# Chapter 5

# Lindblad Master Equation

## 5.1 The Master Equation

For a closed quantum mechanical system, if we consider pure states $|\psi(t)\rangle$, their time evolution can be described by a unitary transformation $|\psi(t_f)\rangle = U(t_f, t_i)\,|\psi(t_i)\rangle$. If one substitutes that transformation into Schrodinger's equation, an equation for the operator $U$ is obtained:

$$i\frac{dU(t,t_0)}{dt} = H(t)U(t,t_0).$$

For a system that is isolated the time-independent hamiltonian gives us a simple form for the operator (assuming $\hbar = 1$):

$$U(t,t_0) = e^{-iH(t-t_0)}$$

For a density matrix $\rho$, which is a proper way of describing quantum statistical ensembles, and defined as $\rho(t) = \sum_k c_k\,|\psi_k(t)\rangle\,\langle\psi_k(t)|$, the time evolution can be obtained from the time evolution of a state. We then have:

$$\rho(t_f) = \sum_k c_k U(t_f,t_i)\,|\psi_k(t_i)\rangle\,\langle\psi_k(t_i)|U^\dagger(t_f,t_i) = U(t_f,t_i)\rho(t_i)U^\dagger(t_f,t_i).$$

Differentiating both sides with respect to time leads to Liouville - von Neumann equation:

$$\begin{aligned}\frac{d\rho(t)}{dt} &= \frac{dU(t,t_i)}{dt}\rho(t_i)U^\dagger(t,t_i) + U(t,t_i)\rho(t_i)\frac{dU^\dagger(t,t_i)}{dt}\\ &= -iH(t)U(t,t_i)\rho(t_i)U^\dagger(t,t_i) + iU(t,t_i)\rho(t_i)U^\dagger(t,t_i)H(t)\\ &= -i[H(t),\rho(t)].\end{aligned}$$

Deriving master equation for time evolution of density matrix in an open quantum system is a non-trivial problem. The equation is obtained by considering one-parameter dynamical maps $V_t$ acting on algebras of bounded operators on Hilbert space. These dynamical maps are defined as transformations such that:

$$\rho(t) = V_t\rho(0).$$

For example, for a closed system described before

$$V_t\rho(0) = U(t,0)\rho(0)U(t,0)^\dagger.$$

These maps are also defined to be trace-preserving, complete positive and satisfy Markov property ($V_{t_1}V_{t_2} = V_{t_1+t_2}$ for $t_1,\ t_2 \geq 0$). These maps create a Markov semigroup that preserves positivity and normalization of density matrix [10].

We can then write the map in terms of its generators: $V_t = \exp(\mathcal{L}t)$. In that case, for the Liouville - von Neumann equation

$$\mathcal{L}\rho = -i[H,\rho]$$

giving us $\dot{\rho} = \mathcal{L}\rho$ and leading to

$$\rho(t) = \exp(\mathcal{L}t)\rho(0).$$

In general, for infinitesimal time steps $s \to 0$:

$$\rho(t+s) \approx (1+\mathcal{L}s)\rho(t),$$

what leads to the quantum master equation:

$$\frac{d\rho(t)}{dt} = \mathcal{L}\rho(t).$$

The most general form of generators was first determined in [11] and [12]. There, a completely orthonormal operator basis $F_i$ was defined and equipped with a trace norm, i.e. $\langle F_i, F_j\rangle \equiv \mathrm{Tr}\left(F_i^\dagger F_j\right) = \delta_{ij}$. The basis consists of $N^2$ operators for a Hilbert space of dimension $N$. Authors eventually determined that the operator $\mathcal{L}$ can be written as:

$$\mathcal{L}\rho = -i[H,\rho] + \frac{1}{2}\sum_{i,j=1}^{N^2-1}\alpha_{ij}\left([F_i,\rho F_j^\dagger] + [F_i,\rho F_j^\dagger]\right).$$

Because the matrix of coefficients $\alpha_{ij}$ is positive and hermitian, it can diagonalized. We obtain eigenvalues $\gamma_i$ and a different complete set of orthonormal operators $C_i$, which are linear combinations of operators $F_i$. We can then write the quantum master equation in the so-called Lindblad form:

$$\mathcal{L}\rho = -i[H,\rho] + \sum_{i=1}^{N^2-1} \gamma_i \left( C_i \rho C_i^\dagger - \frac{1}{2} \left\{ C_i^\dagger C_i, \rho \right\} \right). \tag{5.1.1}$$

## 5.2 Unitary Transformation

The non-hermitian part of Eq.(5.1.1) is sometimes referred to as dissipator, and from now on we will use $\mathcal{L}$ to refer to it. The equation with redefined $\mathcal{L}$ can be then written as:

$$\frac{d\rho}{dt} = -i\hbar[H,\rho] + \mathcal{L}, \tag{5.2.1}$$

which for a 2-level system can be shown to have the same form as optical Bloch equations. Quite often it is easier to work in a rotating basis, where hamiltonian has a much simpler form. Such a transformation can be described using a unitary matrix $T$.

$$\tilde{\rho} = T^\dagger \rho T$$
$$H' = T^\dagger H T$$
$$\tilde{\mathcal{L}} = T^\dagger \mathcal{L} T$$

We can now look how the whole equation transforms. We have (setting $\hbar = 1$):

$$T^\dagger \frac{d\rho}{dt} T = -i(T^\dagger H \rho T - T^\dagger \rho H T) + T^\dagger \mathcal{L} T,$$

which after inserting an identity matrix $TT^\dagger$ and some algebra gives us:

$$T^\dagger \frac{d\rho}{dt} T = -i[H', \tilde{\rho}] + \tilde{\mathcal{L}}. \tag{5.2.2}$$

Next, we look at the way $\rho$ and its derivative transforms:

$$\frac{d\rho}{dt} = \frac{d}{dt}(T\tilde{\rho}T^\dagger) = \frac{dT}{dt}\tilde{\rho}T^\dagger + T\tilde{\rho}\frac{dT^\dagger}{dt} + T\frac{d\tilde{\rho}}{dt}T^\dagger.$$

This, after performing a unitary transformation gives us:

$$T^\dagger \frac{d\rho}{dt} T = T^\dagger \frac{dT}{dt}\tilde{\rho} + \tilde{\rho}\frac{dT^\dagger}{dt}T + \frac{d\tilde{\rho}}{dt}. \tag{5.2.3}$$

We also know that for unitary matrices

$$\frac{dT^\dagger}{dt}T = -T^\dagger\frac{dT}{dt},$$

which together with Eq.(5.2.3) allows us to write

$$T^\dagger\frac{d\rho}{dt}T = \frac{d\tilde{\rho}}{dt} + \left[T^\dagger\frac{dT}{dt}, \tilde{\rho}\right].$$

We can then substitute the above formula into Eq.(5.2.2) and find

$$\frac{d\tilde{\rho}}{dt} = -i[H', \tilde{\rho}] - \left[T^\dagger\frac{dT}{dt}, \tilde{\rho}\right] + \tilde{\mathcal{L}}.$$

Finally, we can define $\tilde{H} = H' - iT^\dagger dT/dt$, which after substituting into the equation above, gives us an equation in the same form as Eq.(5.2.1):

$$\frac{d\tilde{\rho}}{dt} = -i[\tilde{H}, \tilde{\rho}] + \tilde{\mathcal{L}}. \tag{5.2.4}$$

When solving optical Bloch equations we will in fact be solving equation (5.2.4) in a basis where $\tilde{H}$ has a simple form. Basis that we will be using is a kind of rotating basis, where the unitary transformation for a system with $n$ states can be written as:

$$T = \begin{pmatrix} e^{i\alpha_1 t} & & & & \\ & \ddots & & & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & e^{i\alpha_n t} \end{pmatrix}.$$

It can be easily found that:

$$-iT^\dagger\frac{dT}{dt} = \begin{pmatrix} \alpha_1 & & & & \\ & \ddots & & & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & \alpha_n \end{pmatrix}.$$

To obtain $\tilde{H}$, we just need to look at $H'$. First,

$$(HT)_{il} = \sum_{j=1}^{n} H_{ij}T_{jl} = H_{il}T_{ll} = H_{il}e^{i\alpha_l t},$$

and then

$$H'_{kl} = (T^\dagger(HT))_{kl} = \sum_{i=1}^{n} T^\dagger_{ki}(HT)_{il} = T^\dagger_{kk}(HT)_{kl} = H_{kl}e^{i(\alpha_l-\alpha_k)t}.$$

Finally, combining found expressions, we see that for our choice of unitary transformation

$$\tilde{H}_{kl} = H_{kl}e^{i(\alpha_l-\alpha_k)t} + \alpha_k\delta_{kl}. \tag{5.2.5}$$

The hamiltonian $\tilde{H}$ which we will obtain, will be a hamiltonian of the system in interaction picture. We will always try to find such unitary transformation that eliminates all the oscillatory time dependence from the hamiltonian.

## 5.3 Dissipator

One of the valid choices of Lindblad operators in Eq.(5.1.1) are the projection operators $P_i$ having properties $P_i^2 = P_i$ and $P_i^\dagger = P_i$. These are used for example in [13]. Authors there divide the dissipator into two parts. The first one contains dephasing effects, which include influence of finite laser linewidths and various broadening effects such as transit time broadening and collision-induced broadening, and is evaluated using the mentioned projection operators. The second one is related to spontaneous decay processes where the diagonal terms describe transfer between populations, and the off-diagonal terms are related to decoherence effects. Here, only the second part of the dissipator will be included, and it will be evaluated in a similar manner as in [13] - using quantum jump operators.

For states $i$ and $j$ we define quantum jump operators as $C_{i\to f} = |f\rangle\langle i|$. For spontaneous decays from $i$ to $f$ with $\Gamma_{i\to f}$ we will be using operators $G_{i\to f} = \sqrt{\Gamma_{i\to f}}C_{i\to f} = \sqrt{\Gamma_{i\to f}}\,|f\rangle\langle i|$. In matrix form these will look like:

$$G_{i\to f} = \begin{array}{c} \\ f \end{array} \begin{array}{c} \begin{array}{ccccc} & & & & i \end{array} \\ \begin{pmatrix} 0 & \cdots & & & 0 \\ & & & \sqrt{\Gamma_{i\to f}} & \\ \vdots & & & & \vdots \\ 0 & \cdots & & & 0 \end{pmatrix} \end{array}.$$

Because $G_{i\to f}$ is real, $G^{\dagger}_{i\to f} = G^{\mathrm{T}}_{i\to f}$, and (dropping lower indices)

$$G^{\dagger}G = \begin{array}{c} \\ i \end{array} \overset{\qquad\qquad i}{\begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ & & \Gamma_{i\to f} & \\ 0 & \cdots & & 0 \end{pmatrix}}.$$

With these, we can easily evaluate the term of $\mathcal{L}$ related to spontaneous emission, which will include both decoherence and population transfer. First, let's look at the anticommutator part $\{G^{\dagger}G, \rho\}$ knowing that the only nonzero element of $G^{\dagger}G$ is $(G^{\dagger}G)_{ii}$. The first term is:

$$(G^{\dagger}G\rho)_{kl} = \sum_j (G^{\dagger}G)_{kj}\rho_{jl} = \delta_{ki}(G^{\dagger}G)_{ki}\rho_{il},$$

so nonzero elements are indexed $\{i, l\}$ for $1 \leq l \leq n$ - we obtain a nonzero row:

$$G^{\dagger}G\rho = \begin{array}{c} \\ i \end{array} \overset{\qquad\qquad\qquad\qquad}{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ \Gamma_{i\to f}\rho_{i1} & \cdots & \Gamma_{i\to f}\rho_{in} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}.$$

Analogically, the second term is:

$$(\rho G^{\dagger}G)_{kl} = \sum_j \rho_{kj}(G^{\dagger}G)_{jl} = \rho_{ki}(G^{\dagger}G)_{il}\delta_{li},$$

giving us a nonzero column:

$$\rho G^{\dagger}G = \overset{\qquad\qquad i}{\begin{pmatrix} 0 & \cdots & \Gamma_{i\to f}\rho_{1i} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & \Gamma_{i\to f}\rho_{ni} & \cdots & 0 \end{pmatrix}}.$$

Now we just need to evaluate the second term in $\mathcal{L}$ knowing that only nonzero element of $G$ is $G_{fi}$ and only nonzero element of $G^\dagger$ is $G^\dagger_{if}$:

$$\begin{aligned}(G\rho G^\dagger)_{rs} &= \sum_j G_{rj}(\rho G^\dagger)_{js} = \delta_{rf} G_{ri}(\rho G^\dagger)_{is} \\ &= \delta_{rf} G_{ri} \left( \sum_m \rho_{im}(G^\dagger)_{ms} \right) = \delta_{rf} G_{ri} \rho_{ii} (G^\dagger)_{if} \delta_{sf} \\ &= \delta_{rf}\delta_{sf}\Gamma_{i\to f}\rho_{ii},\end{aligned}$$

which gives a matrix with only one nonzero element:

$$G\rho G^\dagger = \begin{array}{c} \\ f \end{array} \begin{array}{c} \begin{array}{ccc} f & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 \\ & \Gamma_{i\to f}\rho_{ii} & \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \end{array}.$$

This finally allows us to get a matrix that represents contribution to $\mathcal{L}$ from spontaneous decay from state $i$ to $f$:

$$\mathcal{L}_{i\to f} = -\frac{1}{2}\{G^\dagger_{i\to f}G_{i\to f}, \rho\} + G_{i\to f}\rho G^\dagger_{i\to f} = \begin{array}{c} \\ f \\ \\ \\ \\ i \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & & f & & i & \end{array} \\ \begin{pmatrix} 0 & & & & -\frac{\Gamma_{i\to f}}{2}\rho_{1i} & & 0 \\ & \Gamma_{i\to f}\rho_{ii} & & & \vdots & & \\ & & & & & & \\ & & & 0 & \vdots & & \\ -\frac{\Gamma_{i\to f}}{2}\rho_{i1} & \cdots & & \cdots & -\Gamma_{i\to f}\rho_{ii} & \cdots & -\frac{\Gamma_{i\to f}}{2}\rho_{in} \\ & & & & \vdots & & \\ 0 & & & & -\frac{\Gamma_{i\to f}}{2}\rho_{ni} & & 0 \end{pmatrix} \end{array}.$$

We see that we properly obtain a decrease of population $\rho_{ii}$ with rate $\Gamma_{i\to f}$ with simultaneous increase of population $\rho_{ff}$ equal to that loss, which results in the matrix having a zero trace. That

is equivalent to a closed system, where the trace of density matrix $\rho$ is preserved. The off-diagonal terms correspond to evolving coherence between two states. Despite the fact that this method gives us a correct result, computationally it can be quite challenging. For large systems the initial state $i$ can have multiple decay paths. For a single initial state, we'd have to calculate matrices above for every final state separately. However, the calculation can simplified.

A single initial state decaying into multiple different states will have a total decay rate $\Gamma_i = \sum_f \Gamma_{i\to f}$, so for a single final state, the corresponding decay rate is $\Gamma_{i\to f} = b_{fi}\Gamma_i$, where $b_{fi}$ is the branching ratio. We also note that all the coherence (off-diagonal) terms were generated by the anticommutator, which is a bilinear operator. So:

$$\sum_f \{G^\dagger_{i\to f} G_{i\to f}, \rho\} = \left\{ \sum_f G^\dagger_{i\to f} G_{i\to f}, \rho \right\} \equiv \{M_i^\dagger M_i, \rho\},$$

where we have defined a operator $M_i$

$$(M_i)_{kl} = \begin{cases} \sqrt{\Gamma_i} & \text{if } k = l = i \\ 0 & \text{otherwise} \end{cases}$$

for which $M_i^\dagger = M_i$. We can do that, because every matrix $G^\dagger_{i\to f} G_{i\to f}$ will have $b_{fi}\Gamma_{i\to f}$ in its $\{i,i\}$-th cell, so after the summation the term in that cell will be $\sum_f b_{fi}\Gamma_{i\to f} = \Gamma_i$. Using matrix $M$ instead of $G$ will, however, not give us the same result in the second term of $\mathcal{L}_i \equiv \sum_f \mathcal{L}_{i\to f}$. Instead, we will obtain:

$$M_i \rho M_i^\dagger = \begin{array}{c} \\ i \end{array} \begin{array}{c} \begin{array}{ccc} & i & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ & \Gamma_i \rho_{ii} & \\ 0 & \cdots & 0 \end{pmatrix} \end{array}$$

so, in the end:

$$\mathcal{L}_i = -\frac{1}{2}\{M_i^\dagger M_i, \rho\} + M_i \rho M_i^\dagger = \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & & i & & \end{array} \\ \begin{pmatrix} 0 & & -\frac{\Gamma_i}{2}\rho_{1i} & & 0 \\ & & \vdots & & \\ -\frac{\Gamma_i}{2}\rho_{i1} & \cdots & 0 & \cdots & -\frac{\Gamma_i}{2}\rho_{in} \\ & & \vdots & & \\ 0 & & -\frac{\Gamma_i}{2}\rho_{ni} & & 0 \end{pmatrix} \end{array}.$$

We, therefore, obtain correct decoherence terms, but no population transfer terms. However, having branching ratios we can quite easily fill in the diagonal terms without performing any matrix operations. Finally, if we have several initial states we can create operator $M = \sum_i M_i$, for which $M^2 = \sum_i M_i^2$, because $M_i M_j$ is zero for $i \neq j$. Again, the off-diagonal terms will be determined correctly, because of the linearity of anticommutator. The other term, however, has to be determined separately for every initial state. Hence, the fastest way to calculate all the off-diagonal contributions to $\mathcal{L}$ from spontaneous decays is:

$$\mathcal{L}_\Gamma = \sum_i \mathcal{L}_i = -\frac{1}{2}\left\{\sum_i M_i^2, \rho\right\} + \sum_i M_i \rho M_i = -\frac{1}{2}\{M^2, \rho\} + \sum_i M_i \rho M_i \tag{5.3.1}$$

## 5.4 Example - EIA in a Four-level system

The modeling and simulations of toy models and experimental procedures required solving the equations multiple times. To streamline the process, I have written a package in MATLAB that allows for a quick setup of a quantum system and appropriate equations and fast solutions to the master equation.[1] As a benchmark test and an example of setting up the master equation using the described method we chose a four-level system in $N$-configuration, which in weak probe limit should exhibit electromagnetically induced absorption (EIA). In that system we have two lower lying states $|A\rangle$ and $|B\rangle$ with energies $\omega_A$ and $\omega_B$ that are connected to a common excited state $|E_1\rangle$ with energy $\omega_{E_1}$. The laser connecting state $|B\rangle$ with $|E_1\rangle$ this time is the probe laser with frequency $\omega_p$ and Rabi frequency $\Omega_p$. In a general case it is assumed to be detuned from the

[1] Code is available at: https://github.com/kwenz/OBE-solver.

transition by $\Delta_p$. The transition from $|A\rangle$ to $|E_1\rangle$ is due to the control laser 1, and transition from $|B\rangle$ to $|E_2\rangle$ is due the control laser 2. Both of them are in general case detuned by $\Delta_{c_1}$ and $\Delta_{c_2}$ respectively. The state $|E1\rangle$ has a total decay rate $\Gamma_A + \Gamma_{B_1}$ and state $|E2\rangle$ decays into state $|B\rangle$ with rate $\Gamma_{B_2}$. The system is shown in the figure 5.4.1.

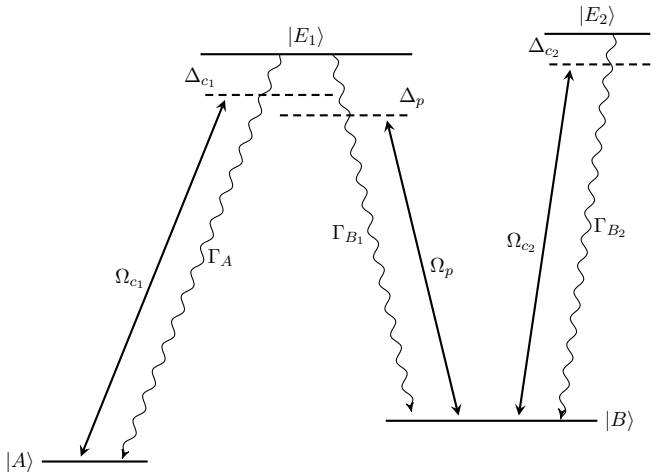


Figure 5.4.1: Diagram of four-level system in $N$ configuration.

For such system we obtain hamiltonian of the form:

$$H = \begin{array}{c} \begin{array}{cccc} |A\rangle & |B\rangle & |E_1\rangle & |E_2\rangle \end{array} \\ \begin{pmatrix} \omega_A & 0 & \frac{\Omega_{c_1}}{2}e^{i\omega_{c_1}t} & 0 \\ 0 & \omega_B & \frac{\Omega_p}{2}e^{i\omega_p t} & \frac{\Omega_{c_2}}{2}e^{i\omega_{c_2}t} \\ \frac{\overline{\Omega}_{c_1}}{2}e^{-i\omega_{c_1}t} & \frac{\overline{\Omega}_p}{2}e^{-i\omega_p t} & \omega_{E_1} & 0 \\ 0 & \frac{\overline{\Omega}_{c_2}}{2}e^{-i\omega_{c_2}t} & 0 & \omega_{E_2} \end{pmatrix} \end{array},$$

where we have already used rotating wave approximation ($\Omega = \Omega_0 \cos\omega t \approx \Omega_0 e^{i\omega t}/2$). We then perform unitary transformation, where we find coefficients $\alpha_i$ in Eq.(5.2.5) by trying to remove oscillating terms from off-diagonal elements of the hamiltonian. We get a set of linear equations:

$$\begin{cases} \alpha_3 - \alpha_1 = & -\omega_{c_1} \\ \alpha_3 - \alpha_2 = & -\omega_p \\ \alpha_4 - \alpha_2 = & -\omega_{c_2} \end{cases},$$

which is solved for

$$\begin{cases} \alpha_1 = & \omega_{c_1} - \omega_p \\ \alpha_2 = & 1 \\ \alpha_3 = & -\omega_p \\ \alpha_4 = & -\omega_{c_2} \end{cases}.$$

This allows us to use unitary transformation with matrix

$$T = \begin{pmatrix} e^{i(\omega_{c_1}-\omega_p)t} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-i\omega_p t} & 0 \\ 0 & 0 & 0 & e^{-i\omega_{c_2} t} \end{pmatrix},$$

giving us

$$\tilde{H} = \begin{pmatrix} \Delta_p - \Delta_{c_1} + \omega_B & 0 & \Omega_{c_1}/2 & 0 \\ 0 & \omega_B & \Omega_p/2 & \Omega_{c_2}/2 \\ \overline{\Omega}_{c_1}/2 & \overline{\Omega}_p/2 & \Delta_p + \omega_B & 0 \\ 0 & \overline{\Omega}_{c_2}/2 & 0 & \Delta_{c_2} + \omega_B \end{pmatrix}.$$

The diagonal terms of hamiltonian $\tilde{H}$ show us that the natural choice for the energy scale will be $\omega_B = 0$. To find matrix $\mathcal{L}$ we first define two matrices $M_{E_1}$ and $M_{E_2}$, where:

$$M_{E_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\Gamma_A + \Gamma_{B_1}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; M_{E_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\Gamma_{B_2}} \end{pmatrix}.$$

We can then also find

$$M^2 = M_{E_1}^2 + M_{E_2}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Gamma_A + \Gamma_{B_1} & 0 \\ 0 & 0 & 0 & \Gamma_{B_2} \end{pmatrix}$$

and plug those three matrices into Eq.(5.3.1) to obtain decoherence terms. Population transfer (diagonal) terms can be easily added by hand. Eventually, we obtain:

$$\mathcal{L} = \begin{pmatrix} \Gamma_A \rho_{e_1 e_1} & 0 & -\frac{\Gamma_A+\Gamma_{b_1}}{2}\rho_{ae_1} & -\frac{\Gamma_{B_2}}{2}\rho_{ae_2} \\ 0 & \Gamma_{B_1}\rho_{e_1e_1} + \Gamma_{B_2}\rho_{e_2e_2} & -\frac{\Gamma_A+\Gamma_{B_1}}{2}\rho_{be_1} & -\frac{\Gamma_{B_2}}{2}\rho_{be_2} \\ -\frac{\Gamma_A+\Gamma_{B_1}}{2}\rho_{e_1a} & -\frac{\Gamma_A+\Gamma_{B_1}}{2}\rho_{e_1b} & -(\Gamma_A+\Gamma_{B_1})\rho_{e_1e_1} & -\frac{\Gamma_A+\Gamma_{B_1}+\Gamma_{B_2}}{2}\rho_{e_1e_2} \\ -\frac{\Gamma_{B_2}}{2}\rho_{e_2a} & -\frac{\Gamma_{B_2}}{2}\rho_{e_2b} & -\frac{\Gamma_A+\Gamma_{B_1}+\Gamma_{B_2}}{2}\rho_{e_2e_1} & -\Gamma_{B_2}\rho_{e_2e_2} \end{pmatrix}.$$

To look at the accuracy of numerical solutions, we solved the equations in the weak probe limit for which analytical solutions exist. In such limit the system should exhibit the mentioned EIA effect. First, however, we looked at the evolution of populations. Even for zero probe laser detuning $\Delta_p$, we expect that the population in the steady state will consist of only states $|B\rangle$ and $|E_2\rangle$. Indeed, that is the case as shown in figure 5.4.2 depicting populations as function of time.

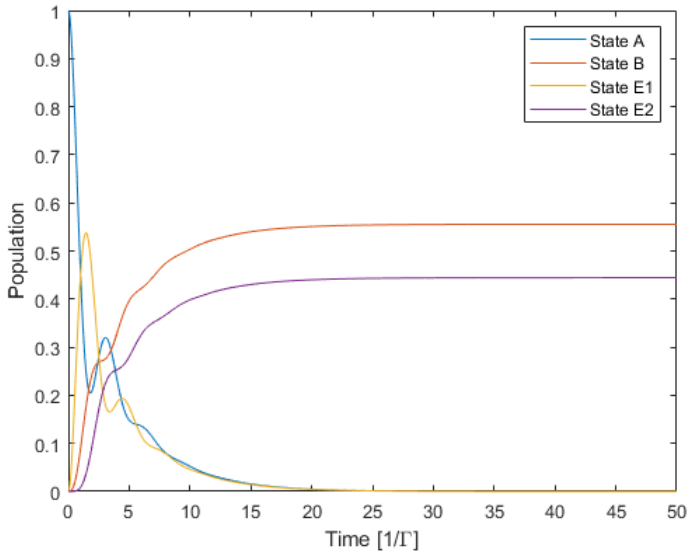


Figure 5.4.2: Populations as function of time for a four-level system in weak probe limit. Solution obtained for $2\Gamma_A = 2\Gamma_{B_1} = \Gamma_{B_2} \equiv \Gamma$, $\Omega_{c_1} = \Omega_{c_2} = 2\Gamma$ and $\Omega_p = 0.00025\Gamma$ with $\delta_{c_1} = \delta_{c_2} = \delta_p = 0$.

Finally, we were able to check existence of EIA. This effect is visible once we inspect imaginary part of $\rho_{be_1}$. In general, the analytical solution is given by (found in [14]):

$$\rho_{be_1} = -\frac{i\Omega_p\rho_{bb}}{2\beta_{be_1}\alpha_2} + \frac{i\Omega_p\Omega_{c_2}^2(\rho_{bb}-\rho_{e_2e_2})}{8\beta_{e_2e_1}\beta_{be_1}\beta_{e_2b}\alpha_2} - \frac{i\Omega_p\Omega_{c_1}^2\Omega_{c_2}^2(\rho_{bb}-\rho_{e_2e_2})}{32\beta_{e_2e_1}\beta_{be_1}\beta_{e_2a}\beta_{e_2b}\alpha_1\alpha_2}\left(\frac{1}{\beta_{ba}}+\frac{1}{\beta_{e_2e_1}}\right),$$

where

$$\begin{aligned}
\rho_{e_2e_2} &= \frac{\Omega_{c_2}^2}{\Gamma^2+2\Omega_{c_2}^2}\\
\rho_{bb} &= 1-\rho_{e_2e_2}\\
\beta_{ba} &= -i\Delta_p\\
\beta_{be_1} &= -\frac{\Gamma}{2}-i\Delta_p\\
\beta_{e_2a} &= -\frac{\Gamma}{2}-i\Delta_p\\
\beta_{e_2e_1} &= -\Gamma-i\Delta_p\\
\beta_{e_2b} &= -\frac{\Gamma}{2}\\
\alpha_1 &= 1+\frac{\Omega_{c_1}^2}{4\beta_{e_2a}\beta_{e_2e_1}}+\frac{\Omega_{c_2}^2}{4\beta_{e_2a}\beta_{ba}}\\
\alpha_2 &= 1+\frac{\Omega_{c_1}^2}{4\beta_{ba}\beta_{be_1}}+\frac{\Omega_{c_2}^2}{4\beta_{e_2e_1}\beta_{be_1}}-\frac{\Omega_{c_1}^2\Omega_{c_2}^2}{16\beta_{be_1}\beta_{e_2a}\alpha_1}\left(\frac{1}{\beta_{ba}}+\frac{1}{\beta_{e_2e_1}}\right)^2.
\end{aligned}$$

We can now compare this ungainly analytical solution with the numerical solution. The comparison is shown in figure 5.4.3. We can see that the two results are identical.

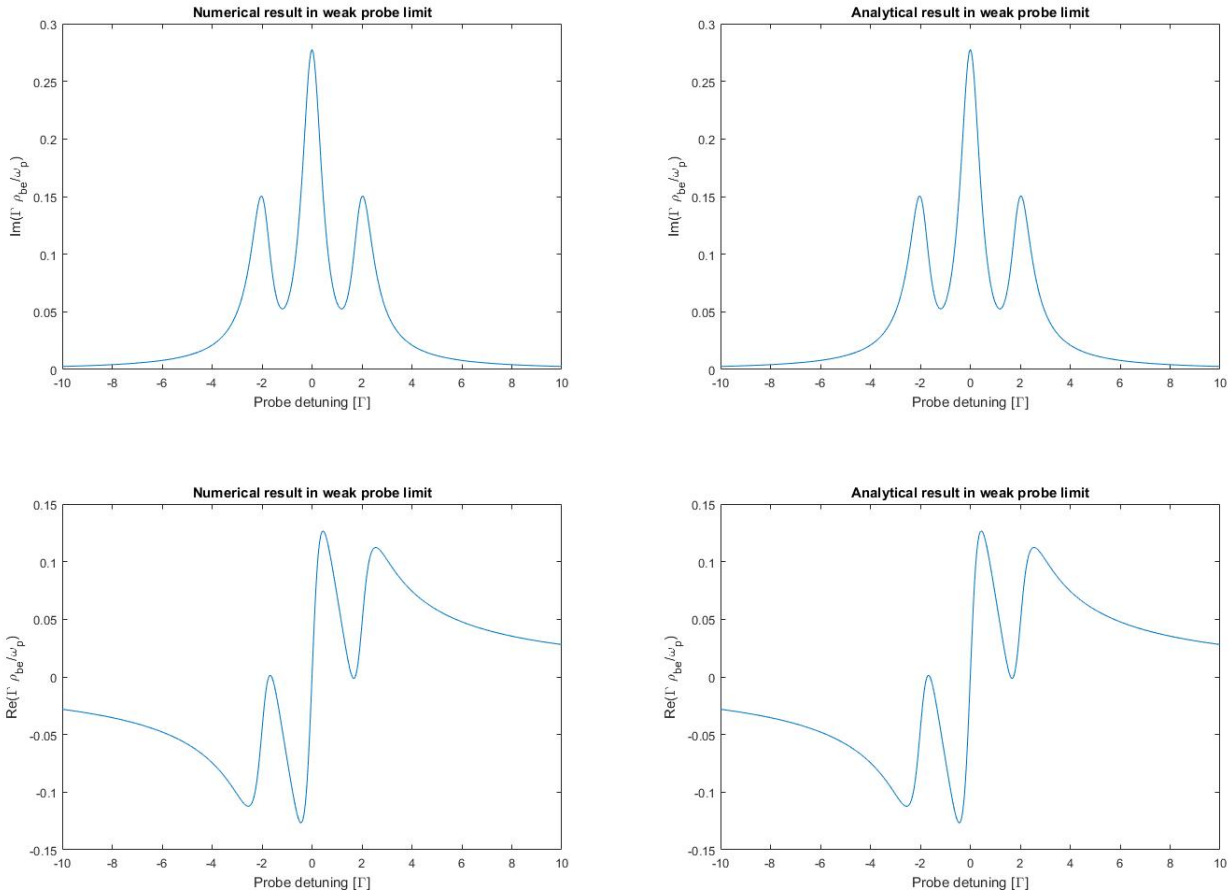


Figure 5.4.3: Numerical (left) and analytical (right) results showing electromagnetically induced absorption by plotting $\Gamma\rho_{be_1}/\Omega_p$ as a function of probe detuning $\Delta_p$. Solution was obtained for $2\Gamma_A = 2\Gamma_{B_1} = \Gamma_{B_2} \equiv \Gamma$, $\Omega_{c_1} = \Omega_{c_2} = 2\Gamma$ and $\Omega_p = 0.00025\Gamma$ with $\delta_{c_1} = \delta_{c_2} = 0$.