Discrete Inverse Spectral Problem - Technical Summary

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1 The Problem

The question in exploration is the 2D discrete inverse spectral problem, namely that can one obtain the metric of a 2D discrete surface from its discrete Laplace-Beltrami spectrum? To make this problem simpler, we assume the surface have the topology of $\mathbb{S}^2$, then its metric would be conformally equivalent to that of a discrete sphere. So we only need to obtain a conformal map that recovers the surface from the sphere.

2 Methods

We approach this problem by naïvely optimizing the'energy' $E(\mathbf{c})$ that represents the squared difference between the discrete spectrum of the current shape $\boldsymbol{\lambda}$, and that of a target shape, $\boldsymbol{\lambda}_T$, where $\mathbf{c}$ is the diagonal conformal factors of the shape in comparison to the round sphere. *i.e.* Let $E(\mathsf{L}(\mathbf{c})) = \frac{1}{2}|\boldsymbol{\lambda} - \boldsymbol{\lambda}_T|^2$, where L is our choice of the Laplace-Beltrami operator of the current surface, and we will compute its gradient $\nabla E(\mathsf{L}(\mathbf{c})) = |\boldsymbol{\lambda} - \boldsymbol{\lambda}_T|\frac{\partial \boldsymbol{\lambda}}{\partial \mathsf{L}}\frac{\partial \mathsf{L}}{\partial \mathbf{c}}$ and do simple gradient descent (or BFGS).

In order to measure success, we first use the modified conformal Mean Curvature Flow (cMCF) of the original mesh to obtain a set of conformal factors to which we compare our optimized factors. Then we also try to use another gradient based method to re-embed our optimized conformal factors into a mesh and visually inspect its differences with the the original target mesh.

2.1 In simplicial basis

If we use triangle shape functions (linear FEM basis functions) and triangular mesh to initialize our sphere, then its LB spectrum can be estimated by the standard weighted *cotan operator*. There are known results about the accuracy of the eigenvalues of this discrete operator with respect to that of the smooth surface, at least in the fine mesh limit. Its spectrum will be called the piecewise linear (PL) spectrum.

2.2 In spherical harmonic basis

We can also use spherical harmonic (SH) functions, Laplacian eigenfunctions of the standard round sphere, as basis functions to decompose the Laplace-Beltrami operator of arbitrary genus 0 surface. This way, there is no dependence on which specific mesh we start with as the final target shape will simply be the sum of the corresponding harmonics with the desired magnitudes. through Wigner 3j symbols

Without giving any proofs, it is intuition to believe that eigenvalues of different harmonics should roughly correspond to features of varying spatial frequencies as only higher and higher degress harmonics have finer and finer 'bumps.'

2.3 Regularization

We can also append a bi-Laplacian regularization term to our energy E to (hopefully) overcome high-frequency discretization noises. This is reasonable because assuming the change in metric is small between each time step, we want to minimize fluctuations of the current Laplace-Beltrami operator, hence its Laplacian (approximated by itself) should be 0 or very small.

2.4 Performance and validation

Since there are additional problems associated with re-embedding a surface based on its conformal factors (see Section A.6), we will measure the metric accuracy by comparing a set of final optimized conformal factors with that obtained by applying cMCF to our target surfaces. Specifically, we will use mean squared difference between the two sets of conformal factors upto rotational invariance (see Section A.1).

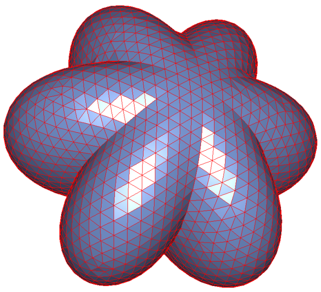
In addition, it might be useful to validate our method further by recomputing the eigenvalues between the target mesh with those of a spherical mesh plus cMCF and final optimized conformal factors, as well as those of the final embedded mesh.

3 Experiments and Results

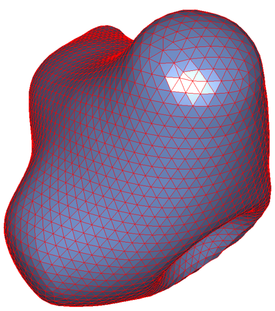
3.1 Tested target shapes

We have tested our methods on the following shapes,

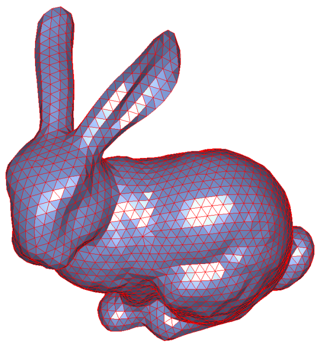
1. "star-shaped" surfaces
   These are surfaces obtained from a spherical mesh flowed along its mesh normals proportional to some scalar spherical harmonic functions ($|Y_3^3|$ on the right).



2. the blob
   To avoid large changes and high frequency data in conformal factor magnitudes yet allow some asymmetry, this blob is used as the general benchmark for our methods.



3. bunny
   Not only the archetypal mesh used by computer graphics literature, the bunny is very useful for tests about robustness. We attempted to recover this mesh from its spectrum in the simplicial setting only for we .



3.2 Simplicial basis inverse problem

3.2.1 raw

1. Performance against feature 'smoothness' of the target mesh

To guarantee higher success rates, the star-shaped surfaces (from $|Y_3^2|$) are used. We vary the craziness of the target mesh by supplying a scaling perturbation constant $p \in \mathbb{R}^{\geq 0}$ where $p = 0$ is unperturbed sphere and $p \to \infty$ limits to the actual spherical harmonic shape with singular points.

2. Performance against the number of optimized eigenvalues

   Fixing the target surface to be the star-shaped surfaces (from $|Y_3^3|$) with a perturbation constant of 0.5,

3.2.2 regularized

To attack the problem for more complicated target shapes such as the bunny, we used a

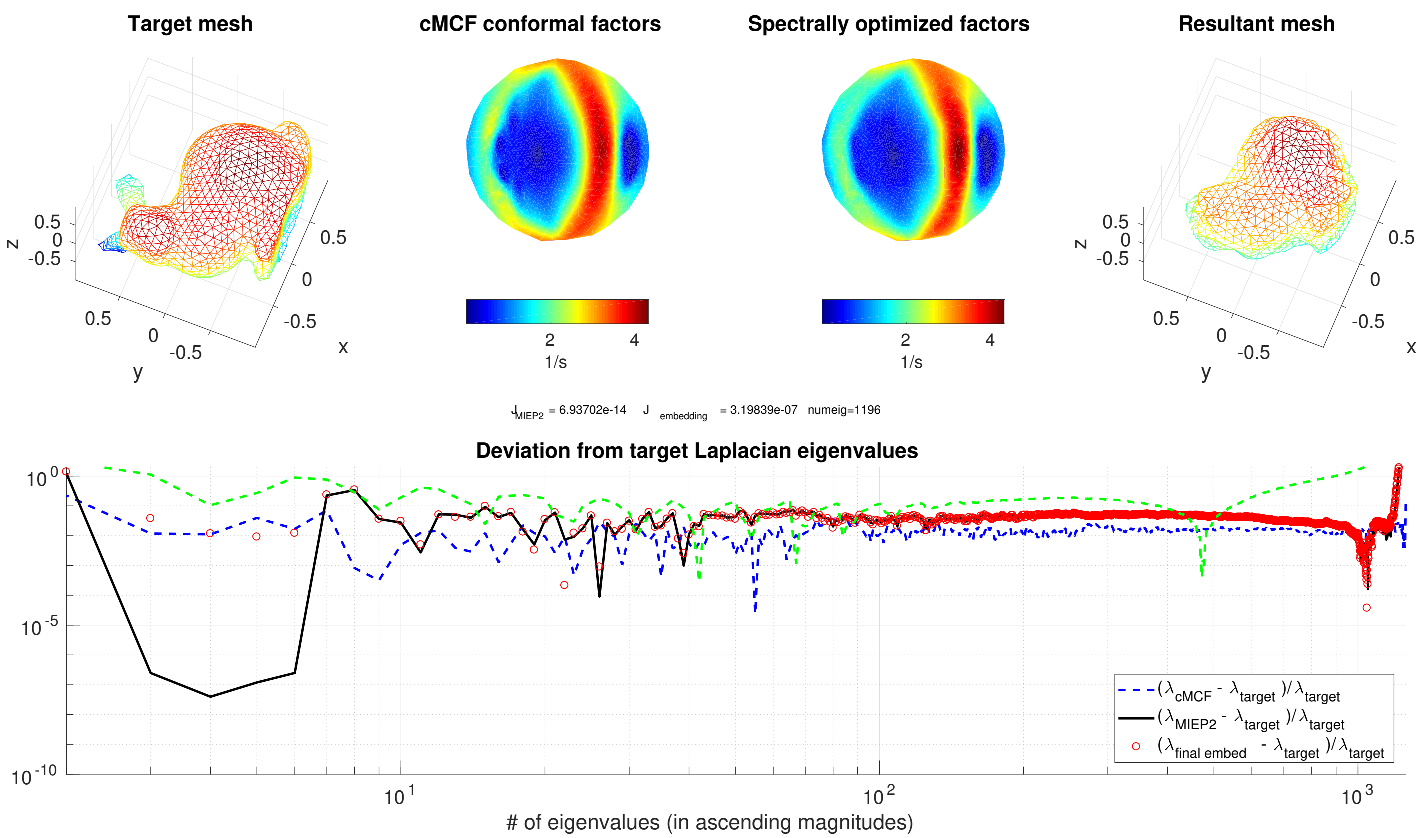


Figure 1: a=1196, r=0.05,
note that the inverse conformal factors are plotted on the hammer projections.

3.2.3 recursive fit

To demonstrate that our optimized conformal factor indeed encode the desired geometric information and even a very simple mesh reconstruction method can yield some visual results despite the large conformal factors at the ears and feet, we passed a resultant mesh from a regularized fit through the algorithm one more time to obtain a new resultant mesh more resemblant to the target bunny as below.

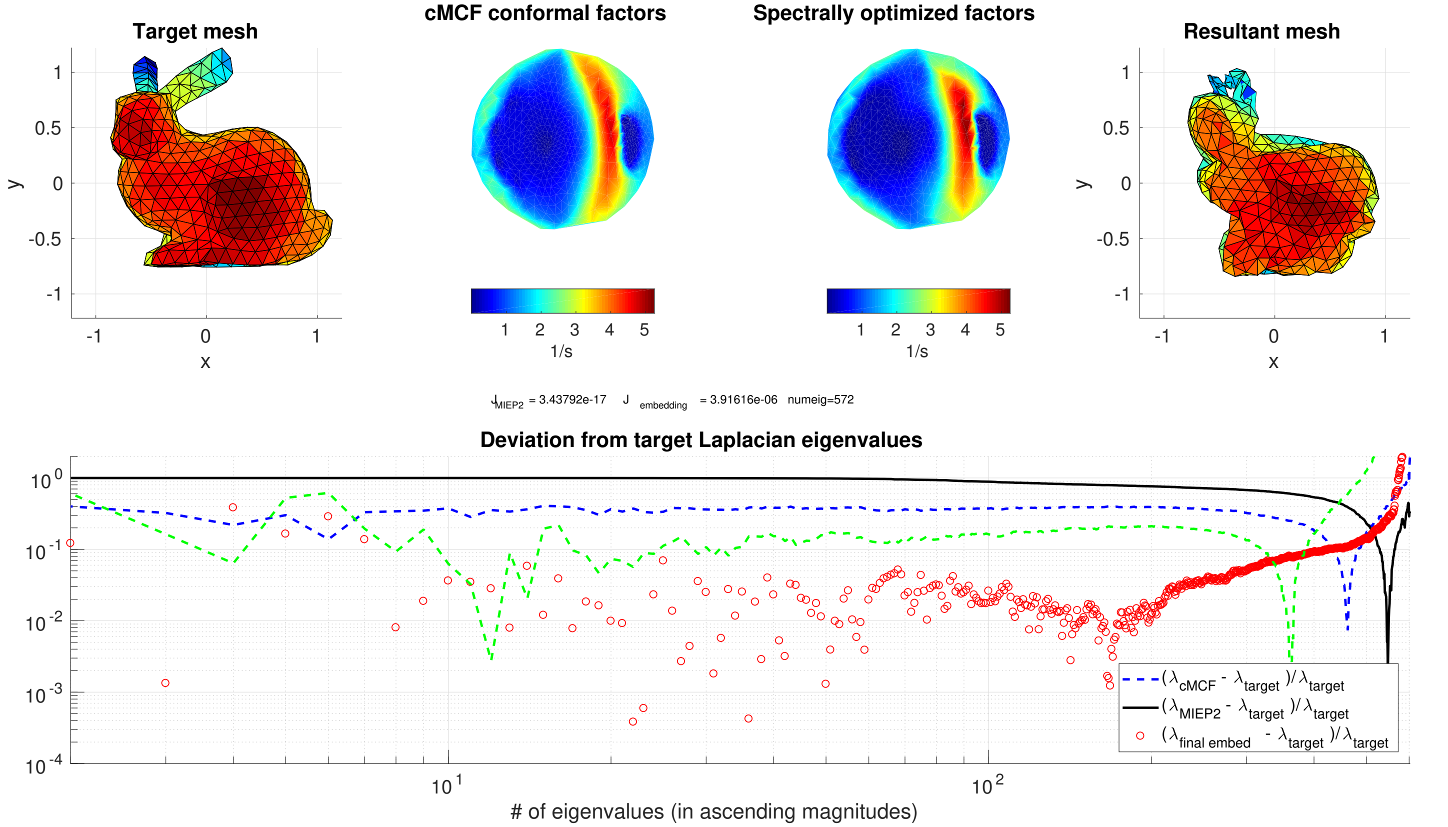


Figure 2: a=572, r=0.1,
note that the inverse conformal factors are plotted on the hammer projections.

An interesting result is that when using our naïve mesh reconstruction, the size of conformal factors can force protrusion of the ears but local errors make them curl pathologically. Also in some cases the features became inverted during the reconstruction (*e.g.* the tail).

3.2.4 Discussion

Since our simplicial basis are attached to some given mesh, we have to work with a PL embedding of the sphere that is the result of the cMCF bunny mesh. This is clearly undesirable as in real life we would not have access to a correctly adaptively refined spherical mesh to begin our gradient descent. To attack this problem, we believe either: 1) we develop a adaptive scheme where we start with a coarse mesh and gradually refine the mesh where the fitted conformal factor is getting large [adaptive mesh], 2) or we have to run structured optimizations that start with lower frequencies and move on to higher frequencies later [recursive optimization]. However, care needs to be taken when doing such schemes as simply running gradient descent on just the first n eigenvalues for $n \in \{1, 2, \dots\}$ tends to give out mostly 'trivial' but highly localized solutions[1].

ts ears require rather large conformal factors that might not work too well with naïve gradient descent type methods.

[1] I think it might have to do with the partitioning eigenvalues

3.3 Spherical harmonics basis forward problem

Before diving into the inverse problem, we need to verify that the LB spectrum can be well represented by a truncated spectrum in spherical harmonic basis.

In Figure ?? below we demonstrate some convergence of the spectrum from above to some value versus the convergence from below when looking at PL spectrum.

convergence w.r.t. maximal degree and w.r.t. the number of free SH basis coefficients

3.4 Spherical harmonics basis inverse problem

3.4.1 raw

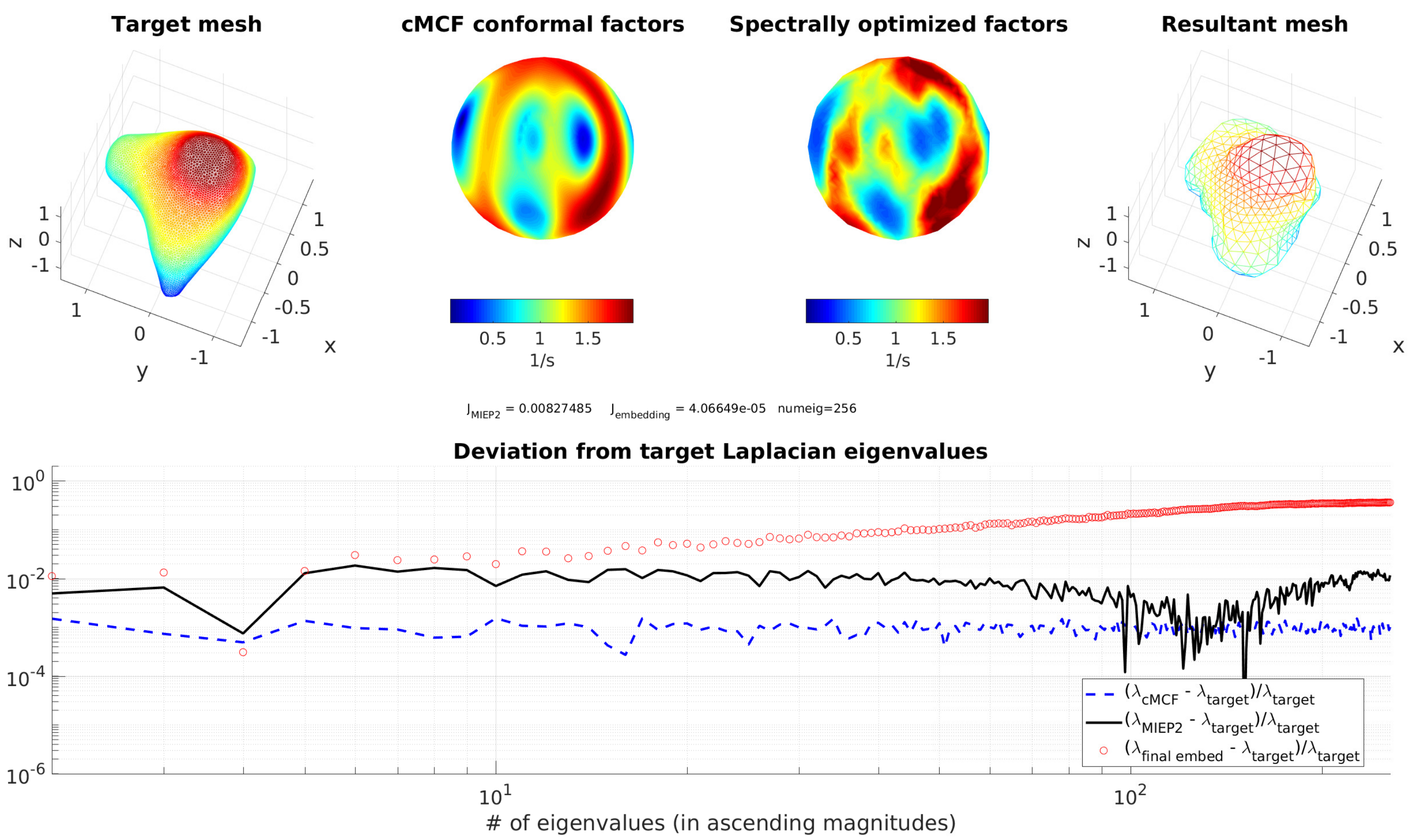


Figure 3: a=256, L=30x30

3.4.2 regularized

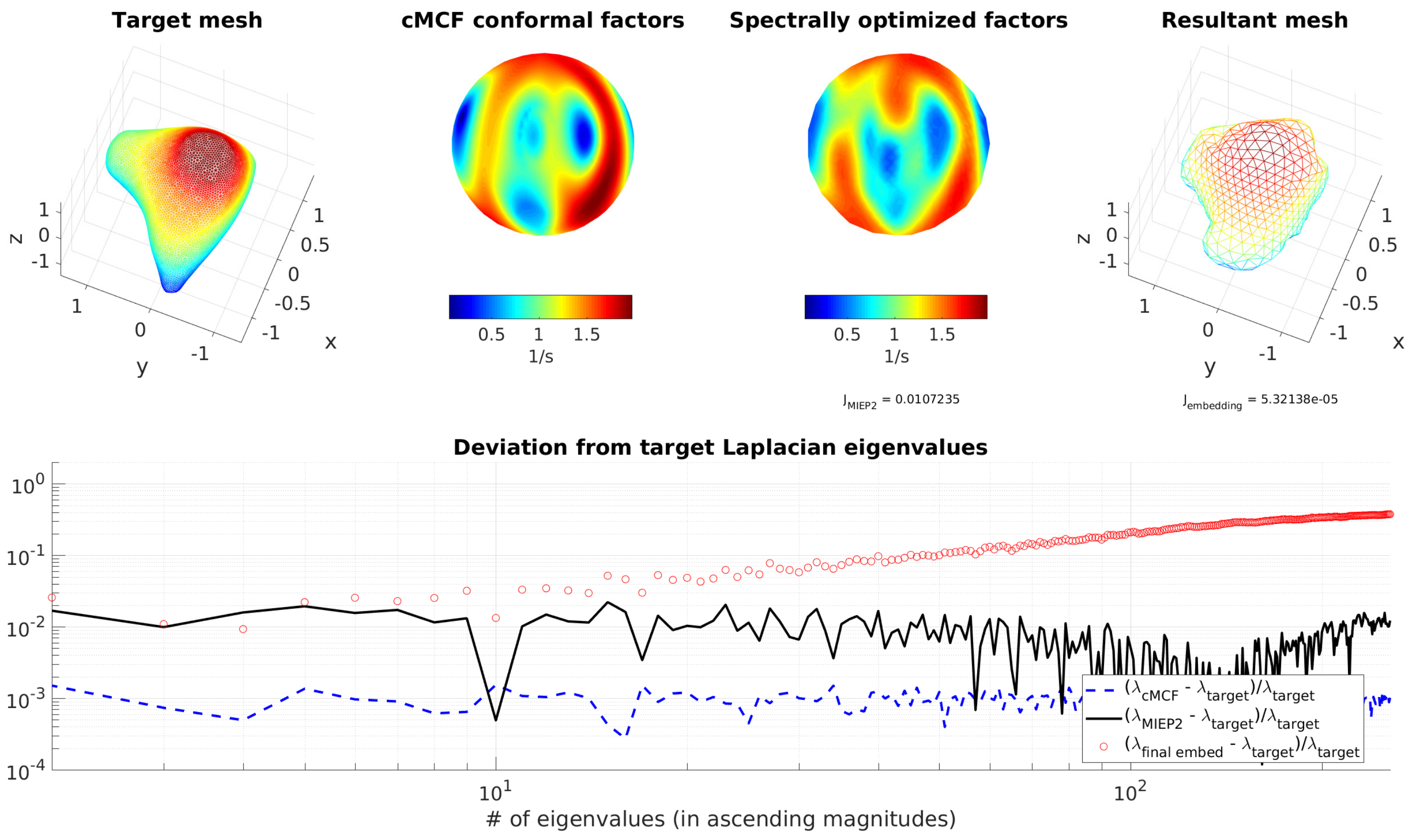


Figure 4: a=256, L=30x30, R=1e-6

APPENDIX

A Questions and a to-do list

A.1 Möbius balancing

Since conformal isometries on a sphere could be any Möbius transformation, we need to mod out the most correct map before comparing conformal factors directly.

For reference, Prof. Keenan Crane have a write-up and working C codes in his geometry processing toolset. The current MATLAB implementation is experiencing (too much?) oscillatory behavior possibly due to some sort of silly bug.

Currently we employ a simple search algorithm to find a rigid body rotation that minimizes mean squared difference between target and final conformal factors. This seems to be both necessary and sufficient for the SH (mesh-independent) problem for now.

A.2 Structured optimization

Assuming the intuitive equivalence between smaller magnitude Laplacian eigenvalues/functions to lower frequency features, it might stand to reason that we can first optimize for lower frequency features (for SH formulations, $a_{l,m}$ where l is small), and employ a 'suitable' method to **keep** the lower frequency result then refine higher frequency eigenvalues/features.

A second natural question is then when should we stop? For simplicial spectrum, we know that the very high frequency eigenvalues associate to noises due to the triangle basis functions. But how does one distinguish a trade-off point where unwanted noise overcomes wanted geometric information? Does a similar threshold even exist for SH formulation?

A.3 Precomputing more 3j symbols

It would probably be worthwhile to utilize some state-of-the-art techniques to calculate as many 3j symbols as one's RAM can handle to speed up further operations. Example reference: `http://epubs.siam.org/doi/abs/10.1137/S1064827503422932`

A.4 Topology

We also would like to calculate the topology of the surface from the spectrum beforehand. In the smooth setting, we can infer these from the **asymptotics** of heat trace from the spectrum. In the discrete case, it is not immediate how to extract these information. A reference is found to mention successfully doing this through some sort of curve fitting. Or is this a suitable task for machine learning? And after knowing a topology with $g \geq 1$, can be also use the corresponding harmonic basis functions?

Reference: Martin Reuter, Franz-Erich Wolter, Niklas Peinecke, "Laplace-Beltrami spectra as 'Shape-DNA' of surfaces and solids"

`http://reuter.mit.edu/blue/papers/reuter-shapeDNA06/reuter-shapeDNA06.pdf`

A.5 Adaptive remeshing for the simplical case

Our method so far only succeeds when we start from the cMCF spherical mesh in the simplicial setting. This is an unfortunate result of the explicit mesh dependence of our simplicial Laplacian operator. In other words, we get what we want if we start with a spherical mesh that is pre-adpated to its features sizes in terms of local sharpness (e.g. more vertices if there is a long thin protusion/. So the conjecture is that if we adapt our mesh at each step based on the magnitude of the current conformal factors, we should be able to recover the correct solution also.

This task is probably better done if the program is independently coded without MATLAB as the fast built-in BFGS optimization algorithm can only used more or less as a blackbox.

A.6 Embedding problem

Our method seems capable of recovering a set of spectrally optimized conformal factors that roughly matches features of factors obtained through cMCF. However, the naïve re-inflating strategy of finding vertex position that yield edge lengths proportional to the set of conformal factors is slow (scales with number of triangles squared) and suffer from local ambiguities (bumps could become dips) and high frequency noises. The issue with ambiguities is most likely a result of this method only working with "intrinsic" stretching constraints without any knowledge about the shape's "extrinsic" bending information. For noise reduction maybe a similar bi-Laplacian regularization term could help this simple gradient descent as before.

If we cook up a set of geometric parameters (some sort of 'extrinsic' angles?) encoding local bending information associated to the extrinsic Dirac operators akin to conformal factors to Laplace-Beltrami, we might be able to use both ends of the operator 'spectrum' to fully recover all necessary information to reconstruct surfaces...

B Nomenclature

In order of appearance:

1. `numeig` - the total number of eigenvalues being optimized
2. `pert` - a scalar proportional to how distorted the target shape is comparing to a round sphere.
3. `numa` - the total number of a
4. `maxL` - the maximal degree used by spherical harmonic basis approximation