- **Probability Density Function (PDF) of a RV X**

$$\text{Probability of } [a \le X \le b] = \int_a^b f_X(x)dx$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

- **Cumulative Density Function (CDF) of a RV X**

$$\text{Probability of } [a \le X \le b] = F_X(b) - F_X(a)$$

$$F_X(x) \equiv \Pr[X \le x] = \int_{-\infty}^{x} f_X(x)\,dx$$

- **Joint PDF of (X, Y)**

If X and Y are independent RVs, then

$$f_{XY} = f_X(x) f_Y(y)$$

Let $X_3 = X_1 + X_2$, and X_1 and X_2 are independent, then PDF of X_3 is given as

$$f_{X_3} = f_{X_1}(x) * f_{X_2}(x)$$

- **Expectation (or called 'Mean' or 'Average')**

Expectation of a RV X is denoted by $E(X)$ or m_X or $\overline{X}$

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

For another RV Y which is a function of X, i.e., $Y = g(X)$

$$\mathrm{E}(Y) = \mathrm{E}\big(g(X)\big) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

- **Variance**

Variance of a RV X is denoted by σ_X^2 or v_{X}

$$\begin{aligned}\sigma_X^2 &\equiv \mathrm{E}((X - m_X)^2) = \int_{-\infty}^{\infty} (x - m_X)^2 f_X(x)\,dx \\ &= \int_{-\infty}^{\infty} (x^2 - 2x m_X + m_X^2) f_X(x)\,dx \\ &= \int_{-\infty}^{\infty} x^2 f_X(x)dx - 2m_X \int_{-\infty}^{\infty} x f_X(x)\,dx \\ &\quad + m_X^2 \int_{-\infty}^{\infty} f_X(x)\,dx = E(X^2) - m_X^2\end{aligned}$$

- **Scaling, constant addition, and sum of RVs**

If $Y = aX + b$, where X is a RV, a and b are constants, then

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty} (ax + b) f_X(x)\,dx = a m_X + b$$

$$\begin{aligned}\sigma_Y^2 &= \int_{-\infty}^{\infty} (ax + b - m_Y)^2 f(x)\,dx \\ &= \int_{-\infty}^{\infty} (ax + b - a m_X - b)^2 f_X(x)\,dx \\ &= \int_{-\infty}^{\infty} a^2 (x - m_Y)^2 f(x)\,dx = a^2 \sigma_X^2\end{aligned}$$

If $Z = aX + bY$, where X and Y are RVs, a and b are constants, then

$$\begin{aligned}\mathrm{E}(Z) &= \mathrm{E}(aX + bY) = \mathrm{E}(aX) + \mathrm{E}(bY) \\ &= a\mathrm{E}(X) + b\mathrm{E}(Y)\end{aligned}$$

$$\begin{aligned}\sigma_Z^2 &= \mathrm{E}(Z^2) - m_Z^2 \\ &= \mathrm{E}(a^2X^2 + 2abXY + b^2Y^2) \\ &\quad - (a^2 m_X^2 + 2ab m_X m_Y + b^2 m_Y^2) \\ &= a^2(\mathrm{E}(X^2) - m_X^2) + b^2(\mathrm{E}(Y^2) - m_Y^2) \\ &\quad + 2ab(\mathrm{E}(XY) - m_X m_Y) \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab(\mathrm{E}(XY) - m_X m_Y)\end{aligned}$$

Where $\mathrm{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dxdy$ and $f_{XY}(x,y)$ denotes the joint PDF of (X,Y). If X and Y independent, $f_{XY}(x,y) = f_X(x) f_Y(y)$.

So,

$$\begin{aligned}\mathrm{E}(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dxdy \\ &= \int_{-\infty}^{\infty} x f_X(x)\,dx \int_{-\infty}^{\infty} y f_Y(y)\,dy = m_X m_Y\end{aligned}$$

Finally,

$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2$$

- **Weighted sum of RVs**

If Y is a weighted sum of RVs, i.e. $Y = \sum_{K=0}^{N} a_k X_k$ where X_k are RVs and a_k are their weights, then

$$\mathrm{E}(Y) = \sum_{k=0}^{N} a_k m_{X_k}$$

if X_k are independent,

$$\sigma_Y^2 = \sum_{k=0}^{N} a_k^2 \sigma_{X_k}^2$$

If X_k are **Independent and Identically Distributed (IID)** RVs and we consider their sample mean $Y = \frac{1}{N}\sum_{k=0}^{N} a_k X_k$ then, this is a special case of weighted sum, i.e., all the weights a_k are equal to $1/N$ and

$$\mathrm{E}(Y) = \sum_{k=0}^{N} a_k m_{X_k} = \sum_{k=0}^{N} \frac{1}{N} m_{X_k} = \frac{1}{N}\sum_{k=0}^{N} m_{X_k} = \frac{1}{N} \cdot N m_{X_k} = m_{X_k}$$

$$\sigma_Y^2 = \sum_{k=0}^{N} a_k^2 \sigma_{X_k}^2 = \sum_{k=0}^{N} \left(\frac{1}{N}\right)^2 \sigma_{X_k}^2 = \left(\frac{1}{N}\right)^2 \sum_{k=0}^{N} \sigma_{X_k}^2 = \left(\frac{1}{N}\right)^2 \cdot N\sigma_{X_k}^2 = \frac{1}{N}\sigma_{X_k}^2$$

- **PDF of a Uniform RV X**

$$f_X(x) = \begin{cases} 1/(A-B), & A \le x \le B \\ 0, & elsewhere \end{cases}$$

- **PDF of a Gaussian RV X**

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right) \equiv X \sim N(m, \sigma^2)$$

- **Rayleigh RV**

Consider a complex Gaussian RV Z, i.e.,

$$Z = X + jY$$

where $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \sigma^2)$ and they are independent.

Let α denote the magnitude of Z, i.e.,

$$\alpha = |Z| = \sqrt{X^2 + Y^2}$$

Then, α has a Rayleigh distribution given as

$$f_\alpha(x) = \begin{cases} \frac{x}{\sigma^2}\exp\left(\frac{-x}{2\sigma^2}\right), & x \ge 0 \\ 0, & x < 0 \end{cases}$$

- **Q-function**

Let X be a Gaussian RV such that $X \sim N(0,1)$, then the $Q(k)$ defined as the probability of X being larger than k is given as

$$Q(k) \equiv \Pr(X \ge k) = \int_k^\infty f_X(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_k^\infty \exp\left(\frac{-x^2}{2}\right) dx = 1 - Q(-k)$$

Let X be a Gaussian RV with general mean and variance, i.e., $X \sim N(m, \sigma^2)$, then probability of X being larger than t is given as

$$\Pr(X \ge t) = \int_t^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-m)^2}{2\sigma^2}\right) dx$$

Replacing x by $\sigma z + m$

$$\Pr(X \ge t) = \int_{(t-m)/\sigma}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right) dz = Q\left(\frac{t-m}{\sigma}\right)$$

The Q-function can be expressed in terms of the error function, as

$$Q(x) = \frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{2}}^\infty \exp(-t^2)\,dt\right) = \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$$

- **Gaussian RV**

Linear operation of Gaussian RV(s) results in another Gaussian RV.

If $Y = \sum_{k=0}^{N} a_k X_k$ where X_k are independent Gaussian RVs, then, Y is the Gaussian RV given as

$$Y \sim N\left(\sum_{k=0}^{N} a_k m_{X_k}, \sum_{k=0}^{N} a_k^2 \sigma_{X_k}^2\right)$$

- **Central Limit Theory (CLT)**

Consider a summation of RVs, i.e., $\mathrm{Y} = \sum_{k=0}^{N} X_k$

According to CLT, as N increases, the PDF of Y converges to Gaussian distribution.

Assume the case when X_k are independent, then

$$f_Y(x) = f_{X_1}(x) * f_{X_2}(x) * \cdots * f_{X_N}(x)$$

- **Random Process $X(t)$**

Each waveform(signal) $x_n(t)$ is corresponds to the outcome(observation) of $X(t)$. So, $x_n(t)$ is the n-th outcome of $X(t)$

- **Ensemble mean of Random Process $X(t)$**

$$E(X(t_0)) = \int_{-\infty}^{\infty} x f_{X(t_0)}(x)\,dx$$

where $f_{X(t_0)}(x)$ denotes the PDF of $X(t)$ at $t = t_0$

- **Ergodic Random Process $X(t)$**

There exist lots of ensemble mean for the different statics, i.e.

$$E(X(t_0)), E(X(t_0)^2), E(X(t_1)X(t_2)), \cdots$$

If all these ensemble means are equal to their time average versions, i.e.

$$E(X(t_0)) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_n(t)\,dt, \forall t_0$$
$$E(X(t_0)^2) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_n(t)^2\,dt, \forall t_0$$
$$\cdots$$

then, the Random Process $X(t)$ is said to be *ergodic*

- **Autocorrelation of a random process $X(t)$**

$$R_X(t_1, t_2) = \mathrm{E}(X(t_1)X^*(t_2))$$

where $\mathrm{E}()$ denotes ensemble mean

- **Wide Sense Stationary (WSS)**

If the followings are satisfied, $X(t)$ is said to be *Wide Sense Stationary.*

1. $\mathrm{E}(X(t_0)) =$ constant irrespective of t_0
2. $R_X(t_1, t_2)$ is determined only by $t_2 - t_1$ not individually by t_1 or t_2, i.e., $R_X(t_1, t_2)$ is simply expressed as $R_X(\tau)$ where $\tau = t_2 - t_1$.

WSS process is not always ergodic, but all ergodic processes are WSS. So, ergodic process is the subset of WSS process.

- **Property of Autocorrelation for WSS process**

  - $R(0) = \mathrm{E}\left((X(t))^2\right) = \text{Power of the process}$ if ergodic.
  - $R(\tau) = \mathrm{E}(X(t_1)X^*(t_1+\tau)) = \mathrm{E}(X(t_1+\tau)X^*(t_1))^*$
    $= R(-\tau)^*$ magnitude: even fn., phase: odd fn.
    $= R(\tau)$ for real-valued processes.

- **Power Spectral Density (PSD)**

Fourier transform of $X(t)$ also randomly changes because $X(t)$ is a random process. So, we can't use Fourier transform of $X(t)$ for analyzing Fourier transform of $X(t)$ in the frequency domain.

However, we can calculate PSD of $X(t)$ as follows

$$\text{PSD } G_X(f) = \text{Fourier transform of } (R_X(\tau))$$

Note that $G_X(f)$ is **deterministic** but describes the spectrum of $X(t)$ in frequency domain.

- **Property of PSD**

  - $\int_{-\infty}^{\infty} G_X(f)\,df = \text{power of } X(t)$

$$\mathcal{F}^{-1}\{G_X(f)\}|_{\tau=0} = \int_{-\infty}^{\infty} G_X(f) e^{j2\pi\tau}\,df\Big|_{\tau=0}$$
$$= \int_{-\infty}^{\infty} G_X(f)\,df = R_X(0) = \text{power of } X(t)$$

  - Assume that a process $X(t)$ passes through a linear system. Then, the PSD of output process $Y(t)$ is given by

$$G_Y(f) = |H(f)|^2 G_X(f)$$

where $H(f) = \text{frequency transform function}$ of the linear system.

- **Gaussian Random Process**

A random process $X(t)$ is a Gaussian random process if for all k and all time instances $(t_1, t_2, \cdots, t_k)$, the random variable set $\{X(t_1), X(t_2), \cdots, X(t_k)\}$ have a jointly Gaussian PDF.

- **Theorem of Gaussian Random Process**

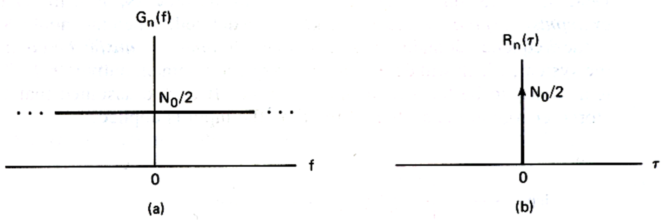
  - Any sample point from a Gaussian random process, i.e., $X(t_0)$ for any time instance t_0 has a Gaussian PDF.

  - For Gaussian processes, knowledge of the mean and autocorrelation fives a complete statistical description of the process.

  - If the Gaussian process is passed through a linear system, then the output process will be another Gaussian process.

- **White Process**

  A Process $N(t)$ is called a white process if it has a flat PSD, i.e., $G_N(f)$ is a constant for all f.



- **Additive White Gaussian Noise (AWGN)**

  The term AWGN self-explains the characteristics of the background noise cause the **background noise** is Additive, White and Gaussian process

- **Decision Variable** z

  Sample $z(t)$ at $t = T$ to generate the decision variable z

$$z(= z(t)|_{t=T}) = \begin{cases} a_1 + n_0, & \text{for bit '1'} \\ a_2 + n_0, & \text{for bit '0'} \end{cases}$$

  Signal component $a_i = a_i(t = T)$, Noise component $n_0 = \mathrm{n}_0(t = T)$ is a Gaussian RV$\sim N(0, \sigma_0^2)$. So,

$$z = \begin{cases} N(a_1, \sigma_0^2), & \text{for bit '1'} \\ N(a_2, \sigma_0^2), & \text{for bit '0'} \end{cases}$$

- **Maximum Likelihood Detection**

$$\hat{d} = \begin{cases} '1', & \text{if } \Pr('1'|z) > \Pr('0'|z) \\ '2', & \text{if } \Pr('1'|z) < \Pr('0'|z) \end{cases}$$
$$= \begin{cases} '1', & \text{if } \Lambda(z) > 1 \\ '2', & \text{if } \Lambda(z) < 1 \end{cases} \quad \text{where } \Lambda(z) \triangleq \frac{\Pr('1'|z)}{\Pr('0'|z)}$$

  $\Lambda(z)$ is called 'Likelihood ratio'

- **Probability of ML Detection Error**

$$P_b = 0.5 \Pr(\hat{d} = '1' | d = '0') + 0.5 \Pr(\hat{d} = '0' | d = '1')$$

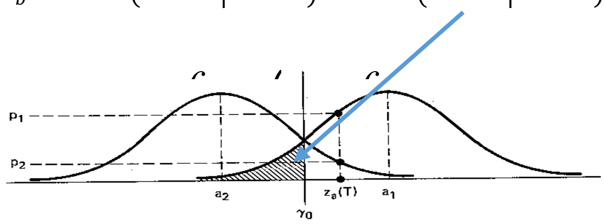


Figure Conditional probability density functions: $p(z|s_1)$ and $p(z|s_2)$.

자세한 건 LAB16 참고

- **Inner Product of Two Waveforms $x(t)$ and $y(t)$**

  over $t = [0 \quad T]$, is

$$\langle x(t), y(t) \rangle = \int_0^T x(t) y^*(t)\, dt$$

- **Orthogonal Signal Set**

  A set of N waveforms, $\{\psi_1(t), \psi_2(t), \cdots, \psi_N(t)\}$ is said to form an orthogonal set over $t = [0 \quad T]$,

$$\langle \psi_j(t), \psi_k(t) \rangle = \int_0^T \psi_j(t) \psi_k^*(t)\, dt = \begin{cases} K_j, & \text{if } j = k \\ x, & \text{if } j \neq k \end{cases}$$

  K_j is the Energy of j-th waveform. $\psi_j(t)$ and $\psi_k(t)$ with $j \neq k$ are said to be orthogonal each other.

- **Orthonormal Set**

  If $K_j = 1, \forall j$ of Orthogonal Set, then the set is said to be and orthonormal set.

- **Signal Generation using Orthonormal Set**

$$s_{\mathrm{i}}(t) = a_{i1}\psi_1(t) + a_{i2}\psi_2(t) + \cdots + a_{iN}\psi_N(t)$$
$$= \sum_{k=1}^{N} a_{ik} \psi_N(t)$$

- **Signal Vector Space**

  The waveform $s_{\mathrm{i}}(t)$ is mapped into a vector $\mathrm{s_i}$ in the N -dimensional space spanned by $\{\psi_1(t), \psi_2(t), \cdots, \psi_N(t)\}$

$$s_{\mathrm{i}}(t) \to \mathrm{s_i} = (a_{i1}, a_{i2}, \cdots, a_{iN})$$

- **Find Axis Components**

$$\langle v, \overrightarrow{u_x} \rangle = \langle a_x \overrightarrow{u_x} + a_y \overrightarrow{u_y} + a_z \overrightarrow{u_z}, \overrightarrow{u_x} \rangle$$
$$= \langle a_x \overrightarrow{u_x}, \overrightarrow{u_x} \rangle + \langle a_y \overrightarrow{u_y}, \overrightarrow{u_x} \rangle + \langle a_z \overrightarrow{u_z}, \overrightarrow{u_x} \rangle$$
$$= a_x \langle \overrightarrow{u_x}, \overrightarrow{u_x} \rangle + a_y \langle \overrightarrow{u_y}, \overrightarrow{u_x} \rangle + a_z \langle \overrightarrow{u_z}, \overrightarrow{u_x} \rangle = a_x$$

$$\therefore a_{ik} = \langle s_{\mathrm{i}}(t), \psi_k(t) \rangle = \int_0^T s_i(t) \psi_k^*(t)\, dt$$

- **Equivalence Properties**

	Waveform Space	Vector Space
Signal representation	$s_i(t)$	$s_i = (a_{i1}, a_{i2}, \cdots, a_{iN})$
Energy	$\int_0^T \|s_i(t)\|^2\, dt$	$\|s_i\|^2$
Difference Energy	$\int_0^T \left\|s_j(t) - s_k(t)\right\|^2 dt$	$\left\|s_j - s_k\right\|^2$
Correlation (Inner product)	$\int_0^T s_i(t)s_k^*(t)\, dt$	$\langle s_j, s_k\rangle$
Orthogonality	$\int_0^T s_i(t)s_k^*(t)\, dt = 0$	$\langle s_j, s_k\rangle = 0$

- **Proof of** $\int_0^T s_i(t)s_k^*(t)\, dt = \langle s_j, s_k\rangle$

$$s_j(t) = a_{j1}\psi_1(t) + a_{j2}\psi_2(t) + \cdots + a_{jN}\psi_N(t)$$
$$s_k(t) = a_{k1}\psi_1(t) + a_{k2}\psi_2(t) + \cdots + a_{kN}\psi_N(t)$$
$$\int_0^T s_j(t)s_k^*(t)\, dt$$
$$= \int_0^T \left(a_{j1}\psi_1(t) + a_{j2}\psi_2(t) + \cdots + a_{jN}\psi_N(t)\right)\left(a_{k1}\psi_1(t) + a_{k2}\psi_2(t) + \cdots + a_{kN}\psi_N(t)\right)^* dt$$
$$= \int_0^T \left(a_{j1}a_{k1}|\psi_1(t)|^2 + a_{j1}a_{k2}\psi_1(t)\psi_2(t)^* + \cdots + a_{j1}a_{kN}(t)\psi_1(t)\psi_N(t)^*\right)$$
$$+ \left(a_{j2}a_{k1}\psi_2(t)\psi_1(t)^* + a_{j2}a_{k2}|\psi_2(t)|^2 + \cdots + a_{j2}a_{kN}(t)\psi_2(t)\psi_N(t)^*\right) + \cdots$$
$$+ \left(a_{jN}a_{k1}\psi_N(t)\psi_1(t)^* + a_{jN}a_{k2}\psi_N(t)\psi_2(t)^* + \cdots + a_{jN}a_{kN}(t)|\psi_N(t)|^2\right)dt$$
$$= a_{j1}a_{k1} + a_{j2}a_{k2} + \cdots + a_{jN}a_{kN} = \langle s_1, s_2\rangle$$

- **Proof of** $E = |s_i|^2$

$$E_i = \int_0^T |s_i(t)|^2\, dt = \int_0^T \left|\sum_j^N a_{ij}\psi_j(t)\right|^2 dt$$
$$= \int_0^T \sum_j^N a_{ij}\psi_j(t)\left(\sum_k^N a_{ik}\psi_k(t)\right)^* dt$$
$$= \sum_j^N \sum_k^N a_{ij}a_{ik}\int_0^T \psi_j(t)\psi_k(t)$$
$$= \sum_j^N \sum_k^N a_{ij}a_{ik}\delta_{jk} = \sum_j^N a_{ij}^2 = |s_i|^2$$

- **ML Detection in Vector Space**

$$\hat{m} = \underset{i}{\operatorname{argmin}}|Z - s_i|$$

- **AWGN in Signal Vector Space**

AWGN signal $n(t)$ is mapped into a vector n in the signal vector space.

$$n(t) \to \mathrm{n} = (n_1, n_2, \cdots, n_N)$$

where $n_i = \int_0^T n(t)\psi_i^*(t)\, dt$

➢ $n_1, n_2, \cdots, n_N$ are Gaussian RVs.

➢ Specifically, $n_k \sim N\left(0, \frac{N_0}{2}\right)$

➢ $n_1, n_2, \cdots, n_N$ are i.i.d.

- **Proof of** $n_k \sim N\left(0, \frac{N_0}{2}\right)$

$$\mathrm{E}[n_k] = \mathrm{E}\left[\int_0^T n(t)\psi_k^*(t)\, dt\right] = \int_0^T \mathrm{E}[n(t)\psi_k^*(t)]\, dt$$
$$= \int_0^T \mathrm{E}[n(t)]\psi_i^*(t)\, dt = \int_0^T 0\cdot\psi^*(t)\, dt = 0$$

$$\mathrm{V}[n_k] = \mathrm{E}[n_k n_k^*]$$
$$= \mathrm{E}\left[\int_0^T n(t)\psi_k^*(t)\, dt \int_0^T n^*(x)\psi_k(x)\, dx\right]$$
$$= \mathrm{E}\left[\int_0^T\int_0^T n(t)\psi_k^*(t)n^*(x)\psi_k(x)\, dtdx\right]$$
$$= \int_0^T\int_0^T \mathrm{E}[n(t)n^*(x)]\psi_k^*(t)\psi_k(x)\, dtdx$$
$$= \int_0^T\int_0^T \frac{N_0}{2}\delta(t-x)\psi_k^*(t)\psi_k(x)\, dtdx$$
$$\because \mathrm{E}[n(t)n^*(x)] = R_n(t-x) = \frac{N_0}{2}\delta(t-x)$$
$$= \frac{N_0}{2}\int_0^T\int_0^T \psi_k^*(x)\psi_k(x)\, dtdx = \frac{N_0}{2}$$

- **Proof of** $n_1, n_2, \cdots, n_N$ independent

$$\mathrm{E}[n_j n_k^*] = \mathrm{E}\left[\int_0^T n(t)\psi_j^*(t)\, dt \int_0^T n^*(x)\psi_k(x)\, dx\right]$$
$$= \left[\int_0^T\int_0^T n(t)\psi_j^*(t)n^*(x)\psi_k(x)\, dtdx\right]$$
$$= \int_0^T\int_0^T \mathrm{E}[n(t)n^*(x)]\psi_j^*(t)\psi_k(x)\, dtdx$$
$$= \int_0^T\int_0^T \frac{N_0}{2}\delta(t-x)\psi_j^*(t)\psi_k(x)\, dtdx$$
$$= \frac{N_0}{2}\int_0^T\int_0^T \psi_j^*(x)\psi_k(x)\, dtdx = 0$$

Independent if Gaussian and no correlation.

- **Correlator-based ML detection**

$$\int_0^T |r(t) - s_k(t)|^2\, dt$$
$$= \int_0^T \left(r(t) - s_k(t)\right)^* \left(r(t) - s_k(t)\right) dt$$
$$= \int_0^T |r(t)|^2 - r^*(t)s_k(t) - r(t)s_k(t)^* + |s_k(t)|^2\, dt$$
$$= \int_0^T |r(t)|^2 - (r(t)s_k(t)^*)^* - r(t)s_k(t)^*$$
$$+ |s_k(t)|^2\, dt$$
$$= \int_0^T |r(t)|^2 - 2\mathrm{Re}(r(t)s_k(t)^*) + |s_k(t)|^2\, dt$$
$$= E_{r(t)} - 2\mathrm{Re}\left(\int_0^T r(t)s_k(t)^*\, dt\right) + E_{s_k(t)}$$

if all M-ary symbols have identical energy, i.e.,

$$E_{s_k(t)} = E \forall k$$

$$\hat{m} = \underset{k}{\operatorname{argmin}} \int_0^T |r(t) - s_k(t)|^2\, dt$$
$$= \underset{k}{\operatorname{argmin}} \left[E_{r(t)} - 2\mathrm{Re}\left(\int_0^T r(t)s_k(t)^*\, dt\right) + E_{s_k(t)}\right]$$
$$= \underset{k}{\operatorname{argmin}} \left[-2\mathrm{Re}\left(\int_0^T r(t)s_k(t)^*\, dt\right)\right] \because E_{s_k(t)} = E\forall k$$
$$= \underset{k}{\operatorname{argmax}}\, \mathrm{Re}\left[\int_0^T r(t)s_k(t)^*\, dt\right]$$
$$= \underset{k}{\operatorname{argmax}} \int_0^T r(t)s_k(t)\, dt = \underset{k}{\operatorname{argmax}} \langle \mathrm{Z}, \mathrm{s}_k \rangle$$

- **Antipodal signaling**

$$s_2(t) = -s_1(t)$$

- ML Detection of Binary Signals

$$\hat{b} = \begin{cases} 0, & \text{if} \int_0^T r(t)s_1^*(x)\, dx > \int_0^T r(t)s_2^*(x)\, dx \\ 1, & \text{if} \int_0^T r(t)s_1^*(x)\, dx < \int_0^T r(t)s_2^*(x)\, dx \end{cases}$$

if $s_2(t) = -s_1(t)$

$$\int_0^T r(t)s_1^*(x)\, dx > \int_0^T r(t)s_2^*(x)\, dx$$
$$= \int_0^T r(t)s_1^*(x)\, dx > -\int_0^T r(t)s_1(x)\, dx$$
$$= \int_0^T r(t)s_1^*(x)\, dx > 0$$

$$\therefore \hat{b} = \begin{cases} 0, & \text{if} \int_0^T r(t)s_1^*(x)\, dx > 0 \\ 1, & \text{else} \end{cases}$$

$$r(t) \to \mathrm{Z} = \begin{cases} (\mathrm{s}_1 + n) \sim N(-\sqrt{E_b}, N_0/2), & \text{for } b = \text{'0'} \\ (\mathrm{s}_2 + n) \sim N(\sqrt{E_b}, N_0/2), & \text{for } b = \text{'1'} \end{cases}$$

- Bit Error Rate (BER) of MLD of Binary Signal

$$P_b = \frac{1}{2}\Pr[\hat{b} = \text{'1'} | \hat{b} = \text{'0'}] \frac{1}{2}\Pr[\hat{b} = \text{'0'} | \hat{b} = \text{'1'}]$$
$$= \frac{1}{2}Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

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