A brief introduction to Euclidean Geometry

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Table of Contents

Isometries of the Euclidean plane

(Standard) inner product

Definition ((Standard) inner product)

The (standard) inner product on $\mathbb{R}^n$ *is defined by*

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

Euclidean Norm

Definition (Euclidean Norm)

The Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$ *is*

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}.$$

This defines a metric on $\mathbb{R}^n$ *by*

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Isometry

Definition (Isometry)

A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an <u>isometry</u> of $\mathbb{R}^n$ if

$$d(f(\mathbf{x}), f(\mathbf{y})) = d(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Orthogonal matrix

Definition (Orthogonal matrix)

An $n \times n$ matrix A is orthogonal if $AA^T = A^TA = I$. The group of all orthogonal matrices is the orthogonal group $\mathrm{O}(n)$.

Theorem of Orthogonal

Theorem

Every isometry of $f : \mathbb{R}^n \to \mathbb{R}^n$ is of the form

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}.$$

for A orthogonal and $\mathbf{b} \in \mathbb{R}^n$.

Proof

Let f be an isometry. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$. Let

$$\mathbf{b} = f(\mathbf{0}), \quad \mathbf{a}_i = f(\mathbf{e}_i) - \mathbf{b}.$$

The idea is to construct our matrix A out of these $\mathbf{a}_i$. For A to be orthogonal, $\{\mathbf{a}_i\}$ must be an orthonormal basis.

Proof

Indeed, we can compute

$$\|\mathbf{a}_i\| = \|\mathbf{f}(\mathbf{e}_i) - f(\mathbf{0})\| = d(f(\mathbf{e}_i), f(\mathbf{0})) = d(\mathbf{e}_i, \mathbf{0}) = \|\mathbf{e}_i\| = 1.$$

For $i \neq j$, we have

$$\begin{aligned}
(\mathbf{a}_i, \mathbf{a}_j) &= -(\mathbf{a}_i, -\mathbf{a}_j) \\
&= -\frac{1}{2}(\|\mathbf{a}_i - \mathbf{a}_j\|^2 - \|\mathbf{a}_i\|^2 - \|\mathbf{a}_j\|^2) \\
&= -\frac{1}{2}(\|f(\mathbf{e}_i) - f(\mathbf{e}_j)\|^2 - 2) \\
&= -\frac{1}{2}(\|\mathbf{e}_i - \mathbf{e}_j\|^2 - 2) \\
&= 0
\end{aligned}$$

So $\mathbf{a}_i$ and $\mathbf{a}_j$ are orthogonal. In other words, $\{\mathbf{a}_i\}$ forms an orthonormal set. It is an easy result that any orthogonal set must be linearly independent. Since we have found n orthonormal vectors, they form an orthonormal basis.

Proof

Hence, the matrix A with columns given by the column vectors $\mathbf{a}_i$ is an orthogonal matrix. We define a new isometry

$$g(\mathbf{x}) = A\mathbf{x} + \mathbf{b}.$$

We want to show $f = g$. By construction, we know $g(\mathbf{x}) = f(\mathbf{x})$ is true for $\mathbf{x} = \mathbf{0}, \mathbf{e}_1, \cdots, \mathbf{e}_n$. We observe that g is invertible. In particular,

$$g^{-1}(\mathbf{x}) = A^{-1}(\mathbf{x} - \mathbf{b}) = A^T\mathbf{x} - A^T\mathbf{b}.$$

Moreover, it is an isometry, since A^T is orthogonal (or we can appeal to the more general fact that inverses of isometries are isometries).

Proof

We define

$$h = g^{-1} \circ f.$$

Since it is a composition of isometries, it is also an isometry. Moreover, it fixes $\mathbf{x} = \mathbf{0}, \mathbf{e}_1, \cdots, \mathbf{e}_n$.
It currently suffices to prove that h is the identity.
Let $\mathbf{x} \in \mathbb{R}^n$, and expand it in the basis as

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i.$$

Let

$$\mathbf{y} = h(\mathbf{x}) = \sum_{i=1}^{n} y_i \mathbf{e}_i.$$

Proof

We can compute

$$
\begin{aligned}
d(\mathbf{x}, \mathbf{e}_i)^2 &= (\mathbf{x} - \mathbf{e}_i, \mathbf{x} - \mathbf{e}_i) = \|\mathbf{x}\|^2 + 1 - 2x_i \\
d(\mathbf{x}, \mathbf{0})^2 &= \|\mathbf{x}\|^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
d(\mathbf{y}, \mathbf{e}_i)^2 &= (\mathbf{y} - \mathbf{e}_i, \mathbf{y} - \mathbf{e}_i) = \|\mathbf{y}\|^2 + 1 - 2y_i \\
d(\mathbf{y}, \mathbf{0})^2 &= \|\mathbf{y}\|^2.
\end{aligned}
$$

Since h is an isometry and fixes $\mathbf{0}, \mathbf{e}_1, \cdots, \mathbf{e}_n$, and by definition $h(\mathbf{x}) = \mathbf{y}$, we must have

$$
d(\mathbf{x}, \mathbf{0}) = d(\mathbf{y}, \mathbf{0}), \quad d(\mathbf{x}, \mathbf{e}_i) = d(\mathbf{y}, \mathbf{e}_i).
$$

The first equality gives $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2$, and the others then imply $x_i = y_i$ for all i. In other words, $\mathbf{x} = \mathbf{y} = h(\mathbf{x})$. So h is the identity.

Isometry group

Definition (Isometry group)

The isometry group $\mathrm{Isom}(\mathbb{R}^n)$ *is the group of all isometries of* $\mathbb{R}^n$*, which is a group by composition.*

Special orthogonal group

Definition (Special orthogonal group)

The *special orthogonal group* *is the group*

$$\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \det A = 1\}.$$

Orientation

Definition (Orientation)

An orientation of a vector space is an equivalence class of bases — let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ *and* $\mathbf{v}'_1, \cdots, \mathbf{v}'_n$ *be two bases and* A *be the change of basis matrix. We say the two bases are equivalent iff* $\det A > 0$*. This is an equivalence relation on the bases, and the equivalence classes are the orientations.*

Definition (Orientation-preserving isometry)

An isometry $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ *is orientation-preserving if* $\det A = 1$*. Otherwise, if* $\det A = -1$*, we say it is orientation-reversing.*

Curves in $\mathbb{R}^n$

Curve

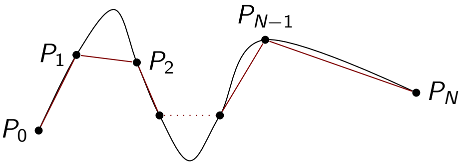
Definition (Curve)

A curve Γ *in* $\mathbb{R}^n$ *is a continuous map* $\Gamma : [a, b] \to \mathbb{R}^n$.

Length of curve

Considering a dissection $\mathcal{D} = a = t_0 < t_1 < \cdots < t_N = b$ of $[a, b]$, and set $P_i = \Gamma(t_i)$, and define

$$\int_a^b \|\Gamma'(t)\| \,\mathrm{d}t,$$



Length of curve

Definition (Length of curve)

The length of a curve $\Gamma : [a, b] \to \mathbb{R}^n$ *is*

$$\ell = \sup_{\mathcal{D}} S_{\mathcal{D}},$$

if the supremum exists.

Alternatively, if we let

$$\mathrm{mesh}(\mathcal{D}) = \max_i (t_i - t_{i-1}),$$

then if ℓ exists, then we have

$$\ell = \lim_{\mathrm{mesh}(\mathcal{D}) \to 0} s_{\mathcal{D}}.$$

How to calculate length of curve?

Proposition

If Γ is continuously differentiable (i.e. C^1), then the length of Γ is given by

$$\text{length}(\Gamma) = \int_a^b \|\Gamma'(t)\| \, \mathrm{d}t.$$

Proof

To simplify notation, we assume $n = 3$. However, the proof works for all possible dimensions. We write

$$\Gamma(t) = (f_1(t), f_2(t), f_3(t)).$$

For every $s \neq t \in [a, b]$, the mean value theorem tells us

$$\frac{f_i(t) - f_i(s)}{t - s} = f_i'(\xi_i)$$

for some $\xi_i \in (s, t)$, for all $i = 1, 2, 3$.

Proof

Now note that f_i' are continuous on a closed, bounded interval, and hence uniformly continuous. For all $\varepsilon \in 0$, there is some $\delta > 0$ such that $|t - s| < \delta$ implies

$$|f_i'(\xi_i) - f'(\xi)| < \frac{\varepsilon}{3}$$

for all $\xi \in (s, t)$. Thus, for any $\xi \in (s, t)$, we have

$$\left\| \frac{\Gamma(t) - \Gamma(s)}{t - s} - \Gamma'(\xi) \right\| = \left\| \begin{pmatrix} f_1'(\xi_1) \\ f_2'(\xi_2) \\ f_3'(\xi_3) \end{pmatrix} - \begin{pmatrix} f_1'(\xi) \\ f_2'(\xi) \\ f_3'(\xi) \end{pmatrix} \right\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

In other words,

$$\|\Gamma(t) - \Gamma(s) - (t - s)\Gamma'(\xi)\| \leq \varepsilon(t - s).$$

We relabel $t = t_i$, $s = t_{i-1}$ and $\xi = \frac{s+t}{2}$.

Proof

Using the triangle inequality, we have

$$(t_i - t_{i-1}) \left\| \Gamma' \left(\frac{t_i + t_{i-1}}{2} \right) \right\| - \varepsilon (t_i - t_{i-1}) < \| \Gamma(t_i) - \Gamma(t_{i-1}) \| < (t_i - t_{i-1}) \left\| \Gamma' \left(\frac{t_i + t_{i-1}}{2} \right) \right\| + \varepsilon (t_i - t_{i-1}).$$

Summing over all i, we obtain

$$\sum_i (t_i - t_{i-1}) \left\| \Gamma' \left(\frac{t_i + t_{i-1}}{2} \right) \right\| - \varepsilon (b - a) < S_{\mathcal{D}} < \sum_i (t_i - t_{i-1}) \left\| \Gamma' \left(\frac{t_i + t_{i-1}}{2} \right) \right\| + \varepsilon (b - a),$$

which is valid whenever $\mathrm{mesh}(\mathcal{D}) < \delta$.

Proof

Since Γ' is continuous, and hence integrable, we know

$$\sum_i (t_i - t_{i-1}) \left\| \Gamma'\left(\frac{t_i + t_{i-1}}{2}\right) \right\| \to \int_a^b \|\Gamma'(t)\| \,\mathrm{d}t$$

as $\mathrm{mesh}(\mathcal{D}) \to 0$, and

$$\text{length}(\Gamma) = \lim_{\mathrm{mesh}(\mathcal{D}) \to 0} S_{\mathcal{D}} = \int_a^b \|\Gamma'(t)\| \,\mathrm{d}t.$$

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