Introduction to Logarithmic Functions

Learning Objectives

- Convert logarithmic equations to exponential equations.
- Convert exponential equations to logarithmic equations.
- Graph logarithmic functions.
- Solve logarithmic equations.

Introduction

A useful family of functions that is related to **exponential functions** is the **logarithmic functions**. You have been calculating the result of b^x, and this gave us the exponential functions. A **logarithm** is a calculation of the **exponent** in the equation $y = b^x$. Put another way, finding a logarithm is the same as finding the exponent to which the given base must be raised to get the desired value. The exponent becomes the *output* rather than the *input*.

Calculating Exponents

Consider these tables of values using a base of 2.

Table 1	
Input x, an exponent x	Output y $y = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

Table 2	
Input x, a number that is a power of 2 $x = 2y$	Output y, the exponent of 2 y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2

$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Note the two tables are the same except the columns are reversed: the point $(1, 2)$ taken from the first table will be the point $(2, 1)$ in the second table.

The graphs of these two relationships should have the same general shape. As shown in the graph, the two curves are symmetrical about the line $y = x$. Another way to put it, if you rotate the red curve about the line $y = x$, it will coincide with the blue curve. (This makes sense, because y in the first table becomes x in the second table, and vice versa.)
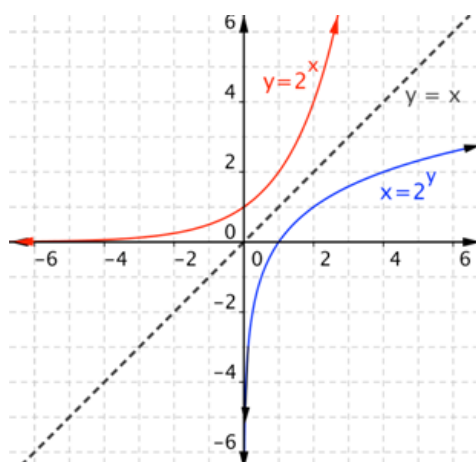
The equation $x = 2^y$ is often written as a logarithmic function (called log function for short). The logarithmic function for $x = 2^y$ is written as $y = \log_2 x$ or $f(x) = \log_2 x$. The number 2 is still called the **base**. In general, $y = \log_b x$ is read, " y equals log to the base b of x," or more simply, "y equals log base b of x." As with exponential functions, $b > 0$ and $b \neq 1$.

You can see from the graph that the range (y values) of the exponential function (in red) is positive real numbers. Since the input and output have been switched, the *domain* (x values) of the logarithmic function (in blue) is positive real numbers.

Similarly, the domain of the exponential function (in red) is all real numbers. The *range* of the logarithmic function (in blue) is all real numbers.

Definition of Logarithm

The logarithm of x in base b is written $\log_b x$ and is defined as:
$\log_b x = y$ if and only if $b^y = x$, where $x > 0$ and $b > 0, b \neq 1$.

Connecting Exponential and Logarithmic Equations

It's important to remember that the result of a logarithm is the exponent! That is, $\log_b x$ asks, "What exponent on the base b will give the result x?"

$$\underbrace{y = \log_b x}_{\substack{\text{logarithmic} \\ \text{form}}} \text{ means } \underbrace{b^y = x}_{\substack{\text{exponential} \\ \text{form}}}$$

Sometimes, you will need to convert $\log_b x = y$ to $b^y = x$. Other times, you will convert $b^y = x$ to $\log_b x = y$. The examples in the table show the logarithmic form and the corresponding exponential form of several equations.

Logarithmic Form	Exponential Form
$\log_2 16 = 4$	$2^4 = 16$
$\log_7 1 = 0$	$7^0 = 1$
$\log_5 5 = 1$	$5^1 = 5$
$\log_4 \left(\frac{1}{4}\right) = -1$	$4^{-1} = \frac{1}{4}$
$\log_{10} 0.01 = \log_{10} \left(\frac{1}{100}\right) = -2$	$10^{-2} = 0.01$

Knowing this connection is essential to understanding, and working with, logarithms.

Example	
Problem **Rewrite $\log_3 9 = 2$ as an exponential equation.**	
$\log_b x = y \Leftrightarrow b^y = x$	You need to change $\log_3 9 = 2$ to the form $b^y = x$.
$\begin{array}{ccc} \log_3 & 9 & = & 2 \\ b = 3 & x = 9 & & y = 2 \end{array}$	Here the base is 3 and the exponent is 2.
$b^y = x \Rightarrow 3^2 = 9$	Substitute for b, y, and x in the exponential equation, $b^y = x$.
Answer $\qquad\qquad 3^2 = 9$	Check the result: Is 3^2 actually equal to 9? Yes!

Although the input of a logarithm must always be positive, the result (an exponent!) can be negative.

Example	
Problem	**Convert $\log_4\left(\frac{1}{16}\right) = -2$ into an exponential equation.**
	$\log_b x = y \Leftrightarrow b^y = x$ You need to rewrite $\log_4\left(\frac{1}{16}\right) = -2$ in the form $b^y = x$.
	$\begin{array}{ccc} \log_4 & \frac{1}{16} & = & -2 \\ b=4 & x=\frac{1}{16} & & y=-2 \end{array}$ Here the base is 4 and the exponent is -2.
	$b^y = x \Rightarrow 4^{-2} = \frac{1}{16}$ Substitute for b, y, and x in the exponential equation, $b^y = x$.
Answer	$4^{-2} = \frac{1}{16}$ Check the result: $4^{-2} = \left(\frac{1}{4}\right)^2 = \frac{1}{4^2} = \frac{1}{16}$

Convert $\log_3\left(\frac{1}{81}\right) = -4$ into an exponential equation.

A) $3^4 = \frac{1}{81}$

B) $81^{\frac{1}{4}} = 3$

C) $3^{-4} = \frac{1}{81}$

D) $3^4 = 81$

[+ Show/Hide Answer](#)

As you have just seen, it is possible to convert log equations into exponential equations. It is also possible to convert exponential equation into log equations.

Example	
Problem	**Rewrite $5^3 = 125$ as a logarithmic equation.**
	$b^y = x \Leftrightarrow \log_b x = y$ You need to rewrite $5^3 = 125$

$$5^3 = 125$$

in the form $\log_b x = y$.

$$\log_b x = y \Rightarrow \log_5 125 = 3$$

Here the base is 5 and the exponent is 3.

Substitute for b, y, and x in the logarithmic equation, $\log_b x = y$.

Answer $\log_b 125 = 3$

Example

Problem	**Convert $10^{-3} = \frac{1}{1,000}$ into a logarithmic equation.**

	You need to rewrite $10^{-3} = \frac{1}{1,000}$ in the form $\log_b x = y$.
$$b^y = x \Leftrightarrow \log_b x = y$$	
$$10^{-3} = \frac{1}{1,000}$$ Where $b = 10^{-3}$ and $y = -3$ and $x = \frac{1}{1,000}$	Here the base is 10 and the exponent is -3.
$$\log_b x = y \Rightarrow \log_{10}\left(\frac{1}{1,000}\right) = -3$$	Substitute for b, y, and x in the logarithmic equation, $\log_b x = y$.

Answer $\log_{10} \frac{1}{1,000} = -3$

Remember that roots are exponents, too.

Example

Problem	**Convert $\sqrt{49} = 7$ into a logarithmic equation.**

$$\sqrt{49} = 49^{\frac{1}{2}}$$

Rewrite the square root using a fractional exponent. You need to rewrite $49^{\frac{1}{2}} = 7$ in the form $\log_b x = y$.

$$b^y = x \Leftrightarrow \log_b x = y$$
$$49^{\frac{1}{2}} = 7$$

Where $b = 49$ and $y = \left(\frac{1}{2}\right)$ and $x = 7$

Here, the base is 49 and the exponent is $\frac{1}{2}$.

$$\log_b x = y \Rightarrow \log_{49} 7 = \frac{1}{2}$$

Substitute for b, y, and x in the logarithmic equation, $\log_b x = y$.

Answer $\log_{49} 7 = \frac{1}{2}$

Convert $11^2 = 121$ into a logarithmic equation.

A) $\log_{11} 2 = 121$

B) $\log_{121} 11 = 2$

C) $\log_{11} 121 = 2$

D) $\log_{121} 2 = 11$

+ Show/Hide Answer

Graphing Logarithmic Functions

Look at the graph below. It should look familiar: you saw it earlier in this topic! The blue graph is the logarithmic function, and the red graph is the corresponding exponential function.
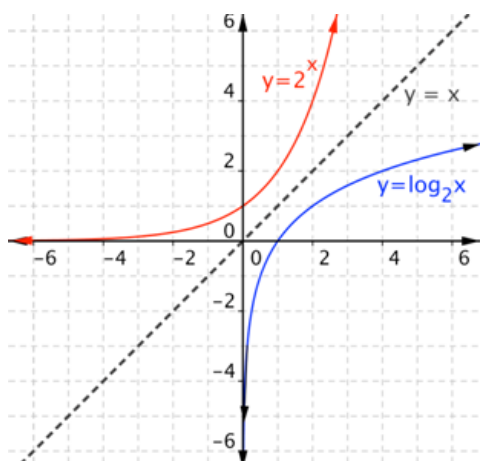
When graphing logarithmic functions, it's important to remember the following:

- The graph can only appear to the right of the y-axis. This is because the domain is restricted to positive values of x.
- The graph gets close to the y-axis for small values of x (close to $x = 0$).

Remember that logarithmic functions almost act like exponential functions. You just have to switch the x and y values. For example, the second bulleted item above is like the exponential function getting close to the x-axis (close to $y = 0$).

Example	
Problem	Graph $f(x) = \log_3 x.$

<table>
<tr>
<td>
Graph $y = \log_3 x$.

This is the same as $3^y = x$.

$$\begin{array}{c|c} x = 3^y & y \\ \hline & -2 \\ & -1 \\ & 0 \\ & 1 \\ & 2 \end{array}$$

$$\begin{array}{c|c} x = 3^y & y \\ \hline \frac{1}{9} & -2 \\ \frac{1}{3} & -1 \\ 1 & 0 \\ 3 & 1 \end{array}$$
</td>
<td>
To make the calculations easier, first make this a logarithmic equation in y. Then convert the logarithm into an exponential equation.

Start with a table of values.

With logarithmic functions, it's usually easier to choose the y values instead of the x values. Be sure to include some negative values for y.
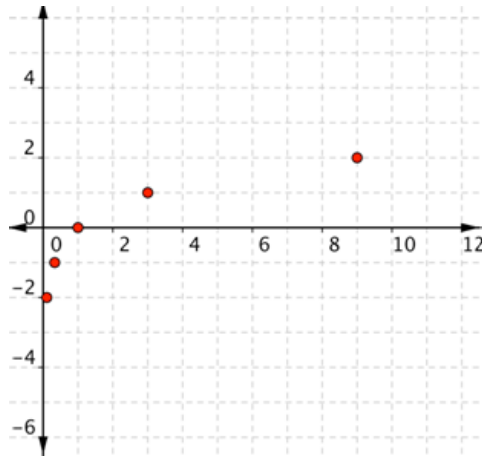
Be careful with the negative exponents!
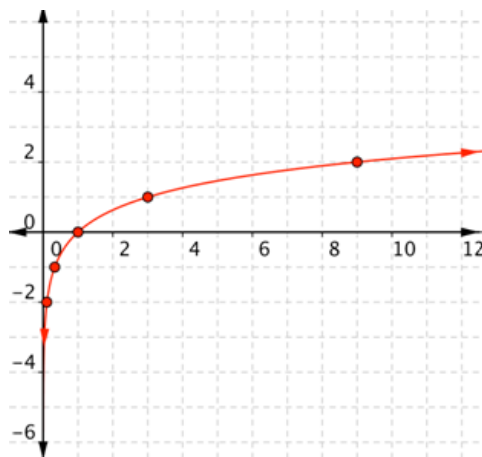
Use the table as ordered pairs. Remember that the
</td>
</tr>
</table>

9 | 2

graph of the function will show all $x - y$ correspondences, so any pair that could be in the table must be on the graph.

Plot the points.

Answer

Since the points are *not* on a line, you can't use a straightedge. Connect the points as best you can, using a *smooth curve* (not a series of straight lines).

Remember that logarithmic functions get close to the y-axis (but won't touch or cross it).

Example

Problem	**Graph $f(x) = \log_4 x$.**

	Graph $y = \log_4 x$. This is the same as $4^y = x$.	To make the calculations easier, first make this a logarithmic equation in y. Then convert the logarithm into an exponential equation.

$x = 4^y$	y
$\frac{1}{16}$	-2
$\frac{1}{4}$	-1
1	0
4	1
16	2

Start with a table of values, choosing the y values and calculating x.

Be careful with the negative exponents!

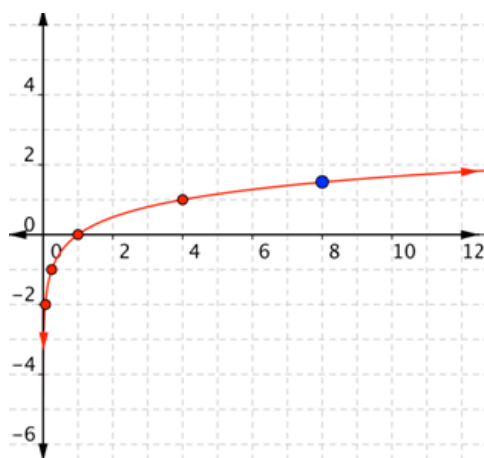Use the table as ordered pairs and plot the points.
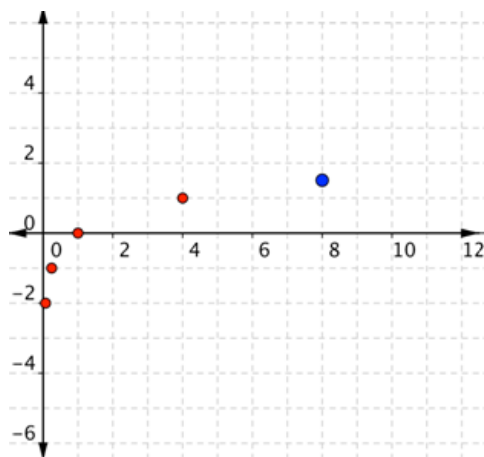
In this case, $(16, 2)$ won't show on the graph. Because you know the square root of 4, try $x = \frac{3}{2}$. In this case, it would be

$$4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8.$$

That point is shown in blue.

Answer

Since the points are *not* on a line, you can't use a straightedge. Connect the points as best you can, using a *smooth curve* (not a series of straight lines).

Remember that logarithmic functions get close to the y-axis (but won't touch or cross it).

Let's compare the logarithmic graphs you've seen: $f(x) = \log_2 x$, $f(x) = \log_3 x$, and $f(x) = \log_4 x$.
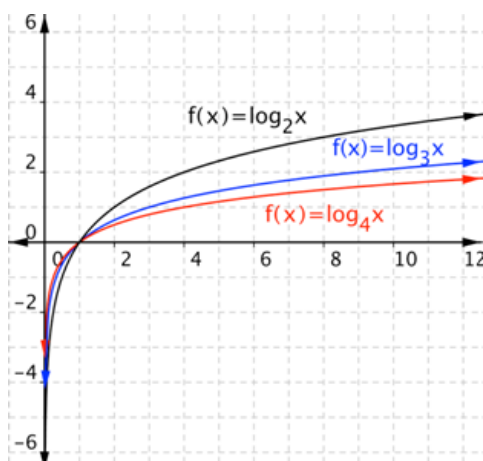
Notice that a larger base makes the graph *less* steep. (This is the opposite from exponential functions, where a larger base meant a steeper graph.) A larger base also makes the graph closer to the x-axis for $y > 0$ (or $x > 1$) and closer to the y-axis for $y < 0$ (or $x < 1$). All the graphs go through $(1, 0)$, while exponential functions always go through $(0, 1)$.

With exponential functions, the graphs were *increasing* (exponential *growth*) when the base b was greater than 1. The graphs were *decreasing* (exponential *decay*) when the base b was less than 1. The logarithmic graphs above are all increasing, and they all have a base greater than 1. Let's see what happens with a base less than 1.

Example	
Problem	Graph $f(x) = \log_{\frac{1}{4}} x$.

$$y = \log_{\frac{1}{4}} x$$
$$\left(\tfrac{1}{4}\right)^y = x$$

Write an exponential equation in y to help you.

| x	y
16	-2
4	-1
1	0
$\frac{1}{4}$	1
$\frac{1}{16}$	2

Create a table of values. Again, remember to be careful with the negative exponents. Remember to take the reciprocal of the base to make the exponent positive.
$$\left(\tfrac{1}{4}\right)^{-2} = \left(\tfrac{4}{1}\right)^2 = 4^2 = 16.$$

Notice that in this table, the x values decrease as the y values increase.

Use the table pairs to plot points. You may want to include new points, especially when one of the points from the table won't fit on your graph. $(16, -2)$ is off the graph. Because you know the square root of 4, try $x = -\frac{3}{2}$:
$$\left(\tfrac{1}{4}\right)^{-\frac{3}{2}} = \left(\tfrac{4}{1}\right)^{\frac{3}{2}} = 4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$$
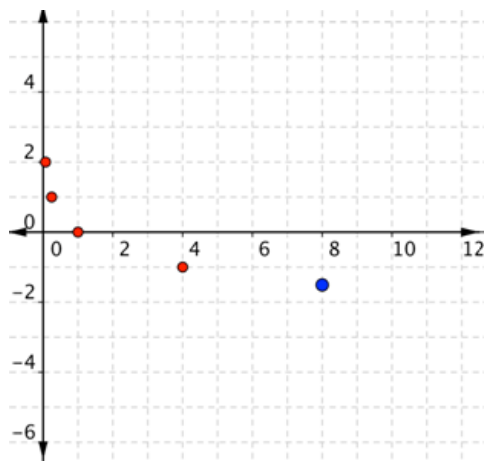
The point $\left(8, -\frac{3}{2}\right)$ has been included in blue. You may not feel it necessary to include additional points. You also may need to use a calculator, depending on the base.
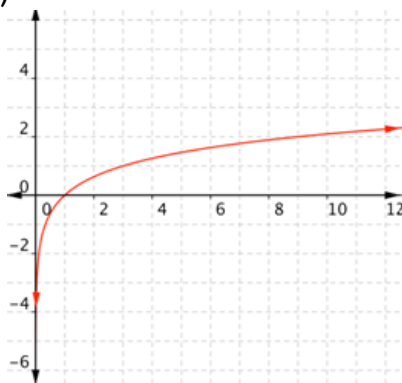
Answer
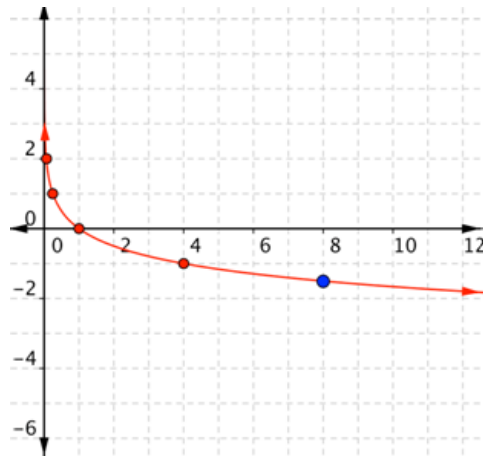
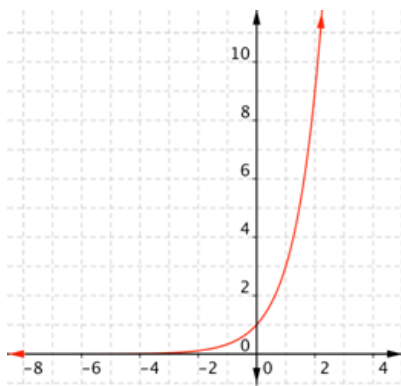

Connect the points as best you can, using a *smooth curve*.

If the base is less than 1, the logarithmic function is decreasing. The graph gets close to the y-axis when x is small, but with *positive* y values instead of negative ones.

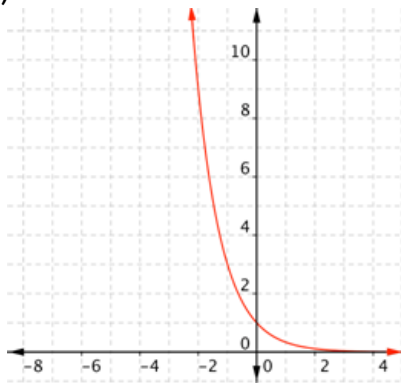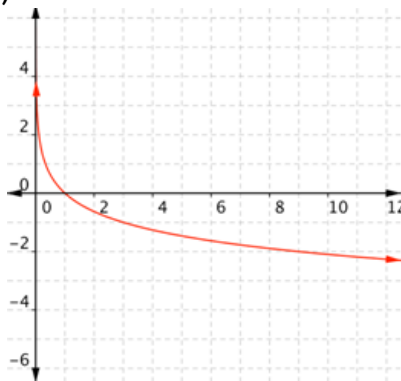Which of the following is a graph for $f(x) = \log_{\frac{1}{3}} x$?

A)

B)

C)

D)

+ Show/Hide Answer

Solving Logarithmic Equations

As you've seen, there are three essential quantities in a logarithmic equation $y = \log_b x$: the base b, the exponent y, and the input x. Any of these could be missing in an equation you need to solve. Often the easiest way to solve these is to convert the logarithmic equation to an exponential equation.

	Example	
	Example	
Problem	Solve $4 = \log_5 x$.	
	$4 = \log_5 x$ means $5^4 = x$	Convert the logarithmic

	equation to an exponential equation.
$$5^4 = 5 \cdot 5 \cdot 5 \cdot 5$$ $$5^4 = 25 \cdot 25$$ $$5^4 = 625$$	In this case, you only need to evaluate 5^4 to solve for x.
Answer $x = 625$	

When solving for b or y, it's helpful to know a lot of exponential relationships, such as $5^3 = 125$ and $2^5 = 32$, off the top of your head. However, even if you don't, you can do some thinking to approximate, or even find exactly, the solutions.

Example	
Problem	Solve $3 = \log_b 64$.

$3 = \log_b 64$ means $b^3 = 64$	Convert the logarithmic equation to an exponential equation.
	You may not know what number to the third power is 64. You can try a few and zero in on it quickly, though.
	1^3 and 10^3 are easy to calculate, so let's start there.
$$1^3 = 1$$ $$10^3 = 1,000$$	64 is much closer to 1 than to $1,000$, so the correct base will be closer to 1 than to 10.
	Let's try some other low values. Since 64 is even, you know you don't need to try any *odd* numbers. An odd number to any power will always be odd, and an even number to any power will always be even.
$$2^3 = 8$$ $$4^3 = 4 \cdot 16 = 64$$	So let's try 2 and 4.
Answer $b = 4$	Found it! Be sure you are providing the correct

number for your answer. It would be easy to mistakenly answer "**64**" here!

Let's try one more:

Example		
Problem	Solve $y = \log_5 125$.	
	$y = \log_5 125$ means $5^y = 125$	Convert the logarithmic equation to an exponential equation.
	$\begin{array}{c\|c} y & 5^y \\ \hline 1 & 5 \\ 2 & 25 \\ 3 & 125 \end{array}$	What power of 5 is 125? You might know, but if you don't, make a table of values for y and 5^y. Look for 125, or close to it, in the 5^y column.
Answer	$y = 3$	Again, be sure you provide the correct number. It would be easy to make a mistake and write 125 here!

Solve $4 = \log_2 x$ for x.

A) $\frac{1}{2}$

B) 2

C) 8

D) 16

+ Show/Hide Answer

Summary

Logarithmic functions are closely related to exponential equations. In fact, $\log_b x = y$ says the same thing as $b^y = x$. The graphs look similar, but with characteristics such as the domain and range switched between the x values and

the y values. Converting a logarithmic equation to the equivalent exponential equation is helpful with both graphing and solving equations.

Continue