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Etude 14 – Super size it

This etude deals with the number of unique combinations of cubes that are possible with the use of any number of colors with a fixated 2x2x2 cube; excluding the cases of reflection and symmetry.

The Burnside's Lemma is used to solve this problem.

The Burnside's theorem can be used for any number of colors, with x as the number of colors. If the cube only has one color, then x = 1. For two colors, i.e. x =2, the number of unique combinations is 23 as proved by Etude 1 manually.

There are 24 total permutations, which can be grouped into categories of symmetry. Let A be the set in which we have unique sets of elements in the set: v,w,x,y,z which are described in the pages below.

A = {v, w, x, y, z};
The frequency of v is 1, w is 6, x is 3, y is 8, z is 6 respective permutations.

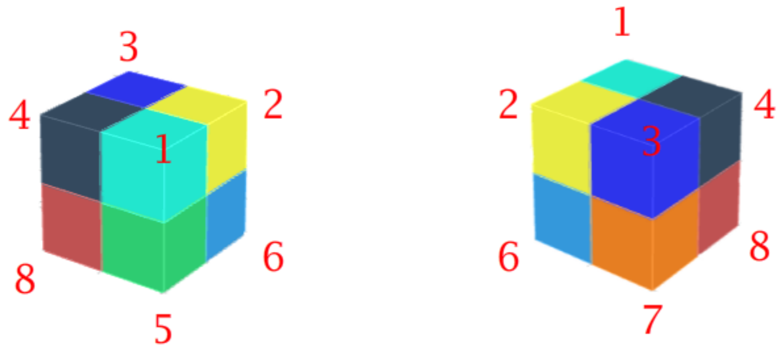
Here we have 8 cubes which are fixed in color. They move around to the positions of other cubes which give us different unique combinations. We only include those combinations and positions which give us the unique positions without reflections and symmetry. The cases are described from the next page.

The formula is:

$$\frac{1}{24}((1 \text{ x } v) + (6 \text{ x } w) + (3 \text{ x } x) + (8 \text{ x } y) + (6 \text{ x } z))$$

$$\frac{1}{24}((1 \text{ x } (x^8) + (6 \text{ x } x^2) + (3 \text{ x } x^4) + (8 \text{ x } x^4) + (6 \text{ x } x^4))$$

Below is a cube.



Left: A whole **8** colored cube, showing 7 visible cubes on one side.
Right: A whole **8** colored cube, showing the 8th cube from the other side

- ## **Case v: Identity –no rotation (1 permutation)**

Each single cube, say 1 indicated on the cube, could be moved to all the sides of the cube. 1 can move to 2,3,4,5,6,7,8. Therefore, there are a total of 8 such positions.

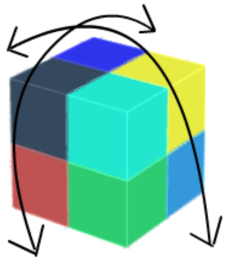


Figure 2: The cube with black lines respectively showing opposite faces pairs. They can rotated in either directions, making a total of 6 permutations.

- ## Case **w: 90° face rotations (6 permutations)**

The consideration is the 90° face rotations, of pairs, that rotate in either directions, as illustrated by the diagram below. We now consider symmetry, the top 4 cubes (Pair 1), in symmetry with bottom 4 cubes (Pair 2).

1st Pair: 1->2, 2->3, 3->4, 4-> 1
2nd Pair: 5->6, 6->7, 7->8, 8->5

Therefore, there are 2 such pairs of unique possibilities.

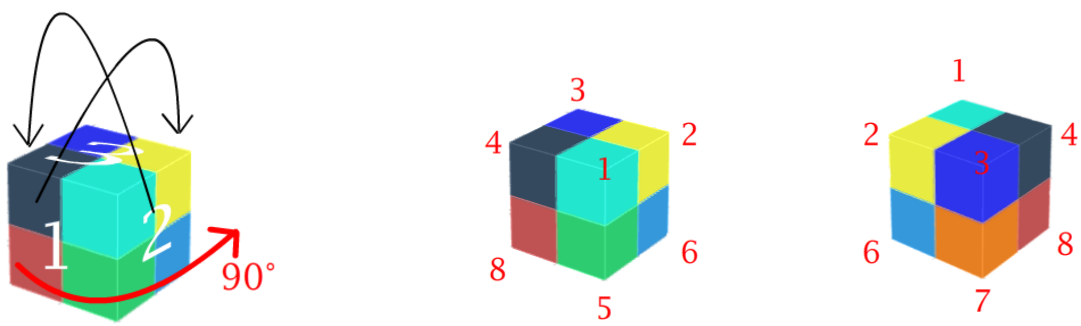


Figure 2: The cube with black lines respectively showing opposite faces pairs. They can rotated in either directions, making a total of 6 permutations.

- ## Case **x**: **180° face rotations (3 permutations)**

  Here,the opposite pairs, rotated 180°. There are only three unique possibilities, 3 permutations. Horizontal rotation plane pairs (4,1,5,8), (1,2,6,5), (3,4,8,7) & vertical (1,2,3,4).There are 4 pairs.

  4,1,5,8 -> 2,3,7,6 (Horizontal)
  1,2,6,5 -> 3,4,8,7 (Horizontal)(in either direction)
  1,2,3,4 -> 5,6,7,8 (Vertical)

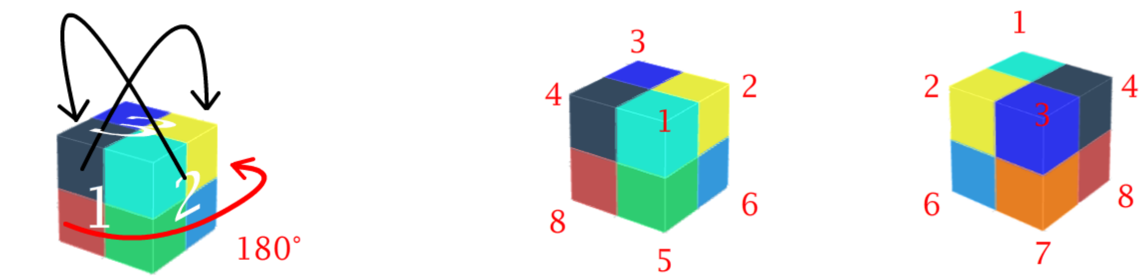


Figure 3: The cube with black lines respectively showing opposite faces pairs. They can rotated 180°, with only three possibilities.

- ## Case **y**: **120° diagonal rotations (8 permutations)**

  Here, the opposite vertices, rotated diagonally in either directions . There are only 8 possibilities, 8 permutations. There are 4 such pairs.

  Cubes (1,7) is symmetrical Cubes(3, 5). Cube(2,8) is symmetrical to (4, 6).
  Cube(1,7) -> Cube(7,1)
  Cube(3,5) -> Cube(5,3)
  Cube(2,8) -> Cube(8,2)
  Cube(4,6) -> Cube(6,4)

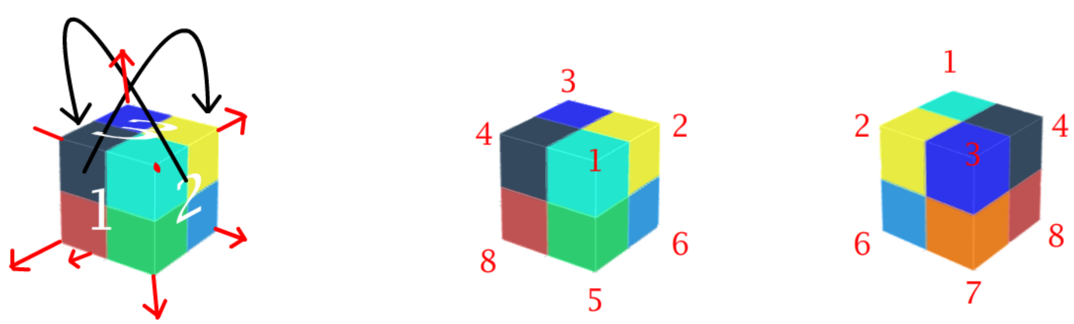


Figure 3: The cube with black lines respectively showing opposite faces pairs. Here the red line crosses through the vertices. They can rotate 120° in either direction, making a total of 8 permutations.

- Case **z**: **180° center edge rotations (6 permutations)**

  Here, the edges swap for a total of 6 times, making a total of 6 permutations.
  There are four such pairs.

  Side(1,3)-> Side(5,7)
  Side(2,4)-> Side(6,8)
  Side(1,2)-> Side(8,7)
  Side(5,8)-> Side(2,3)

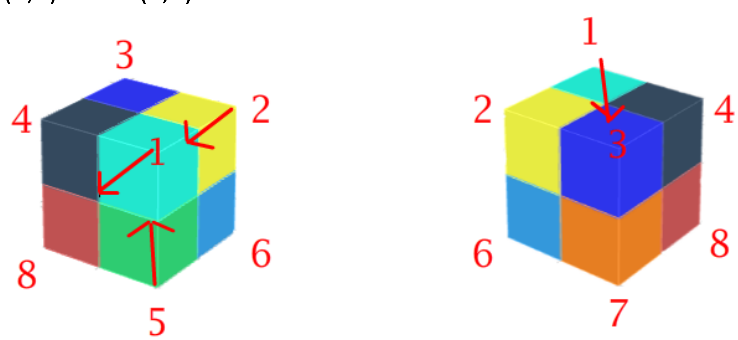


Figure 3: The vertices going from one center edge to another. They can rotate 120° in either

Plug in the value for x is 8, where x represents the number of colors. The calculations for 8 colors are given below.

8 colors: 701968 sets;

Therefore for any number of colors, that is 8 colors, the unique number of combinations would be 701968 sets.

We put the value of x in the following equation.

1/24 * (1*(x^8) + 3*(x^4) + 6*(x^2) + 8*(x^4) + 6*(x^4))