- A **Linear Equation** in the variables $x_1, x_2, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Where b and the coefficients $a_1, a_2, \cdots, a_n$ are real or complex numbers that are usually known in advance.

- A **System of Linear Equations (Linear System)** is a collection of one or more linear equations involving the same variables

$$x_1, x_2, \cdots, x_n$$

The variables in a linear system are called the **Unknowns**.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

System of linear equations has *no solution*, or *exactly one solution*, or *infinitely many solutions*.

- A **Solution** of the system is a list of numbers that makes each equation a true statement when the values $s_1, s_2, \cdots, s_n$ are substituted for $x_1, x_2, \cdots, x_n$ respectively.

The set of all possible solutions is called **Solution Set** of the linear system. Two linear systems are called **Equivalent** if they have the same solution set.

- A system of linear equations is said to be **Consistent** if it has either *one solution* or *infinitely many solutions*.

- A system of linear equations is said to be **Inconsistent** if it has *no solution*.

- **Coefficient Matrix**

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

- **Augmented Matrix**

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- **Elementary Row Operations**

  - **Replacement:** Replace one row by the sum of itself and a multiple of another row.

  - **Interchange:** Interchange two rows.

  - **Scaling:** Multiply all entries in a row by a nonzero constant.

- Two matrices are called **Row Equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

- A rectangular matrix is in **Echelon Form (Row Echelon Form)** if it has the following three properties

  - All nonzero rows are above any rows of all zeros.

  - Each leading entry of a row is in a column to the right of the leading entry of the row above it.

  - All entries in a column below a leading entry are zeros.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

- A rectangular matrix is in **Reduced Echelon Form (Reduced Row Echelon Form)** if it satisfies the following additional conditions.

  - The leading entry in each nonzero row is 1.

  - Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- **Theorem: Uniqueness of the Reduced Echelon Form**

Each matrix is row equivalent to one and only one reduced echelon form.

If a matrix A is row equivalent to an echelon matrix U, we call U an **Echelon Form (Row Echelon Form) of A**; if U is in reduced echelon form, we call U the **Reduced Echelon Form (Reduced Row Echelon Form) of A.**

- A **Pivot Position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- A **Pivot Column** is a column of A that contains a pivot position.

- A variable is a **Basic Variable** if it corresponds to a pivot column. Otherwise, the variable is known as a **Free Variable**.

- **Parametric Description**

$$\begin{aligned} &x_1 = -6x_2 - 3x_4 \\ &x_2 = 4 - x_3 \\ &x_3 \text{ is free} \end{aligned}$$

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

$$\begin{aligned} &x_1 + 5x_2 = 21 \\ &x_2 + x_3 = 4 \\ &x_3 \text{ is free} \end{aligned}$$

When a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has *no parametric representation*.

- **Theorem: Existence and Uniqueness**

  A linear system is consistent if and only if the right most column of the augmented matrix is not a pivot column – i.e., if and only if an echelon form of the augmented matrix has no row of the form $[0 \quad \cdots \quad 0 \quad b]$ with b nonzero.

  If a linear system is consistent, then the solution set contains either

  - a unique solution, when there are no free variables, or
  - infinitely many solutions, when there is at least one free variable.

- A matrix with only one column is called a **Column Vector**, or simply a **Vector**.

$$\mathrm{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

  where w_1 and w_2 are any real numbers.

- The set of all vectors with n entries is denoted by $\mathbb{R}^n$.

- Given vectors $\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \cdots, c_p$ the vector y defined by

$$\mathrm{y} = c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \cdots + c_p\mathrm{v}_p$$

  is called a **Linear Combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ with Weights $c_1, c_2, \cdots, c_p$**.

- If $\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p$ is denoted by $\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ and is called the **Subset of $\mathbb{R}^n$ Spanned (or Generated) by $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$**

  That is, $\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \cdots + c_p\mathrm{v}_p$$

  with $c_1, c_2, \cdots, c_p$ scalars.

  Asking whether a vector b is in $\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p\}$ equivalent to asking whether the vector equation

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \cdots + c_p\mathrm{v}_p = \mathrm{b}$$

  has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\mathrm{v}_1 \quad \mathrm{v}_2 \quad \cdots \quad \mathrm{v}_p \quad \mathrm{b}]$ has a solution.

- If A is an $m \times n$ matrix, with columns $\mathrm{a}_1, \mathrm{a}_2, \cdots, \mathrm{a}_n$, and if x is in $\mathbb{R}^n$, then the product of A and x, denoted by $\boldsymbol{A}\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in x as weights; that is,

$$A\mathrm{x} = [\mathrm{a}_1 \quad \mathrm{a}_2 \quad \cdots \quad \mathrm{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = c_1\mathrm{a}_1 + c_2 a_2 + \cdots + c_n\mathrm{a}_n$$

  $A\mathrm{x}$ is denoted only if the number of columns of A equals the number of entries in x.

- The matrix with 1s on the diagonal and 0s elsewhere is called an **Identity Matrix** and is denoted by I.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- A system of linear equations is said to be **Homogeneous** if it can be written in the form $A\mathrm{x} = 0$, where A is an $m \times n$ matrix and 0 is the zero vector in $\mathbb{R}^{\mathrm{m}}$.

  Such a system $A\mathrm{x} = 0$ always has at least one solution, namely, $\mathrm{x} = 0$. These zero solutions are usually called the **Trivial Solution**. The homogeneous equation $A\mathrm{x} = 0$ has a **Nontrivial Solution** if and only if the equation has at least one free variable.

- The equation of the form $\mathrm{x} = s\mathrm{u} + t\mathrm{v}$ (s, t in $\mathbb{R}$) is called a **Parametric Vector Equation** of the plane.

- **Solution of Nonhomogeneous Systems**

$$A\mathrm{x} = \mathrm{b} \rightarrow \mathrm{x} = \mathrm{p} + t\mathrm{v}$$

  When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

  Suppose the equation $A\mathrm{x} = \mathrm{b}$ is consistent for some given b, and let p be a solution. Then the solution set of $A\mathrm{x} = \mathrm{b}$ is the set of all vectors of the form $\mathrm{w} = \mathrm{p} + \mathrm{v}_h$, where v_h is any solution of the homogeneous equation $A\mathrm{x} = 0$.

  This theorem says that if $A\mathrm{x} = \mathrm{b}$ has a solution, then the solution set is obtained by translating the solution set of $A\mathrm{x} = 0$, using any particular solution p of $A\mathrm{x} = \mathrm{b}$ for the translation.

- An indexed set of vectors $\{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p\}$ in $\mathbb{R}^n$ is said to be **Linearly Independent** if the vector equation

$$x_1\mathrm{v}_1 + x_2\mathrm{v}_2 + \cdots + x_p\mathrm{v}_p = 0$$

  has only the trivial solution.

  The columns of matrix of A are linearly independent if and only if the equation $A\mathrm{x} = 0$ has only the trivial solution.

- The set $\{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p\}$ is said to be **Linearly Dependent** if there exist weights $c_1, c_2, \cdots, c_p$ not all aero, such that

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \cdots + c_p\mathrm{v}_p = 0$$

  It is called a **Linear Dependence Relation** among $\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p$ when the weights are not all zero.

  Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathrm{x} = 0$.

- **Theorem: Characterization of Linearly Dependent Sets**

  An indexed set $S = \{v_1, v_2, \cdots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

  If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, v_2, \cdots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

  If a set $\{v_1, v_2, \cdots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

- The equation $f(x) = 0$ is called an **Algebraic**, if it is purely a polynomial in x.

$$x^3 + 5x^2 - 6x + 3 = 0$$

- It is a **Transcendental** if $f(x)$ contains trigonometric, exponential or logarithmic functions.

$$ax^2 + \log(x-3) + e^x \sin x = 0$$

- **Descartes rule of signs**

  The number of positive roots of an algebraic equation $f(x) = 0$ with real coefficients cannot exceed the number of changes in sign of the coefficients in the polynomial $f(x) = 0$.

  Similarly, the number of negative roots of $f(x) = 0$ cannot exceed the number of changes in the sign of the coefficients of $f(-x) = 0$.

  Consider an equation

$$x^3 - 3x^2 + 4x - 5 = 0$$

  As there are three changes in sign, so, the degree of the equation is three, and hence the given equation will have all the three positive roots.

$$f(-x) = -x^3 - 3x^2 - 4x - 5$$

  There is no change of sign so there will be no negative root of equation.

- **Direct Methods**, which require no knowledge of the initial approximation of a root of the equation $f(x) = 0$.

- **Iterative Methods**, require first approximation to initiate iteration.

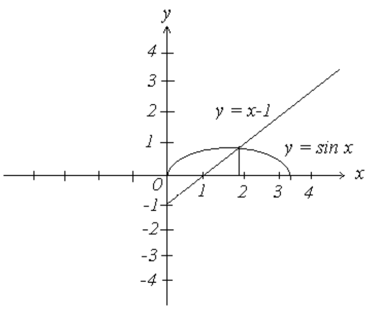
- **Graphical Method**

  For example, consider,

  $f(x) = x - \sin x - 1 = 0$
  $\rightarrow x - 1 = \sin x$

  Now, we shall draw the graphs of

  $y = x - 1, y = \sin x$



  The approximate value of the root is found to be 1.9

- **Analytical Method**

  This method is based on 'intermediate value property'.

$$f(x) = 3x - \sqrt{1 + \sin x} = 0$$

$$f(0) = -1, f(1) = 1.64299$$

  Here $f(0)$ and $f(1)$ are of opposite signs. Therefore, using intermediate value property we infer that there is at least one root between $x = 0$ and $x = 1$.

- **Absolute Relative Approximate Error**

$$|\mathcal{E}| = \frac{|\text{New value} - \text{Previous value}|}{|\text{New value}|} \times 100\%$$

- **Bisection Method**

  An equation $f(x) = 0$, where $f(x)$ is a real and continuous function, has at least one root between x_l and x_u if $f(x_l)f(x_u) < 0$.

  The desired root is approximately defined by the midpoint

$$x_2 = \frac{x_0 + x_1}{2}$$

  If $f(x_2) = 0$, then x_2 is the desired root of $f(x) = 0$.

  However, if $f(x_2) \neq 0$, then the root may be between x_0 and x_2 or x_2 and x_1.

  Pros. Always convergent / The root bracket gets halved with each iteration guaranteed.

  Cons. Slow convergence / If one of the initial guessed is close to the root, the convergence is slower. / It is impossible to find if the function f is tangent to the x-axis. / Discontinuous function changes sign but root does not exist. i.e. $f(x) = 1/x$.

```
f = @(x)( 2*x * cos(2*x) - (x+1).^2 );

x_lo = -1;
x_hi =  0;

x_mid = (x_lo + x_hi) / 2;

i = 0;
while( abs(f(x_mid)) > 0.05 )
    if( f(x_mid) * f(x_hi) < 0 )
        x_lo = x_mid;
    else
        x_hi = x_mid;
    end

    x_mid = (x_lo + x_hi) / 2;

    i = i + 1;
    fprintf('Iterate %2d times | %- 12g | %- 12g
| %- 12g | %- 12g\n', ...
            i, x_lo, x_hi, x_mid, f(x_mid));
end
```

- **Regula-Falsi Method**

Choose two points x_n and x_{n-1} such that $f(x_n)$ and $f(x_{n-1})$ are of opposite signs.

Now, the equation of the chord joining the points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$ is

$$\frac{y - f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n}$$

Setting $y = 0$, we get

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

- **Newton-Raphson Method**

This method is powerful methods for finding a root of an equation in the form $f(x) = 0$.

Suppose x_0 is an approximate root of $f(x) = 0$. Let $x_1 = x_0 + h$, where h is small, be the exact root of $f(x) = 0$, then $f(x_1) = 0$.

Now, expanding $f(x_0 + h)$ by Taylor's theorem we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

Since h is small, we neglect terms containing h^2 and its higher powers, then

$$f(x_0) + hf'(x) = 0 \rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

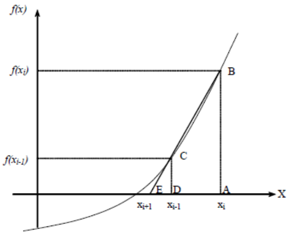
Therefore, a better approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

```
f      = @(x)( exp(-x) * cos(x) );
f_diff = @(x)( -exp(-x) * (sin(x) + cos(x)) );

eps_step = 1e-5;
eps_a    = 1e-5;
N = 100;
x = 1.3;

for i = 1:N
    xn = x - f(x) / f_diff(x);

    if( abs(x-xn) < eps_step && abs(f(xn)) < eps_a )
        break;
    elseif i == N
        error('Newton-Raphson methd did not 
converge');
    end

    fprintf('Iterate %2d times | %- 12g | %- 12g 
| %- 12g | %- 12g\n', ...
           i, x, xn, abs(f(xn)), abs(x-xn));

    x = xn;
end
```

- **Secant Method**

The Geometric Similar Triangles



$$\frac{AB}{AE} = \frac{DC}{DE}$$

can be written as

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-i})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

The Secant method converges faster than linear and slower than Newton's quadratic.

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