1 Double-sided Crystal Ball

PDF and CDF definitions:

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{1}$$

$$F(x) = \int_{-\infty}^{x} f(x')dx' \tag{2}$$

$$y \equiv F(x) \to F^{-1}(y) = x \tag{3}$$

Parameters:

$$\vec{p} = (\mu, \sigma, a_L, n_L, a_R, n_R) \tag{4}$$

$$d_L = n_L/a_L \tag{5}$$

$$d_R = n_R/a_R \tag{6}$$

Parameter conditions:

$$n_L, n_R > 1 \tag{7}$$

$$a_L, a_R > 0 \tag{8}$$

Normalization:

$$N = \frac{1}{\sigma\left[\frac{d_L}{n_L-1}\cdot\exp\left(-\frac{a_L^2}{2}\right) + \sqrt{\frac{\pi}{2}}\left(\operatorname{erf}\left(\frac{a_L}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{a_R}{\sqrt{2}}\right)\right) + \frac{d_R}{n_R-1}\cdot\exp\left(-\frac{a_R^2}{2}\right)\right]} \tag{9}$$

Probability density function:

$$f(x;\vec{p}) = N \cdot \begin{cases} \exp\left(-\frac{a_L^2}{2}\right)\cdot\left[\frac{1}{d_L}\left(d_L - a_L - \frac{x-\mu}{\sigma}\right)\right]^{-n_L} & \text{for } \frac{x-\mu}{\sigma} \leq -a_L \\ \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) & \text{for } -a_L < \frac{x-\mu}{\sigma} < a_R \\ \exp\left(-\frac{a_R^2}{2}\right)\cdot\left[\frac{1}{d_R}\left(d_R - a_R + \frac{x-\mu}{\sigma}\right)\right]^{-n_R} & \text{for } \frac{x-\mu}{\sigma} \geq a_R \end{cases} \tag{10}$$

Cumulative distribution function:

$$F(x;\vec{p}) = \sigma N \cdot \begin{cases} \frac{d_L}{n_L-1}\exp\left(-\frac{a_L^2}{2}\right)\left[\frac{1}{d_L}\left(d_L - a_L - \frac{x-\mu}{\sigma}\right)\right]^{-n_L+1} & \text{for } \frac{x-\mu}{\sigma} \leq -a_L \\ \\ \frac{d_L}{n_L-1}\exp\left(-\frac{a_L^2}{2}\right) + \sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{a_L}{\sqrt{2}}\right) + \sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right) & \text{for } -a_L < \frac{x-\mu}{\sigma} < a_R \\ \\ \frac{d_L}{n_L-1}\exp\left(-\frac{a_L^2}{2}\right) + \sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{a_L}{\sqrt{2}}\right) & \\ +\sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{a_R}{\sqrt{2}}\right) + \frac{d_R}{n_R-1}\exp\left(-\frac{a_R^2}{2}\right) & \\ +\frac{d_R}{1-n_R}\exp\left(-\frac{a_R^2}{2}\right)\left[\frac{1}{d_R}\left(d_R - a_R + \frac{x-\mu}{\sigma}\right)\right]^{-n_R+1} & \text{for } \frac{x-\mu}{\sigma} \geq a_R \end{cases} \tag{11}$$

$$= \sigma N \cdot \begin{cases} B_L\left[\frac{1}{d_L}\left(d_L - a_L - \frac{x-\mu}{\sigma}\right)\right]^{-n_L+1} & \text{for } \frac{x-\mu}{\sigma} \leq -a_L \\ \\ A_L + C_L + \sqrt{\frac{\pi}{2}}\left(1 - \mathrm{erfc}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right) & \text{for } -a_L < \frac{x-\mu}{\sigma} < a_R \\ \\ A_L + C_L + C_R + A_R & \\ +B_R\left[\frac{1}{d_R}\left(d_R - a_R + \frac{x-\mu}{\sigma}\right)\right]^{-n_R+1} & \text{for } \frac{x-\mu}{\sigma} \geq a_R \end{cases} \tag{12}$$

Inverse cumulative distribution function:

$$x = \begin{cases} \mu + \sigma\left(-d_L\left[\frac{y}{\sigma N}/B_L\right]^{\frac{1}{-n_L+1}} - a_L + d_L\right) & \text{for } y < \sigma N A_L \\ \\ \mu + \sigma\sqrt{2}\mathrm{erfc}^{-1}\left[1 - \sqrt{\frac{2}{\pi}}\left(\frac{y}{\sigma N} - A_L - C_L\right)\right] & \text{for } \sigma N A_L \leq y \leq \sigma N(A_L + C_L + C_R) \\ \\ \mu + \sigma\left(d_R\left[\frac{\frac{y}{\sigma N} - A_L - C_L - C_R - A_R}{B_R}\right]^{\frac{1}{-n_R+1}} + a_R - d_R\right) & \text{for } y > \sigma N(A_L + C_L + C_R) \end{cases} \tag{13}$$