■ 라플라스 변환

$$\boldsymbol{F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)dt}$$

| | |
|---|---|
| $\mathbf{t^n}$ | $\dfrac{\boldsymbol{n!}}{\boldsymbol{s^{n+1}}}$ |
| $\boldsymbol{e^{at}}$ | $\dfrac{1}{s-a}$ |
| $\mathbf{cos}\,\boldsymbol{\omega t}$ | $\dfrac{s}{s^2+\omega^2}$ |
| $\mathbf{sin}\,\boldsymbol{\omega t}$ | $\dfrac{\omega}{s^2+\omega^2}$ |
| $\mathbf{cosh}\,\boldsymbol{\omega t}$ | $\dfrac{s}{s^2-\omega^2}$ |
| $\mathbf{sinh}\,\boldsymbol{\omega t}$ | $\dfrac{\omega}{s^2+\omega^2}$ |

■ 라플라스 변환의 선형성

$$\boldsymbol{\mathcal{L}\big(af(t)+bg(t)\big) = a\mathcal{L}\big(f(t)\big) + b\mathcal{L}\big(g(t)\big)}$$

■ 제 1 이동 정리, s-이동

$$\mathcal{L}\big(f(t)\big) = F(s) \rightarrow \boldsymbol{\mathcal{L}\big(e^{at}f(t)\big) = F(s-a)}$$

$$\mathcal{L}\big(f(t)\big) = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$$\begin{aligned}\rightarrow F(s-a) &= \int_0^\infty e^{-(s-a)t} f(t)dt \\ &= \int_0^\infty e^{-st}\big(e^{at}f(t)\big)dt \\ &= \mathcal{L}\big(e^{at}f(t)\big)\end{aligned}$$

■ 도함수의 라플라스 변환

$$\begin{aligned}\boldsymbol{\mathcal{L}\big(f^{(n)}\big)} &\boldsymbol{= s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots} \\ &\boldsymbol{- f^{(n-1)}(0)}\end{aligned}$$

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)dt$$

$$= e^{-st}f(t)\big|_{t=0}^{t=\infty} - (-\mathrm{s})\int_0^\infty e^{-st}f(t)dt$$

$$= s\mathcal{L}(f) - f(0)$$

■ 적분의 라플라스 변환

$$\mathcal{L}\big(f(t)\big) = F(s) \rightarrow \boldsymbol{\mathcal{L}\left(\int_0^\infty f(\tau)d\tau\right) = \frac{1}{s}F(s)}$$

$$\begin{aligned}g(t) = \int_0^t f(\tau)d\tau \rightarrow \mathcal{L}\big(f(t)\big) &= \mathcal{L}\big(g'(t)\big) \\ &= s\mathcal{L}\big(g(t)\big) - g(0) = s\mathcal{L}\big(g(t)\big) \\ &\rightarrow s\mathcal{L}\big(g(t)\big) = \mathcal{L}\big(f(t)\big) \\ &\rightarrow \mathcal{L}\left(\int_0^\infty f(\tau)d\tau\right) = \frac{1}{s}\mathcal{L}\big(f(t)\big)\end{aligned}$$

■ 단위 계단 함수

$$\boldsymbol{u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}}$$

■ 단위 계단 함수의 라플라스 변환

$$\boldsymbol{\mathcal{L}\big(u(t-a)\big) = \frac{e^{-as}}{s}}$$

$$\begin{aligned}\mathcal{L}\big(u(t-a)\big) &= \int_0^\infty e^{-st}u(t-a)dt = \int_a^\infty e^{-st}dt \\ &= -\frac{1}{s}e^{-st}\bigg|_{t=0}^{t=\infty} = \frac{e^{-as}}{s}\end{aligned}$$

■ 제 2 이동 정리, t-이동

$$\mathcal{L}\big(f(t)\big) = F(s)$$
$$\rightarrow \boldsymbol{\mathcal{L}\big(f(t-a)\cdot u(t-a)\big) = e^{-as}F(s)}$$

$$\mathcal{L}\big(f(t)\big) = F(s)$$
$$\begin{aligned}\rightarrow \mathcal{L}\big(f(t-a)\cdot u(t-a)\big) \\ &= \int_0^\infty e^{-st}f(t-a)\cdot u(t-a)dt \\ &= \int_a^\infty e^{-st}f(t-a)dt \\ &= \int_0^\infty e^{-s(t+a)}f(t)dt \\ &= e^{-as}\int_0^\infty e^{-st}f(t)dt \\ &= \mathrm{e}^{-\mathrm{as}}\mathcal{L}\big(f(t)\big) = e^{-as}F(t)\end{aligned}$$

■ Dirac의 델타 함수

$$\boldsymbol{f_k(t-a) = \begin{cases} 1/k, & a \le t \le a+k \\ 0, & other \end{cases}}$$
$$\boldsymbol{\delta(\mathbf{t}-\mathbf{a}) = \lim_{\mathbf{k}\to 0} f_k(t-a)}$$

■ Dirac의 델타 함수의 라플라스 변환

$$\boldsymbol{\mathcal{L}\big(\delta(t-a)\big) = e^{as}}$$

$$\begin{aligned}\mathcal{L}\big(f_k(t-a)\big) &= \mathcal{L}\left(\frac{1}{k}\big[u(t-a) - u\big(t-(a-k)\big)\big]\right) \\ &= \frac{1}{ks}\big[e^{-as} - e^{-(a+k)s}\big] \\ &= e^{-as}\frac{1-e^{-ks}}{ks}\end{aligned}$$

$$\begin{aligned}\lim_{t\to 0}\mathcal{L}\big(f_k(t-a)\big) &= \mathcal{L}\big(\delta(t-a)\big) \\ &= \lim_{t\to 0} e^{-as}\frac{1-e^{-ks}}{ks} = e^{-as}\end{aligned}$$

■ 합성곱

$$(\boldsymbol{f} * \boldsymbol{g})(\boldsymbol{t}) = \int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{f}(\boldsymbol{\tau})\boldsymbol{g}(\boldsymbol{t}-\boldsymbol{\tau})\boldsymbol{d\tau}$$

| | |
|---|---|
| 교환법칙 | $f * g = g * f$ |
| 분배법칙 | $f * (g_1 + g_2)$ $= f * g_1 + f * g_2$ |
| 결합법칙 | $(f * g) * v$ $= f * (g * v)$ |
| 특이성질 | $f * 1 \neq f$ $f * 0 = 0$ |
| 정리 | $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$ |

■ 합성곱 정리 증명

$$\mathcal{L}(\mathrm{f}) = \int_0^\infty f(t)e^{-st}dt$$

$$\mathcal{L}(\mathrm{g}) = \mathrm{e}^{s\tau}\int_0^\infty g(t-\tau)e^{-st}dt$$

$$\begin{aligned} L(f)L(g) &= \int_0^\infty f(\tau)d\tau \int_\tau^\infty g(t-\tau)e^{-st}dt \\ &= \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st}dt\,d\tau \\ &= \int_0^\infty \int_0^t f(\tau)g(t-\tau)e^{-st}d\tau\,dt \\ &= \int_0^\infty \big(f(t) * g(t)\big)e^{-st}dt \\ &= L(f * g) \end{aligned}$$

■ 변환의 미분

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st}f(t)dt$$

$$\rightarrow \boldsymbol{\mathcal{L}}\big(\boldsymbol{tf}(\boldsymbol{t})\big) = -\boldsymbol{F}'(\boldsymbol{s})$$

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st}f(t)dt$$

$$\rightarrow F'(s) = \frac{dF}{ds} = -\int_0^\infty e^{-st}tf(t)dt = -\mathcal{L}(tf)$$

■ 변환의 적분

$$F(\tilde{\boldsymbol{s}}) = \mathcal{L}(f) = \int_0^\infty e^{-\tilde{s}t}f(t)dt$$

$$\rightarrow \boldsymbol{\mathcal{L}}\left(\frac{\mathbf{1}}{\boldsymbol{t}}\boldsymbol{f}(\boldsymbol{t})\right) = \int_{\boldsymbol{s}}^\infty \boldsymbol{F}(\tilde{\boldsymbol{s}})\boldsymbol{d}\tilde{\boldsymbol{s}}$$

$$F(\tilde{\boldsymbol{s}}) = \mathcal{L}(f) = \int_0^\infty e^{-\tilde{s}t}f(t)dt$$

$$\begin{aligned} \rightarrow \int_s^\infty F(\tilde{s})d\tilde{s} &= \int_s^\infty \left(\int_0^\infty e^{-\tilde{s}t}f(t)dt\right)d\tilde{s} \\ &= \int_0^\infty f(t)\left(\int_s^\infty e^{-\tilde{s}t}d\tilde{s}\right)dt \\ &= \int_0^\infty f(t)\left(-\frac{1}{t}e^{-\tilde{s}t}\Big|_{\tilde{s}=0}^{\tilde{s}=\infty}\right)dt \\ &= \int_0^\infty f(t)\left(\frac{1}{t}e^{-st}\right)dt \\ &= \int_0^\infty \left(\frac{1}{t}f(t)\right)e^{-st}dt \\ &= \mathcal{L}\left(\frac{1}{t}f(t)\right) \end{aligned}$$

■ inner product

| | |
|---|---|
| $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ | $\langle f, g\rangle = \int_a^b fg dx$ |
| $\langle \alpha x + \beta y, z\rangle$ $= \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ | $\langle \alpha f + \beta g, h\rangle$ $= \alpha\langle f, h\rangle + \beta\langle g, h\rangle$ |

■ 푸리에 급수

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{a_0} + \sum_{\boldsymbol{n}=\mathbf{1}}^{\boldsymbol{x}} \left(\boldsymbol{a_n}\cos\frac{\boldsymbol{n\pi}}{\boldsymbol{L}}\boldsymbol{x} + \boldsymbol{b_n}\sin\frac{\boldsymbol{n\pi}}{\boldsymbol{L}}\boldsymbol{x}\right)$$

$$\boldsymbol{a_0} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{L}}\int_{-\boldsymbol{L}}^{\boldsymbol{L}} \boldsymbol{f}(\boldsymbol{x})\boldsymbol{dx}$$

$$\boldsymbol{a_n} = \frac{\mathbf{1}}{\boldsymbol{L}}\int_{-\boldsymbol{L}}^{\boldsymbol{L}} \boldsymbol{f}(\boldsymbol{x})\cos\frac{\boldsymbol{n\pi x}}{\boldsymbol{L}}\boldsymbol{dx}$$

$$\boldsymbol{b_n} = \frac{\mathbf{1}}{\boldsymbol{L}}\int_{-\boldsymbol{L}}^{\boldsymbol{L}} \boldsymbol{f}(\boldsymbol{x})\sin\frac{\boldsymbol{n\pi x}}{\boldsymbol{L}}\boldsymbol{dx}$$

$a_0{:}\,\langle f, 1\rangle, a_n{:}\,\langle f, \cos mx\rangle, b_n\langle f, \sin mx\rangle$

■ 푸리에 변환

$$\mathcal{F}\big(f(t)\big) = \begin{cases} F(f) = \int_{-\infty}^\infty f(t)e^{-i2\pi ft}dt \\ F(\omega) = \int_{-\infty}^\infty f(t)e^{-i\omega t}dt \end{cases}$$

$$f(t) = \begin{cases} \int_{-\infty}^\infty \mathcal{F}(f)e^{-i2\pi ft}df \\ \frac{1}{2\pi}\int_{-\infty}^\infty \mathcal{F}(\omega)e^{-i\omega t}d\omega \end{cases}$$

■ 도함수의 푸리에 변환

$$\boldsymbol{\mathcal{F}}\big(\boldsymbol{f}^{(\boldsymbol{n})}\big) = (\boldsymbol{i\omega})^{\boldsymbol{n}}\boldsymbol{\mathcal{F}}\big(\boldsymbol{f}(\boldsymbol{t})\big)$$

$$\begin{aligned} \mathcal{F}\big(f'(t)\big) &= \int_{-\infty}^\infty f'(t)e^{-i2\pi ft}dt \\ &\xrightarrow{\text{부분적분}} i2\pi f\int_{-\infty}^\infty f(t)e^{-i2\pi ft}dt \\ &= i2\pi f\mathcal{F}\big(f(t)\big) \end{aligned}$$

■ 적분의 푸리에 변환

$$\boldsymbol{\mathcal{F}\left(\int_{-\infty}^{t} f(\tau)d\tau\right) = \frac{1}{i\omega}F(\omega)}$$

$$\mathrm{g(t)} = \int_{-\infty}^{t} f(\tau)d\tau \to g'(t) = f(x)$$

$$\mathcal{F}\big(g'(x)\big) = \mathcal{F}\big(f(x)\big) \leftrightarrow j\omega G(\omega) = F(\omega)$$

$$\therefore G(\omega) = \frac{1}{i\omega}F(\omega)$$

■ time shift

$$\boldsymbol{\mathcal{F}\big(f(t-t_0)\big) = e^{-i2\pi f t_0}\mathcal{F}\big(f(t)\big)}$$

$$\begin{aligned}\mathcal{F}\big(f(t-t_0)\big) &= \int_{-\infty}^{\infty} f(t-t_0)e^{-i2\pi f t}dt \\ &\xrightarrow{t'=t-t_0} \int_{-\infty}^{\infty} f(t')e^{-i2\pi f(t'+t_0)}dt \\ &= e^{-i2\pi f t_0}\int_{-\infty}^{\infty} f(t)e^{-i2\pi f t}dt \\ &= e^{-i2\pi f t_0}\mathcal{F}\big(f(t)\big)\end{aligned}$$

■ frequency shift

$$\boldsymbol{\mathcal{F}\left(e^{i2\pi f_0 t}f(t)\right) = F(f-f_0)}$$

$$\begin{aligned}\mathcal{F}\left(e^{i2\pi f_0 t}f(t)\right) &= \int_{-\infty}^{\infty} e^{i2\pi f_0 t}f(t)e^{-i2\pi f t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-i2\pi(f-f_0)t}dt \\ &= F(f-f_0)\end{aligned}$$

■ 변환의 미분

$$\boldsymbol{F'(\omega) = \frac{1}{2\pi}F'(f) = \mathcal{F}\big(-itf(t)\big)}$$

$$\begin{aligned}F'(f) = \frac{d}{df}F(f) &= \frac{d}{df}\left(\int_{-\infty}^{\infty} f(t)e^{-i2\pi f t}dt\right) \\ &= \int_{-\infty}^{\infty} f(t)(-i2\pi t)e^{-i2\pi f t}dt \\ &= -i2\pi\int_{-\infty}^{\infty} tf(t)e^{-i2\pi f t}dt \\ &= -i2\pi\mathcal{F}\big(tf(t)\big)\end{aligned}$$

$$\begin{aligned}F'(\omega) = \frac{d}{d\omega}F(\omega) &= \frac{d}{d\omega}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)(-it)e^{-i\omega t}dt \\ &= -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} tf(t)e^{-i\omega t}dt \\ &= -i\mathcal{F}\big(tf(t)\big)\end{aligned}$$

■ 합성곱

$$\mathcal{F}(f*g) = F(f)G(f) = \sqrt{2\pi}F(\omega)G(\omega)$$

$$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

$$\begin{aligned}\mathcal{F}\big(f(t)*g(t)\big) &= \int_{-\infty}^{\infty}\big(f(t)*g(t)\big)e^{-i2\pi f t}dt \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right)e^{-i2\pi f t}dt \\ &= \int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} g(t-\tau)e^{-i2\pi f t}dt\right)d\tau \\ &\xrightarrow{t'=t-\tau} \int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} g(t')e^{-i2\pi f(t'+\tau)}dt'\right)d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)e^{-i2\pi f\tau}d\tau\int_{-\infty}^{\infty} g(t)e^{-i2\pi f t}dt \\ &= \mathcal{F}\big(f(t)\big)\mathcal{F}\big(g(t)\big) = F(f)G(f)\end{aligned}$$

$$\begin{aligned}\mathcal{F}\big(f(t)*g(t)\big) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\big(f(t)*g(t)\big)e^{-i\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right)e^{-i\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} g(t-\tau)e^{-i\omega t}dt\right)d\tau \\ &\xrightarrow{t'=t-\tau} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\left(\int_{-\infty}^{\infty} g(t')e^{-i\omega(t'+\tau)}dt'\right)d\tau \\ &= \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)e^{-i2\pi f\tau}d\tau\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{-i2\pi f t}dt\right) \\ &= \sqrt{2\pi}\mathcal{F}\big(f(t)\big)\mathcal{F}\big(g(t)\big) = F(\omega)G(\omega)\end{aligned}$$

■ 변환비교

| $\mathcal{L}\big(f(t)\big) = F(s)$ | $\mathcal{F}\big(f(t)\big) = F(s)$ |
|---|---|
| $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ | $\mathcal{F}\big(f^{(n)}\big) = (i\omega)^n\mathcal{F}\big(f(t)\big)$ |
| $\mathcal{L}\left(\int_0^{\infty} f(\tau)d\tau\right) = \frac{1}{s}F(s)$ | $\mathcal{F}\left(\int_{-\infty}^{t} f(\tau)d\tau\right) = \frac{1}{i\omega}F(\omega)$ |
| $\mathcal{L}\big(e^{at}f(t)\big) = F(s-a)$ | $e^{-i2\pi f t_0}\mathcal{F}\big(f(t)\big)$ $= \mathcal{F}\big(f(t-t_0)\big)$ |
| $\mathcal{L}\big(f(t-a)\cdot u(t-a)\big)$ $= e^{-as}F(s)$ | $\mathcal{F}\left(e^{i2\pi f_0 t}f(t)\right) = F(f-f_0)$ |
| $\mathcal{L}\big(tf(t)\big) = -F'(s)$ | $\boldsymbol{\mathcal{F}\big(-itf(t)\big) = F'(\omega)}$ |