Straight Fieldline Coordinate System

$$\mathcal{J} \equiv (\nabla\psi \cdot \nabla\theta \times \nabla\zeta)^{-1}, \quad \mathcal{F}'(\psi,\theta,\zeta) \equiv \frac{\partial \mathcal{F}}{\partial \psi} \tag{1}$$

$$\mathcal{F}(\psi,\theta,\zeta) = \mathcal{F}(\psi,\theta+1,\zeta) = \mathcal{F}(\psi,\theta,\zeta+1) \tag{2}$$

$$|\nabla\phi|^2 = \frac{1}{R^2}, \quad \nabla\phi\cdot\nabla\psi = \nabla\theta\cdot\nabla\phi = 0 \tag{3}$$

$$\phi(\psi,\theta,\zeta) = 2\pi\zeta + \nu(\psi,\theta), \quad \nu(\psi,\theta) = \nu(\psi,\theta+1) \tag{4}$$

Axisymmetric Equilibrium Magnetic Field

$$\mathbf{B} \equiv f\nabla\phi + \frac{\chi'}{2\pi}\nabla\phi\times\nabla\psi = \chi'\left[\nabla\zeta - q(\psi)\nabla\theta\right]\times\nabla\psi \tag{5}$$

$$B_\psi \equiv \mathbf{B}\cdot\nabla\psi = 0, \quad B_\theta \equiv \mathbf{B}\cdot\nabla\theta = \frac{\chi'}{\mathcal{J}}, \quad B_\zeta \equiv \mathbf{B}\cdot\nabla\zeta = q\frac{\chi'}{\mathcal{J}} \tag{6}$$

$$q(\psi) \equiv \frac{\mathbf{B}\cdot\nabla\zeta}{\mathbf{B}\cdot\nabla\theta} = \frac{f}{2\pi\chi'}\int_0^1 d\theta\ \frac{\mathcal{J}}{R^2}, \quad \nu(\psi,\theta) = \frac{f}{2\pi\chi'}\int d\theta\frac{\mathcal{J}}{R^2} - 2\pi q\theta \tag{7}$$

$$\mathbf{B}\cdot\nabla\mathcal{F} = \mathbf{B}\cdot\nabla\theta\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\zeta}\right)\mathcal{F}(\psi,\theta,\zeta) \tag{8}$$

Equilibrium Current and Pressure

$$\mathbf{J} = \nabla\times\mathbf{B} \tag{9}$$

$$J_\psi = \frac{2\pi}{\mathcal{J}}\frac{\partial f}{\partial\theta} = 0, \quad J_\theta = -\frac{2\pi}{\mathcal{J}}\frac{\partial f}{\partial\psi}, \quad J_\phi = \frac{1}{2\pi}\nabla\cdot\left(\frac{1}{R^2}\nabla\chi\right) \tag{10}$$

$$\mathbf{J}\times\mathbf{B} = \nabla P, \quad J_\zeta = qJ_\theta - \frac{P'}{\chi'} = -\frac{2\pi q f'}{\mathcal{J}} - \frac{P'}{\chi'} \tag{11}$$

$$\Delta^*\chi \equiv R^2\nabla\cdot\left(\frac{1}{R^2}\nabla\chi\right) = -\frac{4\pi^2}{\chi'}(ff' + R^2P') \tag{12}$$

Velocity and Vector Potential

$$\mathbf{v} = \mathcal{J}\left(\dot{\psi}\nabla\theta \times \nabla\zeta + \dot{\theta}\nabla\zeta \times \nabla\psi + \dot{\zeta}\nabla\psi \times \nabla\theta\right) \tag{13}$$

$$\mathbf{A} = \Phi\nabla\theta - \chi\nabla\zeta, \quad \Phi' = q\chi' \tag{14}$$

Metric Tensor

$$\begin{aligned} g_{11} &= |\nabla\theta \times \nabla\zeta|^2 & g_{23} = g_{32} &= (\nabla\zeta \times \nabla\psi)\cdot(\nabla\psi \times \nabla\theta) \\ g_{22} &= |\nabla\zeta \times \nabla\psi|^2 & g_{31} = g_{13} &= (\nabla\psi \times \nabla\theta)\cdot(\nabla\theta \times \nabla\zeta) \\ g_{33} &= |\nabla\psi \times \nabla\theta|^2 & g_{12} = g_{21} &= (\nabla\theta \times \nabla\zeta)\cdot(\nabla\zeta \times \nabla\psi) \end{aligned} \tag{15}$$

$$g_{ij}g^{jk} = \delta_i^k \tag{16}$$

Lagrangian

$$\begin{aligned} L =& \frac{1}{2}mv^2 + e(\mathbf{A}\cdot\mathbf{v} - \varphi) \\ =& \frac{m\mathcal{J}^2}{2}\left(g_{11}\dot{\psi}^2 + g_{22}\dot{\theta}^2 + g_{33}\dot{\zeta}^2 + 2g_{12}\dot{\psi}\dot{\theta} + 2g_{23}\dot{\theta}\dot{\zeta} + 2g_{31}\dot{\zeta}\dot{\psi}\right) \\ &+ e\left(\Phi\dot{\theta} - \chi\dot{\zeta} - \varphi\right) \end{aligned} \tag{17}$$

Canonical Momenta

$$\begin{aligned} p_\psi \equiv& \frac{\partial L}{\partial\dot{\psi}} = m\mathcal{J}^2\left(g_{11}\dot{\psi} + g_{12}\dot{\theta} + g_{13}\dot{\zeta}\right) \\ p_\theta \equiv& \frac{\partial L}{\partial\dot{\theta}} = m\mathcal{J}^2\left(g_{21}\dot{\psi} + g_{22}\dot{\theta} + g_{23}\dot{\zeta}\right) + e\Phi \\ p_\zeta \equiv& \frac{\partial L}{\partial\dot{\zeta}} = m\mathcal{J}^2\left(g_{31}\dot{\psi} + g_{32}\dot{\theta} + g_{33}\dot{\zeta}\right) - e\chi \end{aligned} \tag{18}$$

Hamiltonian

$$
\begin{aligned}
H \equiv& \dot{\psi} p_\psi + \dot{\theta} p_\theta + \dot{\zeta} p_\zeta - L \\
=& \frac{m\mathcal{J}^2}{2} \left(g_{11}\dot{\psi}^2 + g_{22}\dot{\theta}^2 + g_{33}\dot{\zeta}^2 + 2g_{12}\dot{\psi}\dot{\theta} + 2g_{23}\dot{\theta}\dot{\zeta} + 2g_{31}\dot{\zeta}\dot{\psi} \right) + e\varphi \\
=& \frac{1}{2m\mathcal{J}^2} \left\{ g^{11}p_\psi^2 + g^{22}(p_\theta + e\Phi)^2 + g^{33}(p_\zeta - e\chi)^2 \right. \\
& \quad + 2p_\psi \left[g^{12}(p_\theta - e\Phi) + g^{31}(p_\zeta + e\chi) \right] \\
& \quad \left. + 2g^{23}(p_\theta - e\Phi)(p_\zeta + e\chi) \right\} + e\varphi
\end{aligned}
\tag{19}
$$

Hamilton's Equations

$$
\begin{aligned}
\dot{\psi} =& \frac{\partial H}{\partial p_\psi} = \frac{1}{m\mathcal{J}^2} \left[g^{11}p_\psi + g^{12}\left(p_\theta - e\Phi\right) + g^{13}\left(p_\zeta + e\chi\right) \right] \\
\dot{\theta} =& \frac{\partial H}{\partial p_\theta} = \frac{1}{m\mathcal{J}^2} \left[g^{21}p_\psi + g^{22}\left(p_\theta - e\Phi\right) + g^{23}\left(p_\zeta + e\chi\right) \right] \\
\dot{\zeta} =& \frac{\partial H}{\partial p_\zeta} = \frac{1}{m\mathcal{J}^2} \left[g^{31}p_\psi + g^{32}\left(p_\theta - e\Phi\right) + g^{33}\left(p_\zeta + e\chi\right) \right]
\end{aligned}
\tag{20}
$$

$$
\dot{p}_\psi = -\frac{\partial H}{\partial \psi}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}, \quad \dot{p}_\zeta = -\frac{\partial H}{\partial \zeta}
\tag{21}
$$

Matrix Formulation

$$\mathbf{x} \equiv \begin{pmatrix} \psi \\ \theta \\ \zeta \end{pmatrix}, \quad \dot{\mathbf{x}} \equiv \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\zeta} \end{pmatrix}, \quad \mathbf{v} \equiv \begin{pmatrix} \dot{R} \\ \dot{Z} \\ R\dot{\phi} \end{pmatrix} \tag{22}$$

$$\mathbf{p} \equiv \begin{pmatrix} p_\psi \\ p_\theta \\ p_\zeta \end{pmatrix}, \quad \mathbf{A} \equiv \begin{pmatrix} 0 \\ \Phi \\ -\chi \end{pmatrix} \tag{23}$$

$$\mathsf{J} \equiv \begin{pmatrix} \partial R/\partial\psi & \partial R/\partial\theta & 0 \\ \partial Z/\partial\psi & \partial Z/\partial\theta & 0 \\ R\partial\phi/\partial\psi & R\partial\phi/\partial\theta & 2\pi R \end{pmatrix}, \quad \mathsf{g} = \mathsf{g}^T = \frac{\mathsf{J}^T\mathsf{J}}{\mathcal{J}^2}, \quad \mathsf{G} \equiv \mathsf{g}^{-1} \tag{24}$$

$$\mathbf{v} = \mathsf{J}\dot{\mathbf{x}}, \quad \mathbf{p} - e\mathbf{A} = m\mathcal{J}^2\mathsf{g}\dot{\mathbf{x}} = m\mathsf{J}^T\mathsf{J}\dot{\mathbf{x}} = m\mathsf{J}^T\mathbf{v} \tag{25}$$

$$\begin{aligned} H =& \frac{mv^2}{2} \\ =& \frac{1}{2m\mathcal{J}^2}(\mathbf{p} - e\mathbf{A})^T\mathsf{G}(\mathbf{p} - e\mathbf{A}) \\ =& \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^T\mathsf{J}^{-1}\mathsf{J}^{-1T}(\mathbf{p} - e\mathbf{A}) \end{aligned} \tag{26}$$

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{1}{m\mathcal{J}^2}\mathsf{G}(\mathbf{p} - e\mathbf{A}) = \frac{1}{m}\mathsf{J}^{-1}\mathsf{J}^{-1T}(\mathbf{p} - e\mathbf{A}) = \mathsf{J}^{-1}\mathbf{v} \tag{27}$$

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{28}$$