The direct criterion of Newcomb for the stability of an axisymmetric toroidal plasma

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A new method is introduced for determining the magnetohydrodynamic stability of an axisymmetric toroidal plasma, based on a toroidal generalization of the method developed by Newcomb for cylindrical plasmas. For toroidal mode number $n \neq 0$, the problem is reduced to the solution of a high-order complex system of ordinary differential equations, the Euler-Lagrange equation for minimizing the potential energy, and the evaluation of a real Critical Determinant whose poles indicate the presence of ideal instabilities. Coupling to a vacuum region outside the plasma incorporates the effects of free-boundary instabilities. The asymptotic behavior of the solutions in the neighborhood of singular surfaces provides the necessary information for coupling to singular layers incorporating resistivity, inertia, and rotation. The method lends itself to efficient implementation using adaptive numerical integration.

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I. INTRODUCTION

A key step in understanding the behavior of tokamaks is the determination of the stability of the confined plasma to magnetohydrodynamic (MHD) perturbations. Such instabilities are believed to set limits on the achievable current density and plasma pressure and to influence the onset of disruptions which can damage the device. Stability studies constitute a major effort in interpretation of experiments and design studies for new tokamaks.

Most existing MHD stability codes[1–6] are based on a procedure in which the perturbed potential and kinetic energies are expanded in a set of 2-dimensional basis functions, reducing the numerical problem to the solution of a large matrix eigenvalue problem for the growth rates, frequencies, and eigenfunctions of the entire MHD spectrum. These methods can suffer from poor convergence of the basis functions, especially for difficult equilibria with high β (the ratio $2\mu_0 P/B^2$ of plasma pressure to magnetic field energy density), strong noncircularity, and poloidal divertors. They require large expenditures of human and computer time. Much of the effort required by such methods goes into determining eigenvalues and eigenvectors which may not be needed when the real concern is simply the location of the stability boundary and perhaps the distance of a given equilibrium from that boundary.

An alternative procedure was introduced by Newcomb for cylindrical plasmas.[7] He found conditions for the existence of test functions which satisfy the boundary conditions and make the potential energy δW negative, relying on the well-known result that the existence of such test functions implies the existence of exponentially growing perturbations. His condition involves integrating a real 2nd-order ordinary differential equation (ODE), the Euler-Lagrange equation for minimizing the potential energy, and determining whether the solution ξ changes sign between successive singular points. This condition can be interpreted as determining whether there is a mode which fits into the "box" of the equilibrium. We refer to it as the Direct Criterion of Newcomb, or `DCON`.

The purpose of this paper is to generalize Newcomb's criterion to an axisymmetric toroidal plasma with toroidal mode number $n \neq 0$. The key difference between a cylinder and a torus is that θ, the poloidal angle which goes around the short way, is no longer a symmetry, as it is in a cylinder. As a result, one can no longer treat individual Fourier components in this variable, because toroidicity and noncircularity couple different components and thus require that they be treated simultaneously. The generalization of Newcomb's 2nd-order real Euler-Lagrange equation is a complex system of ODEs of order $2M$, where M is the number of poloidal Fourier components retained in the treatment. This system can be solved numerically with a very efficient adaptive ODE solver.

In order to generalize Newcomb's sign-change criterion, it is necessary to identify a real scalar quantity whose sign change plays the same role in the toroidal case as that of Newcomb's displacement ξ in the cylindrical case. This quantity has been found as a critical determinants constructed from the Euler-Lagrange solutions with different initial conditions. Its reality is a consequence of the Hamiltonian symmetry of the Euler-Lagrange equation, which is in turn a consequence of the well-known self-adjointness of the linearized ideal MHD equations. Its role in the theory is demonstrated by a generalization of Newcomb's use of the Hilbert Invariant Integral. Another important difference between the cylindrical and toroidal cases is that in the former, successive intervals between singular surfaces can be treated independently, while in the latter they are coupled by nonresonant components which must be continuous across singular surfaces.

Bineau[8] attempted a treatment similar to ours but for general, non-axisymmetric toroidal plasmas. His approach has many of the same features as this one, but it is more formal, it is not clear how to apply it as a practical stability program, and he does not obtain our key result, the Critical Determinant. Section VI discusses the difficulties of extending our treatment to nonaxisymmetric systems. Connor *et al.*[9] and Fitzpatrick *et al.*[10] treat the equivalent of our Euler-Lagrange equation in the context of a large aspect ratio expansion. Their interest is in toroidal resistive tearing modes; they do not discuss the symmetries of the equations or the generalization of Newcomb's criterion for ideal stability. Dewar and Pletzer[11,12] discuss a generalization of Newcomb's Euler-Lagrange equation to axisymmetric toroidal plasmas, but they do not express its properties as a coupled system of ODEs for the complex Fourier amplitudes, nor do they generalize Newcomb's criterion. Their concern is with resistive stability, and their numerical treatment uses an expansion in finite elements in the radial direction.

The remainder of this paper is organized as follows. In Section II we introduce a coordinate system and a vector representation used to evaluate and minimize the potential energy δW, derive the Euler-Lagrange equation, and discuss the symmetries of this equation. In Section III we discuss the behavior of the Euler-Lagrange equation in the neighborhood of its singular points at the resonant surfaces, the magnetic axis, and the separatrix. In Section IV we prove the toroidal generalization of Newcomb's criterion in terms of the the critical determinant. In Section V we extend the treatment to include a vacuum region surrounding the plasma region, enabling us to treat external instabilities. In Section VI, we discuss our conclusions. Appendix A gives detailed expressions for the coefficient matrices which appear in the Euler-Lagrange equation. SI units are used.

II. THE EULER-LAGRANGE EQUATION

In this section, we first describe the equilibrium and introduce a coordinate system and a representation for perturbed vectors. We then use these to express the ideal MHD potential energy δW, and derive the Euler-Lagrange equation as the condition that δW be stationary with respect to small changes in the perturbation. We conclude with a discussion of the symmetry properties of this equation.

We treat a stationary equilibrium[13] satisfying the pressure balance equation

$$\mathbf{J} \times \mathbf{B} = \mu_0 \nabla P$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ is the plasma current, $\mathbf{B}$ and $\mathbf{J}$ are divergenceless, and P is the plasma pressure. We assume that the system is axisymmetric, *i.e.* all equilibrium scalars are independent of the azimuthal angle ϕ. With these assumptions, the magnetic field can be represented as

$$\mathbf{B} = f\nabla\phi + \frac{\chi'}{2\pi}\nabla\phi \times \nabla\psi$$

where $f = RB_T$, R is the radius from the axis of symmetry, B_T is the toroidal component of the magnetic field, and χ is the poloidal magnetic flux, and P, χ, and f are constant on a flux surface, *i.e.* a surface of constant ψ, with ψ an arbitrary surface label. The poloidal flux satisfies the Grad-Shafranov equation,

$$\Delta^*\chi \equiv R^2\nabla \cdot \left(\frac{1}{R^2}\nabla\chi\right) = -\frac{4\pi^2}{\chi'}(ff' + \mu_0 R^2 P')$$

This static equilibrium does not allow for strong plasma rotation, velocities of order the sound or Alfvén velocities, for which inertial and anisotropic pressure effects can be important. Such fast rotation can occur in the presence of strong neutral beam injection. However, we do not exclude the case of diamagnetic rotation in which these effects are small and affect primarily the neighborhood of resonant surfaces.

Location within the region of closed magnetic field lines is described in terms of straight-fieldline coordinates. Each flux surface is labeled by an arbitrary coordinate ψ, which increases monitonically from the axis to the plasma edge. It is commonly chosen to be linear in the poloidal flux and normalized to go from 0 at the axis to 1 at the edge. Position on a flux surface is described in terms of the periodic variables θ and ζ, which increase by 1 (not 2π) after one turn around the torus the short way and the long way, respectively. Physical scalars must be single-valued, returning to their original values after θ or ζ increases by one. The system has Jacobian

$$\mathcal{J} \equiv (\nabla\psi \cdot \nabla\theta \times \nabla\zeta)^{-1}.$$

The contravariant representation of the equilibrium magnetic field is given by

$$\mathbf{B} = \chi'(\nabla\zeta - q\nabla\theta) \times \nabla\psi,$$

where q, the safety factor or winding number. is a function of ψ only. Derivatives along the magnetic field, which play a key role in MHD stability theory, have the simple representation

$$\mathbf{B} \cdot \nabla\mathcal{F} = \frac{\chi'}{\mathcal{J}}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\zeta}\right)\mathcal{F}(\psi,\theta,\zeta)$$

Here and subsequently, primes denote derivatives with respect to ψ. The poloidal and toroidal currents constant on a flux surface are given by

$$\mathbf{J} \cdot \nabla\theta = -\frac{2\pi f'}{\mathcal{J}}, \qquad \mathbf{J} \cdot \nabla\zeta = q\mathbf{J} \cdot \nabla\theta - \frac{\mu_0 P'}{\chi'}$$

while the normal current $\mathbf{J} \cdot \nabla\psi$ vanishes.

The perturbed displacement vector $\boldsymbol{\xi}$ is described in terms of its contravariant representation,

$$\boldsymbol{\xi} = \mathcal{J}\left(\xi_\psi \nabla\theta \times \nabla\zeta + \xi_\theta \nabla\zeta \times \nabla\psi + \xi_\zeta \nabla\psi \times \nabla\theta\right) \quad (1)$$

and a useful combination of these components, the surface displacement

$$\xi_s \equiv \mathcal{J}\left(\boldsymbol{\xi} \times \mathbf{B}\right) \cdot (\nabla\theta \times \nabla\zeta) = \chi'(q\xi_\theta - \xi_\zeta) \qquad (2)$$

Likewise, the perturbed magnetic field is expressed in terms of its contravariant components,

$$\begin{aligned} \mathbf{Q} = &\quad \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) = \mathcal{J}\left(Q_\psi \nabla\theta \times \nabla\zeta\right. \\ + &\quad \left. Q_\theta \nabla\zeta \times \nabla\psi + Q_\zeta \nabla\psi \times \nabla\theta\right) \end{aligned} \qquad (3)$$

These components are related to the components of $\boldsymbol{\xi}$ by

$$\begin{aligned} Q_\psi = &\quad \frac{\chi'}{\mathcal{J}}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\zeta}\right)\xi_\psi \\ Q_\theta = &\quad -\frac{1}{\mathcal{J}}\frac{\partial}{\partial\psi}\left(\chi'\xi_\psi\right) + \frac{1}{\mathcal{J}}\frac{\partial\xi_s}{\partial\zeta} \\ Q_\zeta = &\quad -\frac{1}{\mathcal{J}}\frac{\partial}{\partial\psi}\left(q\chi'\xi_\psi\right) - \frac{1}{\mathcal{J}}\frac{\partial\xi_s}{\partial\theta} \end{aligned} \qquad (4)$$

The ideal MHD perturbed potential energy is given by[14]

$$\begin{aligned} \delta W = &\quad \frac{1}{2\mu_0}\int d\mathbf{x}\left[Q^2 + \mathbf{J}\cdot\boldsymbol{\xi}\times\mathbf{B}\boldsymbol{\xi}\times\mathbf{Q}\right. \\ &\left. +\mu_0(\boldsymbol{\xi}\cdot\nabla P)(\nabla\cdot\boldsymbol{\xi}) + \mu_0\gamma P(\nabla\cdot\boldsymbol{\xi})^2\right] \end{aligned} \qquad (5)$$

When Eqs. (1) - (4) are introduced into Eq. (5), we obtain

$$\begin{aligned} \delta W = \frac{1}{2\mu_0}\int d\psi d\theta d\zeta\mathcal{J} &\quad \left\{\left[\nabla\xi_s \times \nabla\psi + (\nabla\theta\times\nabla\zeta)\mathcal{J}\mathbf{B}\cdot\nabla\xi_\psi - \mathbf{B}\xi_\psi' - \left[\chi''\nabla\zeta\times\nabla\psi + (q\chi')'\nabla\psi\times\nabla\theta\right]\xi_\psi\right]^2\right. \\ &+\xi_\psi\mathbf{J}\cdot\nabla\xi_s - \xi_s\mathbf{J}\cdot\nabla\xi_\psi + \left[J_\theta(q\chi')' - J_\zeta\chi''\right]\xi_\psi^2 + \frac{\mu_0 P'}{\mathcal{J}}\left(\mathcal{J}\xi_\psi^2\right)' \\ &\left.+\frac{1}{\mathcal{J}}\frac{\partial}{\partial\theta}\left(\mathcal{J}\mu_0 P'\xi_\psi\xi_\theta\right) + \frac{1}{\mathcal{J}}\frac{\partial}{\partial\zeta}\left(\mathcal{J}\mu_0 P'\xi_\psi\xi_\zeta\right) + \gamma\mu_0 P(\nabla\cdot\boldsymbol{\xi})^2\right\}. \end{aligned} \qquad (6)$$

The first two terms on the last line are perfect derivatives which vanish on integration over a flux surface. The last term is positive-definite and is the only term containing ξ_θ. Minimization with respect to ξ_θ eliminates this term for toroidal mode number $n \neq 0$. The minimizing perturbations are thus divergenceless. This will be assumed henceforth, and the compression term will be dropped.

The remaining terms contain only the normal displacement ξ_ψ and the surface displacement ξ_s. While ξ_ψ enters through its radial derivative ξ_ψ' as well as angular derivatives with respect to θ and ζ, ξ_s enters only through its angular derivatives, which will permit us to eliminate it from the treatment and reduce δW to a form involving only the radial displacement and its radial derivative, as in Newcomb's treatment of the cylindrical plasma.

We now introduce Fourier series for the normal and surface displacements,

$$\begin{aligned} \begin{pmatrix}\xi_s\\ \xi_\psi\end{pmatrix}(\psi,\theta,\zeta) &\quad = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left.\begin{pmatrix}\bar{\xi}_s\\ \bar{\xi}_\psi\end{pmatrix}\right|_{m,n}(\psi) \\ &\quad \times\exp\left[2\pi i\left(m\theta - n\zeta\right)\right] \end{aligned} \qquad (7)$$

In practice, the infinite sums in Eq. (7) are truncated to include a finite number M of components. The reality of the physical perturbations implies that the Fourier coefficients satisfy the reality conditions

$$\begin{pmatrix}\xi_s\\ \xi_\psi\end{pmatrix} = \begin{pmatrix}\xi_s\\ \xi_\psi\end{pmatrix}^* \Rightarrow \left.\begin{pmatrix}\bar{\xi}_s\\ \bar{\xi}_\psi\end{pmatrix}\right|_{-m,-n} = \left.\begin{pmatrix}\bar{\xi}_s\\ \bar{\xi}_\psi\end{pmatrix}\right|_{m,n}^* \qquad (8)$$

A useful, compact notation is to introduce the complex M-vectors of Fourier components of the radial and normal displacements.

$$\begin{pmatrix}\Xi_s\\ \Xi_\psi\end{pmatrix}(\psi) \equiv \left\{\left.\begin{pmatrix}\bar{\xi}_s\\ \bar{\xi}_\psi\end{pmatrix}\right|_{m,n}(\psi),\ m_{\mathrm{low}} \leq m \leq m_{\mathrm{high}}\right\}. \qquad (9)$$

With these definitions, the potential energy can be expressed as

$$\delta W = \frac{1}{2\mu_0}\int d\psi[\Xi_s^\dagger \mathbf{A}\Xi_s + \Xi_s^\dagger(\mathbf{B}\Xi'_\psi + \mathbf{C}\Xi_\psi) + (\Xi'^\dagger_\psi \mathbf{B}^\dagger + \Xi^\dagger_\psi \mathbf{C}^\dagger)\Xi_s + \Xi'^\dagger_\psi \mathbf{D}\Xi'_\psi + \Xi'^\dagger_\psi \mathbf{E}\Xi_\psi + \Xi^\dagger_\psi \mathbf{E}^\dagger \Xi'_\psi + \Xi^\dagger_\psi \mathbf{H}\Xi_\psi], \tag{10}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{H}$ are complex $M \times M$ matrices, dagger denotes Hermitian conjugate, and $\mathbf{A}$, $\mathbf{D}$, and $\mathbf{H}$ are self-adjoint. Detailed expressions for these matrices are given in Eq. (A3) of Appendix A.

Since Ξ_s enters Eq. (10) only algebraically, and since $\mathbf{A}$ is found to be everywhere nonsingular, δW can be minimized with respect to Ξ simply by solving the matrix equation

$$\begin{aligned}\mathbf{A}\Xi_s + \mathbf{B}\Xi'_\psi + \mathbf{C}\Xi_\psi &= 0,\\ \Xi_s &= -\mathbf{A}^{-1}(\mathbf{B}\Xi'_\psi + \mathbf{C}\Xi_\psi)\end{aligned} \tag{11}$$

When this expression is substituted for Ξ_s in Eq. (10), we obtain a form of δW involving only Ξ_ψ and its radial derivative,

$$\delta W = \int d\psi \left[\Xi'^\dagger_\psi \mathbf{F}\Xi'_\psi + \Xi'^\dagger_\psi \mathbf{K}\Xi_\psi + \Xi^\dagger_\psi \mathbf{K}^\dagger \Xi'_\psi + \Xi^\dagger_\psi \mathbf{G}\Xi_\psi\right] \tag{12}$$

in terms of the composite matrices

$$\begin{aligned}\mathbf{F} &\equiv \mathbf{D} - \mathbf{B}^\dagger \mathbf{A}^{-1}\mathbf{B} = \mathbf{F}^\dagger,\\ \mathbf{K} &\equiv \mathbf{E} - \mathbf{B}^\dagger \mathbf{A}^{-1}\mathbf{C} \neq \mathbf{K}^\dagger.\\ \mathbf{G} &\equiv \mathbf{H} - \mathbf{C}^\dagger \mathbf{A}^{-1}\mathbf{C} = \mathbf{G}^\dagger,\end{aligned} \tag{13}$$

Simplified expressions for $\mathbf{F}$ and $\mathbf{K}$ are given in Eqs. (A4) and (A5) of Appendix A.

The Euler-Lagrange equation is the condition that δW be stationary with respect to small changes in $\Xi_V(V)$,

$$\left(\mathbf{F}\Xi'_\psi + \mathbf{K}\Xi_\psi\right)' - \left(\mathbf{K}^\dagger \Xi'_\psi + \mathbf{G}\Xi_\psi\right) = 0 \tag{14}$$

a system of M coupled 2nd-order ODEs. This is a necessary but not sufficient condition that δW be a minimum. Whether the stationary point is a minimum depends on additional considerations discussed in Section IV. However, these conditions are expressed in terms of the properties of the solutions to Eq. (14), which therefore plays a central role in the theory.

The relationship to Newcomb's Euler-Lagrange equation is now straightforward. For the cylindrical case, the Fourier components decouple. The matrices $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{K}$ become diagonal and can be regarded as real scalars. In this case, the term $(\mathbf{K}\Xi_\psi)'$ can be expanded, the contribution from Ξ' cancels the other term in the equation, and the contribution from $\mathbf{K}'$ can be combined with $\mathbf{G}$ to give Newcomb's equation. In the more general toroidal case, $\mathbf{K}$ is not self-adjoint, and no real simplification is obtained by this procedure.

It is useful, for both analytical and numerical purposes, to express Eq. (14) as an equivalent system of $2M$ coupled 1st-order ODEs. This can be done by introducing the complex $2M$-vector

$$\mathbf{u} \equiv \begin{pmatrix}\Xi_\psi \\ \mathbf{F}\Xi'_\psi + \mathbf{K}\Xi_\psi\end{pmatrix} \tag{15}$$

and the complex $2M \times 2M$ matrix

$$\mathbf{L} \equiv \begin{pmatrix}-\mathbf{F}^{-1}\mathbf{K} & \mathbf{F}^{-1} \\ \mathbf{G} - \mathbf{K}^\dagger \mathbf{F}^{-1}\mathbf{K} & \mathbf{K}^\dagger \mathbf{F}^{-1}\end{pmatrix}. \tag{16}$$

in terms of which Eq. (14) can be written

$$\mathbf{u}' = \mathbf{L}\mathbf{u}. \tag{17}$$

The matrix $\mathbf{L}$ has an important symmetry. The off-diagonal blocks are self-adjoint, while the diagonal blocks are the negative of each other's adjoint. More formally, in terms the unit symplectic matrix

$$\mathbf{J} \equiv \begin{pmatrix}\mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0}\end{pmatrix}$$

$\mathbf{L}$ has the Hamiltonian property, $\mathbf{JLJ} = \mathbf{L}^\dagger$. In Hamiltonian terms, the second M components of $\mathbf{u}$ in Eq. (15) can be regarded as the conjugate momenta of the first M components, the displacements. This is understood in abstract terms, since the independent variable ψ is not time, and the variables are complex Fourier components. The Hamiltonian symmetry of $\mathbf{L}$ may be thought of as a manifestation in this representation of the well-known self-adjointness of ideal MHD.

Corresponding to the Hamiltonian symmetry of $\mathbf{L}$, the fundamental matrix of solutions has the property of being symplectic. Equation (17) is a $2M$th-order ODE and therefore has $2M$ independent solutions, corresponding to different initial conditions. These $2M$ column vectors can be assembled into a $2M \times 2M$ matrix $\mathbf{U}$ which also satisfies Eq. (17), with $\mathbf{u}$ replaced by $\mathbf{U}$. $\mathbf{U}$ is a linear canonical transformation, the propagator of the system. Using this equation and its Hermitian conjugate, it is simple to show that $\mathbf{U}^\dagger \mathbf{JU}$ is a conserved quantity; in fact, it is the Hamiltonian of the system. If it is chosen initially to be the identity, then $\mathbf{U}$ everywhere satisfies the equation

$$\mathbf{U}^\dagger \mathbf{JU} = \mathbf{J} \tag{18}$$

We can express $\mathbf{U}$ in terms of $M \times M$ blocks,

$$\mathbf{U} = \begin{pmatrix}\mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22}\end{pmatrix} \tag{19}$$

in which the upper blocks represent displacements, the lower blocks represent their conjugate momenta, and the left and right halves represent different sets of initial conditions. In terms of these blocks, the symplectic property is equivalent to

$$\begin{aligned} \mathsf{U}_{11}^{\dagger}\mathsf{U}_{21} &= \mathsf{U}_{21}^{\dagger}\mathsf{U}_{11} \\ \mathsf{U}_{12}^{\dagger}\mathsf{U}_{22} &= \mathsf{U}_{22}^{\dagger}\mathsf{U}_{12} \\ \mathsf{U}_{11}^{\dagger}\mathsf{U}_{22} - \mathsf{U}_{21}^{\dagger}\mathsf{U}_{12} &= \mathsf{I} \end{aligned} \tag{20}$$

Similarly, one can show that if

$$\mathsf{UJU}^{\dagger} = \mathsf{J} \tag{21}$$

initially, then it retains this property, and from this we obtain the additional independent relations,

$$\begin{aligned} \mathsf{U}_{11}\mathsf{U}_{12}^{\dagger} &= \mathsf{U}_{12}\mathsf{U}_{11}^{\dagger} \\ \mathsf{U}_{21}\mathsf{U}_{22}^{\dagger} &= \mathsf{U}_{22}\mathsf{U}_{21}^{\dagger} \\ \mathsf{U}_{11}\mathsf{U}_{22}^{\dagger} - \mathsf{U}_{21}\mathsf{U}_{12}^{\dagger} &= \mathsf{I}. \end{aligned} \tag{22}$$

These symmetries play a central role in proving the theorems of Section IV.

III. SINGULAR SURFACES

The Euler-Lagrange equation has regular singular points at the resonant surfaces where $q = m/n$, at the magnetic axis, and at a separatrix if one exists. In this section we discuss the asymptotic behavior of the solutions in the neighborhood of each of these singular points.

The form of the Euler-Lagrange equation, Eqs. (16) and (17), shows that the coefficient matrix L becomes singular wherever the determinant of F vanishes. Using the forms of these matrices given in Eqs. (A4) and (A5) of Appendix A, we can write

$$\mathsf{F} = \mathsf{Q}\bar{\mathsf{F}}\mathsf{Q}, \qquad \mathsf{K} = \mathsf{Q}\bar{\mathsf{K}}, \qquad \mathsf{G} = \bar{\mathsf{G}},$$

where $\bar{\mathsf{F}}$, $\bar{\mathsf{G}}$, and $\bar{\mathsf{K}}$ are nonsingular at the resonant surfaces. Since Q becomes singular at the resonant surfaces where $q = m/n$, these are singular points. For an axisymmetric torus, as treated here, the toroidal mode number n is a good quantum number, *i.e.* a fixed integer for each perturbation treated, and therefore the resonant surfaces are discrete. In the presence of a magnetic separatrix, where $q \to \infty$ logarithmically, there is a point of accumulation of the resonant surfaces. While we have not been able to prove it analytically, we find numerically that $\bar{\mathsf{F}}$ vanishes linearly with ψ and $\bar{\mathsf{G}}$ varies as ψ^{-1} as $\psi \to 0$; this corresponds to the behavior of Newcomb's equation for the cylinder, making the axis a singular point. In addition, we find numerically that $\det \bar{\mathsf{F}}$ vanishes at a separatrix, making it another singular point. In certain degenerate cases, *e.g.* if $q = m/n$ at the axis or at an extremum where q' vanishes, two or more singular points can merge to form a singularity of higher rank. We treat only the relatively simple cases for which this does not occur.

In analyzing the asymptotic behavior of the solution in the neighborhood of a resonant surface, we must take account of the full matrix nature of the equations and the coupling between resonant and nonresonant components. To accomplish this, we use a matrix form of the Frobenius analysis due to Turrittin.[15] Before treating the full case, we illustrate the procedure for Newcomb's cylindrical case, for which the Euler-Lagrange equation is

$$(f\xi')' - g\xi = 0, \tag{23}$$

where primes denote derivatives with respect to r. To express this in matrix form, let

$$\mathbf{u} = \begin{pmatrix} \xi \\ f\xi' \end{pmatrix}, \qquad \mathsf{L} = \begin{pmatrix} 0 & 1/f \\ g & 0 \end{pmatrix}.$$

Then Eq. (23) becomes $\mathbf{u}' = \mathsf{L}\mathbf{u}$. Let r_s be the position of the singular surface, let $z \equiv r - r_s$ for $r > r_s$, and reinterpret the primes to denote derivatives with respect to z. The functions $f(r)$ and $g(r)$ can be Taylor expanded about the singular surface as $f = f_0 z^2 + \cdots$, $g = g_0 + \cdots$. Then to lowest order,

$$\mathsf{L} = \begin{pmatrix} 0 & 1/f_0 z^2 \\ g_0 & 0 \end{pmatrix} + \cdots.$$

We now define a new dependent variable vector $\mathbf{v}$ by introducing the shearing transformation,

$$\mathbf{u} = \mathsf{R}\mathbf{v}, \qquad \mathsf{R} \equiv \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix} \tag{24}$$

Then $\mathbf{v}$ satisfies the equation $z\mathbf{v}' = \mathsf{M}\mathbf{v}$ in terms of the transformed matrix

$$\mathsf{M} = z\mathsf{R}^{-1}(\mathsf{L}\mathsf{R} - \mathsf{R}') = \begin{pmatrix} 1/2 & 1/f_0 \\ g_0 & -1/2 \end{pmatrix} + \cdots.$$

If we now seek power-like solutions of the form $\mathbf{v} = z^{\alpha}\mathbf{v}^{(0)} + \cdots$, then to lowest order we obtain the matrix eigenvalue equation

$$(\mathsf{M} - \alpha\mathsf{I})\mathbf{v} = 0 \tag{25}$$

with eigenvalues satisfying the characteristic equation,

$$\begin{aligned} &\det(\mathsf{M} - \alpha\mathsf{I}) = 0, \qquad \alpha_{\pm} = \pm\sqrt{-D_I}, \\ &D_I = -g_0/f_0 - 1/4, \end{aligned} \tag{26}$$

and eigenvectors

$$\mathbf{v}_{\pm} = \begin{pmatrix} 1 \\ (\alpha_{\pm} - 1/2)f_0 \end{pmatrix}.$$

Equation (25) thus plays the role of the indicial equation in the standard Frobenius analysis and also determines the form of the solution vectors. Transforming these solutions back to the original vector of dependent variables $\mathbf{u}$ by means of Eq. (24), we find that the general solution can be expressed as a linear combination of large and small solutions,

$$\begin{aligned}\mathbf{u} = \quad & c_- z^{\alpha_-} \begin{pmatrix} z^{-1/2} \\ z^{1/2}(\alpha_- - 1/2) f_0 \end{pmatrix} \\ & + c_+ z^{\alpha_+} \begin{pmatrix} z^{-1/2} \\ z^{1/2}(\alpha_+ - 1/2) f_0 \end{pmatrix}\end{aligned} \tag{27}$$

The shearing transformation introduced in Eq. (24) is required in order to properly balance the components of the new dependent variable vector $\mathbf{v}$, giving both components the same power-like behavior. If we applied the same shearing transformation at an ordinary point of the equation, or to a nonresonant component of the solution at a resonant surface, then since the components of $\mathbf{u}$ are already in balance, both approaching a constant, the shearing transformation would unbalance them. Thus, in treating the toroidal case, we must choose a shearing transformation which appropriately balances the resonant components without unbalancing the nonresonant components. This introduces half-integral powers into the coupling between resonant and nonresonant components.

In the neighborhood of a resonant surface of a toroidal plasma, let $q(\psi_R) = m/n$, $z \equiv \psi - \psi_R$ Then the dominant terms in L, Eq. (16), vary as

$$\begin{aligned}&\mathsf{F}^{-1} \sim z^{-2}, \\ &\mathsf{F}^{-1}\mathsf{K} \sim \mathsf{K}^\dagger \mathsf{F}^{-1} \sim z^{-1}, \\ &\mathsf{G} \sim \mathsf{K}^\dagger \mathsf{F}^{-1} \mathsf{K} \sim 1\end{aligned} \tag{28}$$

To balance the equations, we introduce the shearing transformation

$$\begin{aligned}\mathbf{u} \quad &= \mathbf{R}\mathbf{v}, \qquad R_{m,m'} = z^{r_m}\delta_{m,m'}, \\ r_m \quad &= -1/2, \qquad r_{m+M} = 1/2 \qquad \text{for} \quad m = nq, \\ r_m \quad &= r_{m+M} = 0 \qquad \text{for} \quad m \neq nq,\end{aligned} \tag{29}$$

where M is the number of retained Fourier components. Then the equation is transformed to

$$z\mathbf{v}' = \mathsf{M}\mathbf{v}. \tag{30}$$

The transformed matrix,

$$\mathsf{M} = z\mathsf{R}^{-1}(\mathsf{L}\mathsf{R} - \mathsf{R}'),$$

can be Taylor expanded about the resonant point as

$$\mathsf{M} = \sum_{k=0}^{\infty} z^{k/2}\mathsf{M}_k. \tag{31}$$

We now seek power-like solutions of the form

$$\mathbf{v} = z^\alpha \sum_{k=0}^{\infty} z^{k/2}\mathbf{v}^{(k)}. \tag{32}$$

Introducing Eqs. (31) and (32) into Eq. (30), we obtain, as the lowest-order terms, the matrix eigenvalue equation, $[\mathsf{M}_0 - \alpha\mathsf{I}]\mathbf{v}^{(0)} = 0$. The eigenvalues α are the solutions to the characteristic equation,

$$\det[\mathsf{M}_0 - \alpha\mathsf{I}] = 0. \tag{33}$$

Higher-order terms in the power series solutions can be obtained iteratively from the higher-order equations,

$$[\mathsf{M}_0 - (\alpha + k/2)\mathsf{I}]\mathbf{v}^{(k)} = -\sum_{l=1}^{k} \mathsf{M}_l \mathbf{v}^{(k-l)}$$

Since this is a regular singular point, the powers series solutions are convergent.

The spectrum of Eq. (33) is simple. Since M_0 is a $2M \times 2M$ matrix, there are $2M$ eigenvalues and eigenvectors. Of these, two represent the resonant solutions corresponding to Eq. (26) for the cylinder. The other $2M-2$ eigenvalues are zero, corresponding to the nonresonant solutions.

Determination of the resonant eigenvalues can be reduced to the solution of a 2×2 characteristic equation. We take two basis vectors, each containing 1 for one of its resonant components and all other components zero. Multiplication of these vectors by M_0 annihilates all contributions from the null space. The resonant components of the resulting vectors form a 2×2 matrix whose eigenvalues are

$$\alpha = \pm\sqrt{-D_I}, \tag{34}$$

where D_I is the well-known expression for the Mercier criterion in axisymmetric toroidal geometry,[16–18]

$$\begin{aligned}D_I = \quad & -\frac{1}{4} + \frac{2\pi f P' V'}{q'\chi'^3}\left\langle \frac{1}{|\nabla\psi|^2}\right\rangle \left[1 - \frac{2\pi f P' V'}{q'\chi'^3}\left\langle \frac{1}{|\nabla\psi|^2}\right\rangle\right] \\ & + \left\langle \frac{B^2}{|\nabla\psi|^2}\right\rangle \frac{P' V'^2}{q'^2\chi'^4}\left\{P'\left[\left\langle \frac{1}{B^2}\right\rangle + \left(\frac{2\pi f}{\chi'}\right)^2 \left\langle \frac{1}{B^2|\nabla\psi|^2}\right\rangle\right] + \frac{\chi''}{\chi'} - \frac{V''}{V'}\right\}\end{aligned} \tag{35}$$

where V is the volume inclosed within a flux surface. While we have not been able to prove analytically that the solutions to Eq. (33) are in fact given by Eqs. (34) and (35), we find numerically that they converge to these values as the number of retained Fourier components increases. Agreement between these two ways of computing D_I can thus serve as a test for adequate Fourier convergence at a resonant surface. The Mercier criterion, that $D_I \leq 0$ is a necessary condition for stability, follows from that fact that if $D_I > 0$, then α is imaginary, the solution vector $\mathbf{v}$ in Eq. (32) is oscillatory, and the generalization of Newcomb's stability criterion, presented in the next section, is repeatedly violated.

It would appear from Eqs. (31), (32), and (33) that the higher-order terms in the power-series solutions contain half-integral powers of z. This is deceptive, because the shearing transformation in Eq. (29) restores the half powers to integral powers for the original variables $\mathbf{u}$. The real significance of the half powers is that determination of higher order terms alternates between the resonant and nonresonant components. For example, the zeroth-order resonant eigenvectors contain only resonant components, the first-order terms are all nonresonant, the second-order terms are resonant, etc. For the nonresonant eigenvectors, this order is reversed. In principle, the power series solutions can be determined to arbitrarily high order; in practice, this is limited by the difficulty of evaluating the higher order terms in the Taylor expansion of $\mathbf{M}$, Eq. (31).

Once all of the independent eigenvectors have been determined to some order, they can be assembled as the columns of a matrix which we denote $\mathbf{V}(\psi)$. We choose a particular order for these vectors, as close as possible to the identity. The subspace of nonresonant eigenvectors, corresponding to the eigenvalues $\alpha = 0$, is spanned by a basis in which each basis vector contains only one nonresonant component to lowest order. We arrange these vectors so that the nonvanishing component appears on the diagonal and normalize them so that the diagonal element is 1. We place the large resonant eigenvector in the resonant column of the left half of $\mathbf{V}$ and the small resonant eigenvector in the resonant column of the left half of $\mathbf{V}$. With a proper choice of normalization of the resonant solutions, $\mathbf{V}$ can be made symplectic.

In principle, the convergent power series could be evaluated to all orders, and $\mathbf{V}$ would then be an exact fundamental solution matrix. Any particular solution vector $\mathbf{u}(\psi)$ of the exact Euler-Lagrange equation, such as one obtained by integrating the equations numerically, could then be expressed as a linear combination of these independent solutions, $\mathbf{u}(\psi) = \mathbf{V}(\psi)\mathbf{c}$, where the constant vector $\mathbf{c}$ consists of the expansion coeffients in a linear combination. In practice, it is often more convenient to truncate the power series after a few terms, in which case $\mathbf{V}(\psi)$ is not an exact fundamental solution matrix. We can still express an exact solution as a linear combination of the truncated power series solutions,

$$\mathbf{u}(\psi) = \mathbf{V}(\psi)\mathbf{c}_A(\psi). \tag{36}$$

Then the vector $\mathbf{c}_A(\psi)$ of expansion coefficients is no longer constant, but asymptotically approaches a limit as the resonant surface is approached, as long as the number of terms in the power series is sufficiently large. We therefore denote the components of this vector asymptotic coefficients.

As $\psi \to \psi_R$, $\mathbf{c}_A$ can be obtained by solving Eq. (36). The matrix $\mathbf{V}(\psi)$ is used in the following section in deriving the proofs of the basic stability theorems. The components of $\mathbf{c}_A(\psi)$ are also involved in the generalization of the quantity Δ' which appears in the study of singular modes. This will be the subject of a later paper.

The two resonant eigenvectors corresponding to the eigenvalues in Eq. (34) have powers differing by $2\sqrt{-D_I}$. If accurate determination of the asymptotic coefficient of the small resonant solution is necessary, as in the treatment of singular modes, then enough terms in the series solution of the large solution must be retained so that its remaining error is smaller than the zeroth-order term in the small solution. If $|D_I|$ is large, this can make the procedure numerically impractical. This typically occurs in regions of very low shear q', for example when q has a turning point, or if β, the ratio of plasma pressure to magnetic energy density, is large.

The axis is a singular point of the Euler-Lagrange equation, just at it is for Newcomb's cylindrical case. For the latter, the solutions for $r\xi_r$ vary as $r^{\pm m}$ in the neighborhood of the axis. Boundary conditions at the axis must exclude the r^{-m} solutions and retain the r^m solutions. Similarly, one would expect that, in the toroidal case, solutions for $\xi \cdot \nabla\psi$ should vary as $\psi^{\pm\mu/2}$, with μ an integer. Determination of the asymptotic behavior at the axis should be reducible to an analysis similar to that presented above for the resonant surfaces, based on a matrix eigenvalue equation of the form

$$\lim_{\psi \to 0} [\psi \mathbf{L} - \alpha \mathbf{I}]\mathbf{u} = 0. \tag{37}$$

There are several complications which make this difficult. The complex form of $\mathbf{F}$ given in Eq. (A4), especially coupling to the $m = 0$ term, make it difficult to extract the asymptotic behavior of $\mathbf{L}$ analytically. Since the eigenvalues differ by integers, there are further complications discussed by Turrittin.

Equation (37) can be treated numerically as the solution to a matrix eigenvalue problem. Doing so, we find that the eigenvalues do indeed approach $\mu/2$, as expected. The eigenvectors obtained by this means are used to initialize the integration near the axis. In general, a pure eigenvector, corresponding to pure $\psi^{\mu/2}$ behavior, contains a mixture of all Fourier components m, due to toroidal (± 1) and elliptic (± 2) coupling in the neighborhood of the axis.

If a separatrix bounds the region of confined plasma, it has two effects on the distribution of singular surfaces. First, it causes $q \to \infty$, introducing an accumulation point of singular surfaces due to the $\mathbf{Q}$ factors in $\mathbf{F}$, Eq. (A4). Second, we have found numerically that the remaining factors of $\mathbf{F}$ have a vanishing determinant at the axis. If we treat a truncated Fourier series with some maximum value of m, then this additional singularity can be regarded to play a role similar to that of the singular point at the axis. A full understanding of this behavior remains to be achieved.

At each resonant point, one eigenvalue of $\mathbf{F}$ has a quadratic zero; it does not change sign. The only other singularities of $\mathbf{F}$ are at the axis and the separatrix. Thus, throughout the region of closed field lines between the axis and the separatrix, all eigenvalues of $\mathbf{F}$ are non-negative. This property is used in the next section to generalize the proofs of the stability theorems.

IV. FIXED-BOUNDARY MODES

Our goal in this paper is to determine whether there exist test functions $\Xi_\psi(\psi)$ which make δW negative while satisfying the boundary conditions. In our analysis so far, we have derived the Euler-Lagrange equation which makes δW stationary with respect to variations of Ξ_ψ, and we have studied its symmetry properties and its singular points. Now we must understand the relationship between the solutions to this equation and the existence of test functions which make δW negative.

Here is an outline of the proofs in this section. We first consider the case of a plasma bounded by flux surfaces ψ_1 and ψ_2 which are both ordinary points of the Euler-Lagrange equation, with no resonant points between them. We study a 2-point boundary value problem on the interval $[\psi_1, \psi_2]$. We use this to find the slope of the Euler-Lagrange solution passing through any point in the space of solutions. This motivates the definition of the plasma response matrix $\mathbf{W}_P(\psi)$ and the critical determinant $D_C(\psi) \equiv \det \mathbf{W}_P(\psi)$. If D_C has no poles in $[\psi_1, \psi_2]$, we use the results of the 2-point boundary value problem to construct a Hilbert invariant integral and prove that it depends only on the boundary conditions and that it constitutes a lower bound on δW. The lower bound is achieved if and only if $\Xi(\psi)$ is an Euler-Lagrange solution satisfying the boundary conditions. If ψ_1 and ψ_2 are perfect conductors, this lower bound is shown to vanish, proving that the absence of poles in D_C is a sufficient condition for stability. If D_C has a pole in $[\psi_1, \psi_2]$, we prove by construction that there exists a test function $\Xi(\psi)$ which makes $\delta W < 0$, and thus that the absence of poles in D_C is a necessary condition for stability. Next we generalize the proofs to the case where the plasma is bounded by the axis, $\psi = 0$; and by a resonant surface where $q(\psi_R) = m/n$. Finally, we consider the case of one or more resonant points $\psi_R \in [\psi_1, \psi_2]$. We show that if D_C has no pole in $[\psi_1, \psi_R)$ or in $(\psi_R, \psi_2]$, then the lower bound on δW is achieved if and only if $\Xi(\psi)$ is an Euler-Lagrange solution in each nonresonant subinterval, contains no large resonant solution at ψ_R, and has nonresonant contributions which are continuous with continuous derivatives. Generalizations of the necessary and sufficient conditions then follow easily. The case where the plasma is bounded by a vacuum region is considered in the next section.

As a preliminary, we treat a 2-point boundary value problem. Consider an interval $\psi_1 \le \psi \le \psi_2$ containing no singular points where $\det \mathbf{F} = 0$. The solutions to the Euler-Lagrange equation in this interval constitute a $2M$-dimensional complex vector space. A unique solution can be selected by imposing $2M$ boundary conditions. These could be imposed entirely at ψ_1, entirely at ψ_2, or any combination thereof. We impose half of the conditions at ψ_1 and half at ψ_2, with all conditions imposed on Ξ, the upper half of $\mathbf{u}$ as defined in Eq. (15). We define a fundamental matrix of solutions satisfying

$$\mathbf{U}'(\psi) = \mathbf{L}(\psi)\mathbf{U}(\psi), \qquad \mathbf{U}(\psi_1) = \mathbf{I}. \tag{38}$$

Any particular solution $\mathbf{u}$ can be expressed as a linear combination of the columns of $\mathbf{U}$,

$$\mathbf{u}(\psi) = \mathbf{U}(\psi)\mathbf{c}, \tag{39}$$

where $\mathbf{c}$ is a constant vector of expansion coefficients. Determining $\mathbf{u}$ is equivalent to determining $\mathbf{c}$. The boundary conditions are given by $\Xi(\psi_1) = \mathbf{u}_1(\psi_1) = \Xi_1$, $\Xi(\psi_2) = \mathbf{u}_1(\psi_2) = \Xi_2$, where $\mathbf{u}_1$ refers to the upper half of $\mathbf{u}$ and the Ξ's are constant M-vectors. Using the decomposition of $\mathbf{U}$ given in Eq. (19), we can express the boundary conditions in terms of $\mathbf{c}$ as $\mathbf{c}_1 = \Xi_1$, $\mathbf{U}_{11}(\psi_2)\mathbf{c}_1 + \mathbf{U}_{12}(\psi_2)\mathbf{c}_2 = \Xi_2$, where $\mathbf{c}_1$ and $\mathbf{c}_2$ are the upper and lower halves of $\mathbf{c}$. Since the boundary conditions at ψ_1 determine $\mathbf{c}_1$, the boundary conditions at ψ_2 must be satified by solving for $\mathbf{c}_2$,

$$\mathbf{c}_2 = \mathbf{U}_{12}^{-1}(\psi_2)[\Xi_2 - \mathbf{U}_{11}(\psi_2)\Xi_1]. \tag{40}$$

From this it is clear that the existence of a solution to the boundary value problem depends on whether $\mathbf{U}_{12}(\psi_2)$ is invertible, *i.e.* has a nonvanishing determinant. If it does not, then the solution exists only if the quantity in brackets is orthogonal to the null space of $\mathbf{U}_{12}^\dagger$, in which case the solutions are not unique, but there is a family of solutions passing through the same point.

Following Newcomb, we wish to define a Hilbert invariant integral and determine when it exists and constitutes a lower bound on δW. For the cylindrical case, Newcomb defines the Hilbert invariant integral as a contour integral in a 2-dimensional real vector space whose axes are ξ and r; we define our generalization as a contour integral in a space which is a cartesian product of one real dimension for ψ and an M-dimensional complex vector space of Ξ's. Newcomb covers his 2-space with Euler-Lagrange solutions which all satisfy a given left boundary condition $\xi(r_1) = \xi_1$; we cover our space with solutions which all satisfy a given left boundary condition $\Xi(\psi_1) = \Xi_1$. Newcomb defines his contour integral in terms of the slope of the Euler-Lagrange solution passing through a given point in ξ-r space. In our space, this involves finding the Euler-Lagrange solution which passes through Ξ_1 at ψ_1 and through Ξ at ψ, the solution to our the 2-point boundary value problem. Its slope is given by

$$\Xi^*(\psi, \Xi_1, \Xi) \equiv \mathbf{F}^{-1} \left\{ [\mathbf{U}_{21}(\psi)\Xi_1 - \mathbf{K}(\psi)\Xi] + \mathbf{U}_{22}(\psi)\mathbf{U}_{12}^{-1}(\psi)[\Xi - \mathbf{U}_{11}(\psi)\Xi_1] \right\}. \quad (41)$$

We define the plasma response matrix $\mathbf{W}_P$ and the critical determinant D_C as

$$\mathbf{W}_P(\psi) \equiv \mathbf{U}_{22}(\psi)\mathbf{U}_{12}^{-1}(\psi), \quad D_C(\psi) \equiv \det \mathbf{W}_P(\psi). \quad (42)$$

The plasma response matrix occurs in the expression for Ξ^* in Eq. (41). Since $\det \mathbf{F}$ never vanishes in the interval $[\psi_1, \psi_2]$, the existence of Ξ^* throughout the interval depends only on whether $\mathbf{W}_P$ is well-behaved, *i.e.* whether its determinant D_C has poles in the interval. From the symplectic symmetry properties in Eqs. (19) and (20), we find that $\mathbf{W}_P$ is self-adjoint and therefore D_C is real. We shall see that poles in D_C constitute the appropriate generalization of Newcomb's criterion involving zeroes of his real scalar Euler-Lagrange solution $\xi(r)$. We can also express our generalized criterion in a manner more similar to Newcomb's by noting that the smallest eigenvalue of $\mathbf{W}_P^{-1}$ has a zero wherever D_C has a pole. In fact, this is a more useful numerical diagnostic.

If D_C has no poles in the interval $[\psi_1, \psi_2]$, we define our Hilbert invariant integral as a contour integral in $\Xi - \psi$ space,

$$\begin{aligned} W^\dagger(C) \equiv & \int_C \Big\{ [\Xi^{*\dagger}(\mathbf{F}\Xi^* + \mathbf{K}\Xi) + \Xi^\dagger(\mathbf{K}^\dagger\Xi^* + \mathbf{G}\Xi)] d\psi \\ & + (d\Xi^\dagger - \Xi^{*\dagger} d\psi)(\mathbf{F}\Xi^* + \mathbf{K}\Xi) \\ & + (\Xi^{*\dagger}\mathbf{F} + \Xi^\dagger\mathbf{K}^\dagger)(d\Xi - \Xi^* d\psi) \Big\} \\ = & \int_C \left[w_\psi d\psi + d\Xi^\dagger \mathbf{w}_{\Xi^\dagger} + \mathbf{w}_\Xi d\Xi \right]. \end{aligned} \quad (43)$$

where C is any contour connecting the boundary conditions in our $(M+1)$-dimensional $\Xi - \psi$ space and

$$w_\psi \equiv \Xi^\dagger \mathbf{G}\Xi - \Xi^{*\dagger}\mathbf{F}\Xi^*, \quad \mathbf{w}_{\Xi^\dagger} = \mathbf{w}_\Xi^\dagger = \mathbf{F}\Xi^* + \mathbf{K}\Xi. \quad (44)$$

To prove that $W^\dagger(C)$ is independent of the contour and depends only on the endpoints, we must prove that the integrand is curl-free in this multidimension space. This follows from the conditions

$$\frac{\partial w_\psi}{\partial \Xi^\dagger} = \frac{\partial w_{\Xi^\dagger}}{\partial \psi}, \quad \frac{\partial w_{\Xi^\dagger}}{\partial \Xi} = \frac{\partial w_\Xi}{\partial \Xi^\dagger},$$

which follow easily from Eq. (44). To prove that $W^\dagger(C)$ constitutes a lower bound for δW, we note that

$$\delta W - W^\dagger(C) = \int_C d\psi (\Xi' - \Xi^*)^\dagger \mathbf{F} (\Xi' - \Xi^*) \geq 0,$$

because $\mathbf{F}$ is non-negative, as discussed at the end of the previous section. Equality holds if and only if Ξ is an Euler-Lagrange solution.

Having shown that δW is minimized by the unique Euler-Lagrange solution satisfying the boundary condition if D_C has no poles, we now find the value of that minimum. Starting with Eq. (12), we first symmetrize the expression to obtain

$$\begin{aligned} \delta W = & \frac{1}{2} \int_{\psi_1}^{\psi_2} d\psi \big[\Xi'^\dagger (\mathbf{F}\Xi' + \mathbf{K}\Xi) + \Xi^\dagger(\mathbf{K}^\dagger \Xi' + \mathbf{G}\Xi) \\ & + (\Xi'^\dagger \mathbf{F} + \Xi^\dagger \mathbf{K}^\dagger)\Xi' + (\Xi'^\dagger \mathbf{K} + \Xi^\dagger \mathbf{G})\Xi \big]. \end{aligned} \quad (45)$$

Next we use the Euler-Lagrange equation, Eq. (14), and its Hermitian conjugate, to eliminate terms in Eq. (45) containing $\mathbf{G}$ and obtain

$$\begin{aligned} \delta W = & \frac{1}{2} \int_{\psi_1}^{\psi_2} d\psi \Big[\Xi'^\dagger (\mathbf{F}\Xi' + \mathbf{K}\Xi) + \Xi^\dagger (\mathbf{F}\Xi' + \mathbf{K}\Xi)' \\ & + (\Xi'^\dagger \mathbf{F} + \Xi^\dagger \mathbf{K}^\dagger)\Xi' + (\Xi'^\dagger \mathbf{F} + \Xi^\dagger \mathbf{K}^\dagger)'\Xi \Big] \\ = & \frac{1}{2} \int_{\psi_1}^{\psi_2} d\psi \Big[\Xi^\dagger (\mathbf{F}\Xi' + \mathbf{K}\Xi) + (\Xi'^\dagger \mathbf{F} + \Xi^\dagger \mathbf{K}^\dagger)\Xi \Big]' \\ = & \frac{1}{2} \Big[\Xi^\dagger (\mathbf{F}\Xi' + \mathbf{K}\Xi) + (\Xi'^\dagger \mathbf{F} + \Xi^\dagger \mathbf{K}^\dagger)\Xi \Big] \Big|_{\psi_1}^{\psi_2} \\ = & \frac{1}{2} \left(\mathbf{u}_1^\dagger \mathbf{u}_2 + \mathbf{u}_2^\dagger \mathbf{u}_1 \right) \Big|_{\psi_1}^{\psi_2} \end{aligned} \quad (46)$$

The unique Euler-Lagrange solution satisfying the boundary conditions is the solution to the 2-point boundary value problem above. If the plasma response matrix is nonsingular throughout the interval, then $\mathbf{u}_2(\psi)$ is bounded everywhere. If $\mathbf{u}_1 = \Xi$ vanishes at the boundaries, as it must at perfectly conducting walls, then this minimizing δW vanishes, proving that there are no test functions which make δW negative. This proves the sufficient condition for stability, that in the absence of poles in D_C, the system is stable. So far the proof applies only to the case where ψ_1 and ψ_2 are conducting walls. This is further generalized below.

Next we prove, for the same case, that the absence of poles in D_C is a necessary condition for stability, by showing that if D_C has a pole, we can construct a test function which makes δW negative. If $\mathbf{c}_1 = \Xi_1 = 0$, then $\mathbf{u}_1(\psi) = \mathbf{U}_{21}(\psi)\mathbf{c}_2$, $\mathbf{u}_2(\psi) = \mathbf{U}_{22}(\psi)\mathbf{c}_2 = \mathbf{U}_{22}(\psi)\mathbf{U}_{21}^{-1}(\psi)\mathbf{u}_1(\psi) = \mathbf{W}_P(\psi)\mathbf{u}_1(\psi)$, and $\mathbf{u}_1(\psi) = \mathbf{W}_P(\psi)^{-1}\mathbf{u}_2(\psi)$. If $D_C(\psi)$ has a pole at $\psi = \psi_P$, then $\mathbf{W}_P^{-1}(\psi_P)$ has a vanishing determinant and hence a nontrivial null space. If we choose $\mathbf{u}_2(\psi_P)$ to be in that null space, the corresponding $\mathbf{u}_1(\psi_P)$ vanishes. We may therefore construct a test function which is nonzero in $[\psi_1, \psi_P]$, for which $\mathbf{u}_1(\psi)$ vanishes at both ψ_1 and ψ_P, and therefore, by Eq. (46), the contribution of this interval to δW vanishes. If we make the test function vanish identically on $[\psi_P, \psi_2]$, then the contribution from that interval also vanishes. This test function is an Euler-Lagrange solution on each of the subintervals, but not on the whole interval $[\psi_1, \psi_2]$. In particular, it is not an Euler-Lagrange solution on any subinterval containing ψ_P. We may choose a small subinterval containing ψ_P and connect the points on the original test function on either side of ψ_P with an Euler-Lagrange solution between those points. This must give a lower contribution to δW than the original test function. Since δW vanishes for the original test function, the new trial function makes δW negative. This completes the proof.

To generalize the proofs to the case where $\psi_1 = 0$, the magnetic axis, we replace the initial conditions on $\mathbf{U}$. Rather than initialize to the identity, as in Eq. (38), we construct an initial $\mathbf{U}$ from the asymptotic solutions in the neighborhood of the axis, the solutions to Eq. (37). Let the left half of $\mathbf{U}$ contain the solutions which are asymptotically large at the axis and the right half contain those which are small. Then the constant vector $\mathbf{c}$ defined in Eq. (39) constitutes the expansion coefficients of these asymptotic solutions, the top half $\mathbf{c}_1$ the coefficients of the irregular solutions and the bottom half $\mathbf{c}_2$ the coefficients of the regular solutions. The boundary conditions at the axis require that $\mathbf{c}_1 = 0$. The rest of the proofs follow unchanged.

To understand the behavior as ψ_2 approaches a resonant surface ψ_R, we first consider the behavior of the plasma response matrix $\mathbf{W}_P$. For the cylindrical case, with $M = 1$, $\mathbf{W}_P$ is a scalar, the ratio of the second to the first component of the 2-vector $\mathbf{u}$ which is small at the axis. From the shearing transformation in Eq. (24) or (29), we see that this vanishes, $\mathbf{W}_P \sim z = \psi_R - \psi$ as $z \to 0$. From Eq. (46), with $\Xi_1 = 0$ because ψ_1 is either a conducting wall or the axis, we find

$$\delta W = \frac{1}{2}\left(\mathbf{u}_1^\dagger \mathbf{u}_2 + \mathbf{u}_2^\dagger \mathbf{u}_1\right)\Big|_{\psi_1}^{\psi_2} = \Xi^\dagger \mathbf{W}_P \Xi\big|_{\psi_2}. \tag{47}$$

Thus, if Ξ were bounded at the resonant surface, δW would vanish. To keep δW bounded, it is overly restrictive to require that Ξ bounded; it would be sufficient to let $\Xi \sim z^{-1/2}$. Since the minimizing test function is an Euler-Lagrange solution, we may take Ξ to be a linear combination of the large and small resonant solutions, varying as $z^{-1/2 \pm \sqrt{-D_I}}$. Since the large solution diverges more rapidly than $z^{-1/2}$, it would contribute a positive-definite infinite term to δW, precluding the determination of a minimum, and must therefore be excluded. Then only the small resonant solution can be allowed, and its contribution to δW vanishes. For the general case $M > 1$, $\mathbf{W}_P$ has one vanishing eigenvalue and $M - 1$ eigenvalues approaching constants. The contribution from the large resonant solution is required to vanish, as a boundary condition at the resonant surface. In the cylindrical case, the only solution which is small at both the axis and the resonant surfaces is the trivial solution which is identically zero throughout the first interval. In the general case, there may be an arbitrary linear combination of nonresonant terms, spanning a space of dimension $M - 1$, which make finite contributions to δW, according to Eq. (47). The vector Ξ may be bounded or may have an unbounded contribution from the small resonant solution, and is otherwise arbitrary. Only the nonresonant terms make a finite contribution to δW.

To treat the boundary conditions at a resonant surface $\psi_2 = \psi_R$, we note that a particular solution can be expressed in terms of two different bases, $\mathbf{u}(\psi) = \mathbf{U}(\psi)\mathbf{c} = \mathbf{V}(\psi)\mathbf{d}$. Here, $\mathbf{U}$ is the fundamental solution matrix defined above, initialized to the large and small solutions at the axis; $\mathbf{V}$ is the fundamental solution matrix defined in Section III above Eq. (36), constructed from the full convergent power series solutions about the resonant surface; $\mathbf{c}$ is a constant vector whose upper and lower halves $\mathbf{c}_1$ and $\mathbf{c}_2$ contain the expansion coefficients of solutions which are large and small at the axis, respectively; and $\mathbf{d}$ is a constant vector whose upper half $\mathbf{d}_1$ contains the components of the nonresonant displacements and the coefficient of the large resonant solution at the resonant surface, and whose lower half $\mathbf{d}_2$ contains the second components of the nonresonant solutions and the coefficient of the small resonant solution at the resonant surface. The boundary condition at the axis is $\mathbf{c}_1 = 0$. We wish to determine $\mathbf{c}_2$ by imposing boundary conditions at the resonant surface, specifying $\mathbf{d}_1$, imposing an arbitrary value for the nonresonant displacements and zero for the coefficient of the nonresonant solution. This can be done by letting $\mathbf{d} = \mathbf{V}^{-1}\mathbf{U}\mathbf{c}$, letting $d_1 = \Xi_R$ with vanishing resonant component, using the symplectic property, Eq.(18), to write $\mathbf{V}^{-1} = -\mathbf{J}\mathbf{V}^\dagger\mathbf{J}$, and inverting to obtain

$$\mathbf{c}_2 = (\mathbf{V}_{22}^\dagger \mathbf{U}_{12} - \mathbf{V}_{12}^\dagger \mathbf{U}_{22})^{-1}\Xi_R, \tag{48}$$

If the power series in $\mathbf{V}$ are truncated to some order, then $\mathbf{c}_2$ may be evaluated as the limit as $\psi \to \psi_R$. Equation (48) may be regarded as a generalization of Eq. (40) to a resonant surface, with $\Xi_1 = 0$ and $\mathbf{U}_{12}$ replaced by the appropriate limiting matrix. The nonresonant case is recovered by replacing $\mathbf{V}_{22}$ with $\mathbf{I}$ and $\mathbf{V}_{12}$ with $\mathbf{0}$. In the cylindrical case, the only solution is the trivial one, $\mathbf{c}_2 = \mathbf{0}$. In the toroidal case, the nonresonant components of Ξ remain arbitrary, and thus $\mathbf{c}_2$ spans a space of dimension $M - 1$. Equation (48) is a general result for any value of Ξ_R, including a nonzero resonant component. With a vanishing resonant component, it can be further simplified to

$$\mathbf{c}_2 = \lim_{\psi \to \psi_R} \mathbf{U}_{12}^{-1}(\psi)\Xi_R, \tag{49}$$

the same form as for the nonresonant case.

If $[\psi_1, \psi_2]$ contains one or more resonant surfaces, the above proof of the sufficient condition is no longer valid. We can no longer construct Ξ^*, Eq. (41); or $W^\dagger$, Eq. (43); or the Euler-Lagrange solutions, Eq. (38); on the whole interval because $\mathbf{F}^{-1}$ diverges at each resonant point. We consider first the case of one resonant point $\psi_R \in [\psi_1, \psi_2]$ and assume that $\Xi_1 = 0$ because ψ_1 is either the axis or a conducting wall. In order to determine the minimizing test function, we divide the interval into two subintervals $[\psi_1, \psi_R]$ and $[\psi_R, \psi_2]$. The perturbation $\Xi(\psi_R) = \Xi_R$ may be bounded or it may have an unbounded contribution from the small resonant solution. The nonresonant terms must be continuous across the singular surface in order to avoid positive infinite contributions to δW, but are otherwise arbitrary. We compute the minimizing contributions to δW from each subinterval as a function of Ξ_R. We then perform a formal minimization with respect to Ξ_R and show that the minimizing solution is the one for which the nonresonant solutions have continuous derivatives as well as values. Since the contribution from the small resonant solution in the neighborhood of ψ_R vanishes, it remains arbitrary, to be determined by boundary conditions for $\psi > \psi_R$.

For simplicity, we first study the procedure for the case where ψ_R is an arbitrary reference point, an ordinary point, and then consider how it must be modified if ψ_R is a resonant point. We expect that for the ordinary point, the results should show that the solution is a single, continuous Euler-Lagrange solution, and this is indeed the case.

In the subinterval $[\psi_1, \psi_R]$, we initialize the fundamental solution matrix $\mathbf{U}(\psi)$ to the identity at ψ_1 if ψ_1 is a conductor; or to the asymptotic solutions in the neighborhood of the axis if $\psi_1 = 0$, with the large solutions in the left half and the small solutions in the right half. In either case, $\mathbf{c}_1 = 0$, $\mathbf{u}_1(\psi) = \mathbf{U}_{12}(\psi)\mathbf{c}_2$, and $\mathbf{u}_2(\psi) = \mathbf{U}_{22}(\psi)\mathbf{c}_2$. The boundary conditions at ψ_R then require that $\mathbf{u}_1(\psi_R^-) = \Xi_R$, $\mathbf{c}_2 = \mathbf{U}_{12}^{-1}(\psi_R^-)\Xi_R$, and $\mathbf{u}_2(\psi_R^-) = \mathbf{U}_{22}(\psi_R^-)\mathbf{U}_{12}^{-1}(\psi_R^-)\Xi_R$. To simplify notation, we define $\mathbf{S} \equiv \mathbf{U}(\psi_R^-)$; thus $\mathbf{u}_2(\psi_R^-) = \mathbf{S}_{22}\mathbf{S}_{12}^{-1}\Xi_R$.

In the subinterval $[\psi_R, \psi_2]$, we initialize the fundamental solution matrix $\mathbf{U}(\psi)$ to the identity at ψ_R. The general solution in this subinterval is $\mathbf{u}_1 = \mathbf{U}_{11}\mathbf{d}_1 + \mathbf{U}_{12}\mathbf{d}_2$, $\mathbf{u}_2 = \mathbf{U}_{21}\mathbf{d}_1 + \mathbf{U}_{22}\mathbf{d}_2$. The boundary condition at ψ_R^+ determines that $\mathbf{d}_2 = \Xi_R$, while that at ψ_2 determines that $\mathbf{d}_2 = \mathbf{U}_{12}^{-1}(\psi_2)(\Xi_2 - \mathbf{U}_{11}\Xi_R)$. Again to simplify notation, we define $\mathbf{T} \equiv \mathbf{U}(\psi_2)$. Then $\mathbf{u}_2(\psi_R^+) = \mathbf{T}_{12}^{-1}(\Xi_2 - \mathbf{T}_{11}\Xi_R)$ and $\mathbf{u}_2(\psi_2) = \mathbf{T}_{21}\Xi_R + \mathbf{T}_{22}\mathbf{T}_{12}^{-1}(\Xi_2 - \mathbf{T}_{11}\Xi_R)$.

Using the symplectic symmetry properties in Eqs. (20) and (22), we can express the total potential energy in $[\psi_1, \psi_2]$ as

$$\begin{aligned}
\delta W = \; & \frac{1}{2}\big[\mathbf{u}_1^\dagger(\psi_R^-)\mathbf{u}_2(\psi_R^-) + \mathbf{u}_2^\dagger(\psi_R^-)\mathbf{u}_1(\psi_R^-) - \mathbf{u}_1^\dagger(\psi_R^+)\mathbf{u}_2(\psi_R^+) - \mathbf{u}_2^\dagger(\psi_R^+)\mathbf{u}_1(\psi_R^+) \\
& + \mathbf{u}_1^\dagger(\psi_2)\mathbf{u}_2(\psi_2) + \mathbf{u}_2^\dagger(\psi_2)\mathbf{u}_1(\psi_2)\big] \\
= \; & \Xi_R^\dagger\mathbf{S}_{22}\mathbf{S}_{12}^{-1}\Xi_R - \frac{1}{2}\Xi_R^\dagger\mathbf{T}_{12}^{-1}(\Xi_2 - \mathbf{T}_{11}\Xi_R) - \frac{1}{2}(\Xi_2^\dagger - \Xi_R^\dagger\mathbf{T}_{11}^\dagger)\mathbf{T}_{12}^{-1\dagger}\Xi_R \\
& + \frac{1}{2}\Xi_2^\dagger[\mathbf{T}_{21}\Xi_R + \mathbf{T}_{22}\mathbf{T}_{12}^{-1}(\Xi_2 - \mathbf{T}_{11}\Xi_R)] + \frac{1}{2}[\Xi_R^\dagger\mathbf{T}_{21}^\dagger + (\Xi_2^\dagger - \Xi_R^\dagger\mathbf{T}_{11}^\dagger)\mathbf{T}_{12}^{-1\dagger}\mathbf{T}_{22}^\dagger]\Xi_2 \\
= \; & \Xi_R^\dagger(\mathbf{S}_{22}\mathbf{S}_{12}^{-1} + \mathbf{T}_{12}^{-1}\mathbf{T}_{11})\Xi_R + \Xi_2^\dagger\mathbf{T}_{22}\mathbf{T}_{12}^{-1}\Xi_2 \\
& - \frac{1}{2}\Xi_R^\dagger(\mathbf{T}_{12}^{-1} - \mathbf{T}_{21}^\dagger + \mathbf{T}_{11}^\dagger\mathbf{T}_{12}^{-1\dagger}\mathbf{T}_{22}^\dagger)\Xi_2 - \frac{1}{2}\Xi_2^\dagger(\mathbf{T}_{12}^{-1\dagger} - \mathbf{T}_{21} + \mathbf{T}_{22}\mathbf{T}_{12}^{-1}\mathbf{T}_{11})\Xi_R \\
= \; & \Xi_R^\dagger(\mathbf{S}_{22}\mathbf{S}_{12}^{-1} + \mathbf{T}_{12}^{-1}\mathbf{T}_{11})\Xi_R - \Xi_R^\dagger\mathbf{T}_{12}^{-1}\Xi_2 - \Xi_2^\dagger\mathbf{T}_{12}^{-1\dagger}\Xi_R + \Xi_2^\dagger\mathbf{T}_{22}\mathbf{T}_{12}^{-1}\Xi_2.
\end{aligned} \tag{50}$$

Minimizing this with respect to $\Xi_R^\dagger$, we obtain

$$(\mathbf{S}_{22}\mathbf{S}_{12}^{-1} + \mathbf{T}_{12}^{-1}\mathbf{T}_{11})\Xi_R = \mathbf{T}_{12}^{-1}\Xi_2. \tag{51}$$

This is just the condition that $\mathbf{u}_2(\psi_R^-) = \mathbf{u}_2(\psi_R^+)$, *i.e.* that the whole solution be continuous at Ξ_R. It is also the condition that $\mathbf{u}(\psi)$ be the same Euler-Lagrange solution on both sides of the reference point.

Substituting Eq. (51) into Eq. (49), we find

$$\begin{aligned}
\delta W = \; & \Xi_2^\dagger\left[\mathbf{T}_{22}\mathbf{T}_{12}^{-1}\right. \\
& \left. - \mathbf{T}_{12}^{-1\dagger}\left(\mathbf{S}_{22}\mathbf{S}_{12}^{-1} + \mathbf{T}_{12}^{-1}\mathbf{T}_{11}\right)^{-1}\mathbf{T}_{12}^{-1}\right]\Xi_2.
\end{aligned} \tag{52}$$

Again using the Eqs. (20) and (20), we obtain finally

$$\delta W = \Xi_2^\dagger\left(\mathbf{T}_{21}\mathbf{S}_{12} + \mathbf{T}_{22}\mathbf{S}_{22}\right)\left(\mathbf{T}_{11}\mathbf{S}_{12} + \mathbf{T}_{12}\mathbf{S}_{22}\right)^{-1}\Xi_2. \tag{53}$$

The matrix product in Eq. (53) is just the plasma response matrix obtained by multiplying the fundamental solution matrices to the left and right sides of ψ_R.

Now consider how this procedure may be adapted to the case where ψ_R is a resonant point. We use Eq. (49) for the behavior to the left of the resonant point. To the right, we initialize $\mathbf{U}(\psi)$ to $\mathbf{V}(\psi)$, the matrix of power series solutions to the right of the resonant point. On both sides we take the resonant contribution of Ξ_R to vanish. The small resonant component on either side makes no contribution to δW. Then the problem has the same form as for the nonresonant case except for the vanishing of the resonant component. There are $M - 1$ rather than M components of Ξ to determine by minimization of δW. The result of this constrained minization procedure is the same as for the nonresonant case, that the minimizing value of Ξ is the one that makes the derivatives of the nonresonant components continuous. This allows us to continue the minimization procedure into the next interval by analytically continuing the nonresonant components of $\mathbf{U}(\psi)$ and by restarting the small nonresonant component with an arbitrary initial value, to be determined by boundary conditions further to the right. The proof of the necessary condition for stability follows unmodified, using this continued matrix. Continuation past successive singular surfaces follows the same procedure.

An effective numerical procedure for implementing continuation past each singular surface is based on an adaptation of Gaussian elimination applied to the matrix of solutions. Each independent solution vector is dominated by its admixture of the large resonant solution on approach to a resonant surface. We choose one of these and subtract the multiple of it from each of the other solution vectors which eliminates its first resonant component. This leaves all but the first solution vector effectively free of the large resonant solution. The first solution vector is then reinitialized to the small resonant solution on the other side of the singular surface. Since the coefficient of this small solution is arbitrary until it is determined by boundary conditions further to the right, there is no loss of generality in eliminating the second resonant component from each of the other vectors. The nonresonant solution vectors are then continued unmodified on the other side of the singular surface.

The results of this section show that the equilibrium is stable to internal modes if and only if the critical determinant D_C has no poles. In that case, the potential energy in the plasma region is given by

$$\delta W_P = \Xi^\dagger \mathbf{W}_P \Xi, \tag{54}$$

evaluated at the plasma-vacuum interface. This justifies the name of the plasma response matrix. If the plasma is surrounded by a vacuum, there may still be external, or free-boundary, instabilities involving motion of the boundary. In that case, the potential energy in the vacuum region may be similarly expressed in terms of a vacuum response matrix evaluated on the same surface. The total potential energy is the sum of the two, and the existence of free boundary modes is determined by the lowest eigenvalue of the sum. This is the topic of the next section.

V. FREE-BOUNDARY MODES

Section IV describes a procedure for determining the stability of the plasma to internal modes. If the critical determinant D_C has poles in the plasma region, then the potential energy δW is unbounded below and can therefore always be made negative. If D_C has no poles in the plasma region, then δW is bounded below by the Hilbert invariant integral, or equivalently by the value of δW for the Euler-Lagrange solution constructed in Section IV. If the plasma is bounded by a conducting wall, this lower bound is zero and the plasma is therefore stable. This is the only case treated by Newcomb, extended to include a perfectly-conducting pressureless plasma but not a vacuum region outside the plasma.

The purpose of this section is to extend the analysis to include free-boundary modes, in which the plasma region is bounded by a vacuum region rather than a conducting wall. The plasma contribution to δW is given by Eq. (54) in terms of the plasma response matrix. In this section we compute a vacuum response matrix. The total perturbed potential energy is the sum of the two. If a solution exists, satisfying the boundary conditions, for which the plasma energy is negative and exceeds the positive-definite vacuum energy, then δW can be made negative and there is an unstable free-boundary mode. Much of this section is based on an extension of the work of Chance.[19]

If the plasma region is bounded by a separatrix, then the vacuum region contains open field lines and cannot be treated by the methods of the previous sections, which rely on Hamada coordinates defined only on closed field lines. We therefore require a treatment of the vacuum region which does not rely on such coordinates outside the plasma region. However, we make use of Hamada coordinates at the plasma-vacuum interface. If this interface is a separatrix, we take the boundary to be just inside the separatrix.

We begin with a brief summary of the Green's function formalism. The perturbed magnetic field $\mathbf{b}$ in the vacuum region, where $\nabla \times \mathbf{b} = 0$, is expressed in terms of a scalar magnetic potential, $\mathbf{b} = -\nabla\varphi$. $\nabla \cdot \mathbf{b} = 0$ then implies that $\varphi(\mathbf{x})$ satisfies Laplace's equation, $\nabla^2\varphi = 0$.

The scalar magnetic potential must in general be allowed to be non-single-valued. This follows from the integral form of Ampére's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = -\int_0^L \frac{\partial\varphi}{\partial l} dl = -\varphi\big|_0^L = \mu_0 I$$

for any closed loop of length L, where I is the current passing throught the closed loop. For example, if the loop encircles the plasma the short way, then I is the toroidal current through the plasma. However, we use the magnetic scalar potential only for the perturbed magnetic field $\mathbf{b}$ with toroidal mode number $n \neq 0$, and for this case, there is no contribution from the total current.[1] Thus, those perturbed scalar potential φ is single-valued.

The Green's function method is used to derive an integral equation for a solution $\varphi(\mathbf{x})$ satisfying Laplace's equation and the boundary conditions. The derivation is most easily done by starting with the more general Poisson equation,

$$\nabla^2\varphi(\mathbf{x}) = -4\pi\rho(\mathbf{x})$$

and then setting the source term $\rho(\mathbf{x})$ to zero. The Green's function is defined as the solution to Poisson's equation with a delta-function source,

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = \nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'), \tag{55}$$

which vanishes at infinity. We define the integral

$$\begin{aligned} \mathcal{I} &\equiv \int_V d\mathbf{x}' G(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}') \\ &= -\frac{1}{4\pi}\int_V d\mathbf{x}' G(\mathbf{x}, \mathbf{x}')\nabla'^2\varphi(\mathbf{x}'), \end{aligned} \tag{56}$$

where V is the volume of the vacuum region. We now perform two integrations by parts to transfer the derivates from φ to G, obtaining

$$\begin{aligned}\mathcal{I} = \quad & -\frac{1}{4\pi}\int_V d\mathbf{x}'\nabla'^2 G(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}') \\ & +\frac{1}{4\pi}\int_S d\mathbf{x}'\,[\hat{\mathbf{n}}\cdot\nabla' G(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}') \\ & -G(\mathbf{x},\mathbf{x}')\hat{\mathbf{n}}\cdot\nabla'\varphi(\mathbf{x}')] \end{aligned} \tag{57}$$

where S is the boundary of the vacuum region and $\mathbf{n}$ is the unit outward normal to the vacuum region. Now we use Eq. (55) to obtain

$$\begin{aligned} & -\frac{1}{4\pi}\int_V d\mathbf{x}'\nabla'^2 G(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}') \\ & = \int_V d\mathbf{x}'\delta(\mathbf{x}-\mathbf{x}')\varphi(\mathbf{x}') = \Omega\varphi(\mathbf{x}), \end{aligned} \tag{58}$$

where $\Omega = 1$ if $\mathbf{x}$ is in the vacuum region V, $\Omega = 1/2$ if $\mathbf{x}$ is on the boundary S of the vacuum region, and $\Omega = 0$ if $\mathbf{x}$ is outside the vacuum region. Finally, we let $\mathbf{x}$ be on the boundary, let $\rho(\mathbf{x}') = 0$, and obtain an integral equation for φ on the boundary,

$$\begin{aligned}\mathcal{I} = \varphi(\mathbf{x}) + \frac{1}{2\pi}\int_S d\mathbf{x}'\,[\hat{\mathbf{n}}\cdot\nabla' G(\mathbf{x},\mathbf{x}')\varphi(\mathbf{x}') \\ -G(\mathbf{x},\mathbf{x}')\hat{\mathbf{n}}\cdot\nabla'\varphi(\mathbf{x}')] = 0. \end{aligned} \tag{59}$$

Since φ is the scalar magnetic potential, $\hat{\mathbf{n}}\cdot\nabla\varphi$ is the normal magnetic field and is known from boundary conditions. Thus, Eq. (59) may be regarded as an equation for φ, with the normal derivative as a known inhomogeneity. Since the vacuum energy is the integral of the perturbed magnetic energy density over the vacuum region, which can also be expressed as an integral over S,

$$\delta W_V = \int_V d\mathbf{x}|\mathbf{b}|^2 = \int_V d\mathbf{x}|\nabla\varphi|^2 = \int_S d\mathbf{x}\varphi\hat{\mathbf{n}}\cdot\nabla\varphi, \tag{60}$$

the whole vacuum problem can be expressed in terms of φ and its normal derivative on S, with no further reference to φ inside the vacuum region.

The surface S bounding the vacuum region may consist of various noncontiguous segments, including the vacuum-plasma interface, a conducting wall or the surface at infinity, and various internal metallic conductors such as passive stabilizers and active coils. The variables $\mathbf{x}$ and $\mathbf{x}'$ vary over all of these surfaces. Since the normal magnetic field vanishes at the surface of a perfect conductor, the inhomogeneity vanishes there, as it does also at infinity. Likewise, the surface integral for δW_V is nonvanishing only over the plasma-vacuum interface. The role of the external conductors is to modify the homogeneous terms in Eq. (59). In the following discussion, we consider only the simplest case, with no external conductors and the vacuum region extending to infinity. Extension of the treatment to include external conductors is straightforward.

At the plasma-vacuum interface, the normal magnetic field is known in terms of its Fourier components in Hamada coordinates. We wish to express the integral equation and the vacuum energy, Eqs. (59) and (60), in terms of these Fourier components. We therefore write

$$\varphi(\mathbf{x}) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\varphi_{m,n}(V)\exp[2\pi i(m\theta - n\zeta)].$$

As for the displacement vector in Eq. (9), we define the complex vector

$$\Phi(V) \equiv \{\varphi_{m,n}(V),\ m_{min} \le m \le m_{max}\}$$

The surface integral can be expressed in Hamada coordinates as

$$\int_S d\mathbf{x}' = \int_0^1 d\theta\int_0^1 d\zeta|\nabla V|.$$

The boundary condition at the plasma-vacuum interface is that the normal components of the magnetic field $\mathbf{Q}$ in the plasma and the magnetic field $\mathbf{b}$ in the vacuum must match. Noting that the outward normal to the vacuum is inward to the plasma, we can write $\hat{\mathbf{n}} = -\nabla V/|\nabla V|$. Then the boundary condition is given by $\mathbf{b}\cdot\nabla V = |\nabla V|\hat{\mathbf{n}}\cdot\nabla\varphi = \mathbf{Q}\cdot\nabla V = \mathbf{B}\cdot\nabla\xi_V$.

We now multiply the integral equation in Eq. (59) by $\exp[-2\pi i(m\theta - n\zeta)]$ and integrate over θ and ζ to convert it to a matrix equation,

$$(\mathsf{I}+\mathsf{L})\Phi = 2\pi i\chi'\mathsf{RQ}\Xi \tag{61}$$

where the diagonal matrix Q is defined in Eq. (A2) and

$$\begin{aligned} L_{m,m'} &= -\frac{1}{2\pi}\int_0^1 d\theta\int_0^1 d\theta'\int_0^1 d\zeta\int_0^1 d\zeta'\exp[2\pi i(m'\theta' - m\theta)]\nabla' V\cdot\nabla' G(V,\theta,\zeta;V,\theta',\zeta')\exp[2\pi ni(\zeta-\zeta')], \\ R_{m,m'} &= -\frac{1}{2\pi}\int_0^1 d\theta\int_0^1 d\theta'\int_0^1 d\zeta\int_0^1 d\zeta'\exp[2\pi i(m'\theta' - m\theta)]G(V,\theta,\zeta;V,\theta',\zeta')\exp[2\pi ni(\zeta-\zeta')]. \end{aligned} \tag{62}$$

The toroidal straight-fieldline coordinate can be expressed as

$$\zeta = \frac{\phi}{2\pi} + \nu(\theta, V),$$

where $\nu(\theta, V)$ is a periodic function of θ. Using this, the ζ integrals over the Green's function G can be expressed in terms of associated Legendre functions. The solution to Eq. (55) is

$$G(\mathbf{x},\mathbf{x}') = |\mathbf{x}-\mathbf{x}'|^{-1} = \left[R^2 + R'^2 + (Z-Z')^2 - 2RR'\cos\psi\right]^{-1/2} \quad (63)$$

where (R, Z, ϕ) are cylindrical coordinates and $\psi = \phi - \phi'$. We define

$$\mathcal{G}^n(\mathbf{x},\mathbf{x}') \equiv \frac{1}{2\pi}\int_0^{2\pi} d\psi e^{-in\psi} G(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\int_0^{2\pi} d\psi \cos n\psi \times \left[R^2 + R'^2 + (Z-Z')^2 - 2RR'\cos\psi\right]^{-1/2}. \quad (64)$$

Defining the quantities α and η by

$$\begin{aligned} \alpha^2\cosh\eta &= R^2 + R'^2 + (Z-Z')^2, \\ \alpha^2\sinh\eta &= -2RR', \\ \alpha^4 &= \left[R^2 + R'^2 + (Z-Z')^2\right]^2 - (2RR')^2 \\ &= \left[(R-R')^2 + (Z-Z')^2\right] \\ &\quad \times \left[(R+R')^2 + (Z-Z')^2\right], \end{aligned} \quad (65)$$

we can express the Green's function as[22]

$$\begin{aligned} \mathcal{G}^n(\mathbf{x},\mathbf{x}') &= \frac{1}{2\pi\alpha}\int_0^{2\pi} d\psi \cos n\psi \\ &\quad \left[\cosh\eta + \sinh\eta\cos\psi\right]^{-1/2} \\ &= \frac{(-1)^n}{\alpha\sqrt{\pi}}\Gamma(1/2-n)P^n_{-1/2}(\cosh\eta) \end{aligned} \quad (66)$$

where $P^n_{-1/2}(\cosh\eta)$ is an associated Legendre function. Finally, this allows us to simplify Eq. (62) to

$$\begin{aligned} L_{m,m'} &= -\frac{1}{2\pi}\int_0^1 d\theta \int_0^1 d\theta' \exp[2\pi i(m'\theta' - m\theta)] \exp\{2\pi n i[\nu(\theta) - \nu(\theta')]\} \nabla' V \cdot \nabla' \mathcal{G}^n(V,\theta;V,\theta'), \\ R_{m,m'} &= -\frac{1}{2\pi}\int_0^1 d\theta \int_0^1 d\theta' \exp[2\pi i(m'\theta' - m\theta)] \exp\{2\pi n i[\nu(\theta) - \nu(\theta')]\} \mathcal{G}^n(V,\theta;V,\theta'). \end{aligned}$$

The remaining integrals over θ and θ' depend on the shape of the outermost flux surface and must be performed numerically.

Once $\mathbf{L}$ and $\mathbf{R}$ are evaluated, Eq. (61) can be solved for Φ and used to evaluate δW_V, Eq. (60), as

$$\begin{aligned} \delta W_V &= -\int_0^1 d\theta \int_0^1 d\zeta \varphi(V,\theta,\zeta)\nabla V \cdot \nabla\varphi(V,\theta,\zeta) \\ &= 2\pi i\chi' \sum_{m=-\infty}^{\infty}(m-nq) \\ &\quad \left[\varphi^*_{m,n}(V)\xi_{m,n}(V) - \xi^*_{m,n}(V)\varphi_{m,n}(V)\right] \\ &= 2\pi i\chi'(\Phi^\dagger\mathbf{Q}\Xi - \Xi^\dagger\mathbf{Q}\Phi). \end{aligned} \quad (67)$$

Finally, using Eq. (61), we can express this as $\delta W_V = \Xi^\dagger \mathbf{W}_V \Xi$, with

$$\mathbf{W}_V = (2\pi\chi')^2\mathbf{Q}\left[(\mathbf{I}+\mathbf{L})^{-1}\mathbf{R} + \mathbf{R}^\dagger(\mathbf{I}+\mathbf{L}^\dagger)^{-1}\right]\mathbf{Q}.$$

The total potential energy is given by

$$\delta W = \delta W_P + \delta W_V = \Xi^\dagger(\mathbf{W}_P + \mathbf{W}_V)\Xi = \Xi^\dagger\mathbf{W}\Xi.$$

This is expressed in terms of a particular solution vector Ξ. We must still determine whether any linear combination of the solutions for Ξ satisfying the boundary conditions at the magnetic axis can make δW negative. This is the case if and only if the lowest eigenvalue of the total response matrix $\mathbf{W}$ is negative. The eigenvectors corresponding to the unstable eigenvalues represent the spatial structure of the unstable modes. This completes the determination of stability.

VI. DISCUSSION AND CONCLUSIONS

We have introduced a new method for determining the ideal MHD stability of axisymmetric toroidal plasmas to nonaxisymmetric perturbations. We have generalized the Direct Criterion of Newcomb for the stability of internal modes from his original cylindrical treatment to a full toroidal treatment. We have further extended the treatment to apply to external, or free-boundary modes.

The overall procedure involves a sequence of criteria. First we test for Mercier stability, $D_I < 0$, Eq. (35), on all flux surfaces. If this criterion is violated, the equilibrium is unstable. If it is satisfied, we then test for stability to perturbations of each toroidal mode number $n \neq 0$. We integrate the Euler-Lagrange Equations from the axis to the plasma-vacuum interface, initializing it at the axis and crossing each resonant surface according to the procedures discussed in Section IV. We monitor the critical determinant D_C, Eq. (42). If it has a pole in the plasma region, there is an ideal internal instability, regardless of conditions at or beyond the plasma-vacuum interface. If there are no poles, we compute the plasma and vacuum contributions to the potential energy matrix described in Section V and determine whether it has any negative eigenvalues. If so, there are external, or free-boundary, ideal instabilities. Otherwise, the system is stable to all ideal MHD perturbations.

If the system is stable to ideal modes, there may still be singular modes, involving more subtle effects such as plasma resistivity and diamagnetic rotation.[18,20] The procedure presented here can be further extended to provide information about the behavior of the perturbations in the ideal regions. The asymptotic coefficients defined in Eq. (36) provide this information, generalizing the quantity Δ' used in the cylindrical case, which must be matched to the corresponding inner region information to determine the stablity to singular modes. This will be the subject of a later publication.

It would be difficult to attempt a further generalization of this approach to nonaxisymmetric toroidal systems such as stellarators and torsatrons, as attempted by Bineau.[8] For an axisymmetric system, in which the toroidal mode number is a good quantum number, the resonant surfaces at $q = m/n$ are discrete and well-separated, except in unusual cases in which the axis is also a resonant surface or q' vanishes at a rational surface. In a nonaxisymmetric system, since there are in principle infinitely many toroidal and poloidal harmonics, the singular surfaces would be dense everywhere, and this would appear to make the procedure impractical. Perhaps it could be shown that convergence is achieved for a moderate number of toroidal harmonics. Even then, the situation would be further complicated by the existence of stochastic regions, especially in the neighborhood of low-order resonant surfaces, as occur in most nonaxisymmetric configurations, where flux surfaces, and hence flux coordinates, fail to exist.

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APPENDIX A: COEFFICIENT MATRICES

In this appendix we give detailed expressions for the coefficient matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{K}$.

Before expressing these principal matrices, we define some more fundamental matrices in which are used in their definition. We define the following scalar components of the metric tensor:

$$\begin{aligned}
g_{11} &\equiv |\nabla\theta \times \nabla\zeta|^2, \\
g_{22} &\equiv |\nabla\zeta \times \nabla\psi|^2, \\
g_{33} &\equiv |\nabla\psi \times \nabla\theta|^2, \\
g_{23} &\equiv (\nabla\zeta \times \nabla\psi) \cdot (\nabla\psi \times \nabla\theta), \\
g_{31} &\equiv (\nabla\psi \times \nabla\theta) \cdot (\nabla\theta \times \nabla\zeta), \\
g_{12} &\equiv (\nabla\theta \times \nabla\zeta) \cdot (\nabla\zeta \times \nabla\psi).
\end{aligned} \tag{A1}$$

For any scalar $f(\theta)$, we define an associated matrix $< f >$ as the matrix of Fourier coefficients,

$$< \mathcal{F} >_{mm'} \equiv \int_0^1 d\theta \mathcal{J} \exp[2\pi i(m' - m)\theta]\mathcal{F}(\theta)$$

Note that if f is real, then $< f >$ is Hermitian. The following quantities are required when the Jacobian is not uniform:

$$J_{mm'} \equiv < 1 >_{mm'}, \qquad J'_{mm'} \equiv < \mathcal{J}'/\mathcal{J} >_{mm'}$$

Next we define the matrices

$$(G_{ij})_{mm'} \equiv < g_{ij} >_{mm'}$$

Finally, we define the diagonal matrices $\mathbf{M}$ and $\mathbf{Q}$ in terms of their diagonal components,

$$M_{m,m'} \equiv m\delta_{m,m'}, \qquad Q_{m,m'} \equiv (m - nq)\delta_{m,m'}. \tag{A2}$$

In terms of these fundamental matrices, the coefficient matrices appearing in Eq. (10) are given by

$$
\begin{aligned}
\mathbf{A} =\ & (2\pi)^2 \left[n(n\mathbf{G}_{22} + \mathbf{G}_{23}\mathbf{M} + \mathbf{M}(n\mathbf{G}_{23} + \mathbf{G}_{33}\mathbf{M})\right] \\
\mathbf{B} =\ & -2\pi i\chi' \left[n(\mathbf{G}_{22} + q\mathbf{G}_{23}) + \mathbf{M}(\mathbf{G}_{23} + q\mathbf{G}_{33})\right] \\
\mathbf{C} =\ & -2\pi i \left[\chi''(n\mathbf{G}_{22} + \mathbf{M}\mathbf{G}_{23}) + (q\chi')'(n\mathbf{G}_{23} + \mathbf{M}\mathbf{G}_{33})\right] \\
& -(2\pi)^2\chi'(n\mathbf{G}_{12} + \mathbf{M}\mathbf{G}_{31})\mathbf{Q} + 2\pi i(2\pi f'\mathbf{Q} - nP'/\chi'\mathbf{J}) \\
\mathbf{D} =\ & \chi'^2 \left[(\mathbf{G}_{22} + q\mathbf{G}_{23}) + q(\mathbf{G}_{23} + q\mathbf{G}_{33})\right] \\
\mathbf{E} =\ & \chi' \left[\chi''(\mathbf{G}_{22} + q\mathbf{G}_{23}) + (q\chi')'(\mathbf{G}_{23} + q\mathbf{G}_{33})\right] \\
& -2\pi i\chi'^2(\mathbf{G}_{12} + q\mathbf{G}_{31})\mathbf{Q} + P'\mathbf{J} \\
\mathbf{H} =\ & \chi'' \left[\chi''\mathbf{G}_{22} + (q\chi')'\mathbf{G}_{23}\right] + (q\chi')' \left[\chi''\mathbf{G}_{23} + (q\chi')'\mathbf{G}_{33}\right] \\
& +2\pi i\chi' \left[\chi''(\mathbf{M}\mathbf{G}_{12} - \mathbf{G}_{12}\mathbf{M}) + (q\chi')'(\mathbf{M}\mathbf{G}_{31} - \mathbf{G}_{31}\mathbf{M})\right] \\
& +(2\pi\chi')^2\mathbf{Q}\mathbf{G}_{11}\mathbf{Q} + \left[P'\left(\mathbf{J}\chi''/\chi' + \mathbf{J}'\right) - 2\pi f'q'\chi'\mathbf{I}\right].
\end{aligned}
\tag{A3}
$$

Note that all of the off-diagonal components of these matrices arise from the metric tensor quantities $\mathbf{G}_{ij}$, which in turn arise from the stabilizing Q^2 term in Eq. (5), and thus all potentially destabilizing terms in δW are diagonal. This is a useful property of Hamada coordinates. Note also that the matrices $\mathbf{A}$, $\mathbf{D}$, and $\mathbf{H}$, which couple like quantities in Eq. (10), are Hermitian, while $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{E}$, which couple unlike quantities, are not.

Two of the composite matrices defined in Eq. (13) contain important cancellations. Thus, we can express

$$
\begin{aligned}
\mathbf{F} \equiv\ & \mathbf{D} - \mathbf{B}^\dagger\mathbf{A}^{-1}\mathbf{B} \\
=\ & (\chi'/n)^2\mathbf{Q}\left[\mathbf{G}_{33} - (n\mathbf{G}_{23} + \mathbf{G}_{33}\mathbf{M})\right. \\
& \left.\times(2\pi)^2\mathbf{A}^{-1}(n\mathbf{G}_{23} + \mathbf{M}\mathbf{G}_{33})\right]\mathbf{Q}
\end{aligned}
\tag{A4}
$$

Note the presence of the factor $\mathbf{Q}$ on either side of $\mathbf{F}$. This is responsible for the vanishing of $\det\mathbf{F}$ at the resonant surfaces, discussed in detail in Section III. Similarly, we can express

$$
\begin{aligned}
\mathbf{K} =\ & \mathbf{E} - \mathbf{B}^\dagger\mathbf{A}^{-1}\mathbf{C} \\
=\ & \mathbf{Q}(\chi'/n)\left\{2\pi i(n\mathbf{G}_{23} + \mathbf{G}_{33}\mathbf{M})\mathbf{A}^{-1}\mathbf{C}\right. \\
& \left.-[\chi''\mathbf{G}_{23} + (q\chi')'\mathbf{G}_{33} - 2\pi i\chi'\mathbf{G}_{31}\mathbf{Q} + J_\theta\mathbf{I}]\right\}
\end{aligned}
\tag{A5}
$$

No such simplification has been found for the remaining matrix $\mathbf{G}$ defined in Eq. (13).

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