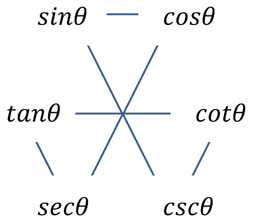
Derivatives (도함수)

$$\frac{df(x)}{dx} = lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

Natural logarithm (자연 로그)

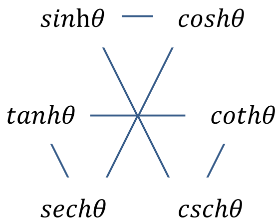
$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Trigonometric Function (삼각 함수)



$sin^2\theta + cos^2\theta = 1$
$tan^2\theta + 1 = sec^2\theta$
$1 + cot^2\theta = csc^2\theta$
$sin(A\pm B) = sin\,A\,cos\,B \pm cos\,A\,sin\,B$
$cos(A\pm B) = cos\,A\,cos\,B \mp \sin A \sin B$
$tan(A\pm B) = \dfrac{tan\,A \pm tan\,B}{1 \mp tan\,A\,tan\,B}$
$sin(A+B) + sin(A-B) = 2\,sin\,A\,cos\,B$
$sin(A+B) - sin(A-B) = 2\,cos\,A\,sin\,B$
$cos(A+B) + cos(A-B) = 2\,cos\,A\,cos\,B$
$cos(A+B) - cos(A-B) = -2\,sin\,A\,sin\,B$

Hyperbolic Function (쌍곡선 함수)



$sinh\,\theta = \dfrac{e^\theta - e^{-\theta}}{2}$
$cosh\,\theta = \dfrac{e^\theta + e^{-\theta}}{2}$
$tanh\,\theta = \dfrac{sinh\,\theta}{cosh\,\theta}$
$-sinh^2\theta + cosh^2\theta = 1$
$-tanh^2\theta + 1 = sech^2\theta$
$-1 + coth^2\theta = csch^2\theta$
$sinh(A\pm B) = sinh\,A\,cosh\,B \pm cosh\,A\,sinh\,B$
$cosh(A\pm B) = cosh\,A\,cosh\,B \pm \sinh A \sinh B$
$tanh(A\pm B) = \dfrac{tanh\,A \pm tanh\,B}{1 \pm tanh\,A\,tanh\,B}$
$sinh(A+B) + sinh(A-B) = 2\,sinh\,A\,cosh\,B$
$sinh(A+B) - sinh(A-B) = 2\,cosh\,A\,sinh\,B$
$cosh(A+B) + cosh(A-B) = 2\,cosh\,A\,cosh\,B$
$cosh(A+B) - cosh(A-B) = 2\,sinh\,A\,sinh\,B$
$sinh'\,x = cosh\,x$
$cosh'^x = sinh\,x$
$tanh'^x = sech^2\,x$

Gamma Function

$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx\ (n>0)$
$\Gamma(n+1) = n\Gamma(n)$
$\Gamma(n) = (n-1)!$
$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
$\int_0^\infty e^{-\alpha x}x^n dx = \dfrac{\Gamma(n+1)}{\alpha^{n+1}}$
$\int_0^\infty e^{-\alpha x^2}x^n dx = \dfrac{1}{2}\dfrac{\Gamma(\frac{n+1}{2})}{\alpha^{\frac{n+1}{2}}}$

Complex Number

$e^{iy} = \cos y + i\sin y$
$\sin z = \dfrac{e^{iz}+e^{-iz}}{2i}$
$\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$
$\sinh z = \dfrac{e^{z}-e^{-z}}{2}$
$\cosh z = \dfrac{e^{z}+e^{-z}}{2}$

1계 상미분방정식

변수 분리법

$g(y)y' = f(x) \to g(y)dy = f(x)dx$
$\to \int g(y)dy = \int f(x)dx + c$

확장방법

$u = \dfrac{y}{x} \to y = ux \to y' = u'x + u$
$y' = f\left(\dfrac{y}{x}\right) \to u'x + u = f(u) \to u'x = f(u) - u$
$\to \dfrac{du}{f(u)-u} = \dfrac{dx}{x} \to \int \dfrac{1}{f(u)-u}du = \int \dfrac{1}{x}dx + c$

전미분, 완전미분방정식

$$du = f_x(x,y)dx + f_y(x,y)d = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

완전미분방정식의 필요충분조건

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

적분인자

$Pdx + Qdy = 0$
$F(x) = exp\left(\int \dfrac{1}{Q}\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right)dx\right)$

완전미분방정식의 해법

$du = M(x,y)dx + N(x,y)dy = 0$ 에서 필요충분조건 검사

완전미분방정식이 아닐 때, 적분인자를 곱함.

$\frac{\partial u}{\partial x} = M(x,y) \xrightarrow{x\text{에 대하여 적분}} u(x,y) = \int M(x,y)dx + k(y)$

$\frac{\partial u}{\partial y} = N(x,y) \to \frac{dk}{dy} \to k(y)$

$\therefore u(x,y) = \int M(x,y)dx + k(y)$

$\xrightarrow{\text{검증}} du = \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy = 0$

$\to du = M(x,y)dx + N(x,y)dy = 0$

제차 미분방정식의 해법(변수 분리법)

$y' + p(x)y = 0 \to \frac{\mathrm{dy}}{\mathrm{dx}} - p(x)y \to \frac{dy}{y} - p(x)dx$

$\to ln|y| = -\int p(x)dx + c_0 \to y = c_1 e^{-\int p(x)dx}$

비제차 미분방정식의 해법(완전미분방정식의 해법 응용)

$y' + p(x)y = r(x) \to \frac{dy}{dx} + p(x)y - r(x) = 0$

$\to dy + (py - r)dx = 0 \to Q = 1, P = py - r$

$\to \frac{\partial P}{\partial y} = p \neq 0 = \frac{\partial Q}{\partial x}\left(\therefore \text{완전미분형이 아님}\right)$

-적분인자 구하기

$\mathrm{R} = \frac{1}{\mathrm{Q}}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = p(\because Q = 1) \to F = e^{\int pdx}$

-적분인자 곱하여 해 구하기

$(\mathrm{py} - \mathrm{r})\mathrm{dx} + \mathrm{dy} = 0$

$\xrightarrow{\text{적분인자 곱하기}} e^{\int pdx}(py - r)dx + e^{\int pdx}dy = 0 = du$

$\frac{\partial u}{\partial y} = e^{\int pdx} \xrightarrow{y\text{에 대하여 적분}} u(x,y) = ye^{\int pdx} + l(x)$

$\frac{\partial u}{\partial x} = pye^{\int pdx} + l'(x) = e^{\int pdx}(py - r)$

$\to l'(x) = -re^{\int pdx} \to l(x) = -\int re^{\int pdx}dx + c$

$\therefore u = ye^{\int pdx} - \int re^{\int pdx}dx = c$

$\to ye^{\int pdx} = \int re^{\int pdx}dx + c$

$\therefore y = e^{-\int pdx}(\int re^{\int pdx}dx + c)$

<u>2계 선형상미분방정식</u>

차수 축소법

$y'' + p(x)y' + q(x)y = 0 \xrightarrow{when\ know\ y_1} y = y_2 = uy_1$

$\to y' = y_2' = y'u_1 + uy_1'$

$\to y'' = y_2'' = u''y_1 + 2u'y_1' + uy_1''$

$\xrightarrow{\text{대입}} (u''y_1 + 2u'y_1' + uy_1'') + p(u'y_1 + uy_1') + quy_1 = 0$

$\xrightarrow{divide\ y_1} u'' + \left(2\frac{y_1'}{y_1} + p\right)u' = 0$

$\xrightarrow{U=u',\ U'=u''} U' + \left(2\frac{y_1'}{y_1} + p\right)U = 0$

$\to \frac{dU}{U} = -\left(2\frac{y_1'}{y_1} + p\right)dx \to ln|U| = -2\,ln|y_1| - \int pdx$

$\therefore U = \frac{1}{y_1^2}e^{-\int pdx}, \qquad y_2 = uy_1 = y_1\int Udx$

상수계수를 갖는 2계 제차 선형상미분방정식

$\mathrm{y}'' + \mathrm{ay}' + \mathrm{by} = 0 \xrightarrow{\text{특성방정식}} \lambda^2 + a\lambda + b = 0$

$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$case\ I)\ if\ a^2 - 4b > 0 \to two\ real\ roots$

$case\ II)\ if\ a^2 - 4b = 0 \to a\ real\ double\ root$

$case\ III)\ if\ a^2 - 4b < 0 \to complex\ conjugate\ roots$

Summary of Cases I-III

Case	Basis of	General Solution of
I	$\mathrm{e}^{\lambda_1 x}, e^{\lambda_2 x}$	$\mathrm{y} = \mathrm{c_1 e}^{\lambda_1 x} + \mathrm{c_2 e}^{\lambda_2 x}$
II	$\mathrm{e}^{-\lambda}, xe^{-\lambda}$	$\mathrm{y} = (\mathrm{c_1} + c_2 x)\mathrm{e}^{\lambda x}$
III	$\mathrm{e}^{-\frac{a}{2}x}\cos\omega x$ $e^{-\frac{a}{2}x}\sin\omega x$	$y = \mathrm{e}^{-\frac{a}{2}x}(A\cos\omega x + \mathrm{B}\sin\omega x)$

$\omega = \frac{\mathrm{i}}{2}\sqrt{a^2 - 4b}$

비감쇠 시스템

$my'' + ky = 0 \to \mathrm{y(t)} = \mathrm{A}\cos\omega_0 t + B\sin\omega_0 t$

$= \mathrm{C}\cos(\omega_0 t - \delta), \qquad \omega_0^2 = \frac{k}{m}$

감쇠 시스템

$my'' + cy' + ky = 0$

$case\ I)\ if\ c^2 - 4mk > 0 \to$ *과감쇠*

$case\ II)\ if\ c^2 - 4mk = 0 \to$ *임계감쇠*

$case\ III)\ if\ c^2 - 4mk < 0 \to$ *저감쇠*

비제차방정식과 일반해

$y'' + p(x)y' + q(x)y = r(x)$

$y(x) = y_h(x) + y_p(x)$

미정계수법

$r(x)$ 의 항	y_p에 대한 선택
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_0$
$k\,cos\,\omega x, k\,sin\,\omega x$	$K\,cos\,\omega x + M\,sin\,\omega x$
$ke^{\alpha x}cos\,\omega x, ke^{\alpha x}sin\,\omega x$	$e^{\alpha x}(K\,cos\,\omega x + M\,sin\,\omega x)$

미정계수법을 위한 법칙

기본법칙 - 비제차방정식에서 $r(x)$가 미정계수법의 열에 있는 함수 중의 하나라면, 대응하는 함수 y_p를 선택하고, y_p와 그 도함수를 비제차방정식에 대입하여 미정계수를 결정.

변형법칙 - y_p로 선택된 항이 비제차 방정식에 대응하는 제차방정식의 해가 된다면, x또는 x^2을 곱한다.

합법칙 - $r(x)$가 첫번째 열에 있는 함수의 합일 경우, 두 번째 열의 대응하는 줄에 있는 함수들의 합으로 ky_p를 선택한다.

비감쇠강제진동

주기적인 외력

$my'' + ky = F_0 \cos \omega t$

$\rightarrow y = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \,, \omega_0^2 = \frac{k}{m}$

맥놀이 현상

$my'' + ky = F_0 \cos \omega t$

$\rightarrow y = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$

$\rightarrow y = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$

공진

$my'' + ky = F_0 \cos \omega_0 t$

$\rightarrow y = C \cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t \sin \omega t$

오일러-코시 방정식

$\boldsymbol{x^2 y'' + axy' + by = 0}$

$y = x^m, y' = mx^{m-1}, y'' = m(m-1)x^{m-2}$

$x^2 y'' + axy' + by = \boldsymbol{m(m-1)x^m + amx^m + bx^m = 0}$

$\xrightarrow{divide\ x^m} m(m-1) + am + b = \boldsymbol{m^2 + (a-1)m + b = 0}$

$case\ I)\ if\ two\ real\ roots$

$y = c_1 x^{m_1} + c_2 x^{m_2}$

$case\ II)\ if\ a\ real\ double\ root$

$y = (c_1 + c_2 \ln x) x^m$

$case\ III)\ if\ complex\ conjungate\ roots$

$y = x^\mu [A \cos(v \ln x) + B \sin(v \ln x)], m = \mu \pm iv$

전기회로

$E_R = Ri, \qquad E_L = L\frac{di}{dt}, \qquad E_C = \frac{1}{C}\int idt$

$E_R + E_L + E_C = E(t)$

$\rightarrow L\frac{dl}{dt} + RI\frac{1}{C}\int Idt = E(t)$

$\rightarrow L\frac{d^2 l}{dt^2} + R\frac{dl}{dt} + \frac{1}{C}I = E'(t)$

매개변수 변환법

$y'' + p(x)y' + q(x)y = r(x)$

$\rightarrow y_h = c_1 y_1 + c_2 y_2, \qquad y_p = u(x)y_1 + v(x)y_2$

$\xrightarrow{\boldsymbol{u'y_1 + v'y_2 = 0}} u'y_1' + v'y_2' = r(x)$

$\xrightarrow{\boldsymbol{W = y_1 y_2' - y_2 y_1'}} u' = -\frac{y_2 r}{W}, \qquad v' = \frac{y_2 r}{W}$

$u = -\int \frac{y_2 r}{W} dx\,, \qquad v = \int \frac{y_1 r}{W} dx$

$\therefore y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

고계 선형상미분방정식