- **Correlation Function**

$$r_{fg}(\tau) = \int_{-\infty}^{\infty} f^*(t)g(t+\tau)dt = \int_{-\infty}^{\infty} g(t)f^*(t-\tau)dt$$

- **Parseval's Theorem**

$$\begin{aligned}\text{Energy of } f(t) \;\; E &= \int_{-\infty}^{\infty} |f(t)^2|dt = \int_{-\infty}^{\infty} f(t)f^*(t)dt \\ &= \int_{-\infty}^{\infty} f(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-j\omega t}d\omega\right)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(t)\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)F(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)^2|d\omega\end{aligned}$$

- **Energy Spectral Density (ESD)**

$$\text{Energy Spectral Density of } f(t) = |F(\omega)^2|$$

- **ESD = FT of Auto-Correlation Function (ACF)**

$$\begin{aligned}F\{r_f(\tau)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(t)f(t+\tau)dt\, e^{-j\omega\tau}d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t+\tau)e^{-j\omega\tau}d\tau\, f^*(t)dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}f^*(t)\,dt \\ &= F(\omega)\int_{-\infty}^{\infty} f^*(t)e^{j\omega t}dt \\ &= F(\omega)\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right)^* \\ &= F(\omega)F^*(\omega) = |F(\omega)|^2\end{aligned}$$

- **Properties of ACF and CCF**

$$r_f(0) = \int_{-\infty}^{\infty} f(t)^* f(t+0)dt = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$\begin{aligned}r_f(\tau) &= \int_{-\infty}^{\infty} f^*(t)f(t+\tau)dt = \int_{-\infty}^{\infty} f^*(x-\tau)f(x)dx \\ &= \left(\int_{-\infty}^{\infty} f^*(x)f(x-\tau)dx\right)^* = r_f^*(-\tau)\end{aligned}$$

- **Power Signal**

$$\text{Power} = \lim_{\mathrm{T}\to\infty}\frac{1}{\mathrm{T}}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

If Power is not 0, we call $f(t)$ is power signal

- **ACF of Power Signal**

$$R_f(\tau) = \lim_{\mathrm{T}\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)f(t+\tau)dt$$

- **Power Spectral Density (PSD)**

$$\text{Power Spectral Density } \; S(\omega) = \text{FT of } \; R_f(\tau)$$

- **ACF of Power Signal of Periodic Function**

$$\begin{aligned}R_f(\tau) &= \lim_{\mathrm{T}\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)f(t+\tau)dt \\ &= \lim_{\mathrm{n}\to\infty}\frac{1}{nP}\int_{-n\frac{P}{2}}^{n\frac{P}{2}} f^*(t)f(t+\tau)dt \\ &= \lim_{\mathrm{n}\to\infty}\frac{1}{nP}n\int_{-\frac{P}{2}}^{\frac{P}{2}} f^*(t)f(t+\tau)dt \\ &= \lim_{\mathrm{n}\to\infty}\frac{1}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f^*(t)f(t+\tau)dt \\ &= \frac{1}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f^*(t)f(t+\tau)dt\end{aligned}$$

$$\begin{aligned}R_f(\tau+P) &= \lim_{\mathrm{T}\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)f(t+\tau+P)dt \\ &= \lim_{\mathrm{T}\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^*(t)f(t+\tau)dt = R_f(\tau)\end{aligned}$$

- **Double Side-Band Suppressed Carrier (DSB-SC)**

$$\begin{aligned}&\Phi(t) = f(t)\cos(\omega_c t) \\ &g(t) = \Phi(t)\cos(\omega_c t) = f(t)\cos^2(\omega_c t) \\ &\qquad = f(t)\frac{1+\cos(2\omega_c t)}{2} \\ &\qquad = \frac{1}{2}f(t) + \frac{1}{2}f(t)\cos(2\omega_c t) \\ &G(\omega) = \frac{1}{2}F(\omega) + \frac{1}{4}F(\omega - 2\omega_c) + \frac{1}{4}F(\omega + 2\omega_c) \\ &\mathrm{LPF}\{G(\omega)\} = \frac{1}{2}F(\omega)\end{aligned}$$

- **DSB-SC Phase Error**

$$\begin{aligned}g(t) = \Phi(t)\cos(\omega_c t + \theta) &= f(t)\cos(\omega_c t)\cos(\omega_c t + \theta) \\ &= f(t)\cdot\frac{1}{2}(\cos(2\omega_c t + \theta) + \cos(-\theta)) \\ &= \frac{1}{2}f(t)\cos(2\omega_c t + \theta) \\ &\quad + \frac{1}{2}f(t)\cos(\theta)\end{aligned}$$

- **DSB-SC Phase Error with Noise**

$$\begin{aligned}g(t) &= \Phi(t)\cos(\omega_c t + \theta) \\ &= \big(f(t)\cos(\omega_c t) + n(t)\big)\cos(\omega_c t + \theta) \\ &= (f(t)\cos(\omega_c t))\cos(\omega_c t + \theta) \\ &\quad + n(t)\cos(\omega_c t + \theta)\end{aligned}$$

$$\begin{aligned}R_{n_\theta}(\tau = 0) &= \lim_{\mathrm{T}\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |(n(t)\cos(\omega_c t + \theta)|^2 dt \\ &= \frac{1}{2}R_n(\tau)\end{aligned}$$

- **PSD of $n_\theta(t)$**

$$S_{n_\theta} = \mathcal{F}\{R_{n_\theta}(\tau)\} = \mathcal{F}\left\{\frac{1}{2}R_n(\tau)\right\} = \mathcal{F}\left\{\frac{N_0}{4}\delta(\tau)\right\} = \frac{N_0}{4}$$

- **Signal to Noise Power Ratio (SNR)**

$$\begin{aligned}\mathrm{SNR}(\theta) &= \frac{\text{Signal Power}}{\text{Noise Power}} = \frac{\text{Power of } \frac{1}{2}f(t)\cos(\theta)}{\text{Power of } \ \mathrm{LPF}\{n\_\theta(t)\}} \\ &= \left(\frac{\cos(\theta)}{2}\right)^2 \frac{\text{Power of } f(t)}{\text{Power of } \ \mathrm{LPF}\{n_\theta(t)\}}\end{aligned}$$

- **DSB-SC Frequency Error**

$$\begin{aligned}g(t) &= \Phi(t)\cos\big((\omega_c + \Delta\omega)t\big) = \Phi(t)\cos(\omega_c t + \Delta\omega t) \\ &= \Phi(t)\cos(\omega_c t + \theta)|_{\theta=\Delta\omega t}\end{aligned}$$

- **Generating AM signal without Oscillator**

Modulate by Sampling and Demodulate by BPF

- **Quadrature Multiplexing (QM)**

$$\begin{aligned}\Phi(t) &= f_1(t)\cos(\omega_c t) + f_2(t)\sin(\omega_c t) \\ g_1(t) &= \Phi(t)\cos(\omega_c t) \\ &= (f_1(t)\cos(\omega_c t) \\ &\quad + f_2(t)\sin(\omega_c t))\cos(\omega_c t) \\ &= f_1(t)\cos^2(\omega_c t) \\ &\quad + f_2(t)\sin(\omega_c t)\cos(\omega_c t) \\ &= f_1(t)\frac{1+\cos(2\omega_c t)}{2} \\ &\quad + f_2(t)\frac{\sin(2\omega_c t)}{2} \\ G_1(\omega) &= \frac{1}{2}F_1(\omega) + \frac{1}{4}\big(F_1(\omega - 2\omega_c) + F_1(\omega + 2\omega_c)\big) \\ &\quad + \frac{1}{j4}\big(F_2(\omega - 2\omega_c) - F_2(\omega + 2\omega_c)\big) \\ \mathrm{LPF}\{G_1(\omega)\} &= \frac{1}{2}F_1(\omega) \\ g_2(t) &= \Phi(t)\sin(\omega_c t) \\ &= (f_1(t)\cos(\omega_c t) \\ &\quad + f_2(t)\sin(\omega_c t))\sin(\omega_c t) \\ &= f_1(t)\cos(\omega_c t)\sin(\omega_c t) \\ &\quad + f_2(t)\sin^2(\omega_c t) \\ &= f_1(t)\frac{\sin(2\omega_c t)}{2} \\ &\quad + f_2(t)\frac{(1-\cos(2\omega_c t))}{2} \\ G_2(\omega) &= \frac{1}{2}F_2(\omega) + \frac{1}{\mathrm{j}4}\big(F_1(\omega - 2\omega_c) + F_1(\omega + 2\omega_c)\big) \\ &\quad - \frac{1}{4}\big(F_2(\omega - 2\omega_c) - F_2(\omega + 2\omega_c)\big) \\ \mathrm{LPF}\{G_2(\omega)\} &= \frac{1}{2}F_2(\omega)\end{aligned}$$

- **QM Phase Error**

$$\begin{aligned}g_1(t) &= \Phi(t)\cos(\omega_c t + \theta) \\ &= (f_1(t)\cos(\omega_c t) \\ &\quad + f_2(t)\sin(\omega_c t))\cos(\omega_c t + \theta) \\ &= f_1(t)\cos(\omega_c t)\cos(\omega_c t + \theta) \\ &\quad + f_2(t)\sin(\omega_c t)\cos(\omega_c t + \theta) \\ &= f_1(t)\frac{\cos(2\omega_c t + \theta) + \cos(-\theta)}{2} \\ &\quad + f_2(t)\frac{\sin(2\omega_c t + \theta) + \sin(-\theta)}{2} \\ \mathrm{LPF}\{g_1(t)\} &= \frac{1}{2}f_1(t)\cos(\theta) - \frac{1}{2}f_2(t)\sin(\theta)\end{aligned}$$

- **QM Frequency Error**

$$\begin{aligned}g_1(t) &= \Phi(t)\cos\big((\omega_c + \Delta\omega)t\big) = \Phi(t)\cos(\omega_c t + \Delta\omega t) \\ &= \Phi(t)\cos(\omega_c t + \theta)|_{\theta=\Delta\omega t} \\ \mathrm{LPF}\{g_1(t)\} &= \frac{1}{2}f_1(t)\cos(\Delta\omega t) - \frac{1}{2}f_2(t)\sin(\Delta\omega t)\end{aligned}$$

- **Frequency Division Multiplexing (FDM)**

$$\begin{aligned}\Phi(t) &= f_1(t)\cos(\omega_1 t) + f_2(t)\cos(\omega_2 t) \\ &\quad + f_3(t)\cos(\ \omega_3 t) + \cdots\end{aligned}$$

- **Hilbert Transform (HT)**

$$H(\omega) = j \cdot \mathrm{sgn}(\omega) = \begin{cases} -j, & \omega > 0 \\ \ \ j, & \omega < 0 \end{cases}$$

$$f(t) \xrightarrow{\text{Hilbert Transform}} \hat{f}(t)$$

$$F(\omega) \xrightarrow{\text{Hilbert Transform}} \hat{F}(\omega)$$

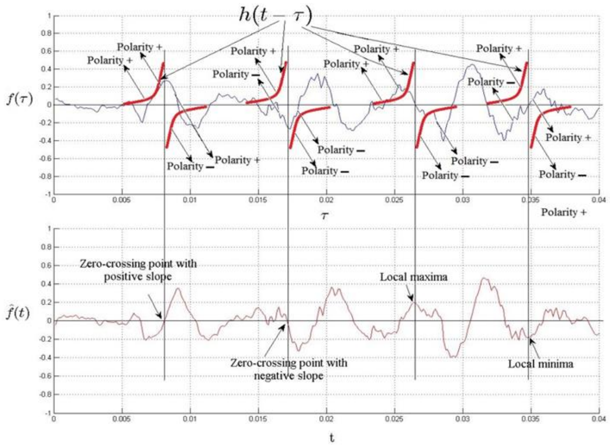
$$\hat{F}(\omega) = H(\omega)F(\omega) = \begin{cases} -jF(\omega), & \omega > 0 \\ \ \ jF(\omega), & \omega < 0 \end{cases}$$

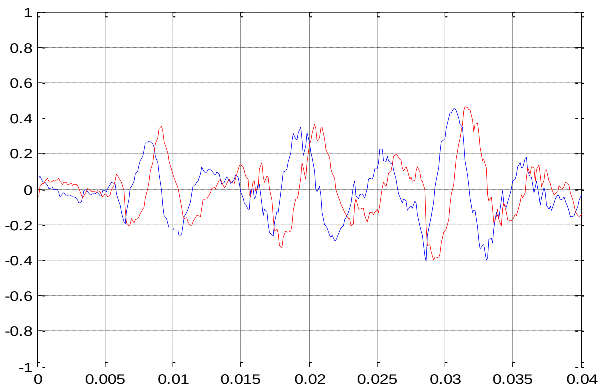
- **HT in Time Domain**

$$\begin{aligned}h(t) = \mathcal{F}^{-1}\{H(\omega)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} je^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} -je^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\left(\lim_{a\to+0}\int_{-\infty}^{0} je^{j\omega t}d\omega\right. \\ &\quad \left.+ \lim_{a\to-0}\int_{0}^{\infty} -je^{j\omega t}d\omega\right) \\ &= \frac{1}{2\pi}\left(\lim_{a\to+0}\frac{j}{jt+a}(1-0)\right. \\ &\quad \left.+ \lim_{a\to-0}\frac{-j}{jt+a}(0-1)\right) = \frac{1}{2\pi}\left(\frac{1}{t} + \frac{1}{t}\right) \\ &= \frac{1}{\pi t}\end{aligned}$$

$$\hat{F}(\omega) = H(\omega)F(\omega) \leftrightarrow \hat{f}(t) = h(t) * f(t) = \frac{1}{\pi t} * f(t)$$

- **HT is $\frac{\pi}{2}$ Phase Shifter**





- **Analytic Signal**

$$z(t) \triangleq \frac{1}{2}\left(f(t)+j\hat{f}(t)\right) \xrightarrow{\text{FT}} Z(\omega)=\mathcal{F}\{z(t)\}$$
$$=\frac{1}{2}\left(F(\omega)+j\hat{F}(\omega)\right)$$
$$=\begin{cases}\frac{1}{2}\left(F(\omega)+j\left(-jF(\omega)\right)\right), & \omega>0\\ \frac{1}{2}\left(F(\omega)+j\left(jF(\omega)\right)\right), & \omega<0\end{cases}$$
$$=\begin{cases}F(\omega), & \omega>0\\ 0, & \omega<0\end{cases}$$

- **Single-Side Band (SSB)**

  - **Upper SSB (USSB)**

$$\Phi_{USSB}(t)=2\,\mathrm{Re}\{z(t)e^{j\omega t}\}$$
$$=2\,\mathrm{Re}\left\{\frac{1}{2}\left(f(t)+j\hat{f}(t)\right)(\cos(\omega_c t)+j\sin(\omega_c t))\right\}$$
$$=f(t)\cos(\omega_c t)-\hat{f}\sin(\omega_c t)$$

  - **Lower SSB (LSSB)**

$$\Phi_{LSSB}(t)=2\,\mathrm{Re}\{z(t)e^{-j\omega t}\}$$
$$=2\,\mathrm{Re}\left\{\frac{1}{2}\left(f(t)+j\hat{f}(t)\right)(\cos(\omega_c t)-j\sin(\omega_c t))\right\}$$
$$=f(t)\cos(\omega_c t)+\hat{f}\sin(\omega_c t)$$

- **Demodulation of SSB**

$$g(t)=\mathrm{LPF}\{\Phi_{USSB}(t)\cos(\omega_c t)\}$$
$$=\mathrm{LPF}\{(f(t)\cos(\omega_c t)-\hat{f}\sin(\omega_c t))\cos(\omega_c t)\}$$
$$=\mathrm{LPF}\{f(t)\cos^2(\omega_c t)-\hat{f}\sin(\omega_c t)\cos(\omega_c t)\}$$
$$=\mathrm{LPF}\left\{f(t)\frac{1-\cos(2\omega_c t)}{2}-\hat{f}\frac{\sin(2\omega_c t)}{2}\right\}=\frac{1}{2}f(t)$$

- **Instantaneous Frequency**

$$\omega_i(t)=\frac{d}{dt}\theta(t)$$

$$\int_0^t \omega_i(\tau)d\tau=\theta(t)-\theta(0)\rightarrow\theta(t)=\int_0^t\omega_i(\tau)d\tau+\theta(0)$$

- **Voltage Controlled Oscillator (VCO)**

Frequency of VCO output $VCO_O(t)$
= Quiescent Frequency f_Q
+ Input Sensitivity k_v × VCO Input x

Phase of $VCO_O(t)$ $\theta_{VCO}(t)=\int_0^t\omega_{VCO,i}(\tau)d\tau+\theta_0$
$$=\int_0^t\left(2\pi f_Q+2\pi k_v x(\tau)\right)d\tau+\theta_0$$
$$=2\pi f_Q t+2\pi k\int_0^t x(\tau)d\tau+\theta_0$$

$$\therefore VCO_O(t)=\cos\left(\theta_{VCO}(t)\right)$$
$$=\cos\left(2\pi f_Q t+2\pi k_v\int_0^t x(\tau)d\tau+\theta_0\right)$$
$$=\cos\left(\omega_c t+k_f\int_0^t f(\tau)d\tau+\theta_0\right)$$

- **Narrow Band FM (NBFM)**

$$\Phi_{FM}(t)=A\cos\left(\omega_c t+k_f\int_0^t f(\tau)d\tau\right)$$
$$=A\cos(\omega_c t)\cos\left(k_f\int_0^t f(\tau)d\tau\right)$$
$$-A\sin(\omega_c t)\sin\left(k_f\int_0^t f(\tau)d\tau\right)$$
$$\Phi_{NBFM}(t)=A\cos(\omega_c t)-Ak_f\sin(\omega_c t)\int_0^t f(\tau)d\tau$$
$$\because k_f\int_0^t f(\tau)d\tau\ll 1\ \ \text{(Carson's rule)}$$

- **Demodulation of NBFM**

$$g(t) = \frac{d}{dt}\mathrm{LPF}\{\Phi_{NBFM}(t)(-\sin(\omega_c t))\}$$

$$\begin{aligned}\Phi_{NBFM}(t)(-\sin(\omega_c t)) \\ &= -A\cos(\omega_c t)\sin(\omega_c t) \\ &+ Ak_f\sin^2(\omega_c t)\int_0^t f(\tau)d\tau \\ &= -\frac{1}{2}\sin(2\omega_c t) \\ &+ Ak_f\frac{1-\cos(2\omega_c t)}{2}\int_0^t f(\tau)d\tau \\ &= -\frac{1}{2}\sin(2\omega_c t) + \frac{1}{2}Ak_f\int_0^t f(\tau)d\tau \\ &-\frac{1}{2}Ak_f\cos(2\omega_c t)\int_0^t f(\tau)d\tau\end{aligned}$$

$$\therefore \mathrm{LPF}\{\Phi_{NBFM}(t)(-\sin(\omega_c t))\} = \frac{1}{2}Ak_f\int_0^t f(\tau)d\tau$$

$$\begin{aligned}g(t) &= \frac{d}{dt}\mathrm{LPF}\{\Phi_{NBFM}(t)(-\sin(\omega_c t))\} \\ &= \frac{d}{dt}\left(\frac{1}{2}Ak_f\int_0^t f(\tau)d\tau\right) = \frac{1}{2}Ak_f f(t)\end{aligned}$$

- **Phase Detector (PD)**

$$\begin{aligned}\mathrm{PD_O}(t) &= \mathrm{LPF}\{\mathrm{PLL_{IN}}(t)\,\mathrm{VCO_O}(t)\} \\ &= \mathrm{LPF}\{\sin(\omega_0 t + \theta_{IN}) \\ &\cdot\cos(\omega_0 t + \theta_{VCO})\} \\ &= \mathrm{LPF}\left\{\frac{1}{2}\sin(2\omega_0 t + \theta_{IN} + \theta_{VCO})\right. \\ &\left.+\frac{1}{2}\sin(\theta_{IN} - \theta_{VCO})\right\} \\ &= \frac{1}{2}\sin(\theta_{IN} - \theta_{VCO}) \approx \theta_{IN} - \theta_{VCO}\end{aligned}$$

- **Phase Locked Loop (PLL) with Frequency Error**

$$\begin{aligned}\mathrm{PD_O}(t) &= \mathrm{LPF}\{\mathrm{PLL_{IN}}(t)\,\mathrm{VCO_O}(t)\} \\ &= \mathrm{LPF}\{\sin\big((\omega_0 + \Delta\omega)t\big)\cdot\cos(\omega_0 t)\} \\ &= \mathrm{LPF}\left\{\frac{1}{2}(\sin(2\omega_0 t + \Delta\omega t)\right. \\ &\left.+\sin(\Delta\omega t))\right\} = \frac{1}{2}\sin(\Delta\omega t) \approx \Delta\omega t\end{aligned}$$

$$\int_0^t \Delta\omega\tau d\tau =$$

- **PLLL as an FM Demodulator**

$$\mathrm{VCO\ input} = \frac{\mathrm{Frequency\ Error}}{\mathrm{Input\ Sensitivity}} = \frac{k_f}{2\pi k_v}f(t)$$

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