# Solution to Cal State Hayward Problem of the month March 2017

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March 9, 2017

**Problem:** Suppose $f : P(S) \to P(S)$ satisfies $A \subset B \Rightarrow f(A) \subset f(B)$. Show there is $X \subset S$ such that $f(X) = X$.

**Solution:** Consider $\mathcal{A} = \{A \subset S | A \subset f(A)\}$ and let $X = \bigcup\limits_{\mathcal{A}} A$.

I claim that $f(X) = X$.

First note that $X$ is non-empty since it contains $f(\emptyset)$. If $f(\emptyset) = \emptyset$ we're done. $\emptyset \subset A$ for all $A \in P(S)$, so $\emptyset \subset f(\emptyset) \in P(S)$. Furthermore $\emptyset \subset f(\emptyset) \Rightarrow f(\emptyset) \subset f(f(\emptyset))$, so $f(\emptyset) \in \mathcal{A}$ which implies $f(\emptyset) \subset X$ by construction of $X$. Therefore, $\emptyset \neq f(\emptyset) \subset X$ and $X$ is not empty.

To see that $X \subset f(X)$, pick $x \in X$. Then $x \in A$ for some $A \in \mathcal{A}$. $A \subset X$ implies $f(A) \subset f(X)$, but $A \subset f(A) \Rightarrow f(A) \in f(f(A))$ so $f(A) \in \mathcal{A}$. So $f(A) \subset f(X)$. Furthermore, by construction of $\mathcal{A}$, $A \subset f(A)$ so

$$x \in A \subset f(A) \subset f(X) \to X \subset f(X).$$

To see that $f(X) \subset X$ note that $X \subset f(X) \Rightarrow f(X) \subset f(f(X))$ by construction of $f$ hence $f(X) \in \mathcal{A}$. All members of $\mathcal{A}$ are subsets of $X$ by construction of $X$, so $f(X) \subset X$.

So $X \subset f(X)$ and $f(X) \subset X$ so $f(X) = X$ as desired.