- **Gaussian Elimination Method**

1. Reduction to upper-triangular form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases}$$
$$\rightarrow \begin{cases} a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n = b'_1 \\ a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2 \\ \vdots \\ a'_{n2}x_2 + \cdots + a'_{nn}x_n = b'_n \end{cases} \rightarrow \cdots$$
$$\rightarrow \begin{cases} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n = d_1 \\ c_{22}x_2 + \cdots + c_{2n}x_n = d_2 \\ \vdots \\ c_{nn}x_n = d_n \end{cases}$$

2. Backward substitution

The Gaussian elimination method fails if any one of the pivot elements becomes zero. In such a situation, we rewrite the equations in a different order to avoid zero pivots. Changing the order of equations is called **pivoting**.

- **Gauss-Jordon Elimination Method**

Partial pivoting is also used whenever the pivot element becomes zero.

Example

$$\begin{cases} 10x + y + z = 12 \\ 2x + 10y + z = 13 \\ x + y + 5z = 7 \end{cases}$$

$$(A|B) = \left(\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \\ 0 & 9 & 49 & 58 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 26 & 89 & 115 \\ 0 & 9 & 49 & 58 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 1 & -8 & -44 & -51 \\ 0 & 1 & 58 & 59 \\ 0 & 9 & 49 & 58 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 420 & 421 \\ 0 & 1 & 58 & 59 \\ 0 & 0 & -473 & -473 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \therefore x = y = z = 1$$

- **Crout's Reduction Method**

$$[A][X] = [C]$$

1. Decompose $[A]$ into $[L]$ and $[U]$.

2. Solve $[L][Z] = [C]$ for $[Z]$.

3. Solve $[U][X] = [Z]$ for $[X]$.

In this method, the coefficient matrix $[A]$ of the system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases}$$

is decomposed into the product of two matrices $[U]$ and $[L]$.

$$[A] = [L][U] = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

Where, $[U]$ is an *upper-triangular matrix* and $[L]$ is a *lower-triangular matrix* with 1's on its main diagonal.

$$[U] = \begin{pmatrix} c_{11}x_1 & c_{12}x_2 & c_{13}x_3 \\ 0 & c_{22}x_2 & c_{23}x_3 \\ 0 & 0 & c_{33}x_3 \end{pmatrix}$$

$[U]$ is the same as the coefficient matrix at the end of forward elimination step.

$$[L] = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}/a_{11} & 0 & 0 \\ a_{31}/a_{11} & a'_{23}/a'_{22} & 0 \end{pmatrix}$$

$[L]$ is obtained using the multipliers that were used in the forward elimination process.

- **Jacobi's Method**

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum\nolimits_{j=1, j\neq i}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(r)}$$

$$\begin{cases} x_1^{(r+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(r)} \\ x_2^{(r+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(r)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(r)} \\ \vdots \\ x_n^{(r+1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(r)} - \cdots - \frac{a_{n(n-1)}}{a_{nn}} x_{n-1}^{(r)} \end{cases}$$

Example

Find the solution to the following system of equations using Jacobi's iterative method. Taking the initial starting of solution vector as $(0,0,0)^T$

$$\begin{cases} 83x + 11y - 4z = 95 \\ 7x + 52y + 13z = 104 \\ 3x + 8y + 29z = 71 \end{cases}$$

$$\rightarrow \begin{cases} x^{(1)} = \frac{95}{83} - \frac{11}{83} y^{(0)} + \frac{4}{83} z^{(0)} \\ y^{(1)} = \frac{104}{52} - \frac{7}{52} x^{(0)} - \frac{13}{52} z^{(0)} \\ z^{(1)} = \frac{71}{29} - \frac{3}{29} x^{(0)} - \frac{8}{29} y^{(0)} \end{cases} \rightarrow$$

$$\rightarrow \begin{cases} x^{(1)} = \frac{95}{83} - \frac{11}{83} \cdot 0 + \frac{4}{83} \cdot 0 \\ y^{(1)} = \frac{104}{52} - \frac{7}{52} \cdot 0 - \frac{13}{52} \cdot 0 \\ z^{(1)} = \frac{71}{29} - \frac{3}{29} \cdot 0 - \frac{8}{29} \cdot 0 \end{cases} = \begin{cases} x^{(1)} = 1.1446 \\ y^{(1)} = 2.0000 \\ z^{(1)} = 2.4483 \end{cases}$$

$$\rightarrow \begin{cases} x^{(2)} = \frac{95}{83} - \frac{11}{83} y^{(1)} + \frac{4}{83} z^{(1)} \\ y^{(2)} = \frac{104}{52} - \frac{7}{52} x^{(1)} - \frac{13}{52} z^{(1)} \\ z^{(2)} = \frac{71}{29} - \frac{3}{29} x^{(1)} - \frac{8}{29} y^{(1)} \end{cases}$$

$$= \begin{cases} x^{(2)} = 0.9976 \\ y^{(2)} = 1.2339 \\ z^{(2)} = 1.7424 \end{cases} \rightarrow \cdots$$

- **Gauss-Seidel Iteration Method**

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum\nolimits_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(r+1)} - \sum\nolimits_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(r)}$$

$$\begin{cases} x_1^{(r+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(r)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(r)} \\ x_2^{(r+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(r+1)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(r)} \\ \vdots \\ x_n^{(r+1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(r+1)} - \cdots - \frac{a_{n(n-1)}}{a_{nn}} x_{n-1}^{(r+1)} \end{cases}$$

Example

Find the solution of the following system of equations using Gauss-Seidel method. Taking the initial starting of solution vector as $(0,0,0)^T$

$$\begin{cases} 4x_1 - x_2 - x_3 = 2 \\ -x_1 + 4x_2 - x_4 = 2 \\ -x_1 + 4x_3 - x_4 = 1 \\ -x_2 - x_3 + 4x_4 = 1 \end{cases} \rightarrow \begin{cases} x_1^{(1)} = \frac{2}{4} + \frac{1}{4} x_2^{(0)} + \frac{1}{4} x_3^{(0)} \\ x_2^{(1)} = \frac{2}{4} + \frac{1}{4} x_1^{(1)} + \frac{1}{4} x_4^{(0)} \\ x_3^{(1)} = \frac{1}{4} + \frac{1}{4} x_1^{(1)} + \frac{1}{4} x_4^{(0)} \\ x_4^{(1)} = \frac{1}{4} + \frac{1}{4} x_2^{(1)} + \frac{1}{4} x_3^{(1)} \end{cases}$$

$$\rightarrow \begin{cases} x_1^{(1)} = \frac{2}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 \\ x_2^{(1)} = \frac{2}{4} + \frac{1}{4} x_1^{(1)} + \frac{1}{4} \cdot 0 \\ x_3^{(1)} = \frac{1}{4} + \frac{1}{4} x_1^{(1)} + \frac{1}{4} \cdot 0 \\ x_4^{(1)} = \frac{1}{4} + \frac{1}{4} x_2^{(1)} + \frac{1}{4} x_3^{(1)} \end{cases}$$

$$\xrightarrow{x_1^{(1)}=0.5} \begin{cases} x_1^{(1)} = \frac{2}{4} + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 \\ x_2^{(1)} = \frac{2}{4} + \frac{1}{4} \cdot 0.5 + \frac{1}{4} \cdot 0 \\ x_3^{(1)} = \frac{1}{4} + \frac{1}{4} \cdot 0.2 + \frac{1}{4} \cdot 0 \\ x_4^{(1)} = \frac{1}{4} + \frac{1}{4} x_2^{(1)} + \frac{1}{4} x_3^{(1)} \end{cases}$$

$$\xrightarrow[x_3^{(1)}=0.375]{x_2^{(1)}=0.625} \begin{cases} x_1^{(1)} = 0.5 \\ x_2^{(1)} = 0.625 \\ x_3^{(1)} = 0.375 \\ x_4^{(1)} = \frac{1}{4} + \frac{1}{4} \cdot 0.625 + \frac{1}{4} \cdot 0.375 \end{cases}$$

$$\rightarrow \begin{cases} x_1^{(1)} = 0.5 \\ x_2^{(1)} = 0.625 \\ x_3^{(1)} = 0.375 \\ x_4^{(1)} = 0.5 \end{cases} \rightarrow \cdots$$

- **Matrix Inversion**

The matrix A^{-1} is called the invers of A. A matrix without an inverse is called **singular** (or **noninvertible**).

For any nonsingular $n \times n$ matrix A,

- A^{-1} is unique.
- A^{-1} is nonsingular and $(A^{-1})^{-1} = \mathrm{A}$
- If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$

- **Gaussian Elimination Method**

$$(A_{n\times n}|I_{n\times n}) = \left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$
$$\to \left(\begin{array}{ccc|ccc} a'_{11} & a'_{12} & a'_{13} & b_{11} & b_{12} & b_{13} \\ 0 & a'_{22} & a'_{23} & b_{21} & b_{22} & b_{23} \\ 0 & 0 & a'_{33} & b_{31} & b_{32} & b_{33} \end{array}\right) = (A^U|B)$$
$$A^U A^{-1} = B$$

- **Gauss-Jordan Method**

$$(A_{n\times n}|I_{n\times n}) \to (I_{n\times n}|A^{-1}_{n\times n})$$

- **Eigenvalues and Eigenvectors**

Let A be an $n \times n$ square matrix. Suppose, there exists a scalar λ and a vector

$$\mathrm{x} = (x_1 \quad x_2 \quad \cdots \quad x_n)^T$$

Such that

$$A\mathrm{x} = \lambda\mathrm{x}$$

Then λ is the **eigenvalue** and x is the corresponding **eigenvector** of the matrix $[A]$.

- If $[A]$ is a $n \times n$ triangular matrix -upper, lower or diagonal, the eigenvalues of $[A]$ are the diagonal of $[A]$.
- $\lambda = 0$ is an eigenvalue of $[A]$ if $[A]$ is a singular (noninvertible) matrix.
- $[A]$ and $[A]^T$ have the same eigenvalues.
- $|\det(A)|$ is the product of the absolute values of the eigenvalues of $[A]$.

- **Power Method**

It is used to find the largest eigenvalue in an absolute sense.

1. Choose the initial vector such that the largest element is unity (or 1).
2. The normalized vector $\mathrm{v}^{(0)}$ is pre-multiplied by matrix $[A]$.
3. The resultant vector is again normalized.
4. This process of iteration is continued, and the new normalized vector is repeatedly pre-multiplied by the matrix $[A]$ until the required accuracy is obtained.

Example

Find the eigenvalue of largest modulus, and the associated eigenvector of the matrix. Choose an initial vector $\mathrm{v}^{(0)}$ as $(1 \quad 1 \quad 1)^T$

$$[A] = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{pmatrix}$$

$$\mathrm{u}^{(1)} = [A]\mathrm{v}^{(0)} = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \\ 14 \end{pmatrix}$$
$$= 14\begin{pmatrix} 7/14 \\ 12/14 \\ 1 \end{pmatrix} = \lambda_1 \mathrm{v}^{(1)}$$
$$\mathrm{u}^{(2)} = [A]\mathrm{v}^{(1)} = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{pmatrix}\begin{pmatrix} 7/14 \\ 12/14 \\ 1 \end{pmatrix} = \begin{pmatrix} 39/7 \\ 67/7 \\ 171/14 \end{pmatrix}$$
$$= 12.2143\begin{pmatrix} 0.456140 \\ 0.783626 \\ 1 \end{pmatrix} = \lambda_2 \mathrm{v}^{(2)}$$
$$\mathrm{u}^{(3)} = [A]\mathrm{v}^{(1)} = \begin{pmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{pmatrix}\begin{pmatrix} 0.456140 \\ 0.783626 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 5.263158 \\ 9.175438 \\ 11.935672 \end{pmatrix} = 11.935672\begin{pmatrix} 0.44096 \\ 0.776874 \\ 1 \end{pmatrix} = \lambda_3 \mathrm{v}^{(3)}$$

$$\therefore \lambda = 11.84, \mathrm{x} = \begin{pmatrix} 0.44 \\ 0.78 \\ 1 \end{pmatrix}$$

- **The Absolute Relative Approximate Error**

$$|\varepsilon_a| = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

- **Forward Differences**

For a given table of values

$$(x_k, y_k), k = 0, 1, 2, \cdots, n$$

with equally spaced abscissas of a function $y = f(x)$ we define the forward difference operator Δ as follows

$$\Delta y_i = y_{i+1} - y_i, i = 0, 1, 2, \cdots, n-1$$

To be explicit, we write

$$\begin{gathered} \Delta y_0 = y_1 - y_0 \\ \Delta y_1 = y_2 - y_1 \\ \vdots \\ \Delta y_{n-1} = y_n - y_{n-1} \end{gathered}$$

These differences are called *first differences* of the function y and are denoted by the symbol Δy_i.

Similarly, the differences of the *first differences* are called *second differences*, defined by

$$\begin{gathered} \Delta^2 y_0 = \Delta y_1 - \Delta y_0 \\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1 \\ \vdots \\ \Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1} \end{gathered}$$

Thus, in general

$$y_n = (1 + \Delta)^n y_0 = \sum\nolimits_{i=0}^{n} \mathrm{C}(n, i)\, \Delta^i y_0$$

- **Backward Differences**

For a given table of values

$$(x_k, y_k), k = 0, 1, 2, \cdots, n$$

of a function $y = f(x)$ with equally spaced abscissas, the first backward differences are usually expressed in terms of the backward difference operator ∇ as

$$\nabla y_i = y_i - y_{i-1}, i = n, n-1, \cdots, 1$$

To be explicit, we write

$$\begin{gathered} \nabla y_1 = y_1 - y_0 \\ \nabla y_2 = y_2 - y_1 \\ \vdots \\ \nabla y_n = y_n - y_{n-1} \end{gathered}$$

In general

$$y_{n-r} = \sum\nolimits_{i=0}^{r} (-1)^i\, \mathrm{C}(r, i)\, \nabla^i y_n$$

- **Central Differences**

We use the symbol δ to represent central difference operator and the subscript of δy for any difference as the average of the subscripts.

$$\delta y_i = y_{i+1/2} - y_{i-1/2}$$

- **Shift Operator, E**

Let $y = f(x)$ be a function of x, and let x takes the consecutive values x, $x + h$, $x + 2h$, etc.

We then define an operator having the property

$$E^n f(x) = f(x + nh), \text{or } E^n y_x = y_{x+nh}$$

Thus, when E operates on $f(x)$, the result is the next value of the function.

- **Average Operator, μ**

$$\begin{aligned} \mu f(x) &= \frac{1}{2}\left(f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right) \\ &= \frac{1}{2}\left(y_{x+h/2} + y_{x-h/2}\right) \end{aligned}$$

- **Differential Operator, D**

$$\begin{aligned} Df(x) &= \frac{d}{dx} f(x) = f'(x) \\ D^2 f(x) &= \frac{d^2}{dx^2} f(x) = f''(x) \end{aligned}$$

- **Few Results Using Δ, ∇, δ, E, μ and D**

From the definition of operator Δ and E, we have

$$\Delta y_x = y_{x+h} - y_x = E y_x - y_x = (E-1) y_x$$
$$\therefore \Delta = E - 1$$

From the definition of operator ∇ and E^{-1}, we have

$$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1 - E^{-1}) y_x$$
$$\therefore \nabla = 1 - E^{-1}$$

From the definition of operator δ and E, we have

$$\begin{aligned} \delta y_x &= y_{x+h/2} - y_{x-h/2} = E^{1/2} y_x - E^{-1/2} y_x \\ &= \left(E^{1/2} - E^{-1/2}\right) y_x \end{aligned}$$
$$\therefore \delta = E^{1/2} - E^{-1/2}$$

From the definition of operators μ and E, we have

$$\begin{aligned} \mu y_x &= \frac{1}{2}\left(y_{x+h/2} - y_{x-h/2}\right) = \frac{1}{2}\left(E^{1/2} y_x + E^{-1/2} y_x\right) \\ &= \left(E^{1/2} - E^{-1/2}\right) y_x \end{aligned}$$
$$\therefore \mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

Using Taylor series expansion, we have

$$\begin{aligned} E y_x &= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots \\ &= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \cdots \\ &= \left(1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + \cdots\right) f(x) = e^{hD} y_x \end{aligned}$$
$$\therefore hD = \log E$$

Example: Prove that

$$hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$$

$$\begin{aligned} hD &= \log E = \log(1+\Delta) \\ hD &= \log E = -\log E^{-1} = -\log(1-\nabla) \\ \mu\delta &= \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} + E^{-1/2}\right) = \frac{1}{2}(E - E^{-1}) \\ &= \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD) \therefore \sinh^{-1}(\mu\delta) \end{aligned}$$

- **Newton's Forward Difference Interpolation**

$$\begin{aligned} f(x_0 + ph) &= E^p f(x_0) = (1+\Delta)^p f(x_0) \\ &= \sum_{i=0}^{p} \mathrm{C}(n,i)\, \Delta^i f(x_0) \text{ where } p = \frac{x - x_0}{h} \\ &= f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) \\ &+ \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \cdots \\ &+ \frac{p(p-1)\cdots(p-n+1)}{n!}\Delta^n f(x_0) + \mathrm{Error} \end{aligned}$$

This formula is also known as Newton-Gregory forward difference interpolation formula.

- **Newton's Backward Difference Interpolation**

$$\begin{aligned} f(x_n + ph) &= E^p f(x_0) = (E^{-1})^{-p} f(x_n) \\ &= (1-\nabla)^{-p} f(x_n) = \sum_{i=0}^{p} \mathrm{C}(n,i)\, \nabla^i f(x_n) \\ &= f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) \\ &+ \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n) + \cdots \\ &+ \frac{p(p+1)\cdots(p+n-1)}{n!}\nabla^n f(x_n) + \mathrm{Error} \end{aligned}$$

- **Lagrange's Interpolation Formula**

$$\begin{aligned} y &= f(x) \\ &= \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 \\ &+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \cdots \\ &+ \frac{(x-x_0)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} y_i \\ &+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n \end{aligned}$$

- **Divided Differences**

$$\begin{aligned} y[x_0] &= y_0 \\ y[x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0} \\ y[x_0, x_1, x_2] &= \frac{y[x_1, x_2] - y[x_0, x_1]}{x_2 - x_0} \\ &= \frac{1}{x_2 - x_0}\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right) \\ &= \frac{y_0}{(x_0-x_1)(x_0-x_2)} + \frac{y_1}{(x_1-x_0)(x_1-x_2)} \\ &+ \frac{y_2}{(x_2-x_0)(x_2-x_1)} \\ &\vdots \\ y[x_0, x_1, \cdots, x_n] &= \frac{y[x_1, x_2, \cdots, x_n] - y[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \\ &= \frac{y_0}{(x_0-x_1)\cdots(x_0-x_k)} + \frac{y_1}{(x_1-x_0)\cdots(x_1-x_k)} \\ &+ \cdots + \frac{y_k}{(x_k-x_0)\cdots(x_k-x_{k-1})} = \sum_{i=0}^{k} \frac{y_i}{\prod_{\substack{i=0 \\ i\neq k}}^{k}(x_i - x_j)} \end{aligned}$$

- **Newton's Divided Difference Interpolation**

$$
\begin{aligned}
y &= f(x) \\
&= y_0 + (x - x_0)y[x_0, x_1] \\
&+ (x - x_0)(x - x_1)y[x_0, x_1, x_2] + \cdots \\
&+ (x - x_0)(x - x_1)\cdots(x - x_{n-1})y[x_0, x_1, \cdots x_n]
\end{aligned}
$$

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