Question 1

a)

(i)

The z-transform of a discrete-time signal $x[n]$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \tag{1}$$

Where z is a complex variable. Set $z = e^{j\omega}$ and substitute into Eq. 1, yields

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= \cdots h[-2]e^{j2\omega} + h[-1]e^{j\omega} + h[0]e^{j\omega 0} + h[1]e^{-j\omega} + h[2]e^{-j2\omega} \cdots \end{aligned}$$

Apply time shift property to find out the time domain representation

$$h[n] = \cdots h[-2]\delta[n+2] + h[-1]\delta[n+1] + h[0]\delta[n] + h[1]\delta[n-1] + h[2]\delta[n-2] \cdots$$

Now, if we set the complex variable $z_1 = z^D = e^{j\omega D}$. Then

$$\begin{aligned} H(e^{j\omega D}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n D} \\ &= \cdots h[-2]e^{j2\omega D} + h[-1]e^{j\omega D} + h[0]e^{j\omega 0D} + h[1]e^{-j\omega D} + h[2]e^{-j2\omega D} \cdots \end{aligned}$$

Convert it into time domain,

$$h[n] = \cdots h[-2]\delta[n+2D] + h[-1]\delta[n+D] + h[0]\delta[n] + h[1]\delta[n-D] + h[2]\delta[n-2D] \cdots$$

Obviously, the time domain signal has horizontally stretched out D times.

(ii)

Comparing $H(e^{j\omega})$ and $H(e^{j\omega D})$, it is clear that the x-axis variable ω becomes $D\omega$, which means the spectrum in frequency domain has shrunk by D times.

b)

Prior to answering (i), (ii) and (iii), we firstly sketch the magnitude response of $H(z)$ and $H(-z)$.

$$H(-z) = H(-e^{j\omega}) = H((-1)e^{j\omega}) = H(e^{j\pi}e^{j\omega}) = H(e^{j(\omega+\pi)}) \tag{2}$$

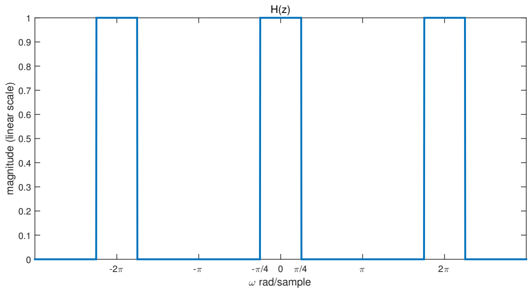

Figure 1: Magnitude response of $H(z)$

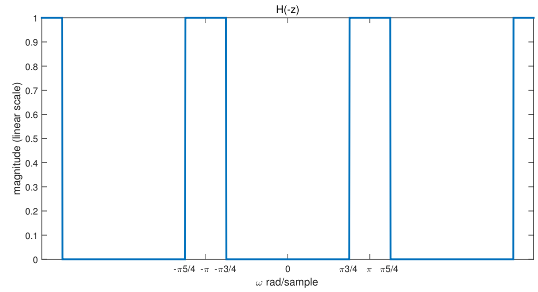

Figure 2: Magnitude response of $H(-z)$

(i)

Using the conclusion from Question 1(i), when $z \to z^3$, the spectrum will shrink by a factor of 3.

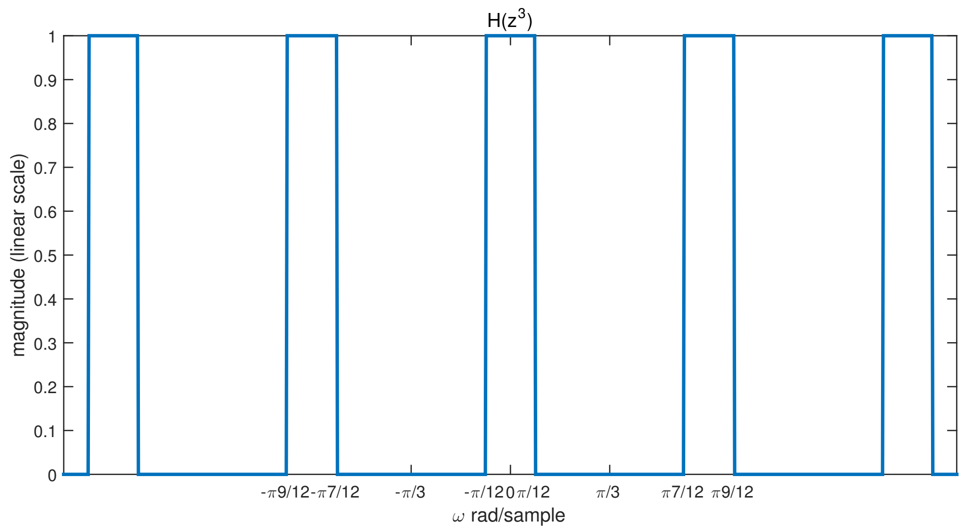

Figure 3: Magnitude response of $H(z^3)$

(ii)

Based on magnitude responses of $H(z)$ (Figure. 1) and $H(z^3)$ (Figure. 3), the product of these two can be depicted.

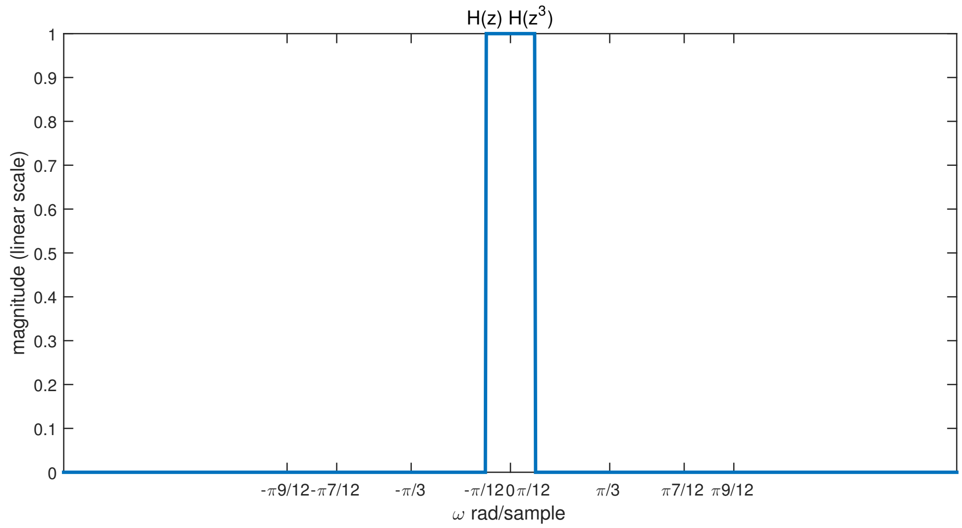

Figure 4: Magnitude response of $H(z)H(z^3)$

(iii)

Based on magnitude responses of $H(-z)$ (Figure. 2) and $H(z^3)$ (Figure. 3), the product of these two can be depicted.

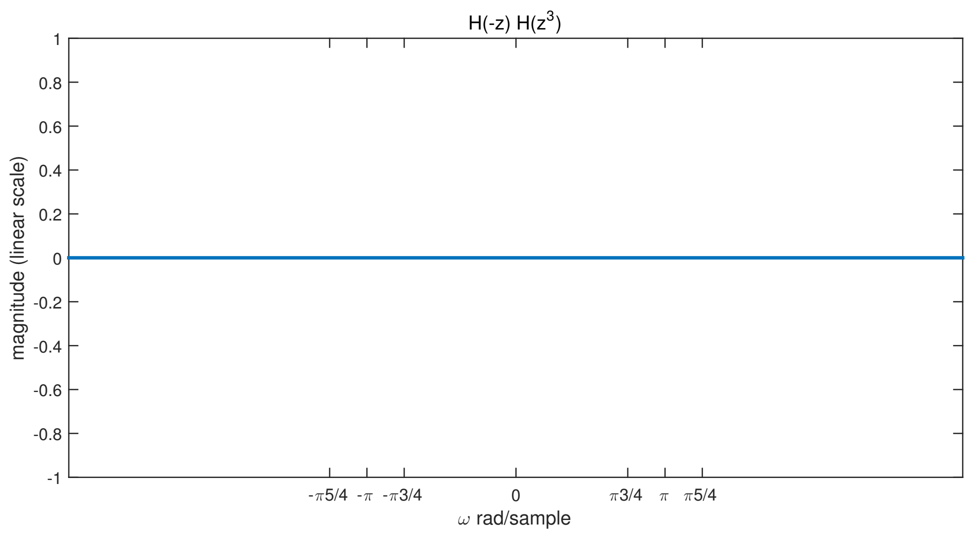

Figure 5: Magnitude response of $H(-z)H(z^3)$

c)

Errors perhaps occur at the edge of low pass filter. For example, when $\omega = \frac{3\pi}{4}$, $|H(-z)| = |H(z^3)| = 1$ and $|H(-z)H(z^3)| = 1$, there should be a delta impulse at $\omega = \frac{3\pi}{4}$.

The impulse response of an ideal low pass filter:

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1e^{j\omega n}d\omega \\ &= \frac{1}{\pi n \times 2j}[e^{j\omega_c n} - e^{-j\omega_c n}] \\ &= \frac{\sin(\omega_c n)}{\pi n} \end{aligned}$$

Ideal low pass filter is impossible to be implemented in practice. $h[n]$ is infinitely long in time. It is not causal and cannot be shifted to make it causal because the impulse response extends all the way to time $-\infty$.

Question 2

$$H_1(z) = 1 + \alpha z^{-1} \tag{3}$$

a)

$$H_1(z) = 1 + \alpha z^{-1} = \sum_{n=0}^{1} \alpha^n z^{-n} \tag{4}$$

Impulse response:

$$h_1[n] = \sum_{i=0}^{1} \alpha^i \delta[n-i] \tag{5}$$

Impulse response coefficients:

$$b_i = \begin{cases} \alpha^i & i = 0,1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

b)

$$H_1(z^D) = 1 + \alpha z^{-D} = \frac{\alpha + z^D}{z^D}, \alpha > 0 \tag{7}$$

$$\begin{aligned} H_1(e^{j\omega D}) &= 1 + \alpha e^{-j\omega D} \\ &= 1 + \alpha\cos(\omega D) - j\alpha\sin(\omega D) \\ |H_1(e^{j\omega D})| &= \sqrt{(1+\alpha\cos(\omega D))^2 + (\alpha\sin(\omega D))^2} \\ &= \sqrt{1 + 2\alpha\cos(\omega D) + \alpha^2\cos^2(\omega D) + \alpha^2 sin^2(\omega D)} \\ &= \sqrt{1 + \alpha^2 + 2\alpha\cos(\omega D)} \end{aligned}$$

When $\cos(\omega D) = 1$,

$$|H_1(e^{j\omega D})|_{max} = \sqrt{1 + \alpha^2 + 2\alpha} = 1 + \alpha \tag{8}$$

When $\cos(\omega D) = -1$

$$|H_1(e^{j\omega D})|_{min} = \sqrt{1 + \alpha^2 - 2\alpha} = |1 - \alpha| \tag{9}$$

As $|H_1(e^{j\omega D})|$ has a period of $T = \frac{2\pi}{D}$, when ω ranges in $0 \le \omega \le 2\pi$, $\frac{2\pi}{\frac{2\pi}{D}} = D$ peaks and dips occur.

Peaks happen at $\omega = \frac{2k\pi}{D}$ $(k = 0, 1, 2, \cdots, D-1)$
Dips happen at $\omega = \frac{(2k+1)\pi}{D}$ $(k = 0, 1, 2, \cdots, D-1)$

From Eq. 7,

$$\begin{aligned} 1 + \alpha z^{-D} &= 0 \\ z^D &= \alpha(-1) \\ &= \alpha e^{j(\pi + 2k\pi)} \\ z &= \alpha^{\frac{1}{D}} e^{j\frac{(2k+1)\pi}{D}} (k = 0, 1, \cdots, D-1) \end{aligned}$$

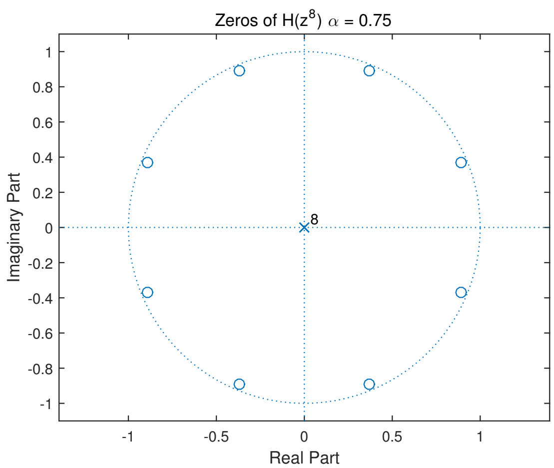

Figure 6: The zeros of the transfer function

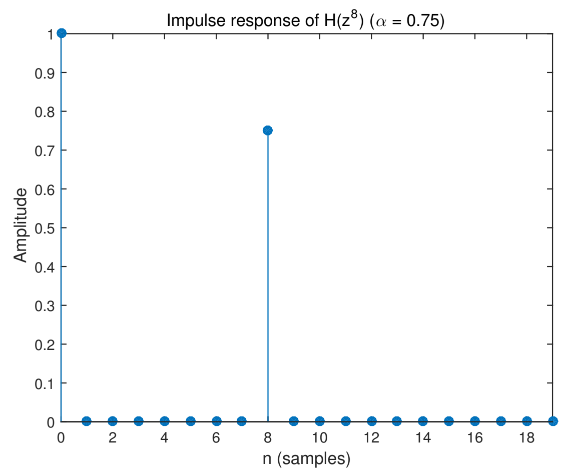

Figure 7: The impulse response

As can be seen from Figure. 6, there are eight zeros located uniformly along a circle. Figure. 7 shows the impulse response has horizontally stretched out 8 times (in terms of the distance between two spikes).

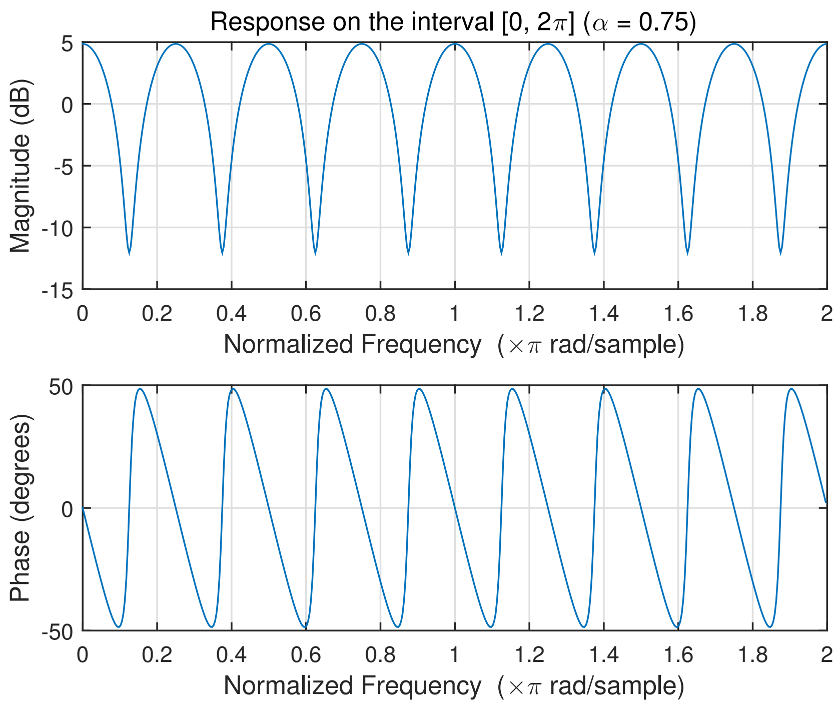

Figure 8: The magnitude and phase response on the interval $[0,\,2\pi)$

Figure. 8 shows 8 peaks and 8 dips at $\frac{k\pi}{4}$ and $\frac{(2k+1)\pi}{8}$ respectively.

Question 3

a)

$H_2(z)$ is a causal filter means its inverse z transform is a right-sided sequence. If we assume the filter is stable ($|\alpha| \leq 1$), then

$$\begin{aligned} H_2(z) &= \frac{1}{1+\alpha z^{-1}} \\ &= \lim_{n\to\infty} \frac{1-(-\alpha z)^n}{1-(-\alpha z)} \text{ (assuming } |\alpha| < 1) \\ &= \sum_{n=0}^{\infty} (-\alpha)^n z^{-n} \end{aligned}$$

The region of convergence of $H_2(z)$ is exterior to a circle $|z| = |\alpha|$.

Frequency response:

$$H_2(e^{j\omega}) = \frac{1}{1+\alpha e^{-j\omega}}$$

Impulse response:

$$h_2[n] = \sum_{i=0}^{\infty} (-\alpha)^i \delta[n-i] \tag{10}$$

Impulse response coefficients:

$$b_i = \begin{cases} (-\alpha)^i & i \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

b)

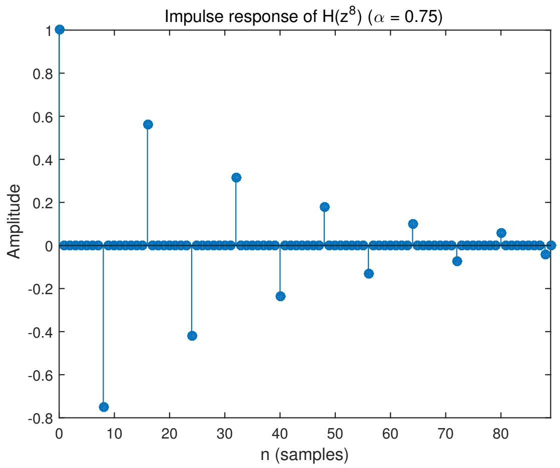

Figure 9: The impulse response

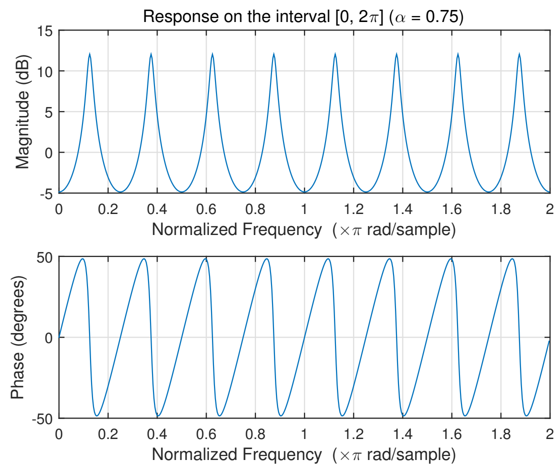

Figure 10: The magnitude and phase response

Figure. 9 shows the impulse response of $H_2(z^8)$, the distance between any two adjacent spikes is 8. Also, the amplitude of the spike decreases as n increases (predicted by Eq. 11). Figure. 10 shows the magnitude and phase response in the range of $[0, 2\pi]$.

Question 4

a)

$$\begin{aligned} |H_3(e^{j\omega})|^2 &= \left|\frac{e^{-j\omega-\alpha}}{1-\alpha e^{-j\omega}}\right|^2 \\ &= \left|\frac{\cos\omega - j\sin\omega - \alpha}{1-(\alpha\cos\omega - \alpha j\sin\omega)}\right|^2 \\ &= \frac{(\cos\omega-\alpha)^2 + (\sin\omega)^2}{(1-\alpha\cos\omega)^2 + (\alpha\sin\omega)^2} \\ &= \frac{(\cos\omega)^2 + (\sin\omega)^2 - 2\alpha\cos\omega + \alpha^2}{1 - 2\alpha\cos\omega + \alpha^2[(\cos\omega)^2 + (\sin\omega)^2]} \\ &= 1 \end{aligned}$$

Note that $(\cos\omega)^2 + (\sin\omega)^2 = 1$.

b)

First define

$$h_0[n] = \begin{cases} (\alpha)^n & n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Take the z-transform

$$H_0(z) = \sum_{n=0}^{\infty} h_0[n] z^{-n} = \sum_{n=0}^{\infty} (\alpha)^n z^{-n} \tag{13}$$

$$\begin{aligned} H_3(z) &= \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \\ &= \frac{z^{-1}}{1 - \alpha z^{-1}} - \frac{\alpha}{1 - \alpha z^{-1}} \\ &= z^{-1} \sum_{n=0}^{\infty} (\alpha)^n z^{-n} - \alpha \sum_{n=0}^{\infty} (\alpha)^n z^{-n} \\ &= z^{-1} H_0(z) - \alpha H_0(z) \end{aligned}$$

Take inverse z transform on both sides and apply time shift and linearity property,

$$h_3[n] = h_0[n-1] - \alpha h_0[n] = \alpha^{n-1} - \alpha \cdot \alpha^n = (1 - \alpha^2)\alpha^{n-1} \text{ (for } n > 0)$$

Since the filter is causal, $h_3[n] = 0$ for $n < 0$.

When $n = 0$, $h_3[0] = h_0[-1] - \alpha h_0[0] = -\alpha h_0[0] = -\alpha$.

c)

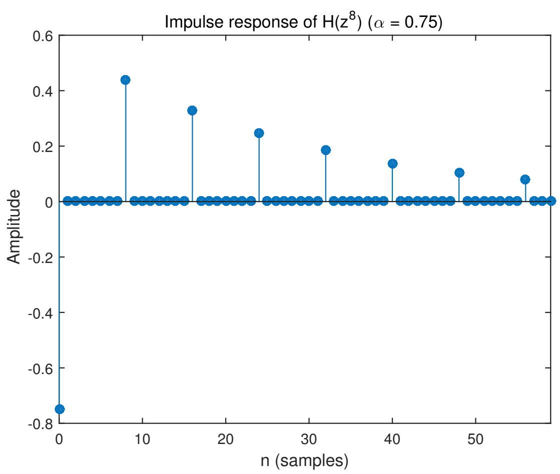

Figure 11: The impulse response

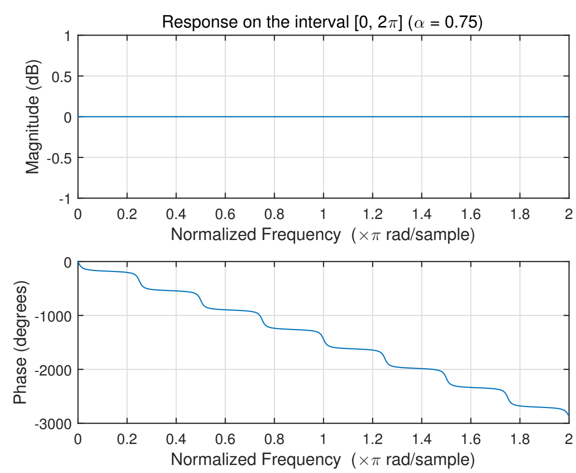

Figure 12: The magnitude and phase response

Question 5

$$\omega_0 = \frac{2\pi \times 330 \text{ Hz}}{44.1 \times 10^3 \text{ sample/s}} = 0.04701 \text{ rad/sample} \tag{14}$$

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \tag{15}$$

where $\beta = \cos(\omega_0) = \mathbf{0.9989}$.

Poles at $re^{\pm j\phi}$, a stable system requires $r = \sqrt{\alpha} < 1$, i.e. $0 < \alpha < 1$.

$$B_w = \cos^{-1}(\frac{2\alpha}{1+\alpha^2}) \tag{16}$$

Given $0 < \alpha < 1$

$$\alpha = \frac{1}{\cos(B_w)} - \sqrt{\frac{1}{(\cos(B_w))^2} - 1} \tag{17}$$

(i) $B_w = 0.1\pi$

When $B_w = 0.1\pi$, from Eq. 17

$$\alpha = 0.726543 \tag{18}$$

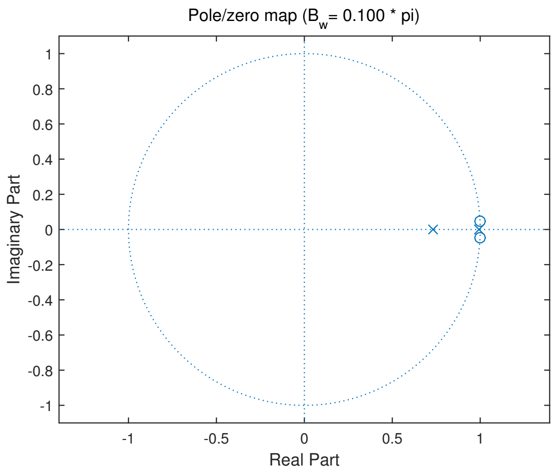

Figure 13: The pole / zero map

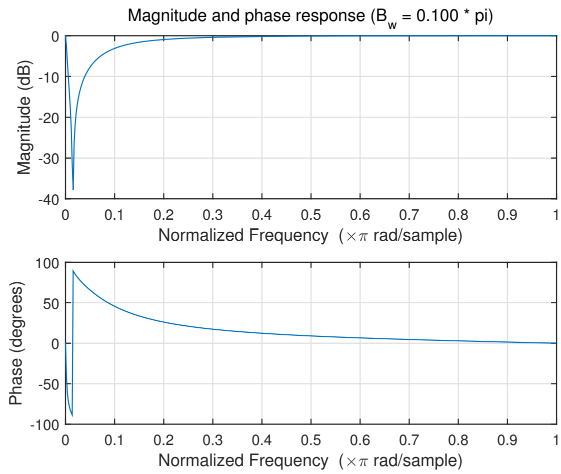

Figure 14: The magnitude and phase response

(ii) $B_w = 0.01\pi$

When $B_w = 0.01\pi$, from Eq. 17

$$\alpha = 0.969067 \tag{19}$$

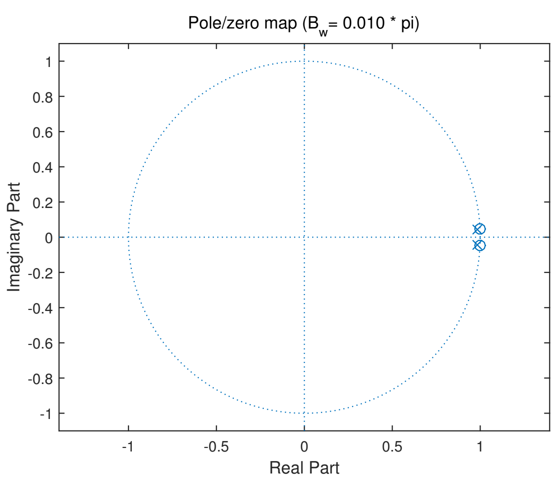

Figure 15: The pole / zero map

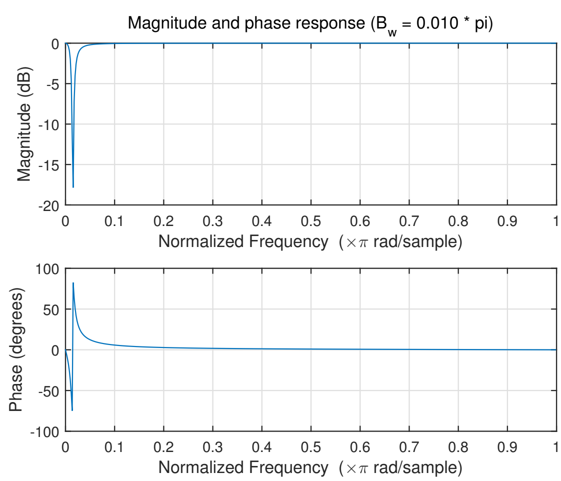

Figure 16: The magnitude and phase response

(iii) $B_w = 0.005\pi$

When $B_w = 0.005\pi$, from Eq. 17

$$\alpha = 0.984414 \tag{20}$$

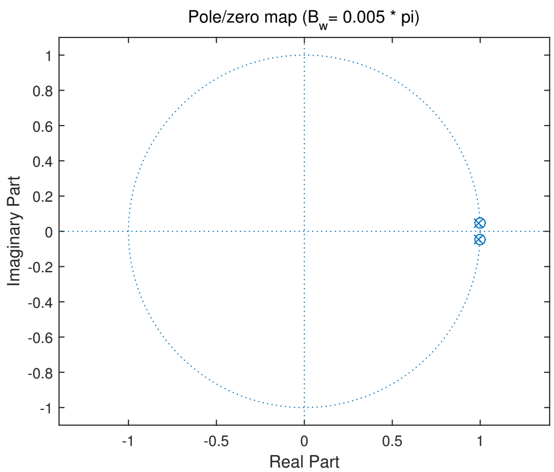

Figure 17: The pole / zero map

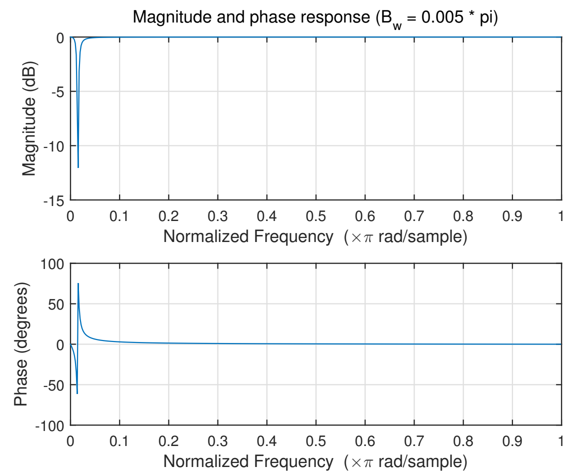

Figure 18: The magnitude and phase response

As the bandwidth is reduced, the notch filter will more concentrate on the specific frequency, in other words, rejective band becomes narrower. However, poles go closer to zeros and the unit circle, which makes it hard to stabilize the filter system.

MATLAB function

```
% bw     3dB bandwidth in Hz
% F_s    sampling frequency in Hz
% f_0    notch frequency in Hz

function [alpha, beta] = A5_function(bw, F_s, f_0)
    omega0 = f_0 * 2 * pi / F_s;
    beta = cos(omega0);

    bw = bw * 2 * pi / F_s;

    cosine = cos(bw);
    alpha = 1/cosine - sqrt(1/cosine^2 - 1);
end
```

Appendix

Question 1 b)

```
clear;
close all;

omega = -3*pi:0.01:3*pi;

% H(z)
y1 = square(omega+pi/4, 100/4)/2 + 0.5;

figure;
plot(omega, y1, 'LineWidth', 2);

set(gca,...
    'xlim', [min(omega) max(omega)],...
    'xtick', [-2*pi -pi -pi/4 0 pi/4 pi 2*pi],...
    'xticklabel', {'-2\pi' '-\pi' '-\pi/4' '0' '\pi/4' '\pi' '2\pi'});
set (gcf, 'Position', [500 500 900 420]);
title('H(z)');
xlabel('\omega rad/sample');
ylabel('magnitude (linear scale)');

% H(z^3)
y2 = square((omega+pi/12)*3, 100/12*3)/2 + 0.5;

figure;
plot(omega, y2, 'LineWidth', 2);

set(gca,...
    'xlim', [min(omega)/2 max(omega)/2],...
    'xtick', [-pi*9/12 -pi*7/12 -pi/3 -pi/12 0 pi/12 pi/3 pi*7/12 pi*9/12],...
    'xticklabel', {'-\pi9/12' '-\pi7/12' '-\pi/3' '-\pi/12' '0' '\pi/12' '\pi/3' '\pi7/12' '
        \pi9/12'});
set (gcf, 'Position', [500 500 900 420]);
title('H(z^3)');
xlabel('\omega rad/sample');
ylabel('magnitude (linear scale)');

% H(z) H(z^3)
figure;
plot(omega, y1 .* y2, 'LineWidth', 2);

set(gca,...
    'xlim', [min(omega)/2 max(omega)/2],...
    'xtick', [-pi*9/12 -pi*7/12 -pi/3 -pi/12 0 pi/12 pi/3 pi*7/12 pi*9/12],...
    'xticklabel', {'-\pi9/12' '-\pi7/12' '-\pi/3' '-\pi/12' '0' '\pi/12' '\pi/3' '\pi7/12' '
        \pi9/12'});
set (gcf, 'Position', [500 500 900 420]);
title('H(z) H(z^3)');
xlabel('\omega rad/sample');
ylabel('magnitude (linear scale)');

% H(-z)
y3 = square((omega+pi/4)+pi, 100/4)/2 + 0.5;

figure;
plot(omega, y3, 'LineWidth', 2);

set(gca,...
```

```
    'xlim', [min(omega) max(omega)],...
    'xtick', [-pi*5/4 -pi -pi*3/4 0 pi*3/4 pi pi*5/4],...
    'xticklabel', {'-\pi5/4' '-\pi' '-\pi3/4' '0' '\pi3/4' '\pi' '\pi5/4'});
set (gcf, 'Position', [500 500 900 420]);
title('H(-z)');
xlabel('\omega rad/sample');
ylabel('magnitude (linear scale)');

% H(-z) H(z^3)
figure;
plot(omega, y3 .* y2, 'LineWidth', 2);

set(gca,...
    'xlim', [min(omega) max(omega)],...
    'xtick', [-pi*5/4 -pi -pi*3/4 0 pi*3/4 pi pi*5/4],...
    'xticklabel', {'-\pi5/4' '-\pi' '-\pi3/4' '0' '\pi3/4' '\pi' '\pi5/4'});
set (gcf, 'Position', [500 500 900 420]);
title('H(-z) H(z^3)');
xlabel('\omega rad/sample');
ylabel('magnitude (linear scale)');
```

Question 2 b)

```
clear;
close all;

D = 8;
alpha = 0.75;

num = zeros(1, D+1);
num(1) = 1;
num(D+1) = alpha;

den = 1;

figure;
zplane(num, den);
title('Zeros of H(z^8) \alpha = 0.75');

figure;
impz(num, den, 20);
title('Impulse response of H(z^8) (\alpha = 0.75)');

figure;
freqz(num, den, 'whole');
title('Response on the interval [0, 2\pi] (\alpha = 0.75)');
```

Question 3 b)

```
clear;
close all;

D = 8;
alpha = 0.75;

den = zeros(1, D+1);
den(1) = 1;
den(D+1) = alpha;

num = 1;

figure;
impz(num, den, 90);
title('Impulse response of H(z^8) (\alpha = 0.75)');

figure;
freqz(num, den, 'whole');
title('Response on the interval [0, 2\pi] (\alpha = 0.75)');
```

Question 4 c)

```
clear;
close all;

D = 8;
alpha = 0.75;

num = zeros(1, D+1);
num(1) = -alpha;
num(D+1) = 1;

den = zeros(1, D+1);
den(1) = 1;
den(D+1) = -alpha;

figure;
impz(num, den, 60);
title('Impulse response of H(z^8) (\alpha = 0.75)');

figure;
freqz(num, den, 'whole');
title('Response on the interval [0, 2\pi] (\alpha = 0.75)');
```

Question 5 plot

```
clear;
close all;

omega0 = 330 * 2 * pi / 44.1E3;
beta = cos(omega0);


for bw = [0.1 0.01 0.005]
    cosine = cos(bw * pi);
    alpha = 1/cosine - sqrt(1/cosine^2 - 1);

    fprintf('alpha=%f (Bw = %.3fpi)\n', alpha, bw);

    num = ((1+alpha) / 2) * [1 -2*beta 1];
    den = [1 -beta*(1+alpha) alpha];

    figure;
    zplane(num, den);
    title(sprintf('Pole/zero map (B_w= %.3f * pi)', bw));

    figure;
    freqz(num, den);
    title(sprintf('Magnitude and phase response (B_w = %.3f * pi)', bw));
end
```

Question 5 sound

```
clear;

load('testsignal');
t1 = clock;
sound(y);
% at the default sample rate of 8192 hertz

bw = 30;
F_s = 8192;
f_0 = 550;

[alpha, beta] = A5_function(bw, F_s, f_0);

fprintf('alpha=%f\n', alpha);
fprintf('beta=%f\n', beta);

num = ((1+alpha) / 2) * [1 -2*beta 1];
den = [1 -beta*(1+alpha) alpha];
output = filter(num, den, y);

% wait for the sound to finish
time = length(y)/F_s - etime(clock, t1);
if (time > 0)
    pause(time);
end

sound(output);
```