Lecture 9a

Tuesday, October 31, 2006
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Lecture 9a: 2nd Order Systems

... in which we determine the step response of an ideal 2^{nd} order system. This will allow us to talk very precisely about performance measures.

I. The Archetypical System

Consider the following system

$$T(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}.$$

This is the general form of a system with 2 poles and no zeros.

The poles are

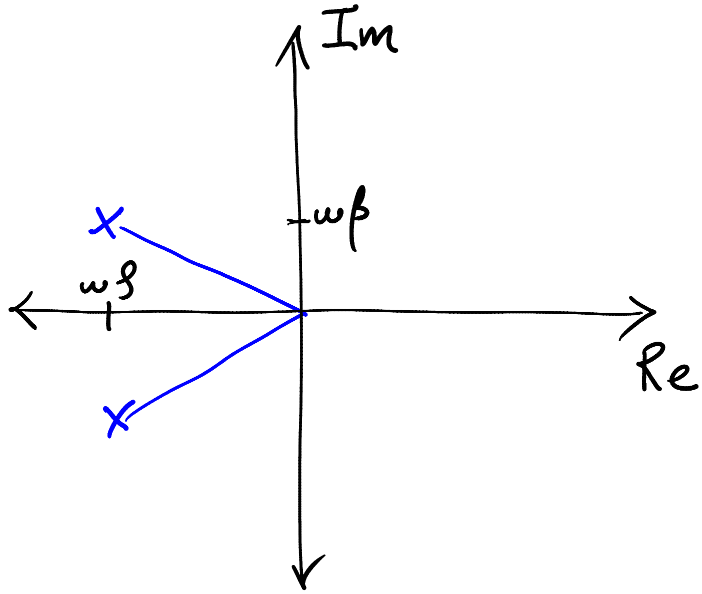
$$\frac{-2\zeta\omega \pm \sqrt{4\zeta^2\omega^2 - 4\omega^2}}{2}$$

which simplifies to

$$-\omega\zeta \pm \omega\sqrt{\zeta^2-1}$$

$$= -\omega\zeta \pm \omega\beta j \quad \left(\text{where } \beta = \sqrt{1-\zeta^2}\right)$$

In the s-plane, this looks like



The parameter ζ is called the <u>damping ratio</u>:

$\zeta = 0 \Rightarrow$ pure oscillations

$\zeta = 1 \Rightarrow$ critically damped

The parameter ω is called the <u>natural frequency</u>.

II. The Step Response

To learn more about the behavior of this system, we'll find its step response. We will:

1. Put the system in state space
2. Find $e^{At} = P e^{\Gamma t} P^{-1}$
3. Compute $x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu d\tau$

   and $y(t) = C x(t)$
4. Simplify the expression

1. State Space

Choose $A = \begin{pmatrix} -\omega\zeta & -\omega\beta \\ \omega\beta & -\omega\zeta \end{pmatrix}$.

$$B = \begin{pmatrix} \omega \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & \frac{1}{\beta} \end{pmatrix}$$

Then $C(sI - A)^{-1} B = T(s)$.

② e^{At}

To find e^{At}, we first diagonalize.

Eigenvalues: $-\omega\zeta \pm \omega\beta j$

Eigenvectors: $$P = \begin{array}{cc} \begin{pmatrix} -j & j \\ 1 & 1 \end{pmatrix} \\ \begin{array}{cc} -\omega\zeta-\omega\beta j & -\omega\zeta+\omega\beta j \end{array} \end{array}$$

$$P^{-1} = \begin{pmatrix} j/2 & 1/2 \\ -j/2 & 1/2 \end{pmatrix}$$

Thus, we get that

$$e^{At} = P \overbrace{\begin{pmatrix} e^{t(-\omega\zeta-\omega\beta j)} & 0 \\ 0 & e^{t(-\omega\zeta+\omega\beta j)} \end{pmatrix}}^{\text{call this } e^{\Gamma(t)}} P^{-1}$$

③ Find $y(t)$

First, assume that $\vec{x}(0)=0$. Then,

$$y(t) = C\vec{x}(t)$$

$$= C\int_0^t e^{A(t-\tau)} B\underbrace{u(t)}_{=1 \text{ (step input)}} d\tau$$

$$= \int_0^t C P e^{\Gamma(t-\tau)} P^{-1} B \, d\tau$$

So let's figure out $CPe^{\Gamma(t)}P^{-1}B$ first

$$CP = \begin{pmatrix} 0 & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} -j & j \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta} & \frac{1}{\beta} \end{pmatrix}.$$

$$P^{-1}B = \frac{1}{2}\begin{pmatrix} j & 1 \\ -j & 1 \end{pmatrix}\begin{pmatrix} \omega \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} j\omega \\ -j\omega \end{pmatrix}.$$

Then

$$CPe^{\Gamma(t)}P^{-1}B = \frac{1}{2}\begin{pmatrix} \frac{1}{\beta} & \frac{1}{\beta} \end{pmatrix}\begin{pmatrix} e^{t(-\omega\zeta-\omega\beta j)} & 0 \\ 0 & e^{t(-\omega\zeta+\omega\beta j)} \end{pmatrix}\begin{pmatrix} j\omega \\ -j\omega \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \frac{1}{\beta} & \frac{1}{\beta} \end{pmatrix}\begin{pmatrix} j\omega e^{t(-\omega\zeta-\omega\beta j)} \\ -j\omega e^{t(-\omega\zeta+\omega\beta j)} \end{pmatrix}$$

$$= \frac{\omega}{2\beta} e^{-\omega\zeta t}\left(j e^{-\omega\beta j t} - j e^{\omega\beta j t}\right).$$

$$= \frac{\omega}{\beta} e^{-\omega\zeta t}\sin(\omega\beta t).$$

Thus,

$$y(t) = \frac{\omega}{\beta}\int_0^t e^{-\omega\zeta(t-\tau)}\sin[\omega\beta(t-\tau)]\,d\tau$$

$\Downarrow$ you do the math

$$\boxed{= 1 - \frac{1}{\beta}e^{-\zeta\omega t}\left[\zeta\sin\omega\beta t + \beta\cos\omega\beta t\right].}$$

④ A Simplification

You might remember that

$$\sin(a+\theta) = \underbrace{\sin a}\,\underbrace{\cos\theta} + \sin\theta\,\underbrace{\cos a}$$

$$\sin(\omega\beta t)\ \zeta + \sqrt{1-\zeta^2}\,\cos(\omega\beta t)$$

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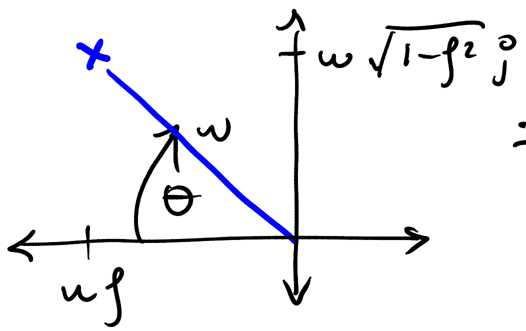
What θ gives

$$\cos\theta = \zeta$$

$$\sin\theta = \beta = \sqrt{1-\zeta^2}$$



$$\Rightarrow \theta = \cos^{-1}\zeta$$

Using this, we get

$$\boxed{y(t) = \frac{1}{\beta} e^{-\zeta\omega t} \sin(\omega\beta t + \theta)}$$

$\omega\beta$ = the damped frequency

θ = the offset

III. The Impulse Response

Note that

$$y(t) \longleftrightarrow Y(s) = \frac{1}{s} T(s)$$

So

$$\dot{y}(t) \longleftrightarrow sY(s) = s\frac{1}{s} T(s)$$

$$= 1 \cdot T(s)$$

The Laplace Transform of an Impulse.

Therefore, the time response of $T(s)$ to an impulse is

$$\dot{y}(t) = \frac{\omega}{\beta} e^{-\zeta\omega t} \sin(\omega\beta t)$$

which was the integrand in the preceding section.