Solving Radical Equations

Learning Objectives

- Solve equations containing radicals.
- Recognize extraneous solutions.
- Solve application problems that involve radical equations as part of the solution.

Introduction

An equation that contains a **radical expression** is called a **radical equation**. Solving radical equations requires applying the rules of exponents and following some basic algebraic principles. In some cases, it also requires looking out for errors generated by raising unknown quantities to an even power.

Squaring Both Sides

A basic strategy for solving radical equations is to isolate the radical term first, and then raise both sides of the equation to a power to remove the radical. (The reason for using powers will become clear in a moment.) This is the same type of strategy you used to solve other, non-radical equations: rearrange the expression to isolate the variable you want to know, and then solve the resulting equation.

There are two key ideas that you will be using to solve radical equations. The first is that if $a = b$, then $a^2 = b^2$. (This property allows you to square both sides of an equation and remain certain that the two sides are still equal.) The second is that if the square root of any nonnegative number x is squared, then you get x: $\left(\sqrt{x}\right)^2 = x$. (This property allows you to "remove" the radicals from your equations.)

Let's start with a radical equation that you can solve in a few steps: $\sqrt{x} - 3 = 5$.

Example

Problem	**Solve.** $\sqrt{x} - 3 = 5$	
	$\begin{array}{r} \sqrt{x}-3= \;\; 5 \\ +3 \quad +3 \\ \hline \end{array}$	Add 3 to both sides to isolate the variable term on the left side of the equation.
	$\sqrt{x} = 8$	Collect like terms.
	$\left(\sqrt{x}\right)^2 = 8^2$ $x = 64$	Square both sides to remove the radical, since $\left(\sqrt{x}\right)^2 = x$. Make sure to square the 8 also! Then simplify.
Answer	$x = 64$ is the solution to $\sqrt{x} - 3 = 5$.	

To check your solution, you can substitute 64 in for x in the original equation. Does $\sqrt{64} - 3 = 5$? Yes—the square root of 64 is 8, and $8 - 3 = 5$.

Notice how you combined like terms and then squared both *sides* of the equation in this problem. This is a standard method for removing a radical from an equation. It is important to isolate a radical on one side of the equation and simplify as much as possible *before* squaring. The fewer terms there are before squaring, the fewer additional terms will be generated by the process of squaring.

In the example above, only the variable x was underneath the radical. Sometimes you will need to solve an equation that contains multiple terms underneath a radical. Follow the same steps to solve these, but pay attention to a critical point: square both *sides* of an equation, not individual *terms*. Watch how the next two problems are solved.

	Example
Problem	**Solve.** $\sqrt{x+8}=3$
$\left(\sqrt{x+8}\right)^2=(3)^2$	Notice how the radical contains a binomial: $x+8$. Square both sides to remove the radical.
$x+8=9$ $x=1$	$\left(\sqrt{x+8}\right)^2=x+8$. Now simplify the equation and solve for x.
$\sqrt{1+8}=3$ $\sqrt{9}=3$ $3=3$	Check your answer. Substituting 1 for x in the original equation yields a true statement, so the solution is correct.
Answer	$x=1$ is the solution to $\sqrt{x+8}=3$.

	Example
Problem	**Solve.** $1+\sqrt{2x+3}=6$
$1+\sqrt{2x+3}-1=6-1$ $\sqrt{2x+3}=5$ $\left(\sqrt{2x+3}\right)^2=(5)^2$	Begin by subtracting 1 from both sides in order to isolate the radical term. Then square both sides to remove the binomial from the radical.
$2x+3=25$ $2x=22$ $x=11$	Simplify the equation and solve for x.

$$\begin{aligned} 1+\sqrt{2(11)+3} &= 6 \\ 1+\sqrt{22+3} &= 6 \\ 1+\sqrt{25} &= 6 \\ 1+5 &= 6 \\ 6 &= 6 \end{aligned}$$

Check your answer. Substituting 11 for x in the original equation yields a true statement, so the solution is correct.

Answer $x=11$ is the solution for $1+\sqrt{2x+3}=6$.

Solving Radical Equations

Follow the four steps to solve radical equations.

1. Isolate the radical expression.
2. Square both sides of the equation: If $x=y$ then $x^2=y^2$.
3. Once the radical is removed, solve for the unknown.
4. Check all answers.

Solve. $\sqrt{3x+22}=4$

A) $x=2$

B) $x=\frac{16}{3}$

C) $x=-2$

D) $x=-6$

+ Show/Hide Answer

Extraneous Solutions

Following rules is important, but so is paying attention to the math in front of you, especially when solving radical equations. Take a look at this next problem that demonstrates a potential pitfall of squaring both sides to remove the radical.

Example

Problem	**Solve.** $\sqrt{a-5}=-2$	
	$\left(\sqrt{a-5}\right)^2=(-2)^2$	Square both sides to remove the term $a-5$ from the radical.
	$a-5=4$ $a=9$	Write the simplified equation, and solve for a.
	$\sqrt{9-5}=-2$ $\sqrt{4}=-2$ $2\neq-2$	Now check the solution by substituting $a=9$ into the original equation. It does not check!
Answer	No solution.	

The answer $a=9$ does not produce a true statement when substituted back into the original equation. What happened?

Check the original problem: $\sqrt{a-5}=-2$. Notice that the radical is set equal to -2, and recall that the principal square root of a number can only be *positive*. This means that no value for a will result in a radical expression whose positive square root is -2! You might have noticed that right away and concluded that there were no solutions for a. But why did the process of squaring create an answer, $a=9$, that proved to be incorrect?

The answer lies in the process of *squaring* itself. When you raise a number to an even power, whether it is the second, fourth, or 50^{th} power, you can introduce a false solution because the result of an even power is always a positive number. Think about it: 3^2 and $(-3)^2$ are both 9, and 2^4 and $(-2)^4$ are both 16. So when you squared -2 and got 4 in this problem, you artificially turned the quantity positive. This is why you were still able to find a value for a; you solved the problem as if you were solving $\sqrt{a-5}=2$! (The correct solution to $\sqrt{a-5}=-2$ is actually "no solution.")

Incorrect values of the variable, such as those that are introduced as a result of the squaring process are called **extraneous solutions**. Extraneous solutions may look like the real solution, but you can identify them because they will not create a true statement when substituted back into the original equation. This is one of the reasons why checking your work is so important. If you do not check your answers by substituting them back into the original equation, you may be introducing extraneous solutions into the problem.

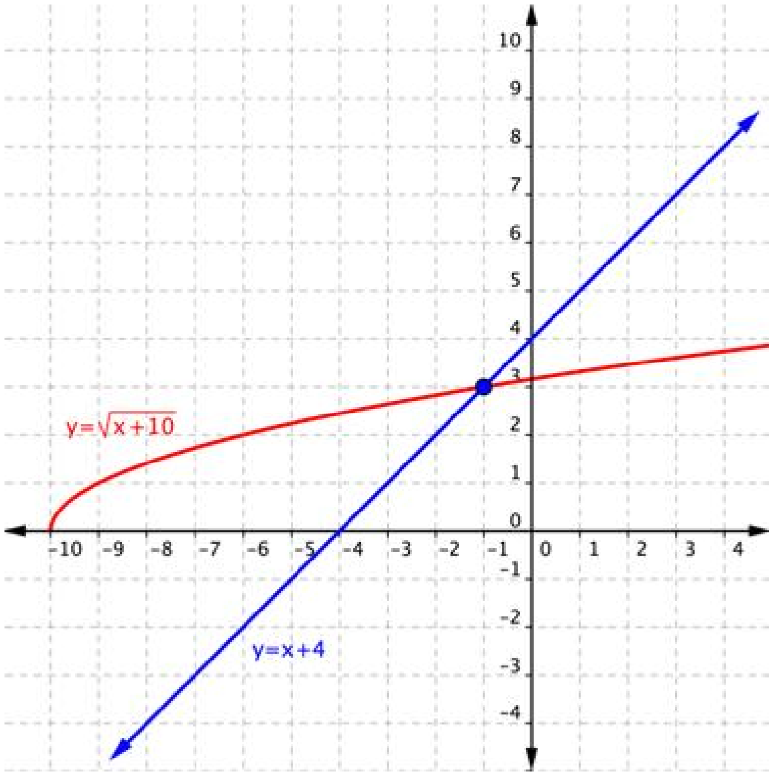
Have a look at the following problem. Notice how the original problem is $x+4=\sqrt{x+10}$, but after both sides are squared, it becomes $x^2+8x+16=x+10$. Squaring both sides may have introduced an extraneous solution.

Problem	**Example** **Solve.** $x+4=\sqrt{x+10}$	
	$(x+4)^2=\left(\sqrt{x+10}\right)^2$	Square both sides to remove the term $x+10$ from the radical.
	$(x+4)(x+4)=x+10$ $x^2+8x+16=x+10$ $x^2+8x-x+16-10=0$ $x^2+7x+6=0$	Now simplify and solve the equation. Combine

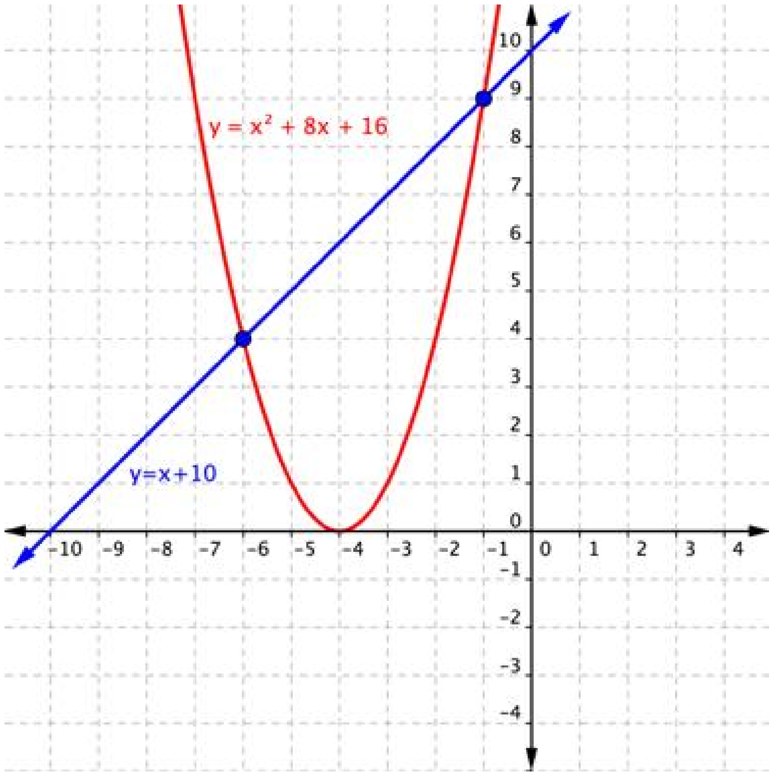
$(x+6)(x+1)=0$	like terms, and then factor.
$(x+6)=0$ or $(x+1)=0$ $x=-6$ or $x=-1$	Set each factor equal to zero and solve for x.
$-6+4=\sqrt{-6+10}$ $\quad$ $-1+4=\sqrt{-1+10}$ $-2=\sqrt{4}$ $\quad$ $3=\sqrt{9}$ $-2=2$ $\quad$ $3=3$ FALSE! $\quad$ TRUE!	Now check both solutions by substituting them into the original equation. Since $x=-6$ produces a false statement, it is an extraneous solution.
Answer	$x=-1$ is the only solution

It may be difficult to understand why extraneous solutions exist at all. Thinking about extraneous solutions by graphing the equation may help you make sense of what is going on.

You can graph $x+4=\sqrt{x+10}$ on a coordinate plane by breaking it into a system of two equations: $y=x+4$ and $y=\sqrt{x+10}$. The graph is shown below. Notice how the two graphs intersect at one point, when the value of x is -1. This is the value of x that satisfies both equations, so it is the solution to the system.



Now, following the work we did in the example problem, let's square both of the expressions to remove the variable from the radical. Instead of solving the equation $x+4=\sqrt{x+10}$ we are now solving the equation $(x+4)^2=\left(\sqrt{x+10}\right)^2$, or $x^2+8x+16=x+10$. The graphs of $y=x^2+8x+16$ and $y=x+10$ are plotted below. Notice how the two graphs intersect at two points, when the values of x are -1 and -6.



Although $x = -1$ is shown as a solution in both graphs, squaring both sides of the equation had the effect of adding an extraneous solution, $x = -6$. Again, this is why it is so important to check your answers when solving radical equations!

	Example	
Problem	**Solve.** $4 + \sqrt{x+2} = x$	
	$\sqrt{x+2} = x - 4$	Isolate the radical term.
	$\left(\sqrt{x+2}\right)^2 = (x-4)^2$	Square both sides to remove the term $x+2$ from the radical.
	$\begin{aligned} x + 2 &= x^2 - 8x + 16 \\ 0 &= x^2 - 8x - x + 16 - 2 \\ 0 &= x^2 - 9x + 14 \\ 0 &= (x-7)(x-2) \end{aligned}$	Now simplify and solve the equation. Combine like

terms, and then factor.

$$(x-7)=0 \quad \text{or} \quad (x-2)=0$$
$$x=7 \quad \text{or} \quad x=2$$

Set each factor equal to zero and solve for x.

$$4+\sqrt{7+2}=7 \qquad 4+\sqrt{2+2}=2$$
$$4+\sqrt{9}=7 \qquad 4+\sqrt{4}=2$$
$$4+3=7 \qquad 4+2=2$$
$$7=7 \qquad 6=2$$

TRUE! FALSE!

Now check both solutions by substituting them into the original equation.

Since $x=2$ produces a false statement, it is an extraneous solution.

Answer $x=7$ is the only solution.

Solve. $x-3=\sqrt{4x+9}$

A) $x=3,0$

B) $x=0,10$

C) $x=0$

D) $x=10$

+ Show/Hide Answer

Solving Application Problems with Radical Equations

Radical equations play a significant role in science, engineering, and even music. Sometimes you may need to use what you know about radical equations to solve for different variables in these types of problems.

Example	
Problem	**One way to measure the amount of energy that a moving object (such as a car) possesses is by finding its Kinetic Energy. The Kinetic Energy (E_k, measured in Joules) of an object depends on the object's mass (m, measured in kg) and velocity (v, measured in meters per second) and can be written as $v=\sqrt{\frac{2E_k}{m}}$.** **What is the Kinetic Energy of an object with a mass of $1,000$ kilograms that is traveling at 30 meters per second?**
$E_k = \text{unknown}$ $m = 1000$ $v = 30$	Identify variables and known values.
$30=\sqrt{\frac{2E_k}{1{,}000}}$	Substitute values into the formula.
$(30)^2=\left(\frac{\sqrt{2E_k}}{1{,}000}\right)^2$ $900=\frac{2E_k}{1{,}000}$ $900\cdot 1,000=\frac{2E_k}{1{,}000}\cdot 1,000$ $900,000=2E_k$	Solve the radical equation for E_k.

$$\frac{900{,}000}{2} = \frac{2E_k}{2}$$
$$450{,}000 = E_k$$

$$30 = \sqrt{\frac{2 \cdot 450{,}000}{1{,}000}}$$
$$30 = \sqrt{\frac{900{,}000}{1{,}000}}$$
$$30 = \sqrt{900}$$
$$30 = 30$$

Now check the solution by substituting it into the original equation.

Answer The Kinetic Energy is $450,000$ Joules.

Summary

A common method for solving radical equations is to raise both sides of an equation to whatever power will eliminate the radical sign from the equation. But be careful: when both sides of an equation are raised to an *even* power, the possibility exists that extraneous solutions will be introduced. When solving a radical equation, it is important to always check your answer by substituting the value back into the original equation. If you get a true statement, then that value is a solution; if you get a false statement, then that value is not a solution.

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