4}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{pmatrix}.$$

(4) Use the trick

$$e^A = Pe^D P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} e^5 & 0 \\ 0 & e^0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} e^5 & -2 \\ 2e^5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} e^5+4 & 2e^5-2 \\ 2e^5-2 & 4e^5+1 \end{pmatrix}.$$

B) Unfortunately, A is not always diagonalizable. In this case we have to expand e^A.

Example: Find e^{At} when $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$

First, compute the terms in e^A

$$A^2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix}$$

$$\vdots$$

$$A^k = \begin{pmatrix} (-1)^{k-1} & 0 \\ k(-1)^k & (-1)^{k-1} \end{pmatrix}$$

Then compute sum for each element in the matrix:

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} t + \frac{t^2}{2!}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} + \cdots$$

$$\left(e^{At}\right)_{11} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots = e^{-t}.$$

Same for $\left(e^{At}\right)_{22}$.

$$\left(e^{At}\right)_{12} = 0.$$

$$\left(e^{At}\right)_{21} = 0 + t - 2\frac{t^2}{2!} + 3\frac{t^3}{3!} - 4\frac{t^4}{4!} + \cdots$$

$$= t\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = te^{-t}.$$

So we get

$$e^{At} = \begin{pmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{pmatrix}.$$

For more info, see the supplimental material posted on the course web page from the robotics course I teach.

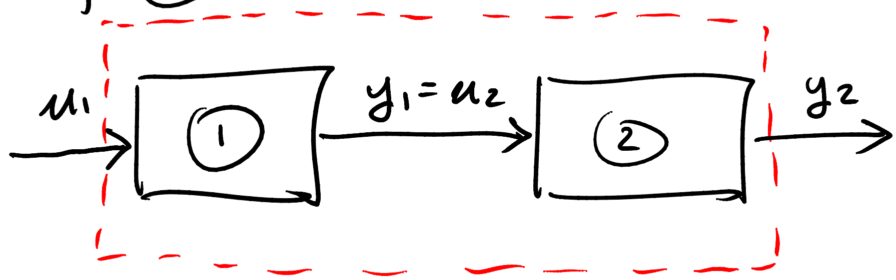
II. Combining Systems

Consider two systems of the form:

$$\text{①} \begin{cases} \dot{\vec{x}}_1 = A_1 \vec{x} + B_1 u_1 \\ y_1 = C_1 \vec{x} \end{cases}$$

$$\text{②} \begin{cases} \vec{x}_2 = A_2 \vec{x}_2 + B_2 u_2 \\ y_2 = C_2 \vec{x}. \end{cases}$$

We can combine these systems in various ways. For example, we can use the output of ① as the input of ②:



The result is a new system with equations

$$\dot{\vec{x}}_1 = A_1\vec{x}_1 + B_1 u_1$$

$$\dot{\vec{x}}_2 = A_2\vec{x}_2 + B_2 u_2 = A_2\vec{x}_2 + B_2 C_1 \vec{x}_1$$

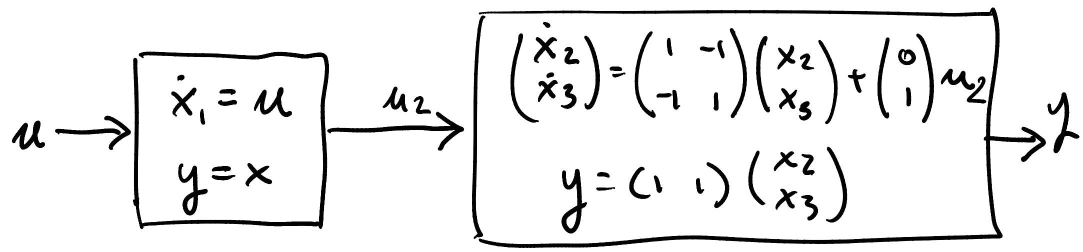
$$y_2 = C_2 \vec{x}_2 .$$

Or

$$\begin{pmatrix} \dot{\vec{x}}_1 \\ \dot{\vec{x}}_2 \end{pmatrix} = \overbrace{\begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix}}^{\vec{x}} + \overbrace{\begin{pmatrix} B_1 \\ 0 \end{pmatrix}}^{B} u_1$$

$$y_2 = \underbrace{\begin{pmatrix} 0 & C_2 \end{pmatrix}}_{C} \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix} .$$

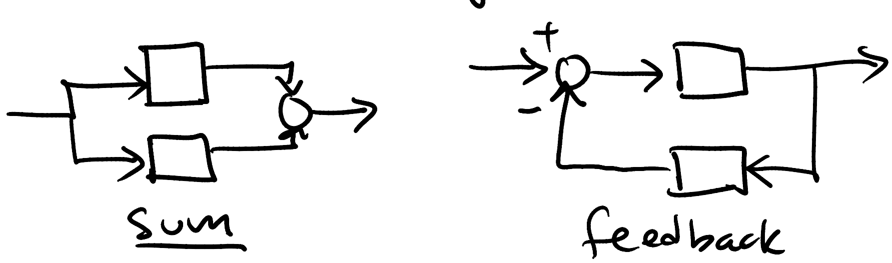
Example: What is the state space form for:



$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & -1\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}u$$

$$y = (0 \;\; 1 \;\; 1)\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}.$$

There are other ways to combine systems:-



But we will see that these are easier to think about in the Frequency Domain.