Lecture 2a: Linearization and Eigenvalues

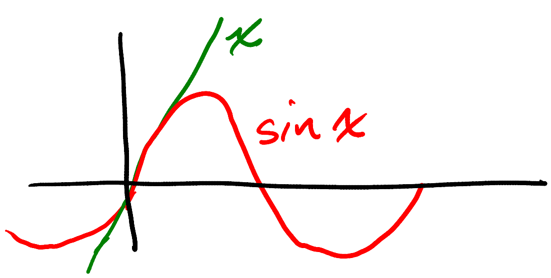
... in which we see that non-linear systems look linear locally.

I. Linearization w/o Control

Say f is a scalar function. Then near zero

$$f(x) = f(0) + \underbrace{f'(0)\,x}_{\text{linear part}} + \underbrace{\text{higher order terms}}_{\text{"HOT"}}$$

Ex. $\sin(x) = 0 + x + \text{H.O.T.}$



If f is a vector valued function, you can do the same

$$f(\vec{x}) = \vec{0} + \left.\frac{\partial f}{\partial \vec{x}}\right|_{\vec{x}=\vec{0}} \cdot \vec{x}$$

where

$$\frac{\partial f}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial f_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} = A$$

is an $n \times n$ matrix called the Jacobian.

Example: $\begin{pmatrix} \dot\theta \\ \dot\omega \end{pmatrix} = \begin{pmatrix} \omega \\ \sin\theta - \omega \end{pmatrix}$

(suppose $g/\ell = b = 1$)

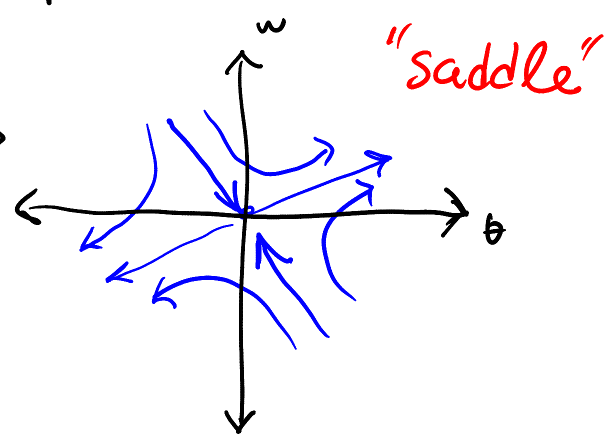
$$\left.\frac{\partial f}{\partial \vec{x}}\right|_{\vec{x}=0} = \left.\begin{pmatrix} \frac{\partial \omega}{\partial \theta} & \frac{\partial \omega}{\partial \omega} \\ \frac{\partial(\sin\theta - \omega)}{\partial \theta} & \frac{\partial(\sin\theta-\omega)}{\partial \omega} \end{pmatrix}\right|_{\substack{\omega=0\\ \theta=0}}$$

$$= \left.\begin{pmatrix} 0 & 1 \\ \cos\theta & -1 \end{pmatrix}\right|_{\substack{\omega=0\\ \theta=0}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

So, near $\vec{0}$, the system looks like

$$\begin{pmatrix} \dot\theta \\ \dot\omega \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \theta \\ \omega \end{pmatrix} \rightsquigarrow$$

$$= \begin{pmatrix} \omega \\ \theta - \omega \end{pmatrix}$$



At other equilibrium points, it is convient to change coordinates, so that the point in question is at $\vec{0}$.

① Say $\vec{x}^* =$ equilibrium point

② Define $\vec{x}_e = \vec{x} - \vec{x}^*$ [error coordinates]

Then $\dot{\vec{x}}_e = \dot{\vec{x}} = f(\vec{x})$

$$= f(\vec{x}_e + \vec{x}^*).$$

For example, let's look at the behavior of the pendulum model at $(\theta\ \omega) = (\pi, 0)$. In this case, we linearize

$$f\begin{pmatrix} \theta_e + \pi \\ \omega_e \end{pmatrix} = \begin{pmatrix} \omega_e \\ \sin(\theta_e + \pi) - \omega_e \end{pmatrix}$$

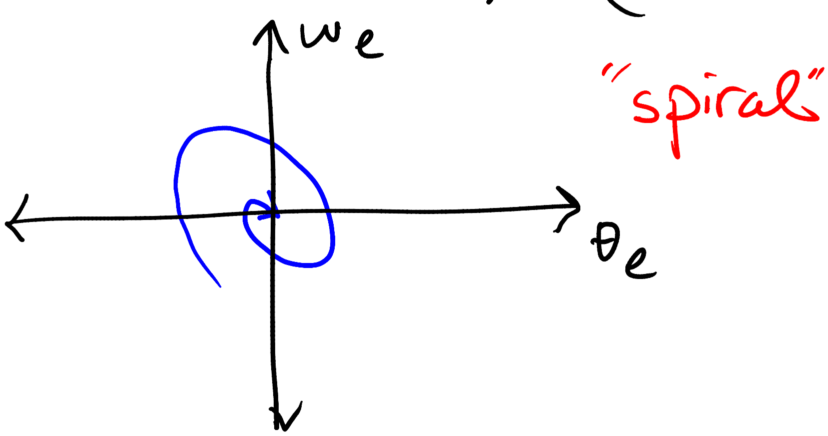
The A matrix is then

$$A = \begin{pmatrix} \dfrac{\partial \omega_e}{\partial \theta_e} & \dfrac{\partial \omega_e}{\partial \omega_e} \\ \dfrac{\partial \sin(\theta_e+\pi) - \omega_e}{\partial \theta_e} & \dfrac{\partial \sin(\theta_e+\pi) - \omega_e}{\omega_e} \end{pmatrix}_0$$

$$= \begin{pmatrix} 0 & 1 \\ \cos(\theta_e + \pi) & -1 \end{pmatrix}_0 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

So, near $(\theta\ \omega) = (\pi\ 0)$, the system looks like

$$\begin{pmatrix} \dot\theta_e \\ \dot\omega_e \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \theta_e \\ \omega_e \end{pmatrix} = \begin{pmatrix} \omega_e \\ -\theta_e - \omega_e \end{pmatrix}.$$



II. Linear Systems & Eigenvalues

We now look more closely at systems of the form

$$\dot{\vec{x}} = A\vec{x}$$

where $\vec{x}(t) \in \mathbb{R}^2$. They can be characterized by their eigenvalues and eigenvectors.

Note: Eigenvalues are also known as poles.

Recall that an eigenvalue is a solution to

see section 5.8

$$A\vec{x} = \underset{\text{value}}{\lambda}\,\underset{\text{vector}}{\vec{x}}$$

This can be written $A\vec{x} = \lambda I \vec{x}$

Identity

or $(\lambda I - A)\vec{x} = 0.$

This implies that

$$\det(\lambda I - A) = 0.$$

Ex. $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$$|\lambda I - A| = \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right|$$

$$= \begin{vmatrix} \lambda & -1 \\ -1 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1) - 1$$

$$= \lambda^2 + \lambda - 1$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2} \approx -1.62, 0.62$$

The eigenvectors are solutions to $A\vec{x} = \lambda\vec{x}$

$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \vec{v} = \lambda \vec{v} \iff \begin{array}{l} x_2 = \lambda x_1 \\ x_1 - x_2 = \lambda x_1 \\ x_1 - \lambda x_1 = \lambda x_1 \\ x_1 = \text{anything} \\ \quad = 1 \text{ (for example)} \end{array}$$

$$\lambda = \frac{-1-\sqrt{5}}{2}$$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1.62 \end{pmatrix}$$

$$\lambda = \frac{-1+\sqrt{5}}{2} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 0.62 \end{pmatrix}.$$

Note that knowing the "eigenstructure" of a system $\dot{x} = Ax$ makes it easy to draw the phase portrait.

This is because for an eigenvector $\vec{x}$,

$$\dot{\vec{x}} = A\vec{x} = \lambda\vec{x}.$$

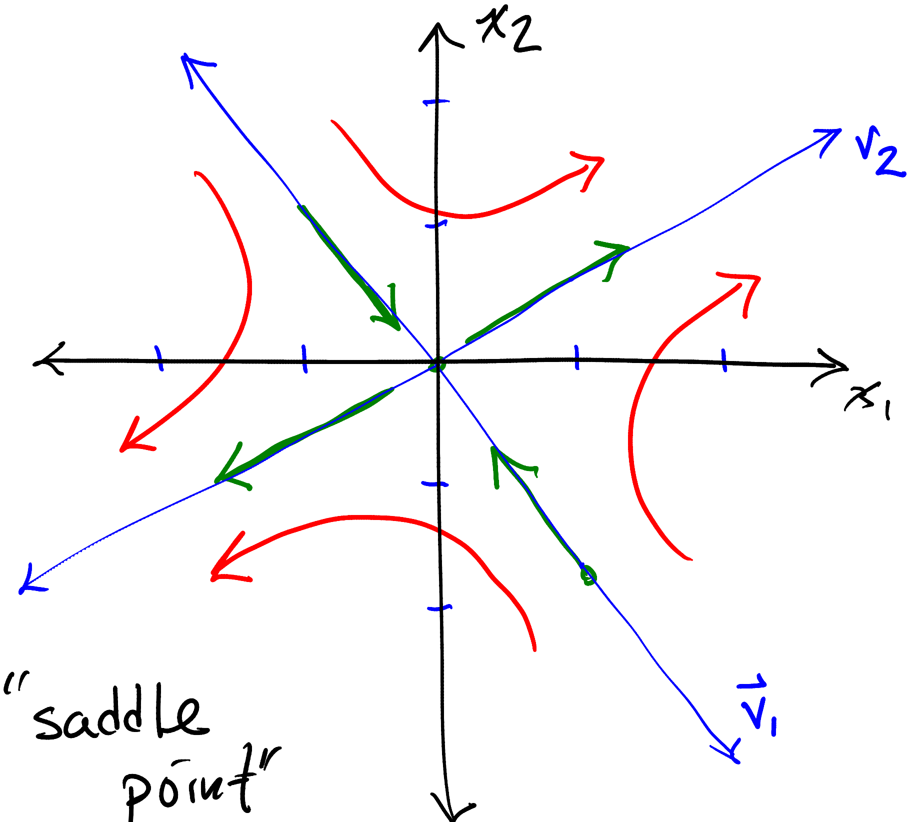
EX. Draw the phase portrait for

$\dot{\vec{x}} = A\vec{x}$ when $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

① Draw eigenvector subspaces

② Draw $\dot{\vec{x}}$ on subspaces

③ Fill in the rest



A "saddle point"

III. Similarity

Say $\dot{\vec{x}} = A\vec{x}$ and $\lambda_1, \vec{v}_1$, $\lambda_2, \vec{v}_2$ are eigenvalue, eigenvector pairs. Then

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$

or $$\begin{pmatrix} A\vec{v}_1 & A\vec{v}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \vec{v}_1 & \lambda_2 \vec{v}_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

or $$A \underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{D}$$

or $AP = PD \iff A = PDP^{-1}$.

A and D are called "similar".

Note: When A has repeated eigenvalues you need to be more careful.

The main point is that if $A = PDP^{-1}$: $\Leftrightarrow D = P^{-1}AP$

① A & D have the same eigenvalues.

② P is a "change of coordinates":

$$\vec{x} = P\vec{z} \qquad \vec{z} = P^{-1}\vec{x}$$

$$\dot{\vec{z}} = P^{-1}\dot{\vec{x}} = P^{-1}A\vec{x} = P^{-1}AP\vec{z}$$

$$= D\vec{z}.$$

③ The phase portrait of $\dot{\vec{x}} = A\vec{x}$ just a squashed, rotated version of the phase portrait of $\dot{\vec{z}} = D\vec{z}$.

④ To understand the behavior of an equilibrium point, we only need to what kind of eigenvalues the system has there.