Lecture 14b

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Lecture 14b : Controllability

... in which we learn to test whether a system can be completely controlled using full-state feedback.

I. Last time

Last time we took a system

$$\dot{\vec{x}} = A\vec{x} + Bu$$

and defined $u = -K\vec{x} + r$ to get

$$\dot{\vec{x}} = (A - BK)\vec{x} + Br.$$

In particular, we saw how to define K so that the poles of $A-BK$ are whatever you want!

We also discovered that some systems are not amenable to this treatment: For example,

$$\dot{\vec{x}} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

is a system whose first state cannot be affected by u. This is called not completely controllable.

II. A Formal Notion of Controllability

A system is controllable if for any $\vec{x}(0)$ and $\vec{x}^*$ there is some $u(t)$ that will drive the system from $\vec{x}(0)$ to $\vec{x}^*$ in finite time.

- For linear systems, this means K can be defined to arbitrarily place the poles of $A-BK$.

A test of controllability is the following fundamental result:

Theorem: A system $\dot{\vec{x}} = A\vec{x} + Bu$ is controllable if and only if the matrix

$$M = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}$$

has rank n (where n is the dimension of $\vec{x}$). ∎

This would take all day to prove. See Chen: Linear Systems.

Example: The previous system has

$$M = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$

which has rank $1 \neq 2 \Rightarrow$ not controllable.

Example: Determine whether

$$\dot{\vec{x}} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u$$

from Lecture 15b is controllable.

$$AB = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$A^2B = A(AB) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

So

$$M = \begin{bmatrix} 0 & 3 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The determinant is

$$\left.\begin{aligned} |M| &= 0 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ &= 0 - 3(1-1) + 2(1+1) \\ &= 4 \end{aligned}\right\} \Rightarrow \text{controllable}$$

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Note: The rank of a matrix is the largest number of linearly independent columns. You can find the rank by finding the largest non-singular submatrix (which means determinant $\neq 0$). If the matrix itself has non zero determinant, then it has "full rank".

III. Properties of M

The coolest thing about M (besides that it tells you if your system is controllable) is that it helps you build K. Recall that K is easy to build when $A\vec{x} + Bu$ is in phase variable form, but not otherwise.

Say

$$\dot{\vec{x}} = A\vec{x} + Bu$$

is <u>not</u> in phase variable form.
Say that

$$\vec{x} = P\vec{z}$$

transforms $\vec{z}$ from phase variable form into $\vec{x}$ variables. Then

$$\dot{\vec{z}} = P^{-1}\dot{\vec{x}} = P^{-1}(A\vec{x} + Bu)$$

$$= P^{-1}(AP\vec{z} + Bu)$$

$$= \underbrace{P^{-1}AP}_{A_z}\vec{z} + \underbrace{P^{-1}B}_{B_z}u.$$

Now the $\vec{x}$ system has

$$M_x = \left(B \quad AB \quad \cdots \quad A^{n-1}B \right)$$

and the z system has

$$M_z = \left(P^{-1}B \quad (P^{-1}AP)P^{-1}B \quad \cdots \quad (P^{-1}AP)^{n-1}P^{-1}B \right)$$

$$= P^{-1}\left(B \quad AB \quad \cdots \quad A^{n-1}B \right)$$

$$= P^{-1} M_x .$$

Thus,

$$\boxed{P = M_x M_z^{-1}}$$

After transforming the system into phase variables we design a controller $u = -K_z z + r$ and get

$$\dot{\vec{z}} = (A_z - B_z K_z)\vec{z} + B_z r$$

$$= (P^{-1}AP - P^{-1}BK_z)\vec{z} + P^{-1}Br.$$

Therefore, since $\vec{x} = P\vec{z}$:

$$\dot{\vec{x}} = P\dot{z} = P(P^{-1}AP - P^{-1}BK_z)P^{-1}\vec{x} + PP^{-1}Br$$

$$= (A - BK_zP^{-1})\vec{x} + Br.$$

$$= (A - BK_x)\vec{x} + Br.$$

So $\boxed{K_x = K_z P^{-1}}$.

Example: Say

$$\dot{\vec{x}} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u.$$

Design a full state feedback controller to place the poles at $(-5, -6)$.

Solution: First, we find $M_x = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$ which has $|M_x| = -2$, so the system is controllable. The $\vec{z}$ coordinates are obtained from

$$|sI - A| = \begin{vmatrix} s+1 & 1 \\ 0 & s+2 \end{vmatrix} = (s+1)(s+2)$$

$$= s^2 + \underset{a_1}{3}s + \underbrace{2}_{a_0}$$

So

$$\dot{\vec{z}} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

To control this system with full state feedback, we set

$$s^2 + (3 + k_2)s + (2 + k_1)$$

$$= (s+5)(s+6) = s^2 + 11s + 30$$

$$\Rightarrow \quad K_z = (k_1 \ k_2) = (28 \ 8).$$

The controllability matrix for z is

$$M_z = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}.$$

So $P = M_x M_z^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} +3 & +1 \\ +1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$

And

$$K_x = K_z P^{-1} = (28 \ 8) \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \frac{1}{2}$$

$$= (20 \ -4) = (10 \ -2)$$

Checking this

$$A - BK_x = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 10 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} - \begin{pmatrix} 10 & -2 \\ 10 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -11 & 3 \\ -10 & 0 \end{pmatrix}$$

Which has characteristic equation

$$(s+11)s + 30 = s^2 + 11s + 30$$

$$= (s+5)(s+6)$$

as desired.